Course - COMMUNICATION SYSTEMS (ETEC-212)

Prepared By:
Dr. Himanshu Khanna
Asst. Professor, ECE Dept.

Table of contents I

Lesson Plan

UNIT- I Introduction to Communication systems, Random variables:
Introduction: Overview of Communication system, Communication channels, Mathematical Models for Communication Channels.
Introduction of random Variables: Definition of random variables, PDF, CDF and its properties, joint PDF, CDF, Marginalized PDF, CDF, WSS wide stationery, strict sense stationery, non stationery signals, UDF, GDF, RDF, Binomial distribution, White process, Poisson process, Wiener process.

UNIT- II Amplitude Modulation: Need for modulation, Representation of Band Pass signals and systems: Hilbert Transform, In-phase, Quad-phase representations, Power relation, modulation index, Bandwidth efficiency, AM: modulation and demodulation, DSB-SC: Modulation and demodulation, SSB: modulation and demodulation, VSB: modulation and demodulation.

Lesson Plan

UNIT- III:
Angle Modulation Systems: Frequency Modulation, Types of Frequency Modulation, Generation of NBFM, WBFM, Transmission BW of FM Signal, Phase Modulation, Relationship between PM & FM.
Radio Receivers: Functions & Classification of Radio Receivers, Tuned Radio Frequency (TRF) Receiver, Superheterodyne Receiver, Basic Elements, Receiver Characteristics, Frequency Mixers, AGC Characteristics.

UNIT- IV: Noise Theory: Noise, Types of noise, Addition of Noise due to several sources in series and parallel, Generalized Nyquist Theorem for Thermal Noise, Calculation of Thermal Noise for a Single Noise Source, RC Circuits & Multiple Noise sources. Equivalent Noise Bandwidth, Signal to Noise Ratio, Noise-Figure, Noise Temperature, Calculation of Noise Figure.

Prepared By: Dr. Himanshu Khanna, Asst. Professor, ECE Dept., MAIT

Performance of Communication Systems: Receiver Model, Noise in DSB-SC Receivers, Noise in SSB-SC Receivers, Noise in AM receiver (Using Envelope Detection), Noise in FM Receivers, FM Threshold Effect, Threshold Improvement through Pre-Emphasis and De-Emphasis, Noise in PM system Comparison of Noise performance in PM and FM, Link budget analysis for radio channels.

Prepared By: Dr. Himanshu Khanna, Asst. Professor, ECE Dept., MAIT

Text Books/ Reference Books

Text Books:

- John G. Proakis & Masoud Salehi, "Communication System Engineering," Pearson Education.
- Simon Haykin, "Communication Systems," John Wiley & Sons, Inc. 4th Edition.

Reference Books:

- Taub, H., "Principles of Communication Systems," McGraw-Hill (2008) 3rd ed.
- Kennedy, G., "Electronic Communication Systems," McGraw-Hill (2008) 4th ed.
- V. ChandraSekar, "Analog Communication," Oxford University Press, Incorporated, 2010
- John G Proakis, M.Salehi and G.Bauch, "Modern Communication System Using MATLAB," Cengage Learning, 3rd edition, 2013
- J. C. Hancock, "An Introduction to the Principles of Communication Theory," TMH, 1998.
- Peebles, "Probability and Stochastic Process," Prentice Hall; 3rd edition

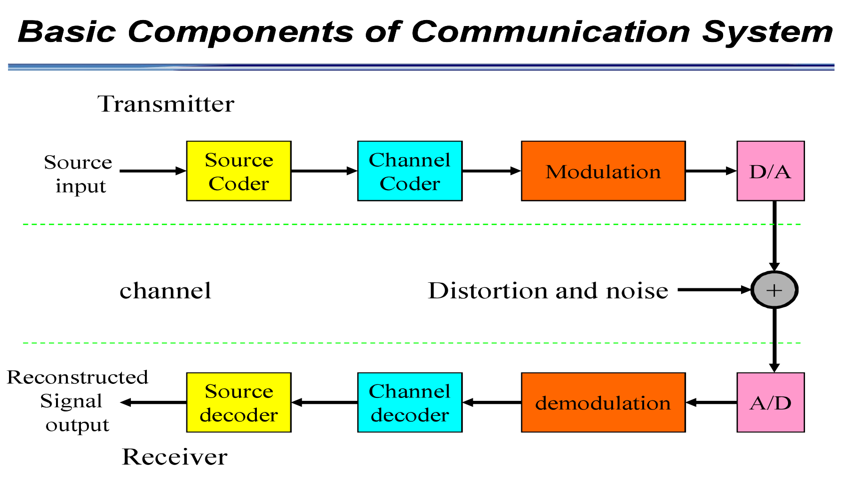


Figure: Basic elements of communication system.

- **Modes of communication:**
  - **Broadcasting**- Involves the use of single powerful transmitter & numerous receivers which are relatively inexpensive to build. Information bearing signals flow only in one direction.
  - **Point to Point**- Communication is bet. a single transmitter & receiver. Usually has a bidirectional flow.
- **Primary communication resources:**
  - Transmitter power
  - Channel bandwidth

One resource may be considered important than another. Accordingly channel may be classified as **power limited** (e.g. space comm. link or satellite channel) or **bandlimited** (e.g. telephone circuit).

- Noise which may be internal or external affects the system performance. A quantitative way to account for the effect of noise is signal-to-noise-ratio (SNR).

