Classification of linear second order partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0 \quad \text{------ (1)}$$

where A, B, C, D, E and F are functions of x, y or are real constants. The p.d.e. is said to be a

parabolic equ. if $B^2 - 4AC = 0$

hyperbolic equ. if $B^2 - 4AC > 0$

elliptic equ. if $B^2 - 4AC < 0$

Some simple examples of the above equ's are the following

(i) One dimensional heat equ. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is parabolic.

(ii) One dimensional wave equ. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is hyperbolic.

(iii) Two dimensional Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is elliptic.

Principle of superposition If $c_1 u_1, c_2 u_2, c_3 u_3, \ldots$ are solutions of equ.(1) then the complete sol. of (1) is

$$u(x, y) = \sum_{n=1}^{\infty} c_n u_n$$

Method of separation of variables

Que Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad \text{[I}^{\text{st}}\text{ term Feb.15, 2.5 marks]}$$

where $u(x, 0) = 6e^{-3x}$, $x > 0$, $t > 0$

Sol Let $u(x, t) = X(x).T(t)$

$\therefore \frac{\partial u}{\partial x} = X'T, \ \frac{\partial u}{\partial t} = XT'$

where dashes denote derivatives w.r.t. their variables.

$\therefore$ The given equ. becomes

$$X'T = 2XT' + XT$$

$$\Rightarrow \frac{X'}{X} = \frac{2T'+T}{T} = \text{constant } (\lambda) \text{ say}$$

$$\Rightarrow X' - \lambda X = 0 \text{ and } T' - \frac{(\lambda-1)}{2}T = 0$$

$$\Rightarrow X = Ae^{\lambda x}, \quad T = Be^{\frac{(\lambda-1)}{2}t}$$

$$\therefore u(x,t) = Ce^{\lambda x} \cdot e^{\frac{(\lambda-1)}{2}t} \quad \text{where } AB = C$$

Given $u(x,0) = 6e^{-3x}$

$$\therefore \quad Ce^{\lambda x} = 6e^{-3x} \Rightarrow C = 6, \lambda = -3$$

Hence $\boxed{u(x,t) = 6e^{-(3x+2t)}}$ A

Que (for practice) Use the method of separation of variables to solve the p.d.e. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$

[A $u(x,y) = 4e^{-x}e^{3y/2}$]

Que Solve the equation $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ given $u = 3e^{-y} - e^{-5y}$ when $x = 0$.

Sol Let $u(x,y) = X(x)Y(y)$

$$\therefore \frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY'$$

where dashes denote derivatives w.r.t. their variables.

$\therefore$ The given equation becomes

$$4X'Y + XY' = 3XY$$

$$\Rightarrow \frac{4X'}{X} = \frac{-Y'+3Y}{Y} = \text{constant } (\lambda) \text{ say)}$$

$$\therefore X' - \frac{\lambda}{4}X = 0, \quad Y' - (3-\lambda)Y = 0$$

$$\therefore X = Ae^{\frac{\lambda}{4}x}, \quad Y = Be^{(3-\lambda)y}$$

$$\therefore u(x,y) = Ce^{\frac{\lambda}{4}x}e^{(3-\lambda)y} \quad \text{where } AB = C$$

Now $u(0,y) = 3e^{-y} - e^{-5y}$

$\therefore$ $u(x,y)$ is sum of two solutions as

$$u(x,y) = C_1 e^{\frac{\lambda_1}{4}x} e^{(3-\lambda_1)y} + C_2 e^{\frac{\lambda_2}{4}x} e^{(3-\lambda_2)y}$$

$$u(0,y) = C_1 e^{(3-\lambda_1)y} + C_2 e^{(3-\lambda_2)y} = 3e^{-y} - e^{-5y} \text{ (given)}$$

$\therefore$ Either $C_1 = 3,\ \lambda_1 = 4,\ C_2 = -1,\ \lambda_2 = 8$

or $C_1 = -1,\ \lambda_1 = 8,\ C_2 = 3,\ \lambda_2 = 4$

In both cases, solution is $\boxed{u(x,y) = 3e^{x-y} - e^{2x-5y}}$

Que Use the method of separation of variables to solve the equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$ given that $V = 0$ when $t \to \infty$ as well as $V = 0$ at $x = 0$ and $x = l$.

Sol Let $V = X(x).T(t)$

$\therefore$ $\frac{\partial^2 V}{\partial x^2} = X''T$, $\frac{\partial V}{\partial t} = XT'$ where dashes denote derivatives w.r.to their variables

$\therefore$ The given equ. becomes $X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = \text{constant } (\lambda) \text{ say}$

$\therefore$ $X'' - \lambda X = 0$, $T' - \lambda T = 0$

Now, $T' - \lambda T = 0 \Rightarrow T = Ae^{\lambda t}$

As $V = XT = 0$ when $t \to \infty$ so λ must be negative.

Take $\lambda = -p^2$, $p > 0$

$\therefore$ $T = Ae^{-p^2 t}$

Now, $X'' - \lambda X = 0 \Rightarrow X'' + p^2 = 0$

$\Rightarrow X = B\cos px + C\sin px$

$\therefore$ $V = (D\cos px + E\sin px)e^{-p^2 t}$ where $AB = D$ and $AC = E$

$V(0,t) = 0 \Rightarrow De^{-p^2 t} = 0 \Rightarrow D = 0$

$\therefore$ $V = E\sin px\, e^{-p^2 t}$

Now $V(l,t) = 0 \Rightarrow E\sin pl\, e^{-p^2 t} = 0 \Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l},\ n = 1,2,3,\ldots$ $(\because p > 0)$

$\therefore$ Solutions are $V(x,t) = E_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2}{l^2}t}$, $n = 1, 2, 3, \ldots$

$\therefore$ By principle of superposition, complete solution is

$$\boxed{V(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2}}}$$