

The Mathematical Association of Victoria

Trial Examination 2022

MATHEMATICAL METHODS

Trial Written Examination 2 - SOLUTIONS

SECTION A: Multiple Choice

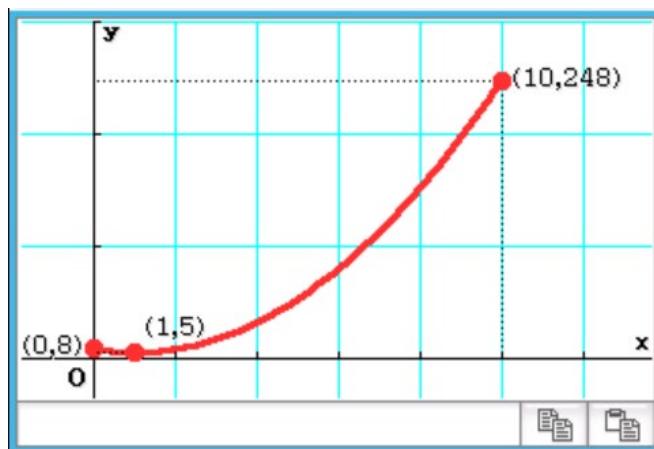
Question	Answer	Question	Answer
1	B	11	C
2	A	12	D
3	E	13	C
4	E	14	D
5	B	15	E
6	C	16	A
7	A	17	A
8	D	18	C
9	B	19	E
10	D	20	B

Question 1

Answer B

$f : [0,10] \rightarrow R, f(x) = 3(x-1)^2 + 5$ has restricted domain $[0,10]$.

Minimum turning point is at $(1,5)$ and the y -intercept is at $(0, 8)$



range $[5, 248]$

Question 2**Answer A**

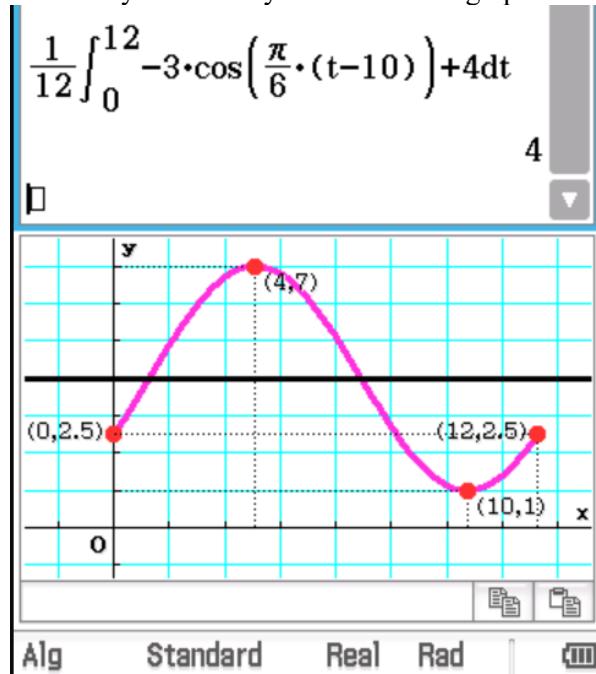
$$h(t) = -3 \cos\left(\frac{\pi}{6}(t-10)\right) + 4$$

The average height of water from 9 am ($t = 0$) to 9 pm ($t = 12$) is found by

$$\frac{1}{12-0} \int_0^{12} h(t) dt = 4$$

Alternatively

For one cycle of the symmetric cosine graph the vertical translation of 4 becomes the average value.

**Question 3****Answer E**

For $y = -\pi \sin(2\pi x) - \pi$, the amplitude is π

Question 4**Answer E**

$$f(x) = 2^{x-1}$$

$$f(x) \times f(y)$$

$$= 2^{x-1} \times 2^{y-1}$$

$$= 2^{x+y-2}$$

$$= 2^{x+y-1-1}$$

$$= f(x+y-1)$$

Question 4 (continued)

The screenshots show a CAS interface with three windows:

- Top Window:** Shows the definition $f(x) := 2^{x-1}$ and the result of the judge command: $f(x) \cdot f(y) = f(x+y-1)$ is labeled "true".
- Middle Left Window:** Shows the judge command for $f(x) \cdot f(y) = f(x+y)$. It shows the equation $\frac{e^{\ln(2) \cdot (y+x)}}{4} = \frac{2^{x+y}}{2}$.
- Middle Right Window:** Shows the judge command for $f(x) \cdot f(y) = f(x+y-2)$. It shows the equation $\frac{e^{\ln(2) \cdot (y+x)}}{4} = \frac{e^{\ln(2) \cdot (y+x)}}{8}$.
- Bottom Window:** Shows the history of commands entered:
 - define $f(x) := 2^{x-1}$ (done)
 - judge($f(x)f(y) = f(xy)$) (Undefined)
 - judge($f(x)f(y) = f(x+y)$) (FALSE)
 - judge($f(x)f(y) = f(x)+f(y)$) (Undefined)
 - judge($f(x)f(y) = f(x+y-2)$) (FALSE)
 - judge($f(x)f(y) = f(x+y-1)$) (TRUE)

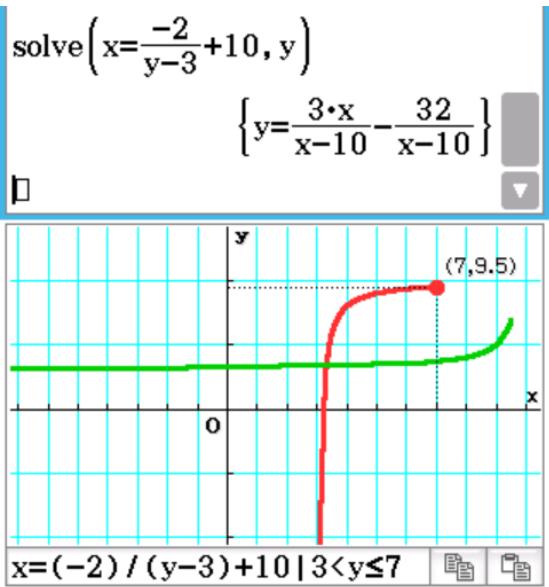
Question 5**Answer B**

$$g : (3, 7] \rightarrow R, g(x) = \frac{-2}{x-3} + 10 \text{ with domain } (3, 7] \text{ and range } (-\infty, 9.5]$$

Using CAS to swap x and y to get the rule for inverse

$$g^{-1}(x) = \frac{3x}{x-10} - \frac{32}{x-10}$$

$$\text{Dom } g^{-1}(x) = \text{Ran } g(x) = (-\infty, 9.5]$$

Question 5 (continued)**Question 6****Answer C**

$T: R^2 \rightarrow R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ has been applied to get the image $y = e^{2x}$.

We need to find the original exponential rule.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 3y \end{bmatrix}$$

Giving $x' = -x$ and $y' = 3y$

Substitute into image equation $y' = e^{2x'}$

$$3y = e^{-2x}$$

$$y = \frac{1}{3}e^{-2x}$$

Question 7**Answer A**

The gradient of both equations is 1

$$2x - 2y = m \text{ and } 6x - 6y = m$$

$$y = x - \frac{m}{2} \text{ and } y = x - \frac{m}{6}$$

Hence parallel lines

Question 8**Answer D**

$$y = a \log_e(bx + c)$$

$$(3, -6 \log_e(2)) \text{ and } \left(\frac{2}{3}, 0\right)$$

From the graph the vertical asymptote is at $x = \frac{1}{3}$ and the graph is dilated by $\frac{1}{3}$ from the y -axis to get

$$\text{the } x\text{-intercept of } \left(\frac{2}{3}, 0\right)$$

So the rule is $y = a \log_e(3x - 1)$

Question 8 (continued)

Substitute $(3, -6 \log_e(2))$

$$-6 \log_e(2) = a \log_e(8)$$

$$a = \frac{-6 \log_e(2)}{\log_e(8)}$$

$$a = \frac{-6 \log_e(2)}{\log_e(2)^3}$$

$$a = \frac{-6 \log_e(2)}{3 \log_e(2)}$$

Giving $a = -2$

Equation of graph is $y = -2 \log_e(3x - 1)$

Question 9**Answer B**

A and B are two independent events.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{\Pr(A) \times \Pr(B)}{\Pr(B)} \text{ for independent events}$$

$$= \Pr(A) = \frac{1}{2}$$

$$\Pr(A') = \frac{1}{2}$$

$$\Pr(A' \cap B') = \Pr(A') \times \Pr(B') = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

$$\Pr(B') = \frac{2}{5}$$

$$\Pr(A \cap B') = \Pr(A) \times \Pr(B') = \frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$$

OR

$$\Pr(A' \cap B') = \frac{1}{5} \text{ given}$$

$$\Pr(A' \cap B) = 1 - \left(\frac{1}{2} + \frac{1}{5} \right) = \frac{3}{10}$$

Question 9 (continued)

