Ques 1 The function $f(x)$ is given by.

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$f(x+2\pi) = f(x)$$

Draw its graph and find its fourier series and hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Ans. Fourier - Series expansion of $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$

$$= \frac{1}{\pi}\left[(-\pi x)\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \boxed{-\frac{\pi}{2}}$$

$$a_n + i b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi e^{inx}\,dx + \int_{0}^{\pi} x e^{inx}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{i\pi}{n} e^{inx}\Big|_{-\pi}^{0} + \left\{x\left(\frac{-i e^{inx}}{n}\right) - \left(-\frac{e^{inx}}{n^2}\right)\right\}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi i}{n}\left(1 - (-1)^n\right) - \frac{\pi i}{n}(-1)^n + \frac{1}{n^2}\left((-1)^n - 1\right)\right]$$

$$= \frac{i}{n}\left[1 - 2(-1)^n\right] - \frac{1}{\pi n^2}\left[1 - (-1)^n\right]$$

Equate real and imaginary parts.

$$a_n = -\frac{1}{\pi n^2}\left[1 - (-1)^n\right], \quad b_n = \frac{1 - 2(-1)^n}{n}$$

$$a_{2n} = 0, \quad a_{2n-1} = \frac{-2}{\pi(2n-1)^2}; \quad n = 1, 2, 3, \cdots$$

$$b_{2n} = \frac{-1}{2n}, \quad b_{2n-1} = \frac{3}{2n-1}, \quad n = 1, 2, 3, \cdots$$

$\therefore$ Fourier-Series expansion of $f(x)$ is

(10)

$$f(x) \sim -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty}\left[\frac{3\sin(2n-1)x}{2n-1} - \frac{\sin 2nx}{2n}\right]$$

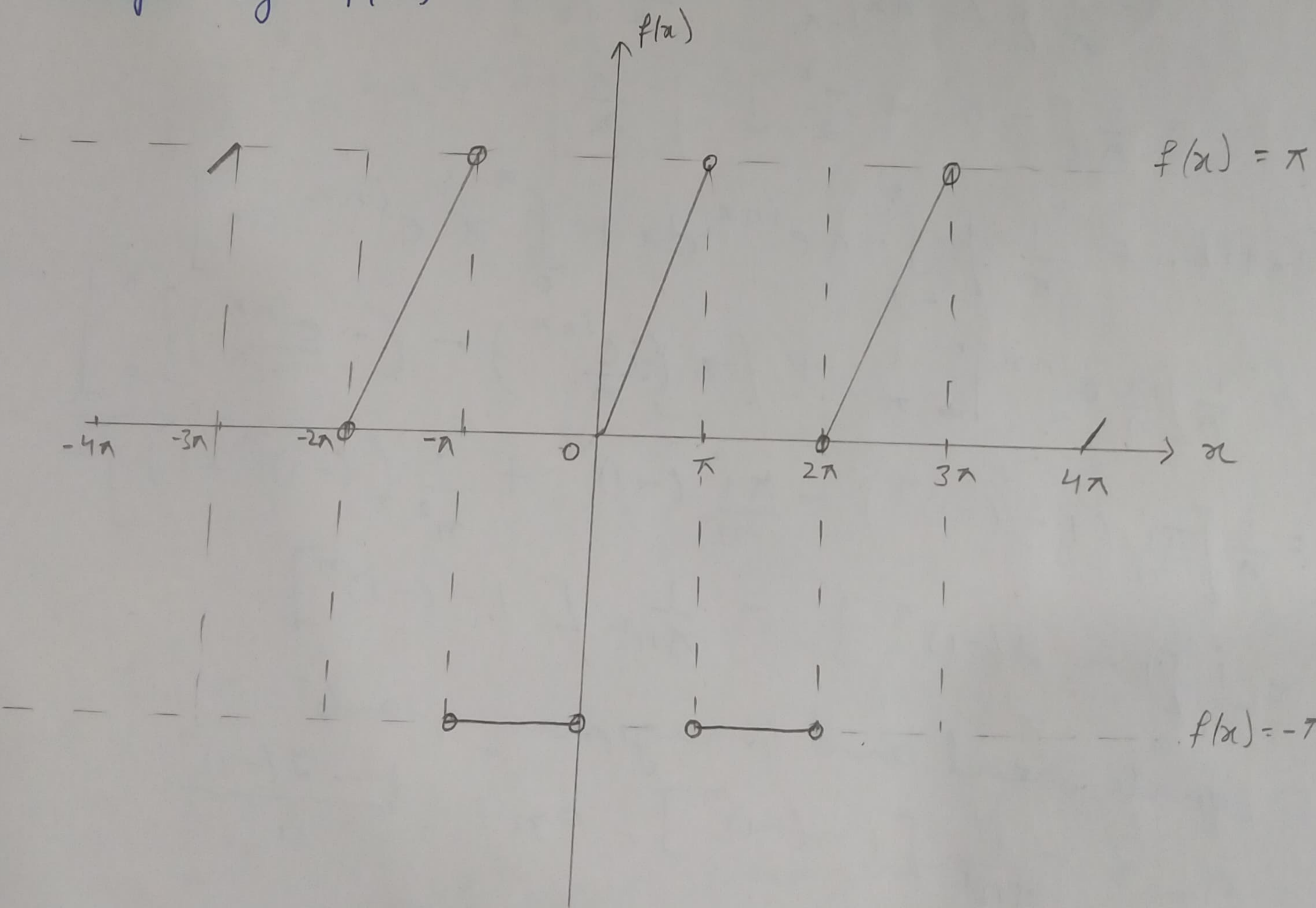
$$f(0) = \frac{f(0+0)+f(0-0)}{2} = \frac{0+(-\pi)}{2} = -\frac{\pi}{2}$$

