UNIT III

Analysis of Digital Receivers

1→ Earlier :

(1) analog w/f's (waveforms) ⟶ digital data → electrical w/f's or symbols ⟶ (2)

(2) detection of symbols and recovery of digital data from received w/f.

⇓ Problems in recovering.

NOISE

2→ NOISE (unwanted electrical sig.)

- Man-made
  - → Switching Transients
  - → Other radiating EM signals
- Natural
  - → Electrical signals in electrical ckt and component noise
  - → atmospheric disturbances

↓

can be eliminated through filtering and shielding.

3→ Noise that cannot be eliminated

→ Thermal or Johnson noise :

↓

caused by random motion of $e^{\ominus}s$ in all components eg: resistors, wires, etc

↓

Thermal noise = zero mean Gaussian random process

→ in a digital tx^n sys, one of M possible wf's are txed.

$$s_i(t) \in \{ s_1(t), s_2(t), \ldots, s_M(t) \} \text{ for } 0 \leq t \leq T$$

At the Rx:

$$r(t) = s_i(t) + \underbrace{n(t)}_{\text{causes a corruption in the original signal}}$$

$r(t)$ → received sig.
$s_i(t)$ → original txed sig
$n(t)$ → noise (AWGN)

A ⇒ Additive (adds 'noise' onto the signal)
W ⇒ White (the noise has a flat spectrum $\bar{c}$ PSD $= \frac{N_0}{2}$ (W/Hz) from $-\infty$ to ∞ Hz)
G ⇒ Gaussian (amplitude of the noise voltage fluctuations follows a Gaussian distribution)

→ Now, Noise ⇒ Random process

∴ it can be described in terms of:-
→ Mean (or avg.) value
→ Variance (or std. deviation)
→ pdf (prob. density fun^n)

→ Now, noise is assumed to be AWGN, so its pdf is:

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2}$$

$n(t)$ has mean $= 0$
& variance $= \sigma^2$

→ The expression for pdf shows that the noise voltage amplitudes are distributed according to Gaussian distribution.

→ The most probable amplitudes of noise are those c̄ small +ve or −ve values.

→ Since, $n(t)$ → randomly varying quantity,
So, $r(t)$ → will also be randomly varying quantity.

∴ there will be uncertainity in finding the value of $r(t)$ ⇒ lead to error in recovery of digital data.

→ The fun^n of a digital Rx is to find which (and not "what") of the signalling w/fs (or symbols) was txed in any given signalling interval.

* Analog Rx → what signal was txed
* Digital Rx → which " " "

→ In a digital Rx,
set of txed signals $\{s_1(t), s_2(t), \ldots s_M(t)\}$ is known a priori.

→ The Rx knows what to expect & it has to find out c̄ sig. ~~was~~ is being txed in any given time interval.

→ Digi. Rx has superior noise performance over analog Rx.

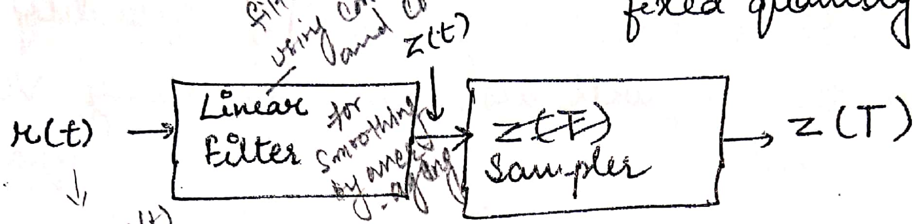
⇒ How to detect a Digital Signal ?

2 basic steps :

Step 1 :

reduce $r(t) \xrightarrow{\text{to}} z(T)$ (Test statistic) fixed quantity

filtering using convolution and correlation



$s_i(t) + n(t)$
original msg signal
AWGN

Step 2 : Compare $z(T)$ $\bar{c}$ a reference value or threshold γ

(This will find out which sig. was txed)

$$z(T) \underset{s_2}{\overset{s_1}{\gtrless}} \gamma$$

if $z(T) > \gamma$, ⇒ $s_1(t)$ is considered to have been txed

& if $z(T) < \gamma$, ⇒ $s_2(t)$ - - - - - " - - -

* Sometimes, due to noise, we can have errors i.e wrong estimation of signal.

Since, $n(t)$ = rand. var ⇒ $r(t)$ & hence $z(t)$ or $z(T)$ is also random.

∴ $z(T)$ can be considered by its mean or variance or pdf's.

→ The analysis of digit. Rx begins $\bar{c}$ the concept of matched filter.

⇒ MATCHED FILTER (Design and Property)

→ It is a linear time-invariant (LTI) filter that leads to the optimum detection of a signal w/f that is immersed in AWGN.

* optimum = min. prob. of an error occuring.

→ Here, we match the impulse response of the filter (H(f) or h(t)) to the sig. w/f. ($s_i(t)$)

→ The filter is designed to detect the presence of signal $s_i(t)$ (which is buried in the noisy recd signal $r(t)$).

→* More precisely; Matched filter is designed to maximise the SNR at the filter o/p, for a given w/f $s_i(t)$ at sampling instant $t=T$.

→ $r(t)$ is filter i/p.

$$r(t) = \underbrace{s_i(t)}_{\text{(original signal)}} + \underbrace{n(t)}_{\text{(noise)}}$$

→ Since, the filter is linear

⇒ the o/p at $t=T$ is :-

$$z(T) = a_i + n_0$$

a_i → (signal component) → Deterministic

n_0 → (noise component) → random

→ Now, the o/p Noise power (variance or avg. power)

$$= \sigma^2$$

→ SNR of matched filter at sampling time instant t=T is :-

$$\left(\frac{S}{N}\right)_{t=T} = \frac{a_i^2}{\sigma^2}$$

Aim : we have to find the optimum filter transfer funⁿ $H_0(f)$ that maximises $(S/N)_{t=T}$. This will minimise the probability of making an incorrect decision i.e to minimise the prob. of error.

→ Expⁿ for SNR can be re-written as :-

$$\left(\frac{S}{N}\right)_{t=T} \leq \frac{2E}{N_0}$$

$\left(\frac{S}{N}\right)$ dep. on → sig. energy 'E'

and

→ Noise power spectral Density ($N_0/2$)

→ Now, max. value of ~~A~~ (S/N) is at :

$$\boxed{\frac{S}{N} = \frac{2E}{N_0}}$$

Correspondingly, the max. value of $H(f)$, denoted $H_{opt}(f)$ occurs when :

impulse response in freq. domain

$$\boxed{H_{opt}(f) = k\, S_i^*(f)\, e^{-j2\pi f T}}$$

k → an arbitrary constt.

$*$ → denotes complex conjugate

$S_i(f)$ → Fourier Transform of sig. w/f $s_i(t)$.

$t = T$

→ In time domain, the impulse response of the optimum filter $h_{opt}(t)$ is given by inverse FT of $H_{opt}(f)$ i.e

$$h_{opt}(t) = \overset{\text{IFT or}}{\mathcal{F}^{-1}}\{H(f)\}$$

$$= \overset{\text{IFT or}}{\mathcal{F}^{-1}}\{k S_i^*(f) e^{-j2\pi f T}\}$$

$$= k s_i(T-t)$$

→ So, the impulse response of the optimum filter is a time-inversed and delayed version of the i/p sig $s_i(t)$.

→ In other words, $h_{opt}(t)$ is "matched" to the i/p signal $s_i(t)$.

→ An LTI sys defined in this way is termed as a matched filter.

→ Use of this filter will result in the reception and detection of digital signals $\bar{c}$ smallest prob. of error.

