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ult - Relation b/w μ, δ, Δ and ∇

We have $\mu\delta = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}[(1+\Delta) - (1-\nabla))$

$$\mu\delta = \frac{1}{2}(\Delta + \nabla)$$

Result - $\Delta^n y_r = \nabla^n y_{n+r}$

we have $\Delta^n y_r = (E-1)^n y_r \quad \because \Delta = E-1$

$= y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

$= (E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - \cdots + (-1)^n) y_r$

$= E^n y_r - {}^nC_1 E^{n-1} y_r + {}^nC_2 E^{n-2} y_r - \cdots + (-1)^n y_r$

$= y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^n y_r$

Also $\nabla^n y_{n+r} = (1 - E^{-1})^n y_{n+r} \quad \because \nabla = 1 - E^{-1}$

$= (1 - {}^nC_1 E^{-1} + {}^nC_2 E^{-2} + \cdots + (-1)^n E^{-n}) y_{n+r}$

$= y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^n y_r$

$\therefore \Delta^n y_r = \nabla^n y_{n+r}$

Q1. Evaluate the following :-

(i) Δe^x

(ii) $\Delta^2 e^x$

(iii) $\Delta \tan^{-1} x$

(iv) $\Delta\left(\dfrac{x+1}{x^2-3x+2}\right)$

(v) $\Delta f_K^2 = (f_K + f_{K+1})\,\Delta f_K$

Ans. i) $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

$\Delta e^x = e^x(e-1)$, if $h = 1$

ii) $\Delta^2 e^x = \Delta(\Delta e^x)$

$= \Delta[e^x(e^h - 1)]$

$= (e^h - 1)\,\Delta e^x$

$= (e^h - 1)(e^{x+h} - e^x)$

$= (e^h - 1)\,e^x(e^h - 1) = e^x(e^h - 1)^2$

iii) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$= \tan^{-1}\left(\dfrac{x+h-x}{1+(x+h)x}\right) = \tan^{-1}\dfrac{h}{1+(x+h)x}$

iv) $\Delta\left(\dfrac{x+1}{x^2-3x+2}\right) = \Delta\left(\dfrac{x+1}{(x-1)(x-2)}\right)$

$= \Delta\left(\dfrac{-2}{x-1} + \dfrac{3}{x-2}\right) = \Delta\left(\dfrac{-2}{x-1}\right) + \Delta\left(\dfrac{3}{x-2}\right)$

$= -2\left(\dfrac{1}{x+1-1} - \dfrac{1}{x-1}\right) + 3\left(\dfrac{1}{x+1-2} - \dfrac{1}{x-2}\right)$

$= -2\left(\dfrac{1}{x} - \dfrac{1}{x-1}\right) + 3\left(\dfrac{1}{x-1} - \dfrac{1}{x-2}\right)$

$= -\dfrac{(x+4)}{x(x-1)(x-2)}$

$\Delta f_k^2 = f_{k+1}^2 - f_k^2 = (f_{k+1} + f_k)(f_{k+1} - f_k)$ (25)

$= (f_k + f_{k+1}) \Delta f_k$

Q. Evaluate $\Delta^4 [(1-2x)(1-3x)(1-4x)(1-x)]$, where interval of differencing is one.

Ans. $\Delta^4[(1-2x)(1-3x)(1-4x)(1-x)]$

$= \Delta^4 [24x^4 + \cdots + 1] = 24 \cdot 4! \, 1^4 = 576$

$[\because \Delta^n f(x) = a_0 n! h^n$ and $\Delta^4 x^n = 0$ when $n < 4$.

Que. Evaluate $\Delta^3 ((1-x)(1-2x)(1-3x))$ Ans. -36.

$-6 \times 3!$

Que. P.T. $\Delta^3 y_3 = \nabla^3 y_6$

Ans. $\Delta^3 y_3 = (E-1)^3 y_3$ $\quad \because \Delta = E - 1$

$= (E^3 - 1 - 3E^2 + 3E) y_3$ $\quad E^3 - 1 - 3E(E-1)$

$= E^3 y_3 - y_3 - 3E^2 y_3 + 3E y_3$

$= y_6 - y_3 - 3y_5 + 3y_4$

Also, $\nabla^3 y_6 = (1 - E^{-1})^3 y_6$ $\quad \because \nabla = 1 - E^{-1}$

$= (1 - E^{-3} - 3E^{-1} + 3E^{-2}) y_6$

$= y_6 - y_3 - 3y_5 + 3y_4$

Que. P.T. (i) $\Delta - \nabla = \delta^2$

(ii) $\mu = \sqrt{1 + \frac{1}{4}\delta^2} = \left(1 + \frac{\Delta}{2}\right)(1+\Delta)^{-1/2}$

Ans. (i) $\delta^2 = (E^{1/2} - E^{-1/2})^2 = E + E^{-1} - 2$ $\quad \because \delta = E^{1/2} - E^{-1/2}$