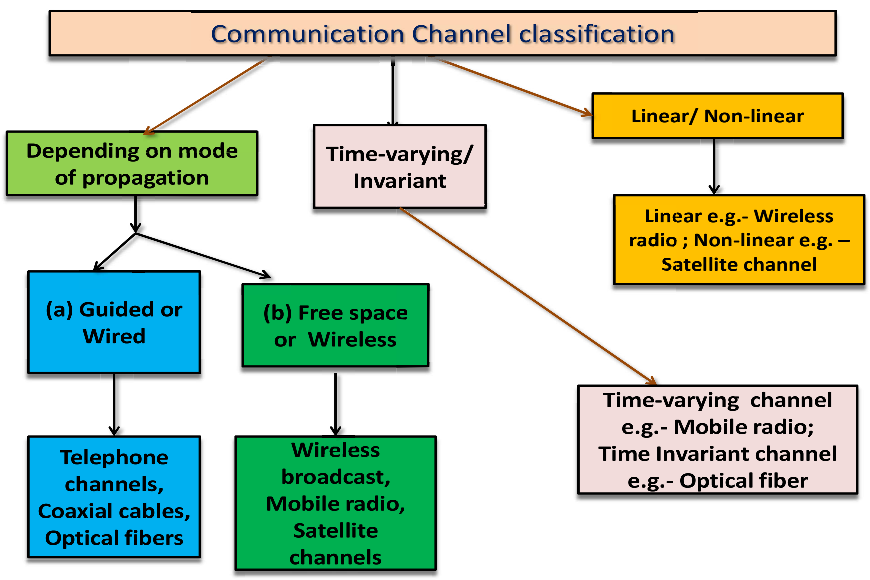
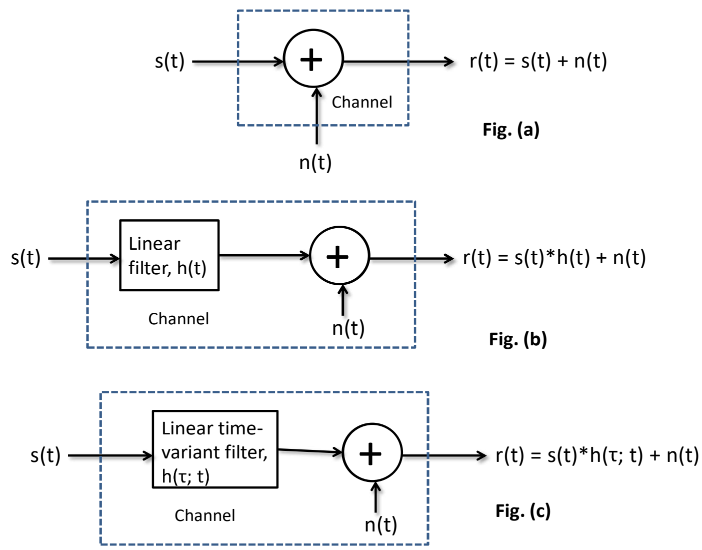


Figure: Classification of communication channels.

Mathematical models for communication channels



Random variables

Definition

A random variable (rv) is a function that maps an event from the sample space $\mathcal{S}$ to a real number: $X : \omega \rightarrow \mathbb{R}$, where $\omega \in \mathcal{S}$.

Discrete & Continuous rv's

- A **discrete** random variable takes finite or countable number of values.
- A **continuous** random variable takes values in an interval of the real line or all of the real line.

Discrete Random variables: Statistical characteristics

The **probability density function** (pdf) or **probability mass function** of a discrete random variable is defined for every possible x by $p(x) = \mathbb{P}(X = x) = \mathbb{P}(X(s) = x :$ for all $s \in \mathcal{S})$.

- **Some properties:**
  - $p(x) \geq 0$ for all $X = x$
  - $\sum_x p(x) = 1$

Example: The pdf of Bernoulli random variable - Consider a coin-tossing exp. in which probability of heads is p. For a *Bernoulli rv X* which assumes value 0 on occurrence of tails & 1 for appearance of heads, the pdf is defined as

$$P(X = x) = \begin{cases} 1 - p, & \text{x} = 0 \\ p, & \text{x} = 1 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Example: Let X represent a discrete rv that represents the sum of nos. appearing on two dice. The sample space in this case is given as

$$\mathcal{S} = [(1,1); (1,2),(2,1); (2,2),(1,3),(3,1); (1,4),(4,1),(2,3),(3,2); (1,5),(5,1),(2,4), (4,2),(3,3); (1,6),(6,1),(2,5),(5,2),(3,4),(4,3); (2,6),(6,2),(3,5),(5,3),(4,4); (3,6),(6,3),(4,5),(5,4); (4,6),(6,4),(5,5); (5,6),(6,5); (6,6)]$$

The **pmf** in this case is given as

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Discrete Random variables: Statistical characteristics

The **cumulative distribution function** (cdf) $F_X(x)$ of a rv X with pdf $p(x)$ is defined as $F_X(x) = \mathbb{P}(X \leq x) = \sum_{i=1}^{x} p(i)$. So for any number x, $F_X(x)$ is the probability that the observed value of X is at most x.

- **Some properties:**
  - The distribution function is bounded between 0 and 1, i.e. $0 \leq F_X(x) \leq 1$.
  - $F_X(x)$ is a monotone non-decreasing function of x, i.e. $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$.

Example: The cdf of Bernoulli random variable -

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 0 + 1 - p = 1 - p, & 0 \leq x < 1 \\ 0 + 1 - p + p = 1, & x \geq 1 \end{cases} \tag{2}$$

Discrete Random variables: Statistical averages (Moments)

n^{th} moment of a random variable

- Let $Y = X^n$ be a discrete rv with a set of possible values D and pdf of rv X is $p(x)$.
- The n^{th}-moment of rv X is the **expected** or **mean** value of X^n given by

$$\mathbb{E}[Y] = \mathbb{E}[X^n] = \mu_Y = \sum_{x \in D} x^n \cdot p(x). \tag{3}$$

n^{th} central-moment of a random variable

- The n^{th} central-moment of rv X is the **expected** or **mean** value of $(X - \mu_X)^n$ given by

$$\mathbb{E}[(X - \mu_X)^n] = \sum_{x \in D} (x - \mu_X)^n \cdot p(x). \tag{4}$$

Discrete Random variables: Statistical averages (Special cases)

1^{st} moment (n = 1): Expectation

Let X be a discrete rv with a set of possible values D and pdf $p(x)$. The **expected** or **mean** value of X is $\mathbb{E}[X] = \mu_x = \sum_{x \in D} x \cdot p(x)$.

2^{nd} central-moment (n = 2): Variance

Let X be a discrete rv with a set of possible values D and pdf $p(x)$. The **variance** of X is

$$\begin{aligned} Var[X] &= \sigma_x^2 \\ &= \mathbb{E}[(X - \mu_x)^2] \\ &= \sum_{x \in D} (x - \mu_x)^2 \cdot p(x). \end{aligned}$$

The **standard deviation** $\sigma_x = \sqrt{\sigma_x^2}$.

