Linear Equations

Newton's method has a second order of quadratic convergence.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{— (A)}$$

Suppose x_n differs from the root α by a small quantity ϵ_n so that $x_n = \alpha + \epsilon_n$ & $x_{n+1} = \alpha + \epsilon_{n+1}$ then

eq (A) becomes.

$$\cancel{\alpha} + \epsilon_{n+1} = \cancel{\alpha} + \epsilon_n - \frac{f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$= \epsilon_n - \frac{f(\alpha) + \epsilon_n f'(\alpha) + \frac{1}{2!}\epsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots}$$

(By Taylor's exp)

$$= \epsilon_n - \frac{\epsilon_n f'(\alpha) + \frac{1}{2!}\epsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots} \quad [\because f(\alpha) = 0]$$

$$= \frac{\epsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \epsilon_n f''(\alpha)]} = \frac{\epsilon_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}$$

This shows that the subsequent error at each step, is proportional to the square of the previous error & as such the cgs is quadratic.