Properties of Fourier Transform

1. **Linearity property:-** If $F(s)$ and $G(s)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, then

$$F[a f(x) + b g(x)] = a F(s) + b G(s)$$

where a and b are constants.

Proof:- By the defination of Fourier transform

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$\& \; G(s) = G(g(x)) = \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$\therefore F[af(x) + bg(x)] = \int_{-\infty}^{\infty} [a f(x) + b g(x)]\, e^{isx}\, dx$$

$$= a\int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b\int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$= a F(s) + b G(s)$$

2. **Change of scale property**

If $F(s)$ is the complex Fourier transform of $f(x)$, then $F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right);\; a \neq 0$

Proof we have

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \quad -①$$

$$\therefore F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx \quad -②$$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$\Rightarrow dx = \frac{dt}{a}$

$$F[f(ax)] = \int_{-\infty}^{\infty} e^{ist/a} f(t) \frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)t} f(t)\, dt$$

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

3) Shifting Theorem

If $F(s)$ is a complex Fourier transform of $f(x)$, then
$$F\{f(x-a)\} = e^{ias} f(s)$$

Proof :- we have $F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$\therefore F(f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$

put $x-a = t \Rightarrow dx = dt$
$x = a+t$

$$F[f(x-a)\} = \int_{-\infty}^{\infty} e^{is(a+t)} f(t) dt$$
$$= e^{isa}\int_{-\infty}^{\infty} e^{ist} f(t)\, dt =$$
$$F\{f(x-a)\} = e^{isa} F(s)$$

4) Modulation Theorem

If $F(s)$ is complex Fourier transform of $f(x)$, then
$$F\{f(x).\cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)]$$

Proof We have
$$F(s) = F(f(x)) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$
$$\therefore F[f(x).\cos ax] = \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx$$
$$= \int_{-\infty}^{\infty} f(x)\, e^{isx}\left(\frac{e^{iax} + e^{-iax}}{2}\right) dx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[f(x)\, e^{i(s+a)x} + f(x)\, e^{i(s-a)}\right] dx$$
$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

|||ly, $F\{f(x).\sin ax\} = \frac{1}{2i}\left[F(s-a) - F(s+a)\right]$

(iii) Here

5. If $F_s(s)$ and $F_c(s)$ are Fourier Sine and Cosine transforms of $f(x)$ resp, then

i) $F_s(x f(x)) = -\frac{d}{ds}[F_c(s)]$

ii) $F_c(x f(x)) = \frac{d}{ds}[F_s(s)]$

<u>Proof</u> (i) $\frac{d}{ds}[F_c(s)] = \frac{d}{ds}\int_0^\infty f(x)\cos sx\,dx$

$= \int_0^\infty f(x)\left[\frac{d}{ds}\cos sx\right]dx$

$= \int_0^\infty f(x)[-x\sin sx]\,dx$

$= -\int_0^\infty \sin sx\,\{x f(x)\}\,dx$

$\frac{d}{ds}[F_c(s)] = -F_s\{x f(x)\}$

$\Rightarrow F_s[x f(x)] = -\frac{d}{ds}[F_c(s)]$

Illy, we can prove (ii)