

Course - COMMUNICATION SYSTEMS (ETEC-212)

Prepared By:
Dr. Himanshu Khanna
Asst. Professor, ECE Dept.



Table of contents I

- 1 Lesson Plan
- 2 Text Books/ Reference Books
- 3 UNIT-I: Introduction
- 4 UNIT-II: Random Signal Theory
 - Random variables
 - Random Processes
- 5 UNIT-II: Amplitude Modulation

Lesson Plan

UNIT- I Introduction to Communication systems, Random variables:

Introduction: Overview of Communication system, Communication channels, Mathematical Models for Communication Channels.

Introduction of random Variables: Definition of random variables, PDF, CDF and its properties, joint PDF, CDF, Marginalized PDF, CDF, WSS wide stationery, strict sense stationery, non stationery signals, UDF, GDF, RDF, Binomial distribution, White process, Poisson process, Wiener process.

UNIT- II Amplitude Modulation: Need for modulation, Representation of Band Pass signals and systems: Hilbert Transform, In-phase, Quad-phase representations, Power relation, modulation index, Bandwidth efficiency, AM: modulation and demodulation, DSB-SC: Modulation and demodulation, SSB: modulation and demodulation, VSB: modulation and demodulation.

Lesson Plan

UNIT- III:

Angle Modulation Systems: Frequency Modulation, Types of Frequency Modulation, Generation of NBFM, WBFM, Transmission BW of FM Signal, Phase Modulation, Relationship between PM & FM.

Radio Receivers: Functions & Classification of Radio Receivers, Tuned Radio Frequency (TRF) Receiver, Superheterodyne Receiver, Basic Elements, Receiver Characteristics, Frequency Mixers, AGC Characteristics.

UNIT- IV: Noise Theory: Noise, Types of noise, Addition of Noise due to several sources in series and parallel, Generalized Nyquist Theorem for Thermal Noise, Calculation of Thermal Noise for a Single Noise Source, RC Circuits & Multiple Noise sources. Equivalent Noise Bandwidth, Signal to Noise Ratio, Noise-Figure, Noise Temperature, Calculation of Noise Figure.

Performance of Communication Systems: Receiver Model, Noise in DSB-SC Receivers, Noise in SSB-SC Receivers, Noise in AM receiver (Using Envelope Detection), Noise in FM Receivers, FM Threshold Effect, Threshold Improvement through Pre-Emphasis and De-Emphasis, Noise in PM system Comparison of Noise performance in PM and FM, Link budget analysis for radio channels.

Text Books/ Reference Books

Text Books:

- John G. Proakis & Masoud Salehi, "Communication System Engineering," Pearson Education.
- Simon Haykin, "Communication Systems," John Wiley & Sons, Inc. 4th Edition.

Reference Books:

- Taub, H., "Principles of Communication Systems," McGraw-Hill (2008) 3rd ed.
- Kennedy, G., "Electronic Communication Systems," McGraw-Hill (2008) 4th ed.
- V. ChandraSekar, "Analog Communication," Oxford University Press, Incorporated, 2010
- John G Proakis, M.Salehi and G.Bauch, "Modern Communication System Using MATLAB," Cengage Learning, 3rd edition, 2013
- J. C. Hancock, "An Introduction to the Principles of Communication Theory," TMH, 1998.
- Peebles, "Probability and Stochastic Process," Prentice Hall; 3rd edition

Communication Systems & Channels: Overview

Basic Components of Communication System

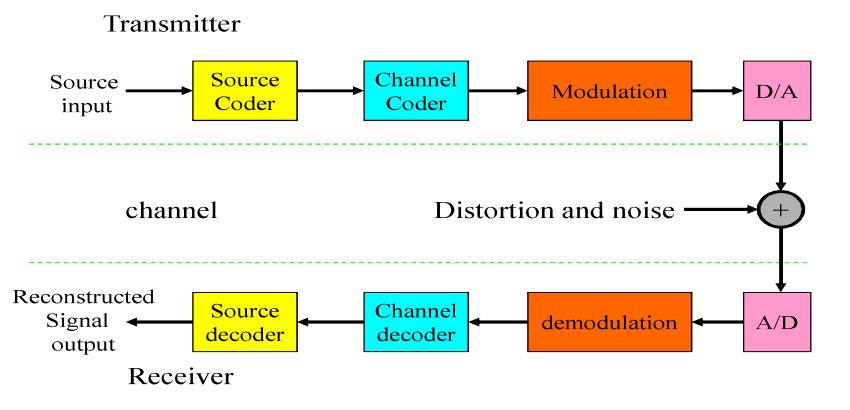


Figure: Basic elements of communication system.

