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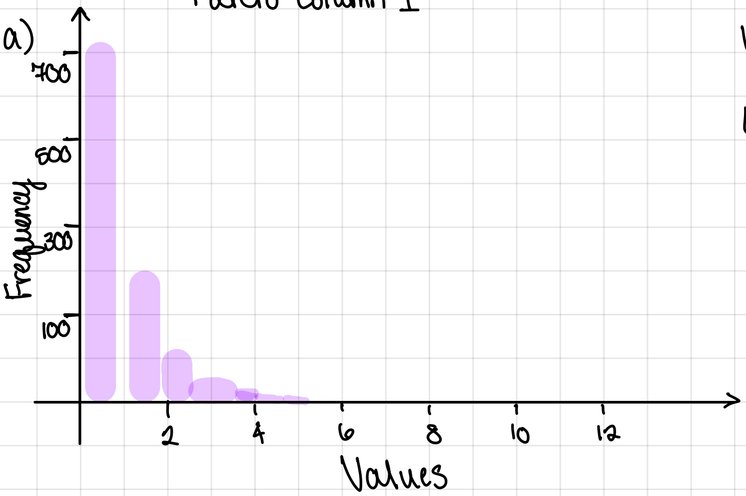
QBIO 310

1.

a) Pareto Column 1



b) MLE

$L(\alpha)=\prod_{i=1}^{n}\alpha x_i^{-(\alpha+1)}=x_i^{-(\alpha+1)} \xrightarrow{\log} \ell(\alpha)=n\ln\alpha-(\alpha+1)\sum_{i=1}^{n}\ln x_i$

$\hookrightarrow \frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln x_i=0 \rightarrow \hat{\alpha}_{MLE}=\frac{n}{\sum_{i=1}^{n}\ln x_i}$

$\sum\ln x_i=\text{logsum}=\frac{n}{\alpha_{MLE}}=\frac{1000}{2.972494}\approx 336.594$ ✓

c) $\ln f(x_i)=\ln\alpha-(\alpha+1)\ln x_i \rightarrow \frac{\partial\ln f}{\partial\alpha^2}=\frac{1}{\alpha}-\ln x \rightarrow \frac{\partial^2\ln f}{\partial\alpha^2}=-\frac{1}{\alpha^2} \rightarrow I(\alpha)=\frac{\overset{=1000}{n}}{\alpha^2}=\frac{1000}{\alpha^2}$

$\hookrightarrow I(2.972494)=\frac{1000}{(2.972494)^2}\approx 113.181$

Standard error: $SE(\hat{\alpha}_{MLE})=\sqrt{\frac{1}{I(\hat{\alpha}_{MLE})}}=\frac{\hat{\alpha}_{MLE}}{\sqrt{n}}=\frac{2.972494}{31.622778}\approx 0.09397$

$CI=2.972494\pm 1.96\cdot 0.09397$
$\hookrightarrow 2.972494\pm 0.184182$
$\approx(2.788312, 3.156676)$

d) $E[X]=\frac{\alpha}{\alpha-1} \rightarrow \bar{x}=\frac{\hat{\alpha}_{MOM}}{\hat{\alpha}_{MOM}-1} \rightarrow \bar{x}\hat{\alpha}_{MOM}-\hat{\alpha}_{MOM}=\hat{\alpha}_{MOM} \rightarrow \hat{\alpha}_{MOM}=\frac{\bar{x}}{\bar{x}-1}$

$\hat{\alpha}_{MOM}=1.509695=\frac{\bar{x}}{\bar{x}-1} \rightarrow \bar{x}(1-1.509695)=-1.509695\approx 1.509695$

e) $\hat{\alpha}_{boot}=\frac{\bar{x}^*}{\bar{x}^*-1} \rightarrow CI_{boot}=(\hat{\alpha}_{boot}, 0.025, \hat{\alpha}_{boot}, 0.975) \rightarrow (1.482697, 1.537991)$

f) $\alpha=0.5$ $\quad z=\frac{0.4933142-0.5}{0.015597}\approx\frac{-0.0066858}{0.015597}\approx -0.4286$ $\quad p=2\cdot P(Z>0.4286)\approx 2\cdot 0.3341=0.6682$

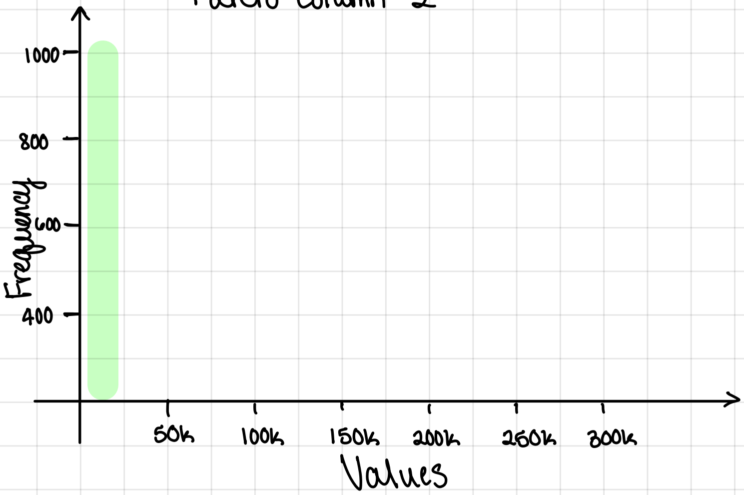
g) $\ell(\hat{\alpha}_{MLE})=n\ln\hat{\alpha}_{MLE}-(\hat{\alpha}_{MLE}+1)\cdot\frac{n}{\hat{\alpha}_{MLE}} \rightarrow \ln(0.4933142)\approx -0.7067$

