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# APPLIED MATHEMATICS - III
## Assignment - 1

Ques 1. State Dirichlet's condition for convergence of fourier series and check whether the func$^n$ $\frac{1}{3x}$, $0 < x < 2\pi$ satisfies Dirichlet's conditions or not?

Ans Dirichlet's conditions of fourier series $f(x)$ in $[c, c+2l]$

a) $f(x)$ is single - valued and periodic with period $2l$.
b) $f(x)$ is piecewise continuous in $[c, c+2l]$.
c) $f(x)$ has finite number of maxima or minima in $[c, c+2l]$

$$f(x) = \frac{1}{3-x}, \quad 0 < x < 2\pi$$

i) $f(x)$ is not periodic in $(0, 2\pi)$

ii) $f(x)$ is not defined at $x = 3$ and $0 < 3 < 2\pi$ NOW, the func$^n$ $f(x)$ is not piecewise continuous because

$$f(3-0) = \lim_{h \to 0} f(3-h) = \lim_{h \to 0} \frac{1}{3-(3-h)} = \lim_{h \to 0} \frac{1}{h} = \infty$$

$$f(3+0) = \lim_{h \to 0} f(3+h) = \lim_{h \to 0} \frac{1}{3-(3+h)} = -\infty$$

Since both limits do not exists.

$\therefore$ $f(x)$ is not piecewise continuous at $x = 3$.

Hence, its fourier series expansion is not possible.

Ques 2. Find a fourier series to represent $x - x^2$ from $-\pi$ to $\pi$.
Hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{12}$

Ans $f(x) = x - x^2$

Let the fourier series is given by.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi^2}{2} - \frac{\pi^3}{3}\right\} - \left\{\frac{\pi^2}{2} + \frac{\pi^3}{3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{-2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \cos nx\, dx - \int_{-\pi}^{\pi} x^2 \cos nx\, dx\right]$$

$= \frac{1}{\pi}\left[0 - 2\int_0^\pi x^2 \cos nx\,dx\right] = -\frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx$

$a_n = -\frac{2}{\pi}\left[(x)^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$

$= -\frac{2}{\pi}\left[2\pi \frac{\cos n\pi}{n^2} + 0\right]$

$\boxed{a_n = \frac{-4}{n^2}(-1)^n = \frac{4}{n^2}(-1)^{n+1}}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \sin nx\,dx - \int_{-\pi}^{\pi} x^2 \sin nx\right.$

$= \frac{1}{\pi}\left[2\int_0^\pi x \sin nx\,dx - 0\right]$

$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^\pi$

$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} - 0\right]$

$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$

$\boxed{f(x) = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx}$

Put $x = 0$ on both sides

$f(0) = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$

$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$ Ez

Ques 3. Express $f(x) = |x|$, $-\pi < x < \pi$ as fourier series

Ans $f(-x) = |-x| = x = f(x)$ $\therefore f(-x) = f(x)$

$\therefore f(x)$ is an even function

Of fourier series is given by $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where, $a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi$

$\boxed{a_0 = \pi}$

$a_n = \frac{2}{\pi}\int_0^\pi x \cos nx\,dx = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$

$= \frac{2}{\pi}\left[\left(\frac{(-1)^n}{n^2}\right) - \frac{1}{n^2}\right]$

$\boxed{a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]}$

$$a_{2n} = \frac{2}{\pi(2n)^2}\left[(-1)^{2n}-1\right] = 0$$

$$a_{2n-1} = \frac{2}{\pi(2n-1)^2}\left[(-1)^{2n-1}-1\right] = -\frac{4}{\pi(2n-1)^2}$$

$$\therefore f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n}\cos 2nx + \sum_{n=1}^{\infty} a_{2n-1}\cos(2n-1)x$$

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Ques 4. Express $f(x) = x^2$ as a half-range cosine series for $0 < x < 2$

A Let the half-range cosine series of $f(x)$ be

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where, $a_0 = \frac{2}{2}\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$ $\boxed{a_0 = \frac{8}{3}}$

$$a_n = \int_0^2 x^2 \cos\frac{n\pi x}{2}\,dx = \left[x^2\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - 2x\left\{-\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right.\right.$$

$$\boxed{a_n = \frac{16}{n^2\pi^2}(-1)^n}$$

$$\therefore f(x) \sim \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$$

Ques 5. Obtain the fourier sine series for $f(x)$ containing three non-zero terms where $f(x)$ is given in the following table:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 10 | 15 | 8 | 5 | 3 |

