# Analog To Digital Conversion : Sampling Theorm

Sampling Process is used to convert a continuous time signal into a discrete time signal. No of samples to be taken depends upon max. signal frequency present in the signal.

Sampling Theorm :-

A Continuous time signal may be completely represented in its samples and recovered back if Sampling Frequency is $F_s \geq 2 F_m$.

$F_s \rightarrow$ Sampling Freq.

$F_m \rightarrow$ Max Freq. Present in the signal.

## Proof of Sampling Theorm :-

Consider a continuous time Signal $x(t)$ whose spectrum is band-limited to $f_m$ Hz.
This means signal $x(t)$ has no frequency components beyond $f_m$ Hz.

$\therefore$ $X(\omega)$ is zero for $|\omega| > \omega_m$

$$\omega_m = 2\pi f_m$$

Fig 1 shows a continuous time Signal $x(t)$.

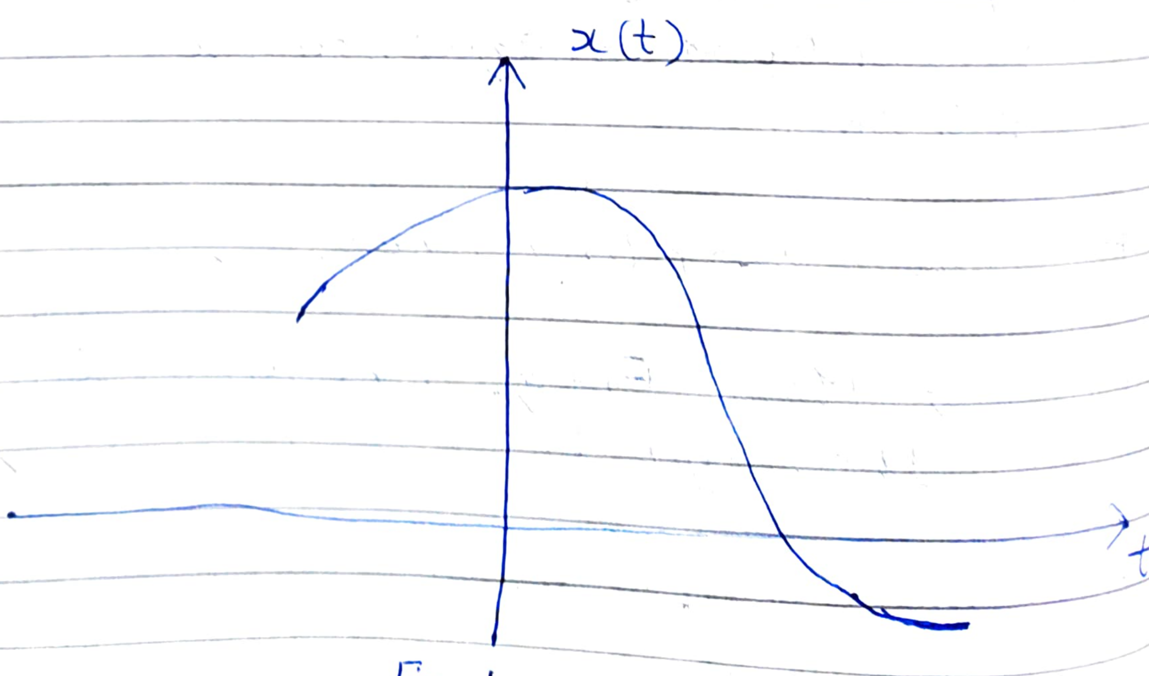



Fig 1

Let $X(\omega)$ represents its Fourier Transform or frequency spectrum as shown in Fig 2.

