LECTURE-14 POSET,HASSE DIAGRAM & EXTREMEL ELEMENTS

- **Definition: A Partially Ordered Set (Poset) is a set with a relation "$\leq$" satisfying**

1. $x \leq x$ for all $x \alpha X$ (Reflexive Property)
2. $x \leq y$ and $y \leq x$ implies $x = y$ (Anti-Symmetric)
3. $x \leq y$, $y \leq z$ implies $x \leq z$ (Transitive Property)

FORMALLY, A PARTIAL ORDER IS ANY BINARY RELATION THAT IS REFLEXIVE (EACH ELEMENT IS COMPARABLE TO ITSELF), ANTISYMMETRIC (NO TWO DIFFERENT ELEMENTS PRECEDE EACH OTHER), AND TRANSITIVE (THE START OF A CHAIN OF PRECEDENCE RELATIONS MUST PRECEDE THE END OF THE CHAIN).

EXAMPLE :

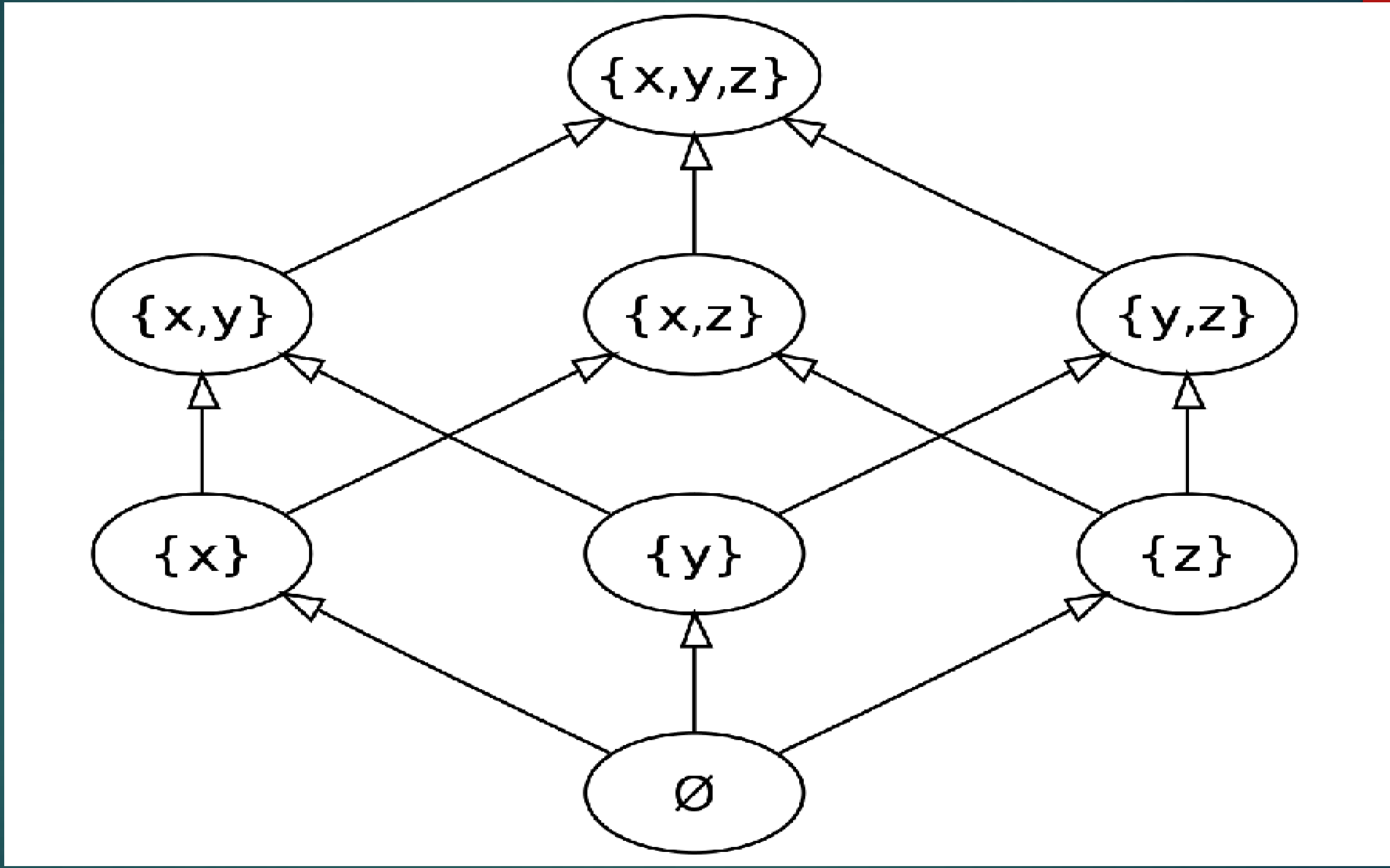
1. THE REAL NUMBERS ORDERED BY THE STANDARD LESS-THAN-OR-EQUAL RELATION ≤ (A TOTALLY ORDERED SET AS WELL).