Expectation of a function of a discrete rv

- Let X be a discrete rv with a set of possible values D and pdf $p(x)$.
- Let Y be another rv where one-to-one transformation exists bet. Y and X such that $Y = h(X)$.
- The expectation of Y is the expectation of a function $h(X)$, i.e.

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \sum_{x \in D} h(x) \cdot p(x).$$

Linearity of expectation

$$
\begin{aligned}
\mathbb{E}[aX + b] &= \sum_{x \in D} (ax + b) \cdot p(x), \\
&= \sum_{x \in D} ax \cdot p(x) + \sum_{x \in D} b \cdot p(x), \\
&= a \sum_{x \in D} x \cdot p(x) + b \sum_{x \in D} p(x), \\
&= a\,\mathbb{E}[X] + b, \\
&= a\mu_X + b.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[aX + bY] &= a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] \\
&= a\mu_X + b\mu_Y.
\end{aligned}
$$

Properties of variance

$$
\begin{aligned}
Var[X] &= \mathbb{E}[(X-\mu_x)^2] \\
&= \mathbb{E}[(X^2 - 2X\mu_x + \mu_x^2)], \\
&= \mathbb{E}[X^2] - 2\,\mathbb{E}[X]\mu_x + \mu_x^2, \\
&= \mathbb{E}[X^2] - 2\mu_x^2 + \mu_x^2, \\
&= \mathbb{E}[X^2] - \mu_x^2, \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2.
\end{aligned}
$$

$$
\begin{aligned}
Var[aX+b] &= \mathbb{E}[(aX+b)^2] - (\mathbb{E}[aX+b])^2. \\
&= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mu + b)^2. \\
&= \mathbb{E}[a^2X^2] + \mathbb{E}[2abX] + b^2 - a^2\mu^2 - b^2 - 2\mu ab. \\
&= a^2\,\mathbb{E}[X^2] + 2ab\,\mathbb{E}[X] + b^2 - a^2\mu^2 - b^2 - 2\mu ab. \\
&= a^2\,\mathbb{E}[X^2] + 2ab\mu + b^2 - a^2\mu^2 - b^2 - 2\mu ab. \\
&= a^2\,\mathbb{E}[X^2] - a^2\mu^2, \\
&= a^2\,Var[X].
\end{aligned}
$$

Properties of variance

$$
\begin{aligned}
Var[CX] &= \mathbb{E}[(CX - C\mu_x)^2] \\
&= C^2\,\mathbb{E}[(X - \mu_x)^2] \\
&= C^2\,Var[X]
\end{aligned}
$$

$$
\begin{aligned}
Var[X + Y] &= Var[X] + Var[Y] \\
Var[X - Y] &= Var[X] + Var[Y]
\end{aligned}
$$

Continuous Random variables: Statistical characteristics

The **probability density function** (pdf) of a continuous random variable is defined for every possible $x \in \{-\infty, \infty\}$ by $p(x)$ or $f(x)$ where $\mathbb{P}(a \leq X \leq b) = \int_a^b p(x)dx$.

- **Some properties:**
  - $p(x) \geq 0$ for all X
  - $\int_{-\infty}^{\infty} p(x)dx = 1$
- **Example:** A uniformly-distributed rv X with $a \leq x \leq b$ is having pdf given as

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

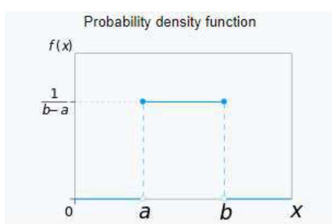


Figure: Uniform-distribution

Continuous Random variables: Statistical characteristics

The **cumulative distribution function** (cdf) $F_X(x)$ of a rv X with pdf $f_X(x)$ is defined as $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(\zeta) d\zeta$.

- **Some properties:**
  - The distribution function is bounded between 0 and 1, i.e. $0 \leq F_X(x) \leq 1$.
  - $F_X(x)$ is a monotone non-decreasing function of x, i.e. $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$.

Example: The cdf of Uniformly-distributed random variable -

$$F_X(x) = \begin{cases} 0, & x < a \\ \int_a^x \frac{1}{b-a} d\zeta = \frac{x-a}{b-a}, & a \leq x < b \\ \int_a^b \frac{1}{b-a} d\zeta = 1, & x \geq b \end{cases} \tag{6}$$

Continuous Random variables: Statistical characteristics

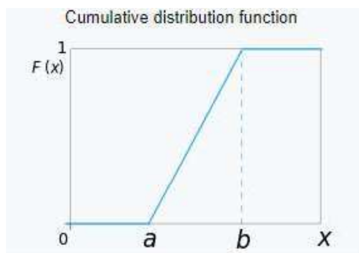


Figure: **cdf** of Uniform-distribution

Relationship between pdf & cdf

$$\frac{d}{dx} F_X(x) = f_X(x) \tag{7}$$

Prepared By: Dr. Himanshu Khanna, Asst. Professor, ECE Dept., MAIT

Two-Dimensional Random Variables: Joint pmf & cdf

Joint pmf (for discrete rv)

- Let $(X,\ Y)$ be a discrete 2-D rv & $(X,\ Y)$ takes on the values $(x_j,\ y_k)$ where $\{j, k\} \in$ integers.
- The joint pmf of $(X,\ Y)$ is given as $p_{XY}(x_j, y_k) = \mathbb{P}(X = x_j, Y = y_k)$.
- **Properties:**
  - $p_{XY}(x_j, y_k) \geq 0$ for all $(X = x_j,\ Y = y_k)$.
  - $\sum_{x_j} \sum_{y_k} p(x_j, y_k) = 1$

Joint cdf (for discrete rv)

The joint cdf, i.e. $F_{XY}(x, y)$ of a 2-D rv $(X,\ Y)$ with joint pmf $p_{XY}(x_j, y_k)$ is defined as $F_{XY}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \sum_{x_j \leq x} \sum_{y_k \leq y} p_{XY}(x_j, y_k)$.

