Change of interval

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha+2c)$. To change the problem to period 2π.

put $z = \frac{\pi x}{c}$ or $x = \frac{cz}{\pi}$ — (1)

So when $x = \alpha$ $\quad z = \frac{\alpha\pi}{c} = \beta$ (say)

& when $x = \alpha + 2c$ $\quad z = \frac{(\alpha+2c)\pi}{c} = \beta + 2\pi$

Thus, the function $f(x)$ of period $2c$ in $(\alpha, \alpha+2c)$ is transformed to the function $f\left(\frac{cz}{\pi}\right)$ [$= F(z)$ say] of period 2π in $(\beta, \beta+2\pi)$.

Hence $f\left(\frac{cz}{\pi}\right)$ can be expressed as the F.S.

$$F(z) = f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \quad \text{— (2)}$$

where
$$a_0 = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz$$

$$a_n = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz \, dz \quad \text{— (3)}$$

$$b_n = \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz \, dz$$

Making the inverse substitutions $z = \frac{\pi x}{c}$, $dz = \left(\frac{\pi}{c}\right) dx$ in (2) & (3), the Fourier exp. of $f(x)$ in the interval $(\alpha, \alpha+2c)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where $a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$, $a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) \cos \frac{n\pi x}{c} dx$

$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) \sin \frac{n\pi x}{c} dx.$ — (4)

Putting $\alpha = 0$ in (4), we get result for interval $(0, 2c)$ &

put $\alpha = -c$ in (4) " " " " $(-c, c)$

Q1. Find the fourier series for the function $f(x) = 2x - x^2$, $0 < x < 3$ and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(15)

An. Length of period $= 2l$ $\therefore$ $l = \frac{3}{2}$

Fourier. series of

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{3} + b_n \sin \frac{2n\pi x}{3}\right)$$

where $a_0 = \frac{2}{3} \int_0^3 (2x - x^2)\, dx = \frac{2}{3}\left(x^2 - \frac{x^3}{3}\right)_0^3 = 0$

$$a_n + i b_n = \frac{2}{3} \int_0^3 (2x - x^2)\, e^{i\frac{2n\pi x}{3}}\, dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(\frac{-3i\, e^{i\frac{2n\pi x}{3}}}{2n\pi}\right) - (2 - 2x)\left(-\frac{9\, e^{i\frac{2n\pi x}{3}}}{4n^2\pi^2}\right) + (-2)\left(\frac{27i}{8n^3\pi^3}\, e^{i\frac{2n\pi x}{3}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9i}{2n\pi} - \frac{9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right] = \frac{3i}{n\pi} - \frac{9}{n^2\pi^2}$$

Equating real and imaginary.

$$a_n = \frac{-9}{n^2\pi^2}, \quad b_n = \frac{3}{n\pi}; \quad n = 1, 2, 3, \ldots$$

F.S. of $f(x)$ is

$$f(x) \sim \frac{3}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin \frac{2n\pi x}{3} - \frac{3}{\pi n^2} \cos \frac{2n\pi x}{3}\right)$$

For $x = 3$ $\quad \frac{f(3-0) + f(3+0)}{2} = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\Rightarrow \quad \frac{f(3-0) + f(3+0)}{2} = -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{-\pi^2}{9}\left(\frac{-3+0}{2}\right) = \frac{\pi^2}{6}$$

Q2. Find the F.S. of

$$f(x) = \begin{cases} \pi x & ; 0 \le x < 1 \\ 0 & ; x = 1 \\ \pi(x-2) & ; 1 < x \le 2 \end{cases}$$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

A. $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$

where $a_0 = \int_0^1 \pi x \, dx + \int_1^2 \pi(x-2)\, dx$

$= \frac{\pi}{2}(x^2)_0^1 + \pi\left(\frac{x^2}{2} - 2x\right)_1^2 = \frac{\pi}{2} + \pi\left[-2 - \frac{1}{2} + 2\right) = 0$

$a_n + i b_n = \int_0^1 \pi x e^{in\pi x} dx + \int_1^2 \pi(x-2) e^{in\pi x} dx$

$= \pi\left[x\left(-\frac{i}{n\pi} e^{in\pi x}\right) - \left(-\frac{1}{n^2\pi^2} e^{in\pi x}\right)\right]_0^1$

$+ \pi\left[(x-2)\left(\frac{-i}{n\pi} e^{in\pi x}\right) - \left(-\frac{1}{n^2\pi^2} e^{in\pi x}\right)\right]_1^2$

$= \pi\left[\frac{-i(-1)^n}{n\pi} - \frac{1}{n^2\pi^2}(1-(-1)^n) - \frac{i(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}(1-(-1)^n)\right]$

$= -\frac{2i}{n}(-1)^n = \frac{2i}{n}(-1)^{n+1}$

Eq. real & img.

$a_n = 0, \; b_n = \frac{2}{n}(-1)^{n+1}; \; n = 1, 2, 3, \ldots$

$\therefore$ F.S. of $f(x)$ is

$f(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)$

$f(\frac{1}{2}) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2}$

Let $K_n = \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2}$, $K_{2n} = 0$, $K_{2n-1} = \frac{(-1)^{n+1}}{2n-1}$; $n = 1, 2, 3 \ldots$

$\therefore \frac{\pi}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$