Solution of Two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

Let $u(x,y) = X(x).Y(y)$ ——— (2)

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$

where $X(x)$ is function of x only & $Y(y)$ is function of y only.

$\therefore$ The p.d.e. (1) becomes

$$X''Y + XY'' = 0$$

$$\text{or} \quad \frac{X''}{X} = \frac{-Y''}{Y} = \text{constant} = \lambda \text{ (say)} \quad \text{——— (3)}$$

<u>Case I:</u> If $\lambda = 0$, then from (3),

$$X'' = 0, \quad Y'' = 0$$

$$\therefore X = Ax + B, \quad Y = Cy + D$$

$\therefore$ By (2), $\boxed{u(x,y) = (Ax+B)(Cy+D)}$ ——— (4)

<u>Case II:</u> If $\lambda > 0$ i.e., $\lambda = p^2;\ p > 0$, we have from (3),

$$X'' - p^2 X = 0, \quad Y'' + p^2 Y = 0$$

$$\therefore \quad X = Ae^{px} + Be^{-px}, \quad Y = C\cos py + D\sin py$$

$\therefore$ By (2), $\boxed{u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)}$

<u>Case III:</u> If $\lambda < 0$ i.e., $\lambda = -p^2;\ p > 0$, we have from (3),

$$X'' + p^2 X = 0, \quad Y'' - p^2 Y = 0$$

$$\therefore \quad X = (A\cos px + B\sin px), \quad Y = (Ce^{py} + De^{-py})$$

$\therefore$ By (2), $\boxed{u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})}$

Among these solutions, we are to find those solutions which satisfy initial and boundary conditions consistent with the physical nature.

Remark

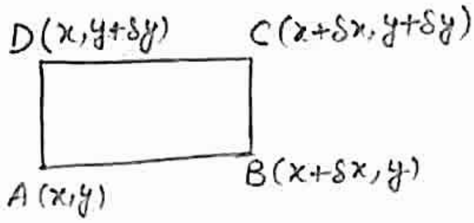
(1) In particular, if $u \to 0$ as $y \to \infty$ for all x, then solution must be

$$\boxed{u(x,y) = e^{-py}\left(E\cos px + F\sin px\right)} \; ; \; p>0$$

and if $u \to 0$ as $x \to \infty$ for all y, then solution must be

$$\boxed{u(x,y) = e^{-px}\left(E\cos py + F\sin py\right)} \; ; \; p>0$$

(2) Two dimensional heat equation is



$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state it reduces to Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Ques Find the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which satisfies the conditions

(i) $u \to 0$ as $y \to \infty$ for all x (ii) $u = 0$ at $x = 0$ for all y

(iii) $u = 0$ at $x = l$ for all y (iv) $u = lx - x^2$ if $y = 0$ for all $x \in (0, l)$

Solution Since $u \to 0$ as $y \to \infty$ for all x

$\therefore$ Solution is of the form

$$u(x,y) = e^{-py}\left(A\cos px + B\sin px\right) \; ; \; p>0 \quad \text{——— (1)}$$

$u(0,y) = 0 \;\; \forall \, y$ (given)

$\therefore$ By (1), $\quad Ae^{-py} = 0 \;\; \forall\, y \Rightarrow A = 0$

$\therefore$ By (1), $\quad u(x,y) = Be^{-py}\sin px \; ; \; p>0 \quad \text{——— (2)}$

Now, $u(l,y)=0 \quad \forall y$

$\therefore$ By ②, $\quad Be^{-py}\sin pl=0 \quad \forall y$

$\Rightarrow \sin pl=0 \Rightarrow p=\frac{n\pi}{l}$; $n=1,2,3,\ldots$

$\therefore$ By ②, $\quad u(x,y) = B_n e^{-\frac{n\pi y}{l}} \sin\frac{n\pi x}{l}$; $n=1,2,3,\ldots$

$\therefore$ By principle of superposition, solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi y}{l}} \sin\frac{n\pi x}{l} \qquad \text{—— ③}$$

Now, $u(x,0) = lx-x^2 \quad \forall x \in (0,l)$ (given)

$\therefore$ By ③, $\quad \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = lx-x^2$

which is half range Fourier sine series in $(0,l)$

$$\therefore B_n = \frac{2}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-(l-2x)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$B_n = \frac{-4l^2}{n^3\pi^3}\left[(-1)^n-1\right]$$

$$\therefore \left.\begin{aligned} B_{2n} &= 0 \\ B_{2n-1} &= \frac{8l^2}{(2n-1)^3\pi^3} \end{aligned}\right\} n=1,2,3,\ldots$$

$\therefore$ By ③, solution is

$$u(x,y) = \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} e^{-\frac{(2n-1)\pi y}{l}} \sin\frac{(2n-1)\pi x}{l}$$