Property / Summary:

1) A filter matched to an i/p sig. $s_i(t)$ of duration 'T' is char by an impulse resp that is time reversed and time delayed

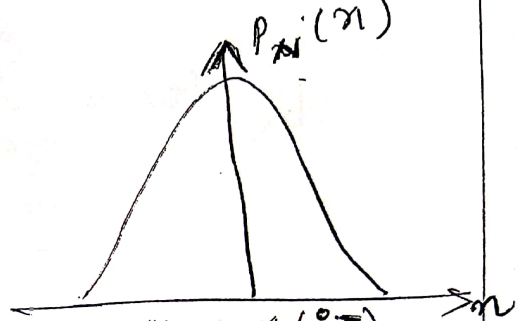
$$\boxed{h_{opt}(t) = k s_i(T-t)}$$

2) In freq domain, the matched filt. is char. by a freq resp. i.e a complex conjugate of the FT of i/p sig $s_i(t)$

$$\boxed{H_{opt}(f) = k S_i^*(f) e^{-j2\pi f T}}$$

3 → Max. SNR deps. upon → E [sig. energy]
and
→ $\left(\frac{N_0}{2}\right)$ [PSD of white Noise]

$$\left(\frac{S}{N}\right)_{t=T} = \frac{2E}{N_0}$$



pdf of $x(iT)$

⇒ **Derivation for Prob. of error (P_e) for the Matched filter**

P_e for optimum filter is :

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad \text{—— (1)}$$

$\left.\begin{matrix} x_{01}(T) \\ x_{02}(T) \end{matrix}\right\}$ o/p of the Rx in the absence of noise $n(t)$.

Here, we have

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df \quad \text{—— (2)}$$

$S_{ni}(f)$ → i/p PSD (of noise)

Here, ($S_{ni}(f)$) is considered as AWGN)

∴ psd of this noise = $S_{ni}(f) = \frac{N_0}{2}$ —— (3)

Hence,

$$\left[\frac{x_{o1}(T) - x_{o2}(T)}{T}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\left(\frac{N_0}{2}\right)} df$$

$$= \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \quad \text{——— (4)}$$

Also, according to Parseval's theorem:

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t) dt = \int_{0}^{T} x^2(t) dt \quad \text{——— (5)}$$

limits 0 to T ($\because$ x(t) exists from 0 to T only)

Now,
we know that:

$$x(t) = x_1(t) - x_2(t)$$

$\therefore$ the above eq^n (5) becomes:

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{0}^{T} [x_1(t) - x_2(t)]^2 dt$$

$$= \int_{0}^{T} [x_1^2(t) + x_2^2(t) - 2x_1(t)x_2(t)] dt$$

$$= \int_{0}^{T} x_1^2(t) dt + \int_{0}^{T} x_2^2(t) dt - 2\int_{0}^{T} x_1(t) x_2(t) dt \quad \text{——— (6)}$$

Now,

$\int_0^T x_1^2(t)\, dt = E_1$ i.e energy of $x_1(t)$

$\quad$ ||ly $\int_0^T x_2^2(t) dt = E_2$

and $\int_0^T x_1(t)\,x_2(t) = E_{12}$ (energy due to autocorrelation b/w $x_1(t)$ & $x_2(t)$)

Now, if we choose $x_1(t) = -x_2(t)$, then:

$$E_1 = E_2 = -E_{12} = E$$

Put in ⑥

$$\therefore \int_{-\infty}^{\infty} |X(f)|^2 df = [E + E - 2(-E)] = 4E \quad \text{—— (7)}$$

Put ⑦ in ④

$$\Rightarrow \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \cdot 4E = \frac{8E}{N_0}$$

Taking √ on b/s

$$\left.\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = 2\sqrt{2}\sqrt{\frac{E}{N_0}} \quad \text{—— (8)}$$

Put eqⁿ ⑧ in ①

$\therefore$ prob. of error P_e for matched filter is :-

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left[\sqrt{\frac{E}{N_0}}\right]}$$

MAXIMUM LIKELIHOOD DETECTOR

Concept of "Detection of signals in noise"

A source tx's M diff signals i.e $S_1(t)$, $S_2(t)$, ---$S_M(t)$ etc.

↓

All these are equally likely

↓

each one has a probability of $\left(\frac{1}{M}\right)$

Now, let us consider a time slot of 'T' sec.

↓

if one of these 'M' possible signals be txed in one ('T' sec) duration

↓

Then, for an AWGN channel

$$\boxed{x(t) = s_i(t) + W(t)} \quad --- \quad 0 \leq t \leq T$$

where $i = 1, 2, 3, ---, M$

$x(t)$ → Rxed sig.

$s_i(t)$ → ~~m~~ i^{th} msg sig.

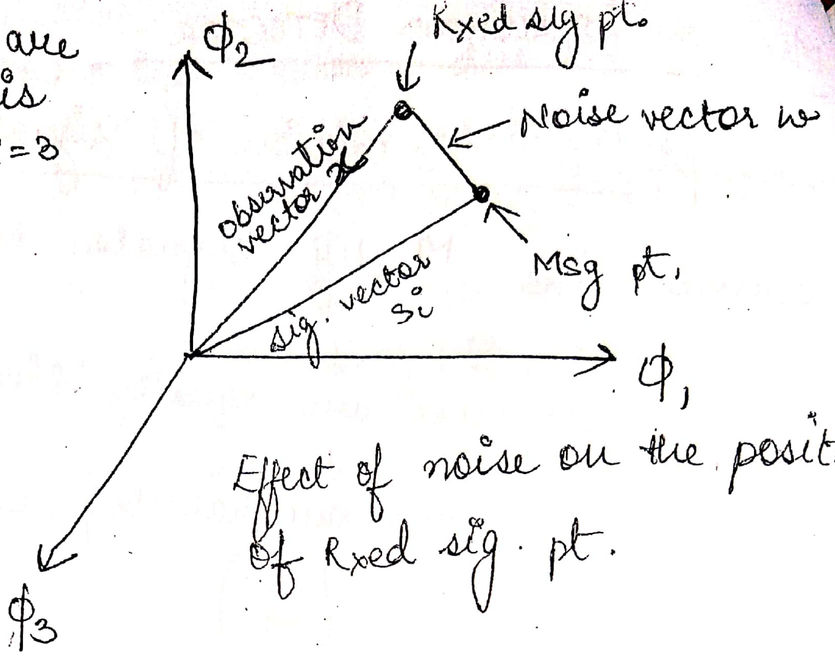
$W(t)$ → sample funⁿ of white noise process

$w(t)$ ⇒ zero mean value

psd $= \frac{N_0}{2}$

* **Note**: After Rxing the sig. $x(t)$, the R_x has to make an estimate of the txed signal $s_i(t)$ to decide ≤ symbol was txed.

* Here, $\phi_1, \phi_2, \phi_3 \rightarrow$ are N orthonormal basis functions where $N=3$



Effect of noise on the position of Rxed sig. pt.

In short:

→ ~~signals~~ The above fig. illustrates relationship b/w the observation vector x, sig vector s_i and noise vector w for $N=3$.

MLD

→ Here, we are dealing $\bar{c}$ the detection problem.

→ if observat" vector 'x' is given then :-
we have to perform a mapping from x to get an estimate $\hat{m}$ of the txed ~~sig~~ symbol m_i, in a way the avg. P_e is minimum
(av. prob. of symbol error)

This is called Max. Likelihood Detection.

Now, $m_i \rightarrow$ original txed sig.

$\hat{m}_i \rightarrow$ estimated sig.

Assume that x = Observation vector
then,
Rx will make decision of $\underline{\hat{m}_i = m_i}$

Now, avg. prob. of symbol error in such a decision is:- (27)

$$\boxed{P_e(m_i, x) = P(m_i \text{ not sent}/x) = P(m_i \text{ sent}/x)} \quad (1)$$

$P(m_i \text{ sent}/x) \Rightarrow$ conditional prob. that m_i was sent provided x is rxed.

