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Maths Assignment - 1

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4C1 CSE

Q1 Solve the partial differential equation.

$$(D^2 - DD' - 2D'^2)z = (y-1)e^x$$

$D = m$

$D' = 1$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2$ & $m = -1$

Roots are real and distinct

CF: $f_1(y+2x) + f_2(y-x)$

PI: $\dfrac{1}{D^2 - DD' - 2D'^2} \cdot (y-1)e^x$

$\Rightarrow \dfrac{1}{(D-2D')(D+D')} * (y-1)e^x$

$\rightarrow \left[\dfrac{1}{D-2D'}\right]\left[\dfrac{1}{(D+D')} \cdot (y-1)e^x\right]$ —— ①

$D+D' = D - mD'$

$m = -1$

$y = c - mx \rightarrow y = c + x$ —— ②

put (2) in (1)

$$\frac{1}{(D-2D')}\left[\int (c+x-1)\, e^{x}\, dx\right]$$

$$\Rightarrow \frac{1}{(D-2D')}\left[(c+x-1)\, e^{x} - e^{x}\right]$$

$$\rightarrow \frac{1}{(D-2D')}\left[(y-1)\, e^{x} - e^{x}\right] \longleftarrow (3)$$

$$D-mD' = D-2D'$$

$$m=2$$

$$y = c-mx \quad \Rightarrow \quad y = c-2x \longleftarrow (4)$$

put (4) in (3)

$$\int \left[(c-2x-1)\, e^{x} - e^{x}\right] dx$$

$$\Rightarrow (c-2x-1)\, e^{x} - (-2)\cdot e^{x} - e^{x}$$

$$\Rightarrow (y-1)\, e^{x} + 2e^{x} - e^{x}$$

$$\Rightarrow y e^{x}$$

$$z = CF + PI$$

$$\Rightarrow f_1(y+2x) + f_2(y-x) + ye^{x}$$

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Q. Solve the partial differential equation.

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$$

→ Now, $D = m$ & $D' = -1$

$$m^3 - 7m - 6 = 0$$
$$(m-3)(m+2)(m+1) = 0$$
$$m = 3 \text{ or } -2 \text{ or } -1$$

roots are distinct

CF: $f(y+3x) + f(y-2x) + f(y-x)$

PI: $\dfrac{1}{D^3 - 7DD'^2 - 6D'^3} \left(\sin(x+2y) + e^{2x+y}\right)$

$\Rightarrow \dfrac{\sin(x+2y)}{D^3 - 7DD'^2 - 6D'^3} + \dfrac{e^{2x+y}}{D^3 - 7DD'^2 - 6D'^3}$

(1) (2)

put $D = 1$ & $D' = 2$ in (1)
put $D = 2$ & $D' = 1$ in (2)

$\rightarrow \dfrac{\sin(x+2y)}{1^3 - 7(1)(2)^2 - 6(2)^3} + \dfrac{e^{2x+y}}{2^3 - 7(2)1 - 6}$

$$\Rightarrow -\frac{\cos(x+2y)}{75} + \frac{e^{2x+y}}{(-12)}$$

$z = CF + PI$

$$\Rightarrow f(y+3x) + f(y-2x) + f(y-x)$$

$$-\frac{1}{75}\cos(x+2y) + \frac{e^{2x+y}}{(-12)}$$

Q8 Solve the partial differential equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$$

$f(D, D') = (D+D')(D+D'-2)$

the solution corresponding to the factor

$D - mD' - c$ is known to be

$$z = e^{cx}\,\phi(y+mx)$$

$$CF = \phi(y-x) + e^{2x} f_2(y-x)$$

$$PI = \frac{1}{D^2+2DD'+D'^2-2D-2D'} \times \sin(x+2y)$$

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$$\rightarrow \frac{1}{-1+2(-2)+(-4)-2D-2D'} \times \sin(x+2y)$$

$$\rightarrow \frac{1}{2(D+D'+9)} \sin(x+2y)$$

$$\rightarrow \frac{2(D+D')-9}{4(D^2+2DD'+D'^2)-81} \times \sin(x+2y)$$

$$\rightarrow \frac{1}{39}(2\cos(x+2y)-3\sin(x+2y))$$

Hence the complete solution is

$$Z = \phi(y-x) + e^{2x}\cdot\phi_2(y-x) + \frac{1}{39}\times \left[2\cos(x+2y) - 3\sin(x+2y)\right]$$