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B) = \frac{1}{2} \times \left(\frac{3}{10} + \Pr(A \cap B) \right)$$

$$\Pr(A \cap B) = \frac{3}{10}$$

$$\Pr(A \cap B') = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}$$

	A	A'	
B	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{5}$
B'	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
	$\frac{1}{2}$	$\frac{1}{2}$	1

Question 10**Answer D**

$$f(x) = \begin{cases} \frac{x}{2} + k, & 0 \leq x \leq k \\ 0, & \text{otherwise} \end{cases} \quad \text{where } k \in R^+.$$

Firstly, find constant k

For PDF let $\int_0^k \left(\frac{x}{2} + k \right) dx = 1$

For $k \in R^+$, $k = \frac{2\sqrt{5}}{5}$

To find the median of $f(x)$ solve

$$\int_0^m \left(\frac{x}{2} + \frac{2\sqrt{5}}{5} \right) dx = 0.5$$

$$m = \frac{\sqrt{130} - 4\sqrt{5}}{5}$$

Question 10 (continued)

solve $\left(\int_0^m \frac{x}{2} + \frac{2\sqrt{5}}{5} dx = 0.5, m\right)$
 $\left\{m = \frac{\sqrt{130} - 4\sqrt{5}}{5}, m = \frac{-(\sqrt{130} + 4\sqrt{5})}{5}\right\}$

Median is $\frac{\sqrt{130} - 4\sqrt{5}}{5}$

Question 11**Answer C**

To find the mean of the sample, find the midpoint of the confidence interval (0.3346, 0.4654)

$$\frac{0.3346 + 0.4654}{2} = 0.4$$

Using the left edge of the CI solve for n where

$$0.4 - 1.96 \sqrt{\frac{0.4(1-0.4)}{n}} = 0.3346$$

Giving $n = 215.5598\dots$

Number of students in the random sample is closest to 216

solve $\left(0.4 - 1.96 \cdot \sqrt{\frac{0.4 \cdot (1-0.4)}{n}} = 0.3346, n\right)$
 $\{n=215.5598575\}$

Question 12**Answer D**

To find k solve $0.2 + \frac{1+k^2}{10} + \frac{4-k}{10} + \frac{k}{20} = 1$

gives $k = 2, k = -\frac{3}{2}$

As $k \in R^+$, $k = 2$

To find the mean evaluate $0 \times 0.2 + 1 \times \frac{1+k^2}{10} + 2 \times \frac{4-k}{10} + 3 \times \frac{k}{20}$ where $k = 2$

Mean = 1.2

Question 12 (continued)

TI-Nspire CX CAS Solution for Question 12 (continued)

Solve $0.2 + \frac{1+k^2}{10} + \frac{4-k}{10} + \frac{k}{20} = 1$ for k .

The solutions are $k=2$ and $k=-\frac{3}{2}$.

Input: $\frac{1+k^2}{10} + 2(\frac{4-k}{10}) + 3(\frac{k}{20})$

Question 13**Answer C**

$$f(x) = x^4 + bx^3 + x^2 - 2$$

Solve $\frac{d}{dx}(x^4 + bx^3 + x^2 - 2) = 0$ for x

$$x = \frac{\pm\sqrt{9b^2 - 32} + 3b}{8} \text{ or } x = 0$$

There will be three solutions when $9b^2 - 32 > 0$.

$$b < -\frac{4\sqrt{2}}{3} \text{ or } b > \frac{4\sqrt{2}}{3}$$

TI-Nspire CX CAS Calculations for Question 13

1. Solve $\frac{d}{dx}(x^4 + bx^3 + x^2 - 2) = 0$ for x . The solutions are $x = \frac{-\sqrt{9b^2 - 32} + 3b}{8}$ and $x = \frac{\sqrt{9b^2 - 32} - 3b}{8}$, plus $x = 0$.

2. The discriminant condition is $b^2 - 32 > 0$, which simplifies to $9b^2 - 32 > 0$.

3. Solve $9b^2 - 32 > 0$ for b . The solution is $b < -\frac{4\sqrt{2}}{3}$ or $b > \frac{4\sqrt{2}}{3}$.

Question 13 (continued)

```

define f(x)=x^4+bx^3+x^2-2
done
solve(d(f(x))/dx=0, x)
{x=0, x=-(-3+b-sqrt(9*b^2-32))/8, x=-(-3+b+sqrt(9*b^2-32))/8}

```

Question 14**Answer D**

$$h(x) = \sqrt{1 - 3x}$$

The gradient of the line perpendicular to the line $y = 2x$ is $-\frac{1}{2}$.

Solve $h'(x) = -\frac{1}{2}$ for x

$$x = -\frac{8}{3}$$

The tangent is $y = -\frac{x}{2} + \frac{5}{3}$.

```

h(x):=sqrt(1-3*x)
Done
solve(d(h(x))/dx=-1/2,x)
x=-8/3
tangentLine(h(x),x,-8/3)
5-x
--- - --
3 2

```

Question 14

The calculator screen shows the following steps:

```

define h(x)=sqrt(1-3x)
done
solve(d/dx(h(x))=-1/2,x)
{x=-8/3}
tanLine(h(x),x,-8/3)
-x/2+5/3

```

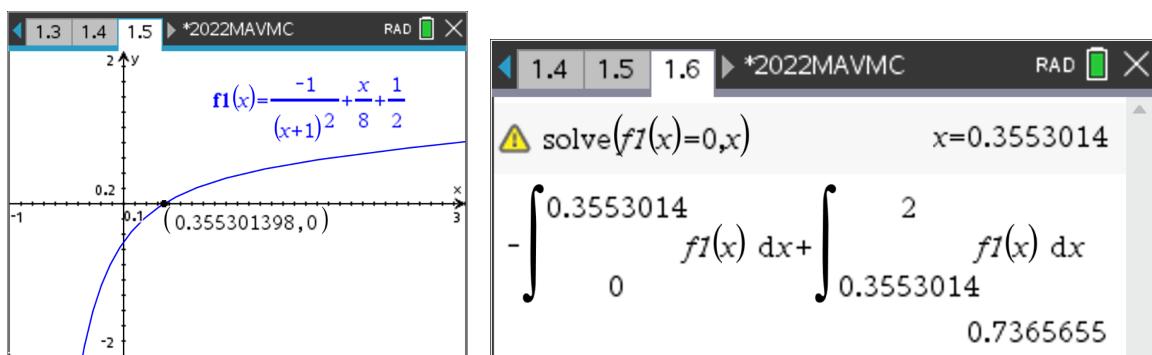
Question 15**Answer E**

$$v = \frac{-1}{(t+1)^2} + \frac{t}{8} + \frac{1}{2}$$

Solve $v(t) = 0$ for t

$$t = 0.3553\dots$$

$$-\int_0^{0.3553\dots} (v(t)) dt + \int_{0.3553\dots}^2 (v(t)) dt$$

 $= 0.737$ correct to three decimal places


Question 15 (continued)

```

define v(t)=-1/(t+1)^2+x/8+1/2
done
solve(v(t)=0,t)
{t=0.3553013976}
-∫_0^0.3553013976 v(t)dt+∫_0.3553013976^2 v(t)dt
0.7365655325

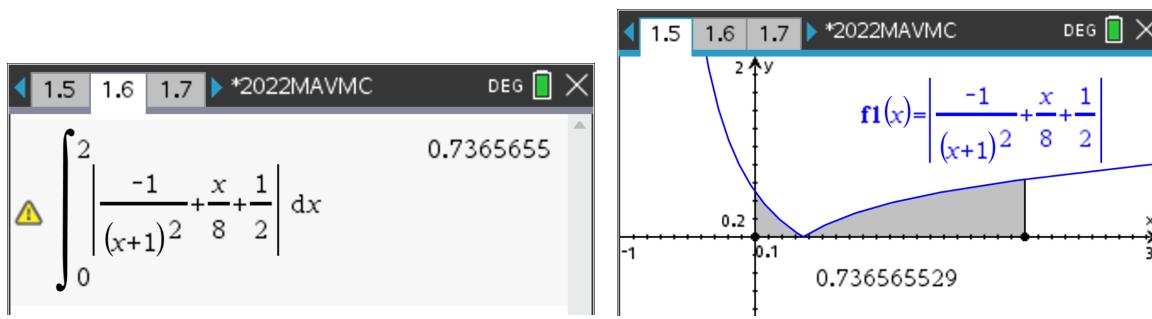
```

Alg Standard Real Rad

OR

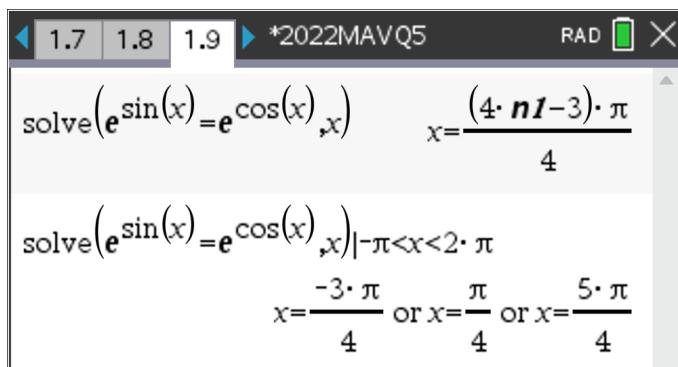
Please note that absolute values are not on the course but can be used for these types of questions.

