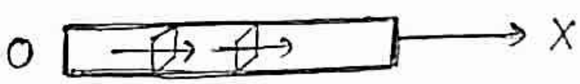
One Dimensional Heat Flow Equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Consider the flow of heat by conduction in a uniform bar. We assume the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible.

Take one end of the bar as origin and x-axis along the direction of flow of heat. Let $u(x,t)$ be temperature at a point distance x from origin at time t and the temperature at all points of a cross section is same. (Also, we know that in a body heat flows in a direction of decreasing temperature).



Then $u(x,t)$ is given by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

where c^2 is diffusivity of bar. (the amount of heat that passes through a given area in unit time).

Let its solution be given by

$$u(x,t) = X(x).T(t) \quad \text{——— (2)}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = X''T \text{ and } \frac{\partial u}{\partial t} = XT'$$

$\therefore$ The p.d.e. (1) becomes,

$$XT' = c^2 X'' T$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \text{constant} = \lambda \text{ (say)} \quad \text{——— (3)}$$

Case I If $\lambda = 0$, then from (3),

$$X'' = 0, \quad T' = 0$$

$$\therefore X = Ax + B, \quad T = C$$

$\therefore$ By (2), $\quad u(x,t) = C(Ax+B)$

$\therefore$ $\boxed{u(x,t) = Dx + E}$ where $D = CA$, $E = CB$ ——— (4)

Case II If $\lambda > 0$ i.e., $\lambda = p^2$, $p > 0$, we have from (3)

$$X'' - p^2 X = 0 \text{ and } T' - p^2c^2 T = 0$$

$\therefore$ $X = (Ae^{px} + Be^{-px})$ and $T = Ce^{p^2c^2t}$

$\therefore$ By (2), $\quad u(x,t) = (Ae^{px} + Be^{-px})\, C e^{p^2c^2t}$

$\therefore$ $\boxed{u(x,t) = (De^{px} + Ee^{-px})\, e^{p^2c^2t}}$ ——— (5)

Case III If $\lambda < 0$ i.e. $\lambda = -p^2$, $p > 0$ then from (3)

$$X'' + p^2 X = 0, \quad T' + p^2 c^2 T = 0$$

$\therefore$ $X = A\cos px + B\sin px$ and $T = Ce^{-p^2c^2t}$

$\therefore$ By (2), $\quad u(x,t) = (A\cos px + B\sin px)Ce^{-p^2c^2t}$

$\therefore$ $\boxed{u(x,t) = (D\cos px + E\sin px)\, e^{-p^2c^2t}}$ ——— (6)

Out of these three solutions, we are to choose solutions satisfying initial and boundary conditions. Since the temperature $u(x,t)$ decreases as the time t increases (as we are dealing with problems on heat conduction, it must be a transient solution), therefore the solution given by (5) can be rejected.

Remark (1) The solution $Dx+E$ is the solution of $\frac{\partial^2 u}{\partial x^2} = 0$ and hence $\frac{\partial u}{\partial t} = 0$ in heat equation (1). Thus, solution $Dx+E$ is steady state solution.

(2) If $u(0,t) = u(l,t) = 0 \ \forall t$, then there is no steady state solution and $u(x,t)$ decreases as time t increases and hence in this case solution is given by equ. (6).

[by (4), $u(0,t) = 0 \Rightarrow E = 0$ and $u(l,t) = 0 \Rightarrow Dl + E = 0$
$\therefore$ $D = E = 0$, So no steady state solution exist.]

Que) A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0^\circ c$, and are kept at that temperature. Find the temperature $u(x,t)$.

Sol The temperature $u(x,t)$ is given by the solution of one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{—— (1)}$$

Given $u(x,0) = u_0$, $u(0,t) = 0$, $u(l,t) = 0$

Since $u(0,t) = u(l,t) = 0$, therefore the solution of equ.(1) is given by $u(x,t) = (A\cos px + B\sin px)\, e^{-p^2c^2t}$; $p > 0$ —— (2)

Now, $u(0,t) = 0 \Rightarrow A e^{-p^2c^2t} = 0 \Rightarrow A = 0$

$\therefore$ By (2), $u(x,t) = B\sin px\, e^{-p^2c^2t}$; $p > 0$ —— (3)

Now, $u(l,t) = 0 \Rightarrow B\sin pl\, e^{-p^2c^2t} = 0$

$\Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

$\therefore$ By (3), $u(x,t) = B_n \sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2}{l^2}t}$; $n = 1, 2, 3, \ldots$