Q4 Solve the partial differential equation

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy$$

Sol. $(D+D')(D-2D'+2)z = e^{2x+3y} + xy$

$$CF = e^{0x}\phi(y-x) + e^{-2x}\phi_2(y-2x)$$

Next PI corresponding to e^{2x+3y}

$$\rightarrow \frac{1}{(D+D')(D-2D'+2)} e^{2x+3y}$$

$$\rightarrow \frac{1}{(2+3)(2-6+2)} e^{2x+3y} \Rightarrow \frac{-1}{10}\left(e^{2x+3y}\right)$$

P.I corresponding to xy

$$\rightarrow \frac{1}{(D+D')(D-2D'+2)} \times xy$$

$$\rightarrow \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left(1+\frac{1}{2}D-D'\right)^{-1} xy$$

$$\rightarrow \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left[1-\frac{D}{2}+D^2+\left(\frac{1}{4}D^2+D'^2-DD'\right)\right] \times xy$$

$$\rightarrow \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left[xy-\frac{y}{2}+x-1\right]$$

$$\rightarrow \frac{1}{2D}\left[1-\frac{D'}{D}+\frac{D'^2}{D}\cdots\right]\left(xy-\frac{y}{2}+x-1\right)$$

$$\rightarrow \frac{1}{2D}\left(1-\frac{D'}{D}\right)\left(xy-\frac{y}{2}+x-1\right)$$

$$\rightarrow \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{x^2}{2}+\frac{x}{2}\right]$$

$$\rightarrow \frac{1}{2}\left[\frac{x^2y}{2}-\frac{xy}{2}+\frac{x^2}{2}-x-\frac{x^3}{6}+\frac{x^2}{4}\right]$$

$$\rightarrow \frac{1}{2}\left[\frac{1}{2}x^2y-\frac{1}{2}xy-\frac{x^3}{6}+\frac{3}{4}x^2-x\right]$$

$$\rightarrow -\frac{1}{24}\left[2x^3 - 6x^2y - 9x^2 + 6xy + 12x\right)$$

$$Z = CF + PI$$

$$e^{-2x}\,\phi_2\,(y+2x) + \phi_1\,(y-x) - \frac{1}{10}\,e^{2x+3y}$$

$$-\frac{1}{24}\,(2x^3 - 6x^2y - 9x^2 + 6xy + 12x)$$

Q5 Using the method of separation of variables, solve

$$\frac{du}{dx} = \frac{2du}{dt} + u \quad , \text{ where } u(x,0) = 6e^{-3}$$

$$x > 0,\ t > 0$$

$$\frac{du}{dx} = \frac{2du}{dt} + u$$

Let the solution of this equation be

$$u(x,t) = X(x) \cdot T(t)$$

$$\rightarrow X'T = 2XT' + XT$$

$$(X'-X)T = 2XT'$$

$$\frac{(X'-X)}{X} = \frac{2T'}{T} = K$$

Solving $\frac{X'-X}{X} = K \quad \Rightarrow \frac{X'}{X} = K+1$

on integrating, $\log x = (K+1)x + \log C_1$

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$$\log X = (K+1)x + \log c_1$$

$$\log X = (k+1)x \log_e + \log c_1$$

$$\log X = \log_e (K+1)x + \log c_1$$

$$X = c_1 e^{(K+1)x}$$

Solving, $\frac{T'}{T} = K/2$

on integrating, $\log T = \frac{Kt}{2} + \log c_2$

$$T = c_2 e^{Kt/2}$$

$$\therefore u(x,t) = XT = c_1 e^{(K+1)x} \cdot c_2 e^{Kt/2}$$

$$= c_1 c_2 e^{(K+1)x + Kt/2}$$

when $t=0$

$$u(x,0) = c_1 c_2 e^{(K+1)x} = 6e^{-3x}$$

on comparing $c_1 c_2 = 6$

$$K+1 = -3$$

$$K = -4$$

Hence $u(x,t) = 6e^{-3x-2t}$

$$= 6e^{-(3x+2t)}$$

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Q6 Solve the equation $4\frac{du}{dx} + \frac{du}{dy} = 3u$

