Fourier Integrals and Fourier Transform

Fourier series treating various problems involving periodic functions.
Since, many practical problems involve non periodic functions. So, we extend the method of Fourier series to Fourier integrals for such functions.

Fourier Transform

The integral transform of a function $f(x)$ denoted by $I\{f(x)\}$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x)\, K(s, x)\, dx$$

where $K(s, x)$ is called the Kernel of the transform and is a function of s and x.
The funtion $f(x)$ is called the inverse transform of $\bar{f}(s)$.

(i) when $K(s, x) = e^{-sx}$

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx$$

ii) when $K(s, x) = e^{isx}$

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = F(s)$$

$\therefore$ $F\{f(x)\}$ is called the Fourier transform of $f(x)$.

Fourier Sine transform of $f(x)$ is

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx = F_s(s)$$

The Fourier cosine transform of $f(x)$ is

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx = F_c(s)$$

Fourier Integral Theorem

If $f(x)$ is piecewise cts in every finite interval & LHD, R.H.D at every pt

Theorem:- If (i) $f(x)$ satisfies Dirichlet's conditions in every interval $(-l, l)$ however large.

(ii) $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges; and $\int_{-\infty}^{\infty} |f(x)|\,dx$ exist, then $f(x)$ can be repted by F. I

then $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$,

The integral on the right hand side is called the Fourier integral.

Proof:- Consider a function $f(x)$ which satisfies the Dirichlet's conditions in every interval $(-l, l)$, so that we have.

$$f(x) = \frac{a_0}{2} + \Sigma\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \quad -① $$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

(2) Substituting the values of a_0, a_n & b_n in eq (1)

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\left\{\cos\frac{n\pi x}{l}\cdot\cos\frac{n\pi t}{l} + \sin\frac{n\pi x}{l}\cdot\sin\frac{n\pi t}{l}\right\}dt$$

$$\Rightarrow f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\sum_{n=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{n\pi(t-x)}{l}\,dt \quad -(2)$$

If we assume that $\int_{-\infty}^{\infty} |f(x)|$ converges, the first term on the R.H.S. of eq (2) approaches to 0 as $l \to \infty$, since

$$\left|\frac{1}{2l}\int_{-l}^{l} f(t)\,dt\right| \le \left|\frac{1}{2l}\int_{-\infty}^{\infty} f(t)\,dt\right|$$

Now, let us consider $\lambda_n = \frac{n\pi}{l}$,

hence $\lambda_{n+1} = \frac{(n+1)\pi}{l}$

then $\delta\lambda = \lambda_{n+1} - \lambda_n = \frac{\pi}{l} \Rightarrow \frac{1}{l} = \frac{\delta\lambda}{\pi}$

eq (2) becomes.

$$f(x) = 0 + \frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-l}^{l} f(t)\cos\lambda(t-x)\,dt$$

on $l \to \infty$

$$\boxed{f(x) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{t=-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda} \quad -(3)$$

Fourier Sine and Cosine Integral

By eq ③

$$f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{t=-\infty}^{\infty} f(t)\left[\cos \lambda t \cdot \cos \lambda x + \sin \lambda t \sin \lambda x\right] dt\, d\lambda \quad -④$$

If $f(t)$ is an odd function, then $f(t)\cos \lambda t$ is also an odd function while $f(t) \cdot \sin \lambda t$ is even, then

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \sin \lambda x \int_{t=0}^{\infty} \sin \lambda t\, f(t)\, dt\, d\lambda \quad -⑤$$

which is known as fourier sine integral.

Illy, if $f(t)$ is an even function, then

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \cos \lambda x \int_{t=0}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda \quad -⑥$$

which is known as fourier cosine integral.

$\int_0^{\infty} A(\lambda) \cos \lambda x\, d\lambda$, where $A(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(t) \cos \lambda t\, dt$

$\int_0^{\infty} A(\lambda) \cos \lambda x\, d\lambda = \frac{1}{2}\left[f(x+0) + f(x-0)\right]$, if f is piecewise cts on $[0, \infty)$

$= f(x)$, if f is cts on $[0, \infty)$

$\int_0^{\infty} B(\lambda) \sin \lambda x\, d\lambda = \frac{1}{2}\left[f(x+0) + f(x-0)\right]$, if f is piecewise cts on $[0, \infty)$