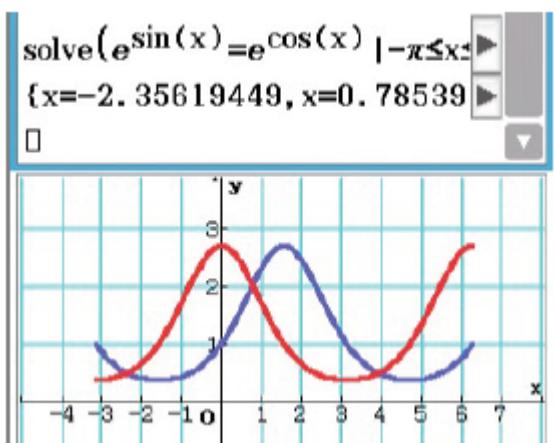
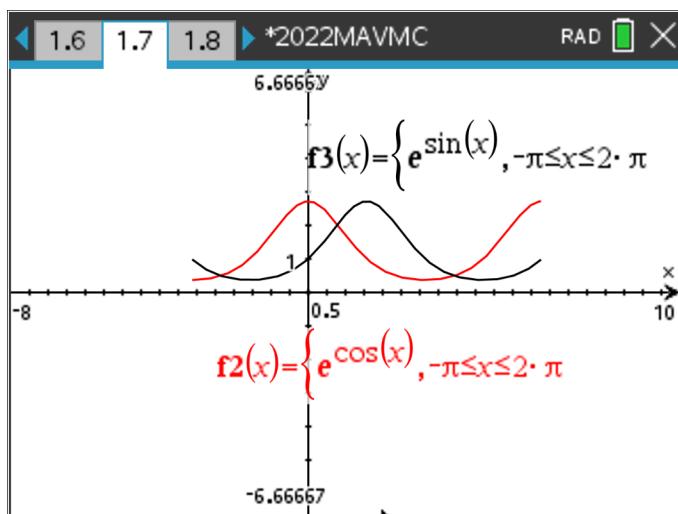
$$\int_0^2 (|v(t)|) dt = 0.737$$

**Question 16****Answer A**

$$f(x) = e^{\sin(x)} \text{ and } g(x) = e^{\cos(x)}$$

$$\begin{aligned} \text{Area} &= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (g(x) - f(x)) dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (f(x) - g(x)) dx \\ &= 2 \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (g(x) - f(x)) dx \end{aligned}$$



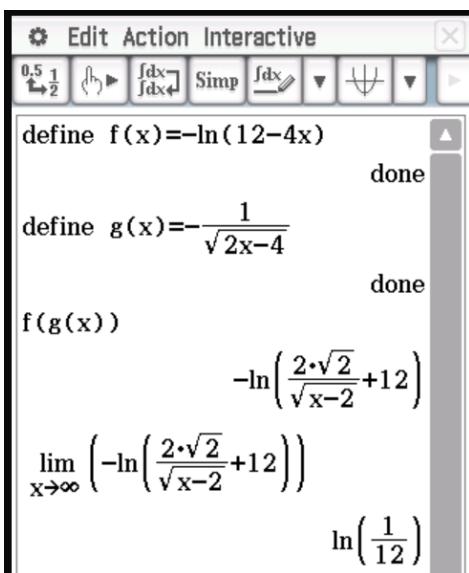
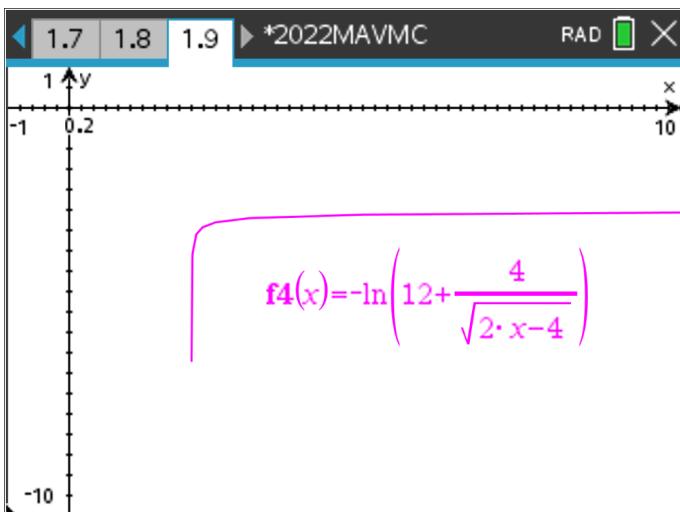
Question 16**Question 17****Answer A**

$$f(x) = -\log_e(12 - 4x) \text{ and } g(x) = -\frac{1}{\sqrt{2x - 4}}$$

$$f(g(x)) = -\log_e\left(12 + \frac{4}{\sqrt{2x - 4}}\right)$$

Domain of $f(g(x))$ is $x > 2$

Range is $(-\infty, -\log_e(12))$

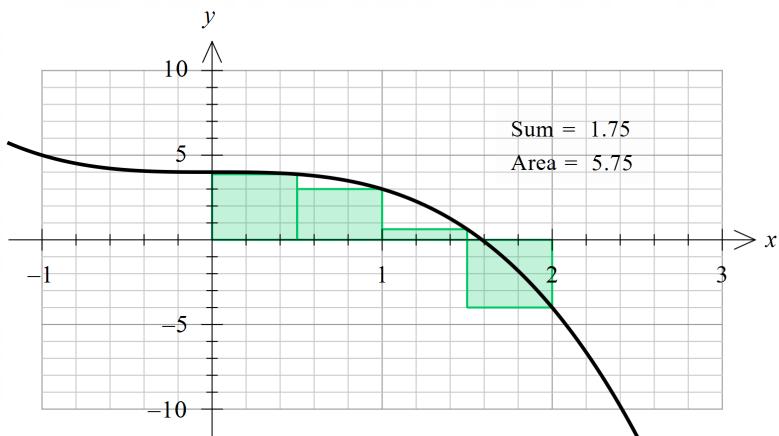
Question 17 (continued)**Question 18****Answer C**

$$y = f(x) = -x^3 + 4$$

$$\int_0^2 (-x^3 + 4) dx$$

$$\approx \frac{1}{2} \left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right)$$

$$= \frac{7}{4}$$

Question 18 (continued)

1.9 | 1.10 | 1.11 | *2022MAVMC | RAD | Done | X

$$f(x) := -x^3 + 4$$

$$\frac{1}{2} \cdot \left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right) = \frac{7}{4}$$
Question 19 **Answer E**

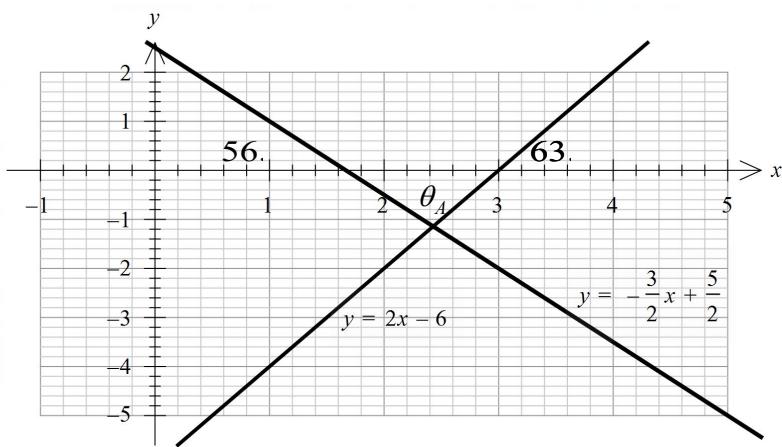
$$2x - y = 6 \text{ and } 3x + 2y = 5$$

$$y = 2x - 6 \text{ and } y = -\frac{3}{2}x + \frac{5}{2}$$

$$\tan(\theta_1) = 2 \text{ and } \tan(\theta_2) = -\frac{3}{2}$$

$$\theta_1 = \tan^{-1}(2) = 63.43\dots, \quad \theta_2 = \tan^{-1}\left(-\frac{3}{2}\right) = -56.30\dots$$

Acute angle $= \theta_A = 180 - (63.43\dots + 56.30\dots) = 60.26$ correct to two decimal places



Question 19 (continued)