given $u = 3e^{-y} - e^{-5y}$ when $x = 0$

let the solution be $u(x,y) = X(x).Y(y)$

$$4X'Y + XY' = 3XY$$

$$4X'Y = (3Y - Y')X$$

$$\frac{4X'}{X} = \frac{3Y - Y'}{Y} = K$$

Solving $\frac{X'}{X} = \frac{K}{4}$

on integrating $\log X = \frac{Kx}{4} + \log C_1$

$$\Rightarrow X = C_1 e^{Kx/4}$$

Solving $\frac{3 - Y'}{Y} = K \Rightarrow \frac{Y'}{Y} = 3 - K$

on integrating $\log Y = (3-K)y + \log C_2$

$$\Rightarrow Y = C_2 e^{(3-K)y}$$

$$\therefore u(x,y) = C_1 C_2 e^{[Kx/4 + (3-K)y]}$$

when $x = 0$

$$\Rightarrow u(0,y) = C_1 C_2 e^{(3-K)y} = 3e^{-y} - e^{-5y}$$

on comparing $C_1 C_2 e^{(3-K)y} = 3e^{-y}$

$\Rightarrow c_1 c_2 = 3 \quad \& \quad 3-K = -1 \Rightarrow K = 4$

$$u(x,y) = 3e^{x-y} \quad \text{—— (I)}$$

Also comparing

$$c_1 c_2 e^{(3-K)y} = -e^{-5y}$$

$c_1 c_2 = -1 \quad \& \quad 3-K = -5 \Rightarrow K = 8$

$$u(x,y) = -e^{2x-5y} \quad \text{—— (II)}$$

from (I) and (II)

$$u(x,y) = 3e^{x-y} - e^{2x-5y}$$

Q. A thin uniform tightly stretched vibrating string fixed at the points $x=0$ & $x=l$ satisfies the equation $\frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2}$; $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

released from rest from this position. find the displacement $y(x,t)$ at any x & any time t.

boundary condition & initial conditions are

1) $y(0,t)$ 2) $y(l,t) = 0$

3) $\frac{dy}{dt} = 0$ at $t=0$ & 4) $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$

let us take $u = X(x) \cdot T(t)$

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos pt + c_4 \sin pt) \quad \text{—— (I)}$$

at $y(0,t) = 0$

$0 = (C_1 + 0)(C_3 \cos apt + C_4 \sin apt) \Rightarrow C_1 = 0$

① becomes $y = \sin px (C_3 \cos apt + C_4 \sin apt)$ — ②

now $y(l, t) = 0$

$0 = \sin pl (C_3' \cos apt + C_4' \sin apt)$

$\Rightarrow \sin pl = 0$

Hence $p = \frac{n\pi}{l}$ thus ② becomes

$y = \sin \frac{n\pi x}{l} \left(C_3' \cos \frac{an\pi t}{l} + C_4' \sin \frac{an\pi t}{l} \right)$ — ③

from ③ we can write

$\frac{dy}{dt} = \sin \frac{n\pi x}{l} \left[-\frac{an\pi}{l} C_3' \sin \frac{an\pi t}{l} + \frac{an\pi}{l} C_4' \cos \frac{an\pi t}{t} \right]$

Since $\frac{dy}{dt} = 0$ at $t = 0$, we get

$0 = C_4' \frac{an\pi}{l} \sin \frac{n\pi x}{l} \Rightarrow C_4' = 0$

$\therefore$ ③ becomes $y = C_3' \sin \frac{n\pi x}{l} \cos \frac{an\pi t}{l}$

$n = 1, 2, 3$

general solution is

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{a n \pi t}{l} \longrightarrow (4)$$

Now apply conditions (iv) on (4)

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

using $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$ we get

$$\frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + \cdots$$

This will be satisfied if

$$a_1 = \frac{3y_0}{4}, \quad a_2 = 0, \quad a_3 = -y_0/4, \quad a_4 = a_5 \cdots = 0$$

So final solution

$$y = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l}\cos\frac{c\pi t}{l} - \sin\frac{3\pi x}{l} \times \cos\frac{3\pi t c}{l}\right)$$

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Q. Solve the differential equation $\frac{d^2u}{dt^2} = 4\frac{d^2y}{dx^2}$ subject to the conditions $u = \sin t$ at $x=0$ and $\frac{du}{dx}$ $= \sin t$ at $x=0$

Ans $\frac{d^2y}{dt^2} = 2^2 \frac{d^2y}{dx^2}$

the suitable solution is

$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos 2pt + c_4 \sin 2pt)$

Applying initial conditions

$y(0,t) = \sin t$

$u(0,t) = c_1 (c_3 \cos 2pt + c_4 \sin 2pt)$

$\sin t = c_1 (c_3 \cos 2pt) + c_1 c_4 \sin 2pt$

$\Rightarrow c_3 = 0$

$p = 1/2 \quad , c_1 = 1 \quad \& \quad c_4 = 1$

now, $u(x,t) = (\cos px + c_2 \sin px)(\sin 2pt)$

$\frac{du}{dx} = -p \sin px \sin 2pt + pc_2 \cos px \sin 2pt$

$\sin t = \frac{c_2}{2} \sin 2pt$

$c_2 = 2$

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Therefore, we have

$u(x,t) = \sin t \left(\cos x/2 + 2 \sin x/2 \right)$

Q9 Sol: Temperature function $u(x,t)$ equation

$$\frac{du}{dt} = c^2 \frac{d^2u}{dx^2}$$ satisfies by solution

$$u(x,t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t}$$

now, we know, $u(0,t) = 0 = u(l,t)$

we have,

$$C_1 e^{-c^2p^2t} = 0 \Rightarrow C_1 = 0$$

&

$$C_2 \sin pl\, e^{-c^2p^2t} = 0$$

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$\therefore u = C_2 \sin \frac{n\pi x}{l}\, e^{-c^2p^2t}$$