- **Modes of communication:**

- **Broadcasting**- Involves the use of single powerful transmitter & numerous receivers which are relatively inexpensive to build. Information bearing signals flow only in one direction.
- **Point to Point**- Communication is bet. a single transmitter & receiver. Usually has a bidirectional flow.

- **Primary communication resources:**

- Transmitter power
- Channel bandwidth

One resource may be considered important than another. Accordingly channel may be classified as **power limited** (e.g. space comm. link or satellite channel) or **bandlimited** (e.g. telephone circuit).

- Noise which may be **internal** or **external** affects the system performance. A quantitative way to account for the effect of noise is **signal-to-noise-ratio (SNR)**.

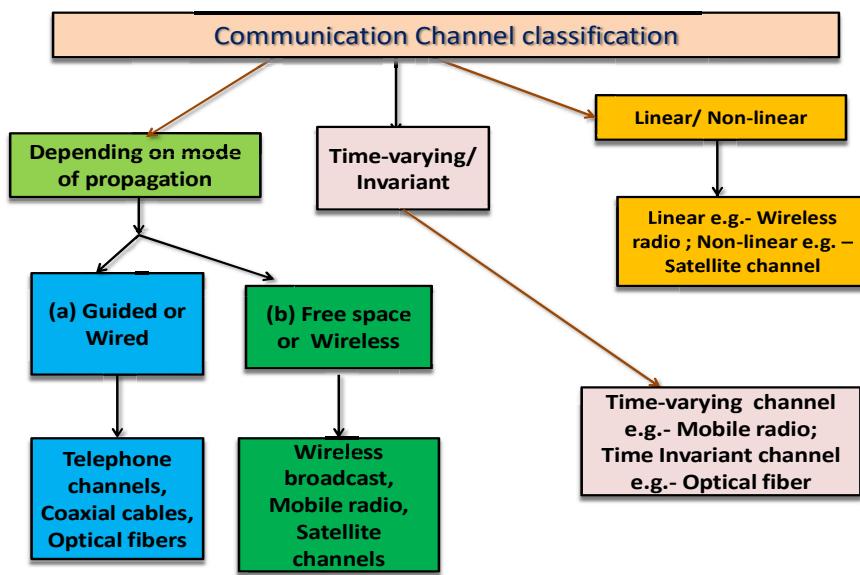


Figure: Classification of communication channels.

Mathematical models for communication channels

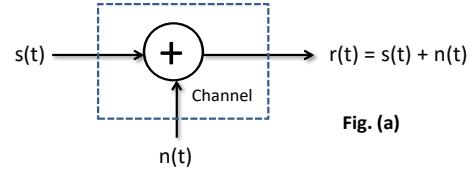


Fig. (a)

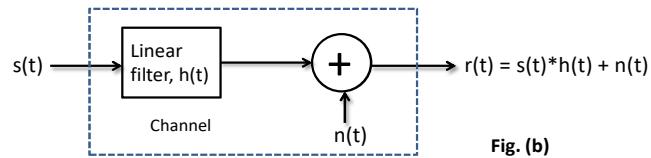


Fig. (b)

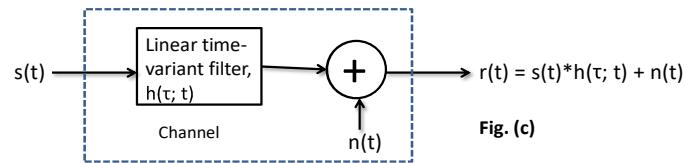


Fig. (c)

Random variables

Definition

A random variable (rv) is a function that maps an event from the sample space \mathcal{S} to a real number: $X : \omega \rightarrow \mathbb{R}$, where $\omega \in \mathcal{S}$.

Discrete & Continuous rv's

- A **discrete** random variable takes finite or countable number of values.
- A **continuous** random variable takes values in an interval of the real line or all of the real line.

Discrete Random variables: Statistical characteristics

The **probability density function** (pdf) or **probability mass function** of a discrete random variable is defined for every possible x by $p(x) = \mathbb{P}(X = x) = \mathbb{P}(X(s) = x : \text{for all } s \in \mathcal{S})$.