Now, Optimum Decision Rule

aim $\Rightarrow$ minimize the prob. of error in mapping 'x' into a decision.

Hence, from eq^n (1), we have to deduce an optimum decision rule.

The estimate $\hat{m}_i = m_i$ if

$$\boxed{P(m_i \text{ sent}/x) \geq P(m_k \text{ sent}/x)} \text{ for all } k \neq i \text{ where } K = 1, 2, \ldots, M. \quad (2)$$

This rule is called as the maximum a posterior prob.

Now, applying Bayes rule to eq^n (2). we restate the decision rule as:-

The estimate $\hat{m}_i = m_i$ if

$$\left\{ \frac{p_k f_x(x/m_k)}{f_x(x)} \right\} \text{ is max. for } k = i \quad (3)$$

$p_k \rightarrow$ prior prob. of occurence of symbol m_k

$f_x(x/m_k) \rightarrow$ likelihood fun^n of corresponding to t_x of symbol m_k

$f_X(x) \rightarrow$ unconditional joint pdf of rand. variable X.

Observations from eqⁿ (3)

(i) denominator term $f_X(x)$ is indep. of txed sig.

(ii) There, a prior prob. $p_K = p_i$

∴ all txed sigs are equally likely.

So, simplifying the decision rule :-

The estimate $\hat{m}_i = m_i$, if

$$\boxed{f_X(x/m_k) \text{ is maximum for } k = i} \quad (4)$$

Generally, using natural logarithm of likelihood funⁿ is easier than the likeli. fun's itself.

⇓

natural logarithm of likelih. funⁿ = metric

⇓

$f_X(x/m_k) \geq 0$ (always non-negative)

⇓

Now, if $X > Y > 0$, then $\log_e X > \log_e Y$

∴ decision rule can be restated as :-

The estimate $\hat{m}_i = m_i$ if

$$\boxed{\log_e[f_X(x/m_k)] \text{ is max. for } k = i} \quad (5)$$

eqⁿ (5) is called the max. likeli. rule & device used for its implementatⁿ is called max. likelihood Detector.

Conclusion :-

From eqⁿ (5) :
The MLD computes the metric of each txed sig. m_K $K=1,2,\ldots$

⇓

compares ~~it $\bar{c}$ mi~~ these values

⇓

then, makes a decision based on the max. of them.

Scrambling :- Pg - 788.

⇒ INTERSYMBOL INTERFERENCE (ISI)

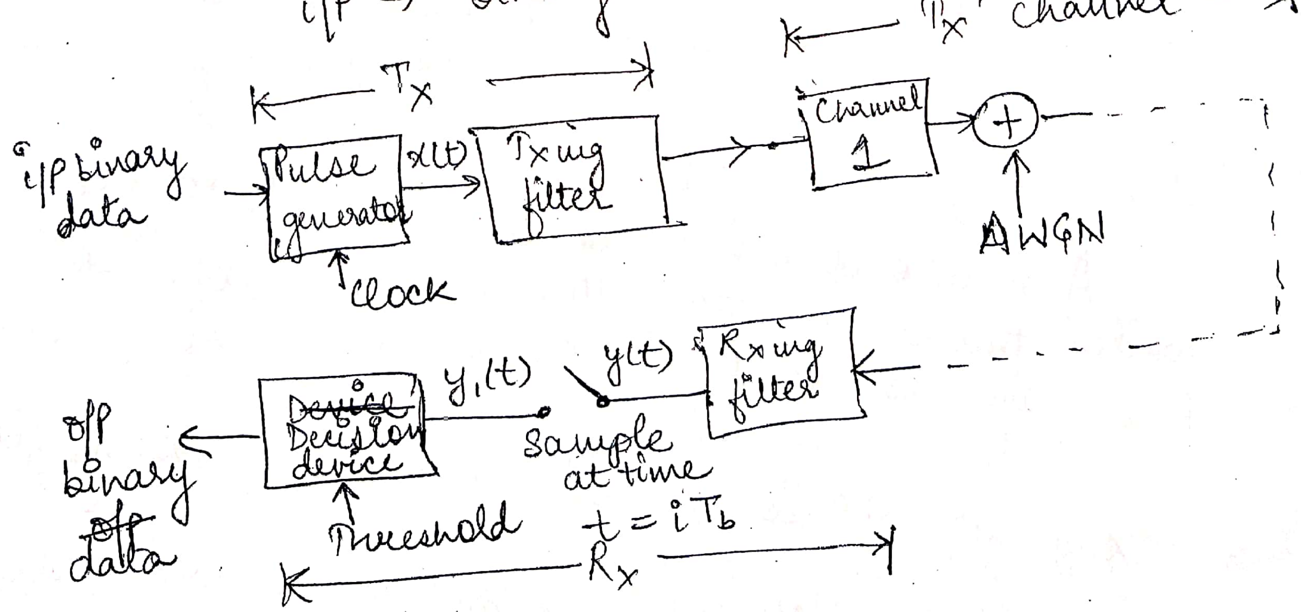
→ ISI defiⁿ

Data is txed in form of bits or pulses, the o/p produced at the R_x due to other bits or symbols interferes $\bar{c}$ the o/p produced by the desired bit. This is ISI.

→ ISI introduces errors in the detected sig.

→ ~~impulse~~ binary PAM sys.

Elements of binary data seq. $= b_K$ $\bar{c}$ bit durat' T_b se
i/p → binary data



i/p → binary data seq $\{b_k\}$

The seq. is applied to a pulse generator to produce a discrete PAM signal & is given by :-

$$x(t) = \sum_{k=-\infty}^{\infty} a_k v(t - kT_b) \quad \text{——— (1)}$$

o/p of pulse generator

$v(t)$ → basic pulse, (normalized such that $V(0) = 1$)

→ The 1st block of the sys. i.e PAM ⇒ converts i/p seq $\{b_k\}$ into polar form i.e

if $b_k = 1 \Rightarrow a_k = 1$

if $b_k = 0 \Rightarrow a_k = -1$

⇓

This PAM sig. $x(t)$ is then fed to txing filter.

⇓

its o/p is fed to the channel. (impulse response of this channel be $h(t)$)

⇓

A random noise is then added to the txed while travelling over the txⁿ channel. So, the Rxed sig. is contaminated c̄ noise.

⇓

channel o/p is passed through a Rxing filter

⇓

This filter o/p is sampled synchronously c̄ the tx.

(4) / (9)

Seq. of samples obtained at o/p of Rxing is used to reconstruct the original data seq. $\bar{c}$ the help of a decision making device

⇓

Each sample is compared to a threshold level in the decision making device.

if amplitude is > threshold level ⇒ symbol 1 is Rxed.

if amp. is < threshold level ⇒ it is decided that symbol 0 is Rxed

⇓

Rxing filter o/p ⇒ $y(t) = \mu \sum_{k=-\infty}^{\infty} a_k\, p(t - kT_b) + n$

$\mu \rightarrow$ scaling factor

$n(t) \rightarrow$ noise

$p(t-kT_b) \rightarrow$ combined impulse response of the Rxing fil

at time $t = t_i = iT_b$ ($i \rightarrow$ any integer)

So,

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k\, p(iT_b - kT_b) + n(t_i) \quad \text{--- (2)}$$

Or

$$y(t_i) = \mu a_i + \mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k\, p(iT_b - kT_b) + n(t_i) \quad \text{--- (3)}$$

Eqn (3) shows the Rx o/p $y(t)$ at instant $t = t_i$

Now, from eqn (3) :-

i) The first term μa_i is produced by i^{th} txed bit. Only this term should be present.

(ii) 2^{nd} term represents the residual effect of txed bits, obtained at the time of sampl ith bit. This residual effect = ISI.