By the principle of superposition,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{n^2c^2\pi^2}{l^2}t} \quad \text{——— (4)}$$

Now, $u(x,0) = u_0 \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = u_0$

which is half range Fourier sine series of u_0 in $(0,l)$.

$$\therefore\ B_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2u_0}{l}\cdot\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore\ \left.\begin{aligned} B_{2n} &= 0 \\ B_{2n-1} &= \frac{4u_0}{(2n-1)\pi}\end{aligned}\right\}\ n = 1,2,3,---$$

$\therefore$ By (4),

$$\boxed{u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}\, e^{-\frac{(2n-1)^2c^2\pi^2}{l^2}t}}$$

Que2 (a) An insulated rod of length l has its ends A and B maintained at 0° and 100°c respectively until steady state conditions prevail. If B is suddenly reduced to 0°c and are maintained at 0°c, find the temperature at a distance x from A at time t.

(b) Solve the above problem, if the change consists of raising the temperature of A to 20°c and reducing that of B to 80°c.

Sol (a) The temperature $u(x,t)$ is given by the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

Before changing the temperature at the end B,
When $t=0$, the heat flow was in steady state (i.e, independent of time) i.e, $\frac{\partial u}{\partial t}=0$. So by (1), $\frac{\partial^2 u}{\partial x^2}=0$

$$\therefore u(x,0)=Ax+B$$

Now $u(0,0)=0$ and $u(l,0)=100$

$$\therefore B=0 \text{ and } A=\frac{100}{l}$$

$$\therefore u(x,0)=\frac{100x}{l}$$

Now, for the subsequent flow,

$$u(0,t)=0 \ \forall t \text{ and } u(l,t)=0 \ \forall t$$

$\therefore$ Solution of (1) is of the form

$$u(x,t)=(A\cos px+B\sin px)e^{-p^2c^2t};\ p>0 \quad \text{——} (2)$$

[Now, do the same steps upto equ.(4) as in above que.(1)]

$$\therefore u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2c^2\pi^2}{l^2}t} \quad \text{——} (4)$$

Now, $u(x,0)=\frac{100x}{l}$

$$\therefore \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}=\frac{100x}{l}$$

which is half range Fourier sine series of $\frac{100x}{l}$ in $(0,l)$

$$\therefore B_n=\frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{200}{l^2}\left[x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$=\frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right]=\frac{200}{n\pi}(-1)^{n+1}$$

$$\therefore \text{By (4)},\quad u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{n^2c^2\pi^2}{l^2}t}$$

Ans

(b) Here the Initial condition

$$u(x,0) = \frac{100x}{l}$$

remains the same as in (a) part.

Also, the boundary conditions are

$$u(0,t) = 20 \ \forall t \quad \text{and} \quad u(l,t) = 80 \ \forall t$$

Since the boundary values $u(0,t)$ and $u(l,t)$ are given to be non-zero, so we split the temperature function $u(x,t)$ into two parts as

$$u(x,t) = u_s(x) + u_t(x,t) \quad \text{———— (2)}$$

where $u_s(x) = Ax+B$ s.t. $u_s(0) = 20$ and $u_s(l) = 80$

$$\therefore B = 20, \ A = \frac{60}{l}$$

$$\therefore u_s(x) = \frac{60x}{l} + 20 \quad \text{———— (3)}$$

Put $x = 0$ in (2), we have

$$u_t(0,t) = u(0,t) - u_s(0) = 20 - 20 = 0 \quad \text{———— (4)}$$

Put $x = l$ in (2), we have

$$u_t(l,t) = u(l,t) - u_s(l) = 80 - 80 = 0 \quad \text{———— (5)}$$

Also, by (2), $u_t(x,0) = u(x,0) - u_s(x)$

$$= \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \quad \left(\because u(x,0) = \frac{100x}{l}\right)$$

$$= \frac{40x}{l} - 20 \quad \text{———— (6)}$$

Now $u_t(x,t)$ is given by

$$u_t(x,t) = (A\cos px + B\sin px)\, e^{-p^2c^2t} \ ; \ p > 0$$

where $u_t(0,t)=0$, $u_t(l,t)=0$ and $u_t(x,0)=\frac{40x}{l}-20$

[Now, do the same steps as in que (1) upto equation (4)]

$$\therefore \quad u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2c^2\pi^2}{l^2}t} \qquad \text{--- (7)}$$