Prepared By: Dr. Himanshu Khanna, Asst. Professor, ECE Dept., MAIT

Two-Dimensional Random Variables: Joint pdf & cdf

Joint pdf & cdf for continuous rv, and their Relationship

- Let $(X,\ Y)$ be a continuous 2-D rv with joint pdf, i.e. $f_{XY}(x,y)$ and cdf, i.e. $F_{XY}(x,y)$, then

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \tag{8}$$

- Integrating (8) results in

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\zeta,\eta) d\zeta d\eta \tag{9}$$

- **Properties:**
  - $f_{XY}(x,y) \geq 0$ for all $(X,\ Y)$.
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) dxdy = 1$.

Two-Dimensional Random Variables: Marginal pdf & Statistical independence

Marginal pdf's for continuous 2-D rv

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x, y)dy \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x, y)dx
\end{aligned} \tag{10}
$$

- If X & Y are statistically independent, then the joint pdf can be expressed in terms of marginal pdf's as

$$
f_{XY}(x, y) = f_X(x)f_Y(y) \tag{11}
$$

Two-Dimensional Random Variables: Conditional pdf & Statistical independence

- The conditional pdf of X given the event $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}, \text{where } f_Y(y) \neq 0 \tag{12}$$

- For statistical independence, the conditional pdf of X is independent of the occurrence of the event $\{Y = y\}$, i.e.

$$f_X(x) = \frac{f_{XY}(x,y)}{f_Y(y)} \tag{13}$$

Two-Dimensional Random Variables: Joint moments

Joint simple-moment of a 2-D random variable

- Let $Z = X^j Y^k$ be a 2-D rv.
- The joint simple-moment of rv Z is the **expected** or **mean** value of $X^j Y^k$ given by:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[X^j Y^k] = \mu_Z = \sum_i \sum_l x^j y^k p(x_i, y_l). \tag{14}$$

For Continuous case:

$$\mathbb{E}[Z] = \mathbb{E}[X^j Y^k] = \mu_Z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^j y^k f_{XY}(x, y) dxdy. \tag{15}$$

Joint central-moment of a 2-D random variable

- Let $Z = (X - \mu_X)^j (Y - \mu_Y)^k$ be a 2-D rv.
- The joint central-moment of rv Z is the **expected** or **mean** value of $(X - \mu_X)^j (Y - \mu_Y)^k$ given by:

  For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)^j (Y - \mu_Y)^k] = \mu_Z = \sum_i \sum_l (x - \mu_X)^j (y - \mu_Y)^k p(x_i, y_l). \tag{16}$$

  For Continuous case:

$$\begin{aligned} \mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)^j (Y - \mu_Y)^k] &= \mu_Z \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)^j (y - \mu_Y)^k f_{XY}(x, y) dxdy. \end{aligned} \tag{17}$$

Joint simple-moment: Special cases (Correlation)

Correlation bet. two rv's

- For $j = k = 1$, $Z = XY$ is a 2-D rv.
- The joint simple-moment of rv Z in this case which is the **expected** or **mean** value of XY gives **correlation** as:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[XY] = Corr[XY] = \sum_i \sum_l xy \ p(x_i, y_l). \tag{18}$$

For Continuous case:

$$\mathbb{E}[Z] = \mathbb{E}[XY] = Corr[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{XY}(x, y)dxdy. \tag{19}$$

Joint central-moment: Special cases (Covariance)

Covariance of X & Y

- For $j = k = 1$, $Z = (X - \mu_X)(Y - \mu_Y)$ is a 2-D rv.
- The joint central-moment of rv Z which is the **expected** or **mean** value of $(X - \mu_X)(Y - \mu_Y)$ gives the **covariance** as:

For Discrete case:

$$\mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = Cov[XY] = \sum_i \sum_l (x - \mu_X)(y - \mu_Y)p(x_i, y_l). \quad (20)$$

For Continuous case:

$$\begin{aligned} \mathbb{E}[Z] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] &= Cov[XY] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f_{XY}(x, y) dxdy. \end{aligned} \quad (21)$$

Joint central-moment: Special cases (Covariance), & Correlation coeff.

- From (20) & (21),

Covariance of X & Y

$Cov[XY] = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)]$ can be simplified & expressed in terms of Corr[XY], as

$$\begin{aligned} Cov[XY] &= \mathbb{E}[(XY - Y\mu_X - X\mu_Y + \mu_X\mu_Y)] \\ &= \mathbb{E}[XY] - \mu_X\,\mathbb{E}[Y] - \mu_Y\,\mathbb{E}[X] + \mu_X\mu_Y \\ &= \mathbb{E}[XY] - \mu_X\mu_Y \\ &= Corr[XY] - \mu_X\mu_Y \qquad (22) \end{aligned}$$

Correlation-coefficient, ρ

$$\rho = \frac{Cov[XY]}{\sigma_X\sigma_Y}, \qquad (23)$$

where σ_X & σ_Y are the standard deviations for rv's X & Y, resp., & $-1 \le \rho \le 1$.

Random Processes: Need

- In a radio comm. system, the received signal usually consists of an information bearing signal component, a random interference component, & channel noise.
- Hence, the received time-varying signal is random in nature.

- Random processes combines the concepts of time variation & random variables, and describes the signal in terms of the statistical parameters such as avg. power & power spectral density (p.s.d.).

- Random processes represents formal mathematical model of these random signals & have the following properties:
  - They are functions of time.
  - They are random in the sense that it is not possible to predict exactly the waveform that will be observed in the future.

Random Processes: Definition

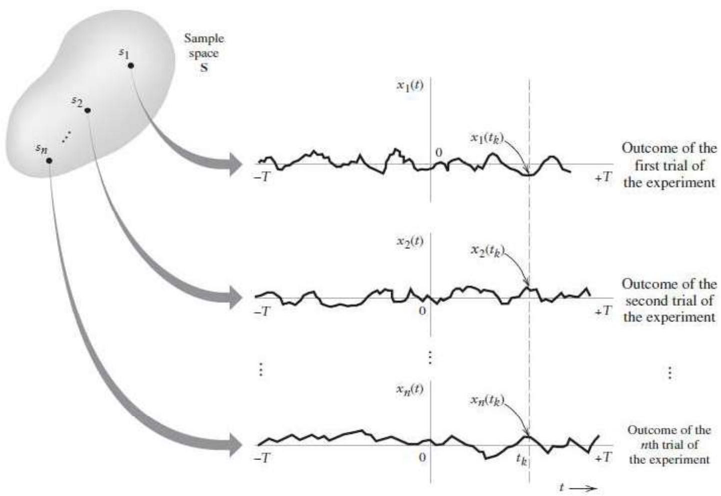


Figure: Relationship bet. sample space & ensemble of sample functions.

