Gauss - Seidal Iteration Method

This method is modification of Jacobi's method. Here, the most recent approximations of the unknowns are used in the next step, the convergence in the Gauss - Seidal method is twice as fast as in Jacobi's method.

Qus. Apply Gauss - Seidal iteration method to solve the eqs.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Ans. Rewrite the given eqs as.

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad — (1)$$

We start from the approx. $x_0 = y_0 = z_0 = 0$.
Substituting $y = y_0$, $z = z_0$ in the right side of the first of eq (1), we get

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500.$$

Put $x = x_1$, $z = z_0$ in the second of eq (1), we get.

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

Illy $z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$

For second iteration

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

Third iteration

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

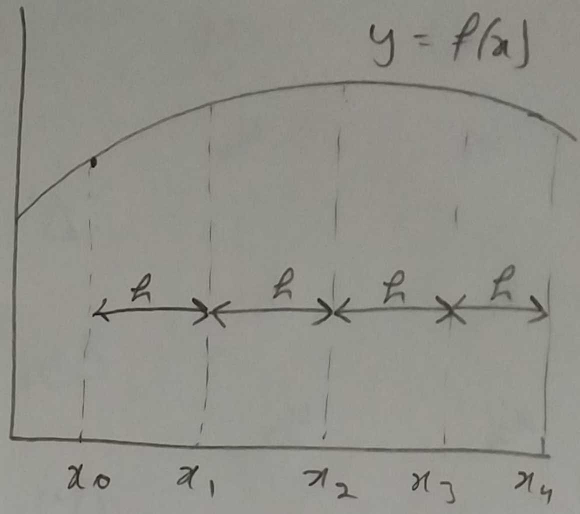
$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the 2^{nd} & 3^{rd} iterations being the same,

Hence the sol^n is $x = 1$, $y = -1$, $z = 1$

Finite Differences

Introduction



For a fn $y = f(x)$, finite differences refer to changes in values of y (dept. var) for any finite (equal or unequal) variation in x (indept var).

Shift or Increment Operator (E)

$$E f(x) = f(x+h)$$

Differencing operators

If $y_0, y_1, y_2, \cdots y_n$ be the values of y for corresponding values of $x_0, x_1, x_2, \cdots x_n$ then the differences of y are defined by $(y_1 - y_0), (y_2 - y_1), \cdots (y_n - y_{n-1})$ & are denoted by difference operators.

Forward Difference Operator Δ

$$\Delta f(x) = f(x+h) - f(x),$$ h is called interval of differencing

$$\therefore \Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

Also $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$

$$= y_2 - 2y_1 + y_0$$

$$\vdots$$

$$\Delta^n y_r = y_{n+r} - {}^nC_1\, y_{n+r-1} + {}^nC_2\, y_{n+r-2} \cdots + (-1)^n y_r$$

Forward Differences

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The arrow indicates the direction of differences from top to bottom.

Relation between Δ and E

$$\Delta \equiv E - 1 \quad \text{or} \quad E = 1 + \Delta$$

Proof
$$\Delta y_n = y_{n+1} - y_n$$
$$= E y_n - y_n$$
$$\Delta y_n = (E - 1) y_n$$
$$\Rightarrow \Delta \equiv E - 1 \quad \text{or} \quad E = 1 + \Delta$$

Backward Difference Operator ∇

$$\nabla f(x) = f(x) - f(x - h)$$

or
$$\nabla y_n = y_n - y_{n-1}$$

Backward Differences

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

$\nabla y_1 = y_1 - y_0$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$, ... $\nabla^5 y_5 = \nabla^4 y_5 - \nabla^4 y_4$

Relation between ∇ and E

$$\nabla \equiv 1 - E^{-1}$$

Proof
$$\nabla y_n = y_n - y_{n-1}$$
$$= y_n - E^{-1} y_n$$
$$\nabla y_n = (1 - E^{-1}) y_n$$
$$\Rightarrow \nabla = 1 - E^{-1}$$

Central difference operator δ

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\delta y_n = y_{n+1/2} - y_{n-1/2}$$

Central Differences

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Relation between δ and E

$$\delta \equiv E^{1/2} - E^{-1/2}$$

Proof

$$\delta y_n = y_{n+\frac{1}{2}} - y_{n-1/2}$$

$$= E^{1/2} y_n - E^{-1/2} y_n$$

$$\delta y_n = (E^{1/2} - E^{-1/2}) y_n$$

$$\therefore \delta = (E^{1/2} - E^{-1/2})$$

observation :- It is only the notation which changes & not the difference.

$$\because y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}$$

Mean Value operator or Averaging operator μ

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

or $$\mu y_x = \frac{1}{2}\left[y_{x+h/2} + y_{x-\frac{h}{2}}\right],$$ h is interval of differencing

Relation between μ and E

$$\mu y_n = \frac{1}{2}\left(y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(E^{1/2} y_n + E^{-1/2} y_n\right)$$

$$\mu y_n = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) y_n$$

$$\therefore \mu \equiv \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

Result :- Relation b/w E and D, where $D \equiv \frac{d}{dx}$

we know $y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \cdots$ By T. thm

$$= y(x) + h D y(x) + \frac{h^2}{2!} D^2 y(x) + \cdots$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 + \cdots\right) y(x)$$

$$E y(x) = e^{hD} y(x)$$

$$\therefore E = e^{hD}, \quad D \equiv \frac{d}{dx}$$

~~Unit III Numerical Methods~~

~~Solution of algebraic and transcedental equation~~

Result Relation b/w Δ and D, where $D \equiv \frac{d}{dx}$

we know that $\Delta \equiv E - 1$

$$\Delta = e^{hD} - 1 \qquad \because E = e^{hD}$$

Result: Relation b/w ∇ & D

we know that $\nabla \equiv 1 - E^{-1} = 1 - e^{-hD}$

Relation:- Relation b/w Δ and ∇

we know that $E \equiv 1 + \Delta$ — (1)

Also $E^{-1} \equiv 1 - \nabla$

$$\Rightarrow E = \frac{1}{1-\nabla} \quad \text{— (2)}$$

$$1 + \Delta = \frac{1}{1-\nabla} \quad \text{from (1) \& (2)}$$

$$\Delta = \frac{1}{1-\nabla} - 1$$

$$\Delta = \frac{\nabla}{1-\nabla}$$

Result:- Relation b/w μ, δ and E

we have $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

Also $\delta = (E^{1/2} - E^{-1/2})$

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$\mu\delta = \frac{1}{2}(E - E^{-1})$$