Unit III Numerical Methods

Numerical methods are often, of a repetitive nature. These consists in repeated execution of the same process where at each step the result of the preceding step is used. This is known as iteration process and is repeated till the result is obtained to a desired degree of accuracy.

Algebraic and Transcendental Equations

An equation $f(x) = 0$ is called an algebraic eq. of degree n, if $f(x)$ is a polynomial of degree n. If $f(x)$ contains some other fns such as trigonometric, logarithmic, exponential etc, then $f(x) = 0$ is called the transcendental eq.

Some useful results

1) If α is root of the eq^n $f(x) = 0$, then $f(\alpha) = 0$
2) Every eq. of 'n' degree has exactly n roots (real or imaginary)
3) Intermediate Value property

If $f(x)$ is continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$, i.e $f(a)$ and $f(b)$ have opposite signs, then the eq $f(x) = 0$ has atleast one real root in (a, b).

Further, if $|f(a)| < |f(b)|$, then, in general, root is near a, as compared to b.

Now, we shall be dealing some methods to find roots of a given eq^n.

Bisection Method or Bolzano Method or Halving Method.

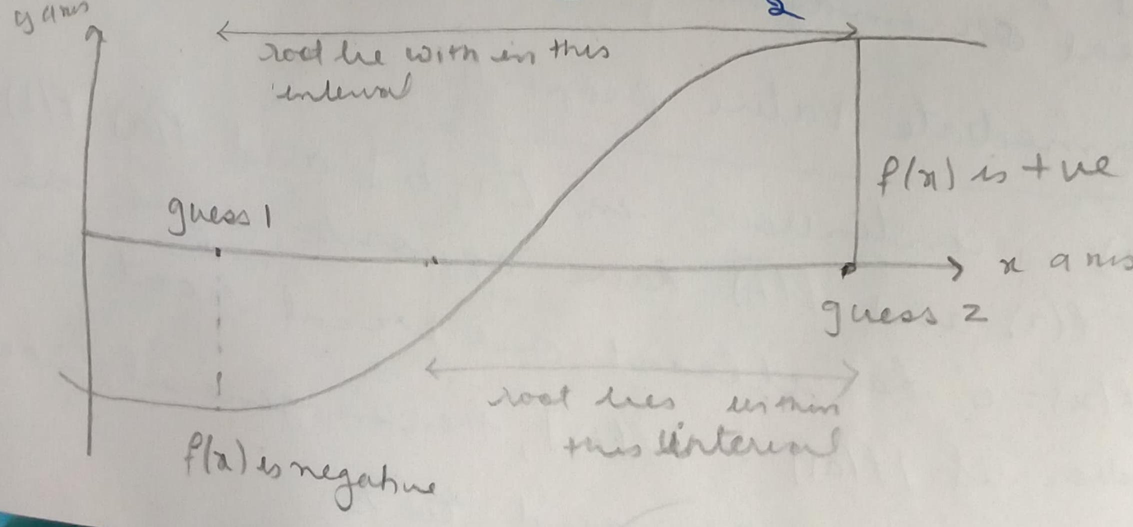
This method is based on the repeated application of intermediate value property.

Suppose we are to find real root of the equation $f(x)=0$, where $f(x)$ is a cts fn.

Let a and b be real numbers s.t. $f(a)$ & $f(b)$ have opposite signs, then the first approximation to the root is $x_1 = \frac{1}{2}(a+b)$. If $f(x_1)=0$ then x_1 is root.

If $f(x_1) \neq 0$, then either $f(a)$ and $f(x_1)$ have opposite signs in which case second approximation will be $x_2 = \frac{a+x_1}{2}$ or $f(x_1)$ and $f(b)$ have opposite signs in which case second approximation will be $x_2 = \frac{x_1+b}{2}$.

Now, replace a or b by x_1 as the case be, then next approximation will be $x_3 = \frac{x_1+x_2}{2}$ & so on.



$b-a$

Number of Iterations Required to reach accuracy ϵ.

Suppose M is the length of interval (a, b), then after first approx. x_1 the root will lie in $(a, \frac{a+b}{2})$ or in $(\frac{a+b}{2}, b)$ or $x_1 = \frac{a+b}{2}$ is root. & thus the root will lie in the interval of length $\frac{M}{2}$. Thus, at every step, the new interval containing the root is exactly half the length of the previous one.

At the end of n steps when we obtain x_n, the root will lie in an interval of length $\frac{b-a}{2^n}$. Thus, the no. of iterations n reqd to reach ~~to~~ accuracy ϵ must satisfy.

$$\frac{b-a}{2^n} \leq \epsilon$$

$$\log(b-a) - n \log 2 \leq \log \epsilon$$

$$n \geq \frac{\log(b-a) - \log \epsilon}{\log 2}.$$

Smallest natural no. n satisfying this inequality gives the no. of iterations reqd to reach accuracy ϵ.

