CS170 Computation Theory

Lecture 8

September 28, 2023

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Review of Last Lecture

- Notation for Encodings and TMs
- Decision procedures for DFAs

Today's Topics

- ~~Notation for Encodings and TMs~~
- ~~Decision procedures for DFAs~~

- Decision procedures for CFGs
- The reducibility method

Recall: T-recognizability and Decidability

Definition (p. 170): Let M be a TM. Let $A = \{w \mid M \text{ accepts } w\}$. Then, A is the language recognized by M, i.e., $A = L(M)$.

Definition (p. 170): A language is T-recognizable if there is a TM that recognizes it.

Definition (p. 170): A TM M is a decider if it halts on all inputs.

Definition (p. 170): A language A is T-decidable (or just decidable) if $A = L(M)$ for some TM decider M.

Acceptance Problem for CFGs

Theorem (p.198): The language $A_{\text{CFG}} = \{\langle G, w \rangle \mid G \text{ is a CFG and } w \in L(G)\}$ is decidable.

Proof Attempt 1: Let $D'_{A_{\text{CFG}}}$ be the following TM.

$D'_{A_{\text{CFG}}}$ = "On input $\langle B, w \rangle$ where B is a CFG and w is a string,

1. Convert B to an equivalent PDA B' using the procedure from Lecture 4.
2. Simulate the computation of B' on w.
3. If B' ends in an accept state, *accept*. Otherwise, *reject.*"

$D'_{A_{\text{CFG}}}$ decides A_{CFG}.

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(ϵ-transitions make argument tricky.)

Chomsky Normal Form

Definition (p.109): A CFG is in Chomsky normal form if every rule is:

$$T \to UV$$

$$T \to a$$

where U and V are not the start variable S. (With the one exception: $S \to \epsilon$.)

Lemma 1: Every CFL can be generated by a CFG in Chomsky normal form.

Proof in Sipser Book (p. 109).

Lemma 2

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(2) It takes $n - 1$ steps to “expand” the start variable into n variables $V_1 V_2 \ldots V_n$.

(3) For each variable V_i, it takes one step to substitute it for a terminal symbol.

Q.E.D.

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2. Try all derivations of length $2\,|\,w\,|\,-1$.

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3. If any derivation generates w, *accept*. Otherwise, *reject.*"

$D_{A_{\text{CFG}}}$ decides A_{CFG}.

Each derivation takes $O(|w|)$ sub-steps.

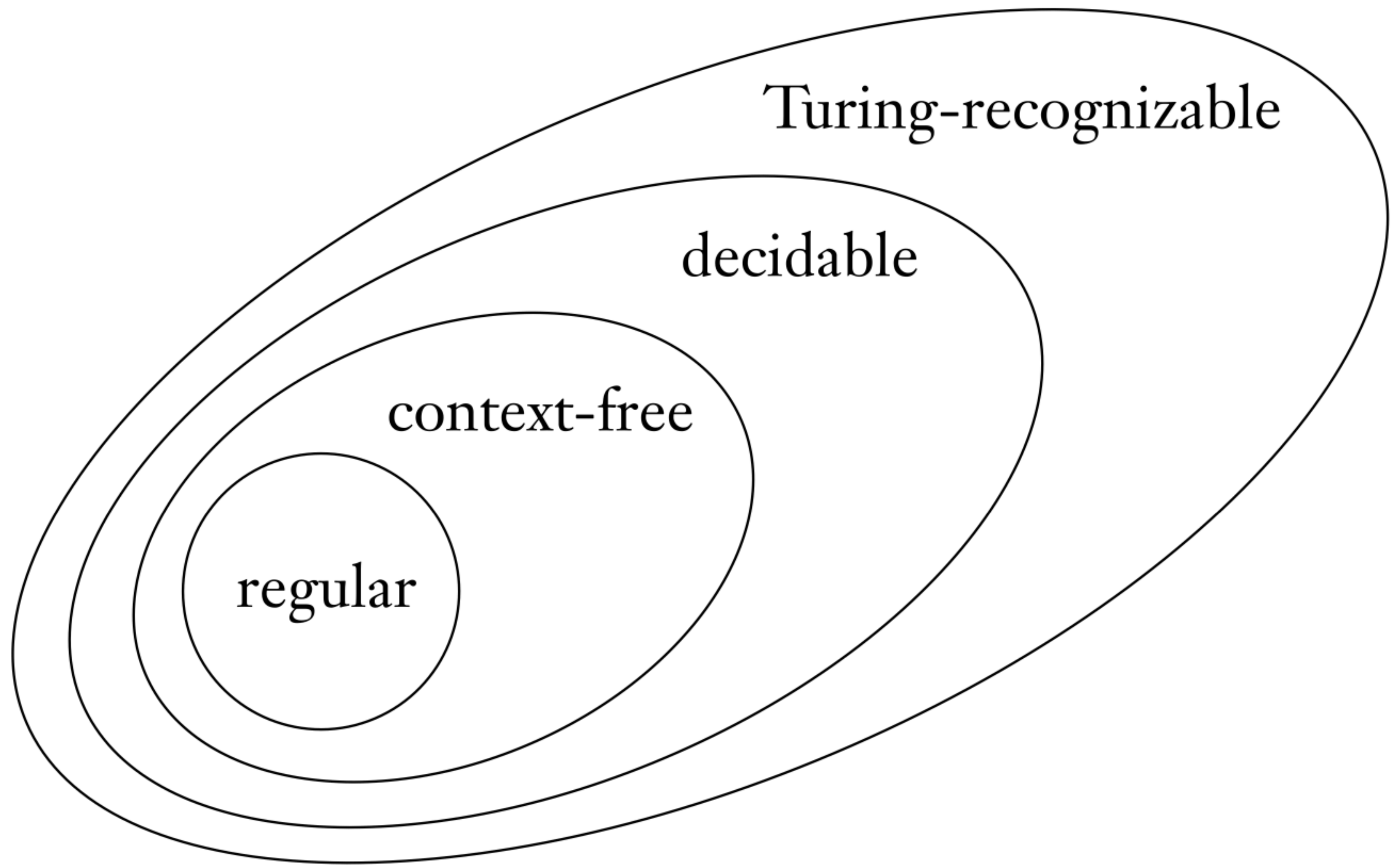
Q.E.D.

