Semester - 3

Unit I

Fourier series

Periodic Function

If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself,
i.e $f(x) = f(x+\alpha)$, for all x, then $y = f(x)$ is called a periodic function having period α,

Period of the function
real
fixed

e.g. $\sin x$, $\cos x$ are periodic fns. having a period 2π.

$\sin x = \sin(x+2\pi)$, $\cos x = \cos(x+2\pi)$

$\tan\theta$, $\cot\theta$ are periodic fns having period π.

$\tan(\theta+\pi) = \tan\theta$, $\cot(\theta+\pi) = \cot\theta$

Properties of integration

1) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even
$= 0$ if $f(x)$ is odd.

even function :- $f(-x) = f(x)$

a) $f(x) = x^2$
$f(-x) = (-x)^2 = x^2 = f(x)$

b) $f(x) = \cos x$
$f(-x) = \cos(-x) = \cos x$.

odd function :- $f(-x) = -f(x)$

a) $f(x) = x^3$
$f(-x) = (-x)^3 = -x^3 = -f(x)$

b) $f(x) = \sin x$, $f(-x) = \sin(-x) = -\sin x = -f(x)$

2) By - Part

$$\int f_1(x) f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int\left[\frac{d}{dx} f_1(x) \int f_2(x)\,dx\right]$$

Limitation :- not work in every situation

Chain Rule of Integration

$$\int \underset{I}{x^2} \underset{II}{\sin 2x}\,dx$$

$$x^2\left(-\frac{\cos 2x}{2}\right) - 2x\left(-\frac{\sin 2x}{4}\right) + 2\left(\frac{\cos 2x}{8}\right)$$

$$= -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

Note :- Only used in the questions where I st integrand is algebraic like

$x^n \cos ax$, $x^n \sin ax$, $x^n e^{ax}$.

Basics

$\sin 0 = \sin \pi = 0$

$\cos 0 = \cos 2n\pi = (-1)^{2n} = 1$

$\cos n\pi = (-1)^n$

$$* \int e^{ax} \sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$* \int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

$\int \sin x\,dx = -\cos x$

$\frac{d}{dx}\sin x = \cos x$

$\int \cos x\,dx = \sin x$

$\frac{d}{dx}\cos x = -\sin x$

If $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} , \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} , \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ of period 2π in terms of cosine and sine functions. These series are trigonometric series whose coeffients are determined by Euler formulae.

Euler's Formulae

The fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\, dx$$

These values of a_0, a_n, b_n are known as Euler's formulae.

Q1 Find a Fourier series to represent $x - x^2$ from $-\pi$ to π. Hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Ans Let $x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \left(\cancel{\frac{\pi^2}{2}} - \frac{(-\pi)^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi^3}{3}\right] = \boxed{-\frac{2}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx \, dx$$

$$= -\frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx \qquad \left(\because \begin{array}{l} x\cos nx \text{ is odd \&} \\ x^2 \cos nx \text{ is even} \end{array}\right.$$

$$= -\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2}{n^3}\sin nx\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \boxed{\frac{4}{n^2}(-1)^{n+1}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx \qquad \left[\because \begin{array}{l} x \sin nx \text{ is even} \\ x^2 \sin nx \text{ is odd} \end{array}\right.$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n}\right] = \boxed{-\frac{2}{n}(-1)^n}$$

∴ Fourier-series expansion of $f(x)$ is

$$\boxed{f(x) \sim -\frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}(2\cos nx + n\sin nx)}$$

Taking $x = 0$

$$0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

Q2. Expand $f(x) = x\sin x$, $0 < x < 2\pi$ as a fourier series.