$\tan^{-1}(2)$ 63.43495
 $\tan^{-1}\left(\frac{-3}{2}\right)$ -56.30993
 $180 - (63.434948822922 + 56.30993247402)$ 60.25512

$\tan^{-1}(2) - \tan^{-1}\left(-\frac{3}{2}\right)$ 119.7448813
 $-ans + 180$ 60.2551187

OR

Please note absolute values are not on the course but can be used for these types of questions.

$$\tan(\theta) = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{2 + \frac{3}{2}}{1 - 3} \right|$$

Acute angle = 60.26 correct to two decimal places

$\tan^{-1}\left(\left|\frac{2 + \frac{3}{2}}{1 - 3}\right|\right)$ 60.25512

Question 20**Answer B**

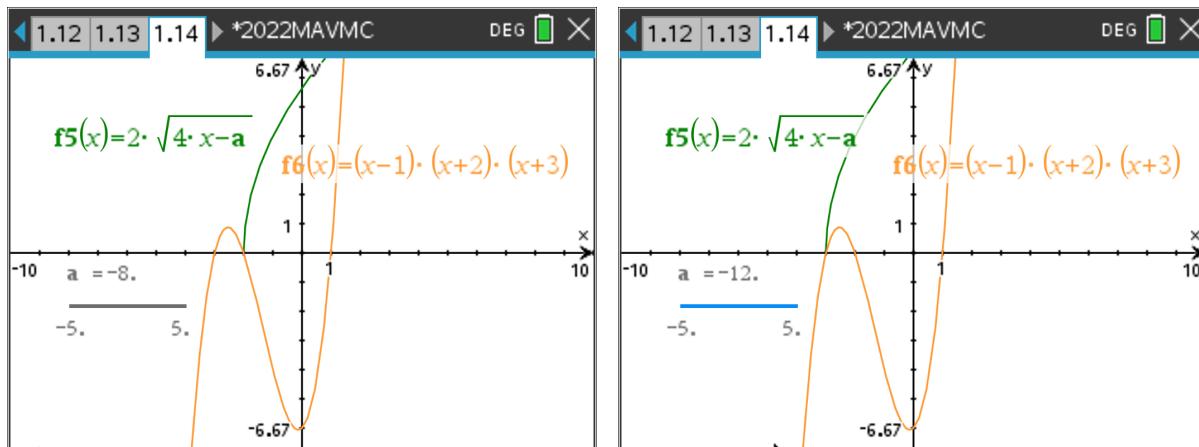
$$f(x) = 2\sqrt{4x-a} \text{ and } g(x) = (x-1)(x+2)(x+3)$$

The maximum number of points of intersection is two.

$$\text{When } x = -2, 2\sqrt{4x-2-a} = 0, a = -8$$

$$\text{When } x = -3, 2\sqrt{4x-3-a} = 0, a = -12$$

So two points of intersection occur when $-12 \leq a \leq -8$.



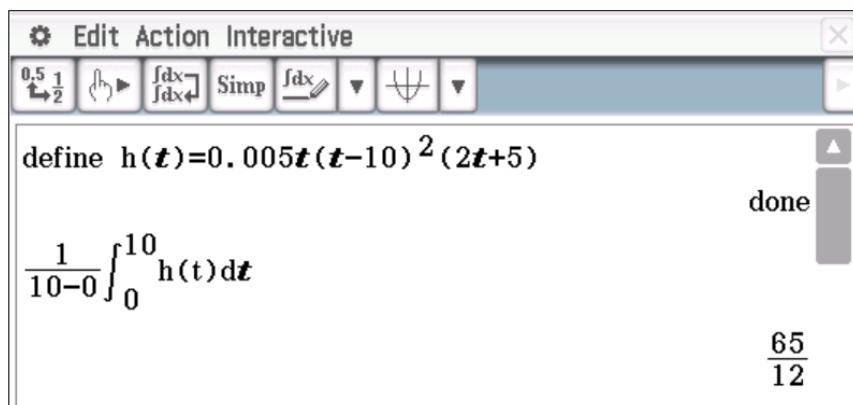
END OF SECTION A SOLUTIONS

SECTION B**Question 1**

$$h: [0, 10] \rightarrow R, h(t) = 0.005t(t-10)^2(2t+5)$$

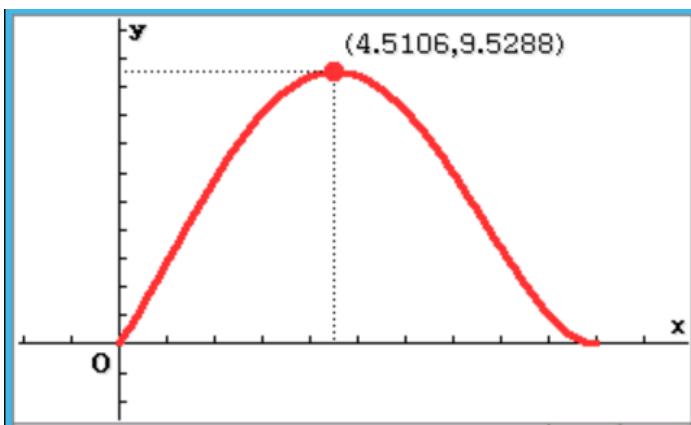
a. Average height = $\frac{1}{10-0} \int_0^{10} h(t) dt$
 $= \frac{65}{12}$ cm

1A



b. Maximum height = 9.5 cm correct to one decimal place

1A



c. Average rate of change of h for $t \in [0, 4.5106...]$ or $t \in \left[0, \frac{25+5\sqrt{89}}{16}\right]$

$$\begin{aligned} &= \frac{h\left(\frac{25+5\sqrt{89}}{16}\right) - h(0)}{\frac{25+5\sqrt{89}}{16} - 0} \\ &= 2.11 \text{ cm/min} \end{aligned}$$

1M

1A

Question 1 (continued)

- d. From the gradient graph the height of liquid is increasing at its maximum rate at $t = 1.09$ min correct to two decimal places.

1A

e. Volume of cone $= \frac{1}{3}\pi r^2 h$ where $r = \frac{1}{4}h$

1M

$$V = \frac{1}{3}\pi \left(\frac{h}{4}\right)^2 h = \frac{1}{3}\pi \frac{h^3}{16}$$

$$V = \frac{\pi}{48}h^3 \quad \text{as required}$$

1M Show that

- f. Let glass have half volume.

$$\text{When full, volume of glass} = \frac{1}{3}\pi \times 3^2 \times 12 = 36\pi \text{ cm}^3$$

$$\text{Solve for } t, \frac{\pi}{48}(h(t))^3 = 18\pi$$

1M

Glass is half full at $t = 4.43, 4.59$ mins correct to two decimal places

1A

Question 1 (continued)

```

define h(t)=0.005t(t-10)^2(2t+5)
solve(π/48 · (h(t))^3=18·π | 0≤t≤10, t)
{t=4.428042507, t=4.593252896}

```

g. $h_1 : [a, 10+a] \rightarrow R, h_1(t) = 0.005(t-a)(t-a-10)^2(2(t-a)+5)$

Recognise $h_1(t) = h(t-a)$ for $t \in [a, 10+a]$

Equate $\frac{h_1(2+a) - h_1(a)}{2+a-a} = h_1'(4.428\dots)$ 1M

$a = 2.73$ or $a = 3.92$ correct to 2 decimal places 1A

```

define h(t)=0.005t(t-10)^2(2t+5)
done
define H(t)=h(t-a)
done
solve(H(2+a)-H(a)=diff(H(t), t, 1, 4.4280425), a)
{a=-6.486105018, a=2.729710444, a=3.915522073}

```

Question 2

w : [-4, 4] → R, $w(x) = \sqrt{r^2 - x^2} + c$ where $r > 0$ and $c \in R^+ \cup \{0\}$.

a. Radius = 4 dm 1A

b. $w(x) = \sqrt{16 - x^2} + 1$ 1A

c. Area of the rectangle is $A_r = 2x_1(y_1 - 1)$.

$$A_r = 2x_1(\sqrt{16 - x^2} + 1 - 1)$$

$$A_r = 2x_1\sqrt{16 - x_1^2}$$
 as required 1M Show that

d. $A_r = 2x_1\sqrt{16 - x_1^2}$

Question 2 (continued)

Solve $\frac{d}{dx_1}(A_r) = 0$

Gives $x_1 = \pm 2\sqrt{2}$

for $x_1 > 0, x_1 = 2\sqrt{2}$

maximum area of rectangle = 16 dm².

