Linear p.d.e's with constant coefficients, non-homogeneous in partial derivatives

$$f(D, D')z = F(x, y) \quad \text{——— (1)}$$

where $f(D, D')$ is not homogeneous i.e., sum of powers of D and D' in terms may not be equal.

Here we also have complete solution = C.F. + P.I.

Rules for finding C.F.

Case I When $f(D, D')$ can be factorized into linear factors of the type $(D - mD' - c)$ where m, c be any constant (may be zero) and factors are not repeated.

$\therefore$ C.F. (corresponding to this factor) $= e^{cx}\phi(y+mx)$

The solution corresponding to various factors added up, give the C.F. of (1).

Case II When factors are repeated. Suppose $(D - mD' - c)$ is repeated twice. Then C.F. (corresponding to these two factors)

$$= e^{cx}[\phi_1(y+mx) + x\,\phi_2(y+mx)]$$

Similarly C.F. $= e^{cx}[\phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx)]$

when factor $(D - mD' - c)$ is repeated thrice.

Case III If $f(D, D')$ cannot be factorized into linear factors. Then C.F. (corresponding to non-linear factor)

$$= \sum_{i=1}^{\infty} c_i e^{h_i x + k_i y} \quad \text{where } f(h_i, k_i) = 0$$

Rules for finding P.I.

Formulae ① to ④ of homogeneous form can be used in the same way for finding P.I. for non-homogenous form.

Remark: We cannot use the formula ⑤ of homogeneous form as $f(D, D')$ is not homogeneous here.

General Formula

$$\frac{1}{(D-mD'-c)} F(x,y) = e^{cx} \int e^{-cx} F(x, a-mx)\, dx$$

where $y = a - mx$

After integration, we substitute $a = y + mx$

Que Find the solutions of the following p.d.e's

(1) $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y) + e^{x+2y}$

(2) $(D^2-D')z = xe^{x+y}$

(3) $(D^2-D'^2+3D'-3D)z = xy$

(4) $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

Sol (1) $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y) + e^{x+2y}$

$\Rightarrow (D+D')(D+D'-2)z = \sin(x+2y) + e^{x+2y}$

$\therefore$ C.F. $= \phi_1(y-x) + e^{2x}\phi_2(y-x)$

$$\text{P.I.} = \frac{1}{(D^2+2DD'+D'^2-2D-2D')}\sin(x+2y) + \frac{1}{(D+D')(D+D'-2)}e^{x+2y}$$

$$= \frac{1}{-1+2(-2)+(-4)-2D-2D'}\sin(x+2y) + \frac{1}{(1+2)(1+2-2)}e^{x+2y}$$

$$= \frac{-1}{(2D+2D'+9)}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= -\frac{(2D+2D'-9)}{(2D+2D'+9)(2D+2D'-9)}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= -\frac{(2D+2D'-9)}{(4D^2+4D'^2+8DD')-81}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= -\frac{(2D+2D'-9)}{-4-16-16-81}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= \frac{(2D+2D'-9)}{117}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= \frac{1}{117}\left[2\cos(x+2y)+4\cos(x+2y)-9\sin(x+2y)\right] + \frac{1}{3}e^{x+2y}$$

$$= \frac{1}{39}\left[2\cos(x+2y)-3\sin(x+2y)\right] + \frac{1}{3}e^{x+2y}$$

$\therefore$ The complete solution is

$$z = \phi_1(y-x) + e^{2x}\phi_2(y-x) + \frac{1}{39}\left[2\cos(x+2y)-3\sin(x+2y)\right] + \frac{1}{3}e^{x+2y}$$

(2) $(D^2-D')z = xe^{x+y}$

Here D^2-D' cannot be factorized into linear factors in D and D'.

$$\therefore \text{C.F.} = \sum_{i=1}^{\infty} c_i e^{h_i x + k_i y} \quad \text{where } h_i^2 - k_i = 0 \text{ i.e., } k_i = h_i^2$$

$$= \sum_{i=1}^{\infty} c_i e^{h_i x + h_i^2 y}$$

$$\text{P.I.} = \frac{1}{(D^2-D')} xe^{x+y}$$

$$= \frac{e^{x+y}}{(D+1)^2-(D'+1)} x$$

$$= \frac{e^{x+y}}{(D^2+2D-D')} x$$

$$= e^{x+y}\frac{1}{(-D')}\left[1-\left(\frac{2D}{D'}+\frac{D^2}{D'}\right)\right]^{-1}. x$$

$$= e^{x+y}\frac{1}{(-D')}\left[1+\frac{2D}{D'}+\frac{D^2}{D'}+\cdots\right]. x$$

$$= e^{x+y}\frac{1}{(-D')}(x+2y) = e^{x+y}(-xy-y^2)$$

$\therefore$ The complete sol. is

$$z = \sum_{i=1}^{\infty} c_i e^{h_i x + h_i^2 y} - e^{x+y}(xy+y^2)$$

where c_i and h_i are arbitrary constants.

(3) $(D^2 - D'^2 + 3D' - 3D)z = xy$

or $(D-D')(D+D'-3)z = xy$

$\therefore$ C.F. $= \phi_1(y+x) + e^{3x}\phi_2(y-x)$

P.I. $= \dfrac{1}{(D^2-D'^2+3D'-3D)} xy$

$= \dfrac{1}{(D-D')(D+D'-3)} xy$

$= -\dfrac{1}{3D}\left(1-\dfrac{D'}{D}\right)^{-1}\left[1-\left(\dfrac{D}{3}+\dfrac{D'}{3}\right)\right]^{-1} \cdot xy$

$= -\dfrac{1}{3D}\left(1+\dfrac{D'}{D}+\cdots\right)\left(1+\dfrac{D}{3}+\dfrac{D'}{3}+\dfrac{2}{9}DD'+\cdots\right)xy$

$= -\dfrac{1}{3D}\left(1+\dfrac{D}{3}+\dfrac{2D'}{3}+\dfrac{D'}{D}+\dfrac{2}{9}DD'+\cdots\right)xy$

$= -\dfrac{1}{3D}\left(xy+\dfrac{y}{3}+\dfrac{2x}{3}+\dfrac{1}{D}x+\dfrac{2}{9}\right) = -\dfrac{1}{3}\left(\dfrac{1}{2}x^2y+\dfrac{xy}{3}+\dfrac{x^2}{3}+\dfrac{x^3}{6}+\dfrac{2x}{9}\right)$

$\therefore$ The complete solution is

$$\boxed{z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{54}(9x^2y+6xy+6x^2+3x^3+4x)}$$

(4) $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

C.F. $= e^{2x}\phi_1(y+3x) + xe^{2x}\phi_2(y+3x)$

P.I. $= \dfrac{1}{(D-3D'-2)^2} 2e^{2x}\tan(y+3x)$

$= 2e^{2x}\dfrac{1}{(D-3D')^2}\tan(y+3x)$

$= 2e^{2x}\dfrac{1}{(D-3D')}\int \tan(a-3x+3x)\,dx$ where $a-3x=y$

$= 2e^{2x}\dfrac{1}{(D-3D')} x\tan a = 2e^{2x}\dfrac{1}{(D-3D')} x\tan(y+3x)$

$= 2e^{2x}\int x\tan a\,dx$ where $a-3x=y$

$= 2e^{2x}\cdot\dfrac{x^2}{2}\tan a = x^2e^{2x}\tan(y+3x)$

$\therefore$ The complete sol. is

$$\boxed{z = e^{2x}\phi_1(y+3x) + xe^{2x}\phi_2(y+3x) + x^2e^{2x}\tan(y+3x)}$$

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