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Fourier Transforms of the Derivatives of a fn

If $u(x,t)$ is a function of two independent variables x and t, then we denote the Fourier transform of $u(x,t)$ by $\bar{u}(s,t)$;

so that $\boxed{\bar{u}(s,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{isx}\, dx}$

F(s)

(i) Fourier Transform of $\frac{\partial u}{\partial x}$, if $u \to 0$ as $x \to \pm\infty$

$$F\left(\frac{\partial u}{\partial x}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \cdot e^{isx}\, dx \qquad \text{(integ by parts)}$$

$$= e^{isx} u \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} is\, e^{isx} . u\, dx$$

$$= 0 - is \int_{-\infty}^{\infty} e^{isx} u(x,t)\, dx, \quad \left[\begin{array}{l}\text{as } u \to 0 \\ \text{as } x \to \pm\infty\end{array}\right.$$

$$\boxed{F\left(\frac{\partial u}{\partial x}\right) = -is\, \bar{u}(s,t)}$$

ii) Fourier Transform of $\frac{\partial^2 u}{\partial x^2}$ as $u \to 0$;

$\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\, e^{isx}\, dx$$

$$= \left| e^{isx} \frac{\partial u}{\partial x} \right|_{-\infty}^{\infty} - is \int_{-\infty}^{\infty} e^{isx} \frac{\partial u}{\partial x}\, dx$$

$$= 0 - is \left| e^{isx} u \right|_{-\infty}^{\infty} - is \int_{-\infty}^{\infty} e^{isx} . u\, dx$$

$$= (-is)^2 F[u(x,t)]$$

$$\boxed{F\left(\frac{\partial^2 u}{\partial x^2}\right) = (-is)^2\ \bar{u}(s,t)}$$

In general

$$\boxed{F\left\{\frac{\partial^n u}{\partial x^n}\right\} = (-is)^n\ \bar{u}(s,t)}$$

(iii) Fourier transform of $\partial u/\partial t$

$$F\left(\frac{\partial u}{\partial t}\right) = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{isx}\,dx = \frac{d}{dt}\int_{-\infty}^{\infty} u \cdot e^{isx}\,dx$$

$$\boxed{F\left(\frac{\partial u}{\partial t}\right) = \frac{d}{dt}\bar{u}(s,t)}$$

iv) Fourier Sine and Cosine transform of $\frac{\partial^2 u}{\partial x^2}$

$$F_s\{u(x,t)\} = \int_0^{\infty} u \cdot \sin sx\,dx = \bar{u}_s(s,t)$$

$$F_c\{u(x,t)\} = \int_0^{\infty} u \cdot \cos sx\,dx = \bar{u}_c(s,t)$$

a) Fourier Sine transform of $\partial^2 u/\partial x^2$, if

$u \to 0$, $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin sx\,dx$$

$$= \left[\sin sx\,\frac{\partial u}{\partial x}\Big|_0^{\infty} - \int_0^{\infty} s\cos sx\,\frac{\partial u}{\partial x}\,dx\right.$$

$$\left.= 0 - s\left[(\cos sx \cdot u)\Big|_0^{\infty} + s\left\{\int_0^{\infty}\sin sx\ u\,dx\right\}\right]\right]$$

$$= -s\left[0 - u(0,t)\right] - s^2\,\bar{u}_s(s,t)$$

$$\boxed{F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = s\,u(0,t) - s^2\,\bar{u}_s(s,t)}$$

(7) b) Fourier cosine-transform of $\frac{\partial^2 u}{\partial x^2}$ if

$u \to 0$, $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$$F_c\left(\frac{\partial^2 u}{\partial x^2}\right) = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \cos sx \, dx$$

$$= \left(\cos sx \frac{\partial u}{\partial x}\right|_0^\infty - \int_0^\infty (-s) \sin sx \frac{\partial u}{\partial x} dx$$

$$= \left[0 - \left(\frac{\partial u}{\partial x}\right)_{x=0}\right] + s\left[\left(\sin sx \cdot u\right|_0^\infty - \int_0^\infty s \cos sx \cdot u \, dx\right]$$

$$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + s\left[(0-0) - s F_c\{u(s,t)\}\right]$$

$$\boxed{F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2 F_c\left[u(s,t)\right]}$$

Applications of Fourier transform to Boundary value problems :-

(i) If the interval is $-\infty < x < \infty$ and if boundary conditions are

$u \to 0$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \pm\infty$

Use infinite Fourier transform.

