Iterative Methods for solving Simultaneous Linear Equations

Consider a system of linear eqs.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} - (1)$$

We have been using direct methods for solving a system of linear eqs. Direct method produce exact sol. after a finite no. of steps whereas iterative method give a seq. of approximate sols until sol^n is obtained upto desired accuracy.

Common iterative methods for solving system of linear eqs. are.

1) Gauss - Jacobi's iteration method

2) Gauss - Seidal's iteration method.

Remark

1) In general, we prefer a direct method for solving system of linear eqs but for large systems, iterative methods maybe be faster than the direct methods.

2) If the coefficient matrix A of given system is diagonally dominant i.e $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \ \forall i$

then the iterative process is sure to converge

Qus: Solve by Jacobi's iteration, the eqs.

$10x + y - z = 11.19$

$x + 10y + z = 28.08$

$-x + y + 10z = 35.61$ correct to two decimal places.

Ans: Rewrite

$x = \frac{1}{10}(11.19 - y + z)$, $y = \frac{1}{10}(28.08 - x - z)$

$z = \frac{1}{10}(35.61 + x - y)$

We start from ~~the~~ an approx., $x_0 = y_0 = z_0 = 0$

First iteration

$x_1 = \frac{11.19}{10} = 1.119$, $y_1 = \frac{28.08}{10} = 2.808$, $z_1 = \frac{35.61}{10} = 3.561$

Second iteration

$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$

$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$

$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$

Third iteration

$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$

$y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = 2.35$

$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$

Fourth iteration

$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$

$y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = 2.34$

$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$

Fifth iteration

$$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23$, $y = 2.34$, $z = 3.45$

Qus. Compute 4 iterations to find an approximate sol. of the given system of eqs. using Gauss Jacobi's method.

$$x + y + 5z = -1$$
$$5x - y + z = 10$$
$$2x + 4y = 12$$

Ans. Rearranging the eqs.

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1$$

$$\therefore x = \frac{1}{5}(10 + y - z)$$

$$y = \frac{1}{4}(12 - 2x)$$

$$z = \frac{1}{5}(-1 - x - y)$$

we start from an approximation

$$x_0 = y_0 = z_0 = 0$$

First iteration

$x_1 = \frac{10}{5} = 2$, $y_1 = \frac{12}{4} = 3$, $z_1 = -\frac{1}{5}$

Second iteration

$x_2 = \cancel{\frac{1}{10}} \frac{1}{5}(10 + y_1 - z_1) = 2.64$

$y_2 = \frac{1}{4}(12 - 2x_1) = 2$

$z_2 = \frac{1}{5}(-1 - x_1 - y_1) = -1.2$

Third iteration

$x_3 = \frac{1}{5}(10 + y_2 - z_2) = 2.64$

$y_3 = \frac{1}{4}(12 - 2x_2) = 1.68$

$z_3 = \frac{1}{5}(-1 - x_2 - y_2) = -0.928$

Fourth iteration

$x_4 = \frac{1}{5}(10 + y_3 - z_3) = 2.52$

$y_4 = \frac{1}{4}(12 - 2x_3) = 1.68$

$z_4 = \frac{1}{5}(-1 - x_3 - y_3) = -1.064$

Approximate sol^n after 4 iterations is given by

$x = 2.52$, $y = 1.68$, $z = -1.064$