Ans Here $l = 6$

$\therefore$ fourier sine series is given by $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{6}$

when $b_n = \frac{2}{6}\left[f_0 \sin\frac{n\pi x_0}{6} + f_1 \sin\frac{n\pi x_1}{6} + f_2 \cdots + f_k \sin\frac{n\pi x_{k+1}}{6}\right]$

| $x$ | $f(x)$ | $\theta = \frac{\pi x}{6}$ | $\sin\theta$ | $\sin 2\theta$ | $\sin 3\theta$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 10 | $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | 1 |
| 2 | 15 | $\pi/3$ | $\sqrt{3}/2$ | $\sqrt{3}/2$ | 0 |
| 3 | 8 | $\pi/2$ | 1 | 0 | -1 |
| 4 | 5 | $2\pi/3$ | $\sqrt{3}/2$ | $-\sqrt{3}/2$ | 0 |
| 5 | 3 | $5\pi/6$ | $1/2$ | $-\sqrt{3}/2$ | 1 |

$$b_1 = \frac{2}{6} \times \sum f(x) \sin\theta = \frac{1}{3}\left[\frac{\sqrt{3}}{2}(15+5) + \frac{1}{2}(10+3) + 8\right]$$

$$= \frac{1}{3} \times [17.32 + 6.5 + 8] = 10.607$$

$$b_2 = \frac{1}{3} \times \sum f(x) \sin 2\theta = \frac{1}{3} \times \left[\frac{\sqrt{3}}{2}(10+15-5-3)\right]$$

$$= \frac{\sqrt{3}}{6} \times 17 = 4.907$$

$$b_3 = \frac{1}{3} \times \sum f(x) \sin 3\theta = \frac{1}{3}[10-8+3] = \frac{5}{3} = 1.667$$

$$f(x) \sim 10.607 \sin\frac{\pi x}{6} + 4.907 \sin\frac{2\pi x}{6} + 1.667 \sin\frac{3\pi x}{6} + \dots$$

Ques 6. Obtain fourier sine integral of the function

$$f(x) = \begin{cases} x \ ; & 0 < x < 1 \\ 2-x \ ; & 1 < x < 2 \\ 0 \ ; & x > 2 \end{cases}$$

Ans fourier sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi} \int_{t=0}^{\infty} \int_{w=0}^{\infty} f(t) \sin wt \sin wx \, dw \, dt$$

$$= \frac{2}{\pi} \int_{w=0}^{\infty} \left[\int_0^1 t \sin wt \, dt + \int_1^2 (2-t) \sin wt \, dt\right] \sin wx \, dw$$

$$= \frac{2}{\pi} \int_{w=0}^{\infty} \left[\left\{t\left(\frac{-\cos wt}{w}\right) - \left(\frac{-\sin wt}{w^2}\right)\right\}_0^1 + \left\{(2-t)\left(\frac{-\cos wt}{w}\right) + 1\left(\frac{-\sin wt}{w^2}\right)\right\}_1^2\right]$$

$$= \frac{2}{\pi} \int_{w=0}^{\infty} \left[\cancel{\frac{-\cos w}{w}} + \frac{\sin w}{w^2} - \frac{\sin 2w}{w^2} + \cancel{\frac{\cos w}{w}} + \frac{\sin w}{w^2}\right] \sin wx \, dw$$

$$f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \left(\frac{2\sin w - \sin 2w}{w^2}\right) \sin wx \, dw$$

Ques 7. Find the fourier transfer of $f(x) = e^{-|x|}$

Ans

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-|x|} e^{-iwx} \, dx$$

$$= \int_{-\infty}^{0} e^{x} e^{-iwx} \, dx + \int_0^{\infty} e^{-x} e^{-iwx} \, dx$$

$$= \int_{-\infty}^{0} e^{(1-iw)x} \, dx + \int_0^{\infty} e^{-(1+iw)x} \, dx$$

$$= \left[\frac{e^{(1-iw)x}}{1-iw}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+iw)x}}{-(1+iw)}\right]_0^{\infty}$$

$$= \frac{1}{1-iw} + \frac{1}{1+iw} = \frac{1+iw+1-iw}{1+w^2}$$

$$\boxed{F\{f(x)\} = \frac{2}{1+w^2}}$$

value

Ques 8. Find the fourier cosine transform of $f(x) = \frac{1}{a^2+x^2}$. Hence derive fourier sine transform of $\phi(x) = \frac{x}{a^2+x^2}$

Solⁿ. $F_c\{f(x)\} = \int_0^\infty \frac{1}{a^2+x^2} \cos wx \, dx$ — $I$ (say) — ①

$$\frac{dI}{dw} = \int_0^\infty -\frac{x \sin wx}{a^2+x^2} dx = \int_0^\infty \frac{(a^2+x^2-a^2)}{x(a^2+x^2)} \sin wx \, dx$$

$$= -\int_0^\infty \frac{\sin wx}{x} dx + a^2 \int_0^\infty \frac{\sin wx}{x(a^2+x^2)} dx$$

$$\frac{dI}{dw} = -\frac{\pi}{2} + a^2 \int_0^\infty \frac{\sin wx \, dx}{x(a^2+x^2)} \quad — ②$$

$$\frac{d^2I}{dw^2} = a^2 \int_0^\infty \frac{\cos wx}{a^2+x^2} dx$$

$$\frac{d^2I}{dw^2} - a^2 I = 0 \Rightarrow \left(\frac{d^2}{dw^2} - a^2\right) I = 0 \quad — ③$$