(ii) If the interval is $0 < x < \infty$ and

a) boundary conditions are $u \to 0$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and

$u(x,t) = 0$ or $f(t)$ at $x = 0 \ \forall t$

Use Fourier sine transform

b) boundary conditions are $u \to 0$ and $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$ and

$\frac{\partial u}{\partial x} = 0$ or $f(t)$ at $x = 0 \ \forall t$

Use Fourier cosine transform

(iii) If the interval is $0 < x < L$ and

a) boundary conditions are $u(0,t) = u(L,t) = 0$ for all t, use Fourier sine transform

b) boundary conditions are $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) =$ for all t, use Fourier cosine transform.

Q1 Solve the equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$; $0<x<\infty$
$t>0$ where

(i) $u(0,t)=0$, $t>0$

(ii) $u(x,0)=e^{-x}$, $x>0$

(iii) u and $\frac{\partial u}{\partial x} \to 0$ as $x\to\infty$

Ans. Given $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$; $x>0$, $t>0$ — (A)

Boundary condition is $u(0,t)=0$,

Initial condn is $u(x,0)=e^{-x}$ and

$u\to 0$, $\frac{\partial u}{\partial x}\to 0$ as $x\to\infty$

$\because$ $u(0,t)$ is given, we take Fourier Sine transform of (A). Thus

$$F_s\left(\frac{\partial u}{\partial t}\right) = 2F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\,dx$$

$$\Rightarrow \frac{d}{dt}\int_0^\infty u.\sin sx\,dx = 2\left[\left.\frac{\partial u}{\partial x}\sin sx\right|_0^\infty - \int_0^\infty s\cos sx\frac{\partial u}{\partial x}dx\right]$$

$$-2s\int_0^\infty \underset{\mathrm{I}}{\cos sx}\,\underset{\mathrm{II}}{\frac{\partial u}{\partial x}}\,dx$$

$$\Rightarrow \frac{d}{dt}\bar{u}_s(s,t) = -2s^2\,\bar{u}_s(s,t)$$

$$-2s\left[\underbrace{\cos sx\cdot u\Big|_0^\infty}_{=0} + \int_0^\infty u.s\sin sx\,dx\right]$$

$$\Rightarrow \frac{d\bar{u}_s}{dt} = -2s^2\bar{u}_s$$

$$-2s^2\underbrace{\int_0^\infty u\sin sx\,dx}_{=\bar{u}_s(s,t)}$$

$$\Rightarrow \frac{d\bar{u}_s}{\bar{u}_s} = -2s^2\,dt$$

$$\Rightarrow \log\bar{u}_s = -2s^2t + \log A$$

$$\Rightarrow \bar{u}_s = A\,e^{-2s^2t} \quad \text{—(1)}$$

Given that $u(x,0) = e^{-x}$, $x > 0$

$$\bar{u}_s(x,0) = \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left\{ \frac{e^{-x}}{1+s^2} \left[-\sin sx - s\cos sx \right] \right\}_0^\infty$$

$$\bar{u}_s(x,0) = \frac{s}{1+s^2} \quad \text{—} \quad (2)$$

By eq (1)

At $t=0$, $\bar{u}_s = A \cdot e^{-2s^2(0)} = A$ — (3)

From (2) & (3) , $A = \dfrac{s}{1+s^2}$

$$\therefore \bar{u}_s(s,t) = \frac{s}{1+s^2} e^{-2s^2 t}$$

Taking inverse Fourier sine transform

$$\boxed{u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} e^{-2s^2 t} \sin sx \, ds}$$