Check-In 1 (Break)

Prove that every context-free language is decidable.

(*Hint*: Consider what we just saw in the last slide: A_{CFG}.)

Check-In 1 (Break)



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1. Mark all terminals.

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1. Mark all terminals.
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$D_{E_{\text{CFG}}}$ decides E_{CFG}.

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3. If start variable is not marked, accept. Else, reject.”

$D_{E_{\text{CFG}}}$ decides E_{CFG}.

$D_{E_{\text{CFG}}}$ always halts within $O(|V|)$ sub-steps.

Q.E.D.

Check In 2 (Break)

Fake Theorem: The language
$ALL_{\text{CFG}} = \{\langle G \rangle \,|\, G$ is a CFG such that $L(G) = \Sigma^*\}$ is decidable.

Fake Proof: Let $D_{ALL_{\text{CGF}}}$ be the following TM.

$D_{ALL_{\text{CGF}}}$= "On input $\langle G \rangle$ where G is a CFG,

1. Determine the CFG $\overline{G}$ that generates the language $\overline{L(G)}$.
2. Run $D_{E_{\text{CFG}}}$ (from previous slide) on $\langle \overline{G} \rangle$.
3. If $D_{E_{\text{CFG}}}$ accepts, accept. Else, reject."

$D_{ALL_{\text{CGF}}}$ decides ALL_{CFG}.

What's wrong with this proof?

Summary Table of Decidability

	Acceptance Problem	Emptiness Problem	Equivalence Problem
DFA			
CFG			
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Roadmap for Proving Undecidable Problems

To show that a language B is undecidable by a reduction:

1. Show that there is an undecidable problem: A (next lecture).
2. Show that A reduces to B: we can decide A by using a decider for B as a subroutine.
   - **Implication:** B decidable $\Longrightarrow$ A decidable

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  (3) Construct a procedure D_A that uses D_B to decide A.

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  (4) Conclude that A is decidable.

  (5) Line (4) contradicts assumption that A is undecidable. **Q.E.D.**

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Not a legal TM action.

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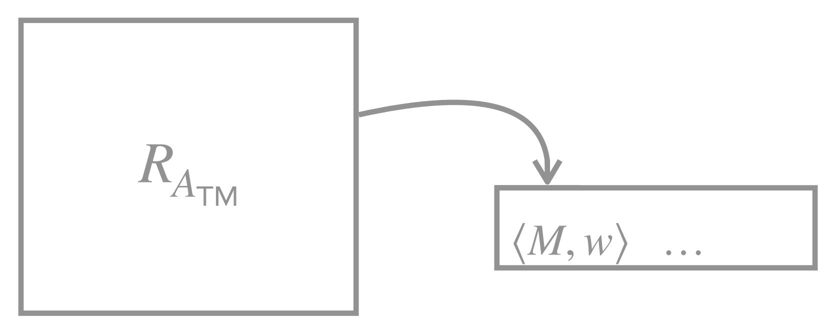

Fig. Turing's original universal TM, capable of simulating any other TM.

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Proof: Next lecture.

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Theorem (p.207): The language $A_{\mathrm{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } w \in L(M)\}$ is undecidable.

Proof: Next lecture.

Theorem (p.216): The language $HALT_{\mathrm{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and halts on input } w\}$ is undecidable.

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Proof: Next lecture.

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Proof by reduction.

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Proof: Next lecture.

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1. Run $D_{HALT_{\mathsf{TM}}}$ on $\langle M, w \rangle$.

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3. If $D_{HALT_{\text{TM}}}$ accepts, simulate M on w until it halts.
4. If M accepts, accept. Otherwise, reject."

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(1) For the sake of reaching a contradiction, assume $HALT_{\text{TM}}$ is decidable.

(2) So, there exists a decider $D_{HALT_{\text{TM}}}$ for $HALT_{\text{TM}}$.

(3) Let $D_{A_{\text{TM}}}$ be the following TM that uses $D_{HALT_{\text{TM}}}$ to decide A_{TM}:

"On input $\langle M, w \rangle$ where M is a TM and w is a string,

1. Run $D_{HALT_{\text{TM}}}$ on $\langle M, w \rangle$.
2. If $D_{HALT_{\text{TM}}}$ rejects (i.e., M doesn't halt on w), reject.
3. If $D_{HALT_{\text{TM}}}$ accepts, simulate M on w until it halts.
4. If M accepts, accept. Otherwise, reject."

(4) So, we've reached a contradiction, namely that A is decidable. **Q.E.D.**

Summary of Today's Lecture

- Decision procedures for CFGs
- The reducibility method

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- **Errata:** If you let us know of any errors in the slides, we'll fix them and acknowledge you here!