2. THE SET OF SUBSETS OF A GIVEN SET (ITS POWER SET) ORDERED BY INCLUSION. SIMILARLY, THE SET OF SEQUENCES ORDERED BY SUBSEQUENCE, AND THE SET OF STRINGS ORDERED BY SUBSTRING.

3. THE SET OF NATURAL NUMBERS EQUIPPED WITH THE RELATION OF DIVISIBILITY.

4. THE VERTEX SET OF A DIRECTED ACYCLIC GRAPH ORDERED BY REACHABILITY.

5. THE SET OF SUBSPACES OF A VECTOR SPACE ORDERED BY INCLUSION.


{x,y,z}
{x,y}
{x,z}
{y,z}
{x}
{y}
{z}
Ø

HASSE DIAGRAMS : A PARTIAL ORDER, BEING A RELATION, CAN BE REPRESENTED BY A DI-GRAPH. BUT MOST OF THE EDGES DO NOT NEED TO BE SHOWN SINCE IT WOULD BE REDUNDANT.

1. FOR INSTANCE, WE KNOW THAT EVERY PARTIAL ORDER IS REFLEXIVE, SO IT IS REDUNDANT TO SHOW THE SELF-LOOPS ON EVERY ELEMENT OF THE SET ON WHICH THE PARTIAL ORDER IS DEFINED.

2. EVERY PARTIAL ORDER IS TRANSITIVE, SO ALL EDGES DENOTING TRANSITIVITY CAN BE REMOVED.

3. THE DIRECTIONS ON THE EDGES CAN BE IGNORED IF ALL EDGES ARE PRESUMED TO HAVE ONLY ONE POSSIBLE DIRECTION, CONVENTIONALLY UPWARDS.

IN GENERAL, A PARTIAL ORDER ON A FINITE SET CAN BE REPRESENTED USING THE FOLLOWING PROCEDURE –

(A) REMOVE ALL SELF-LOOPS FROM ALL THE VERTICES. THIS REMOVES ALL EDGES SHOWING REFLEXIVITY.