$= (E-1) - (1 - E^{-1}) = \Delta - \nabla$

$\because E - 1 = \Delta$ & $1 - E^{-1} = \Delta$

(ii) $\sqrt{1+\frac{1}{4}\delta^2} = \sqrt{1+\frac{1}{4}(E^{1/2}-E^{-1/2})^2}$ $\quad \because \delta = (E^{1/2}-E^{-1/2})$

$= \sqrt{1+\frac{1}{4}(E+E^{-1}-2)}$

$= \sqrt{\frac{1}{4}(E+E^{-1}+2)} = \sqrt{\frac{1}{4}(E^{1/2}+E^{-1/2})^2}$

$= \frac{1}{2}(E^{1/2}+E^{-1/2}) = \mu \quad \because \mu = \frac{1}{2}(E^{1/2}+E^{-1/2})$

Also, $\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-1/2} = \left(1+\frac{E-1}{2}\right)(1+E-1)^{-1/2}$

$= \left(\frac{E+1}{2}\right)E^{-1/2} \quad \because \Delta = E-1$

$= \frac{1}{2}(E^{-1/2}+E^{1/2}) = \mu$

Ques: P.T. i) $D \equiv \frac{1}{h}\log E$

ii) $hD = \log(1+\Delta) = -\log(1-\nabla)$

iii) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 + \cdots$

Ans: i) we know that $E = e^{hD}$

$\log E = \log e^{hD}$

$\log E = hD \log e$

$D = \frac{1}{h}\log E \quad \because \log e = 1$

ii) $hD = \log E$ (from relation i)

$= \log(1+\Delta)$

Also $hD = \log E = -\log E^{-1}$

$= -\log(1-\nabla) \quad \because \nabla = 1-E^{-1}$

iii) We know that $\nabla = 1 - E^{-1}$

$$\nabla = \frac{1}{E}$$

$$= 1 - e^{-hD} \quad \because E = e^{hD}$$

$$\nabla = 1 - \left(1 - hD + \frac{h^2D^2}{2!} - \frac{h^3D^3}{3!} + \cdots\right)$$

$$= hD - \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} - \cdots$$

$$\nabla^2 = \left(hD - \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} - \cdots\right)^2$$

$$\nabla^2 = h^2D^2 + \left(\frac{h^2D^2}{2!}\right) + \cdots \; 2(hD)\left(\frac{h^2D^2}{2!}\right) + 2(hD)\left(\frac{h^3D^3}{3!}\right) - \cdots$$

$$\nabla^2 = h^2D^2 - h^3D^3 + \left(\frac{h^4D^4}{4} + \frac{h^4D^4}{3}\right) - \cdots$$

$$\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \cdots$$

Qus. Form the forward difference table for the function $f(x) = x^3 - 2x^2 - 3x - 1$ for $x = 0, 1, 2, 3, 4$. Hence or otherwise find $\Delta^3 f(x)$. Also show that $\Delta^4 f(x) = 0$

Ans. $f(0) = -1$, $f(1) = -5$, $f(2) = -7$.

$f(3) = -1$, $f(4) = 19$

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0	-1				
		-4			
			2		
1	-5			6	
		-2			0
2	-7		8		
		6		6	
3	-1		14		
		20			
4	19				

From this table, we see that $\Delta^3 f(x) = 6$ and

$$\Delta^4 f(x) = 0$$

(Note :- $\Delta^n f(x) = a_0 \, n! \, h^n$, $\Delta^3 f(x) = 1 \cdot 3! \, 1^n = 6$

Ques. Express $f(x) = x^3 - 2x^2 + x - 1$ ~~in table~~ in factorial notation. S.T. $\Delta^4 f(x) = 0$

$h = 1$

Ans. By synthetic division

$$\begin{array}{c|cccc}
0 & 1 & -2 & 1 & -1 \\
 & \downarrow & 0 & 0 & 0 \\
\hline
1) & 1 & -2 & 1 & \boxed{-1} \\
 & \downarrow & 1 & -1 & \\
\hline
2) & 1 & -1 & \boxed{0} & \\
 & \downarrow & 2 & & \\
\hline
 & 1 & \boxed{1} & &
\end{array}$$

$\Delta f(x) = 3x^2 + 2x$

$\Delta^2 f(x) = 6x$

$\Delta^3 f(x) = 6$

$\Delta^4 f(x) = 0$

$f(x) = x^3 - 2x^2 + x - 1 = x^3 + x^2 - 1$

Note :- $\Delta^n f(x) = a_0 \, n! \, h^n$

$\Delta^4 x^n = 0$ if $n < 4$

Ques If for a polynomial, five observations (29) are recorded as : $y_0 = -8,\ y_1 = -6,\ y_2 = 22,$ $y_3 = 148,\ y_4 = 492$. find y_5.

