LECTURE-16 Composition of Function, Inverse Operations

A **Function** assigns to each element of a set, exactly one element of a related set. Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few. The third and final chapter of this part highlights the important aspects of functions.

Function - Definition

A function or mapping (Defined as $f : X \rightarrow Y$) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Function 'f' is a relation on X and Y such that for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in R$. 'x' is called pre-image and 'y' is called image of function f.

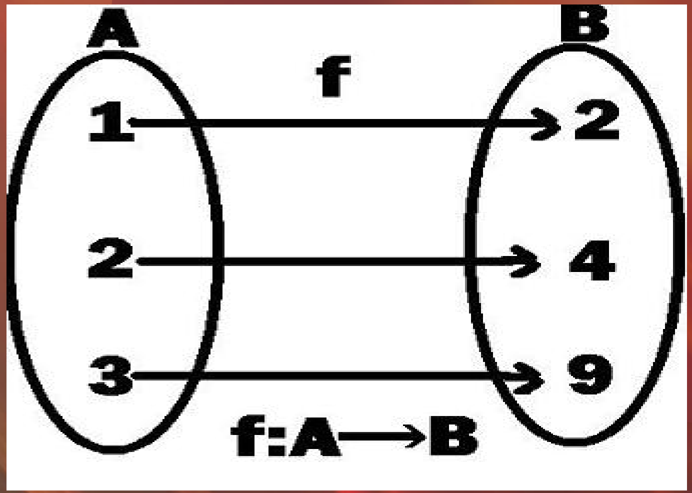
A function can be one to one or many to one but not one to many.

FUNCTIONS ARE OF DIFFERENT TYPES:

- ONE-TO-ONE FUNCTION:

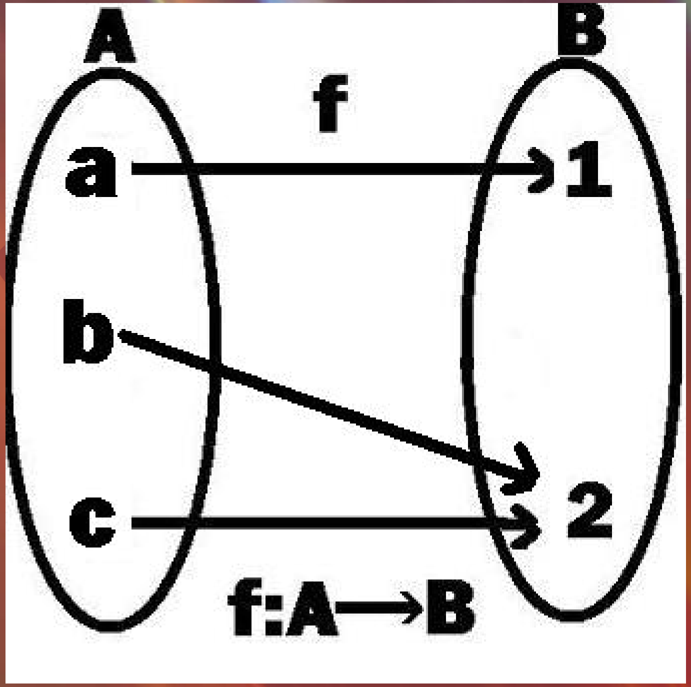
A FUNCTION FROM A TO B IS ONE-TO-ONE OR INJECTIVE, IF FOR ALL ELEMENTS X_1, X_2 IN A SUCH THAT $F(X_1) = F(X_2)$, I.E $X_1 = X_2$.

NO ELEMENTS OF A ARE ASSIGNED TO THE SAME ELEMENT IN B AND EACH ELEMENT OF THE RANGE CORRESPONDS TO EXACTLY ONE ELEMENT IN DOMAIN.



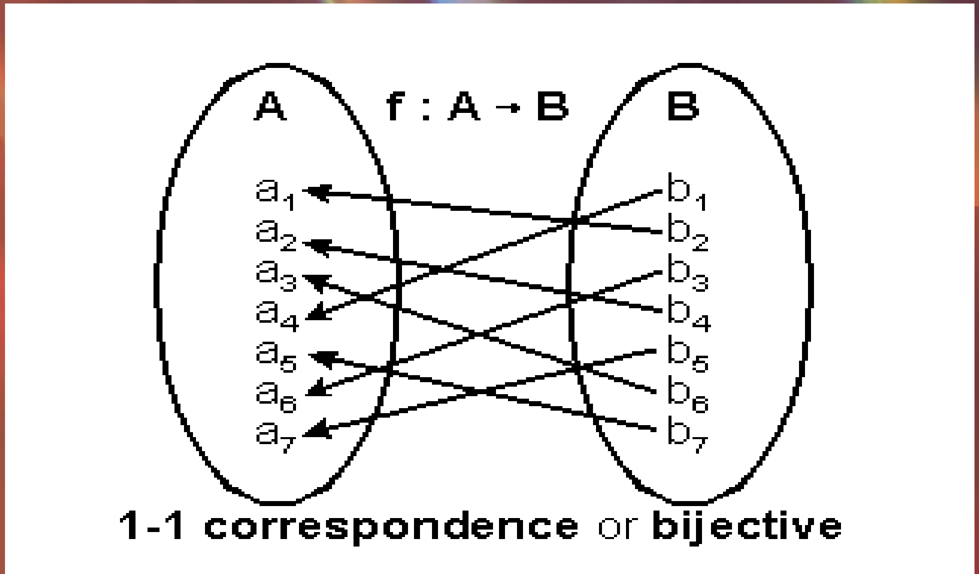
❖ ONTO FUNCTION:

A FUNCTION FROM A TO B IS ONTO OR SURJECTIVE, IF EVERY ELEMENT OF B IS THE IMAGE OF SOME ELEMENT IN A I.E ALL THE ELEMENTS OF B HAS A PRE-IMAGE IN A.