$= f(x)$, if f is cts on $[0, \infty)$

where $B(\lambda) = \frac{2}{\pi} \int_0^{\infty} f(t) \sin \lambda t\, dt$

(3) Q1 Express $f(x) = \begin{cases} 1, & \text{for } 0 \le x \le \pi \\ 0, & \text{for } x > \pi \end{cases}$

as a Fourier Sine Integral and hence evaluate

$$\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda} \cdot \sin(x\lambda)\, d\lambda$$

Ans. The Fourier sine integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \int_0^\pi 1\cdot \sin\lambda t\, dt\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \left[-\frac{\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left(-\frac{\cos\pi\lambda}{\lambda} + \frac{1}{\lambda}\right) d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left(\frac{1-\cos\lambda\pi}{\lambda}\right) d\lambda$$

$$\int_0^\infty \sin\lambda x\left(\frac{1-\cos\pi\lambda}{\lambda}\right) d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \pi/2, & 0 \le x < \pi \\ 0, & x > \pi \end{cases}$$

At $x = \pi$, $f(x)$ has a point of discontinuity.

$$f(x) = \frac{1}{2}\left[f(\pi-0) + f(\pi+0)\right] = \frac{1}{2}[1+0] = \frac{1}{2}$$

$$\int_0^\infty \sin\lambda x\, \frac{(1-\cos\pi\lambda)}{\lambda}\, d\lambda = \frac{\pi}{2}\times\frac{1}{2} = \frac{\pi}{4}$$

Q2. Express the function $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ as a Fourier integral. Hence evaluate

$$\int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda$$

Ans. The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cdot\cos\lambda(t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-1}^{1} 1\cdot\cos\lambda(t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \left(\frac{\sin\lambda(t-x)}{\lambda}\right)\Bigg|_{-1}^{1} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x) - \sin\lambda(-1-x)}{\lambda}\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1+x) + \sin\lambda(1-x)}{\lambda}\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda \qquad \left(\because \sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right.$$

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \pi/2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Put $x=0$

$$\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2} \quad \text{—(1)}$$

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

At $|x|=1$, i.e $x=\pm 1$, $f(x)$ is discontinuous

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda = \frac{1}{2}\left[f(1-0) + f(1+0)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \boxed{\frac{\pi}{4}}$$

(4)

Put $x=0$ in eq (1)

$$\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda = \frac{\pi}{2} \Rightarrow \boxed{\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}}$$

Inversion later

Qus 3 If $f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$, then

prove that $f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos sx + \cos s(\pi - x)}{1 - s^2}\, ds$

and hence find the value of the integral

$$\int_0^\infty \frac{\cos \pi s/2}{1 - s^2}\, ds.$$

Ans. Since the function $f(x)$ is defined in $(-\infty, \infty)$ we find complex fourier transform of $f(x)$

$$F(s) = \bar{f}(s) = \int_{-\infty}^{\infty} e^{ist} f(t)\, dt$$

$$= \int_0^\pi e^{ist} \sin t\, dt$$

$a = is,\ b = 1$

$$= \left[\frac{e^{ist}}{1 - s^2}(is \sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{i\pi s}}{1 - s^2}(0 + 1) - \frac{1}{1 - s^2}(0 - 1)$$

$$= \frac{1}{1 - s^2}(1 + e^{is\pi})$$

Using Inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \underbrace{\bar{f}(s)}_{F(s)}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left(\frac{1 + e^{is\pi}}{1 - s^2}\right) ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isx} + e^{-is(x-\pi)}}{1-s^2}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ \frac{\cos sx - i\sin sx + \cos s(x-\pi) \,\bar{\oplus}\, i\sin s(x-\pi)}{1-s^2} \right\} ds$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_{0}^{\infty} \frac{\cos sx + \cos s(x-\pi)}{1-s^2}\, ds \quad \left[\text{By the prop. of def. integral} \right.$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos sx + \cos s(\pi - x)}{1-s^2}\, ds$$

Also, put $x = \frac{\pi}{2}$

$$1 = \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos s\frac{\pi}{2} + \cos s\frac{\pi}{2}}{1-s^2}\, ds$$

$$1 = \frac{1}{\pi}\int_{0}^{\infty} \frac{2\cos \frac{s\pi}{2}}{1-s^2}\, ds$$

$$\boxed{\int_{0}^{\infty} \frac{\cancel{2}\cos s\pi/2}{1-s^2}\, ds = \frac{\pi}{2}}$$

Fourier Transform and Inversion Formulae

1) Complex form of Fourier integral is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt\, d\lambda$$

Replace λ by s.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left[\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt\right] ds$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, F(s)\, ds \quad \text{— (Inversion formula)}$$

Inverse F.T.

where $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$

$F(s)$ is called the Fourier transform of $f(t)$

2) Fourier Sine integral is

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin\lambda x \left[\int_{0}^{\infty} f(t)\sin\lambda t\, dt\right] d\lambda$$

Replacing λ by s.