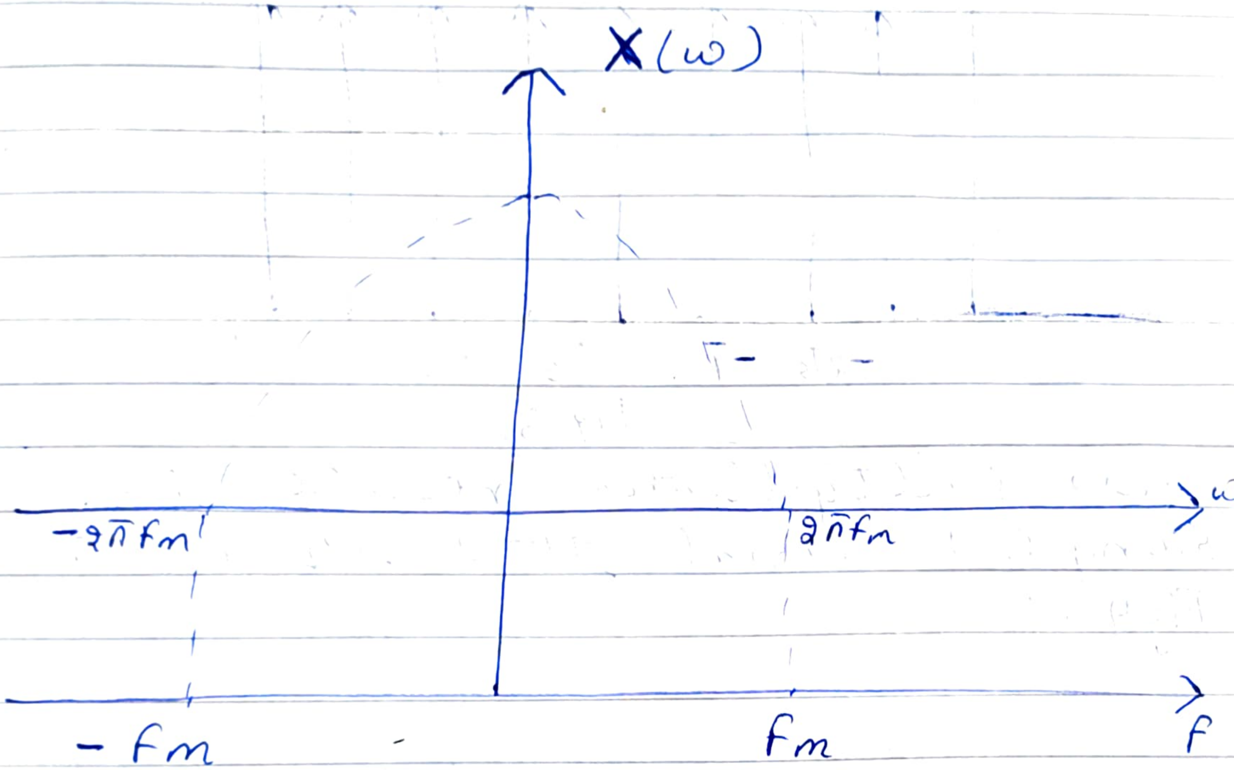



Fig 2

Sampling of $x(t)$ at a rate of $f_s$ Hz may be achieved by multiplying $x(t)$ by an Impulse Train $\delta_{T_s}(t)$. Impulse Train $\delta_{T_s}(t)$

consists of Unit Impulses repeating periodically every $T_s$ secs.

where $T_s = \frac{1}{f_s}$

Fig 3 Shows this Impulse Train.

