Multiplication law of probability or Theorem of compound probability

For two events A and B,

$$P(A\cap B) = P(A).P(B/A), \quad P(A) \neq 0$$
$$= P(B).P(A/B), \quad P(B) \neq 0$$

where $P(A/B)$ means the conditional probability of happening of A when the event B has already happened and $P(B/A)$ means the conditional probability of happening of B when the event A has already happened.

Proof Let N be the total no. of outcomes of which n_1 outcomes are favourable to the event A and n_2 outcomes are favourable to the event B. Let n be the outcomes favourable to the event $A\cap B$. Then

$$P(A) = \frac{n_1}{N}, \quad P(B) = \frac{n_2}{N}, \quad P(A\cap B) = \frac{n}{N}$$

Now the conditional probability $P(A/B)$ refers to the sample space of n_2 outcomes, out of which n outcomes are favourable to event A i.e., when B has already happened.

$$\therefore P(A/B) = \frac{n}{n_2}$$

Similarly, we have $P(B/A) = \frac{n}{n_1}$

Now, $P(A\cap B) = \frac{n}{N} = \frac{n}{n_1}.\frac{n_1}{N} = P(B/A).P(A)$

and $P(A\cap B) = \frac{n}{N} = \frac{n}{n_2}.\frac{n_2}{N} = P(A/B).P(B)$

Hence $P(A\cap B) = P(A).P(B/A), \quad P(A) \neq 0$ ——— (1)

$= P(B).P(A/B), \quad P(B) \neq 0$ ——— (2)

Remark (1) From (1) and (2), we find that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0 \quad \text{and} \quad P(B/A) = \frac{P(A \cap B)}{P(A)}, \; P(A) \neq 0$$

(2) **Extension to n events** For n events $A_1, A_2, \dots$ and A_n, we have

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2) \cdots P(A_n/A_1 \cap A_2 \cap \dots \cap A_{n-1})$$

Independent Events Two events are said to be independent, if happening or failure of one does not affect the happening or failure of the other, otherwise the events are said to be dependent.

Remark (1) If the events A and B are independent, then

$P(B/A) = P(B)$ and $P(A/B) = P(A)$

$$\therefore \quad P(A \cap B) = P(A) \cdot P(B)$$

Also,

$$P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$$

$$P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B)$$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$$

(2) If $A_1, A_2, \dots, A_n$ are independent events then

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n)$$

and $$P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(\bar{A_1}) \cdot P(\bar{A_2}) \cdots P(\bar{A_n})$$

Question A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second to give 4 black balls in each of the following cases:

(i) The balls are not replaced before the second draw.

(ii) The balls are replaced before the second draw.

A. Let us define the events

A: First draw gives 4 white balls

B: Second draw gives 4 black balls

(i) Here we are required to find $P(A \cap B)$

Since the balls are not replaced before the second draw,

So $P(A \cap B) = P(A) \cdot P(B/A)$

$$= \frac{^6C_4}{^{15}C_4} \cdot \frac{^9C_4}{^{11}C_4} = \frac{\frac{6!}{4!\,2!} \cdot \frac{9!}{4!\,5!}}{\frac{15!}{4!\,11!} \cdot \frac{11!}{4!\,7!}} = \frac{6!\,9!\,\cancel{4!}\,7!}{2!\,5!\,15!\,\cancel{4!}}$$

$$= \frac{6.\cancel{5!}\,\cancel{9!}\,.\cancel{7}.\cancel{6}.\cancel{5}.\cancel{4}.\cancel{3}.\cancel{2}.1}{\cancel{2}.\cancel{5!}\cdot\cancel{15}.\cancel{14}.13.\cancel{12}.11.10\,\cancel{9!}} = \frac{3}{715}$$

(ii) Here the balls so drawn are replaced before the second draw. So, $P(A \cap B) = P(A) \cdot P(B)$

$$= \frac{^6C_4}{^{15}C_4} \cdot \frac{^9C_4}{^{15}C_4} = \frac{6!\,9!}{2!\,5!} \cdot \frac{11!}{15!} \cdot \frac{11!}{15!}$$

$$= \frac{\cancel{6}\times9\times8\times\cancel{7}\times6\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1}{\cancel{2}\times\cancel{15}\times\cancel{14}\times13\times\cancel{12}\times15\times14\times13\times12}$$

$$= \frac{6}{5915}$$

Que There are two bags A and B. Bag A contains five red and three black balls whereas Bag B contains 4 red and 4 black balls. A ball is transfered from Bag A to Bag B and then a ball is drawn from Bag B. What is the the probability that the ball so drawn is red?

Sol Two cases arise:

Case I transferred ball is red and the drawn ball is also red.

Case II transferred ball is black and the drawn ball is red.