Ans. $y_5 = E^5 y_0 = (1+\Delta)^5 y_0 \qquad \because E \equiv 1 + \Delta$

$$= y_0 + {}^5C_1 \Delta y_0 + {}^5C_2 \Delta^2 y_0 + {}^5C_3 \Delta^3 y_0 + {}^5C_4 \Delta^4 y_0 + \Delta^5 y_0 \quad \text{--- (1)}$$

Forward Difference table

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	-8				
		2			
			26		
x_1	-6			72	
		28			48
			98		
x_2	22			120	
		126			
x_3	148		218		
		344			
x_4	492				

From table. $\Delta y_0 = 2,\ \Delta^2 y_0 = 26,\ \Delta^3 y_0 = 72,\ \Delta^4 y_0 = 48$ --- (2)

$y_5 = -8 + 5(2) + 10(26) + 10(72) + 5(48) = 1222$

(Using (2) in (1))

Q. Evaluate. $\Delta^3\big((1-x)(1-2x)(1-3x)\big)$ $h=1$

we have $f(x) = (1-x)(1-2x)(1-3x) = -6x^3 + 11x^2 - 6x + 1$

By Synthetic division.

0)	-6	11	-6	1
		0	0	0
1)	-6	11	-6	| 1
		-6	5	
2	-6	5 |	-1	
		-12		
	6	-7		

$f(x) = -6x^3 - 7x^2 - x + 1$

$\Delta f(x) = -6 \times 3x^2$

$\Delta^2 f(x) = -6 \times 3 \times 2x$

$\Delta^3 f(x) = -6(3)(2)(1) = -36$

Missing Values of Data

Qus. Find the missing values in the foll. table.

x	0	5	10	15	20	25
$f(x)$	6	?	13	17	22	?

Ans:

x	y	Δ	Δ^2	Δ^3	Δ^4
0	6				
		$a-6$			
5	a		$19-2a$		
		$13-a$		$3a-28$	
					$38-4a$
10	13		$a-9$		
		4		$10-a$	
					$a+b-38$
15	17		1		
		5		$b-28$	
20	22		$b-27$		
		$b-22$			
25	b				

Since the polynomial represented by the given data is considered to be of 3rd degree, 4th & higher order differences are zero., i.e, $\Delta^4 y = 0$

$\therefore\ 38 - 4a = 0$ & $a + b - 38 = 0$

Solving these 2 eqs, we get $a = 9.5$, $b = 28.5$

Ques: Find the missing entry in the foll. table

x	0	1	2	3	4
$y(x)$	1	3	9	—	81

Ans: Four entries are given

$\therefore$ 4th order differences are zero.

$$\Delta^4 y_0 = 0$$

$$(E-1)^4 y_0 = (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$y_3 = \frac{1}{4}(y_4 + 6y_2 - 4y_1 + y_0)$$

$$= \frac{1}{4}(81 + 6(9) - 4(3) + 1) = 31$$

other method

let $y_3 = a$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
			4		
				$a-19$	
1	3				$124-4a$
		6			
			$a-15$		
2	9			$105-3a$	
		$a-9$			
			$90-2a$		
3	a				
		$81-a$			
4	81				

4th order difference are zero.

$$124 - 4a = 0$$

$$a = 31$$

$$y(3) = y_3 = 31$$

Qus. Find the missing values in the foll. table

x	0	5	10	15	20	25
y	6	10	—	17	—	31

Ans. Let $y(10) = a$, $y(20) = b$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6				
		4			
5	10		$a-14$	$41-3a$	$6a+b-102$
		$a-10$			
10	a		$27-2a$	$3a+b-61$	$143-4a-4b$
		$17-a$			
15	17		$a+b-34$	$82-a-3b$	
		$b-17$			
20	b		$48-2b$		
		$31-b$			
25	31				

As 4 entries are given, so 4^{th} order differences is zero.

$6a+b-102 = 0 \Rightarrow 6a+b = 102$

$143-4a-4b = 0 \Rightarrow 4a+4b = 143$

Cramer Rule.

$$a = \frac{\begin{vmatrix} 102 & 1 \\ 143 & 4 \end{vmatrix}}{\begin{vmatrix} 6 & 1 \\ 4 & 4 \end{vmatrix}} \qquad b = \frac{\begin{vmatrix} 6 & 102 \\ 4 & 143 \end{vmatrix}}{\begin{vmatrix} 6 & 1 \\ 4 & 4 \end{vmatrix}}$$

$a = 13.25$, $b = 22.50$

other method

4 entries are given.

$\Delta^4 y_0 = 0$ & $\Delta^4 y_5 = 0$

Now, $\Delta^4 y_0 = 0$

$(E-1)^4 y_0 = 0$

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$

$y_{20} - 4y_{15} + 6y_{10} - 4y_5 + y_0 = 0$

$6y_{10} + y_{20} = 4y_{15} + 4y_5 - y_0 = 4(17) + 4(10) - 6 = 102$ — (1)

IIly, $\Delta^4 y_5 = 0$

$y_{25} - 4y_{20} + 6y_{15} - 4y_{10} + y_5 = 0$

$4y_{10} + 4y_{20} = y_{25} + 6y_{15} + y_5 = 31 + 6(17) + 10 = 133$ — (2)

Solving (1) & (2), $y_{10} = 13.25$, $y_{20} = 22.50$.