Partial Differential Equations Linear and Homogeneous in Partial Derivatives with Constant Coefficients

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + k_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

is linear and homogeneous in partial derivatives of order n and has constant coefficients.

If we write $\frac{\partial^r}{\partial x^r} \equiv D^r$ and $\frac{\partial^r}{\partial y^r} = D'^r$ the equation can be written as

$$f(D, D')z = F(x,y) \qquad (1)$$

where $f(D, D') = D^n + k_1 D^{n-1} D' + k_2 D^{n-2} D'^2 + \cdots + k_n D'^n$

The complete solution of (1) is given by

$$z = \text{complementary function (C.F.)} + \text{particular integral (P.I.)}$$

Rules to write complementary function (C.F.)

C.F. is the complete solution of

$$f(D, D')z = 0$$

1. Replacing D by m and D' by 1, auxiliary equation is

$$m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n = 0 \qquad (2)$$

Let the roots of this equation be $m_1, m_2, \ldots, m_n$.

Case I When all roots are different

$$\text{C.F.} = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx)$$

Case II If two roots of equation (2) are equal say $m_1 = m_2 = m$ and all others are different

$$\text{C.F.} = \phi_1(y+mx) + x\phi_2(y+mx) + \phi_3(y+m_3x) + \cdots + \phi_n(y+m_nx)$$

Similarly if $m_1 = m_2 = m_3 = m$ and others are different then

$$\text{C.F.} = \phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx) + \phi_4(y+m_4x) + \cdots + \phi_n(y+m_nx)$$

<u>Que</u> Solve the following partial differential equations

(i) $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

(ii) $(D^3 - D^2D' - DD'^2 + D'^3)z = 0$ or $(D_x^3 - D_x^2 D_y - D_x D_y^2 + D_y^3)z = 0$

<u>Sol</u> (i) The given p.d.e. can be written symbolically as

$$(D^4 - D'^4)z = 0$$

A.E. is $m^4 - 1 = 0 \Rightarrow (m^2-1)(m^2+1) = 0$

$\Rightarrow m = \pm 1, \pm i$

$\therefore$ General solution is $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(ii) p.d.e. is $(D^3 - D^2D' - DD'^2 + D'^3)z = 0$

A.E. is $m^3 - m^2 - m + 1 = 0$

$\Rightarrow m^2(m-1) - 1(m-1) = 0$

$\Rightarrow (m-1)(m^2-1) = 0 \Rightarrow m = 1, 1, -1$

$\therefore$ General solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-x)$$

<u>Particular Integral (P.I.)</u>

If $f(D,D')z = F(x,y)$ then

$$P.I. = \frac{1}{f(D,D')} F(x,y)$$

<u>Formulae regarding P.I.</u>

(1) If $F(x,y) = e^{ax+by}$, then

$$P.I. = \frac{1}{f(D,D')} e^{ax+by} = \begin{cases} \frac{1}{f(a,b)} e^{ax+by} & \text{if } f(a,b) \neq 0 \\ \frac{x^k}{\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,\,D'=b}} & \text{if } f(a,b) = 0 \end{cases}$$

where k is the least +ive integer so that it reduces to above form when $\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,\,D'=b} \neq 0$

(2) If $F(x,y) = \sin(ax+by+c)$ or $\cos(ax+by+c)$, then

$$P.I. = \frac{1}{f(D,D')} \underset{\cos}{\sin}(ax+by+c)$$

$$= \frac{1}{[f(D,D')]_{D^2=-a^2,\, D'^2=-b^2,\, DD'=-ab}} \underset{\cos}{\sin}(ax+by+c)$$

provided den $\neq 0$

(3) If $F(x,y)$ = polynomial in x & y

$$P.I. = [f(D,D')]^{-1} \cdot F(x,y)$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and operate on $F(x,y)$.

(4) If $F(x,y) = e^{ax+by} V(x,y)$, then

$$P.I. = \frac{1}{f(D,D')} e^{ax+by} V(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x,y)$$

(5) If $F(x,y) = \phi(ax+by)$ and $f(D,D')$ is a homogeneous function of degree n (say) then

$$P.I. = \frac{1}{f(D,D')} \phi(ax+by) = \frac{1}{f(a,b)} G(ax+by) \quad \text{if } f(a,b) \neq 0$$

where $G(ax+by)$ is obtained after integrating $\phi(z)$ w.r.t. z, n times and then taking $z = ax+by$.

and $$P.I. = \frac{x^k}{\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,\, D'=b}} \quad \text{if } f(a,b) = 0$$

where k is the least positive integer so that it reduces to above form when $\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,\, D'=b} \neq 0$

Now apply the above formula.