- **Some properties:**

- $p(x) \geq 0$ for all $X = x$
- $\sum_x p(x) = 1$

Example: The pdf of Bernoulli random variable - Consider a coin-tossing exp. in which probability of heads is p . For a *Bernoulli rv* X which assumes value 0 on occurrence of tails & 1 for appearance of heads, the pdf is defined as

$$P(X = x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

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Discrete Random variables: Statistical characteristics

Example: Let X represent a discrete rv that represents the sum of nos. appearing on two dice. The sample space in this case is given as

$$\mathcal{S} = [(1, 1); (1, 2), (2, 1); (2, 2), (1, 3), (3, 1); (1, 4), (4, 1), (2, 3), (3, 2); (1, 5), (5, 1), (2, 4), (4, 2), (3, 3); (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3); (2, 6), (6, 2), (3, 5), (5, 3), (4, 4); (3, 6), (6, 3), (4, 5), (5, 4); (4, 6), (6, 4), (5, 5); (5, 6), (6, 5); (6, 6)]$$

The **pmf** in this case is given as

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Discrete Random variables: Statistical characteristics

The **cumulative distribution function (cdf)** $F_X(x)$ of a rv X with pdf $p(x)$ is defined as $F_X(x) = \mathbb{P}(X \leq x) = \sum_{i=1}^x p(i)$. So for any number x , $F_X(x)$ is the probability that the observed value of X is at most x .

- **Some properties:**

- The distribution function is bounded between 0 and 1, i.e. $0 \leq F_X(x) \leq 1$.
- $F_X(x)$ is a monotone non-decreasing function of x , i.e. $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$.

Example: The cdf of Bernoulli random variable -

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 0 + 1 - p = 1 - p, & 0 \leq x < 1 \\ 0 + 1 - p + p = 1, & x \geq 1 \end{cases} \quad (2)$$

Discrete Random variables: Statistical averages (Moments)

n^{th} moment of a random variable

- Let $Y = X^n$ be a discrete rv with a set of possible values D and pdf of rv X is $p(x)$.
- The n^{th} -moment of rv X is the **expected** or **mean** value of X^n given by

$$\mathbb{E}[Y] = \mathbb{E}[X^n] = \mu_Y = \sum_{x \in D} x^n \cdot p(x). \quad (3)$$

n^{th} central-moment of a random variable

- The n^{th} central-moment of rv X is the **expected** or **mean** value of $(X - \mu_X)^n$ given by

$$\mathbb{E}[(X - \mu_X)^n] = \sum_{x \in D} (x - \mu_X)^n \cdot p(x). \quad (4)$$

Discrete Random variables: Statistical averages (Special cases)

1st moment (n = 1): Expectation

Let X be a discrete rv with a set of possible values D and pdf $p(x)$. The **expected** or **mean** value of X is $\mathbb{E}[X] = \mu_x = \sum_{x \in D} x \cdot p(x)$.

2nd central-moment (n = 2): Variance

Let X be a discrete rv with a set of possible values D and pdf $p(x)$. The **variance** of X is

$$\begin{aligned} Var[X] &= \sigma_x^2 \\ &= \mathbb{E}[(X - \mu_x)^2] \\ &= \sum_{x \in D} (x - \mu_x)^2 \cdot p(x). \end{aligned}$$

The **standard deviation** $\sigma_x = \sqrt{\sigma_x^2}$.

Expectation of a function of a discrete rv

- Let X be a discrete rv with a set of possible values D and pdf $p(x)$.
- Let Y be another rv where one-to-one transformation exists bet. Y and X such that $Y = h(X)$.
- The expectation of Y is the expectation of a function $h(X)$, i.e.

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \sum_{x \in D} h(x) \cdot p(x).$$

Linearity of expectation

$$\begin{aligned}\mathbb{E}[aX + b] &= \sum_{x \in D} (ax + b) \cdot p(x), \\ &= \sum_{x \in D} ax \cdot p(x) + \sum_{x \in D} b \cdot p(x), \\ &= a \sum_{x \in D} x \cdot p(x) + b \sum_{x \in D} p(x), \\ &= a\mathbb{E}[X] + b, \\ &= a\mu_X + b.\end{aligned}$$

$$\begin{aligned}\mathbb{E}[aX + bY] &= a\mathbb{E}[X] + b\mathbb{E}[Y] \\ &= a\mu_X + b\mu_Y.\end{aligned}$$