A

Que 2 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions:

$u(0,y) = u(l,y) = u(x,0) = 0$ & $u(x,a) = \sin\frac{n\pi x}{l}$

Sol According to boundary conditions, the solution of given Laplace equation is given by

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \;;\; p>0 \quad \text{——— (1)}$$

Given $u(0,y) = 0$

$\therefore$ By (1) $\quad A(Ce^{py} + De^{-py}) = 0 \Rightarrow A = 0$

$\therefore$ By (1), $\quad u(x,y) = (Ee^{py} + Fe^{-py})\sin px \quad$ ——— (2)

where $BC = E$, $BD = F$

Now, $u(x,0) = 0$ (given)

$\therefore$ By (2) $\quad (E+F)\sin px = 0 \Rightarrow F = -E$

$\therefore$ By (2), solution is given by

$$u(x,y) = E(e^{py} - e^{-py})\sin px$$

$$\therefore u(x,y) = 2E\sinh py \sin px \quad \text{——— (3)}$$

Now, $u(l,y) = 0$ (given)

$\therefore \quad 2E\sinh py \sin pl = 0 \Rightarrow \sin pl = 0 \Rightarrow p = \frac{m\pi}{l};\; m = 1,2,3,\ldots$

$\therefore$ By (3), solutions are $u(x,y) = 2E_m \sin\frac{m\pi x}{l}\sinh\frac{m\pi y}{l};\; m = 1,2,\ldots$

$\therefore$ By principle of superposition, solution is

$$u(x,y) = \sum_{m=1}^{\infty} 2E_m \sin\frac{m\pi x}{l}\sinh\frac{m\pi y}{l} \quad \text{——— (4)}$$

Now, $u(x,a) = \sin\frac{n\pi x}{l}$ (given)

$$\therefore \sum_{m=1}^{\infty} 2E_m \sin\frac{m\pi x}{l}\sinh\frac{m\pi a}{l} = \sin\frac{n\pi x}{l}$$

$\Rightarrow \quad 2E_n \sinh\frac{n\pi a}{l} = 1 \quad$ and $E_m = 0 \;\; \forall\, m \neq n$

$$\therefore E_n = \frac{1}{2\sinh\frac{n\pi a}{l}}$$

$\therefore$ By (4), Solution is

$$u(x,y) = \frac{\sin\left(\frac{n\pi x}{l}\right)\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \qquad \underline{A}$$

Que3 A long rectangular plate of width π cm with insulated surfaces has its temperature equal to zero on both the long sides and one of the short side so that $u(0,y)=0$, $u(\pi,y)=0$, $u(x,\infty)=0$ and $u(x,0)=kx$. Find the steady state temperature within the plate.

Sol The steady state temperature within the plate is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since $u \to 0$ as $y \to \infty$ for all x, therefore solution is given by

$$u(x,y) = e^{-py}(A\cos px + B\sin px) \; ; \; p>0 \quad \text{——— (1)}$$

$u(0,y) = 0 \;\forall\; y$ (given)

$\therefore$ By (1), $\quad Ae^{-py} = 0 \;\; \forall\, y \implies A = 0$

$\therefore$ By (1), $\quad u(x,y) = Be^{-py}\sin px \;;\; p>0$ ——— (2)

Now, $u(\pi, y) = 0 \;\forall\, y$

$\therefore$ By (2), $\quad Be^{-py}\sin p\pi = 0 \;\; \forall\, y$

$\implies \sin p\pi = 0 \implies p = n, \; n = 1,2,3,\ldots$

$\therefore$ By (2), $\quad u(x,y) = B_n e^{-ny}\sin nx \;;\; n = 1,2,3,\ldots$

$\therefore$ By principle of superposition, solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-ny}\sin nx \quad \text{——— (3)}$$

Now, $u(x,0) = kx$

$\therefore$ By ③, $\displaystyle\sum_{n=1}^{\infty} B_n \sin nx = kx$

It is Fourier half range sine series in $[0, \pi]$

$$\therefore \quad B_n = \frac{2}{\pi}\int_0^{\pi} kx \sin nx \, dx$$

$$= \frac{2k}{\pi}\left[x \cdot \left(-\frac{\cos nx}{n}\right) - 1 \cdot \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2k}{\pi} \cdot \frac{\pi}{n}(-1)^{n+1}$$

$$\therefore \quad B_n = \frac{2k}{n}(-1)^{n+1} \quad ; \quad n = 1, 2, 3, \ldots$$

$\therefore$ By ③, solution is

$$\boxed{u(x,y) = 2k \sum_{n=1}^{\infty} (-1)^{n+1} e^{-ny} \frac{\sin nx}{n}}$$