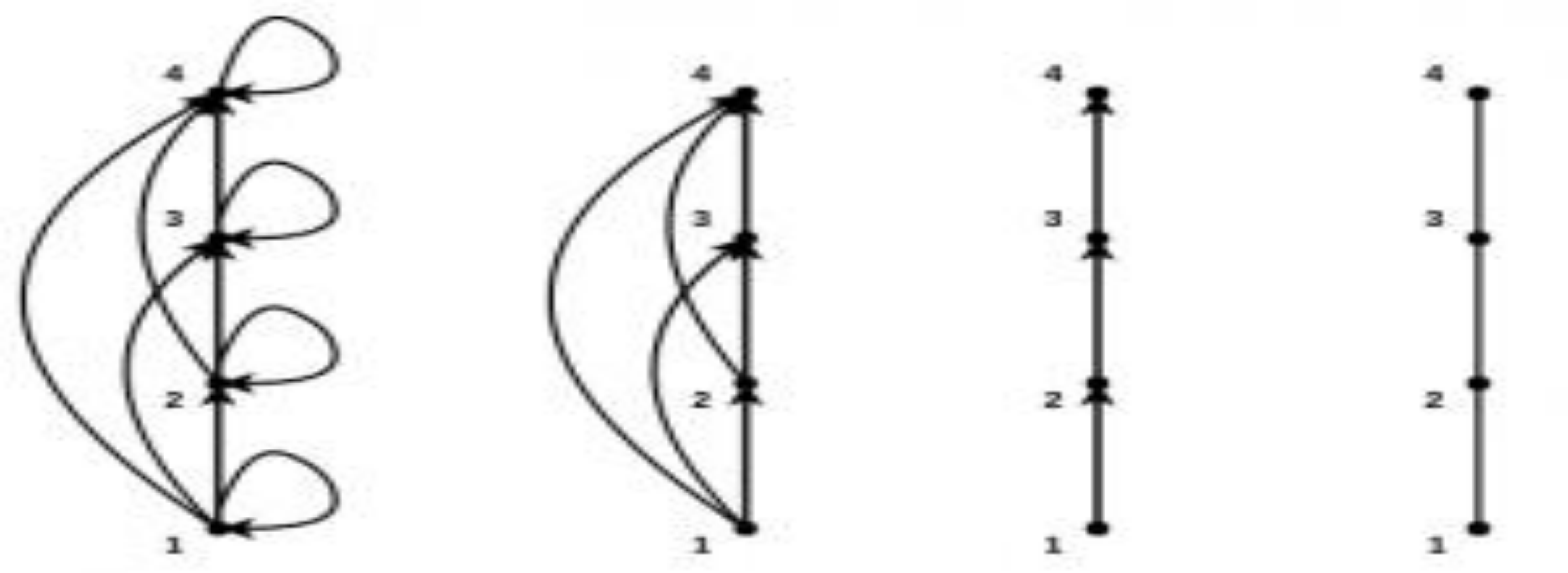
(B) REMOVE ALL EDGES WHICH ARE PRESENT DUE TO TRANSITIVITY I.E. IF (A,B) AND (B, C)

(C) ARE IN THE PARTIAL ORDER, THEN REMOVE THE EDGE (A, C). FURTHERMORE IF (C, D) IS IN THE PARTIAL ORDER, THEN REMOVE THE EDGE (A, D).

(D) ARRANGE ALL EDGES SUCH THAT THE INITIAL VERTEX IS BELOW THE TERMINAL VERTEX.

(E) REMOVE ALL ARROWS ON THE DIRECTED EDGES, SINCE ALL EDGES POINT UPWARDS.

FOR EXAMPLE, THE POSET ({1, 2, 3, 4}, <=) WOULD BE CONVERTED TO A HASSE DIAGRAM LIKE –



EXTREMUMS IN POSETS : ELEMENTS OF POSETS THAT HAVE CERTAIN EXTREMAL PROPERTIES ARE IMPORTANT FOR MANY APPLICATIONS.

- **MAXIMAL ELEMENTS-** AN ELEMENT A IN THE POSET IS SAID TO BE MAXIMAL IF THERE IS NO ELEMENT B IN THE POSET SUCH THAT A < B.
- **MINIMAL ELEMENTS-** AN ELEMENT A IN THE POSET IS SAID TO BE MINIMAL IF THERE IS NO ELEMENT B IN THE POSET SUCH THAT B < A.

MAXIMAL AND MINIMAL ELEMENTS ARE EASY TO FIND IN HASSE DIAGRAMS. THEY ARE THE TOPMOST AND BOTTOMMOST ELEMENTS RESPECTIVELY.