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin sx \underbrace{\left[\int_{0}^{\infty} f(t)\sin st\, dt\right]}_{\text{F. Sine Transf}} ds$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin sx \cdot F_s(s)\, ds \quad \text{— Inversion formula}$$

inverse F. Sine transf

where $F_s(s) = \int_{0}^{\infty} f(t)\sin st\, dt$

$F_s(s)$ is called the Fourier Sine transform

3) Fourier Cosine integral is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t \, dt \, d\lambda$$

Replacing λ by s.

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos sx \left(\int_0^\infty f(t)\cos st \, dt\right) ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos sx \; F_c(s)\, ds \quad - \text{Inversion formula}$$

where $F_c(s) = \int_0^\infty \cancel{\cos sx \; F_c(s)\, ds}$ $\cos st \; f(t)\, dt$

$F_c(s)$ is called Fourier cosine transform.

Qus. Find the Fourier Sine transform of $e^{-|x|}$.
Hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx$

Ans. $F_s(s) = \int_0^\infty f(x)\sin sx \, dx$

$$= \int_0^\infty e^{-|x|}\sin sx \, dx$$

$a=-1,\ b=s$

$$= \left[\frac{e^{-x}}{1+s^2}\left\{-\sin sx - s\cos sx\right\}\right]_0^\infty$$

$$= 0 - \frac{1}{1+s^2}(-s) = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin sx \, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\, ds \Rightarrow \int_0^\infty \frac{s\sin sx}{1+s^2}\,ds = \frac{\pi}{2}e^{-x}$$

Replace 's' by 'x' and 'x' by 'm'

$$\boxed{\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}}$$

Properties of Fourier Transform

1. **Linearity property:-** If $F(s)$ and $G(s)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, then

$$F[a f(x) + b g(x)] = a F(s) + b G(s)$$

where a and b are constants.

Proof:- By the defination of Fourier transform

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$\& \quad G(s) = G(g(x)) = \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$\therefore F[a f(x) + b g(x)] = \int_{-\infty}^{\infty} [a f(x) + b g(x)] e^{isx}\, dx$$

$$= a \int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$= a F(s) + b G(s)$$

2. **Change of scale property**

If $F(s)$ is the complex fourier transform of $f(x)$, then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right); \quad a \neq 0$$

Proof we have

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \quad -①$$

$$\therefore F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx \quad -②$$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$$\Rightarrow dx = \frac{dt}{a}$$

$$F[f(ax)] = \int_{-\infty}^{\infty} e^{ist/a} f(t) \frac{dt}{a} = \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)t} f(t)\, dt$$

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

3) Shifting Theorem

If $F(s)$ is a complex Fourier transform of $f(x)$, then

$$F\{f(x-a)\} = e^{ias} F(s)$$

Proof :- we have $F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\therefore F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$

put $x-a = t \Rightarrow dx = dt$

$x = a+t$

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{is(a+t)} f(t) dt$$

$$= e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t)\, dt \quad =$$

$$F\{f(x-a)\} = e^{isa} F(s)$$

4) Modulation Theorem

If $F(s)$ is complex Fourier transform of $f(x)$, then

$$F\{f(x) \cdot \cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)]$$

Proof We have

$$F(s) = F(f(x)) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$$

$$\therefore F[f(x) \cdot \cos ax] = \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{isx} \left(\frac{e^{iax} + e^{-iax}}{2} \right) dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[f(x)\, e^{i(s+a)x} + f(x)\, e^{i(s-a)} \right] dx$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