Ans. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\Big[x(-\cos x) - 1\cdot(-\sin x)\Big]_0^{2\pi}$

$$= \frac{1}{\pi}\Big[-x\cos x + \sin x\Big]_0^{2\pi} = \frac{1}{\pi}\Big[-2\pi\Big] = \boxed{-2}$$

$$a_n + ib_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\, e^{inx}\,dx = \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{e^{ix} - e^{-ix}}{2i}\right)e^{inx}\,dx$$

$$= \cancel{\frac{1}{\pi}\int_0^{2\pi} x\sin x} = \frac{1}{2\pi i}\int_0^{2\pi} x\left(e^{i(n+1)x} - e^{i(n-1)x}\right)dx \quad \text{— (A)}$$

$$= \frac{1}{2\pi i}\Bigg[x\left(\frac{-ie^{i(n+1)x}}{n+1}\right) - \left(-\frac{e^{i(n+1)x}}{(n+1)^2}\right) - \left\{x\left(\frac{-ie^{i(n-1)x}}{(n-1)}\right) - \left(-\frac{e^{i(n-1)x}}{(n-1)^2}\right)\right\}\Bigg]_0^{2\pi}, \; n \neq 1$$

$$= \frac{1}{2\pi i}\Bigg[\frac{-2\pi i}{n+1} + \cancel{\frac{1}{(n+1)^2}} + \frac{2\pi i}{n-1} - \cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} + \cancel{\frac{1}{(n-}}$$

$n \neq 1$

$$= \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{2}{n^2-1}, \quad n \neq 1$$

equating real and imaginary parts.

$$\boxed{a_n = \frac{2}{n^2-1}, \quad b_n = 0 \; ; \; n \neq 1}$$

From (14)

$$a_1 + i b_1 = \frac{1}{2\pi i} \int_0^{2\pi} x\,(e^{2ix} - 1)\, dx$$

$$= \frac{1}{2\pi i}\left[x\left(\frac{-ie^{2ix}}{2}\right) + \left(\frac{e^{2ix}}{4}\right) - \frac{x^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi i}\left[-\frac{2\pi i}{2} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} - 2\pi^2 \right]$$

$$= -\frac{1}{2} + \pi i$$

equate real & imaginary.

$$a_1 = -\frac{1}{2}, \quad b_1 = \pi$$

$\therefore$ Fourier Series expansion of $f(x)$ is

$$f(x) \sim -1 - \frac{1}{2}\cos x + \pi \sin x + 2\sum_{n=2}^{\infty} \frac{1}{n^2-1} \cos nx$$

Que3 obtain the Fourier Series for
$f(x) = e^{-x}; \quad 0 < x < 2\pi$.

Ans let $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left(\frac{e^{-x}}{-1}\right)_0^{2\pi}$

$= \frac{1}{\pi}\left[-e^{-2\pi} + 1\right] = \frac{1}{\pi}(1 - e^{-2\pi})$

$\cancel{= \frac{1}{\pi}\left[e^{-\pi}(e^{\pi} - e^{-\pi})\right]} = \frac{1}{\pi}\left[e^{-\pi}(e^{\pi} - e^{-\pi})\right]$

$\cancel{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)}$ $= \frac{2e^{-\pi}\sinh\pi}{\pi}$

$a_n + i b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x} e^{inx}\,dx$

$= \frac{1}{\pi} \left.\frac{e^{-x} e^{inx}}{(-1+in)}\right|_0^{2\pi}$

$= -\frac{1}{\pi}\frac{1+in}{1+n^2}(e^{-2\pi} - 1) = \frac{(1+in)(1-e^{-2\pi})}{\pi(1+n^2)}$

$= \frac{(1+in)e^{-\pi}(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$

$= \frac{2(1+in)e^{-\pi}\sinh\pi}{\pi(1+n^2)}$

equate real and imaginary.

$a_n = \frac{2e^{-\pi}\sinh\pi}{\pi(1+n^2)}, \quad b_n = \frac{2n e^{-\pi}\sinh\pi}{\pi(1+n^2)}$

$\therefore$ Fourier series expansion is

$f(x) \sim \frac{1}{\pi} e^{-\pi}\sinh\pi\left[1 + 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}(\cos nx + n\sin nx)\right]$