Probability Probability of a given event is an expression of like or chance of occurrence of an event. Probability is a real number which ranges from 0 to 1; 0 for an event which cannot occur and 1 for an event certain to occur.

Random Experiment An experiment whose results can never be predicted or determined in advance is called a random experiment. Tossing a coin and throwing a dice are examples of random experiments.

Trial and events The experiment is known as a trial and the outcomes are known as events or cases.

Sample space, Sample points and events The set of all possible outcomes of a random experiment is called sample space and each member of this sample space is called sample point. A sample space is usually denoted by S. An event is a subset of sample space.

Ex If a coin is tossed twice then sample space $S = \{HT, TH, TT, HH\}$. If we let A denotes occurence of one head and one tail then only two elements HT and TH belong to event A.

Mutually exclusive events Events are said to be mutually exclusive or incompatible if ~~the happening of any one of~~ no two or more of them can happen simultaneously in the same trial. So if A and B are two mutually exclusive events then $A \cap B = \phi$.

Ex In tossing a coin the events head and tail are mutually exclusive.

Complementary Events Let A and B be two mutually exclusive events then A is called complementary event of event B and vice-versa.

Exhaustive events The total no. of possible outcomes in any trial is known as exhaustive events or exhaustive cases. For ex: In tossing a coin there are two exhaustive cases, viz, head and tail. (the possibility of the coin standing on an edge being ignored)

Equally likely events If one of the events cannot be expected to happen in preference to another then such events are said to be equally likely. For instance, in tossing a coin, the coming of the head or the tail is equally likely.

Mathematical or Classical or 'a priori' Probability

If there are n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to an event A, then the probability 'p' of happening of A is given by

$$p = P(A) = \frac{\text{Favourable no. of cases}}{\text{Exhaustive no. of cases}} = \frac{m}{n} \quad \text{———} (1)$$

Sometimes we express (1) by saying that 'the odds in favour of A are $m:(n-m)$ or the odd against A are $(n-m):m$

Clearly, the probability 'q' of non-happening of A is given by

$$q = P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

$$\therefore \; p + q = 1 \quad \text{or} \quad P(A) + P(A') = 1$$ or we write $P(A) + P(\bar{A}) = 1$

Remark (1) If $P(A) = 1$, A is called a certain event and if $P(A) = 0$, A is called an impossible event.

(2) $P(\phi) = 0$

(3) The above definition of probability fails,

(i) If the exhaustive no. of cases in a trial is infinite.

(ii) If ~~when~~ the various outcomes of the trial are not equally likely.

For ex: The probability that a candidate will pass in a certain test is not 50% since the two possible outcomes; success and failure are not equally likely.

Statistical (or Empirical) definition of probability

If in n trials, an event A happens m times, then the probability 'p' of happening of A is given by

$$p = P(A) = \lim_{n\to\infty} \frac{m}{n}$$

Axioms Let S be a sample space and Σ^* the set of all events (subsets of S) then the probability function $P(A)$, $A \in \Sigma$ satisfies the following three axioms:

(I) $P(A) \geq 0 \;\forall\; A \in \Sigma$

(II) $P(S) = 1$

(III) If $\{A_n\}$ is any finite or infinite sequence of disjoint events in Σ, then $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$

Remark In the above definition Σ is a (σ-algebra or σ-field) i.e, Σ is a non-empty class of sets s.t. $A \in \Sigma \Rightarrow \bar{A} \in \Sigma$ and $A_i \in \Sigma\ (i=1,2,\dots) \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$

Simple results based on above axioms

(i) $P(\phi) = 0$ $[S \cup \phi = S \Rightarrow P(S\cup\phi) = P(S) \Rightarrow P(S) + P(\phi) = P(S) \Rightarrow P(\phi) = 0]$

(ii) $P(\bar{A}) = 1 - P(A)$ $[A \cup \bar{A} = S \Rightarrow P(A) + P(\bar{A}) = P(S) = 1]$

(iii) $0 \leq P(A) \leq 1$ $[P(A) = 1 - P(\bar{A})$ & $P(\bar{A}) \geq 0,\ P(A) \geq 0 \Rightarrow P(A) \leq 1]$