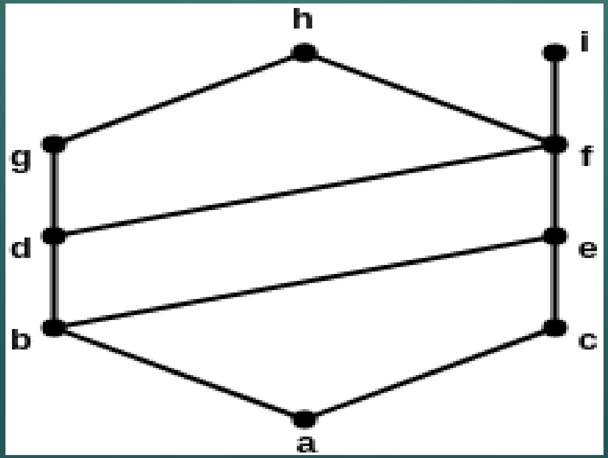
FOR EXAMPLE, IN THE HASSE DIAGRAM DESCRIBED ABOVE, "1" IS THE MINIMAL ELEMENT AND "4" IS THE MAXIMAL ELEMENT. SINCE MAXIMAL AND MINIMAL ARE UNIQUE, THEY ARE ALSO THE GREATEST AND LEAST ELEMENT OF THE POSET.

IMPORTANT NOTE : IF THE MAXIMAL OR MINIMAL ELEMENT IS UNIQUE, IT IS CALLED THE GREATEST OR LEAST ELEMENT OF THE POSET RESPECTIVELY.

IT IS SOMETIMES POSSIBLE TO FIND AN ELEMENT THAT IS GREATER THAN OR EQUAL TO ALL THE ELEMENTS IN A SUBSET A OF POSET (S, <=). SUCH AN ELEMENT IS CALLED THE UPPER BOUND OF A. SIMILARLY, WE CAN ALSO FIND THE LOWER BOUND OF A.

THESE BOUNDS CAN BE FURTHER CONSTRAINED TO GET THE LEAST UPPER BOUND (SUPREMUM) AND THE GREATEST LOWER BOUND (INFIMUM). THESE BOUNDS ARE ELEMENTS WHICH ARE LESS THAN OR GREATER THAN ALL THE OTHER UPPER BOUNDS OR LOWER BOUNDS RESPECTIVELY.

EXAMPLE – FIND THE LEAST UPPER BOUND AND GREATEST LOWER BOUND OF THE FOLLOWING SUBSETS- {B, C}, {G, E, A}, {E, F}



SOLUTION –

- FOR THE SET {B, C}

THE UPPER BOUNDS ARE – E, F, H, I. SO THE LEAST UPPER BOUND IS E.

THE LOWER BOUNDS ARE – A. SO THE GREATEST LOWER BOUND IS A.

- FOR THE SET {G, E, A}

THE UPPER BOUNDS ARE – H. SO THE LEAST UPPER BOUND IS H.

THE LOWER BOUNDS ARE – A. SO THE GREATEST LOWER BOUND IS A.

- FOR THE SET {E, F}

THE UPPER BOUNDS ARE – F, H, I. SO THE LEAST UPPER BOUND IS F.

THE LOWER BOUNDS ARE – E, C, B, A. SO THE GREATEST LOWER BOUND IS E.

QUESTIONS:

1. LET A BE A POSET, A = { 2, 4, 6, 8 } AND THE RELATION A | B IS 'A DIVIDES B. DRAW A HASSE DIAGRAM FOR THE POSET SHOWING ALL THE RELATIONS.
2. LET A BE A POSET, A = { 1, 2, 3,……………,18 } AND THE RELATION A | B IS 'A DIVIDES B. DRAW A HASSE DIAGRAM FOR THE POSET SHOWING ALL THE RELATIONS.
3. DRAW THE HASSE DIAGRAM FOR THE POWER SET OF {X, Y, Z}.
4. LET A BE A POSET, A = { 1, 3, 6, 9,12} AND THE RELATION A | B IS 'A DIVIDES B. DRAW A HASSE DIAGRAM FOR THE POSET SHOWING ALL THE RELATIONS.