Foundation Of Computer Science

Lecture-4
Predicate Logic
&
Quantifiers

Topics

Propositional Logic is not enough

Suppose we have:

- "All men are mortal." "Socrates is a man".
- Does it follow that "Socrates is mortal" ?

This cannot be expressed in propositional logic.

We need a language to talk about objects, their properties and their relations.

Predicate Logic

Extend propositional logic by the following new features.

- Variables: $x, y, z, \ldots$
- Predicates (i.e., propositional functions): $P(x), Q(x), R(y), M(x, y), \ldots$.
- Quantifiers: $\forall, \exists$.

Propositional functions are a generalization of propositions.

- Can contain variables and predicates, e.g., $P(x)$.
- Variables stand for (and can be replaced by) elements from their domain.

Propositional Functions

- Propositional functions become propositions (and thus have truth values) when all their variables are either
  - replaced by a value from their domain, or
  - bound by a quantifier
- $P(x)$ denotes the value of propositional function P at x.
- The domain is often denoted by U (the universe).
- **Example:** Let $P(x)$ denote "$x > 5$" and U be the integers. Then
  - $P(8)$ is true.
  - $P(5)$ is false.

Examples of Propositional Functions

- Let $P(x, y, z)$ denote that $x + y = z$ and U be the integers for all three variables.
  - $P(-4, 6, 2)$ is true.
  - $P(5, 2, 10)$ is false.
  - $P(5, x, 7)$ is not a proposition.
- Let $Q(x, y, z)$ denote that $x - y = z$ and U be the integers.
  - $P(1, 2, 3) \wedge Q(5, 4, 1)$ is true.
  - $P(1, 2, 4) \rightarrow Q(5, 4, 0)$ is true.
  - $P(1, 2, 3) \rightarrow Q(5, 4, 0)$ is false.
  - $P(1, 2, 4) \rightarrow Q(x, 4, 0)$ is not a proposition.

Quantifiers

- We need quantifiers to formally express the meaning of the words "all" and "some".
- The two most important quantifiers are:

  Universal quantifier, "For all". Symbol: $\forall$

  Existential quantifier, "There exists". Symbol: $\exists$
- $\forall x P(x)$ asserts that $P(x)$ is true for **every** x in the domain.
- $\exists x P(x)$ asserts that $P(x)$ is true for **some** x in the domain.
- The quantifiers are said to bind the variable x in these expressions.
- Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables.
- A propositional function that does not contain any free variables is a proposition and has a truth value.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, $P(x)$" or "For every x, $P(x)$".
- The truth value depends not only on P, but also on the domain U.
- **Example:** Let $P(x)$ denote $x > 0$.
  - If U is the integers then $\forall x\, P(x)$ is false.
  - If U is the positive integers then $\forall x\, P(x)$ is true.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, $P(x)$" or "There is an x such that, $P(x)$", or "For at least one x, $P(x)$".
- The truth value depends not only on P, but also on the domain U.
- **Example:** Let $P(x)$ denote $x < 0$.
  - If U is the integers then $\exists x\, P(x)$ is true.
  - If U is the positive integers then $\exists x\, P(x)$ is false.

Uniqueness Quantifier

- $\exists! x\, P(x)$ means that there exists one and only one x in the domain such that $P(x)$ is true.
- $\exists_1 x\, P(x)$ is an alternative notation for $\exists! x\, P(x)$.
- This is read as
  - There is one and only one x such that $P(x)$.
  - There exists a unique x such that $P(x)$.
- **Example:** Let $P(x)$ denote $x + 1 = 0$ and U are the integers. Then $\exists! x\, P(x)$ is true.
- **Example:** Let $P(x)$ denote $x > 0$ and U are the integers. Then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier can be expressed by standard operations. $\exists! x\, P(x)$ is equivalent to $\exists x\, (P(x) \wedge \forall y\, (P(y) \rightarrow y = x))$.

Precedence of Quantifiers

- Quantifiers $\forall$ and $\exists$ have higher precedence then all logical operators.
- $\forall x\ P(x) \wedge Q(x)$ means $(\forall x\ P(x)) \wedge Q(x)$. In particular, this expression contains a free variable.
- $\forall x\ (P(x) \wedge Q(x))$ means something different.