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Properties of variance

$$\begin{aligned} \text{Var}[X] &= \mathbb{E}[(X - \mu_X)^2] \\ &= \mathbb{E}[X^2 - 2X\mu_X + \mu_X^2], \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mu_X + \mu_X^2, \\ &= \mathbb{E}[X^2] - 2\mu_X^2 + \mu_X^2, \\ &= \mathbb{E}[X^2] - \mu_X^2, \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2. \end{aligned}$$

$$\begin{aligned} \text{Var}[aX + b] &= \mathbb{E}[(aX + b)^2] - (\mathbb{E}[aX + b])^2. \\ &= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mu + b)^2. \\ &= \mathbb{E}[a^2X^2] + \mathbb{E}[2abX] + b^2 - a^2\mu^2 - b^2 - 2\mu ab. \\ &= a^2\mathbb{E}[X^2] + 2ab\mathbb{E}[X] + b^2 - a^2\mu^2 - b^2 - 2\mu ab. \\ &= a^2\mathbb{E}[X^2] + 2ab\mu + b^2 - a^2\mu^2 - b^2 - 2\mu ab. \\ &= a^2\mathbb{E}[X^2] - a^2\mu^2, \\ &= a^2\text{Var}[X]. \end{aligned}$$

Properties of variance

$$\begin{aligned} \text{Var}[CX] &= \mathbb{E}[(CX - C\mu_X)^2] \\ &= C^2 \mathbb{E}[(X - \mu_X)^2] \\ &= C^2 \text{Var}[X] \end{aligned}$$

$$\begin{aligned} \text{Var}[X + Y] &= \text{Var}[X] + \text{Var}[Y] \\ \text{Var}[X - Y] &= \text{Var}[X] + \text{Var}[Y] \end{aligned}$$

Continuous Random variables: Statistical characteristics

The **probability density function** (**pdf**) of a continuous random variable is defined for every possible $x \in \{-\infty, \infty\}$ by $p(x)$ or $f(x)$ where $\mathbb{P}(a \leq X \leq b) = \int_a^b p(x)dx$.

- **Some properties:**

- $p(x) \geq 0$ for all X
- $\int_{-\infty}^{\infty} p(x)dx = 1$

- **Example:** A uniformly-distributed rv X with $a \leq x \leq b$ is having pdf given as

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

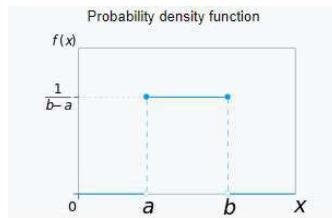


Figure: Uniform-distribution

Continuous Random variables: Statistical characteristics

The **cumulative distribution function (cdf)** $F_X(x)$ of a rv X with pdf $f_X(x)$ is defined as $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f_X(\zeta) d\zeta$.

- **Some properties:**

- The distribution function is bounded between 0 and 1, i.e. $0 \leq F_X(x) \leq 1$.
- $F_X(x)$ is a monotone non-decreasing function of x , i.e. $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$.

Example: The cdf of Uniformly-distributed random variable -

$$F_X(x) = \begin{cases} 0, & x < a \\ \int_a^x \frac{1}{b-a} d\zeta = \frac{x-a}{b-a}, & a \leq x < b \\ \int_a^b \frac{1}{b-a} d\zeta = 1, & x \geq b \end{cases} \quad (6)$$

Continuous Random variables: Statistical characteristics

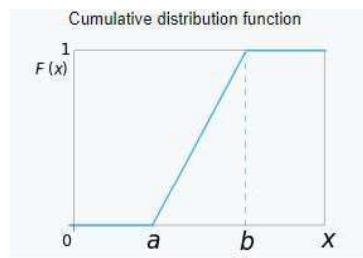


Figure: cdf of Uniform-distribution

Relationship between pdf & cdf

$$\frac{d}{dx} F_X(x) = f_X(x) \quad (7)$$

Two-Dimensional Random Variables: Joint pmf & cdf

Joint pmf (for discrete rv)

- Let (X, Y) be a discrete 2-D rv & (X, Y) takes on the values (x_j, y_k) where $\{j, k\} \in \text{integers}$.
- The joint pmf of (X, Y) is given as $p_{XY}(x_j, y_k) = \mathbb{P}(X = x_j, Y = y_k)$.
- **Properties:**
 - $p_{XY}(x_j, y_k) \geq 0$ for all $(X = x_j, Y = y_k)$.
 - $\sum_{x_j} \sum_{y_k} p(x_j, y_k) = 1$

Joint cdf (for discrete rv)

The joint cdf, i.e. $F_{XY}(x, y)$ of a 2-D rv (X, Y) with joint pmf $p_{XY}(x_j, y_k)$ is defined as $F_{XY}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \sum_{x_j \leq x} \sum_{y_k \leq y} p_{XY}(x_j, y_k)$.