Q2 Use the method of Fourier transform to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$

Ans. The eq^n for the vibration of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—} \quad (1)$$

s.t the initial cond's.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{and} \quad y(x,0) = f(x) \quad \text{—} \quad (2)$$

Taking Fourier transform of ①, we get

$$\frac{d^2}{dt^2}\bar{y} = c^2(-s^2\bar{y})$$

$$\frac{d^2\bar{y}}{dt^2} + c^2s^2\bar{y} = 0$$

Solⁿ is $\bar{y} = A\cos cst + B\sin cst$ — ③

where A & B are arb. constants.

Now taking Fourier transform of ②, we get

$$\frac{\partial\bar{y}}{\partial t} = 0 \text{ and } \bar{y} = F(s) = F\{f(x)\}, \text{ when } t=0$$ — ④

Put $t=0$ in ③, $\boxed{\bar{y}(x,0) = A}$ — ⑤

Using ④ & ⑤, $\underline{A = F(s)}$

$$\frac{\partial\bar{y}}{\partial t} = -csA\sin cst + csB\cos cst$$

Putting $t=0$, $\frac{\partial\bar{y}}{\partial t} = 0 \Rightarrow \boxed{B=0}$

$\therefore \bar{y} = F(s)\cos cst$

Taking Inverse F.T., we get

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cos cst \cdot e^{-isx}\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\left\{\frac{e^{icst} + e^{-icst}}{2}\right\} e^{-isx}\,ds$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[F(s)\,e^{-is(x-ct)} + F(s)\,e^{-is(x+ct)}\right] ds$$

$$\boxed{y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]}$$

(iii)

Q3 Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ subject to the conditions — (1)

i) $u = 0$ when $x = 0$, $t > 0$

ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$; $t = 0$ and

iii) $u(x,t)$ is bounded.

Ans. Since $u(0,t)$ is given, we take Fourier Sine transform on both sides of eq (1),

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

[$\because u(x,t)$ is bounded $\therefore \frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$$\Rightarrow \frac{d}{dt}(\bar{u}_s) = -s^2 \bar{u}_s$$

$$\Rightarrow \frac{d\bar{u}_s}{\bar{u}_s} = -s^2 dt$$

$$\Rightarrow \log \bar{u}_s = -s^2 t + \log A$$

$$\Rightarrow \bar{u}_s = A e^{-s^2 t} \quad \text{— (2)}$$

Put $t = 0$ in eq (2), $A = \bar{u}_s(x,0)$

$$A = \bar{u}_s(x,0) = \int_0^\infty u(x,0) \sin sx \, dx$$

$$= \int_0^1 1.\sin sx \, dx = \left[-\frac{\cos sx}{s}\right]_0^1$$

$$A = \left(\frac{1-\cos s}{s}\right)$$

$$\therefore \bar{u}_s(s,t) = \left(\frac{1-\cos s}{s}\right) e^{-s^2 t}$$

Taking Inverse Fourier transform

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s}\right) e^{-s^2 t} \sin sx \, ds}$$

Inverse Fourier cosi...

Q4. Employ Fourier transform to solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < \infty$, $t > 0$ where $u(x,t)$ satisfies the condⁿ.

i) $\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0$, $t > 0$

ii) $u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

iii) $|u(x,t)| < M$, i.e bounded.

Ans: Since $0 < x < \infty$ and $\frac{\partial u}{\partial x}$ is given at $x = 0$

We take fourier cosine transform

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \frac{d}{dt}(\bar{u}_c) = -s^2 \bar{u}_c(s,t)$$

$$\Rightarrow \bar{u}_c = A e^{-s^2 t} \quad \text{--- (1)}$$

To find A, we take the fourier cosine transform of

$$u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\bar{u}_c(s,0) = \int_0^\infty u(x,0)\cos sx \, dx$$

$$= \int_0^1 x \cos sx \, dx = \frac{s\sin s + \cos s - 1}{s^2} \quad \text{--- (2)}$$