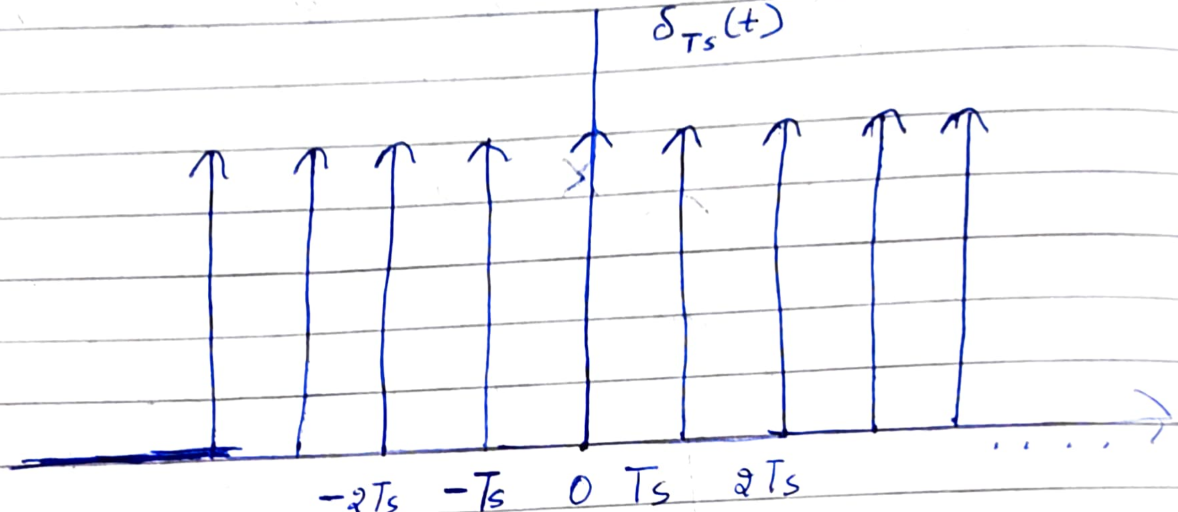



Fig 3.

This multiplication results in sampled signal $g(t)$ shown in Fig 4.

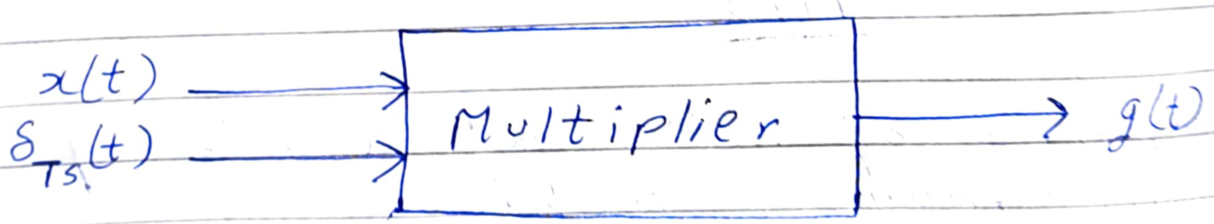



Fig 4

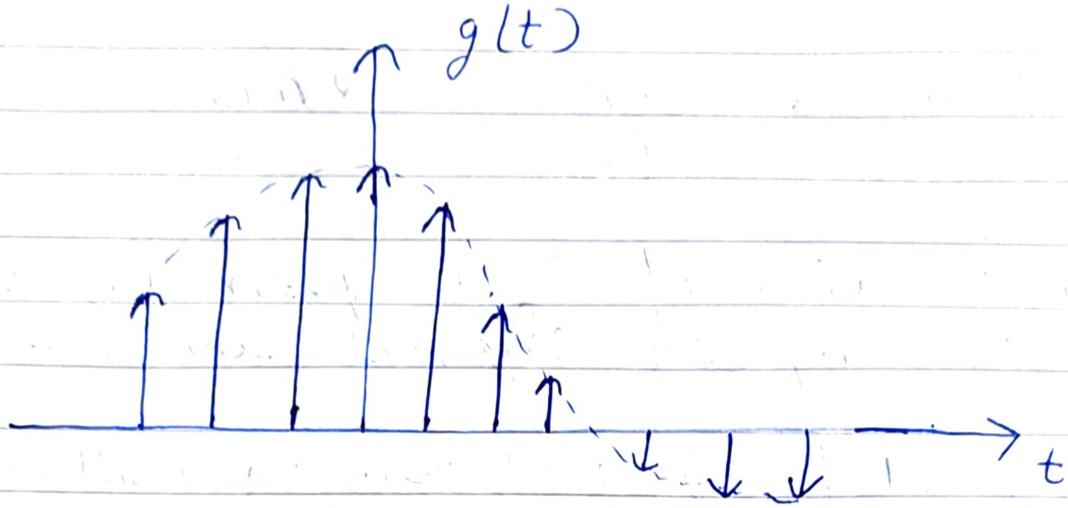



Fig 5

This sampled signal consists of impulses spaced every $T_s$ secs.

