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Q. Find $Z\{u_{n+2}\}$ if $Z\{u_n\} = \frac{z}{z-1} + \frac{z}{z^2+1}$

Ans. Given $Z\{u_n\} = U(z) = \frac{z}{z-1} + \frac{z}{z^2+1}$

From L.S.P. $Z\{u_{n+2}\} = z^2\left[Z\{u_n\} - u_0 - \frac{u_1}{z}\right]$ — (1)

Now, from initial value theorem

$$u_0 = \lim_{z\to\infty} U(z)$$

$$= \lim_{z\to\infty}\left[\frac{z}{z-1} + \frac{z}{z^2+}\right]$$

$$= \lim_{z\to\infty}\left[\frac{1}{1-\frac{1}{z}} + \frac{1/z}{1+\frac{1}{z^2}}\right] = 1+0$$

$\therefore u_0 = 1$ — (2)

Also, from I.V.T. $u_1 = \lim_{z\to\infty} z\left[U(z) - u_0\right]$

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Dem. $z^3 - z^2 + z - 1$

$$z\left(\frac{z^2\left(2 - \frac{1}{z} + \frac{1}{z^2}\right)}{z^3\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3}\right)}\right)$$

$$= \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$= \lim_{z\to\infty} z\left[\frac{2z^2 - z + 1}{(z^2+1)(z-1)}\right] = 2$$

$\therefore u_1 = 2$ — (3)

Using (2) & (3) in (1), we get.

$$Z\{u_{n+2}\} = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

$$\Rightarrow Z\{u_{n+2}\} = \frac{z(z^2 - z + 2)}{(z-1)(z^2+1)}$$

Ques: If $U(z) = \frac{2z^2 + 5z + 14}{(z-1)^4}$, evaluate u_2 & u_3

B.S. grewal

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Some Useful inverse Z- Transforms

	$U(z)$	$u_n = Z^{-1}[U(z)]$
1)	$\frac{1}{z-a}$	a^{n-1}
2)	$\frac{1}{z+a}$	$(-a)^{n-1}$
3)	$\frac{1}{(z-a)^2}$	$(n-1)\, a^{n-2}$
4)	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)\, a^{n-3}$
5)	$\frac{z}{z-a}$	a^n
6)	$\frac{z}{z+a}$	$(-a)^n$
7)	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8)	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)\, a^n u(n)$

Convolution Theorem

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$ then

$$Z^{-1}[U(z) \cdot V(z)] = \sum_{m=0}^{n} u_m \, u_{n-m} = u_n * v_n$$

where the symbol $*$ denotes the convolution operation

Proof We have

$$U(z) = \sum_{n=0}^{\infty} u_n z^{-n}, \quad V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$$

$$\therefore U(z) \cdot V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_n z^{-n} + \cdots \infty) \times$$

$$(v_0 + v_1 z^{-1} + v_2 z^{-2} + \cdots + v_n z^{-n} + \cdots \infty)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots + u_n v_0) z^{-n}$$

$$= Z(u_0 v_n + u_1 v_{n-1} + \cdots + u_n v_0)$$

Qu: Use convolution theorem to evaluate

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$$

Ans. We know that

$$Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n, \quad Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$$

$$\therefore Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-b}\right] = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m \cdot b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is G.P.}$$

$$= b^n \cdot \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

... inverse z-transform of $\dfrac{z^2}{(z-1)(2z-1)}$ using convolution theorem.

Ans. Let $U(z) = Z\{u_n\} = \dfrac{z}{z-1}$

& $V(z) = Z\{v_n\} = \dfrac{z}{2z-1} = \dfrac{1}{2}\left(\dfrac{z}{z-\frac{1}{2}}\right)$

$u_n = Z^{-1}\left(\dfrac{z}{z-1}\right) = 1^n$ & $v_n = Z^{-1}\left\{\dfrac{1}{2}\left(\dfrac{z}{z-\frac{1}{2}}\right)\right\} = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n$

Now, by convolution theorem.

$$Z^{-1}\left\{\frac{z^2}{(z-1)(2z-1)}\right\} = Z^{-1}\left\{\frac{z}{z-1} \cdot \frac{1}{2}\left(\frac{z}{z-\frac{1}{2}}\right)\right\} = u_n * v_n$$

$$= (1)^n * \left(\frac{1}{2}\right)^{n+1}$$

We know that

$$u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m} = \sum_{m=0}^{n} (1)^m \left(\frac{1}{2}\right)^{n+1-m}$$

$$= \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \cdots + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right] \quad \because S_n = \frac{a}{1-r}(1-r^n)$$

$$= \frac{1}{2}\left[2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$= 1 - \left(\frac{1}{2}\right)^{n+1}$$

Evaluation of inverse Z-Transform

1. Power series method.

In this method, we find the inverse Z-transf by expanding $U(z)$ in power series.

Qus. Find u_n if $U(z) = \log \frac{z}{z+1}$

Ans. Given $U(z) = \log \frac{z}{z+1} = \log \left(\frac{z+1}{z}\right)^{-1}$

$$= -\log \frac{z+1}{z} = -\log\left(1+\frac{1}{z}\right)$$

$\therefore U(z) = -\log(1+y)$ Put $\frac{1}{z} = y$

$$= -y + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \cdots$$

$\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\Rightarrow U(z) = -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^3} - \cdots$$

$$\Rightarrow U(z) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}$$

comparing with $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, we get

$$u_n = \begin{cases} 0 & \text{for } n=0 \\ \frac{(-1)^n}{n}, & \text{otherwise.} \end{cases}$$