1A

Justification

x	1	$2\sqrt{2}$	3
$A'_r(x)$	$\frac{28\sqrt{15}}{15}$	0	$\frac{-4\sqrt{7}}{7}$
	/	—	\

1M

OR

$A''_r(2\sqrt{2}) = -8, A''_r < 0$, hence a maximum **1M**

Question 2 (continued)

$$\frac{d^2}{dx^2}(A(x))|_{x=2\sqrt{2}}$$

-8

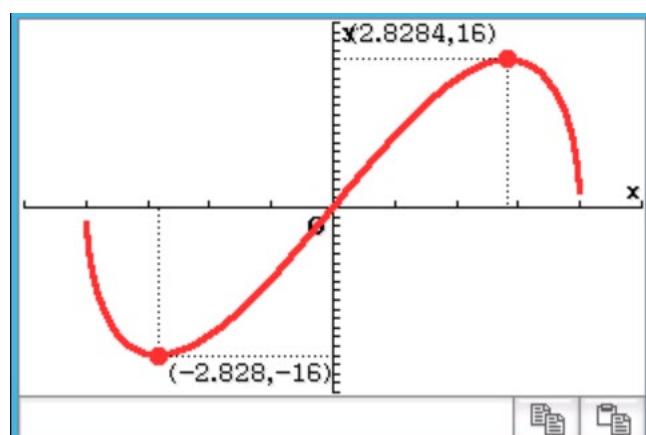
$$\frac{d}{dx}(A(x))|_{x=1}$$

$$\frac{28\cdot\sqrt{15}}{15}$$

$$\frac{d}{dx}(A(x))|_{x=3}$$

$$\frac{-4\cdot\sqrt{7}}{7}$$

□



e. Given radius = 4 dm

$$\text{Area of semicircle} = \frac{1}{2}\pi 4^2 = 8\pi$$

$$\text{Proportion stained glass} = \frac{16}{8\pi} = \frac{2}{\pi} \quad \mathbf{1A}$$

$$\mathbf{f.} \ h(x) = -\frac{1}{2}(e^x + e^{-x}) + d \text{ where } d \in R^+ \cup \{0\}.$$

$$\text{Substitute } (0, 5) \text{ into } h(x) = -\frac{1}{2}(e^x + e^{-x}) + d$$

$$d = 6$$

1A

Question 2 (continued)

```

define h(x)=-1/2(e^x+e^-x)+d
done
solve(h(0)=5, d)
{d=6}

```

g. Solve $-\frac{1}{2}(e^x + e^{-x}) + 6 = 1$

Intersects with line $y = 1$ at

$$\left(\log_e(5 - 2\sqrt{6}), 1\right) \text{ and } \left(\log_e(5 + 2\sqrt{6}), 1\right) \quad \mathbf{1A}$$

```

define h(x)=-1/2(e^x+e^-x)+6
done
solve(h(x)=1, x)
{x=ln(-2*sqrt(6)+5), x=ln(2*sqrt(6)+5)}

```

h. For the rectangle:

$$\text{solve } 16 = 2 \times 2\sqrt{2} \times \text{width}$$

$$\text{width} = 2\sqrt{2}$$

$$\text{Solve } h(x) = 2\sqrt{2} + 1 \text{ for } x$$

$$x = -1.410\dots$$

1M

$$2 \left(\int_{\log_e(5-2\sqrt{6})}^{-1.410\dots} (h(x) - 1) dx + 1.410\dots \times 2\sqrt{2} \right) \quad \mathbf{1M}$$

$$= 10.9 \text{ dm}^2$$

1A

```

solve(16=2*2*sqrt(2)*y, y)
{y=2*sqrt(2)}
solve(h(x)=2*sqrt(2)+1, x)
{x=-1.41079072, x=1.41079072}

```

Question 2 (continued)

$2 \left(\int_{\ln(5-2\sqrt{6})}^{-1.41079} (h(x)-1) dx + \int_{-1.41079}^{1.41079} (h(x)-(2\sqrt{2}+1)) dx \right)$

10.85433313

Alg Standard Real Rad

OR

$$\int_{\ln(5-2\sqrt{6})}^{\ln(5+2\sqrt{6})} (h(x)-1) dx - \int_{-1.41079}^{1.41079} (h(x)-(2\sqrt{2}+1)) dx \quad 2M$$

$$= 10.9 \text{ dm}^2 \quad 1A$$

$\int_{\ln(5-2\sqrt{6})}^{\ln(5+2\sqrt{6})} (h(x)-1) dx - \int_{-1.41079}^{1.41079} (h(x)-(2\sqrt{2}+1)) dx$

10.85433313

Alg Standard Real Rad

Question 3

a.i. $X \sim N(15, 1)$

$$\Pr(X > 17.5) = 0.0062 \quad 1A$$

normCdf(17.5, infinity, 15, 1) 0.0062097

a.ii. $\Pr(X > 17.5 | X \geq 14)$

$$\begin{aligned} &= \frac{\Pr(X > 17.5 \cap X \geq 14)}{\Pr(X \geq 14)} \\ &= \frac{\Pr(X > 17.5)}{\Pr(X \geq 14)} \quad 1M \end{aligned}$$

$$= 0.0074 \text{ correct to four decimal places } 1A$$

normCdf(14, infinity, 15, 1)	0.8413447
normCdf(17.5, infinity, 15, 1)	0.0073807
normCdf(14, infinity, 15, 1)	0.8413447

Question 3 (continued)

b. $M \sim N(\mu, \sigma)$

$$\frac{3.8 - \mu}{\sigma} = -1.666\dots$$

$$\frac{4.2 - \mu}{\sigma} = 1.042\dots \quad \mathbf{1M}$$

$$\mu = 4.05 \text{ kg}, \sigma = 0.15 \text{ kg} \quad \mathbf{1A}$$

The calculator screen shows the following steps:

- invNorm(0.0478,0,1) = -1.66657
- invNorm(1-0.1487,0,1) = 1.042025
- solve($\frac{3.8 - \alpha}{b} = -1.6665696892669$ and $\frac{4.2 - \alpha}{b} = 1.042025$)
- $\alpha = 4.046116$ and $b = 0.147678$

c. $X \sim Bi(25, 0.8035)$

$$p = 1 - (0.0478 + 0.1487) = 0.8035 \quad \mathbf{1M}$$

$$\Pr(X > 20) = \Pr(21 \leq X \leq 25) = 0.4380 \text{ correct to four decimal places} \quad \mathbf{1A}$$

The calculator screen shows the following steps:

- $1 - (0.0478 + 0.1487) = 0.8035$
- binomCdf(25, 0.8035, 21, 25) = 0.4379719

d. $X_1 \sim Bi(n, 0.8035)$

$$\Pr(21 \leq X_1 \leq n) > 0.95 \quad \mathbf{1M}$$

Trial and error

$$n = 30 \quad \Pr(21 \leq X_1 \leq 30) = 0.944\dots$$

$$n = 31 \quad \Pr(21 \leq X_1 \leq 31) = 0.970\dots$$

$$n = 31 \quad \mathbf{1A}$$

The calculator screen shows the following steps:

- binomCdf(n, 0.8035, 21, n) | n=30 = 0.9449049
- binomCdf(n, 0.8035, 21, n) | n=31 = 0.9709725
- invBinomN(0.05, 0.8035, 20, 1)
- $\begin{bmatrix} 30 & 0.0550951 \\ 31 & 0.0290275 \end{bmatrix}$

Question 3 (continued)

$$\text{e. } s(x) = \begin{cases} \frac{x}{4} - 3 & 12 \leq x \leq 14 \\ a(x-16)^2 + b & 14 < x \leq 16 \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{12}^{14} \left(\frac{x}{4} - 3 \right) dx = \frac{1}{2}$$

Solve $\int_{14}^{16} (a(x-16)^2 + b) dx = \frac{1}{2}$ and $a(14-16)^2 + b = \frac{14}{4} - 3$ for a and b 1M

$$a = \frac{3}{32}, \quad b = \frac{1}{8} \quad \text{1A}$$

The calculator screen shows the following input and output:

Input: $\int_{14}^{16} (a \cdot (x-16)^2 + b) dx = \frac{1}{2}$

Output: $a = \frac{3}{32} \text{ and } b = \frac{1}{8}$

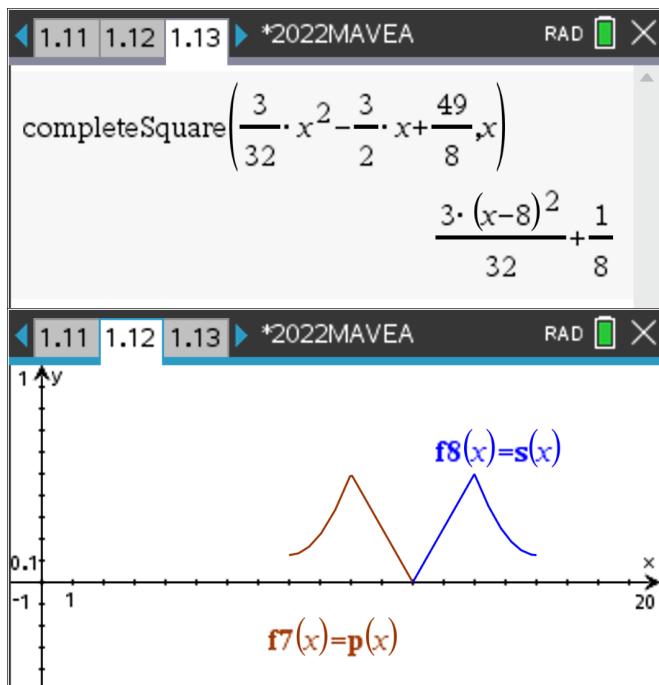
$$\text{f.i. } p(x) = \begin{cases} \frac{3x^2}{32} - \frac{3x}{2} + \frac{49}{8} & 8 \leq x \leq 10 \\ -\frac{x}{4} + 3 & 10 < x \leq 12 \\ 0 & \text{elsewhere} \end{cases}$$