(iv) For any two events A and B

(a) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

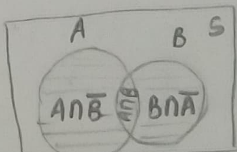
Similarly (b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(c) If $B \subset A$, then

(i) $P(A \cap \bar{B}) = P(A) - P(B)$

(ii) $P(B) \leq P(A)$

(a)

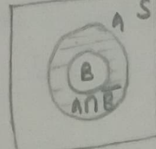


$(\bar{A} \cap B) \cup (A \cap B) = B$ and $(\bar{A} \cap B) \cap (A \cap B) = \phi$

$\therefore$ (a) follows using axiom III

Similarly we can prove (b) result.

c (i)



$(A \cap \bar{B}) \cap B = \phi$

and $(A \cap \bar{B}) \cup B = A$

$\therefore$ c(i) follows from axiom III

(ii) $P(A \cap \bar{B}) \geq 0$

So, $P(A) - P(B) \geq 0$

Two ~~basic~~ fundamental counting principles

(1) Sum rule ('Or' rule)

(2) Product rule ('And' rule)

<u>Sum rule</u> If $A_1, A_2, ---, A_n$ are n disjoint events and can occur in $x_1, x_2, ---, x_n$ ways respectively then total no. of ways so that A_1 or A_2 or A_3 or - or A_n can occur

$$= x_1 + x_2 + -- + x_n \text{ ways}$$

<u>Product rule</u> If $A_1, A_2, --, A_n$ are n disjoint events and can occur in $x_1, x_2, -- x_n$ ways respectively then the sequence of event A_1, followed by A_2, -----, followed by A_n can occur

$$= x_1 . x_2 ---. x_n \text{ ways}$$

<u>Que</u> In how many ways can we draw a king or a queen from an ordinary pack of playing cards?

<u>A</u> Since there are 4 kings and 4 queens, so we may draw a king or a queen in $4+4=8$ ways.

<u>Que</u> Ten students participate in a race competition. In how many ways can the first three prizes be won if a student cannot win more than one prize?

<u>A</u> Here first prize can be won by any one of 10 students by 10 ways,
second " " " " " " the remaining 9 " " 9 ",
and the third " " " " " " 8 " " 8 "

So, by fundamental counting principle,

total no. of ways $= 10 \times 9 \times 8 = 720$ ways

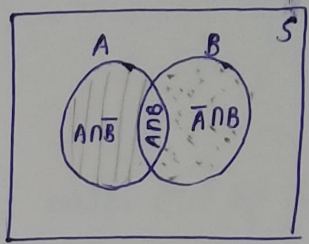
Addition law of probability or Theorem of Total Probability

For any two non-disjoint events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{or} \quad P(A+B) = P(A) + P(B) - P(AB)$$

and if A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$



Proof Clearly

$$A \cap (\bar{A} \cap B) = \phi$$

and $A \cup (\bar{A} \cap B) = A \cup B$

$\therefore \quad P[A \cup (\bar{A} \cap B)] = P(A \cup B)$

$\Rightarrow \quad P(A) + P(\bar{A} \cap B) = P(A \cup B)$

$\Rightarrow \quad \boxed{P(A) + P(B) - P(A \cap B) = P(A \cup B)}$

If $A \cap B = \phi$, then $P(A \cap B) = P(\phi) = 0$

$\therefore \quad P(A \cup B) = P(A) + P(B)$

Remark The above theorem can be extended for any finite no. of events. In particular

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

(if A, B, C are not mutually exclusive events)

and $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

(if A, B, C are mutually exclusive events)

Que A bag contains 5 white, 6 black, and 6 yellow balls. Three balls are drawn at random. Find the chance that of the drawn balls

(i) all are black [A $\frac{{}^6C_3}{{}^{17}C_3} = \frac{1}{34}$]

(ii) exactly two yellow [A $\frac{{}^6C_2 \cdot {}^{11}C_1}{{}^{17}C_3} = \frac{33}{36}$]