Two-Dimensional Random Variables: Joint pdf & cdf

Joint pdf & cdf for continuous rv, and their Relationship

- Let (X, Y) be a continuous 2-D rv with joint pdf, i.e. $f_{XY}(x, y)$ and cdf, i.e. $F_{XY}(x, y)$, then

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \quad (8)$$

- Integrating (8) results in

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(\zeta, \eta) d\zeta d\eta \quad (9)$$

Properties:

- $f_{XY}(x, y) \geq 0$ for all (X, Y) .
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$.

Two-Dimensional Random Variables: Marginal pdf & Statistical independence

Marginal pdf's for continuous 2-D rv

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dy \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y) dx \end{aligned} \quad (10)$$

- If X & Y are statistically independent, then the joint pdf can be expressed in terms of marginal pdf's as

$$f_{XY}(x, y) = f_X(x)f_Y(y) \quad (11)$$

Two-Dimensional Random Variables: Conditional pdf & Statistical independence

- The conditional pdf of X given the event $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}, \text{ where } f_Y(y) \neq 0 \quad (12)$$

- For statistical independence, the conditional pdf of X is independent of the occurrence of the event $\{Y = y\}$, i.e.

$$f_X(x) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad (13)$$

Two-Dimensional Random Variables: Joint moments

Joint simple-moment of a 2-D random variable

- Let $Z = X^j Y^k$ be a 2-D rv.
- The joint simple-moment of rv Z is the **expected** or **mean** value of $X^j Y^k$ given by:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[X^j Y^k] = \mu_Z = \sum_i \sum_l x_i^j y_l^k p(x_i, y_l). \quad (14)$$

For Continuous case:

$$\mathbb{E}[Z] = \mathbb{E}[X^j Y^k] = \mu_Z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{XY}(x, y) dx dy. \quad (15)$$

Two-Dimensional Random Variables: Joint moments

Joint central-moment of a 2-D random variable

- Let $Z = (X - \mu_X)^j(Y - \mu_Y)^k$ be a 2-D rv.
- The joint central-moment of rv Z is the **expected** or **mean** value of $(X - \mu_X)^j(Y - \mu_Y)^k$ given by:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)^j(Y - \mu_Y)^k] = \mu_Z = \sum_i \sum_l (x - \mu_X)^j(y - \mu_Y)^k p(x_i, y_l). \quad (16)$$

For Continuous case:

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[(X - \mu_X)^j(Y - \mu_Y)^k] = \mu_Z \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)^j(y - \mu_Y)^k f_{XY}(x, y) dx dy. \end{aligned} \quad (17)$$

Joint simple-moment: Special cases (Correlation)

Correlation bet. two rv's

- For $j = k = 1$, $Z = XY$ is a 2-D rv.
- The joint simple-moment of rv Z in this case which is the **expected** or **mean** value of XY gives **correlation** as:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[XY] = \text{Corr}[XY] = \sum_i \sum_l xy p(x_i, y_l). \quad (18)$$

For Continuous case:

$$\mathbb{E}[Z] = \mathbb{E}[XY] = \text{Corr}[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy. \quad (19)$$

Joint central-moment: Special cases (Covariance)

Covariance of X & Y

- For $j = k = 1$, $Z = (X - \mu_X)(Y - \mu_Y)$ is a 2-D rv.
- The joint central-moment of rv Z which is the **expected** or **mean** value of $(X - \mu_X)(Y - \mu_Y)$ gives the **covariance** as:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = Cov[XY] = \sum_i \sum_l (x - \mu_X)(y - \mu_Y) p(x_l, y_l). \quad (20)$$

For Continuous case:

$$\begin{aligned} \mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] &= Cov[XY] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f_{XY}(x, y) dx dy. \end{aligned} \quad (21)$$

Joint central-moment: Special cases (Covariance), & Correlation coeff.

- From (20) & (21),

Covariance of X & Y

$\text{Cov}[XY] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$ can be simplified & expressed in terms of $\text{Corr}[XY]$, as

$$\begin{aligned}\text{Cov}[XY] &= \mathbb{E}[(XY - Y\mu_X - X\mu_Y + \mu_X\mu_Y)] \\ &= \mathbb{E}[XY] - \mu_X \mathbb{E}[Y] - \mu_Y \mathbb{E}[X] + \mu_X\mu_Y \\ &= \mathbb{E}[XY] - \mu_X\mu_Y \\ &= \text{Corr}[XY] - \mu_X\mu_Y\end{aligned}\tag{22}$$

Correlation-coefficient, ρ

$$\rho = \frac{\text{Cov}[XY]}{\sigma_X\sigma_Y},\tag{23}$$

where σ_X & σ_Y are the standard deviations for rv's X & Y , resp., & $-1 \leq \rho \leq 1$.