Remark (1) If $F(x,y) = \sinh(ax+by+c)$ or $\cosh(ax+by+c)$ then convert it in exponential forms and use case 1.

(2) Sometimes it is easy to use the formula

General Formula $\quad \frac{1}{(D-mD')} F(x,y) = \int F(x, a-mx)\,dx$

where $y = a - mx$

After integrating it we put $a = y + mx$.

Que Solve the following p.d.e's

(1) $(D^2+4DD'-5D'^2)z = \sin(2x+3y) + 3e^{2x+y}$

(2) $(D^3-7DD'^2-6D'^3)z = \sin(x+2y)$

(3) $(2D^2-5DD'+2D'^2)z = 5\sin(2x+y)$

(4) $4r+12s+9t = e^{3x-2y}$

(5) $r-2s+t = 2x\cos y$

(6) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

(7) $r-s-2t = (y-1)e^x$

Sol (1) $(D^2+4DD'-5D'^2)z = \sin(2x+3y) + 3e^{2x+y}$

A.E. is $\quad m^2+4m-5=0 \Rightarrow (m+5)(m-1)=0$

$\Rightarrow m = 1, -5$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-5x)$

$$\text{P.I.} = \frac{1}{(D^2+4DD'-5D'^2)}\left[\sin(2x+3y) + 3e^{2x+y}\right]$$

$$= \frac{1}{(D^2+4DD'-5D'^2)}\sin(2x+3y) + 3\cdot\frac{1}{(D^2+4DD'-5D'^2)}e^{2x+y}$$

$$= \frac{1}{-4-24+45}\sin(2x+3y) + 3.\frac{1}{(4+8-5)}e^{2x+y}$$

$$= \frac{1}{17}\sin(2x+3y) + \frac{3}{7}e^{2x+y}$$

Hence the complete solution is

$$\boxed{z = \phi_1(y+x) + \phi_2(y-5x) + \frac{1}{17}\sin(2x+3y) + \frac{3}{7}e^{2x+y}}$$ A

(2) $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y)$

A.E. is $m^3 - 7m - 6 = 0$

$m = -1$ satisfies it.

$\therefore\ m^3 - 7m - 6 = m^2(m+1) - m(m+1) - 6(m+1) = 0$

$\Rightarrow (m+1)(m^2 - m - 6) = 0$

$\Rightarrow (m+1)(m-3)(m+2) = 0$

$\Rightarrow m = -2, -1, 3$

$\therefore$ C.F. $= \phi_1(y-2x) + \phi_2(y-x) + \phi_3(y+3x)$

P.I. $= \dfrac{1}{(D^3 - 7DD'^2 - 6D'^3)}\sin(x+2y)$

$= \dfrac{1}{1^3 - 7(1)(2)^2 - 6(2)^3}\cos(x+2y)$

(after integrating $\sin z$ w.r.t. z three times and putting $z = x+2y$, $D = 1$, $D' = 2$)

$= -\dfrac{1}{75}\cos(x+2y)$

Hence the complete solution is

$$\boxed{z = \phi_1(y-2x) + \phi_2(y-x) + \phi_3(y+3x)}\ -\frac{1}{75}\cos(x+2y)$$

Other Method to find P.I.

$\dfrac{1}{(D^3 - 7DD'^2 - 6D'^3)}\sin(x+2y)$

$= \dfrac{1}{(-D + 14D' + 24D')}\sin(x+2y)$

$= \dfrac{(-D - 38D')}{(-D + 38D')(-D - 38D')}\sin(x+2y)$

$= \dfrac{(-D - 38D')}{D^2 - (38)^2 D'^2}\sin(x+2y)$

$= \dfrac{(-D - 38D')}{-1 + 4 \times 38 \times 38}\sin(x+2y)$

$= \dfrac{-\cos(x+2y) - 76\cos(x+2y)}{5775}$

$= -\dfrac{1}{75}\cos(x+2y)$

(3) $(2D^2-5DD'+2D'^2)z = 5\sin(2x+y)$

A.E. is $2m^2-5m+2=0$

$\Rightarrow m = \frac{5\pm\sqrt{25-16}}{4} = \frac{5\pm3}{4} = \frac{1}{2}, 2$

$\therefore$ C.F. $= \phi_1(y+\frac{1}{2}x) + \phi_2(y+2x) = \psi_1(2y+x)+\psi_2(y+2x)$