(iii) one of each colour [A $\frac{{}^5C_1 \cdot {}^6C_1 \cdot {}^6C_1}{{}^{17}C_3} = \frac{9}{34}$]

(iv) no white. [A $\frac{{}^{12}C_3}{{}^{17}C_3} = \frac{11}{34}$]

Sol Total no. of balls = 17

$\therefore$ Total no. of ways of drawing 3 balls out of 17 = $^{17}C_3$

(i) No. of ways of drawing 3 black balls out of 6 = $^6C_3$

$\therefore$ Required probability $= \dfrac{^6C_3}{^{17}C_3} = \dfrac{6!}{3!} \cdot \dfrac{14!}{17!} = \dfrac{\cancel{6}.\cancel{5}.\cancel{4}}{17.\cancel{16}.\cancel{15}} = \dfrac{1}{34}$

(ii) No. of yellow balls = 6

$\therefore$ No. of non-yellow balls = 11

Favourable no. of drawing 2 yellow balls out of 3
(2 yellow and 1 non-yellow) $= {}^6C_2 \times {}^{11}C_1$

$\therefore$ Required prob. $= \dfrac{^6C_2 \cdot {}^{11}C_1}{^{17}C_3} = \dfrac{6!}{2!\,4!} \cdot \dfrac{(11)\ 3!\,.\,14!}{17!}$

$= \dfrac{\cancel{6}.\cancel{5}}{\cancel{2}} \cdot \dfrac{11.\cancel{6}\,3}{17.\cancel{16}.\cancel{15}} = \dfrac{33}{136}$

(iii) No. of favourable cases of one ball of each colour

$= {}^5C_1 \times {}^6C_1 \times {}^6C_1$

$\therefore$ Required prob. $= \dfrac{^5C_1 \cdot {}^6C_1 \cdot {}^6C_1}{^{17}C_3} = \dfrac{\cancel{5}.\cancel{6}.\cancel{6}\ \cancel{3}.\cancel{2}}{17.\cancel{16}.\cancel{15}} = \dfrac{9}{34}$

(iv) No. of favourable cases of getting no white ball

$= {}^{12}C_3 \cdot {}^5C_0$

$\therefore$ Required prob. $= \dfrac{^{12}C_3}{^{17}C_3} = \dfrac{12!\ 14!}{9!\ 17!} = \dfrac{\cancel{12}.11.\cancel{10}}{17.\cancel{16}.\cancel{15}} = \dfrac{11}{34}$

Que A bag contains 150 nuts and 50 bolts. Half of the nuts and bolts are defective. An item is chosen at random. What is the prob. that the item ~~is~~ so chosen is a nut or defective?

Sol Here $n(S) = 200$

Let A: Item is a nut

B: Item is a defective

$\therefore P(A) = \frac{150}{200}$, $P(B) = \frac{100}{200}$, $P(A \cap B) = \frac{75}{200}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{150}{200} + \frac{100}{200} - \frac{75}{200} = \frac{175}{200} = \frac{7}{8} = 0.875$

Que ✓ In a city 3 daily newspaper X, Y and Z are published. 40% of the people of the city read X, 50% read Y, 20% read Z and 20% read both X and Y, 15% read X and Z and 10% read Y and Z and 2.4% read all the three papers. Calculate the percentage of people who do not read any of the three papers.

Sol Let us define the events

A = The event that people read X newspaper

B = The event that people read Y newspaper

C = The event that people read Z newspaper

Here $P(A) = 0.4$ $\quad P(A \cap B) = 0.2$

$P(B) = 0.5$ $\quad P(A \cap C) = 0.15$

$P(C) = 0.2$ $\quad P(B \cap C) = 0.1$

$P(A \cap B \cap C) = 0.024$

Here we are to find $P(\bar{A} \cap \bar{B} \cap \bar{C})$

$\therefore P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(A \cup B \cup C)$

$= 1 - [P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)]$

$= 1 - [0.4 + 0.5 + 0.2 - 0.2 - 0.1 - 0.15 + 0.024$

$= 1 - 0.674 = 0.326$ or 32.6%