LECTURE-12 RELATION, OPERATION & REPRESENTATION OF RELATION

Relation or Binary relation R from set A to B is a subset of A x B which can be defined as a R b ↔ (a,b) € R ↔ R(a,b).

A Binary relation R on a single set A is defined as a subset of A x A. For two distinct set, A and B with cardinalities m and n, the maximum cardinality of the relation R from A to B is mn.

Types of Relation:

1. **Empty Relation:** A relation R on a set A is called Empty if the set A is empty set.
2. **Full Relation:** A binary relation R on a set A and B is called full if AXB.
3. **Reflexive Relation:** A relation R on a set A is called reflexive if (a,a) € R holds for every element a € A .i.e. if set A = {a,b} then R = {(a,a), (b,b)} is reflexive relation.
4. **Irreflexive relation :** A relation R on a set A is called reflexive if no (a,a) € R holds for every element a € A.i.e. if set A = {a,b} then R = {(a,b), (b,a)} is irreflexive relation.
5. **Symmetric Relation:** A relation R on a set A is called symmetric if (b,a) € R holds when (a,b) € R.i.e. The relation R={(4,5),(5,4),(6,5),(5,6)} on set A={4,5,6} is symmetric.

6. **AntiSymmetric Relation:** A relation R on a set A is called antisymmetric if (a,b)€ R and (b,a) € R then a = b is called antisymmetric.i.e. The relation R = {(a,b)→ R|a ≤ b} is anti-symmetric since a ≤ b and b ≤ a implies a = b.
7. **Transitive Relation:** A relation R on a set A is called transitive if (a,b) € R and (b,c) € R then (a,c) € R for all a,b,c € A.i.e. Relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.
8. **Equivalence Relation:** A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive. i.e. relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is equivalence relation as it is reflexive, symmetric, and transitive.
9. **Asymmetric relation:** Asymmetric relation is opposite of symmetric relation. A relation R on a set A is called asymmetric if no (b,a) € R when (a,b) € R.

Important Points:

1. Symmetric and anti-symmetric relations are not opposite because a relation R can contain both the properties or may not.

2. A relation is asymmetric if and only if it is both anti-symmetric and irreflexive.

3. Number of different relation from a set with n elements to a set with m elements is 2mn.

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Ex:
     if R ={r₁, r₂, r₃......rₙ} and S ={s₁, s₂, s₃.....sₘ}
     then Cartesian product of R and S is:
      R X S = {(r₁, s₁), (r₁, s₂), (r₁, s₃)........., (r₁, sₙ),
               (r₂, s₁), (r₂, s₂), (r₂, s₃).........., (r₂, sₙ),
                .................
               (rₙ, s₁),(rₙ, s₂), (rₙ, s₃),........., (rₙ, sₙ)}
This set of ordered pairs contains mn pairs.
Now these pairs can be present in R X S or can be absent.
So total number of possible relation = 2^mn
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4. Number of Reflexive Relations on a set with n elements : $2^{n(n-1)}$.

A relation has ordered pairs (a,b). Now a can be chosen in n ways and same for b. So set of ordered pairs contains n^2 pairs. Now for a reflexive relation, (a,a) must be present in these ordered pairs. And there will be total n pairs of (a,a), so number of ordered pairs will be n^2-n pairs. So total number of reflexive relations is equal to $2^{n(n-1)}$.

5. Number of Symmetric Relations on a set with n elements : $2^{n(n+1)/2}$.

A relation has ordered pairs (a,b). Now for a symmetric relation, if (a,b) is present in R, then (b,a) must be present in R.
In Matrix form, if a_{12} is present in relation, then a_{21} is also present in relation and As we know reflexive relation is part of symmetric relation.
So from total n^2 pairs, only n(n+1)/2 pairs will be chosen for symmetric relation. So total number of symmetric relation will be $2^{n(n+1)/2}$.

