Z - Transforms

Introduction

Z - Transform plays an important role in discrete analysis and may be seen as discrete analogue of Laplace transform. Role of Z - Transforms in discrete analysis is the same as that of Laplace and Fourier transforms in continuous systems.

Defination

If a function u_n is defined for discrete values ($n = 0, 1, 2, \ldots$) and $u_n = 0$ for $n < 0$, then its Z transform is defined to be.

$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ whenever the infinite series converges.

The inverse Z - transform is written as

$$Z^{-1}[U(z)] = u_n$$

Some Standard Z - Transforms

1) $Z\{a^n\} = \dfrac{z}{z-a}$

Proof :- $Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots + \frac{a^n}{z^n} + \cdots$$

$$= \frac{1}{1 - \left(\frac{a}{z}\right)} = \frac{z}{z-a}, \quad \left|\frac{a}{z}\right| < 1$$

2) $Z\{1\} = \frac{z}{z-1}$

$Z\{1\} = \frac{z}{z-1}$, Putting $a=1$ in result 1.

3) $Z\{(-1)^n\} = \frac{z}{z+1}$

$Z\{(-1)^n\} = \frac{z}{z+1}$, putting $a=-1$ in result 1.

4) $Z\{K\} = \frac{Kz}{z-1}$

$$Z\{K\} = \sum_{n=0}^{\infty} K z^{-n} = K \sum_{n=0}^{\infty} z^{-n}$$

$$= K\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^n}\right\} = \frac{Kz}{z-1}$$

5) Recurrence formula for n^p

$$Z\{n^p\} = -z \frac{d}{dz} Z(n^{p-1})$$

$Z\{n^p\} = \sum_{n=0}^{\infty} n^p z^{-n}$, p is a +ve integer — (1)

$Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$ — (2)

Diff. (2) w.r.t z, we get.

$$\frac{d}{dz} Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} (-n) z^{-n-1}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n^p z^{-n}$$

$\Rightarrow \frac{d}{dz} Z(n^{p-1}\} = -z^{-1} Z\{n^p\}$ (using (1))

$$\Rightarrow Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\}$$

6) Multiplication by n

$$Z\{n u_n\} = -z \frac{d}{dz} Z\{u_n\}$$

$$Z\{n u_n\} = \sum_{n=0}^{\infty} n u_n z^{-n} = -z \sum_{n=0}^{\infty} u_n (-n) z^{-n}$$

$$= -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz} (u_n z^{-n})$$

$$= -z \frac{d}{dz} \left(\sum_{n=0}^{\infty} u_n z^{-n} \right)$$

$$= -z \frac{d}{dz} Z(u_n)$$

7) $Z\{n\} = \dfrac{z}{(z-1)^2}$

$$Z\{n\} = -\frac{d}{dz} Z(n^0) \quad (\text{Using } 5 \ \& \ 6)$$

$$= -z \frac{d}{dz} Z\{1\}$$

$$= -z \frac{d}{dz} \frac{z}{z-1} \quad (\text{Using } 2)$$

$$\Rightarrow Z\{n\} = \frac{z}{(z-1)^2}$$

8) $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$

$$Z(n^2) = -z \frac{d}{dz} Z\{n\}, \quad (\text{Using } 5 \ \& \ 6)$$

$$= -z \frac{d}{dz} \frac{z}{(z-1)^2}, \quad \text{Using } 7$$

$$Z(n^2) = z^2+z$$

9) $Z\{u(n)\} = \begin{Bmatrix} 0, & n<0 \\ 1, & n \geq 0 \end{Bmatrix} = \frac{z}{z-1}$

$u(n) = \begin{cases} 0, & n<0 \\ 1, & n \geq 0 \end{cases}$ is unit step sequence.

$$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^n}$$

$$Z\{u(n)\} = \frac{z}{z-1}$$

10) $Z\{\delta(n)\} = \begin{Bmatrix} 1, & n=0 \\ 0, & n \neq 0 \end{Bmatrix} = 1$

$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$ is unit impulse ~~s~~equence.