Resulting or sampled signal is written as :-

$$g(t) = x(t)\,\delta_{Ts}(t) \qquad -(1)$$

Since Impulse Train $\delta_{TS}(t)$ is a periodic signal of period $T_s$, it may be expressed as a Fourier Series.

Trignometric Fourier series expansion of Impulse Train $\delta_{Ts}(t)$ is expressed as :-

$$\delta_{Ts}(t) = \frac{1}{T_s}\left[1 + 2\cos\omega_s t + 2\cos 2\omega_s(t) + 2\cos 3\omega_s t + \ldots\right] \qquad -(2)$$

Here $\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$

Put value of $\delta_{T_s}(t)$ from (2) in (1), the sampled signal is :-

$$g(t) = \frac{1}{T_s}\Big[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + 2x(t)\cos 3\omega_s t + \ldots\Big] \quad (3)$$

To obtain $G(\omega)$; Take F. T. of (3)

F. T. of $x(t)$ is $X(\omega)$.

F. T. of $2x(t)\cos\omega_s t$ is $[X(\omega-\omega_s) + X(\omega+\omega_s)]$

F. T. of $2x(t)\cos 2\omega_s t$ is $[X(\omega - 2\omega_s) + X(\omega+2\omega_s)]$

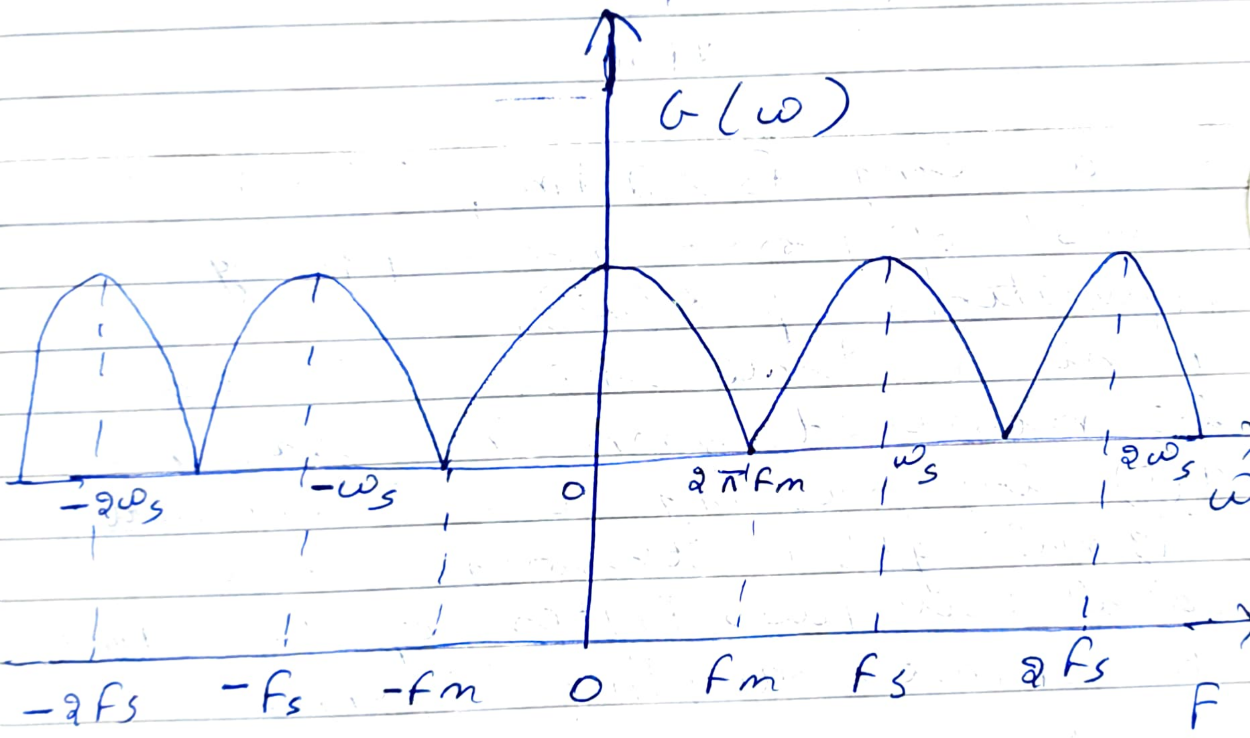
Taking F. T. of (3) we get :-

$$G(\omega) = \frac{1}{T_s}\Big[X(\omega) + X(\omega-\omega_s) + X(\omega+\omega_s) + X(\omega-2\omega_s) + X(\omega+2\omega_s) + X(\omega-3\omega_s) + X(\omega+3\omega_s) + \ldots\Big] \quad - (4)$$

$$G(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \quad - (5)$$

From (4) and (5) it is seen that spectrum $G(\omega)$ consists of $X(\omega)$ repeating periodically with period $\omega_s = \frac{2\pi}{T_s}$ rad/sec or $f_s = \frac{1}{T_s}$ Hz

as shown in Fig 6.




Now if we have to Reconstruct $x(t)$ from $g(t)$, we must be able to recover $X(\omega)$ from $G(\omega)$. This is possible if there is no overlap between successive cycles of $G(\omega)$.