from eqⁿ ①. when $w = 0$

$$I = \int_0^\infty \frac{1}{a^2+x^2} dx = \frac{1}{a}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty = \frac{\pi}{2a}$$

from eqⁿ ② when $w = 0$

$$\frac{dI}{dw} = -\frac{\pi}{2}$$

∴ from eqⁿ ③,

$$I = C_1 e^{aw} + C_2 e^{-aw} \quad — ④$$

$$\frac{dI}{dw} = a C_1 e^{aw} - a C_2 e^{-aw} \quad — ⑤$$

[when $w = 0$]

$$\frac{dI}{dw} = aC_1 - aC_2$$

$aC_1 - aC_2 = -\frac{\pi}{2}$ [from 5]

$$C_1 - C_2 = -\frac{\pi}{2a} \quad — ⑥$$

from ④ when $w = 0$

$$I = C_1 + C_2$$

$$C_1 + C_2 = \frac{\pi}{2a} \quad — ⑦$$

From eqⁿ ⑥ and ⑦

$$C_1 = 0$$

$$C_2 = \frac{\pi}{2a}$$

from eqⁿ ③, solⁿ is

$$I = \frac{\pi}{2a} e^{-aw}$$

$$\therefore F_c\left\{\frac{1}{a^2+x^2}\right\} = \int_0^\infty \frac{\cos wx}{a^2+x^2} dx$$

$$\boxed{F_c\left\{\frac{1}{a^2+x^2}\right\} = \frac{\pi}{2a} e^{-aw}}$$

diff

$$\int_0^\infty \frac{-x \sin wx\, dx}{a^2+x^2} = -a \frac{\pi}{2a} e^{-aw}$$

$$\int_0^\infty \frac{x \sin wx\, dx}{a^2+x^2} = \frac{\pi}{2} e^{-aw}$$

$$\boxed{F_s\left\{\frac{x}{a^2+x^2}\right\} = \frac{\pi}{2} e^{-aw}}$$

Ques 9. Find the inverse fourier transform of func$^n$ $\frac{1}{4+w^2}$

Ans $F^{-1}\left\{\frac{1}{4-w^2}\right\} = F^{-1}\left\{\frac{1}{(2+iw)(2-iw)}\right\}$

$$= \frac{1}{4} F^{-1}\left\{\frac{1}{2-iw} + \frac{1}{2+iw}\right\}$$

$$= \frac{1}{4}\left[e^{2t} U(-t) + e^{-2t} U(t)\right]$$

But, $U(-t) = \begin{cases} 1, & -t \geq 0 \\ 0, & -t < 0 \end{cases} = \begin{cases} 1, & t \leq 0 \\ 0, & t > 0 \end{cases}$

$$U(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$F^{-1}\left\{\frac{1}{4+w^2}\right\} = \begin{cases} \frac{1}{4} e^{2t}, & t < 0 \\ \frac{1}{4} e^{-2t}, & t > 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

Ques 10. The temperature $u$ in a semi-infinite rod $0 \leq x < \infty$ is determined by the differential eq$^n$

$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ subject to the conditions.

i) $u = 0$ when $t = 0$, $x \geq 0$

ii) $\frac{\partial u}{\partial x} = -u$ when $x = 0$, $t > 0$

Show that $u(x,t) = \frac{2u}{\pi} \int_0^\infty \frac{(1-e^{kw^2t})}{w^2} \cos wx\, dx$

Ans Taking fourier cosine transform on both sides of eq$^n$ we get

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = k F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\frac{d}{dt} F_c\{u(x,t)\} = k\left\{-w^2 F_c\{u(x,t)\} - \left(\frac{\partial u}{\partial x}\right)_{(0,t)}\right]$$

let $F_c\{u(x,t)\} = I$

$\Rightarrow \frac{dI}{dt} + kw^2 I = uk$ Integration factor $= e^{\int kw^2 dt} = e^{kw^2t}$

sol$^n$ is, $I \cdot e^{kw^2t} = \int uk e^{kw^2t} dt$

$$I \cdot e^{kwt} = \frac{uk e^{kw^2t}}{kw^2} + A$$

$\bar{u} = \frac{u}{w^2} + Ae^{-kw^2t}$

$F_c\{u(x,t)\} = \frac{u}{w^2} + Ae^{-kw^2t}$ — ①

when $t=0$, $u(x,0) = 0$

$\Rightarrow F_c\{u(x,0)\} = 0$ — ②

from ① and ② when $t=0$

$\frac{u}{w^2} + A = 0$

$\Rightarrow A = \frac{-u}{w^2}$

from eqn ①,

$F_c\{u(x,t)\} = \frac{u}{w^2}(1-e^{-kw^2t})$

∴ Taking inverse fourier cosine transform on both sides,

$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{u}{w^2}(1-e^{-kw^2t})\cos wx\,dw$

$$\boxed{u(x,t) = \frac{2u}{\pi}\int_0^\infty \frac{(1-e^{-kw^2t})}{w^2}\cos wx\,dw}$$