Reflection in the line $x = 12$ 1A

OR

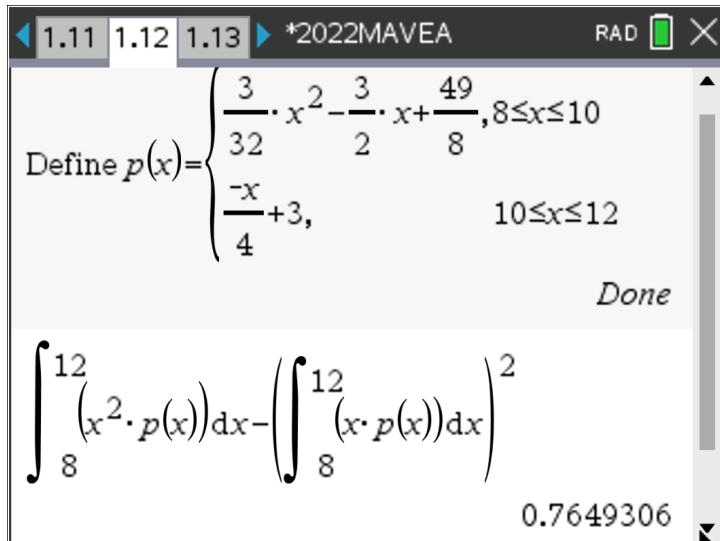
Reflection in the y -axis

Translation of 24 units to the right 1A

Question 3 (continued)

f.ii. $\int_8^{12} (x^2 p(x)) dx - \left(\int_8^{12} (xp(x)) dx \right)^2$ 1M

= 0.765 correct to three decimal places 1A

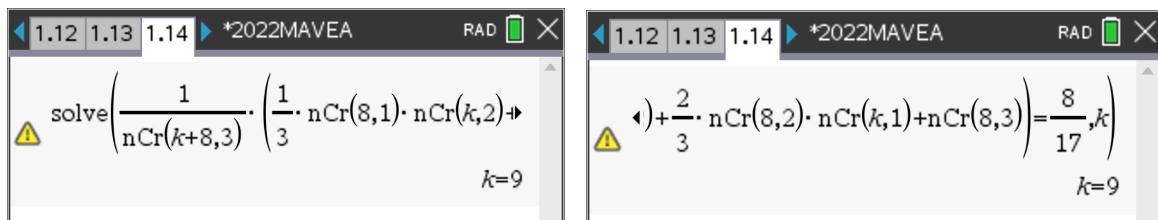


g. Solve $0 \times \binom{8}{0} \binom{k}{3} + \frac{1}{3} \times \binom{8}{1} \binom{k}{2} + \frac{2}{3} \times \binom{8}{2} \binom{k}{1} + 1 \times \binom{8}{3} \binom{k}{0} = \frac{8}{17}$ for k 1M

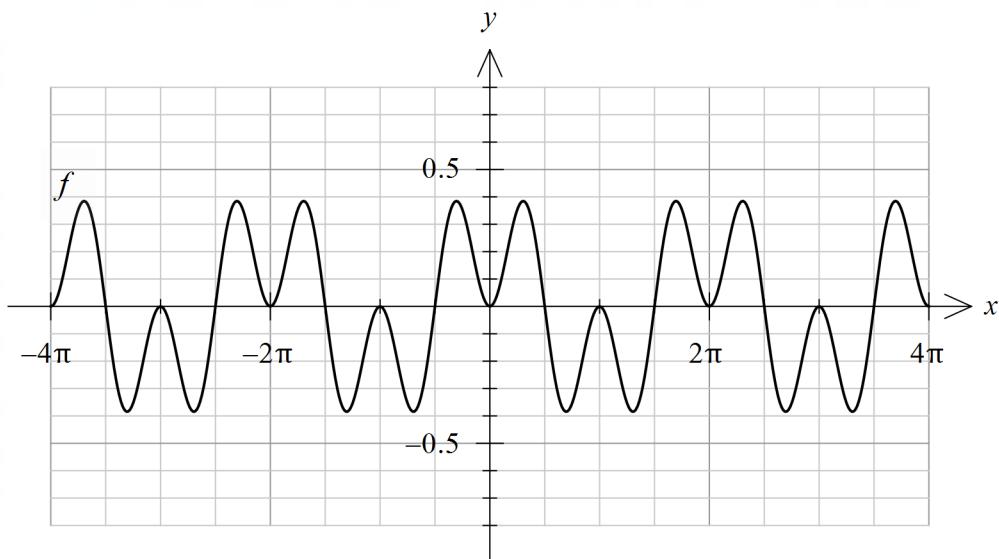
OR

Solve $\frac{3}{3} \times \frac{8}{8+k} = \frac{8}{17}$ for k 1M

$k = 9$ 1A

Question 3 (continued)**Question 4**

$$f : R \rightarrow R, f(x) = \sin^2(x) \cos(x)$$

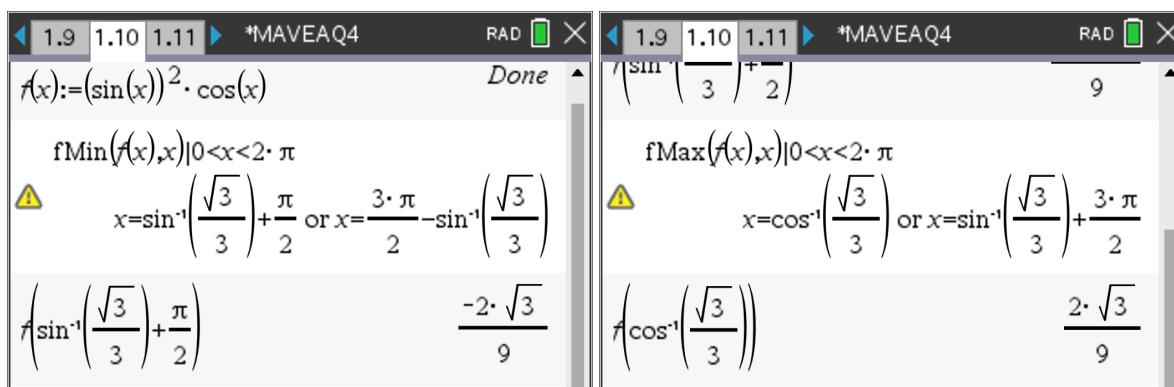


a. 2π

1A

b. $\left[-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9} \right]$

1A



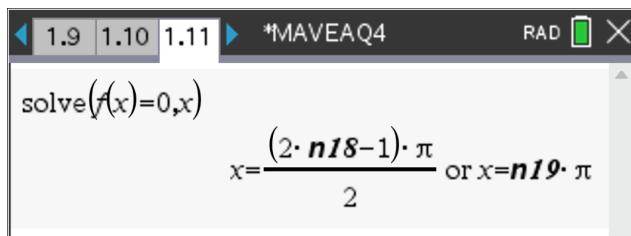
c. $x = \frac{\pi k}{2}, k \in Z$

1A

OR

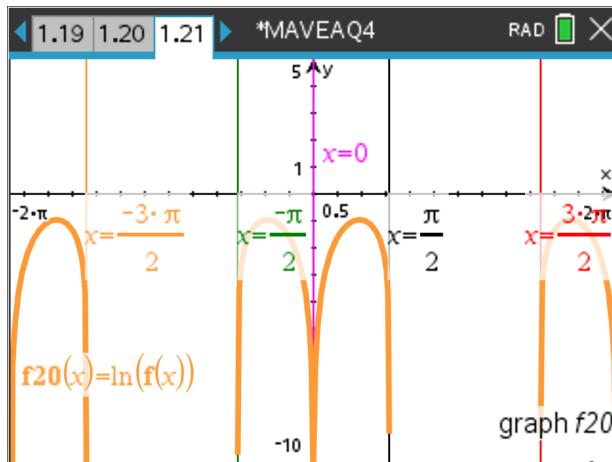
$$x = \frac{(2k-1)\pi}{2}, x = k\pi, k \in Z$$