Pg - 767

→ Cause of ISI

→ Effect of ISI

→ Remedy ~~to~~ Reduce ISI

⇒ EYE DIAGRAM

imp

PREDICTION FILTER

Linear Prediction:

→ used for estimating future samples from the present and past sample values.

→ prediction → linear

iff ⇒ future sample is a linear combinatⁿ of present & past i/p samples.

predicted sample :-

$$\boxed{\hat{x}(n) = \sum_{K=1}^{M} w_K \, x(n-K)} \quad \text{--- (1)}$$

$\hat{x}(n)$ → predicted value of $x(n)$.

$x(n-1), x(n-2), \ldots, x(n-M)$ → past i/p samples

$w_1, w_2, \ldots, w_M$ → set of multipliers called filter coefficien

→ eqⁿ (1) is a linear eqⁿ ∵ it is a combⁿ of $x(n-1), x(n-$ se

Basically, it is a linear convolution of discrete signals.

→ let $x(n)$ → actual sample value

$\hat{x}(n)$ → predicted sample "

So, prediction error, $\boxed{e(n) = x(n) - \hat{x}(n)}$ --- (2)

Not → A prediction filter coefficients $w_1, w_2, \ldots, w_M$ should be selected required such that MSV (mean sq. value) of error $\overline{e^2(n)}$ is minimized.

This condition gives us the eqⁿ :-

$$\boxed{\sum_{j=1}^{M} w_j \, R(k-j) = R(k)} \quad \boxed{\text{and } k = 1, 2, 3, \ldots, M} \quad \text{--- (3)}$$

'w_j' → prediction filter coeff.

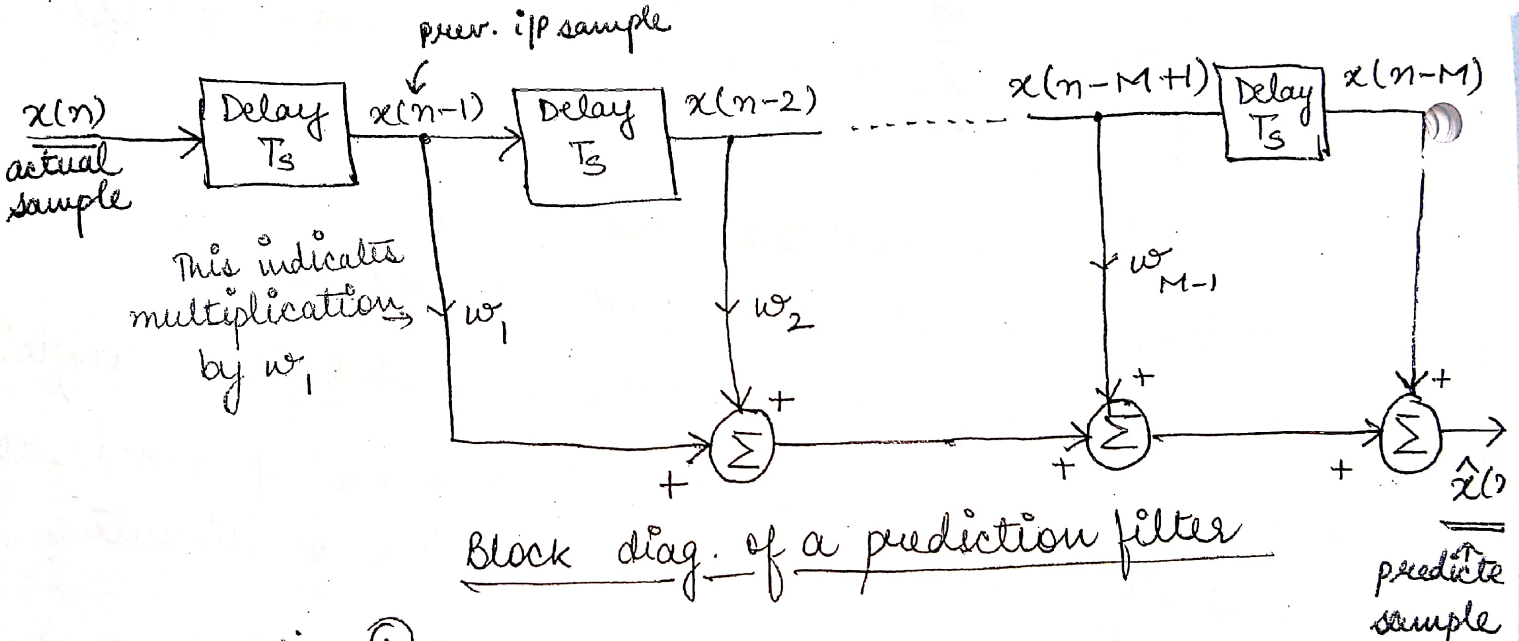
$R(k-j) = \overline{x(n-k) \cdot x(n-j)}$, is autocorrelation of

and

$R(k) = \overline{x(n) \cdot x(n-k)}$, " " " $x(n)$

⇒ Prediction filter :-

→ implementation of eq^n ① is called a prediction filter.



Block diag. of a prediction filter

Equation ① :

$$\hat{x}(n) = \sum_{k=1}^{M} w_k \, x(n-k)$$

→ eq^n ① is called 'filter' ∵ it represents a linear convolut

* Convolution : it is a mathematical way of combining 2 signals to form a third signal.
Linear convolution is the basic operation to calculate the o/p for any linear time invariant sys given its i/p and its impulse resp.
convolution calculates the resp. of an LTI sys.
whereas,
cross-correlation is used for pattern matching.

Hence, $x(n-1), x(n-2), \ldots, x(n-m) \rightarrow$ past i/p's

$x(n) \rightarrow$ present i/p.

→ These i/p samples are used for filtering $\bar{c}$ the help of coeff. $w_1, w_2, \ldots, w_M$. Such filter is of the type of finite impulse response ~~diff~~ digital filters.

→ Filtering operation can be modified by changing the value of filter coeff.

→ Prediction filter is used earlier in DPCM.

→ Prediction error can also be calculated using prediction filter.