IIIly, $F\{f(x) \cdot \sin ax\} = \frac{1}{2i}\left[F(s-a) - F(s+a)\right]$

5. If $F_s(s)$ and $F_c(s)$ are Fourier Sine and Cosine transforms of $f(x)$ resp, then

i) $F_s(x f(x)) = -\frac{d}{ds}[F_c(s)]$

ii) $F_c(x f(x)) = \frac{d}{ds}[F_s(s)]$

<u>Proof</u> (i) $\frac{d}{ds}[F_c(s)] = \frac{d}{ds}\int_0^\infty f(x)\cos sx \, dx$

$$= \int_0^\infty f(x)\left[\frac{d}{ds}\cos sx\right]dx$$

$$= \int_0^\infty f(x)\{-x\sin sx\}\,dx$$

$$= -\int_0^\infty \sin sx\,\{x f(x)\}\,dx$$

$$\frac{d}{ds}\{F_c(s)\} = -F_s\{x f(x)\}$$

$$\Rightarrow F_s\{x f(x)\} = -\frac{d}{ds}\{F_c(s)\}$$

IIly, we can prove (ii)

Que: Find the Fourier transform of e^{-x^2}, Hence find the fourier transform of

i) e^{-ax^2} ii) $e^{-x^2/2}$

iii) $e^{-4(x-3)^2}$ iv) $e^{-x^2}\cos 2x$

Ans. Fourier transform of e^{-x^2} is given by

$$F(e^{-x^2}) = \int_{-\infty}^{\infty} e^{isx} e^{-x^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-(x^2 - isx)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left(x - \frac{is}{2}\right)^2 + \frac{s^2}{4}\right\}} dx \qquad \left[\left\{x^2 - isx + \left(\frac{is}{2}\right)^2\right\} - \left(\frac{is}{2}\right)^2\right] \quad \left(x - \frac{is}{2}\right)^2$$

$$= e^{-s^2/4} \int_{-\infty}^{\infty} e^{-\left(x - \frac{is}{2}\right)^2} dx$$

Put $x - \frac{is}{2} = z$

$dx = dz$

$$= e^{-s^2/4} \int_{-\infty}^{\infty} e^{-z^2} dz = 2 e^{-\frac{s^2}{4}} \int_0^{\infty} e^{-z^2} dz$$

$$\because \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$$

$$= \cancel{2} e^{-\frac{s^2}{4}} \frac{\sqrt{\pi}}{\cancel{2}}$$

$$\boxed{F(e^{-x^2}) = \sqrt{\pi}\, e^{-s^2/4} = F(s)}$$

$$\int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= 2\int_0^{\infty} e^{-z^2} dz$$

Put $z = \sqrt{t}$

$dz = \frac{1}{2\sqrt{t}} dt$

$$= \cancel{2}\int_0^{\infty} e^{-t} \frac{1}{\cancel{2}\sqrt{t}} dt$$

$$= \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

(i) We have $F\{f(bx)\} = \frac{1}{b} F\left(\frac{s}{b}\right)$

$$F(e^{-ax^2}) = F\left\{e^{-(\sqrt{a}\,x)^2}\right\} = \frac{1}{\sqrt{a}}\sqrt{\pi}\, e^{-s^2/4a}$$

$$= \frac{\sqrt{\pi}}{\sqrt{a}} e^{-s^2/4a}$$

ii) Here $a = \frac{1}{2}$

$$= \sqrt{2\pi}\, e^{-s^2/4\times\frac{1}{2}} = \sqrt{2\pi}\, e^{-s^2/2}$$

(iii) Here, $f(x) = -4x^2$

$F(f(x)) = F(e^{-(2x)^2})$

$= \frac{1}{2} F(\frac{s}{2}) = \frac{1}{2}\sqrt{\pi}\, e^{-\frac{s^2}{4\times 4}} = \frac{\sqrt{\pi}}{2} e^{-s^2/16}$

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iv) By Modulation Theorem

$F(f(x)\cos ax) = \frac{1}{2}\{F(s+a) + F(s-a)\}$

$F(e^{-x^2}\cos 2x) = \frac{1}{2}[\sqrt{\pi}\, e^{-\frac{1}{4}(s+2)^2} + \sqrt{\pi}\, e^{-\frac{1}{4}(s-2)^2}]$

Convolution

The Convolution of 2 functions $f(x)$ and $g(x)$ over the integral $(-\infty, \infty)$ is defined as.

