1\. a) Here, $x_0 = 2$, $x_1 = 4$, $x_2 = 6$ $\therefore$ Degree will be 2

$$\text{Now, } V = \begin{bmatrix} 1 & 2^1 & 2^2 \\ 1 & 4^1 & 4^2 \\ 1 & 6^1 & 6^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}$$

$$Y = \begin{bmatrix} 10 \\ 20 \\ 25 \end{bmatrix}$$

We know, $V \cdot A = Y$

$$\Rightarrow A = V^{-1} \cdot Y$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 10 \\ 20 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 35/4 \\ -5/8 \end{bmatrix}$$

$$\text{Now, } P_2(x) = a_0 + a_1 x^1 + a_2 x^2$$

$$= -5 + \frac{35}{4}x - \frac{5}{8}x^2$$

$$\text{Now, } v(7) = P_2(7) = -5 + \frac{35 \times 7}{4} - \frac{5 \times 7^2}{8}$$

$$= 25.625 \, ms^{-1}$$

$$\therefore \text{Acceleration} = \frac{d}{dx}(P_2(x)) = \frac{35}{4} - \frac{5}{4}x \quad \Rightarrow a(7) = \frac{35}{4} - \frac{5}{4} \times 7 = 0 \, ms^{-2}$$

b) Using lagrange method,

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$\text{Now, } l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} = \frac{x-4}{2-4} \times \frac{x-6}{2-6} = \frac{1}{8}(x^2 - 10x + 24)$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_2}{x_1 - x_2} = \frac{x-2}{4-2} \times \frac{x-6}{4-6} = -\frac{1}{4}(x^2 - 8x + 12)$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} = \frac{x-2}{6-2} \times \frac{x-4}{6-4} = \frac{1}{8}(x^2 - 6x + 8)$$

$$\therefore P_2(x) = \frac{1}{8}(x^2 - 10x + 24) \times 10 - \frac{1}{4}(x^2 - 8x + 12) \times 20 + \frac{1}{8}(x^2 - 6x + 8) \times 25$$

$$= -\frac{5x^2}{8} + \frac{35x}{4} - 5$$

c) If new data point is added, Newton's divided difference method should be used. Degree of polynomial will then be 3.

2. Read the following and answer accordingly:
(a) (4 marks) Consider the nodes **[−π/2, 0, π/2]**. Find an interpolating polynomial of appropriate degree by using **Newton's divided-difference** method for **f(x) = x sin(x)** .
(b) (2 marks) Use the interpolating polynomial to find an approximate value at **π/4**, and compute the percentage relative error at **π/4**.
(c) (4 marks) Add a new node **π** to the above nodes, and find the interpolating polynomial of appropriate degree.

2.ⓐ Given, $f(x) = x \sin x$

$x_0 = -\frac{\pi}{2}$ $\quad f(x_0) = -\frac{\pi}{2} \times \sin(-\frac{\pi}{2}) = \pi/2$

$x_1 = 0$ $\quad f(x_1) = 0 \times \sin(0) = 0$

$x_2 = \frac{\pi}{2}$ $\quad f(x_2) = \frac{\pi}{2} \times \sin(\frac{\pi}{2}) = \frac{\pi}{2}$

Using Newton's Divided Difference form,

$x_0 = -\frac{\pi}{2}$ $\quad f[x_0] = \frac{\pi}{2}$

$x_1 = 0$ $\quad f[x_1] = 0$

$x_2 = \frac{\pi}{2}$ $\quad f[x_2] = \frac{\pi}{2}$

$$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{0-\frac{\pi}{2}}{0-(-\frac{\pi}{2})} = \frac{-\frac{\pi}{2}}{\frac{\pi}{2}} = -1$$

$$f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{\frac{\pi}{2}-0}{\frac{\pi}{2}-0} = 1$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{1-(-1)}{\frac{\pi}{2}+\frac{\pi}{2}} = \frac{2}{\pi}$$

Since 3 nodes,

$$P_2(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$= \frac{\pi}{2} + (-1)(x+\frac{\pi}{2}) + \frac{2}{\pi}(x+\frac{\pi}{2})(x-0)$$
$$= \cancel{\frac{\pi}{2}} - x - \cancel{\frac{\pi}{2}} + \frac{2}{\pi}(x^2 + \frac{\pi}{2}x)$$
$$= -\cancel{x} + \frac{2}{\pi}x^2 + \cancel{x} = \frac{2}{\pi}x^2$$

ⓑ At $x = \frac{\pi}{4}$,

$$P_2(\frac{\pi}{4}) = \frac{2}{\pi} \times (\frac{\pi}{4})^2 = \frac{2}{\pi} \times \frac{\pi^2}{16} = \frac{\pi}{8}$$