Translating English to Logic

Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, define a propositional function $J(x)$ denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if U is all people, also define a propositional function $S(x)$ denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

Note: $\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value for every predicate substituted into these statements and for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- Example: $\forall x \neg\neg S(x) \equiv \forall x S(x)$.

Quantifiers as Conjunctions/Disjunctions

- If the domain is finite then universal/existential quantifiers can be expressed by conjunctions/disjunctions.
- If U consists of the integers 1,2, and 3, then

  $\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$

  $\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$
- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

De Morgan's Law for Quantifiers

The rules for negating quantifiers are:

- $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Predicate Calculus

- An assertion in predicate calculus is valid iff it is true
  - for all domains
  - for every propositional functions substituted for the predicates in the assertion.
- An assertion in predicate calculus is satisfiable iff it is true
  - for some domain
  - for some propositional functions that can be substituted for the predicates in the assertion.

Example: $\forall x\,(F(x) \leftrightarrow G(x))$ is not valid, but satisfiable.
Example: $\forall x\,(F(x) \leftrightarrow \neg F(x))$ is unsatisfiable.

Nested Quantifiers

Complex meanings require nested quantifiers.

- "Every real number has an inverse w.r.t. addition."
  Let the domain U be the real numbers. Then the property is expressed by
  $$\forall x\ \exists y\ (x + y = 0)$$
- "Every real number except zero has a multiplicative inverse."
  Let the domain U be the real numbers. Then the property is expressed by
  $$\forall x\,(x \neq 0 \rightarrow \exists y\,(x * y = 1))$$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x, y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - If for some pair of x and y, $P(x, y)$ is false, then $\forall x \forall y P(x, y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x, y)$ is true if the outer loop ends after stepping through each x.

  - To see if $\forall x \exists y P(x, y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x, y)$ has been shown to be false.

  $\forall x \exists y P(x, y)$ is true if the outer loop ends after stepping through each x.

- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Quantifiers can be grouped into blocks

$$\forall x \forall y \ldots \forall z \quad \exists a \exists b \ldots \exists c \quad \forall u \forall v \ldots \forall w \ldots \ldots$$

Quantifiers can be swapped inside a block, but not between blocks.

- Let $P(x, y)$ denote $x + y = y + x$ and U be the real numbers. Then $\forall x \forall y P(x, y)$ is equivalent to $\forall y \forall x P(x, y)$.
- Let $Q(x, y)$ denote $x + y = 0$ and U be the real numbers. Then $\forall x \exists y P(x, y)$ is true, but $\exists y \forall x P(x, y)$ is false.

Quantifications of Two Variables

Statement	**When True?**	**When False**
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Negating Nested Quantifiers

Let $P(x, f)$ denote that person x has taken flight f.
Let $Q(f, a)$ denote that flight f is operated by airline a.
Formulate: "There is no person who has taken a flight on every airline in the world."

$$\neg\exists x\ \forall a\ \exists f\ (P(x,f) \wedge Q(f,a))$$

Now use De Morgan's Laws to move the negation as far inwards as possible.

$$\begin{array}{rl}
\neg\exists x\ \forall a\ \exists f\ (P(x,f) \wedge Q(f,a)) & \\
\forall x\ \neg\forall a\ \exists f\ (P(x,f) \wedge Q(f,a)) & \text{by De Morgan's for } \exists \\
\forall x\ \exists a\ \neg\exists f\ (P(x,f) \wedge Q(f,a)) & \text{by De Morgan's for } \forall \\
\forall x\ \exists a\ \forall f\ \neg(P(x,f) \wedge Q(f,a)) & \text{by De Morgan's for } \exists \\
\forall x\ \exists a\ \forall f\ (\neg P(x,f) \vee \neg Q(f,a)) & \text{by De Morgan's for } \wedge
\end{array}$$

Can you translate the result back into English?
"For every person there is an airline such that for all flights, this person has not taken that flight or that flight is not operated by this airline."

Thank You