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$F[f(x-a)] = e^{ias} f(s)$

$e^{-4(x-3)^2} = \frac{\sqrt{\pi}}{2} e^{3is} e^{-s^2/16}$ Fully

Q. Find the fourier transform of xe^{-ax^2}, $a>0$

Ans $F(e^{-ax^2}) = \frac{\sqrt{\pi}}{\sqrt{a}} e^{-s^2/4a} = F(s)$

Now, if $f(x) = e^{-ax^2}$

$f'(x) = -2ax e^{-ax^2}$

$F[f'(x)] = -is F(s)$

$F(xe^{-ax^2}) = \frac{+is}{+2a} \sqrt{\frac{\pi}{a}} e^{-s^2/4a}$

(A) Qus. Find the Fourier Sine transform of $\frac{e^{-ax}}{x}$)

Ans. $F_s(s) = \int_0^\infty f(x) \sin sx \, dx$

$= \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = I$ (say).

Differentiating under the integral sign w.r.t 's'

$\frac{dI}{ds} = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos sx \,.\, \cancel{x} \, dx$

$= \int_0^\infty e^{-ax} \cos sx \, dx$

$= \left[\frac{e^{ax}}{a^2+s^2} \left\{ -a \cos sx + s \sin sx \right\} \right]_0^\infty$

$= 0 - \frac{1}{a^2+s^2} \{-a\} = \frac{a}{a^2+s^2}$

$dI = \frac{a}{a^2+s^2} ds$

On integ both sides, we get

$I = \tan^{-1}\frac{s}{a} + c$

But $F(s) = 0$

when $s = 0 \Rightarrow I = 0$

$0 = 0 + c \Rightarrow \boxed{c = 0}$

$I = \tan^{-1}\frac{s}{a}$

$$\boxed{F_s(s) = \tan^{-1}\frac{s}{a}}$$

Ques: Find Fourier Sine transform of $\dfrac{1}{x(x^2+a^2)}$

Ans: $F_s(s) = \displaystyle\int_0^{\infty} \frac{1}{x(x^2+a^2)} \sin sx \, ds = I \text{ (say)}$ —— ①

Diff under the integ sign w.r.t 's', we have

$$\frac{dI}{ds} = \int_0^{\infty} \frac{1}{\cancel{x}(x^2+a^2)} \cos sx \cdot \cancel{x} \, ds$$

$$\frac{dI}{ds} = \int_0^{\infty} \frac{\cos sx}{x^2+a^2} \, ds \quad \text{—— ②}$$

diff ② again w.r.t 's'.

$$\frac{d^2I}{ds^2} = \int_0^{\infty} -\frac{x \sin sx}{x^2+a^2} \, ds = \int_0^{\infty} -\frac{x^2 \sin sx}{x(x^2+a^2)} \, ds$$

$$= \int_0^{\infty} \frac{[a^2 - (x^2+a^2)] \sin sx}{x(x^2+a^2)} \, ds$$

$$= a^2 \int_0^{\infty} \frac{\sin sx}{x(x^2+a^2)} \, ds - \int_0^{\infty} \frac{\sin sx}{x} \, ds$$

$$\frac{d^2I}{ds^2} = a^2 I - \frac{\pi}{2} \qquad \left[\because \int_0^{\infty} \frac{\sin \lambda}{\lambda} \, d\lambda = \frac{\pi}{2}\right]$$

$$\frac{d^2I}{ds^2} - a^2 I = -\frac{\pi}{2}$$

$$(D^2 - a^2) I = -\frac{\pi}{2} \, ; \quad \frac{d}{ds} = D$$

For C.F. $D^2 - a^2 = 0$, $D = \pm a$

C.F $= c_1 e^{as} + c_2 e^{-as}$

$$\text{P.I} = \frac{1}{D^2 - a^2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} \frac{1}{(D^2-a^2)} e^{0s} = -\frac{\pi}{2} \cdot \frac{1}{-a^2} = \frac{\pi}{2a^2}$$

$$I = c_1 e^{as} + c_2 e^{-as} + \frac{\pi}{2a^2} \quad \text{—— ③}$$

$$\frac{dI}{ds} = ac_1 e^{as} - ac_2 e^{-as} \quad -(4)$$

From (1), $s=0$, $I=0$

From (2), $s=0$, $\frac{dI}{ds} = \int_0^\infty \frac{1}{x^2+a^2}\,dx = \left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)_0^\infty$

$$\frac{dI}{ds} = \frac{\pi}{2a}$$

At $s=0$, eq^n (3) & (4) gives.

$$0 = c_1 + c_2 + \frac{\pi}{2a^2} \quad -(5)$$

$$\frac{\pi}{2a} = ac_1 - ac_2 \Rightarrow c_1 - c_2 = \frac{\pi}{2a^2} \quad -(6)$$

solving (5) & (6)