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Random Processes: Need

- In a radio comm. system, the received signal usually consists of an information bearing signal component, a random interference component, & channel noise.
 - Hence, the received time-varying signal is random in nature.
- **Random processes** combines the concepts of time variation & random variables, and describes the signal in terms of the statistical parameters such as avg. power & power spectral density (p.s.d.).
- **Random processes** represents formal mathematical model of these random signals & have the following properties:
 - They are functions of time.
 - They are random in the sense that it is not possible to predict exactly the waveform that will be observed in the future.

Random Processes: Definition

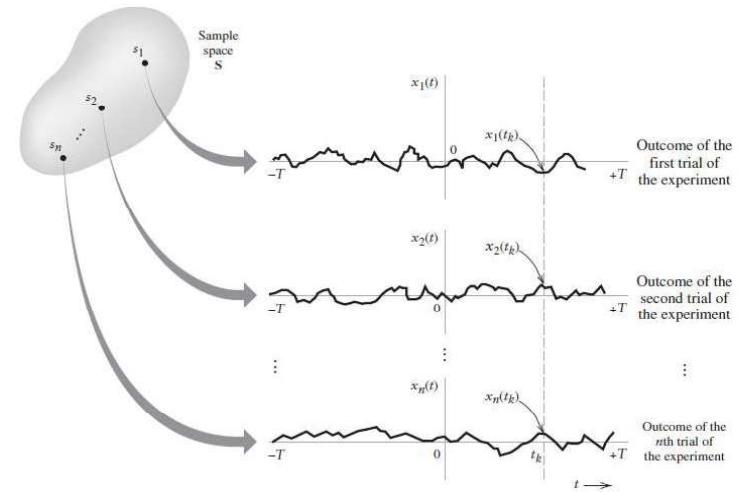


Figure: Relationship bet. sample space & ensemble of sample functions.

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Random Processes: Definition

- For an observation interval of $2T$, we assign to each sample point s a function of time

$$X(t, s), \quad -T < t < T \quad (24)$$

- The set of these **sample functions**, i.e. $\{X(t, s)\}$, together with a probability rule is termed as a **random process**, i.e. $X(t)$.
- For a fixed sample point s_j , $X(t, s_j) = x_j(t)$ denotes the j^{th} sample function of a random process.
- Observing sample functions $\{x_j(t)\}_{j=1,\dots,n}$ at some fixed time instant, say t_k , gives a set of random variables, i.e.

$$\{x_1(t_k), x_2(t_k), \dots, x_n(t_k)\} = \{X(t_k, s_1), X(t_k, s_2), \dots, X(t_k, s_n)\} \quad (25)$$

Random Processes: Types (Stationarity: Strict & Wide-sense)

Stationary random process

- If a random process is divided into a number of time intervals, the various sections of the process exhibit essentially the same statistical properties.
- Otherwise, it is said to be non-stationary.
- If

$$F_{X(t_1+\tau)}(x) = F_{X(t_1)}(x),$$

the process is *stationary* to the *first-order*, and as a consequence, statistical parameters such as mean & variance are also independent of time for such a process.

- If joint-distribution function

$$F_{X(t_1+\tau), X(t_2+\tau)}(x_1, x_2) = F_{X(t_1), X(t_2)}(x_1, x_2),$$

the process is stationary to the *second-order*, and as a consequence, statistical quantities such as covariance & correlation, do not depend upon absolute time.

Strictly-stationary random processes

Strictly-stationary random process: k th-order stationarity

- If the equivalence bet. distribution functions holds for all time shifts, i.e. τ , all k , and all possible observation times t_1, t_2, \dots, t_k , then the process is *strictly-stationary*.
- In other words, for strict-sense-stationarity (SSS), the joint distribution of any set of rv's obtained by observing the random process $X(t)$ is invariant w.r.t. the location of the origin $t = 0$.