Random Processes: Definition

- For an observation interval of $2T$, we assign to each sample point s a function of time

$$X(t,s), \qquad -T < t < T \tag{24}$$

- The set of these sample functions, i.e. $\{X(t,s)\}$, together with a probability rule is termed as a random process, i.e. $X(t)$.
- For a fixed sample point s_j, $X(t,s_j) = x_j(t)$ denotes the j^{th} sample function of a random process.
- Observing sample functions $\{x_j(t)\}_{j=1,...,n}$ at some fixed time instant, say t_k, gives a set of random variables, i.e.

$$\{x_1(t_k), x_2(t_k), ..., x_n(t_k)\} = \{X(t_k,s_1), X(t_k,s_2), ..., X(t_k,s_n)\} \tag{25}$$

Random Processes: Types (Stationarity: Strict & Wide-sense)

Stationary random process

- If a random process is divided into a number of time intervals, the various sections of the process exhibit essentially the same statistical properties.
- Otherwise, it is said to be non-stationary.
- If

  $$F_{X(t_1+\tau)}(x) = F_{X(t_1)}(x),$$

  the process is *stationary* to the *first-order*, and as a consequence, statistical parameters such as mean & variance are also independent of time for such a process.
- If joint-distribution function

  $$F_{X(t_1+\tau),X(t_2+\tau)}(x_1, x_2) = F_{X(t_1),X(t_2)}(x_1, x_2),$$

  the process is stationary to the *second-order*, and as a consequence, statistical quantities such as covariance & correlation, do not depend upon absolute time.

Strictly-stationary random processes

Strictly-stationary random process: kth-order stationarity

- If the equivalence bet. distribution functions holds for all time shifts, i.e. τ, all k, and all possible observation times $t_1, t_2, ..., t_k$, then the process is *strictly-stationary*.
- In other words, for strict-sense-stationarity (SSS), the joint distribution of any set of rv's obtained by observing the random process $X(t)$ is invariant w.r.t. the location of the origin t $= 0$.

Statistical-averages of a random process

Mean

- The ensemble-average mean, i.e. $\mu_X(t)$ or $m_X(t)$, of a random process at instant t is

$$\mu_X(t) = \mathbb{E}[X(t)]$$

Auto-correlation & Auto-covariance

- The ensemble-average autocorrelation, i.e. $R_X(t_1, t_2)$, is obtained by observing the random process at $t = t_1$ and $t = t_2$, and given as

$$R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),X(t_2)}(x, y)\,dxdy$$

- The ensemble-average autocovariance, i.e. $C_X(t_1, t_2)$, is obtained as

$$C_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))] = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Statistical-averages of a random process

Cross-correlation

- The ensemble-average cross-correlation, i.e. $R_{XY}(t_1, t_2)$, is obtained by observing two diff. random processes, i.e. $X(t)$ & $Y(t)$, at $t = t_1$ and $t = t_2$, resp., and given as

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1)Y(t_2)]$$

Wide-sense stationary (WSS)/ Weakly stationary random process

Mean

- The ensemble-average mean, i.e. $\mu_X(t)$ or $m_X(t)$, of a random process is independent of t, and is constant, i.e.

$$\mu_X(t) = \mathbb{E}[X(t)] = \mu = \textit{constant}$$

Auto-correlation & Auto-covariance

- The ensemble-average autocorrelation, i.e. $R_X(t_1, t_2)$, depends only on the time differences of t_1 and t_2, i.e. $t_1 - t_2 = \tau$, and given as

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

- It follows from above, the ensemble-average autocovariance, i.e. $C_X(t_1, t_2)$, depends only on the time differences of t_1 and t_2, i.e. $t_1 - t_2 = \tau$, and given as

$$C_X(t_1, t_2) = C_X(t_1 - t_2) = C_X(\tau) = R_X(\tau) - \mu^2$$

Properties of Autocorrelation for stationary process

Mean-square value in terms of $R_X(\tau)$

- $R_x(0) = \mathbb{E}[X^2(t)]$, i.e. the autocorrelation function for $\tau = 0$ is same as mean-square value of random process, $X(t)$.

  Proof:

$$R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)]$$
$$R_X(0) = \mathbb{E}[X^2(t)]$$

Even symmetry of $R_X(\tau)$

- $R_X(\tau) = R_X(-\tau)$, i.e. the autocorrelation of a real-valued wide-sense stationary process has even symmetry.

  Proof:

$$R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)]$$
$$R_X(-\tau) = \mathbb{E}[X(t)X(t-\tau)]$$

Replacing t with $t+\tau$ above results in

$$R_X(-\tau) = \mathbb{E}[X(t+\tau)X(t)] = R_X(\tau)$$

Properties of Autocorrelation for stationary process

$R_X(\tau)$ has max. magnitude at $\tau = 0$

- $R_X(0) = \mathbb{E}[X^2(t)] \geq \mid R_X(\tau) \mid$.

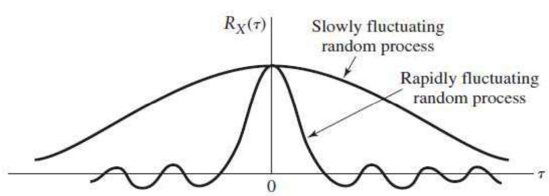


Figure: Illustration of autocorrelation functions of slowly and rapidly fluctuating random processes.

Relationship between $R_X(\tau)$ & p.s.d. $S_X(f)$: Weiner-Khintchine relations

Relationship bet. $R_X(\tau)$ & $S_X(f)$

- The p.s.d., i.e. $S_X(f)$, is the Fourier transform of $R_X(\tau)$, and both form Fourier transform pairs, i.e.

$$
\begin{aligned}
S_X(f) &= \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau, \\
R_X(\tau) &= \mathcal{F}^{-1}\{S_X(f)\} = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df \\
S_X(f) &\overset{\mathcal{F}}{\rightleftharpoons} R_X(\tau)
\end{aligned}
$$

Properties of $S_X(f)$ for stationary process

$S_X(f)$ is always positive for all f

- $S_X(f) \geq 0$, i.e. non-negative for entire range of frequencies, f.