$$\Rightarrow C_2 \sin \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l}}$$

where $n = 1, 2, 3, \ldots$

we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2\pi^2n^2t}{l^2}}$$

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Since $u(x,0) = u_0$ & $u_0 = \Sigma\ b_n \sin \frac{n\pi x}{l}$

which is a half range fourier series of u_0, hence

$$b_n = \frac{2}{l}\int_0^l u_0 \sin \frac{n\pi x}{l}\, dx$$

$$\Rightarrow \frac{-2u_0}{l}\left[\cos \frac{n\pi x}{l}\right]_0^l \cdot \frac{l}{n\pi} \times \frac{2u_0}{n\pi}(1-\cos n\pi)$$

$$\Rightarrow \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi}, & \text{when } n \text{ is odd} \end{cases}$$

Therefore $u(x,t) = \frac{4u_0}{\pi} \Sigma \frac{1}{n} \sin \frac{n\pi x}{l} e^{\frac{-c^2n^2\pi^2 t}{l^2}}$, $n = 1, 2, 3 ..$

or $u(x,t) = \frac{4u_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$

Q10 Sol: Boundary condition suggest that appropriate form of solution

$$u = (c_1 e^{py} + c_2 e^{-py})(c_3 \cos px + c_4 \sin px) \text{ --- } ①$$

according to $u \to 0$ as $y \to \infty$ gives $c_1 = 0$

$u = c_2 e^{-py}[(c_3 \cos px) + (c_4 \sin px)]$ or

$$u = e^{-py}(c_5 \cos px + c_6 \sin px) \text{ --- } ②$$

now, condition (ii)

$0 = e^{-py} (c_5 + 0) \Rightarrow c_5 = 0$

$\therefore$ we have $u = e^{-py} c_6 \sin px$

now,

condition (iii)

$e^{-py} \sin pl = 0 \Rightarrow \sin pl = 0 \rightarrow p = \frac{n\pi}{l}$

$\therefore u = c_6 e^{-n p y/l} \sin \frac{n\pi x}{l}$

for n = positive integer (General Sol)

$u = b_1 e^{-\pi y/l} \sin \frac{\pi x}{l} + b_2 e^{-2\pi y/l} \sin \frac{2\pi x}{l}$

$+ b_3 e^{-3\pi y/l} \sin \frac{3\pi x}{l} + \ldots$

b_1, b_2 are constants to be chosen to satisfy boundary condition

$u(x, 0) = lx - x^2 \quad , \quad 0 < x < l$

$lx - x^2 = \sum_{n=1}^{\infty} b_n \sin (n\pi x / l)$

half range fourier sine series

$b_n = \frac{2}{l} \int_0^l f(x) \sin \left(\frac{n\pi x}{l}\right) dx \; \cancel{= \frac{2}{l} \int_0^l}$

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$$\Rightarrow \frac{2}{l}\int_0^l (lx-x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow \frac{2}{l}\left[-(lx-x^2)\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\right]_0^l$$

$$-(l-2x)\cdot\cos\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\,dx$$

$$\Rightarrow 0+\frac{2}{n\pi}\int_0^l (1-2x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\rightarrow \frac{2}{n\pi}\left[(l-2x)\sin\left(\frac{n\pi x}{l}\right)\cdot\frac{l}{n\pi}\right]_0^l-\frac{2}{n\pi}\int_0^l(-2)\sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}$$

$$\rightarrow 0-\frac{4l}{(n\pi)^2}\left[\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right]_0^l$$

$$\rightarrow \frac{-4l}{(n\pi)^3}(\cos n\pi-1)=\begin{cases}0, & n \text{ is even}\\ \frac{8l}{(n\pi)^3}, & n \text{ is odd}\end{cases}$$

Thus $b_2=b_4=b_6=0$ & $b_1=\frac{8l^2}{\pi^3}$, $b_3=\frac{8l^2}{27\pi^3}$

$$u=\frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l}\cdot e^{-\pi y/l}+\frac{1}{3^2}\sin\frac{3\pi x}{l}e^{-3\pi y/l}\cdots\right.$$

$$\rightarrow \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-\frac{(2n-1)\pi y}{l}}\sin\frac{(2n-1)\pi x}{l}$$