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$

$$= 1 + 0 + 0 + \dots = 1$$

Properties of Z-Transforms

1) Linearity: $Z[au_n + bV_n] = aZ\{u_n\} + bZ(V_n)$

Proof:- $Z\{au_n + bV_n\} = \sum_{n=0}^{\infty} (au_n + bV_n] z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} V_n z^{-n}$$

$$= aZ\{u_n\} + bZ\{V_n\}$$

2) change of scale (or Damping rule)

If $Z\{u_n\} \equiv U(z)$, then $Z\{a^{-n}u_n\} \equiv U(az)$ and

$$Z\{a^n u_n\} \equiv U\left(\frac{a}{z}\right)$$

Proof $Z[a^{-n}u_n] = \sum_{n=0}^{\infty} a^{-n} u_n z^{-n}$

$$= \sum_{n=0}^{\infty} u_n (az)^{-n} \equiv U(az)$$

Ily $Z\{a^n u_n\} = U\left(\frac{z}{a}\right)$

Some standard results

a) $Z\{a^n n\} = \frac{az}{(z-a)^2}$

Proof $z\{n\} = \frac{z}{(z-1)^2} = U(z)$ (say)

$$\therefore z\{a^n z\} \equiv U\left(\frac{z}{a}\right) = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2}$$

b) $Z\{a^n n^2\} = \dfrac{az^2 + a^2 z}{(z-a)^3}$ (c)

Proof :- $Z(n^2) = \dfrac{z^2+z}{(z-1)^3} = U(z)$, say.

$\therefore Z\{a^n n^2\} = U\left(\dfrac{z}{a}\right) = \dfrac{\left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)}{\left[\left(\frac{z}{a}\right) - a\right]^3} = \dfrac{a(z^2+az)}{(z-a)^3}$

c) $Z\{\cos n\theta\} = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$, $Z\{\sin n\theta\} = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

Proof We have

$Z\{e^{-in\theta}\} = Z\{(e^{i\theta})^{-n}\} = Z\{(e^{i\theta})^{-n} \cdot 1\}$

Now $Z(1) = \dfrac{z}{z-1}$

$\therefore Z\{(e^{i\theta})^{-n} \cdot 1\} = \dfrac{ze^{i\theta}}{ze^{i\theta} - 1}$ $\quad \because Z\{a^{-n} u_n\} = U(az)$

$= \dfrac{z}{z - e^{i\theta}} = \dfrac{z(z-e^{i\theta})}{(z-e^{i\theta})(z-e^{i\theta})}$

$= \dfrac{z(z - \cos\theta - i\sin\theta)}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}$ $\quad (\because e^{i\theta} = \cos\theta + i\sin\theta)$

$= \dfrac{z(z - \cos\theta - i\sin\theta)}{z^2 - 2z\cos\theta + 1}$ $\quad \because \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore Z\{e^{-in\theta}\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

$\Rightarrow Z\{\cos n\theta - i\sin n\theta\} = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

$\therefore Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$ (A) ; $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$ (B)

d) $Z(a^n \cos n\theta) = \dfrac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$,

$Z(a^n \sin n\theta) = \dfrac{az\sin\theta}{z^2-2az\cos\theta+a^2}$

By damping rule, replacing z by $\frac{z}{a}$ in (A) & (B), we get.

$$Z\{a^n \cos n\theta\} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$

$$Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$$

Qus. Find the Z-transform of $2n + 3\sin\frac{n\pi}{4} - 5a^4$

Ans. By linearity property

$2Z(n) + 3Z\{\sin\frac{n\pi}{4}\} - \cancel{5a^4 Z\{a^4\}}\ 5a^4 Z\{1\}$

$$= \frac{2z}{(z-1)^2} + \frac{3z\sin\frac{\pi}{4}}{z^2-2z\cos\frac{\pi}{4}+1} - \frac{5a^4 z}{z-1}$$

$$\therefore Z\{n\} = \frac{2z}{(z-1)^2} + \frac{3z}{\sqrt{2}}\Big/ z^2-\sqrt{2}z+1 - \frac{5a^4 z}{z-1}$$

Qus. Find the Z-transform of the sequence $\{4, 8, 16, 32, \text{-----}\}$

Ans. $u_n = 2^{n+2}$, $n = 0, 1, 2, \ldots$

$$Z\{2^{n+2}\} = Z\{2^n \cdot 2^2\} = 4Z\{2^n\}$$

$$= \frac{4z}{z-2}, \quad \left|\frac{2}{z}\right| < 1 \quad \left[\because Z\{a^n\} = \frac{z}{z-a}, \left|\frac{a}{z}\right| < 1\right.$$