6. Number of Anti-Symmetric Relations on a set with n elements: $2^n\ 3^{n(n-1)/2}$.

A relation has ordered pairs (a,b). For anti-symmetric relation, if (a,b) and (b,a) is present in relation R, then a = b.(That means a is in relation with itself for any a).
So for (a,a), total number of ordered pairs = n and total number of relation = 2^n.

if (a,b) and (b,a) both are not present in relation or Either (a,b) or (b,a) is not present in relation. So there are three possibilities and total number of ordered pairs for this condition is n(n-1)/2. (selecting a pair is same as selecting the two numbers from n without repetition) As we have to find number of ordered pairs where a ≠ b. it is like opposite of symmetric relation means total number of ordered pairs = (n^2) – symmetric ordered pairs(n(n+1)/2) = n(n-1)/2. So, total number of relation is $3^{n(n-1)/2}$. So total number

7. Number of Asymmetric Relations on a set with n elements : $3^{n(n-1)/2}$.

In Asymmetric Relations, element a can not be in relation with itself. (i.e. there is no aRa ∀ a∈A relation.) And Then it is same as Anti-Symmetric Relations.(i.e. you have three choice for pairs (a,b) (b,a)). Therefore there are $3^{n(n-1)/2}$ Asymmetric Relations possible.

8. Irreflexive Relations on a set with n elements : $2^{n(n-1)}$.

A relation has ordered pairs (a,b). For Irreflexive relation, no (a,a) holds for every element a in R. It is also opposite of reflexive relation.
Now for a Irreflexive relation, (a,a) must not be present in these ordered pairs means total n pairs of (a,a) is not present in R, So number of ordered pairs will be n^2-n pairs.
So total number of reflexive relations is equal to $2^{n(n-1)}$.

9. Reflexive and symmetric Relations on a set with n elements : $2^{n(n-1)/2}$.

A relation has ordered pairs (a,b). Reflexive and symmetric Relations means (a,a) is included in R and (a,b)(b,a) pairs can be included or not. (In Symmetric relation for pair (a,b)(b,a) (considered as a pair). whether it is included in relation or not) So total number of Reflexive and symmetric Relations is $2^{n(n-1)/2}$.

Representation of Relations :

Relations can be represented in many ways. Some of which are as follows:

1. Relation as a Matrix: Let P = [a1,a2,a3,.......am] and Q = [b1,b2,b3......bn] are finite sets, containing m and n number of elements respectively. R is a relation from P to Q. The relation R can be represented by m x n matrix M = [Mij], defined as

$$M_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

EXAMPLE: Let P = {1, 2, 3, 4}, Q = {a, b, c, d}

and R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.

The matrix of relation R is shown as fig:

Representation of Relations :

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left\{ \begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array}$$

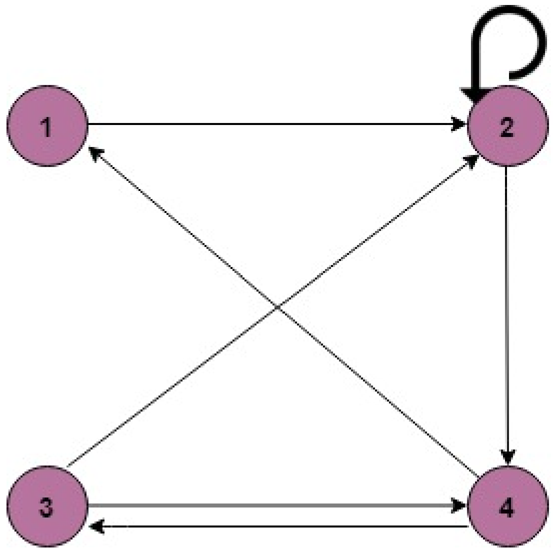
2. **Relation as a Directed Graph**: There is another way of picturing a relation R when R is a relation from a finite set to itself.

Example

A = {1, 2, 3, 4}

R = {(1, 2) (2, 2) (2, 4) (3, 2) (3, 4) (4, 1) (4, 3)}

Representation of Relations :



3. **Relation as an Arrow Diagram**: If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented as an arrow diagram as follows.