Put $t = 0$ in (1) $\Rightarrow \bar{u}_c(s,0) = A$ --- (3)

By (1), (2), (3)

$$\bar{u}_c(s,t) = \left(\frac{s\sin s + \cos s - 1}{s^2}\right) e^{-s^2 t} \quad \text{--- (4)}$$

Taking Inverse Fourier Cosine transform (IV)

$$\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \left(\frac{s\sin s + \cos s - 1}{s^2}\right) e^{-s^2 t}\cos sx\, ds}$$

Q5. Find the solution of the Laplace equation

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ inside the semi-finite strip

$x > 0,\ 0 < y < b$ s.t.

(1) $v = f(x)$ when $y = 0,\ 0 < x < \infty$

(2) $= 0$ when $y = b,\ 0 < x < \infty$

(3) $= 0$ when $x = 0,\ 0 < y < b$

Ans. Since v is given at $x = 0$, use Fourier Sine Transform

$$F_s\left(\frac{\partial^2 v}{\partial x^2}\right) + F_s\left(\frac{\partial^2 v}{\partial y^2}\right) = 0 \quad \text{--- (1)}$$

Let $\bar{V}_s(s,y) = F_s[v(x,y)] = \int_0^\infty v(x,y)\sin sx\, dx$

$$\therefore F_s\left(\frac{\partial^2 v}{\partial x^2}\right) = \int_0^\infty \frac{\partial^2 v}{\partial x^2}\cdot \sin sx\, dx$$

$$= -s^2\bar{V}_s(s,y) \quad \text{--- (2)} \qquad \left[\because v \to 0 \text{ and } \frac{\partial v}{\partial x} \to 0 \text{ as } x \to \infty\right.$$

Using eqn (1), $F_s\left(\frac{\partial^2 v}{\partial x^2}\right) + \frac{d^2}{dy^2}[F_s(v)] = 0$

$\Rightarrow -\lambda^2\bar{V}_s(s,y) + D^2\bar{V}_s(s,y) = 0$, $\left(\frac{d}{dy} = D\right)$

$\Rightarrow (D^2 - \lambda^2)\bar{V}_s(s,y) = 0 \quad \text{--- (3)}$

Sol^n of (3) is

$\bar{V}_s(s,y) = C_1 \cosh sy + C_2 \sinh sy$ — (4)

Take Fourier sine transfom of boundary cond^n (i) and (ii) to find C_1 and C_2.

$\bar{V}_s(s,0) = \int_0^\infty V(x,0) \sin sx \, dx$

$= \int_0^\infty f(x) \sin sx \, dx$ — (5)

and $\bar{V}_s(s,b) = \int_0^\infty V(x,b) \sin sx \, dx = 0$ — (6)

Putting $y=0$ in (4)

$\bar{V}_s(s,0) = C_1 = \int_0^\infty f(t) \sin st \, dt$ (Using (5))

and putting $y=b$ in (4)

$\bar{V}_s(s,b) = C_1 \cosh(sb) + C_2 \sinh(sb) = 0$ (Using (6))

$\therefore \bar{V}_s(s,y) = \cosh(sy) \int_0^\infty f(t) \sin st \, dt$

$- \sinh(sy) \dfrac{\cosh(sb)}{\sinh(sb)} \int_0^\infty f(t) \sin st \, dt$

$\bar{V}_s(s,y) = \left[\cosh(sy) - \sinh(sy) \cdot \dfrac{\cosh(sb)}{\sinh(sb)}\right] \int_0^\infty f(t) \sin st \, dt$

$= \dfrac{\sinh(b-y)}{\sinh(sb)} \int_0^\infty f(t) \sin st \, dt$

Taking Inverse Fourier transfom

$v(x,y) = \dfrac{2}{\pi} \int_0^\infty \bar{V}_s(s,y) \sin sx \, ds$

$$\boxed{= \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \cdot \frac{\sinh(b-y)s}{\sinh(sb)} \sin sx \, dt \, ds.}$$