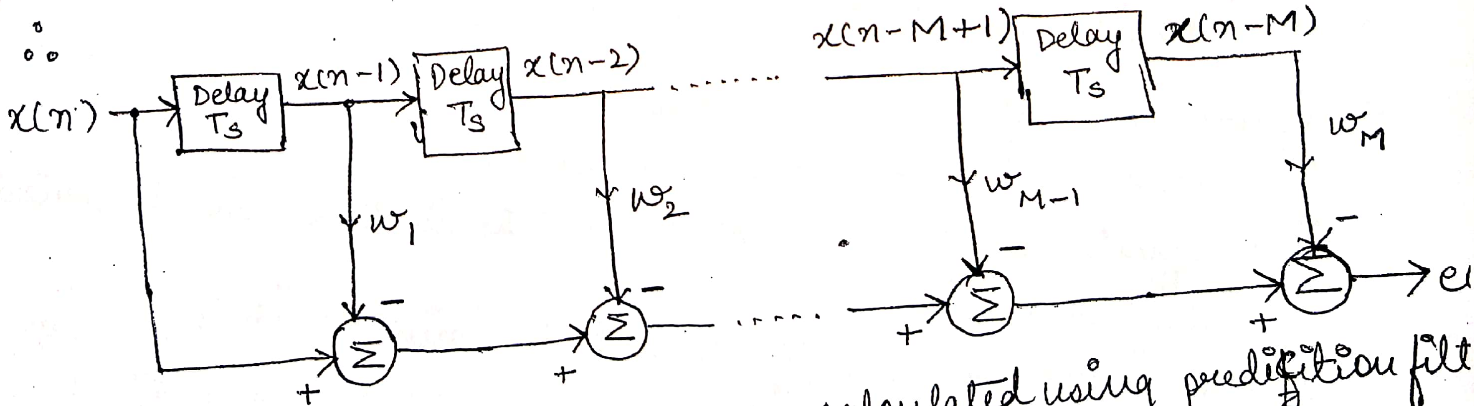
$$e(n) = x(n) - \hat{x}(n)$$

Put ① in above eq^n

$$e(n) = x(n) - \sum_{k=1}^{M} w_k\, x(n-k)$$

$$= x(n) - [w_1 x(n-1) + w_2 x(n-2) + \cdots + w_M x(n-M)]$$

$$= x(n) - w_1 x(n-1) - w_2 x(n-2) - \cdots - w_M x(n-M)$$

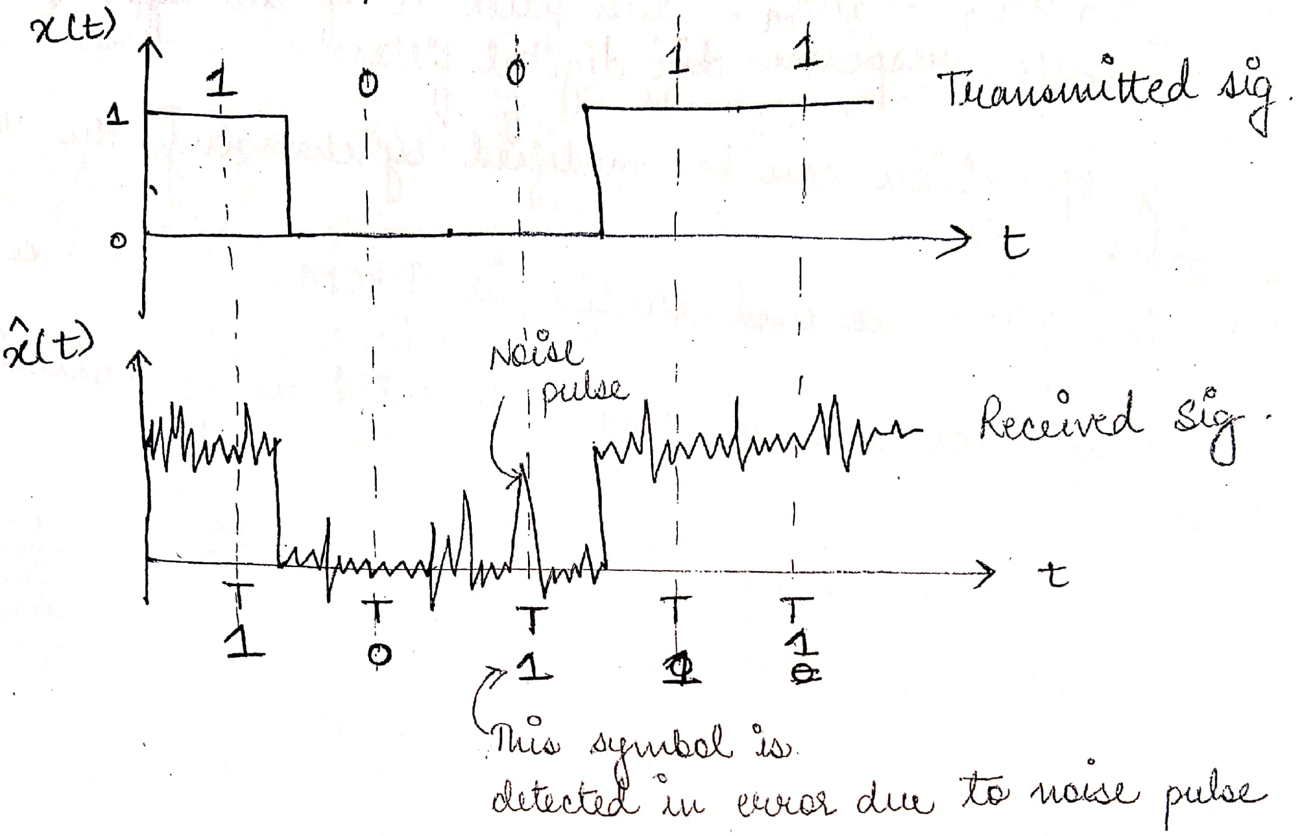


Prediction error calculated using prediction filter

→ in the above diag., $w_1 x(n-1), w_2 x(n-2), \ldots w_M x(n-M)$ are subtracted from $x(n)$. Hence o/p of prediction filter is error $e(n)$.

⇒ MATCHED FILTER

→ used for detection ~~of~~ of signals in baseband and passband tx^n.



→ The above fig. shows txed ~~sig~~ digital sig. and received noisy sig. The txed sig sequence is 1 0 0 1 1. The pulse is checked at the point "T" of every bit period. Because ~~the~~ of noise pulse present in the third bit at the instant 'T' of checking, it is detected in error

→ Requirements of detection error

(i) SNR of Rx must be improved.

(ii) Signal must be checked at the instant in bit period, when SNR is max.

(iii) The error probability should be min.

→ Matched filter

(i) It satisfies all the above requirements.

(ii) It is called matched filter since its impulse response is matched to the shape of i/p signal.

Theorem → Orthog. sets are linearly independent.
(if $S = \{v_1, v_2, ---, v_n\}$ is an orthog. set of non-zero vectors in an inner prod. space V, then S is linearly indep.)

Corollary → If V is an inner prod. space of dimension n, then any orthog. set of 'n' non-zero vectors is a basis for V.

$\boxed{\text{S is a basis of 'V' of dimension } n.}$

This means that we have to check that all the pairs are orthog. or not

Eg:
(2) Show that the following set is a basis for R^4

dimension = 4 & S also has 4 vect

$$S = \{\overset{v_1}{(2,3,2,-2)}, \overset{v_2}{(1,0,0,1)}, \overset{v_3}{(-1,0,2,1)}, \overset{v_4}{(-1,2,-1,1)}\}$$

(2)

$$\left.\begin{aligned} v_1 \cdot v_2 &= 0 \\ v_1 \cdot v_3 &= 0 \\ v_1 \cdot v_4 &= 0 \\ v_2 \cdot v_3 &= 0 \\ v_2 \cdot v_4 &= 0 \\ v_3 \cdot v_4 &= 0 \end{aligned}\right\}$$

So, 1st you have to verify this (check pair-wise orthogonality)

So, S is a basis for R^4.

⇒ Co-ordinates relative to an orthonormal basis.

if $B = \{v_1, v_2, ... v_n\}$ is an orthon. basis for V, then the coord. rep" of a vector 'w' relative to B is :-

$$w = \underbrace{\langle w, v_1\rangle}_{\text{dot prod. of } w \,\&\, v_1} v_1 + \langle w, v_2\rangle v_2 + \dots + \langle w, v_n\rangle v_n \quad \text{——— (1)}$$

(1) Find the coord. matrix of $w = (5, -5, 2)$ relative to the following orthon. basis for R^3

$$B = \{ \underbrace{(\tfrac{3}{5}, \tfrac{4}{5}, 0)}_{v_1}, \underbrace{(-\tfrac{4}{5}, \tfrac{3}{5}, 0)}_{v_2}, \underbrace{(0, 0, 1)}_{v_3} \}$$

(1) in short: find matrix 'w' for orthon. basis B.

→ These coordinates are called Fourier coeff. of 'w' relative to B.

Sol^n: ∵ B is orthon. (as $v_1 \cdot v_2 = 0$, $v_1 \cdot v_3 = 0$, $v_2 \cdot v_3 = 0$)

So, to find coord. of 'w', we use eq^n (1)

$$w \cdot v_1 = (5, -5, 2) \cdot (\tfrac{3}{5}, \tfrac{4}{5}, 0) =$$

$$= 3 - 4 + 0 = -1$$

$$w \cdot v_2 = (5, -5, 2) \cdot (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -4 - 3 = -7$$

$$w \cdot v_3 = (5, -5, 2) \cdot (0, 0, 1) = 2$$

So, the coord. matrix relative to B is :-

$$[w]_B = [-1 \; -7 \; 2]^T$$ ← Transpose

or $\begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$

⇒ Major ques^n

How to generate an orthonormal Basis?