Draw two ellipses for the sets P and Q. Write down the elements of P and elements of Q column-wise in three ellipses. Then draw an arrow from the first ellipse to the second ellipse if a is related to b and a ∈ P and b ∈ Q.

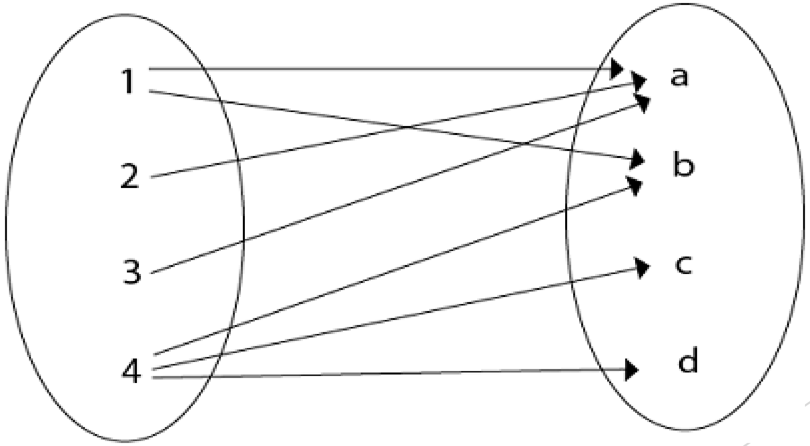
Example:

Let P = {1, 2, 3, 4}

Q = {a, b, c, d}

R = {(1, a), (2, a), (3, a), (1, b), (4, b), (4, c), (4, d)}

The arrow diagram of relation R is shown in fig:



4. **Relation as a Table**: If P and Q are finite sets and R is a relation from P to Q. Relation R can be represented in tabular form.

Make the table which contains rows equivalent to an element of P and columns equivalent to the element of Q. Then place a cross (X) in the boxes which represent relations of elements on set P to set Q.

Example:

Let P = {1, 2, 3, 4}

Q = {x, y, z, k}

R = {(1, x), (1, y), (2, z), (3, z), (4, k)}.

The tabular form of relation as shown in fig:

	x	y	z	k
1	x	x		
2			x	
3			x	
4				x

Composition of Relations

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A × B and S is a subset of B × C. Then R and S give rise to a relation from A to C indicated by R◦S and defined by:

a (R◦S)c if for some b ∈ B we have aRb and bSc.

is,

R ◦ S = {(a, c)| there exists b ∈ B for which (a, b) ∈ R and (b, c) ∈ S}

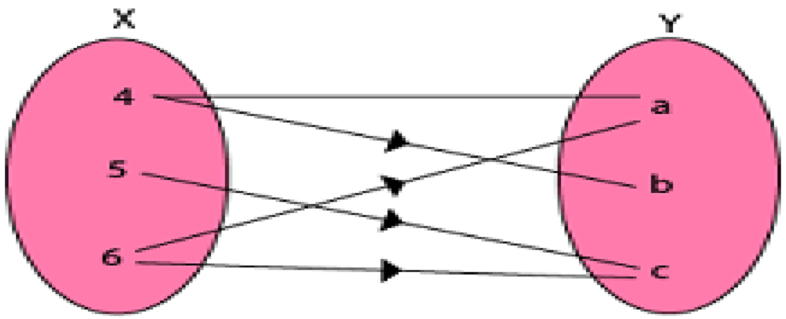
The relation R◦S is known the composition of R and S; it is sometimes denoted simply by RS.