Statistical-averages of a random process

Mean

- The ensemble-average mean, i.e. $\mu_X(t)$ or $m_X(t)$, of a random process at instant t is

$$\mu_X(t) = \mathbb{E}[X(t)]$$

Auto-correlation & Auto-covariance

- The ensemble-average autocorrelation, i.e. $R_X(t_1, t_2)$, is obtained by observing the random process at $t = t_1$ and $t = t_2$, and given as

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{X(t_1), X(t_2)}(x, y)dxdy$$

- The ensemble-average autocovariance, i.e. $C_X(t_1, t_2)$, is obtained as

$$C_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

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Statistical-averages of a random process

Cross-correlation

- The ensemble-average cross-correlation, i.e. $R_{XY}(t_1, t_2)$, is obtained by observing two diff. random processes, i.e. $X(t)$ & $Y(t)$, at $t = t_1$ and $t = t_2$, resp., and given as

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1)Y(t_2)]$$

Wide-sense stationary (WSS)/ Weakly stationary random process

Mean

- The ensemble-average mean, i.e. $\mu_X(t)$ or $m_X(t)$, of a random process is independent of t , and is constant, i.e.

$$\mu_X(t) = \mathbb{E}[X(t)] = \mu = \text{constant}$$

Auto-correlation & Auto-covariance

- The ensemble-average autocorrelation, i.e. $R_X(t_1, t_2)$, depends only on the time differences of t_1 and t_2 , i.e. $t_1 - t_2 = \tau$, and given as

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

- It follows from above, the ensemble-average autocovariance, i.e. $C_X(t_1, t_2)$, depends only on the time differences of t_1 and t_2 , i.e. $t_1 - t_2 = \tau$, and given as

$$C_X(t_1, t_2) = C_X(t_1 - t_2) = C_X(\tau) = R_X(\tau) - \mu^2$$

Properties of Autocorrelation for stationary process

Mean-square value in terms of $R_X(\tau)$

- $R_X(0) = \mathbb{E}[X^2(t)]$, i.e. the autocorrelation function for $\tau = 0$ is same as mean-square value of random process, $X(t)$.

Proof:

$$R_X(\tau) = \mathbb{E}[X(t)X(t + \tau)]$$
$$R_X(0) = \mathbb{E}[X^2(t)]$$

Even symmetry of $R_X(\tau)$

- $R_X(\tau) = R_X(-\tau)$, i.e. the autocorrelation of a real-valued wide-sense stationary process has even symmetry.

Proof:

$$R_X(\tau) = \mathbb{E}[X(t)X(t + \tau)]$$
$$R_X(-\tau) = \mathbb{E}[X(t)X(t - \tau)]$$

Replacing t with $t + \tau$ above results in

$$R_X(-\tau) = \mathbb{E}[X(t + \tau)X(t)] = R_X(\tau)$$

Properties of Autocorrelation for stationary process

$R_X(\tau)$ has max. magnitude at $\tau = 0$

- $R_X(0) = \mathbb{E}[X^2(t)] \geq |R_X(\tau)|.$

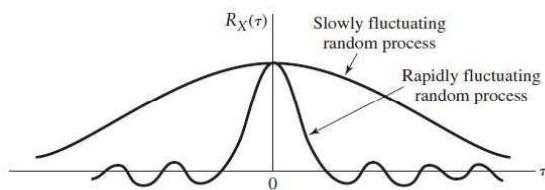


Figure: Illustration of autocorrelation functions of slowly and rapidly fluctuating random processes.

Relationship between $R_X(\tau)$ & p.s.d. $S_X(f)$: Weiner-Khintchine relations

Relationship bet. $R_X(\tau)$ & $S_X(f)$

- The p.s.d., i.e. $S_X(f)$, is the Fourier transform of $R_X(\tau)$, and both form Fourier transform pairs, i.e.

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau,$$

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\} = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f\tau) df$$

$$S_X(f) \stackrel{\mathcal{F}}{\rightleftharpoons} R_X(\tau)$$

Properties of $S_X(f)$ for stationary process

$S_X(f)$ is always positive for all f

- $S_X(f) \geq 0$, i.e. non-negative for entire range of frequencies, f .

$S_X(f)$ at $f = 0$ & $R_X(\tau)$ at $\tau = 0$

- $S_X(f)$ at $f = 0$ is obtained by finding area under $R_X(\tau)$ curve & $R_X(\tau)$ at $\tau = 0$ is obtained by integrating p.s.d. over entire f .

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau,$$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

Properties of $S_X(f)$ for stationary process

Even symmetry of $S_X(f)$

- $S_X(f) = S_X(-f)$, for all f i.e. the p.s.d of a real-valued stationary process has even symmetry.