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Graph of $f(x)$ is shown below.



⊙ denotes that the point is not in the graph.

Q2. Find the Fourier series to represent the function $f(x)$ given by. (11)

$$f(x) = \begin{cases} x & , 0 \le x \le \pi \\ 2\pi - x & , \pi \le x \le 2\pi \end{cases}$$ and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Ans. Fourier-series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\left[\int_0^\pi x\,dx + \int_\pi^{2\pi} (2\pi - x)\,dx\right]$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^\pi + \left(2\pi x - \frac{x^2}{2}\right)_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right] = \boxed{\pi}$$

$$a_n + i b_n = \frac{1}{\pi}\left[\int_0^\pi x e^{inx}\,dx + \int_\pi^{2\pi} (2\pi - x) e^{inx}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-i}{n} e^{inx}\right) - \left(-\frac{e^{inx}}{n^2}\right)\right\}_0^\pi + \left\{(2\pi - x)\left(-\frac{i}{n} e^{inx}\right) - (-1)\left(-\frac{e^{inx}}{n^2}\right)\right\}_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n} i(-1)^n + \frac{1}{n^2}\left[(-1)^n - 1\right) + \frac{i\pi}{n}(-1)^n - \frac{1}{n^2}\left(1 - (-1)^n\right)\right]$$

$$= \frac{-2}{\pi n^2}\left[1 - (-1)^n\right]$$

Equating real and imaginary

$$a_n = \frac{-2}{\pi n^2}\left[1 - (-1)^n\right], \quad b_n = 0; \quad n = 1, 2, 3.$$

$$a_{2n} = 0, \quad a_{2n-1} = \frac{-4}{\pi(2n-1)^2}; \quad n = 1, 2, \cdots$$

Fourier series expansion of $f(x)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$\therefore f(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=}^{\infty} \frac{1}{(2n-1)^2}$$

$$= 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Q3. If $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$

Prove that

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$$

Hence, show that

1) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$

2) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi-2}{4}$

Ans ~~Considering f(x) to~~ Fourier series expansion of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{\pi}(-\cos x\Big|_0^\pi = \frac{1}{\pi}(1+1)$

$$= \boxed{\frac{2}{\pi}}$$

$$a_n + i b_n = \frac{1}{\pi}\int_0^\pi \sin x\, e^{inx}\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{inx}}{1-n^2}(in \sin x - \cos x)\right]_0^\pi, \quad n \neq 1$$

$$\left[\because \int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]\right]$$

$$= \frac{1}{\pi(1-n^2)}\left[1+(-1)^n\right]; \; n \neq 1$$

$$\frac{1}{\pi}\left\{\frac{-1(0-(-1)^n)}{1-n^2} - \frac{1}{1-n^2}(0-1)\right\}$$

$$= \frac{-1}{\pi(n^2-1)}\left[1+(-1)^n\right]$$

$$\frac{(-1)^n+1}{\pi(1-n^2)}$$

Equate real and imaginary parts. (13)

$$a_n = \frac{-1}{\pi(n^2-1)}\left[1+(-1)^n\right], \quad b_n = 0; \quad n = 2,3,\ldots$$

$$a_{2n} = \frac{-2}{\pi(4n^2-1)}; \quad n = 1,2,3,\ldots$$

$$a_{2n-1} = 0, \quad b_n = 0; \quad n = 2,3,\ldots$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx = -\frac{1}{4\pi}(\cos 2x)_0^{\pi} = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{2}$$

$\therefore$ F. S. expansion of $f(x)$ is

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \quad \text{— (1)}$$

Take $x=0$

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$$

In eq (1), take $x = \frac{\pi}{2}$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)}$$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)} = \frac{\pi-2}{2\pi}$$

$$\therefore \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi-2}{4}$$