$S_X(f)$ at $f = 0$ & $R_X(\tau)$ at $\tau = 0$

- $S_X(f)$ at $f = 0$ is obtained by finding area under $R_X(\tau)$ curve & $R_X(\tau)$ at $\tau = 0$ is obtained by integrating p.s.d. over entire f.

$$
\begin{aligned}
S_X(0) &= \int_{-\infty}^{\infty} R_X(\tau) d\tau, \\
R_X(0) &= \int_{-\infty}^{\infty} S_X(f) df
\end{aligned}
$$

Prepared By: Dr. Himanshu Khanna, Asst. Professor, ECE Dept., MAIT

Properties of $S_X(f)$ for stationary process

Even symmetry of $S_X(f)$

- $S_X(f) = S_X(-f)$, for all f i.e. the p.s.d of a real-valued stationary process has even symmetry.

  Proof:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(j2\pi f\tau)d\tau$$

  Replacing τ by $-\tau$ in the above equation results in

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau = S_X(f)$$

Ergodic process

- Its difficult to evaluate the higher order statistical characteristics, i.e. joint pdf & cdf, for a random process on which ensemble averages, i.e. mean, autocorrelation & autocovariance, are based.
- Hence, time-averages are evaluated by taking any sample function $x(t)$ from the ensemble of a random process.
- The time-averaged mean is evaluated as

$$< x(t) > \quad = \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

Ergodic process

- The time-averaged auto-correlation is evaluated as

$$< x(t)x(t+\tau) > \;=\; \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

- If time-averages are equal to ensemble averages, i.e.

$$\begin{aligned} < x(t) > &= \mu_X(t), \\ < x(t)x(t+\tau) > &= R_X(\tau) \end{aligned}$$

- Such processes are termed as ergodic processes, and sample function $x(t)$ provides a possible realization of the entire random process, $X(t)$.

Binomial

The experiment: run the Bernoulli trial n times with each trial independent of the other and count the number of 1's. This count is the random variable.
So the possible values of the rv X are $0, 1, 2, ..., n$.
Say $n = 3$, the outcomes are

$$000, 001, 010, 011, 100, 101, 110, 111$$

this corresponds to X taking

$$0, 1, 1, 2, 1, 2, 2, 3.$$

There are two parameters for this experiment: n the number of trials and p the probability of a 1 for each trial.
What is the pdf ?

Binomial pdf

Theorem

The binomial probability distribution function is

$$\mathbb{P}(X = x) = \text{bin}(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, ..., n.$$

$$\mathbb{P}(\text{getting } x \text{ 1's}) = p^x (1-p)^{n-x}.$$

$$\{\text{number of ways of getting } x \text{ 1's}\} = \binom{n}{x}.$$

Binomial pdf

If $X \sim \text{Bin}(n, p)$ what is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + ... + \mathbb{E}[X_n],$$

where X_i is a Bernoulli random variable.

$$\mathbb{E}[X_i] = p,$$

so

$$\mathbb{E}[X] = np.$$

Binomial pdf

If $X \sim \text{Bin}(n, p)$ what is $\mathbb{V}[X]$?

$$\begin{aligned} \mathbb{V}[X] &= \mathbb{V}[X_1 + X_2 + X_3 + ... + X_n], \\ &= \mathbb{V}[X_1] + \mathbb{V}[X_2] + \mathbb{V}[X_3] + ... + \mathbb{V}[X_n], \end{aligned}$$

where X_i is an independent Bernoulli random variable.

$$\mathbb{V}[X_i] = p(1-p),$$

so

$$\mathbb{V}[X] = np(1-p).$$

Binomial cdf

Theorem

If $X \sim \text{Bin}(n, p)$ the cdf is denoted

$$\mathbb{P}(X \leq x) = \text{Bin}(x; n, p) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i} \quad x = 0, 1, ..., n.$$

Binomial cdf

Example:
I give you an exam and there are $n = 120$ of you. The probability someone fails is $p = .1$.
What is the probability that at most 12 fail the test ?

$$\mathrm{Bin}(12; 120, .1) = \sum_{i=0}^{12} \binom{120}{i} \times (.1)^i \times (.9)^{120-i}.$$

Poisson distribution

Definition

A random variable X is said to have a **Poisson distribution** with parameter $\lambda > 0$ if the pdf of X is

$$\mathrm{Pois}(x; \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

Properties of the Poisson pdf

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$$

By a series expansion

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}.$$

Therefore

$$e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1.$$

Properties of the Poisson pdf

The mean and variance of the Poisson distribution is

$$\mathbb{E}[X] = \mathbb{V}[X] = \lambda.$$

Parameter λ

λ is a positive real number, equal to the expected number of occurrences that occur during the given interval.

For instance, if the events occur on average every 4 minutes, and you are interested in the number of events occurring in a 10 minute interval, you would use as model a Poisson distribution with $\lambda = 10/4 = 2.5$.

Things modeled using Poisson distribution

Examples:

1. The number of spelling mistakes one makes while typing a single page.
2. The number of phone calls at a call center per minute.
3. The number of times a web server is accessed per minute.

Poisson distribution as binomial limit

Theorem

If we take the binomial pdf $\text{bin}(x; n, p)$ and take the limit $\lim_{n \to \infty, p \to 0} np = \lambda > 0$ then

$$\text{bin}(x; n, p) \to \text{Pois}(x; \lambda).$$

Continuous probability distributions: Uniform distribution (Mean & Variance)

- The mean μ of the uniform distribution is given by

$$\mu = E(X) = \int_a^b z\left(\frac{1}{b-a}\right)dz = \frac{b^2 - a^2}{2}\frac{1}{b-a} = \frac{b+a}{2}$$

- The standard deviation σ of the uniform distribution is obtained from the variance σ^2 where

$$\sigma^2 = E((X-\mu)^2) = \int_a^b \left(z - \frac{b+a}{2}\right)^2\left(\frac{1}{b-a}\right)dz = \frac{(b-a)^2}{12}$$

Continuous probability distributions: Rayleigh distribution (RDF)