Proof:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau$$
$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(j2\pi f\tau) d\tau$$

Replacing τ by $-\tau$ in the above equation results in

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau = S_X(f)$$

Ergodic process

- It's difficult to evaluate the higher order statistical characteristics, i.e. joint pdf & cdf, for a random process on which ensemble averages, i.e. mean, autocorrelation & autocovariance, are based.
- Hence, **time-averages** are evaluated by taking any sample function $x(t)$ from the ensemble of a random process.
- The time-averaged mean is evaluated as

$$\langle x(t) \rangle = \frac{1}{2T} \int_{-T}^T x(t) dt$$

$\lim T \rightarrow \infty$

Ergodic process

- The time-averaged auto-correlation is evaluated as

$$\langle x(t)x(t+\tau) \rangle = \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

$\lim T \rightarrow \infty$

- If time-averages are equal to ensemble averages, i.e.

$$\begin{aligned}\langle x(t) \rangle &= \mu_x(t), \\ \langle x(t)x(t+\tau) \rangle &= R_X(\tau)\end{aligned}$$

- Such processes are termed as **ergodic processes**, and sample function $x(t)$ provides a possible realization of the entire random process, $X(t)$.

Binomial

The experiment: run the Bernoulli trial n times with each trial independent of the other and count the number of 1's. This count is the random variable.

So the possible values of the rv X are $0, 1, 2, \dots, n$.

Say $n = 3$, the outcomes are

000, 001, 010, 011, 100, 101, 110, 111

this corresponds to X taking

0, 1, 1, 2, 1, 2, 2, 3.

There are two parameters for this experiment: n the number of trials and p the probability of a 1 for each trial.

What is the pdf ?

Binomial pdf

Theorem

The binomial probability distribution function is

$$\mathbb{P}(X = x) = \text{bin}(x; n, p) = \binom{n}{x} p^x (1 - p)^{n-x} \quad x = 0, 1, \dots, n.$$

$$\mathbb{P}(\text{getting } x \text{ 1's}) = p^x (1 - p)^{n-x}.$$

$$\{\text{number of ways of getting } x \text{ 1's}\} = \binom{n}{x}.$$

Binomial pdf

If $X \sim \text{Bin}(n, p)$ what is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \dots + \mathbb{E}[X_n],$$

where X_i is a Bernoulli random variable.

$$\mathbb{E}[X_i] = p,$$

so

$$\mathbb{E}[X] = np.$$

Binomial pdf

If $X \sim \text{Bin}(n, p)$ what is $\mathbb{V}[X]$?

$$\begin{aligned}\mathbb{V}[X] &= \mathbb{V}[X_1 + X_2 + X_3 + \dots + X_n], \\ &= \mathbb{V}[X_1] + \mathbb{V}[X_2] + \mathbb{V}[X_3] + \dots + \mathbb{V}[X_n],\end{aligned}$$

where X_i is an independent Bernoulli random variable.

$$\mathbb{V}[X_i] = p(1 - p),$$

so

$$\mathbb{V}[X] = np(1 - p).$$

Binomial cdf

Theorem

If $X \sim \text{Bin}(n, p)$ the cdf is denoted

$$\mathbb{P}(X \leq x) = \text{Bin}(x; n, p) = \sum_{i=0}^x \binom{n}{i} p^i (1-p)^{n-i} \quad x = 0, 1, \dots, n.$$

Binomial cdf

Example:

I give you an exam and there are $n = 120$ of you. The probability someone fails is $p = .1$.

What is the probability that at most 12 fail the test ?

$$\text{Bin}(12; 120, .1) = \sum_{i=0}^{12} \binom{120}{i} \times (.1)^i \times (.9)^{120-i}.$$

Poisson distribution

Definition

A random variable X is said to have a **Poisson distribution** with parameter $\lambda > 0$ if the pdf of X is

$$\text{Pois}(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, 3, \dots$$

Properties of the Poisson pdf

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = 1$$

By a series expansion

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}.$$

Therefore

$$e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1.$$

Properties of the Poisson pdf

The mean and variance of the Poisson distribution is

$$\mathbb{E}[X] = \mathbb{V}[X] = \lambda.$$

Parameter λ

λ is a positive real number, equal to the expected number of occurrences that occur during the given interval.

For instance, if the events occur on average every 4 minutes, and you are interested in the number of events occurring in a 10 minute interval, you would use as model a Poisson distribution with $\lambda = 10/4 = 2.5$.

Things modeled using Poisson distribution

Examples:

- ① The number of spelling mistakes one makes while typing a single page.
- ② The number of phone calls at a call center per minute.
- ③ The number of times a web server is accessed per minute.

Poisson distribution as binomial limit

Theorem

If we take the binomial pdf $\text{bin}(x; n, p)$ and take the limit $\lim_{n \rightarrow \infty, p \rightarrow 0} np = \lambda > 0$ then

$$\text{bin}(x; n, p) \rightarrow \text{Pois}(x; \lambda).$$