$\hookrightarrow \ell(0.5)\approx 1000\cdot(-0.693147)-1.5\cdot 2027.65\approx -693.147-3041.475\approx -3734.622$

$LRT=2[-3734.35-(-3734.622)]=2\cdot 0.272\approx 0.544$ $\quad p=P(\chi_1^2>0.544)\approx 0.4607$

2.

Pareto Column 2



b) MLE for a

$\hat{\alpha}_{MLE}=\frac{n}{\sum_{i=1}^{n}\ln x_i} \rightarrow \frac{1000}{2027.45}=0.4933$

c) Fischer Info * 95% CI

$I(\alpha)=n/\alpha^2 \rightarrow I(0.4933)=\frac{1000}{0.4933^2}\approx 4107.87$

Standard error: $0.4933/\sqrt{4107.87}\approx 0.00768$

CI 95% $\hat{\alpha}_{MLE}\pm 1.96 * SE \rightarrow 0.4933\pm 1.96(0.00768)$

$\hookrightarrow (0.4782, 0.5084)$

d) MOM Estimator

$\hat{\alpha}_{MOM}=\frac{\bar{x}}{\bar{x}-1} \rightarrow 2.0033/2.0033-1\approx 1.0017$

e) Bootstrap $\rightarrow$ 1.0007 & 1.0070

f) Wald-test $\alpha=0.5$

$$Z=\frac{\hat{\alpha}_{MLE}-0.5}{SE} \rightarrow \frac{0.4933-0.5}{0.00768} \approx -0.8688$$

g) Likelihood Ratio Test $\alpha=0.5$

$$\log L(\alpha=0.5)=1000\log(0.5)-1.5\cdot\sum\ln x_i$$
$$\log L(\hat{\alpha})=1000\log(0.4933)-1.4933\cdot\sum\ln x_i$$
$$2(\log L(\hat{\alpha})-\log L(0.5))\approx 0.182$$
$$p=P(\chi^2_1>0.182)\approx 0.670$$

3. rv $X\sim NegBin(r,p)$ $x_i\ldots$

a) MLE of p

$$L(p)=\prod_{i=1}^{n}\binom{k_i+r-1}{k_i}(1-p)^{k_i}p^r$$

log it

$$\log L(p)=\sum_{i=1}^{n}\left[\log\binom{k_i+r-1}{k_i}+k_i\log(1-p)+r\log(p)\right]$$
$$\hookrightarrow \log L(p)=\text{const}+\left(\sum_{i=1}^{n}k_i\right)\log(1-p)+nr\log(p)$$

when $S=\sum_{i=1}^{n}k_i$ then $\log L(p)=\text{const}+S\log(1-p)+nr\log(p)$

derivative: $\frac{d}{dp}\log L(p)=\frac{-S}{1-p}+\frac{nr}{p}=\emptyset \rightarrow -\frac{S}{1-p}+\frac{nr}{p}=\emptyset \rightarrow \frac{nr}{p}=\frac{S}{1-p}$

$nr(1-p)=Sp \rightarrow nr-nrp=Sp \rightarrow nr=p(nr+S) \rightarrow \hat{p}=\frac{nr}{nr+\sum k_i}$

$$\hat{p}=\frac{r}{r+\bar{k}}$$

b)

$$\log L(p)=\text{const}+S\log(1-p)+nr\log(p)$$
$$\hookrightarrow \frac{d^2}{dp^2}\log L(p)=-\frac{S}{(1-p)^2}-\frac{nr}{p^2}$$

Fischer info is neg of 2nd der.

$I(p)=E\left[-\frac{d^2}{dp^2}\log L(p)\right] \rightarrow E[S]=\sum E[X_i]=n\cdot\frac{r(1-p)}{p}$

$$\hookrightarrow I(p)=\frac{nr(1-p)}{p^2}+\frac{nr}{p^2}=\frac{nr}{p^2(1-p)}$$

4. Discrete Uniform $(1,k)$

a) MOM Estimator

$E[X]=(k+1)/2$ $\quad \bar{x}=(k+1)/2 \rightarrow k=2\bar{x}-1$ $\quad \hat{k}_{MOM}=2\bar{x}-1$

b) $L(k)=(1/k)^n$ for $k\geq\max(x_i\ldots)$

based on the PMF and likelihood function is a decreasing fxn of k, meaning likelihood gets smaller as k gets larger

if $k<$ max observed value then likelihood $=\emptyset$

if $k>$ mov then likelihood exists but is smaller than necessary... max likelihood occurs @ smallest possible value of k that still makes all observed values possible $\rightarrow$ $\hat{k}_{MLE}=\max(x_i\ldots)$