Sol^n: If we are given a basis $B = \{v_1, v_2, \ldots, v_n\}$ for inner product space V then, how to generate an orthonormal basis B'?

Answer: We use Gram-Schmidt Orthogonalization Process (to calc. an orthon. basis B' from B)

1. Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis for an inner prod space V.

2. Let $B' = \{w_1, w_2, \dots, w_n\}$ when ~~w_n~~ w_i is given by:-

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$\vdots$$

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \cdots - \frac{\langle w_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

- Here, $B' \rightarrow$ Orthogonal basis for V
it is not yet orthonormal. (it is orthog. right now)

The orthonormal basis is $B'' = \{u_1, u_2, \dots, u_n\}$

$u_1, u_2, \dots \rightarrow$ all unit vectors ($\because$ orthonormal)

So,
$$\boxed{u_i = \frac{w_i}{\|w_i\|}}$$

So, B'' is an orthon basis for V.

Moreover,

$$\text{Span}\{v_1, v_2, \dots, v_k\} = \text{span}\{u_1, u_2, \dots, u_k\}$$
for $k = 1, 2, 3, \dots, n$.

Lets do an example.

① Apply Gram-Schmidt Orthog' Process for the following basis for R^3

$$B = \{\underbrace{(1,1,0)}_{v_1}, \underbrace{(1,2,0)}_{v_2}, \underbrace{(0,1,2)}_{v_3}\}$$

① **Step 1**: Find $B' = \{w_1, w_2, w_3\}$

So,
$$w_1 = v_1 = (1,1,0) \quad \text{———— ①}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (1,2,0) - \frac{\{(1,2,0)\cdot(1,1,0)\}}{\{(1,1,0)\cdot(1,1,0)\}}(1,1,0)$$

$$= (1,2,0) - \left[\frac{(1+2)}{(1+1)}\right](1,1,0)$$

$$= (1,2,0) - \left(\frac{3}{2}, \frac{3}{2}, 0\right)$$

$$= \left(-\frac{1}{2}, \frac{1}{2}, 0\right) = w_2 \quad \text{———— ②}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= (0,1,2) - \frac{\{(0,1,2)\cdot(-\frac{1}{2}, \frac{1}{2}, 0)\}}{\{(-\frac{1}{2}, +\frac{1}{2}, 0)\cdot(-\frac{1}{2}, \frac{1}{2}, 0)\}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$- \frac{\{(0,1,2)\cdot(1,1,0)\}}{\{(1,1,0)\cdot(1,1,0)\}}(1,1,0) \qquad \frac{1}{4}+\frac{1}{4}$$

$$= (0,1,2) - \frac{\{\cancel{(\frac{1}{2})}\}}{\{\cancel{\frac{1}{2}}\}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) - \frac{\{(1)\}}{\{2\}}(1,1,0)$$

$$= (0,1,2) - \left(-\frac{1}{2}, \frac{1}{2}, 0\right) - \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$=(0,1,2)-\left\{\left(-\frac{1}{2},\frac{1}{2},0\right)-\left(\frac{1}{2},\frac{1}{2},0\right)\right\}$$

$$=(0,1,2)-\left(-\frac{1}{4},0,0\right)$$

$$=\left(\frac{1}{4},1,2\right)=w_3 \quad \text{——— (3)}$$

<u>Step 2</u> : Calc. $B'' = \{u_1, u_2, u_3\}$

So, $u_1 = \dfrac{w_1}{\|w_1\|}$

$$=\frac{(1,1,0)}{\|(1,1,0)\|}=\frac{(1,1,0)}{\sqrt{(1^2+1^2+0^2)}}=\frac{(1,1,0)}{\sqrt{2}}$$

$$=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) \quad \text{——— (4)}$$

$$u_2=\frac{w_2}{\|w_2\|}$$

$$=\frac{\left(-\frac{1}{2},\frac{1}{2},0\right)}{\sqrt{\left(\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+0^2\right)}}=\frac{\left(-\frac{1}{2},\frac{1}{2},0\right)}{\sqrt{\left(\frac{1}{4}+\frac{1}{4}+0\right)}}=\frac{\left(-\frac{1}{2},\frac{1}{2},0\right)}{\sqrt{(1/2)}}$$

$$=\left(\frac{-1}{\cancel{2}\sqrt{2}},\frac{1}{\cancel{2}\sqrt{2}},0\right) \quad \text{——— (5)}$$

$$u_3=\frac{w_3}{\|w_3\|} \qquad =\frac{(1,1,2)}{\sqrt{6}}=\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{\sqrt{2}}{3}\right).$$

$$=\frac{\left(\frac{1}{4},1,2\right)}{\sqrt{\left(\frac{1}{4}+1+4\right)}}=\cancel{\frac{\left(\frac{1}{4},1,2\right)}{\sqrt{(17/90)}}}=\cancel{\left(\frac{17}{360},\frac{17}{90},\frac{34}{90}\right)} \quad \text{——— (6)}$$

$\frac{1+8}{9}$, $\frac{17}{90}$, $\frac{1}{8}$, $\frac{1}{8}$

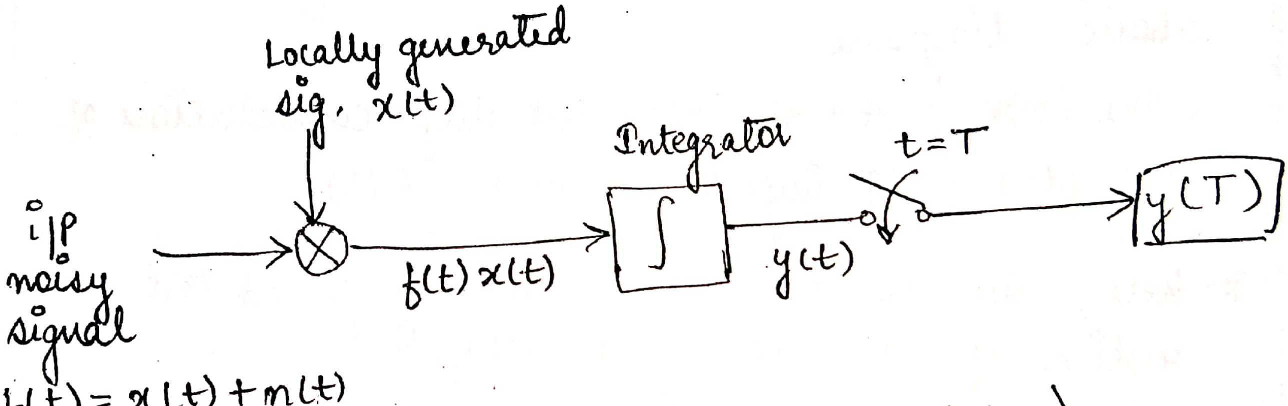
we say that B'' is an orthonormal basis for R^3.

$u_1, u_2, u_3 \rightarrow$ all are unit vectors $\bar{c}$ length 1.

⇒ CORRELATOR RECEIVER

→ Correlation is a pattern matching process where ~~where~~ we find out similarities b/w 2 signals.

→ In practice, a linear matched filter is more often implemented as a correlator. This allows a cheaper implementation of filters.



(Block diag. of a correlator)

Steps :-

working of correlator:

i/p noisy sig. $f(t)$ is multiplied to a replica of i/p sig $x(t)$

↓

Result $(f(t)x(t))$ is fed to an integrator

↓

o/p of integrator is sampled at $t=T$

→ Why it is called a correlator?