Fig 6 shows that this requires

$$f_s > 2f_m$$

But Sampling Interval $T_s = \frac{1}{f_s}$

Hence $T_s < \frac{1}{2f_m}$

So as long $f_s > 2f_m$, $G(\omega)$ will consist of non-overlapping repetitions of $X(\omega)$.
If this is true then $x(t)$ can be recovered from its samples $g(t)$ by passing the sampled Signal $g(t)$ through an Ideal Low Pass Filter of B.W. $f_m$ Hz.
This Proves Sampling Theorm.

Trignometric Fourier Series :-

$$x(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + a_n \cos n\omega t + b_1 \sin\omega t + b_2 \sin 2\omega t + \ldots b_n \sin n\omega t$$

For Even Funcs. Symmetrical about y-axis, only Cosine terms are present.

$$a_0 = \frac{1}{T} \int x(t)\, dt = 1$$

$$a_n = \frac{2}{T} \int x(t) \cos n\omega_0 t \, dt$$

$$\delta_{T_s}(t) \equiv \frac{1}{T_s} \left[ x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + \dots \right]$$

$$2x(t)\cos\omega_s t =$$

$$\cos\omega_s t = \frac{e^{j\omega_s t} + e^{-j\omega_s t}}{2}$$

$$F\left\{ e^{j\omega_s t} \right\} \longrightarrow X(j\omega - j\omega_s)$$

Summary of Sampling Theorm :-

1) When $f_s > 2f_m$, the successive cycles of $G(\omega)$ are not overlapping with each other. So original spectrum $X(\omega)$ can be easily recovered from $G(\omega)$.

2.) When $f_s = 2f_m$, successive cycles of $G(\omega)$ are touching each other. Original Spectrum $X(\omega)$ can be recovered from the sampled spectrum $G(\omega)$ using low pass filter having cutoff freq. $\omega_m$.

3.) When $f_s < 2f_m$, successive cycles of sampled spectrum overlap with each other so original spectrum $X(\omega)$ cannot be extracted from spectrum $G(\omega)$.