1A

Question 4 (continued)

d. $2\pi k < x < \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$ or **1A**

$-\frac{\pi}{2} + 2\pi k < x < 2\pi k, k \in \mathbb{Z}$ **1A**

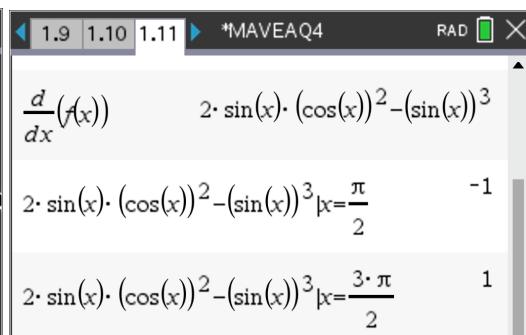
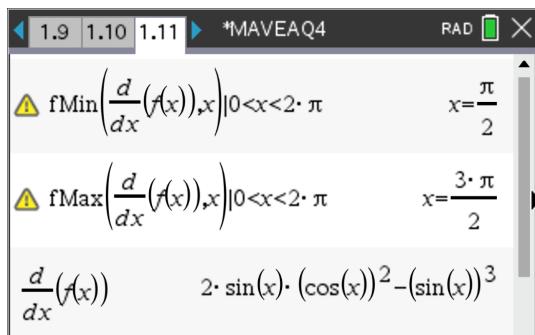


e. $A\left(\frac{\pi}{2}, -1\right), B\left(\frac{3\pi}{2}, 1\right)$ **1A**

$$m = \frac{2}{\pi}$$

$$y + 1 = \frac{2}{\pi} \left(x - \frac{\pi}{2} \right)$$

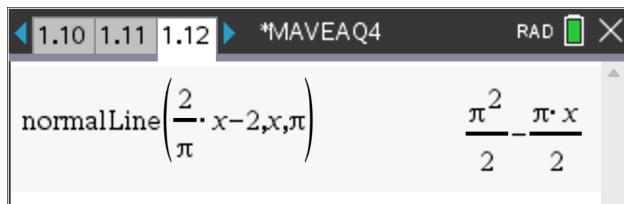
$$y = \frac{2}{\pi}x - 2$$
 1A



f.i. Midpoint $(\pi, 0)$, $m = -\frac{\pi}{2}$

Question 4 (continued)

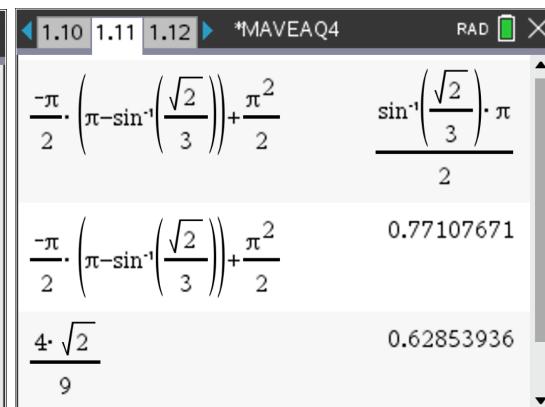
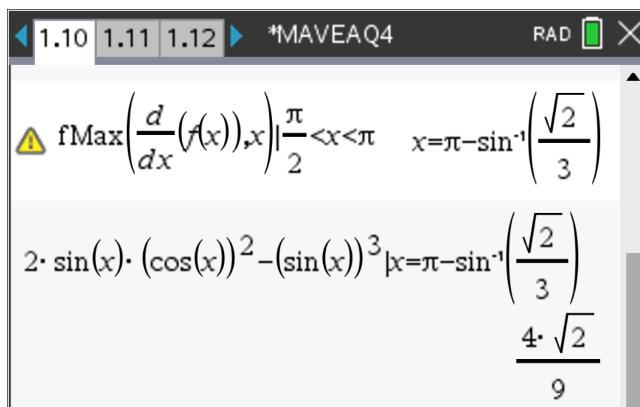
$$y = -\frac{\pi}{2}(x - \pi) = -\frac{\pi}{2}x + \frac{\pi^2}{2} \quad \mathbf{1A}$$



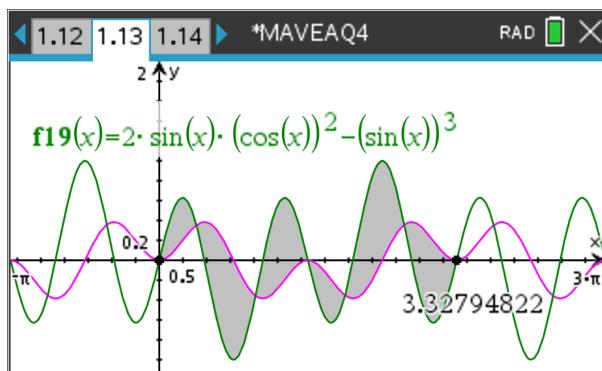
$$\text{f.ii. } C \left(\pi - \sin^{-1} \left(\frac{\sqrt{2}}{3} \right), \frac{4\sqrt{2}}{9} \right) \quad \mathbf{1M}$$

$$\begin{aligned} y &= -\frac{\pi}{2} \times \left(\pi - \sin^{-1} \left(\frac{\sqrt{2}}{3} \right) \right) + \frac{\pi^2}{2} \\ &= -\frac{\sin^{-1} \left(\frac{\sqrt{2}}{3} \right) \pi}{2} \\ &\neq \frac{4\sqrt{2}}{9} \end{aligned}$$

1M Show that



g. Area = 3.33 correct to two decimal places **1A**



Question 4 (continued)

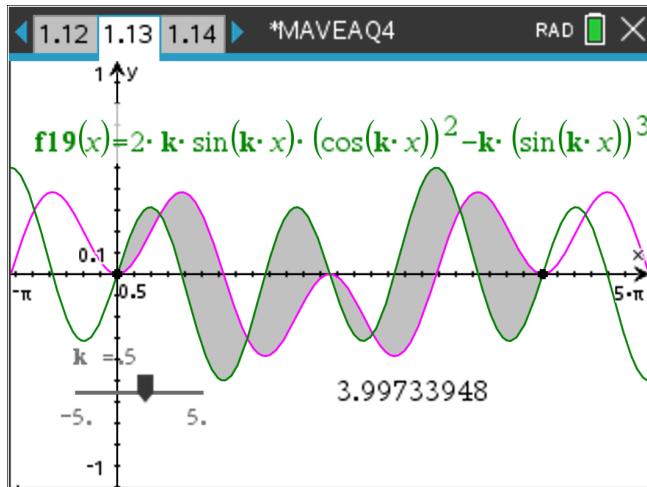
h. $\frac{2\pi}{k} = 4\pi$

$$k = \frac{1}{2}$$

$$g(x) = \sin^2(kx)\cos(kx), \quad g'(x) = 2k\sin(kx)\cos^2(kx) - k\sin^3(kx) \quad \mathbf{1M}$$

Area = 4.00

1A



- i.** Find the bounded area over the interval $\left[0, \frac{2\pi}{k}\right]$.

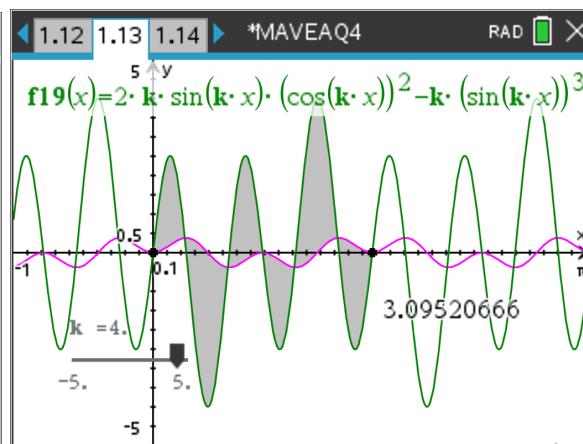
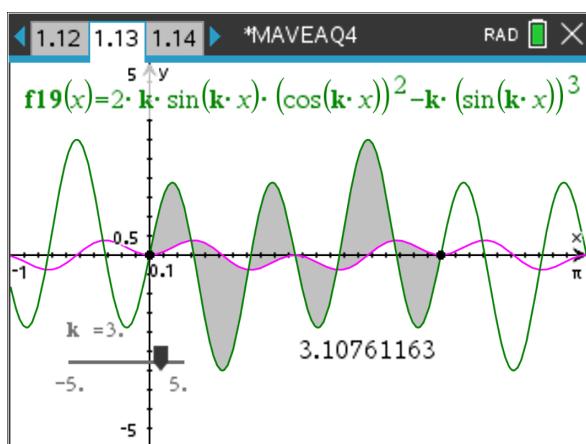
Use trial and error

When $k = 3$, the bounded area is 3.107...