$c_1 = 0$, $\quad c_2 = -\frac{\pi}{2a^2}$

The Sol^n is

$$I = 0 + \left(-\frac{\pi}{2a^2}\right)e^{-as} + \frac{\pi}{2a^2}$$

$$\boxed{F_s(s) = \frac{\pi}{2a^2}\left(1 - e^{-as}\right)}$$

Ques. Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and use it to evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$

Ans. $F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$= \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{-s^2} + (-2)\frac{e^{isx}}{-is^3}\right]_{-1}^{1}$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} - \frac{2x\, e^{isx}}{s^2} + \frac{2\, e^{isx}}{is^3}\right]_{-1}^{1}$$

$$= \left[\left\{0 - \frac{2e^{is}}{s^2} + \frac{2e^{is}}{is^3}\right\} - \left\{0 + \frac{2e^{-is}}{s^2} + \frac{2e^{-is}}{is^3}\right\}\right]$$

$$= -\frac{2}{s^2}\left(e^{is} + e^{-is}\right) + \frac{2}{is^3}\left(e^{is} - e^{-is}\right)$$

$$= -\frac{4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is} - e^{-is}}{2i}\right)$$

$$= \frac{-4}{s^2}\cos s + \frac{4}{s^3}\sin s$$

$$F(s) = -\frac{4}{s^3}(s\cos s - \sin s)$$

Using Inversion formula, for Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, F(s)\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\left\{\frac{-4}{s^3}(s\cos s - \sin s)\right\} ds$$

(c)

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds$$

$$+\frac{2i}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\sin sx\,ds$$

[$\because$ the integrand in first integral is even and in second is odd]

$$\therefore f(x) = \frac{-4}{\pi}\int_{0}^{\infty}\left\{\frac{s\cos s - \sin s}{s^3}\right\}\cos sx\,ds$$

$$\Rightarrow \int_{0}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos sx\,ds = -\frac{\pi}{4}f(x)$$

$$= \begin{cases} -\frac{\pi}{4}(1-x^2), & |x|<1 \\ 0, & |x|>1 \end{cases}$$

For $x=\frac{1}{2}$

$$\int_{0}^{\infty}\left(\frac{s\cos s - \sin s}{s^3}\right)\cos\frac{s}{2}\,ds = -\frac{\pi}{4}\times\frac{3}{4}$$

Replace s by x

$$\int_{0}^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = \frac{-3\pi}{16}$$

Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

Inverse Fourier Sine Transform

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin sx \; F_s(s)\, ds$$

Inverse Fourier Cosine Transform

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos sx \; F_c(s)\, ds$$

Q. Find the Inverse Fourier transform of

$$F(s) = e^{-|s|y}$$

Ans. The inverse fourier transform of F(s) is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, e^{-|s|y}\, ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx} \cdot e^{sy}\, ds + \int_{0}^{\infty} e^{-isx}\, e^{-sy}\, ds\right]$$

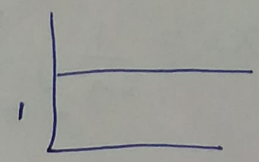
$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{s(y-ix)}\, ds + \int_{0}^{\infty} e^{-s(y+ix)}\, ds\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{e^{s(y-ix)}}{y-ix}\right\}_{-\infty}^{0} + \left\{\frac{e^{-s(y+ix)}}{-(y+ix)}\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y-ix} + \frac{1}{y+ix}\right] = \frac{1}{2\pi}\left\{\frac{2y}{y^2+x^2}\right\} = \boxed{\frac{y}{\pi(y^2+x^2)}}$$

Unit Step function

$$U(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



i) $F\{e^{-ax} U(x)\} = \dfrac{1}{a - is}, \quad a > 0$

ii) $F\{e^{ax} U(-x)\} = \dfrac{1}{a + is}, \quad a > 0$

Q. Find the inverse fourier transform of

i) $\dfrac{e^{4iw}}{3+iw}$

Ans. we have $F^{-1}\left(\dfrac{1}{3+iw}\right) = e^{3t} U(-t)$

By shifting prop. $F[f(x-a)] = e^{ias} F(s)$ $\quad f(x-a) = F^{-1}[e^{ias} F(s)]$

$$F^{-1}\left(\frac{e^{4iw}}{3+iw}\right) = e^{3(t-4)} U(-t+4)$$

$$= e^{3(t-4)} \begin{cases} 1; & -t+4 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$= e^{3(t-4)} \begin{cases} 1, & t \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