Now, $u_t(x,0) = \frac{40x}{l} - 20$

$$\therefore \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{40x}{l} - 20$$

which is half range Fourier sine series of $\frac{40x}{l}-20$ in $(0,l)$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l}-20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{40x}{l}-20\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{40}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left(-\frac{l}{n\pi}\right)\left[20\cos n\pi + 20\right] = \frac{-40}{n\pi}\left[(-1)^n+1\right]$$

$$\therefore \quad B_{2n} = \frac{-80}{2n\pi} \text{ and } B_{2n-1}=0 \quad \Big\} \; n=1,2,3,\ldots$$

$$\therefore \text{ By (7)}, \quad u_t(x,t) = -\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l} e^{-\frac{4n^2c^2\pi^2}{l^2}t}$$

$\therefore$ By (2), (3) and (4),

$$u(x,t) = \frac{60x}{l} + 20 - \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}\, e^{-\frac{4n^2c^2\pi^2}{l^2}t}$$

Que3 A bar of length l with insulated sides is initially at $0^\circ c$ temperature throughout. The end $x=0$ is kept at $0^\circ c$ for all time and heat is suddenly applied such that $\frac{\partial u}{\partial x} = 10$ at $x = l$ for all time. Find the temperature function $u(x,t)$.

Sol The temperature function $u(x,t)$ is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ where c^2 is diffusivity of bar.

Since the temperatures at both end points are not given to be $0^\circ c$, therefore we split $u(x,t)$ into two parts as

$$u(x,t) = u_s(x) + u_t(x,t) \quad \text{———} (1)$$

where $u_s(x) = Ax + B$ s.t. $u_s(0) = 0$ and $\left(\frac{\partial u_s}{\partial x}\right)_{x=l} = 10$

$\therefore \quad B = 0$ and $A = 10$

$\therefore \quad u_s(x) = 10x$

Also, boundary conditions are given as

$u(0,t) = 0 \; \forall t$, $\left(\frac{\partial u}{\partial x}\right)_{x=l} = 10 \quad \forall t$

and initial condition as $u(x,0) = 0 \quad \forall x$

By (1), $\quad u(x,t) = 10x + (A\cos px + B\sin px)e^{-p^2c^2t} \quad (p>0)$ ——— (2)

($\because$ temperature at $x=l$ is not given)

$\therefore \quad u(0,t) = 0 \quad \forall t$

$\Rightarrow \quad A e^{-p^2c^2t} = 0 \; \forall t \Rightarrow A = 0$

$\therefore$ By (2), $\quad u(x,t) = 10x + B\sin px \, e^{-p^2c^2t} \quad (p>0)$ ——— (3)

Now, $\left(\frac{\partial u}{\partial x}\right)_{x=l} = 10$ (given)

$\therefore\ 10 + Bp\cos pl\, e^{-p^2c^2t} = 10$

$\Rightarrow Bp\cos pl\, e^{-p^2c^2t} = 0 \Rightarrow \cos pl = 0$

$\Rightarrow p = \frac{(2n-1)\pi}{2l}$; $n = 1, 2, 3, ---$

$\therefore$ By principle of superposition,

$$u(x,t) = 10x + \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l}\, e^{-\frac{(2n-1)^2\pi^2c^2t}{4l^2}} \quad \text{---} (4)$$

by (3)

Now, $u(x,0) = 0\ \forall x$

$$\therefore\ \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l} = -10x$$

which is half range Fourier sine series of $-10x$ in $[0,l]$

$$\therefore\ B_n = \frac{2}{l}\int_0^l (-10x)\sin\frac{(2n-1)\pi x}{2l}\,dx$$

$$= \frac{-20}{l}\left[x\left\{\frac{-2l}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2l}\right\} - (1).\left\{\frac{-4l^2}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi x}{2l}\right\}\right]_0^l$$

$$= \frac{-20}{l}\cdot\frac{4l^2}{(2n-1)^2\pi^2}\sin(2n-1)\frac{\pi}{2}$$

$$= \frac{80l}{(2n-1)^2\pi^2}(-1)^n$$

$\left[\because \sin\left(n\pi - \frac{\pi}{2}\right) = -\cos n\pi = -(-1)^n\right]$

$\therefore$ Solution is

$$u(x,t) = 10x + \frac{80l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2l}\, e^{-\frac{(2n-1)^2\pi^2c^2t}{4l^2}}$$