LECTURE-33 HOMOMORPHISM, ISOMORPHISM & AUTOMORPHISM

DEFINITION OF GROUP HOMOMORPHISM

LET $< G , * >$ AND $< H , \Delta >$ BE TWO GROUP. A MAPPING $G : G \rightarrow H$ IS CALLED A GROUP HOMOMORPHISM FROM $< G , * >$ TO $< H , \Delta >$ IF FOR ANY $A , B \in G$

- $g(a * b) = g(a) \Delta g(b)$
- $g(e^G) = e^H$
- $g(a^{-1}) = [g(a)]-1$

Group Homomorphism

Let $(G_1, \bullet)$ and $(G_2, \bullet)$ be groups, and let $f : G_1 \rightarrow G_2$ be a function. Then f is said to be a **group homomorphism** if

$$f(a \bullet b) = f(a) \bullet f(b)$$

for all a,b in G_1.

Every isomorphism is an **one-to-one** and **onto** homomorpism.

TYPES OF GROUP HOMOMORPHISM

- **MONOMORPHISM :**

A GROUP HOMOMORPHISM THAT IS INJECTIVE (OR, ONE-TO-ONE); I.E., PRESERVES DISTINCTNESS.

- **EPIMORPHISM :**

A GROUP HOMOMORPHISM THAT IS SURJECTIVE (OR, ONTO); I.E., REACHES EVERY POINT IN THE CODOMAIN.

- **ISOMORPHISM :**

A GROUP HOMOMORPHISM THAT IS BIJECTIVE; I.E., INJECTIVE AND SURJECTIVE. ITS INVERSE IS ALSO A GROUP HOMOMORPHISM. IN THIS CASE, THE GROUPS G AND H ARE CALLED ISOMORPHIC; THEY DIFFER ONLY IN THE NOTATION OF THEIR ELEMENTS AND ARE IDENTICAL FOR ALL PRACTICAL PURPOSES.

- **ENDOMORPHISM :**

A HOMOMORPHISM, H: $G \to G$; THE DOMAIN AND CODOMAIN ARE THE SAME. ALSO CALLED AN ENDOMORPHISM OF G.

- **AUTOMORPHISM :**

AN ENDOMORPHISM THAT IS BIJECTIVE, AND HENCE AN ISOMORPHISM. THE SET OF ALL AUTOMORPHISMS OF A GROUP G, WITH FUNCTIONAL COMPOSITION AS OPERATION, FORMS ITSELF A GROUP, THE AUTOMORPHISM GROUP OF G. IT IS DENOTED BY AUT(G). AS AN EXAMPLE, THE AUTOMORPHISM GROUP OF (Z, +) CONTAINS ONLY TWO ELEMENTS, THE IDENTITY TRANSFORMATION AND MULTIPLICATION WITH -1; IT IS ISOMORPHIC TO $Z/2Z$.

DEFINITION OF GROUP ISOMORPHISM

LET F : $< G , * > \rightarrow < H , \Delta >$.IF F IS ONE TO ONE AND ONTO. THEN GROUP IS CALLED ISOMORPHISM

- A HOMOMORPHISM F : $< G , * > \rightarrow < H , \Delta >$ IS CALLED AN ENDOMORPHISM

- A ISOMORPHISM F : $< G , * > \rightarrow < H , \Delta >$ IS CALLED AN AUTOMORPHISM

Definition Kernal of Homomorphism

- Let $< G , * >$ and $< H , \Delta >$ be two Groups and let f is homomorphism of G into H. The set of elements of G which are mapped into e_H , the identity of H is called the kernal of the homomorphism and is denoted by K_f or Ker(f)

Theorem : The Kernal of homomorphism $f : <G , *> \rightarrow < H , \Delta >$ is sub group of $< G , * >$

Proof :

- Here $f : < G , * > \rightarrow < H , \Delta >$ is homomorphism
- $Ker (f) = \{x \in G \mid f(x) = e_H$ identity element of $H\}$
- $k (f) \neq \emptyset$ because $e_G \in K(f)$ $(f(e_G)=e_H)$
- let $a , b \in K_f$
- $f (a) = e_H$ & $f(b) = e_G$
- Now, $f(ab^{-1}) = f(a) . f(b^{-1})$
- $= f(a) . [f(b)]^{-1}$
- $= e_H . e_H^{-1}$
- $= e_H . e_H$
- $= e_H$

$\Rightarrow$ $ab^{-1} \in K_f$

$\Rightarrow$ K_f is a sub group of $< G , * >$