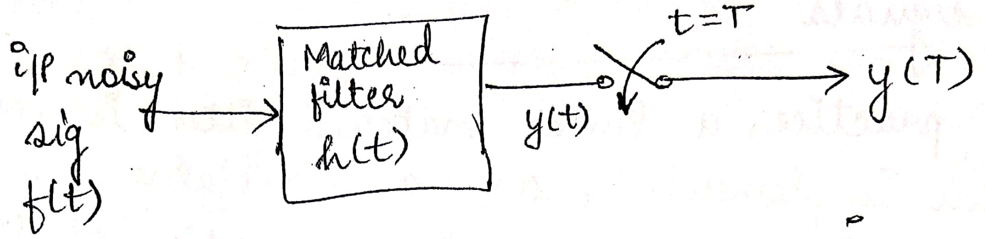
→ ∵ it correlates received sig. $f(t)$ $\bar{c}$ i/p signal $x(t)$.

So, o/p $y(t) = \int_0^T f(t)\,x(t)\,dt$ ——— (1)

Now, at $t=T$

O/P of correlator is :

$$y(T) = \int_0^T f(t)\, x(t)\, dt \quad \text{—— (2)}$$



Now, consider the matched filter ~~acc.~~ in ~~to~~ the above diagram.

→ The o/p y(t) is obtained by convolution of i/p f(t) & its impulse response h(t).

* here, we donot need a locally generated replica of the i/p signal x(t).

∴

$$y(t) = f(t) \otimes h(t)$$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\, d\tau \quad \text{—— (3)}$$

Now, we know that ;
impulse resp. of matched filter, h(t) is :-

$$h(t) = \frac{2k}{N_0} x(T-t) \quad \text{—— (4)}$$

$$\therefore h(t-\tau) = \frac{2k}{N_0} x(T-t+\tau)$$

Now, integration is performed over one bit period.

∴ limits change to 0 to T

So,

$$y(t) = \frac{2k}{N_0}\int_0^T f(\tau)\, x(T-t+\tau)\, d\tau$$

at $t = T$

$$y(T) = \frac{2k}{N_0}\int_0^T f(\tau)\, x(\cancel{T}-\cancel{T}+\tau)\, d\tau$$

$$= \frac{2k}{N_0}\int_0^T f(\tau)\, x(\tau)\, d\tau$$

let $\tau = t$ (only for convenience)

$$\therefore\ y(T) = \frac{2k}{N_0}\int_0^T f(t)\, x(t)\, dt \quad \text{------ (5)}$$

from eqn (2) and (5)

eqn (5) gives the o/p of matched filter

eqn (2) " " " " correlator.

we may say that both are identical

as eqn (2) = (5).

The constt. in eqn (5) $\left(\frac{2k}{N_0}\right)$ can be normalized to 1.

$\therefore$ we may say that both the matched filter & correlator provides the same o/p.

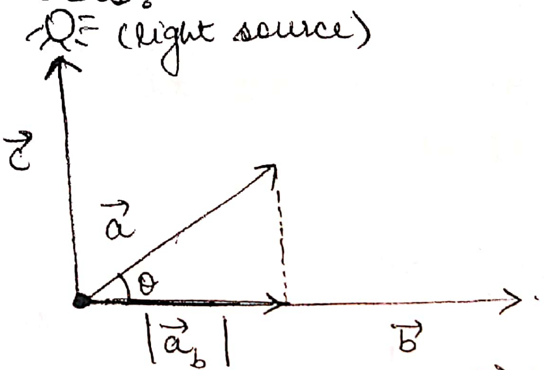
⇒ ORTHOGONAL SIGNALS

Orthogonality: It is the property that allows t_{x}'s of more than one signal over a common channel c̄ successful detection.

Orthogonal Signals: 2 sig.'s are orthog. if they are mutually independent.

How to find out whether 2 signals are orthog.?

→ Orthog. vectors:



(projection of $\vec{a}$ on $\vec{b}$)
shadow of $\vec{a}$ on $\vec{b}$.
$|\,|$ → mod rep. mag. of projⁿ

$$|\vec{a}_b| = |\vec{a}| \cos\theta$$

$$|\vec{a}_b| \cdot |\vec{b}| \neq 0$$

(mag. of projⁿ & mag. of vector $\vec{b}$) ⇒ Their Xⁿ $\neq 0$

$$= |\vec{a}| \cos\theta \cdot |\vec{b}| \neq 0$$

$$\underbrace{|\vec{a}||\vec{b}| \cos\theta}_{\text{dot prod. of } \vec{a} \,\&\, \vec{b}} \neq 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} \neq 0$$

∴ $\vec{a}$ & $\vec{b}$ are not independent & we say that these vectors are not orthog. to each other.

Now, if we have another vector (let it be $\vec{c}$)
and we put it under a light source.
we see that, proj^n of $\vec{c}$ on $\vec{b}$ is NULL.
i.e $\vec{c}$ doesnot have any shadow over $\vec{b}$.

Now, $\theta = 90°$ (b/w $\vec{b}$ & $\vec{c}$)

$\cos 90° = 0$.

$\therefore$ $\vec{b} \cdot \vec{c} = 0$ $\quad$ ($\because bc\cos 90° = 0$)

$\therefore$ we say that $\vec{b}$ & $\vec{c}$ are orthog. (OR mutuall indep.)

Signal space $\rightarrow$ inner prod. (Or dot prod.)
(Definite integral)

let there are 2 sig's $x_1(t)$ & $x_2(t)$

~~2 s~~ The 2 sig's are orthog. if.

$$\boxed{\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = 0}$$ [For non-periodic sig's]

and $$\boxed{\int_{0}^{T} x_1(t)\, x_2(t)\, dt = 0}$$ [For periodic sig's]

over one time period

$\Rightarrow$ <u>Properties of orthog. sig's (4)</u>

(i) 2 harmonics of different frequencies are always $\rightarrow$ orthog.

let $x_1(t) = \sin(n\omega_0 t + \phi_1)$
$x_2(t) = \sin(m\omega_0 t + \phi_2)$ $\Big]$ $n \neq m$ & $\phi_1 \neq \phi_2$

So, both phase & frequencies are different.
$\Rightarrow$ Hence, $x_1(t)$ & $x_2(t) \longrightarrow 0$.

$$\int_0^T \sin(n\omega_0 t+\phi_1)\sin(m\omega_0 t+\phi_2)\,dt = 0.$$

(ii) sine & cosine funⁿ c̄ same freq. & phase → 0. (orthog.)

$$\int_0^T \sin(n\omega_0 t+\phi)\cos(n\omega_0 t+\phi)\,dt = 0$$

f. & φ are same

(iii) dc value & sine funⁿ ⟶ 0

$$\int_0^T \underset{\substack{\uparrow\\ \text{dc value}}}{a}\ \sin(n\omega_0 t+\phi) = 0$$

Imp (iv) Effects of orthogonality on total energy (E) & avg. power (P) calculations.

if we have

$$x_1(t)\ \&\ x_2(t) \longrightarrow 0$$

$$\Rightarrow \int_{-\infty\ 0}^{\infty\ T} x_1(t)\,x_2(t)\,dt = 0$$

& if we have another sig. $y(t)$ c̄ is :-

$$y(t) = x_1(t) + x_2(t)$$

⇒ Then, the av. power $\underbrace{P_y = P_{x_1} + P_{x_2}}_{①}$ (we can use this directly if $x_1(t)$ & $x_2(t)$ are 0)

⇒ Total energy, $\underbrace{E_y = E_{x_1} + E_{x_2}}_{②}$ (" .)

① Total 'E' is ∞ in case of power sig's
we use eqⁿ ① if $x_1(t)$ & $x_2(t)$ → power sig's.