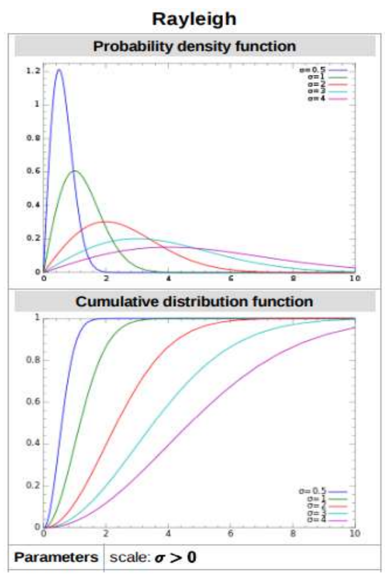
Let $U \sim N(0, \sigma^2)$ and $V \sim N(0, \sigma^2)$ be independent random variables, define $X = \sqrt{U^2 + V^2}$, then X has a Rayleigh distribution with the cumulative probability distribution (c.d.f.) and pprobability density function (p.d.f.) given below.

$$
\begin{aligned}
F(x) &= 1 - e^{-x^2/2\sigma^2}, \quad x > 0 \\
&= 0, \quad x \leq 0
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \tfrac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad x > 0 \\
&= 0, \quad x \leq 0
\end{aligned}
$$

$$
\begin{aligned}
E(X) &= \textstyle\int_0^\infty \frac{x^2}{\sigma^2} e^{-x^2/2\sigma^2} dx = \sqrt{\frac{\pi}{2}}\sigma \\
E(X^2) &= \textstyle\int_0^\infty \frac{x^3}{\sigma^2} e^{-x^2/2\sigma^2} dx = 2\sigma^2\Gamma(2) = 2\sigma^2 \\
Var(X) &= E(X^2) - (E(X))^2 = \tfrac{4-\pi}{2}\sigma^2
\end{aligned}
$$

Continuous probability distributions: Rayleigh distribution (RDF)

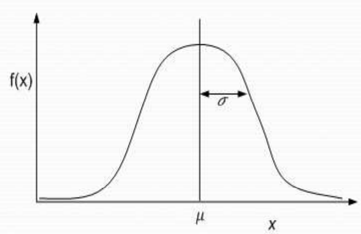


Continuous probability distributions: Gaussian or Normal distribution (GDF)

- The most often used continuous probability distribution is the normal distribution; it is also known as **Gaussian** distribution.
- Its graph called the normal curve is the bell-shaped curve.
- Such a curve approximately describes many phenomenon occur in nature, industry and research.
- Physical measurement in areas such as meteorological experiments, rainfall studies and measurement of manufacturing parts are often more than adequately explained with normal distribution.
- The equivalent distribution of sum of random variables approaches Gaussian or Normal distribution due to Central Limit Theorem.
- A continuous random variable X having the bell-shaped distribution is called a normal random variable.

Continuous probability distributions: Gaussian or Normal distribution (GDF)

- The mathematical equation for the probability distribution of the normal variable depends upon the two parameters μ and σ, its mean and standard deviation.



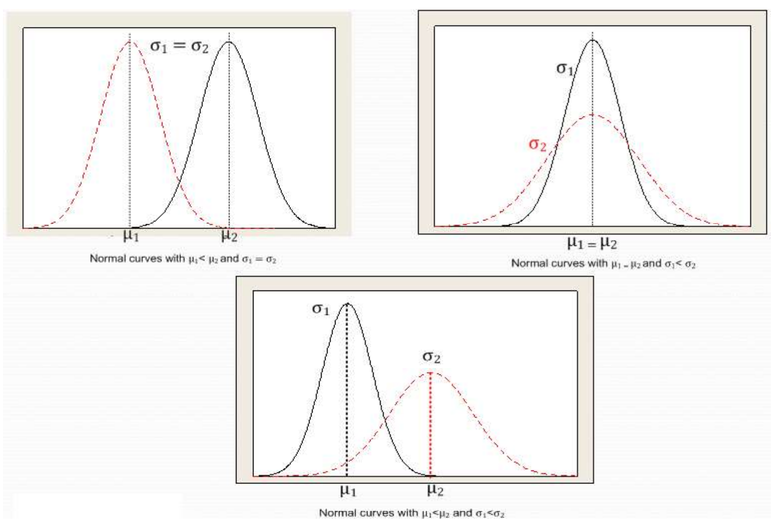
Definition : Normal distribution

The density of the normal variable x with mean μ and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

where $\pi = 3.14159 \ldots$ and $e = 2.71828 \ldots\ldots$, the Naperian constant

Continuous probability distributions: Gaussian or Normal distribution (GDF)



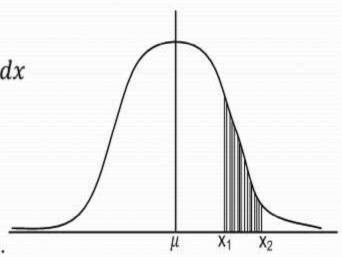
Continuous probability distributions: Gaussian or Normal distribution (GDF)

- The curve is symmetric about a vertical axis through the mean μ.
- The random variable x can take any value from $-\infty\ to\ \infty$.
- The most frequently used descriptive parameter s define the curve itself.
- The mode, which is the point on the horizontal axis where the curve is a maximum occurs at $x = \mu$.
- The total area under the curve and above the horizontal axis is equal to 1.

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx = 1$$

- $\mu = \int_{-\infty}^{\infty} x.f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x.e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$
- $\sigma^2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2.e^{-\frac{1}{2}[(x-\mu)/\sigma^2]}dx$
- $P(x_1 < x < x_2) = \frac{1}{\sigma\sqrt{2\pi}}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$

denotes the probability of x in the interval (x_1, x_2).