2) $F^{-1}\left(\frac{1}{30+11i\omega-\omega^2}\right)$

$$F^{-1}\left(\frac{1}{5+i\omega}-\frac{1}{6+i\omega}\right)$$

$$F^{-1}\left(\frac{1}{5+i\omega}\right)-F^{-1}\left(\frac{1}{6+i\omega}\right)$$

$$e^{5t}u(-t)-e^{6t}u(-t)$$

$$=\begin{cases} e^{5t}-e^{6t}, & -t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$=\begin{cases} e^{5t}-e^{6t}, & t \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

3) $F^{-1}\left\{\frac{1}{4+\omega^2}\right\}$

$$F^{-1}\left\{\frac{1}{(2+i\omega)(2-i\omega)}\right\}$$

$u(-t)=\begin{cases} 1, & -t \geq 0 \\ 0, & -t<0 \end{cases}$

$$F^{-1}\left\{\frac{1}{4}\left[\frac{1}{2+i\omega}+\frac{1}{2-i\omega}\right]\right\}$$

$=\begin{cases} 1, & t \leq 0 \\ 0, & t>0 \end{cases}$

$$=\frac{1}{4}\left[e^{2t}u(-t)+e^{-2t}u(t)\right]$$

$u(t)=\begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases}$

$$=\begin{cases} \frac{1}{4}e^{-2t}, & t>0 \\ \frac{1}{4}e^{2t}, & t<0 \\ \frac{1}{2}, & t=0 \end{cases}$$

$=\frac{1}{4}\left[\frac{1}{4}e^{-2t}, t>\right.$

$s \to -s$

$t \to -t$

Convolution Theorem

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Convolution Theorem

Q1 Using Convolution theorem, find $f(t)$ when

$$F\{f(t)\} = \frac{1}{(1+is)^2}, \quad f(t)=0;\ t \geq 0$$

Ans. $F^{-1}\left\{\frac{1}{1+is}\right\} = e^t H(-t)$

$\therefore$ By convolution thm,

$$F^{-1}\left[\frac{1}{(1+is)^2}\right] = e^t H(-t) * e^t H(-t)$$

$$= \int_{-\infty}^{\infty} e^u H(-u) \cdot e^{(t-u)} H(-t+u)\, du$$

$$= e^t \int_{-\infty}^{\infty} H(-u)\, H(-t+u)\, du$$

Now, $H(-u)\,H(-t+u) = \begin{cases} 1, & -u \geq 0,\ -t+u \geq 0, \text{ i.e. } u \leq 0,\ t \leq u \\ 0, & \text{otherwise} \end{cases}$ $\quad t \leq u \leq 0$

$$\therefore F^{-1}\left\{\frac{1}{(1+is)^2}\right\} = e^t \int_t^0 du = e^t . u \Big|_t^0$$

$$= e^t [0-t] = -te^t$$

$$f(t) = \begin{cases} -te^t, & t \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

Q2. Using convolution theorem, find the inverse fourier transform of $\frac{1}{6+5is-s^2}$

Ans. $\frac{1}{6+5is+i^2s^2} = \frac{1}{(is+2)(is+3)}$

$= \frac{1}{(is+2)} \cdot \frac{1}{(is+3)}$

$F^{-1}\left\{\frac{1}{(is+2)(is+3)}\right\} = e^{2t}H(-t) * e^{3t}H(-t)$

$= \int_{-\infty}^{\infty} e^{2u} H(-u) \cdot e^{3(t-u)} H(-t+u)\, du$

$= e^{3t} \int_{-\infty}^{\infty} e^{-u} H(-u) H(-t+u)\, du$

Now, $H(-u)H(-t+u) = \begin{cases} 1, & t \le u \le 0 \\ 0, & \text{otherwise} \end{cases}$

$= e^{3t} \int_t^0 e^{-u}\, du = e^{3t} \left. \frac{e^{-u}}{-1} \right|_t^0$

$= -e^{3t}\left[1 - e^{-t}\right]$

$= -e^{3t} + e^{2t} = e^{2t}\left[1 - e^{t}\right]$

$f(t) = \begin{cases} e^{2t}(1-e^{t}), & t \le 0 \\ 0, & \text{otherwise} \end{cases}$