Principle of Mathematical Induction

Suppose there is a given statement $P(n)$ involving the natural number n such that

i. The statement is true for $n = 1$, i.e., $P(1)$ is true, and

ii. If the statement is true for $n = k$ (where k is some positive integer), then the statement is also true for $n = k + 1$, i.e., truth of $P(k)$ implies the truth of $P(k + 1)$. Then, $P(n)$ is true for all natural numbers n

Example1

For all $n \geq 1$, prove that $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

Let the given statement be P(n), i.e.,

P(n) : $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

For n = 1,

P(1): 1 = 1(1+ 1)(2* 1+ 1)/ 6

= 1*2*3 / 6 = 1

which is true.

Assume that P(k) is true for some positive integer k, i.e.,

$1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = k(k+1)(2k+1)/6$ ... (1)

We shall now prove that P(k + 1) is also true.

Now, we have $(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2) + (k+1)^2$

$= (k(k+1)(2k+1)/6) + (k+1)^2$ [Using (1)]

$= k(k+1)(2k+1) + 6(k+1)^2 /6$

$= (k+1)(2k^2 + 7k + 6)/6$

$= (k+1)(k+1+1)\{2(k+1)+1\}/6$

Thus P(k + 1) is true, whenever P (k) is true. Hence, from the principle of mathematical induction, the statement P(n) is true for all natural numbers n.

Prove that 2n > n for all positive integers n.

Let P(n): $2^n > n$

When n =1, $2^1 > 1$.

Hence P(1) is true.

Assume that P(k) is true for any positive integer k, i.e.,

$2^k > k$ ... (1)

We shall now prove that P(k +1) is true whenever P(k) is true.

Multiplying both sides of (1) by 2,

we get $2 * 2^k > 2^k$ i.e.,

$2^{k+1} > 2k = k + k > k + 1$

Therefore, P(k + 1) is true when P(k) is true. Hence, by principle of mathematical induction, P(n) is true for every positive integer n.

Questions

1. For every positive integer n, prove that $7^n – 3^n$ is divisible by 4.
2. Prove that $(1 + x)^n \geq (1 + nx)$, for all natural number n. where $x > – 1$.