Assignment 6 Solution

Tuesday, April 9, 2024 1:44 PM

1. (5 marks) Verify that the matrix below for a linear system consists of **orthonormal vectors**.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{2}} \end{bmatrix}$$

① Here, $u_1 = \left(1/\sqrt{2} \quad 1/\sqrt{2} \quad 0\right)^T$, $u_2 = \left(-1/\sqrt{2} \quad 1/\sqrt{2} \quad 0\right)^T$, $u_3 = (0, 0, 1)^T$

Now, $u_1 \cdot u_2 = \left(\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}}\right) + \left(1/\sqrt{2} \times \frac{1}{\sqrt{2}}\right) + (0 \times 0) = -\frac{1}{2} + \frac{1}{2} + 0 = 0$

$u_2 \cdot u_3 = \left(-\frac{1}{\sqrt{2}} \times 0\right) + \left(\frac{1}{\sqrt{2}} \times 0\right) + (0 \times 1) = 0 + 0 + 0 = 0$

$u_3 \cdot u_1 = \left(0 \times \frac{1}{\sqrt{2}}\right) + \left(0 \times \frac{1}{\sqrt{2}}\right) + (1 \times 0) = 0 + 0 + 0 = 0$

Now, $|u_1| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2} = 1$ $\quad |u_2| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2} = 1$

$|u_3| = \sqrt{0^2 + 0^2 + 1^2} = 1$ $\quad \therefore$ Matrix consists orthonormal vectors.

2. Consider a set of four data points: **f(0) = 3, f(4) = −2, f(−1) = 2, f(1) = 1**. In the following, you are asked to find the **best fit polynomial of degree 2** by using the **Discrete square/Least square approximation** method as follows:

a. (2 marks) From the given data, write down the matrices A, b and x.
b. (3 marks) Evaluate A^TA and $\det(A^TA)$.
c. (2 marks) Compute the best-fit polynomial of degree 2.

②

ⓐ $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$

ⓑ $A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & 16 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix}$

$\det(A^TA) = 1448$

ⓒ Using least square approximation,

$(A^T \cdot A) \cdot x = A^T \cdot b$

$$\Rightarrow \begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & 16 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ -29 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 18 \\ 4 & 18 & 64 \\ 18 & 64 & 258 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 4 \\ -9 \\ -29 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.129 \\ -0.383 \\ -0.166 \end{bmatrix}$$

$$\therefore \; P_2(x) = 2.129 - 0.383x - 0.166x^2$$

3. Consider the coordinates: **(x, f(x)) = (0, 1),(0.5, 1.4),(1, 1.7),(1.5, 2)**. In the following, you are asked to construct the **best-fit linear polynomial** by using the **QR-decomposition** method as follows:

a. (2 marks) Construct the matrices A, b and x.
b. (3 marks) Evaluate the orthonormal vectors q1 and q2, and construct the matrix Q.
c. (2 marks) Compute the matrix R.
d. (3 marks) Using Q and R, evaluate the matrix x, and hence compute the best-fit linear polynomial.

3

a $\quad A = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \\ 1 & 1 \\ 1 & 1.5 \end{bmatrix} \qquad x = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1.4 \\ 1.7 \\ 2 \end{bmatrix}$

b $\quad$ Here, $u_1 = (1\ 1\ 1\ 1)^T$, $u_2 = (0, 0.5, 1, 1.5)^T$

$$P_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{P_1}{|P_1|} = \frac{(1\ 1\ 1\ 1)^T}{\sqrt{4}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = u_2 - (u_2^T q_1) q_1$$

$$= u_2 - \left([0\ \ 0.5\ \ 1\ \ 1.5] \times \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \right) \times \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$= u_2 - 1.5 \times \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ 1.5 \end{bmatrix} - \begin{bmatrix} 0.75 \\ 0.75 \\ 0.75 \\ 0.75 \end{bmatrix} = \begin{bmatrix} -0.75 \\ -0.25 \\ 0.25 \\ 0.75 \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{2}{\sqrt{5}} \begin{bmatrix} -0.75 \\ -0.25 \\ 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} -0.67 \\ -0.22 \\ 0.22 \\ 0.67 \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.67 \\ 0.5 & -0.22 \\ 0.5 & 0.22 \\ 0.5 & 0.67 \end{bmatrix}$$

ⓒ $$R = Q^T A = \begin{bmatrix} 0.5 & -0.67 \\ 0.5 & -0.22 \\ 0.5 & 0.22 \\ 0.5 & 0.67 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \\ 1 & 1 \\ 1 & 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1.5 \\ 0 & 1.115 \end{bmatrix}$$

ⓓ $$Rx = Q^T b$$

$$\Rightarrow \begin{bmatrix} 2 & 1.5 \\ 0 & 1.115 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.67 \\ 0.5 & -0.22 \\ 0.5 & 0.22 \\ 0.5 & 0.67 \end{bmatrix}^T \cdot \begin{bmatrix} 1 \\ 1.4 \\ 1.7 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1.5 \\ 0 & 1.115 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3.05 \\ 0.736 \end{bmatrix}$$

$$\therefore a_1 = \frac{0.736}{1.115} = \frac{736}{1115} = 0.66$$

$$\therefore 2a_0 + 1.5a_1 = 3.05 \quad \Rightarrow a_0 = \frac{3.05 - 1.5a_1}{2} = 1.03$$

$$P_1(x) = a_0 + a_1 x = 1.03 + 0.66x$$

4. A function is given by **$f(x) = e^{0.5x} + \sin x$** which is to be integrated on the interval [0, 2].

a. (2 marks) Evaluate the **exact integral I(f)**.
b. (2 marks) Compute the numerical integral by using the **Newton-Cotes formula with n = 2**.
c. (4 marks) Evaluate the numerical integral **$C_{1,4}$** by using the **Composite Newton-Cotes** formula and also find the percentage relative error.

④

ⓐ

$$I(f) = \int_a^b f(x)\,dx$$

$$= \int_0^2 (e^{0.5x} + \sin x)\,dx$$

$$= 4.853$$

(b) $I_2(f) = \frac{b-a}{6}\left[f(a) + 4f(m) + f(b)\right]$

$a = 0$
$b = 2$
$m = \frac{a+b}{2} = 1$

$= \frac{2-0}{6}\left[f(0) + 4f(1) + f(2)\right]$

$= \frac{1}{3}\left[14.588\right]$

$= 4.863$

(c) Here, $h = \frac{b-a}{m} = \frac{2-0}{4} = 1/2$

Now,
$x_0 = 0$
$x_1 = 0 + \frac{1}{2} = 1/2$
$x_2 = 1/2 + 1/2 = 1$
$x_3 = 1 + 1/2 = 3/2$
$x_4 = 3/2 + 1/2 = 2$

$C_{1,4}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$

$= \frac{1}{4}\left[f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + f(2)\right]$

$= \frac{1}{4}\left[19.364\right] = 4.841$

% relative error $= \left|\frac{4.853 - 4.841}{4.841}\right| \times 100\%$

$= 0.248\%$