As the length of interval at each step is $\frac{1}{2}$ the length of interval in the previous step in which root lies, so if ϵ_{n+1} is error in x_{n+1} & ϵ_n is error in x_n, then

$$\epsilon_{n+1} = \frac{1}{2} \epsilon_n$$

Hence, convergence is linear. Also, the convergence is geometric with common ratio $\frac{1}{2} < 1$ and thus, the process must converge to root. Hence, the process is slow but must converge. (5)

Ques 1. Find the root of the eqn $x^3 - 4x - 9 = 0$ using the bisection method correct to three decimal places.

Ans. Let $f(x) = x^3 - 4x - 9$

Since $f(2) = -9$ is $-ve$ & $f(3) = 6$ is $+ve$,

a root lies between 2 and 3.

$\therefore$ the first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375 \quad (-ve)$$

$\therefore$ the root lies b/w x_1 and 3.

Thus, the second approx. to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

$$f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \quad (+ve)$$

$\therefore$ the root lies b/w x_1 & x_2.

Thus, the third approx. to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

$$f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121 \quad (-ve)$$

$\therefore$ the root lies b/w x_2 & x_3.

Thus, the fourth approx. to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

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Repeating this process, the successive (5)
approximations are :

$x_5 = 2.71875$

$x_6 = 2.70313$

$x_7 = 2.71094$

$x_8 = 2.70703$

$x_9 = 2.70508$

$x_{10} = 2.70605$

$x_{11} = 2.70654$

$x_{12} = 2.70642$

Hence, the root is 2.7064.

Or.

Qus. Find a root of the eq^n $x^3 - 4x - 9 = 0$ using the bisection method in four stages.

Ans. $f(x) = x^3 - 4x - 9 = 0$

$f(2) = -9, \quad f(3) = 6$

$|f(2)| > |f(3)|$

$f(2.7) = (2.7)^3 - 4(2.7) - 9 = -0.117$

$f(2.8) = (2.8)^3 - 4(2.8) - 9 = 1.752$

$\therefore$ the root lies b/w 2.7 & 2.8.

$$x_1 = \frac{2.7 + 2.8}{2} = 2.75$$

Approximation root x	$f(x)$	Root b/w	Next approximation
$x_1 = 2.75$	+ve	2.7 and x_1	$\frac{2.7 + 2.75}{2} = 2.725$
$x_2 = 2.725$	+ve	2.7 & x_2	$\frac{2.7 + 2.725}{2} = 2.7125$
$x_3 = 2.7125$	+ve	2.7 & x_3	$\frac{2.7 + 2.7125}{2} = 2.70625$
$x_4 = 2.70625$	-ve	x_3 & x_4	$\frac{2.7125 + 2.70625}{2} = 2.709375$

$\therefore$ approximate root $= 2.71$

Q. Apply bisection method to find a root of the equation $xe^{x} = 1$ correct to three decimal places.

Ans. $f(x) = xe^{x} - 1$

$f(0) = -1$ and $f(1) = e - 1 = 1.718$

$\therefore$ the root lies b/w 0 and 1.

$x_1 = \frac{0+1}{2} = 0.5$, $f(0.5) = -0.1756$ (−ve)

Approximation root x	$f(x)$	Root b/w	Next approximati
$x_1 = 0.5$	(−ve)	$0.5 < 1$	$\frac{0.5+1}{2} = 0.75$
$x_2 = 0.75$	(+ve)	$0.5 < 0.75$	$\frac{0.5+0.75}{2} = 0.62$
$x_3 = 0.625$	(+ve)	$0.5 < 0.625$	$\frac{0.5+0.625}{2} = 0.56$
$x_4 = 0.5625$	(−ve)	$0.5625 < 0.625$	$\frac{0.5625+0.625}{2} = 0.59$
$x_5 = 0.59375$	(+ve)	$0.5625 < 0.59375$	0.5781
$x_6 = 0.5781$	(+ve)	$0.5625 < 0.5781$	0.5703
$x_7 = 0.5703$	(+ve)	$0.5625 < 0.5703$	0.5664
$x_8 = 0.5664$	(−ve)	$0.5664 < 0.5703$	0.5684
$x_9 = 0.5684$	(+ve)	$0.5664 < 0.5684$	0.5674
$x_{10} = 0.5674$	(+ve)	$0.5664 < 0.5674$	0.5669
$x_{11} = 0.5669$	(−ve)	$0.5669 < 0.5674$	0.56715

$\boxed{x_{12} = 0.56715}$ $f(0.56715) = 0.00001 \sim 0$