② Avg. power = 0 (zero) in case of energy sig's.
we use eqⁿ ② if $x_1(t)$ & $x_2(t)$ → E. Sig's.

Example :-

① $y(t) = \overbrace{2\sin(3\omega_0 t + 45^\circ)}^{x_1(t)} + \overbrace{4\sin(4\omega_0 t + 35^\circ)}^{x_2(t)}$.

Cal. avg. power & total energy.

① Solut^n:

(i) Find whether $x_1(t)$ & $x_2(t)$ are orthog. or not.

From the above eq^n.

$f_1 \neq f_2$ (i.e $3 \neq 4$.)

& $\phi_1 \neq \phi_2$ (i.e $45^\circ \neq 35^\circ$)

$\therefore$ $x_1(t)$ & $x_2(t)$ are $\rightarrow$ O. [from prop. (i)]

(ii) Use property (iv) to calc. Avg. P. & total energy (E)

$P_y = P_{x_1} + P_{x_2}$

$= \frac{2^2}{2} + \frac{4^2}{2}$

$= \frac{4}{2} + \frac{16}{2}$

$= 2 + 8$

$\boxed{P_y = 10\ W}$

$E_y = \infty$ ($\because x_1(t)$ & $x_2(t)$ are periodic.
↓
periodic sig's are power sig's
↓
& in case of power sig's energy, $E = \infty$)

& $\boxed{E_y = \infty}$

$\left[\because P = \frac{A_0^2}{2}\right]$

* How to find whether a sig is E.s. or Power-Sig.?

Eg: 1. $x(t) = \underset{x_1(t)}{t}\ \underset{x_2(t)}{u(t)}$

$t \rightarrow$ ramp fun^n

$x_1(t) = t = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$

Sol^n: Simply calc. the avg. power (P) & tot. Energy (E) of that sig.

if $\left.\begin{matrix} E = \text{finite} \\ P = 0 \end{matrix}\right\}$ Energy Sig

if $\left.\begin{matrix} P = \text{finite} \\ E = \infty \end{matrix}\right\}$ Power sig.

$x_2(t) \rightarrow$ unit step funⁿ

$$x_2(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

av. power, $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$ [std. formula to calc. avg. P]

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\int_{-T}^{0} 0 \, dt + \int_{0}^{T} |t \cdot 1|^2 \, dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[0 + \left[\frac{t^3}{3} \right]_0^T \right]$$

$$= \lim_{T \to \infty} \frac{1}{2\cancel{T}} \left[\frac{T^{\cancel{3}\,2}}{3} \right]$$

$$= \lim_{T \to \infty} \frac{T^2}{6} = \infty$$

So, av. Power $= \infty$ for $x(t)$

Tot. energy, $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$= \int_{-\infty}^{0} 0 \, dt + \int_{0}^{\infty} t^2 dt$$

$$= \left[\frac{t^3}{3} \right]_0^{\infty}$$

$$= \infty - 0 = \infty$$

tot, $E = \infty$

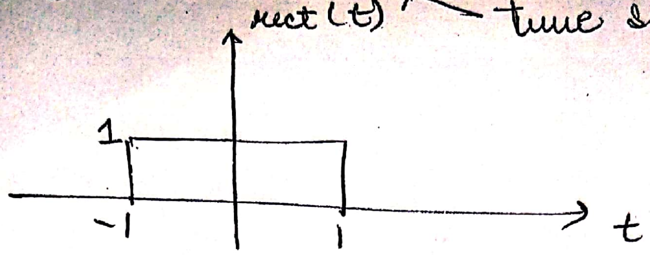
So, sig. $x(t)$ is neither energy sig nor P.S.

(10)

(19)

(2) $y(t) = 2\,\mathrm{rect}(t/2)$ ← time scaling (T.S.)

(2)

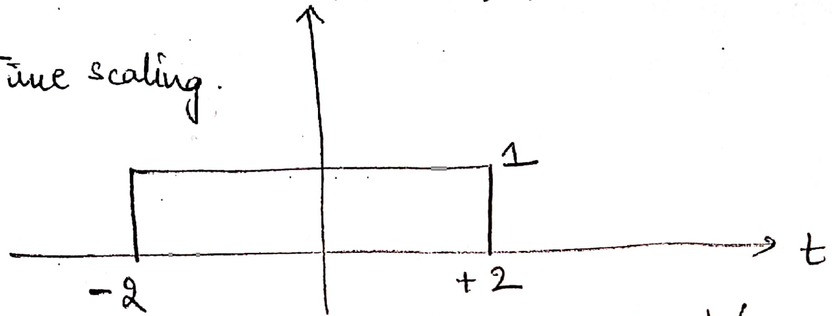


This is a rect. sig.

first, perform time scaling of rect. sig.
2nd, " amplitude " of '2' amplitude

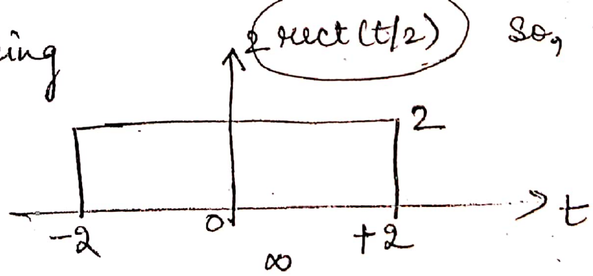
So,

Time scaling.



(divide 1/0.5 on b/s
∵ here we have t/2)
So, time is ÷ed by 0.5

Amp. scaling



(i) cal. tot. E. $= \int_{-\infty}^{\infty} |y(t)|^2 dt$

$= \int_{-2}^{2} (2)^2\, dt = 4\,[t]_{-2}^{2}$ 2+2

$= 16$ Joules (finite)

(ii) cal. avg. P $= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 4\, dt$

$= \frac{1}{2T} 4\,[t]_{-\infty}^{\infty}$ (finite no.)

$= \frac{\infty}{\infty} = 0$ ∴ sig. $y(t)$ = E.S

⇛ # GRAM – SCHMIDT ORTHOGONALIZATION PROCEDURE :–

Goal : To show → ① a set of vectors is orthogonal

⇓

forms orthonormal basis

⇓ +

rep. a vector relative to an orthonormal basis.

② → apply Gram – Schmidt Oⁿ procedure

Definitions :

1. Orthogonal set : A set 'S' of vectors in an inner prod. space V is called orthog. when every pair of vectors in S is orthog.

i.e for every V_i, V_j in 'S' ⇒ $V_i \cdot V_j = 0$ [condⁿ for orthog.] $i \neq j$

2. Orthonormal : if, Each vector in the set is a unit vector then, S is called orthonormal.

Eg: ① Show that set 'S' is an orthon basis for R^3.

$$S = \underbrace{\{V_1, V_2, V_3\}}_{\text{these vector lie on diff. planes}} = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{\sqrt{2}}{6}, +\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right), \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)\right\}$$

① Solution :

(i) Verify that all vectors are pair-wise orthog. & $\|\ \|$ (norm) of each vect. = 1.

∴

$v_1 \cdot V_2 = 0$ and $\|V_1\| = 1 = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$V_1 \cdot V_3 = 0$ $\quad \|V_2\| = 1 = \sqrt{\left(\frac{-\sqrt{2}}{6}\right)^2 + \left(\frac{\sqrt{2}}{6}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2} = \frac{2+2}{36} + \frac{8}{9}$

$= \frac{4}{36} + \frac{8}{9} = \frac{9}{9} =$

$V_2 \cdot V_3 = 0$ $\quad \|V_3\| = 1$

$\hookrightarrow = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$

$= \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = \frac{9}{9} = 1$

∴ we conclude that set 'S' is an orthonormal set and it is also a basis for R^3