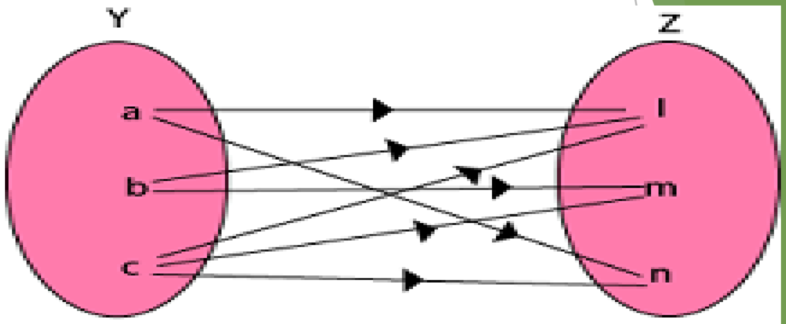
Let R is a relation on a set A, that is, R is a relation from a set A to itself. Then R◦R, the composition of R with itself, is always represented. Also, R◦R is sometimes denoted by R2. Similarly, R3 = R2◦R = R◦R◦R, and so on. Thus Rn is defined for all positive n.

Example1: Let X = {4, 5, 6}, Y = {a, b, c} and Z = {l, m, n}. Consider the relation R1 from X to Y and R2 from Y to Z.

R1 = {(4, a), (4, b), (5, c), (6, a), (6, c)}

R2 = {(a, l), (a, n), (b, l), (b, m), (c, l), (c, m), (c, n)}





Find the composition of relation (i) **R1 o R2** (ii) **R_1o R_1^{-1}** Solution:

(i) The composition relation R1 o R2 as shown in fig:

R1 o R2 = {(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)}

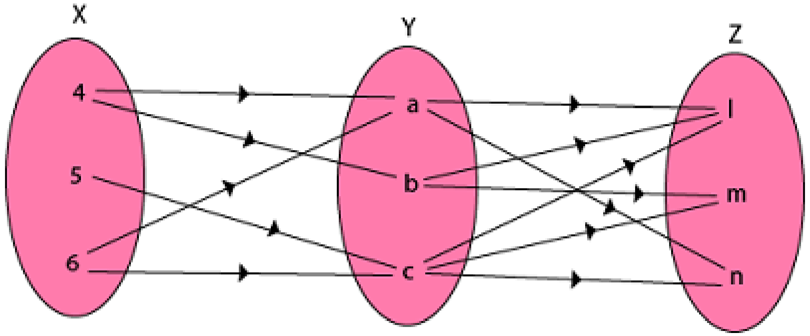


Fig : R_1 o R_2

(ii) The composition relation **R_1o R_1^{-1}** as shown in fig:

R_1o R_1^{-1} = {(4, 4), (5, 5), (5, 6), (6, 4), (6, 5), (4, 6), (6, 6)}

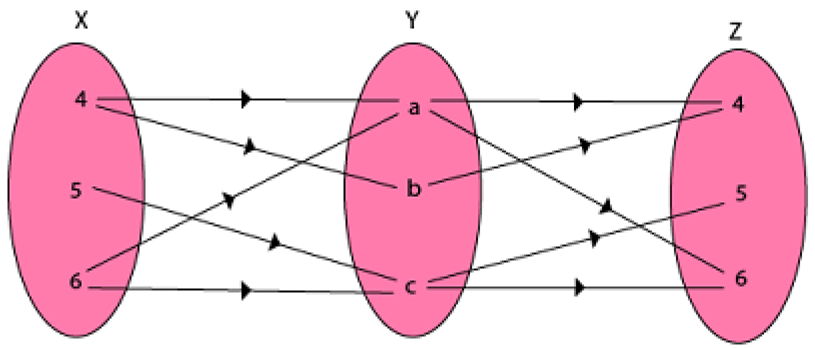


Fig : R_1 o R_1^{-1}

Questions :

1. How many binary relations are there on a set S with 9 distinct elements?

a) 2^{90}
b) 2^{100}
c) 2^{81}
d) 2^{60}

2. __________ number of reflexive relations are there on a set of 11 distinct elements.
a) 2^{110}
b) 3^{121}
c) 2^{90}
d) 2^{132}

3. The number of reflexive as well as symmetric relations on a set with 14 distinct elements is ___________
a) 4^{120}
b) 2^{70}
c) 3^{201}
d) 2^{91}

4. The number of symmetric relations on a set with 15 distinct elements is _______
a) 2^{196}
b) 2^{50}
c) 2^{320}
d) 2^{78}