P.I. $= \frac{1}{(2D^2-5DD'+2D'^2)} \cdot 5\sin(2x+y)$

$= \frac{5x}{(4D-5D')}\sin(2x+y) = 5x\frac{1}{4(2)-5(1)}[-\cos(2x+y)]$

$= \frac{-5x}{3}\cos(2x+y)$

Hence the solution is

$$\boxed{z = \psi_1(2y+x)+\psi_2(y+2x) - \frac{5x}{3}\cos(2x+y)}$$

(4) $4r+12s+9t = e^{3x-2y}$

The given equation can be written as

$(4D^2+12DD'+9D'^2)z = e^{3x-2y}$

A.E. is $4m^2+12m+9=0 \Rightarrow (2m+3)^2=0 \Rightarrow m = -\frac{3}{2}, -\frac{3}{2}$

$\therefore$ C.F. $= \phi_1(y-\frac{3}{2}x) + x\phi_2(y-\frac{3}{2}x) = \psi_1(2y-3x)+x\psi_2(2y-3x)$

P.I. $= \frac{1}{(4D^2+12DD'+9D'^2)}e^{3x-2y}$

$= \frac{1}{(2D+3D')^2}e^{3x-2y}$

$= \frac{x^2}{\frac{d^2}{dD^2}(2D+3D')^2}e^{3x-2y}$ $[\because 2.3+3.(-2)=0]$

$= \frac{x^2}{8}e^{3x-2y}$

$\therefore$ Complete sol. is $\boxed{z = \psi_1(2y-3x)+x\psi_2(2y-3x)+\frac{x^2}{8}e^{3x-2y}}$

(5)

$r - 2s + t = 2x\cos y$

The given equ. can be written as

$(D^2 - 2DD' + D'^2)z = 2x\cos y$

A.E. is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$ C.F. $= \phi_1(y+x) + x\phi_2(y+x)$

$$\text{P.I.} = \frac{1}{(D-D')^2} 2x\cos y$$

$$= 2 \text{ Re part of } \frac{1}{(D-D')^2} x e^{iy}$$

$$= 2 \text{ Re part of } e^{iy} \frac{1}{(D-D'-i)^2} x$$

$$= 2 \text{ Re part of } e^{iy}(-1)[1 + i(D-D')]^{-2} \cdot x$$

$$= 2 \text{ Re part of } (-e^{iy})[1 - 2iD + \cdots]x$$

$$= 2 \text{ Re part of } (-1)(\cos y + i\sin y)(x - 2i)$$

$$= -2(x\cos y + 2\sin y)$$

$\therefore$ The complete solution is

$$\boxed{y = \phi_1(y+x) + x\phi_2(y+x) - 2(x\cos y + 2\sin y)}$$

(6) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

The given equ. can be written as

$(D^2 + DD' - 6D'^2)z = y\cos x$

A.E. is $m^2 + m - 6 = 0 \Rightarrow (m-2)(m+3) = 0 \Rightarrow m = 2, -3$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-3x)$

$$\text{P.I.} = \frac{1}{(D^2 + DD' - 6D'^2)} y\cos x = \frac{1}{(D+3D')(D-2D')} y\cos x$$

$$= \frac{1}{(D+3D')} \int (a-2x)\cos x\, dx \qquad \text{where } y = a-2x$$

$$= \frac{1}{(D+3D')} [(a-2x)\sin x - (-2)(-\cos x)]$$

$$= \frac{1}{(D+3D')} [y\sin x - 2\cos x]$$

$$= \int [(b+3x)\sin x - 2\cos x]\, dx \qquad \text{where } y = b+3x$$

$$= [(b+3x)(-\cos x) - (3)(-\sin x) - 2\sin x]$$

$$= -y\cos x + 3\sin x - 2\sin x = -y\cos x + \sin x$$

$\therefore$ The complete sol. is

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x}$$

(7) $r - s - 2t = (y-1)e^x$

The given eqn. can be written as

$$(D^2 - DD' - 2D'^2)z = (y-1)e^x$$

A.E. is $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m = 2, -1$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-x)$

$$\text{P.I.} = \frac{1}{(D^2 - DD' - 2D'^2)} (y-1)e^x$$

$$= \frac{e^x}{(D+1)^2 - (D+1)D' - 2D'^2} (y-1)$$

$$= \frac{e^x}{D^2 - DD' - 2D'^2 + 2D - D' + 1} (y-1)$$

$$= e^x [1 + (2D - D' - 2D'^2 - DD' + D^2)]^{-1} (y-1)$$

$$= e^x [1 - 2D + D' + \cdots](y-1) = e^x(y-1+1) = ye^x$$

$\therefore$ The complete sol. is

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-x) + ye^x}$$