Now, $f(\frac{\pi}{4}) = \frac{\pi}{4} \times \sin(\frac{\pi}{4}) = \frac{\pi}{4} \times \frac{1}{\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$

$$\text{Percentage relative error} = \left| \frac{f(\frac{\pi}{4}) - p(\frac{\pi}{4})}{f(\frac{\pi}{4})} \right| \times 100\%$$

$$= 29.29\%$$

ⓒ Adding new node x_1

$x_0 = -\frac{\pi}{2}$, $f[x_0] = \frac{\pi}{2}$

$x_1 = 0$, $f[x_1] = 0$

$x_2 = \frac{\pi}{2}$, $f[x_2] = \frac{\pi}{2}$

$x_3 = \pi$, $f[x_3] = 0$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{0 - \frac{\pi}{2}}{0 - (-\frac{\pi}{2})} = \frac{-\frac{\pi}{2}}{\frac{\pi}{2}} = -1$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{\frac{\pi}{2} - 0}{\frac{\pi}{2} - 0} = 1$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{0 - \frac{\pi}{2}}{\pi - \frac{\pi}{2}} = -1$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{2}{\pi}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{-2}{\pi}$$

$f[x_0, x_1, x_2] = \frac{2}{\pi}$

$f[x_1, x_2, x_3] = -\frac{2}{\pi}$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{-\frac{2}{\pi} - \frac{2}{\pi}}{\pi + \frac{\pi}{2}} = \frac{-\frac{4}{\pi}}{\frac{3\pi}{2}} = -\frac{8}{3\pi^2}$$

Since there are 4 nodes.

$$P_3(x) = P_2(x) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= \frac{2}{\pi}x^2 + \left(-\frac{8}{3\pi^2}\right)\left(x + \frac{\pi}{2}\right)(x - 0)\left(x - \frac{\pi}{2}\right)$$

$$= \frac{2}{\pi}x^2 - \frac{8}{3\pi^2}x\left(x^2 - \frac{\pi^2}{4}\right)$$

$$= \frac{2}{\pi}x^2 - \frac{8}{3\pi^2}x^3 + \frac{8\pi^2}{12\pi^2}x$$

$$= \frac{2}{\pi}x^2 - \frac{8}{3\pi^2}x^3 + \frac{2}{3}x$$

3. An interpolating polynomial, $p_1(x) = 1.648(x - 1)$ is derived for the function $f(x) = x \ln x$ at the nodes ($x_0 = 1$, $x_1 = 3$) using the Lagrange method. Answer the following keeping up to 4 significant figures.

a) (1 mark) Explain what you need to do to obtain a **degree** 3 interpolating polynomial for the same function f(x) and for the same nodal points ($x_0 = 1$, $x_1 = 3$).

b) (4 marks) Calculate the bases of the **degree** 3 polynomial.

(a)

To obtain a degree 3 polynomial using the same function $f(x)$ & same nodal points x_0, x_1, we would need to use Hermite interpolation method.

(b)

x	$f(x) = x\ln(x)$	$f'(x) = \ln(x) + 1$
$x_0 = 1$	0	1
$x_1 = 3$	3.296 (4 s.f.)	2.099 (4 s.f.)

2 nodes $\rightarrow n = 1$

$$\therefore P_{2\times 1+1}(x) = P_3(x) = f(x_0)h_0(x) + f'(x_0)\hat{h}_0(x) + f(x_1)h_1(x) + f'(x_1)\hat{h}_1(x)$$

$$= 0 + \hat{h}_0(x) + 3.296 \times h_1 + 2.099 \times \hat{h}_1(x)$$

$$\hat{h}_0(x) = l_0^2(x)(x - x_0)$$

$$= l_0^2(x)(x-1)$$

$$= \left[\frac{(x-3)}{-2}\right]^2 (x-1)$$

$$= \frac{(x-3)^2(x-1)}{4}$$

$$l_0(x) = \frac{(x-x_1)}{(x_0-x_1)} = \frac{(x-3)}{1-3}$$

$$= \frac{(x-3)}{-2}$$

$$h_1(x) = l_1^2(x)\left(1 - 2(x-x_1)\,l_1'(x_1)\right)$$

$$= l_1^2(x)\left(1 - 2(x-3)\,l_1'(x_1)\right)$$

$$= \left[\frac{(x-1)}{2}\right]^2\left(1 - 2(x-3)\left(\tfrac{1}{2}\right)\right)$$

$$= \frac{(x-1)^2}{4}(1 - x + 3)$$

$$= \frac{(x-1)^2(4-x)}{4}$$

$$\hat{h}_1(x) = l_1^2(x)(x - x_1)$$

$$= \left[\frac{(x-1)}{2}\right]^2 (x-3)$$

$$= \frac{(x-1)^2(x-3)}{4}$$

$$l_1(x) = \frac{(x-x_0)}{(x_1-x_0)} = \frac{(x-1)}{(3-1)}$$

$$= \frac{(x-1)}{2}$$

$$l_1'(x) = \frac{1}{2}$$