LECTURE -13 EQUIVALENCE RELATION & ITS CONDITION

Equivalence Relation :

A relation R on a set A is called an equivalence relation if it satisfies following three properties:

1. Relation R is Reflexive, i.e. aRa $\forall$ a$\in$A.
2. Relation R is Symmetric, i.e., aRb $\Longrightarrow$ bRa
3. Relation R is transitive, i.e., aRb and bRc $\Longrightarrow$ aRc.

Example: Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}.

Show that R is an Equivalence Relation.

Solution:

1. Reflexive: Relation R is reflexive as (1, 1), (2, 2), (3, 3) and (4, 4) $\in$ R.

2. Symmetric: Relation R is symmetric because whenever (a, b) $\in$ R, (b, a) also belongs to R.

Example: (2, 4) $\in$ R $\Longrightarrow$ (4, 2) $\in$ R.

3. Transitive: Relation R is transitive because whenever (a, b) and (b, c) belongs to R, (a, c) also belongs to R.

Example: $(3, 1) \in R$ and $(1, 3) \in R \Rightarrow (3, 3) \in R$.

So, as R is reflexive, symmetric and transitive, hence, R is an Equivalence Relation.

Note : If R_1 and R_2 are equivalence relation then $R_1 \cap R_2$ is also an equivalence relation.

Example: $A = \{1, 2, 3\}$

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

$$R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$$

$$R_1 \cap R_2 = \{(1, 1), (2, 2), (3, 3)\}$$

Note : If R_1 and R_2 are equivalence relation then $R_1 \cup R_2$ may or may not be an equivalence relation.

Example: A = {1, 2, 3}

R1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

R2 = {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)}

R1∪ R2= {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)}

Hence, Reflexive or Symmetric are Equivalence Relation but transitive may or may not be an equivalence relation.

Inverse Relation

Let R be any relation from set A to set B. The inverse of R denoted by R-1 is the relations from B to A which consist of those ordered pairs which when reversed belong to R that is:

$R^{-1} = \{(b, a): (a, b) \in R\}$

Example: A = {1, 2, 3}

B = {x, y, z}

Solution: R = {(1, y), (1, z), (3, y)

$R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$

Clearly (R^{-1})-1 = R

Note: Domain and Range of R^{-1} is equal to range and domain of R.

Example : $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 2)\}$

$R^{-1} = \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (2, 3)\}$

Note: If R is an Equivalence Relation then R^{-1} is always an Equivalence Relation.

Partial Order Relations

A relation R on a set A is called a partial order relation if it satisfies the following three properties:

1. Relation R is Reflexive, i.e. aRa $\forall$ a$\in$A.
2. Relation R is Antisymmetric, i.e., aRb and bRa $\Longrightarrow$ a = b.
3. Relation R is transitive, i.e., aRb and bRc $\Longrightarrow$ aRc.

Example : Show whether the relation $(x, y) \in R$, if, $x \geq y$ defined on the set of +ve integers is a partial order relation.

Solution: Consider the set $A = \{1, 2, 3, 4\}$ containing four +ve integers. Find the relation for this set such as $R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (1, 1), (2, 2), (3, 3), (4, 4)\}$.

1. Reflexive: The relation is reflexive as for every a ∈ A. (a, a) ∈ R, i.e. (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

2. Antisymmetric: The relation is antisymmetric as whenever (a, b) and (b, a) ∈ R, we have a = b.

3. Transitive: The relation is transitive as whenever (a, b) and (b, c) ∈ R, we have (a, c) ∈ R.

Example: (4, 2) ∈ R and (2, 1) ∈ R, implies (4, 1) ∈ R.

SOL. As the relation is reflexive, antisymmetric and transitive. Hence, it is a partial order relation.

Example: Show that the relation 'Divides' defined on N is a partial order relation.

Solution:

1. Reflexive: We have a divides a, ∀ a∈N. Therefore, relation 'Divides' is reflexive.

2. Antisymmetric: Let a, b, c ∈N, such that a divides b. It implies b divides a iff a = b. So, the relation is antisymmetric.

3. Transitive: Let a, b, c ∈N, such that a divides b and b divides c.

Then a divides c. Hence the relation is transitive. Thus, the relation being reflexive, antisymmetric and transitive, the relation 'divides' is a partial order relation.

The relation ⊆ of a set of inclusion is a partial ordering or any collection of sets since set inclusion has three desired properties:

$A \subseteq A$ for any set A.

If $A \subseteq B$ and $B \subseteq A$ then $B = A$.

If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

(b) The relation ≤ on the set R of real no that is Reflexive, Antisymmetric and transitive.

(c) Relation ≤ is a Partial Order Relation.

n-Ary Relations

By an n-ary relation, we mean a set of ordered n-tuples. For any set S, a subset of the product set Sn is called an n-ary relation on S. In particular, a subset of S3 is called a ternary relation on S.