Continuous probability distributions: Standard Normal distribution

- The normal distribution has computational complexity to calculate $P(x_1 < x < x_2)$ for any two (x_1, x_2) and given μ and σ
- To avoid this difficulty, the concept of z-transformation is followed.

$$\boxed{z = \frac{x-\mu}{\sigma} \qquad \text{[Z-transformation]}}$$

- X: Normal distribution with mean μ and variance σ^2.
- Z: Standard normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$.
- Therefore, if $f(x)$ assumes a value, then the corresponding value of $f(z)$ is given by

$$f(x: \mu, \sigma): \; P(x_1 < x < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

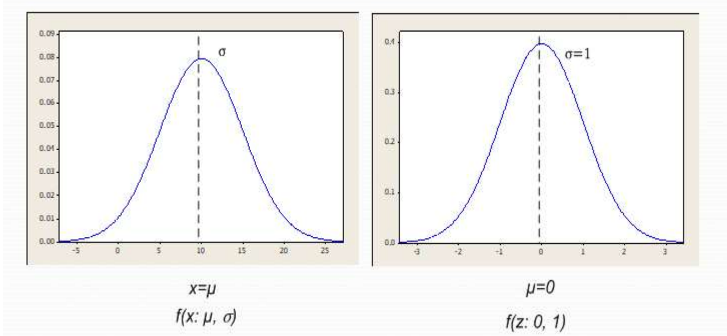
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2} dz$$

$$= f(z: 0, \sigma)$$

Continuous probability distributions: Standard Normal distribution

Definition : Standard normal distribution

The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution.



White Process

W.S.S White Noise Process: If $W(t)$ is a w.s.s white noise process, then

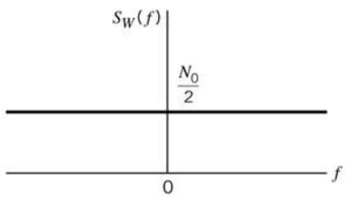
$$R_{WW}(\tau) = q\delta(\tau) \quad \Rightarrow \quad S_{WW}(\omega) = q.$$

Thus the spectrum of a white noise process is flat, thus justifying its name. Notice that a white noise process is unrealizable since its total power is indeterminate.

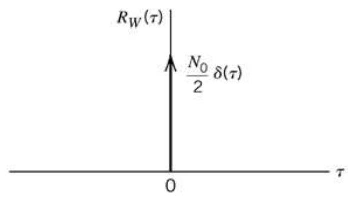
If the input to an unknown system is a white noise process, then the output spectrum is given by

$$S_{YY}(\omega) = q \,|\, H(\omega) \,|^2$$

White Process



(a)



(b)

$$S_W(f) = \frac{N_0}{2} \quad = q$$

$$N_0 = kT_e$$

T_e : equivalent noise temperature of the receiver

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau)$$

White Process

Example : A w.s.s white noise process $W(t)$ is passed through a low pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

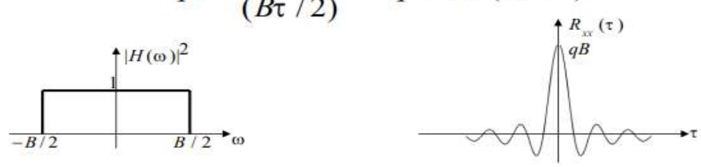
Solution: Let $X(t)$ represent the output of the LPF. Then from

$$S_{XX}(\omega) = q\,|H(\omega)|^2 = \begin{cases} q, & |\omega| \le B/2 \\ 0, & |\omega| > B/2 \end{cases}.$$

Inverse transform of $S_{XX}(\omega)$ gives the output autocorrelation function to be

$$R_{XX}(\tau) = \int_{-B/2}^{B/2} S_{XX}(\omega) e^{j\omega\tau}\, d\omega = q\int_{-B/2}^{B/2} e^{j\omega\tau}\, d\omega$$

$$= qB\,\frac{\sin(B\tau/2)}{(B\tau/2)} = qB\,\mathrm{sinc}(B\tau/2)$$



(a) LPF (b)

Wiener Process

- In mathematics, the Wiener process is a real valued continuous-time stochastic process named in honor of American mathematician Norbert Wiener for his investigations on the mathematical properties of the 1-D Brownian motion.
- It is often also called **Brownian motion** due to its historical connection with the physical process of the same name originally observed by Scottish botanist Robert Brown.
- In applied maths, it is used to represent the integral of a white noise Gaussian process, & so is useful as a model of noise in electronics engg., instrument errors in filtering theory etc.
- The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. This is known as **Donsker's theorem**.

Wiener Process: As a Limit of Random Walk

In mathematics, the limiting processes $\mathsf{W}(t)$ defined by

$$\mathsf{X}_\tau(t) \equiv \sum_{k=0}^{t/\tau} \Delta\mathsf{X}_k \to W(t) \quad \textbf{as} \quad \tau \to 0$$

where $\Delta\mathsf{X}_k$ are i.i.d. random variable with mean 0 and unit variance is called the ***Brownian motion*** or ***Wiener process***.

Wiener Process

Here is the formal definition:

Proposition (Wiener process)

A family of random variables $\{\mathsf{W}(t)\}$ indexed by the continuous variable t ranging over $[0, \infty)$ is called Wiener process (or Brownian motion) if and only if it satisfies the following conditions:

(i) $\mathsf{W}(0) = 0$;

(ii) the increments $\mathsf{W}(t_i + \Delta t_i) - \mathsf{W}(t_i)$ over an arbitrary finite set of disjoint intervals $(t_i, t_i + \Delta t_i)$, are independent random variables;

(iii) for each $s \geq 0$, $t \geq 0$, $\mathsf{W}(t + \Delta t) - \mathsf{W}(t)$ has normal distribution $N(0, \Delta t)$.

For each constant y, the process $\mathsf{W}_y(t) = \mathsf{W}(t) + y$, is called the Wiener process starting at y.

Poisson Process

Poisson process A Poisson process with intensity λ is a process N_t, $t \geq 0$, with state space the positive integers, such that

(i) $N_0 = 0$,

(ii) $N_t - N_s$ is a Poisson distributed random variable with parameter $\lambda(t-s)$ for all $0 \leq s < t$,

(iii) the increments $N_{t_2} - N_{t_1}$ and $N_{t_4} - N_{t_3}$ are independent for all $0 \leq t_1, < t_2 \leq t_3 < t_4$.

Its means, variances and covariances are

$$\mathbb{E}\{N(t)\} = \lambda t \quad , \mathbf{Var}\,(N(t)) = \lambda t \quad C(s,t) = \lambda \min\{s,t\}\,,$$

respectively, for all $s, t > 0$.

A Poisson process is an example of a continuous time stochastic process with independent increments. The process is named after the French mathematician Poisson and is a good model of radioactive decay and many other phenomena which involve the counting of the number of events in a given time interval.