LECTURE 31- CAYLEY'S THEOREM

Every group is isomorphic to a permutation group.

Proof: Let G be any group.

1. For g in G, define Tg(x) = gx.

We show Tg is a permutation on G.

2. Let S = {Tg | for g in G}

We show S is a permutation group.

3. Define the map ϕ:G $\rightarrow$ S by ϕ(g)=Tg We show ϕ is an isomorphism.

1. T_g is a permutation on G

Suppose $T_g(x) = T_g(y)$.

Then $gx = gy$.

By left cancellation, $x=y$.

Hence T_g is 1 to 1.

Choose any y in G. Let $x = g^{-1}y$

Then $T_g(x) = gx = gg^{-1}y = y$

So T_g is onto.

This shows that T_g is a permutation.

2. $\{T_g \mid g \text{ in } G\}$ is a group

The operation is composition. For a, b, x,

$T_aT_b(x) = T_a(bx) = a(bx) = (ab)x = T_{ab}(x)$

So $T_aT_b = T_{ab}$ (*)

From (*), $T_eT_a = T_{ea} = T_a$,

So T_e is the identity in S.

If $b = a^{-1}$ we have, $T_aT_b = T_{ab} = T_e$

So $T_a^{-1} = T_b$ and S has inverses.

Function composition is associative.

Therefore, S is a group.

3. $\phi(g) = T_g$ is isomorphism

1. Choose a, b in G.

   Suppose $\phi(a) = \phi(b)$. Then

   $T_a = T_b$. In particular, for any x in G,

$$T_a(x) = T_b(x)$$

$$ax = bx$$

$$a = b$$

   Therefore ϕ is one-to-one.

2. Choose any Tg in S. Then

$$\phi(g) = Tg$$

   Therefore, ϕ is onto.

3. Choose any a, b in G.

Then $\phi(ab) = T_{ab}$

$= T_a T_b$ by (*)

$= \phi(a)\,\phi(b)$

Therefore, ϕ is Operation Preserving.

It follows that ϕ is an isomorphism.