❖ BIJECTIVE FUNCTION:

A FUNCTION FROM A TO B IS ONE-TO-ONE CORRESPONDENCE OR BIJECTIVE, IF F IS BOTH INJECTIVE(ONE-TO-ONE) AND SURJECTIVE(ONTO).



INVERSE OF A FUNCTION :

Inverse of a Function

The **inverse** of a one-to-one corresponding function $f : A \rightarrow B$, is the function $g : B \rightarrow A$, holding the following property −

$$f(x) = y \Leftrightarrow g(y) = x$$

The function f is called **invertible**, if its inverse function g exists.

Example

- A Function $f : Z \rightarrow Z, f(x) = x + 5$, is invertible since it has the inverse function $g : Z \rightarrow Z, g(x) = x - 5$.
- A Function $f : Z \rightarrow Z, f(x) = x^2$ is not invertiable since this is not one-to-one as $(-x)^2 = x^2$.

COMPOSITION OF FUNCTIONS :

Composition of Functions

Two functions $f : A \to B$ and $g : B \to C$ can be composed to give a composition gof .

This is a function from A to C defined by $(gof)(x) = g(f(x))$

Example

Let $f(x) = x + 2$ and $g(x) = 2x + 1$, find $(fog)(x)$ and $(gof)(x)$.

Solution

$$(fog)(x) = f(g(x)) = f(2x + 1) = 2x + 1 + 2 = 2x + 3$$

$$(gof)(x) = g(f(x)) = g(x + 2) = 2(x + 2) + 1 = 2x + 5$$

Hence, $(fog)(x) \neq (gof)(x)$

Some Facts about Composition

- If f and g are one-to-one then the function (gof) is also one-to-one.
- If f and g are onto then the function (gof) is also onto.
- Composition always holds associative property but does not hold commutative property.

QUESTIONS :

For each of the relations $\{Q, R, S, T, U, V\}$ below, determine whether the relation is a function. If the relation is a function, determine whether the function is injective and/or surjective.

(i) $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$
$Q = \{(1, a), (2, d), (3, b)\}$

(ii) $A = \{1, 2, 3\}$, $B = \{a, b, c\}$
$R = \{(1, a), (2, b), (3, c)\}$

(iii) $A = \{1, 2, 3\}$, $B = \{a, b, c\}$
$S = \{(1, a), (2, b), (3, b)\}$

(iv) $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$
$T = \{(1, a), (2, b), (2, c), (3, d)\}$

(v) $A = \{1, 2, 3\}$, $B = \{a, b\}$
$U = \{(1, a), (2, b), (3, b)\}$

(vi) $A = \{1, 2, 3\}$, $B = \{a, b\}$
$V = \{(1, a), (2, b)\}$

(i) The relation is a function.
The function is injective.
The function is not surjective since c is not an element of the range.

(ii) The relation is a function.
The function is both injective and surjective.

(iii) The relation is a function.
The function is not injective since $f(2) = f(3)$ but $2 \neq 3$.
The function is not surjective since c is not an element of the range.

(iv) The relation is a not a function since the relation is not uniquely defined for 2.

(v) The relation is a function.
The function is not injective since $f(2) = f(3)$ but $2 \neq 3$.
The function is surjective.

(vi) The relation is a not a function since the relation is not defined for 2.

QUESTIONS :

The function $\boldsymbol{f}$ is defined by: $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}: \boldsymbol{x} \mapsto \boldsymbol{x}^2 + 2$.

(i) Give an example to show that f is not injective.

(ii) Give an example to show that f is not surjective.

(i) $f(-1) = f(1) = 3$ but $-1 \neq 1$, therefore the function is not injective.

(ii) There is no real number, x such that $f(x) = 1$ therefore the function is not surjective. Or the range of the function is $y \geq 2$. The range of the function is not $\mathbb{R}$ (the codomain) therefore the function is not surjective.

The function $\boldsymbol{f}$ is defined by: $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}: \boldsymbol{x} \mapsto \boldsymbol{x}^2 - 6\boldsymbol{x}$.

(i) Give an example to show that f is not injective.

(ii) Give an example to show that f is not surjective.

(i) $f(6) = f(0) = 0$ but $6 \neq 0$, therefore the function is not injective.

(ii) $f(x) = (x-3)^2 - 9$ [by completing the square]
There is no real number, x such that $f(x) = -10$ the function is not surjective.
Or the range of the function is $y \geq 2$. The range of the function is not $\mathbb{R}$ (the codomain) therefore the function is not surjective

QUESTIONS :

For each of the functions below determine which of the properties hold, injective, surjective, bijective. Briefly explain your reasoning.

(i) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$.

(ii) The function $f: \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$.

(iii) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x + 1)x(x - 1)$.

(iv) The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 - 9)(x^2 - 4)$.

(i) This function is injective, since e^x takes on each nonnegative real value for exactly one x. However, the function is not surjective, because e^x never takes on negative values. Therefore, the function is not bijective either.

(ii) The function e^x takes on every nonnegative value for exactly one x, so it is injective, surjective, and bijective.

(iii) This function is surjective, since it is continuous, it tends to $+\infty$ for large positive x, and tends to $-\infty$ for large negative x. The function takes on each real value for at least one x. However, this function is not injective, since it takes on the value 0 at $x = -1, x = 0$ and $x = 1$. Therefore, the function is not bijective either.

(iv) This function is not surjective, it tends to $+\infty$ for large positive x, and also tends to $+\infty$ for large negative x. Also this function is not injective, since it takes on the value 0 at $x = 3, x = -3, x = 4$ and $x = -4$. Therefore, the function is not bijective either.