Hence, for Reconstruction without distortion

$$\boxed{f_s \geq 2f_m}$$

# Nyquist Rate and Nyquist Interval

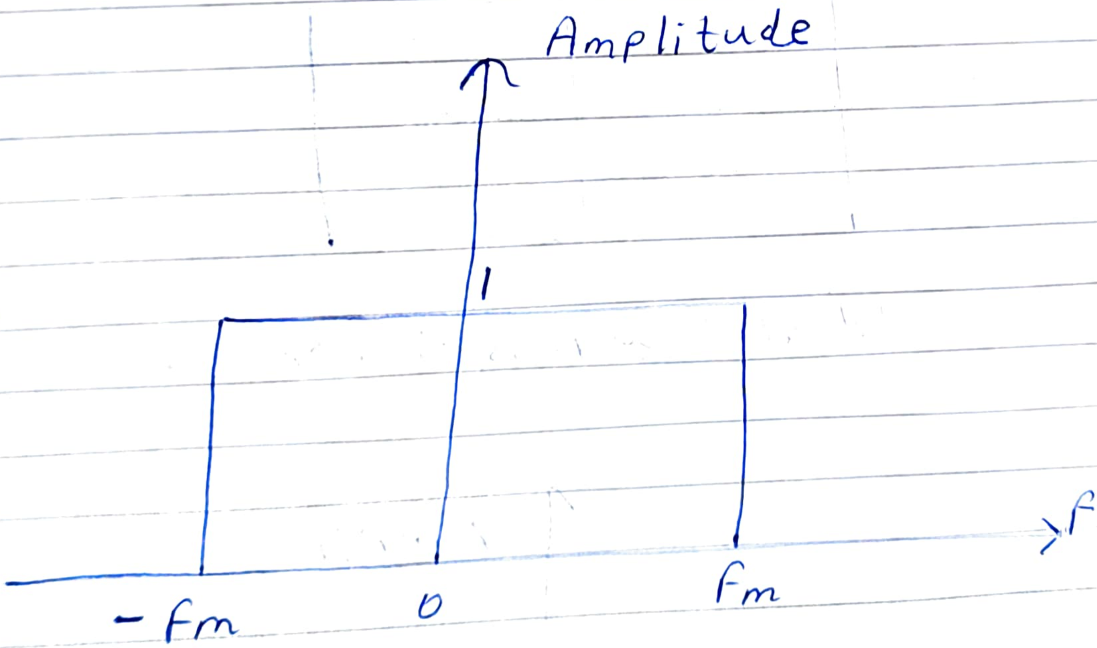
When Sampling Rate becomes exactly equal to $2f_m$ samples per sec, then it is called Nyquist Rate.

It is $f_s = 2f_m$

Nyquist Interval $T_s = \frac{1}{2f_m}$ secs

# Reconstruction Filter (Low Pass Filter)

Low Pass Filter is used to recover original signal from its samples. This is known as Interpolation Filter,




Ideal Low Pass Filter Frequency Response.

It passes only low frequencies upto a specified cutoff freq. and rejects all other frequencies.

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Eg.) Find Nyquist Rate and Nyquist Interval for the signal :-

$$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t)$$

$$= \frac{1}{4\pi} 2\cos(4000\pi t) \cdot \cos(1000\pi t)$$

$$= \frac{1}{4\pi} \left[ \cos(5000\pi t) + \cos(3000\pi t) \right]$$

$$\omega_1 = 5000\pi \qquad \omega_2 = 3000\pi$$

$$2\pi f_1 = 5000\pi \qquad 2\pi f_2 = 3000\pi$$

$$f_1 = 2500 \text{ Hz} \qquad f_2 = 1500 \text{ Hz}$$

Max Freq. $f_m = 2500$ Hz.

Nyquist Rate $f_s = 2 f_m$

$f_s = 5000$ Hz or 5 KHz

Nyquist Interval $T_s = \frac{1}{2 f_m}$

$$T_s = \frac{1}{5000} = 0.2 \times 10^{-3} \text{ sec}$$

$T_s = 0.2$ msec.

# Effect of Under Sampling ; Aliasing

When $f_s < 2f_m$, signal is Under sampled and some amount of aliasing is produced.
In this case, successive cycles of $G(\omega)$ overlap with each other.

Aliasing is a process in which high freq. component in freq. spectrum of the signal takes identity of a low frequency component in spectrum of a sampled signal.

To Avoid Aliasing :-

1.) Prealias Filter usually a low-pass filter limits band of frequencies of the signal upto $f_m$ Hz.

2) $f_s > 2f_m$