LECTURE 19 & 20- RECURRENCE RELATION

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i<n$).

EXAMPLE – FIBONACCI SERIES $F_n = F_{n-1} + F_{n-2}$, TOWER OF HANOI – $F_n = 2F_{n-1} + 1$

Linear Recurrence Relations

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format $x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \ldots A_k x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations −

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas Number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is − $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is −

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots −

Case 1 − If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2 , then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 − If this equation factors as $(x - x_1)^2 = 0$ and it produces single real root x_1 , then $F_n = ax_1^n + bnx_1^n$ is the solution.

Case 3 − If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, then $F_n = r^n(acos(n\theta) + bsin(n\theta))$ is the solution.

Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is –

$$x^2 - 5x + 6 = 0,$$

So, $(x - 3)(x - 2) = 0$

Hence, the roots are –

$x_1 = 3$ and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n \ (As\ x_1 = 3\ and\ x_2 = 2)$

Therefore,

$$1 = F_0 = a3^0 + b2^0 = a + b$$

$$4 = F_1 = a3^1 + b2^1 = 3a + 2b$$

Solving these two equations, we get $a = 2$ and $b = -1$

Hence, the final solution is −

$$F_n = 2.3^n + (-1).\,2^n = 2.3^n - 2^n$$

Problem 2

Solve the recurrence relation – $F_n=10F_{n-1}-25F_{n-2}$ where $F_0=3$ and $F_1=17$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 10x - 25 = 0$$

So $(x - 5)^2 = 0$

Hence, there is single real root $x_1 = 5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is −

$$F_n = ax_1^n + bnx_1^n$$

$$3 = F_0 = a.5^0 + (b)(0.5)^0 = a$$

$$17 = F_1 = a.5^1 + b.1.5^1 = 5a + 5b$$

Solving these two equations, we get $a = 3$ and $b = 2/5$

Hence, the final solution is − $F_n = 3.5^n + (2/5).n.2^n$

PROBLEM 3

Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$ where $F_0 = 1$ and $F_1 = 3$

SOLUTION : The characteristic equation of the recurrence relation is –

$$x^2 - 2x - 2 = 0$$

Hence, the roots are –

$x_1 = 1 + i$ and $x_2 = 1 - i$

In polar form,

$x_1 = r\angle\theta$ and $x_2 = r\angle(-\theta)$, where $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is –

$$F_n = (\sqrt{2})^n (acos(n.\sqcap/4) + bsin(n.\sqcap/4))$$

$$1 = F_0 = (\sqrt{2})^0 (acos(0.\sqcap/4) + bsin(0.\sqcap/4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (acos(1.\sqcap/4) + bsin(1.\sqcap/4)) = \sqrt{2}(a/\sqrt{2} + b/\sqrt{2})$$

Solving these two equations we get $a = 1$ and $b = 2$

Hence, the final solution is –

$$F_n = (\sqrt{2})^n (cos(n.\pi/4) + 2sin(n.\pi/4))$$

NON-HOMOGENEOUS RECURRENCE RELATION AND PARTICULAR SOLUTIONS:

A recurrence relation is called non-homogeneous if it is in the form

$F_n = AF_{n-1} + BF_{n-2} + f(n)$ where $f(n) \neq 0$

Its associated homogeneous recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$

The solution (a_n) of a non-homogeneous recurrence relation has two parts.

First part is the solution (a_h) of the associated homogeneous recurrence relation and the secon

part is the particular solution (a_t) .

$$a_n = a_h + a_t$$

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let $f(n) = cx^n$; let $x^2 = Ax + B$ be the characteristic equation of the associated homogeneous recurrence relation and let x_1 and x_2 be its roots.

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Example

Let a non-homogeneous recurrence relation be $F_n = AF_{n-1} + BF_{n-2} + f(n)$ with characteristic roots $x_1 = 2$ and $x_2 = 5$. Trial solutions for different possible values of $f(n)$ are as follows −

f(n)	Trial solutions
4	A
5.2^n	$An2^n$
8.5^n	$An5^n$
4^n	$A4^n$
$2n^2+3n+1$	An^2+Bn+C

PROBLEM 1

Solve the recurrence relation $F_n = 3F_{n-1} + 10F_{n-2} + 7.5^n$ where f0=4 and f1=3

SOL.

Solution

This is a linear non-homogeneous relation, where the associated homogeneous equation $F_n = 3F_{n-1} + 10F_{n-2}$ and $f(n) = 7.5^n$

The characteristic equation of its associated homogeneous relation is –

$$x^2 - 3x - 10 = 0$$

Or, $(x - 5)(x + 2) = 0$

Or, $x_1 = 5$ and $x_2 = -2$

Hence $a_h = a.5^n + b.(-2)^n$, where a and b are constants.

Since $f(n) = 7.5^n$, i.e. of the form $c.x^n$, a reasonable trial solution of at will be Anx^n

$$a_t = Anx^n = An5^n$$

After putting the solution in the recurrence relation, we get –

$$An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7.5^n$$

Dividing both sides by 5^{n-2} , we get

$$An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7.5^2$$

Or, $25An = 15An - 15A + 10An - 20A + 175$

Or, $35A = 175$

Or, $A = 5$

So, $F_n = An5^n = 5n5^n = n5^{n+1}$

The solution of the recurrence relation can be written as –

$$F_n = a_h + a_t$$

$$= a.5^n + b.(-2)^n + n5^{n+1}$$

Putting values of $F_0 = 4$ and $F_1 = 3$, in the above equation, we get $a = -2$ and $b = 6$

Hence, the solution is –

$$F_n = n5^{n+1} + 6.(-2)^n - 2.5^n$$

PROBLEMS :

1. What is the solution of the recurrence relation
   $$a_n = a_{n-1} + 2a_{n-2}$$
   with $a_0=2$ and $a_1=7$?

2. What is the solution of the recurrence relation
   $$f_n = f_{n-1} + f_{n-2}$$
   with $f_0=0$ and $f_1=1$?

3. What is the solution of the recurrence relation
   $$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$$
   with $a_0=8$, $a_1=6$ and $a_2=26$?