When $k = 4$, the bounded area is 3.095...

$$k = 4$$

1A

**Question 5**

$$p : R \rightarrow R, p(x) = -(x-1)^3$$

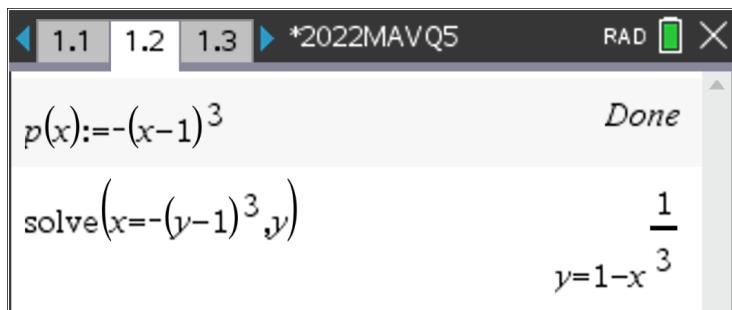
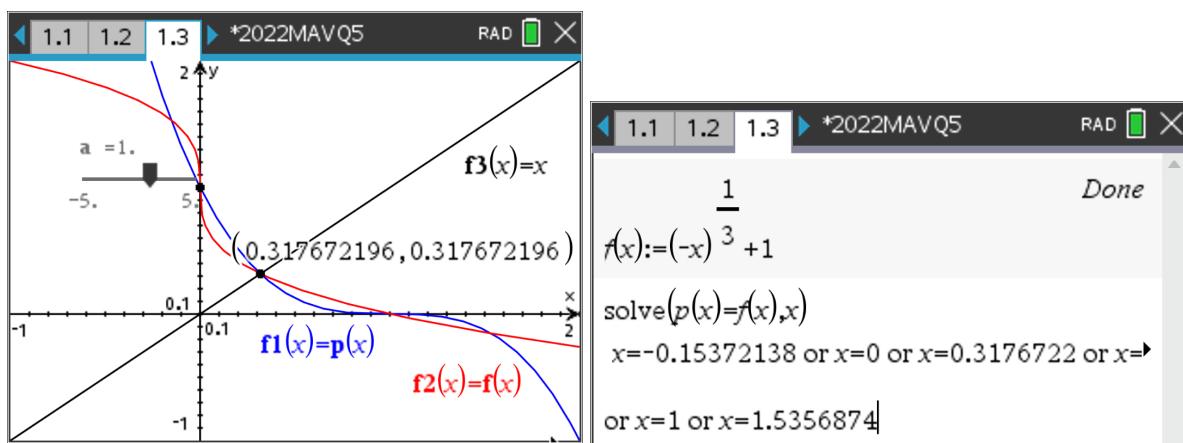
a. Let $y = -(x-1)^3$

Inverse swap x and y

$$x = -(y-1)^3$$

$$p^{-1}(x) = \sqrt[3]{-x} + 1 = 1 - \sqrt[3]{x}$$

1A (either form)

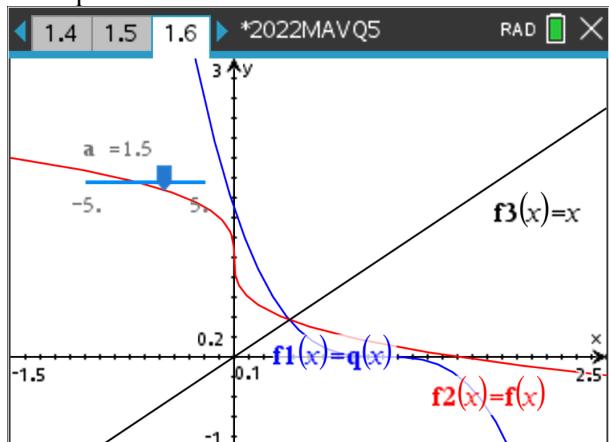
Question 5 (continued)**b. 5****1A**

c. $q : R \rightarrow R, q(x) = -a(x-1)^3, q^{-1} : R \rightarrow R, q^{-1}(x) = 1 - \sqrt[3]{\frac{x}{a}}$

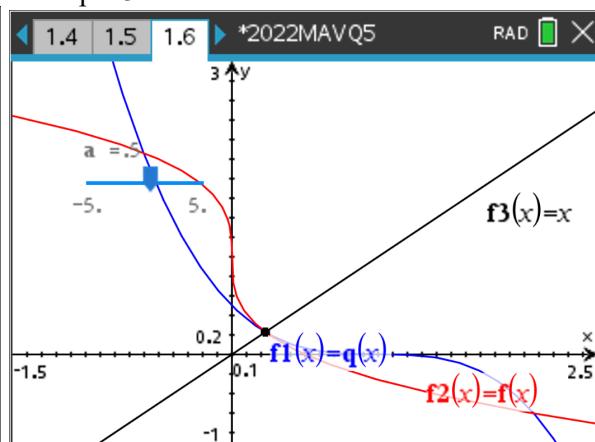
1, 3 or 5

1A

Example 1 solution



Example 3 solutions



d. Solve $q(x) = q^{-1}(x)$ and $\frac{d}{dx}(q(x)) = \frac{d}{dx}(q^{-1}(x))$ **1M**

$$a = \frac{16}{27}$$

1A

Question 5 (continued)

solve $\left(q(x) = f(x) \text{ and } \frac{d}{dx}(q(x)) = \frac{d}{dx}(f(x)), x \right) | a > 0$

$$x = \frac{-(3\sqrt{3} - 5)}{8} \text{ and } a = \frac{32}{27} \text{ or } x = \frac{1}{4} \text{ and } a = \frac{16}{27}$$

The screenshot shows the TI-Nspire CX CAS handheld calculator's interface. The top menu bar includes 'Edit', 'Action', 'Interactive', and various tool icons. The main workspace displays the following steps:

- Step 1:** A user-defined function $q(x) = -\alpha(x-1)^3$ is entered.
- Step 2:** The derivative $\frac{d}{dx}(q(x))$ is calculated as $-3 \cdot \alpha \cdot (x-1)^2$.
- Step 3:** A second derivative $\frac{d}{dx}(f(x))$ is shown, which is $\frac{-1}{3 \cdot \alpha \cdot \left(\frac{-x}{a}\right)^{\frac{2}{3}}}$.
- Step 4:** The equation $q(x) = x$ is solved for α , resulting in $\alpha = \frac{-x}{x^3 - 3x^2 + 3x - 1}$.
- Step 5:** A third derivative $\frac{d}{dx}(q(x)) = \frac{d}{dx}(f(x))$ is solved for a , leading to the equation $x = -0.5, x = 0.25$.

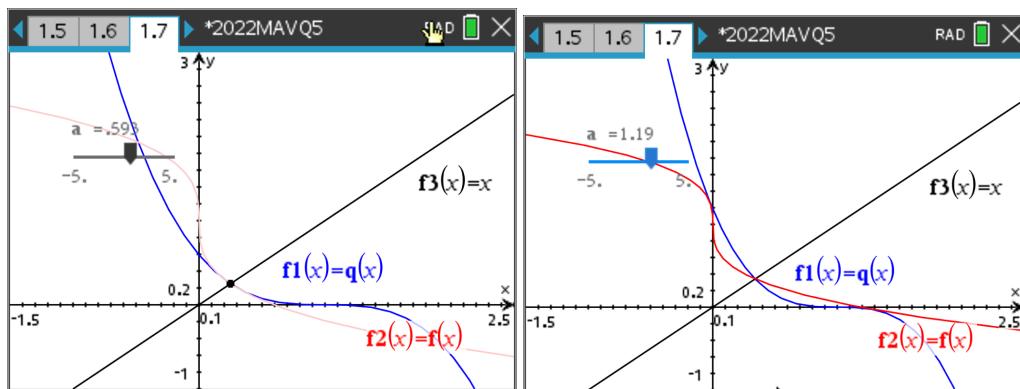
$$0.25 \Rightarrow x = \frac{1}{4}$$

$$a = \frac{-x}{x^3 - 3 \cdot x^2 + 3 \cdot x - 1}$$

$$a = \frac{16}{27}$$

e. Three solutions occur when $0 < a \leq \frac{16}{27}$ and $a = \frac{32}{27}$ 1A

$$a = \frac{16}{27} \qquad a = \frac{32}{27}$$

Question 5 (continued)

f. $t : R \rightarrow R, t(x) = -(x-1)(x^2 + bx + c)$

One solution for $x^2 + bx + c = 0$

$$b^2 - 4c = 0$$

$$b = \pm 2\sqrt{c}, b \neq -2$$

1A

Note if $b = -2$, $t(x) = -(x-1)(x^2 - 2x + 1) = -(x-1)^3$

Two solutions for $x^2 + bx + c = 0$ if one of the factors is $(x-1)$

$$x^2 + bx + c = (x-1)(x-c) = x^2 - (c+1)x + c$$

$$b = -c - 1, b \neq -2$$

1A

END OF SOLUTIONS