Bag A [5 red, 3 black] Bag B [4 red, 4 black]

Let us define the events as

E_1: ball transferred from bag A to B is red

E_2: ball transfered from bag A to B is black

E: ball drawn from bag B is red

Case I In this case we are to find $P(E_1 \cap E)$

$\therefore P(E_1) = \frac{5}{8}$

and $P(E/E_1) = \frac{5}{9}$ ($\because$ 1 red ball is transferred from bag A to B, so bag B will have 5 red and 4 black balls)

$\therefore P(E_1 \cap E) = P(E_1) \cdot P(E/E_1)$

$= \frac{5}{8} \cdot \frac{5}{9} = \frac{25}{72}$

Case II In this case, we are to find $P(E_2 \cap E)$

$\therefore P(E_2) = \frac{3}{8}$

and $P(E/E_2) = \frac{4}{9}$ ($\because$ 1 black ball is transferred from bag A to B, so bag B will have 4 red and 5 black balls)

$\therefore P(E \cap E_2)$

$= P(E_2) \cdot P(E/E_2)$

$= \frac{3}{8} \cdot \frac{4}{9} = \frac{1}{6}$

Hence the required probability $P(E) = \frac{25}{72} + \frac{1}{6} = \frac{37}{72}$

Que The odds against A solving a certain problem are 8 to 6 and the odds in favour of B solving the same problem are 14 to 10. What is the probability that if both of them try, the problem would be solved?

A Let us define the events

E_1: A solves the problem

E_2: B solves the problem

Now, $P(E_1) = \frac{6}{8+6} = \frac{3}{7}$, $P(E_2) = \frac{14}{14+10} = \frac{7}{12}$

$\therefore$ $P(\overline{E_1}) = \frac{4}{7}$, $P(\overline{E_2}) = \frac{5}{12}$

$\therefore$ The probability that none of A and B solve the problem

$$= P(\overline{E_1} \cap \overline{E_2}) = P(\overline{E_1}).P(\overline{E_2}) = \frac{4}{7}\cdot\frac{5}{12} = \frac{5}{21}$$

$\therefore$ The probability that the problem would be solved

$$= 1 - P(\overline{E_1} \cap \overline{E_2}) = 1 - \frac{5}{21} = \frac{16}{21}$$

Que A problem in mathematics is given to two students A and B. Their respective probability of solving it are $\frac{1}{2}$ and $\frac{1}{3}$. If both of them try to solve the problem independently, find the probability that

(a) Exactly one of them solve the problem

(b) None of them solve the problem

(c) Problem is solved

Sol Let us define the following events:

E_1 : The problem is solved by student A

E_2 : The problem is solved by student B

Then $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{3}$

$\therefore$ $P(\overline{E_1}) = 1 - \frac{1}{2} = \frac{1}{2}$, $P(\overline{E_2}) = 1 - \frac{1}{3} = \frac{2}{3}$

(a) P(Exactly one of them solve the problem)

$$= P(E_1 \cap \overline{E_2}) + P(\overline{E_1} \cap E_2) = P(E_1).P(\overline{E_2}) + P(\overline{E_1}).P(E_2)$$

$$= \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{2}$$

(b) P(None of them solve it) $= P(\overline{E_1} \cap \overline{E_2}) = P(\overline{E_1})\cdot P(\overline{E_2}) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$

(c) P (Problem is solved) $= 1 - P(\overline{E_1} \cap \overline{E_2}) = 1 - P(\overline{E_1}).P(\overline{E_2})$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

Que A bag contains 5 red and 4 black balls. Two balls are drawn one by one without replacement, What is the probability that the first ball is red and second is black.

Sol Let us define the events

E_1: Getting a red ball in first draw

E_2: Getting a black ball in second draw

$\therefore$ Required probability $= P(E_1 \cap E_2)$
$= P(E_1) \cdot P(E_2/E_1)$

Now, $P(E_1) = \frac{5}{9}$, $P(E_2/E_1) = \frac{4}{8} = \frac{1}{2}$

$\therefore$ Required probability $= \frac{5}{9} \cdot \frac{1}{2} = \frac{5}{18}$ Ans

Que A can hit a target 3 times in 5 shots, B, 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that

(i) two shots hit (ii) at least two shots hit?

Sol Prob. of A hitting the target $= \frac{3}{5}$

Prob. of B " " " $= \frac{2}{5}$

Prob. of C " " " $= \frac{3}{4}$

(i) In order that two shots may hit the target, the following cases must be considered:

$p_1 =$ Chance that A and B hit and C fails to hit $= \frac{3}{5} \cdot \frac{2}{5} \cdot \left(1 - \frac{3}{4}\right) = \frac{6}{100}$

$p_2 =$ Chance " B and C " " A " " " $= \frac{2}{5} \cdot \frac{3}{4}\left(1 - \frac{3}{5}\right) = \frac{12}{100}$

$p_3 =$ " " A and C " " B " " $= \frac{3}{5} \cdot \frac{3}{4} \cdot \left(1 - \frac{2}{5}\right) = \frac{27}{100}$

$\because$ These are mutually exclusive events, the probability that any two shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = \frac{45}{100} = 0.45$$

(ii) We have p_4 = Chance that A, B, C all hit

$$= \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{18}{100}$$

$\therefore$ The probability that at least two shots hit

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = \frac{63}{100} = 0.63$$

($\because$ all are mutually exclusive events)

Baye's Theorem : If $E_1, E_2, \dots, E_n$ are mutually disjoint events with $P(E_i) \neq 0$ $(i = 1, 2, \dots n)$ then for any arbitrary event $A \subseteq \bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}, \quad i = 1, 2, \dots, n$$

Proof By multiplication theorem, we have

$$P(A \cap E_i) = P(A) \cdot P(E_i/A), \quad P(A) \neq 0 \quad \text{——— (1)}$$

$$= P(E_i) \cdot P(A/E_i), \quad P(E_i) \neq 0 \quad \text{——— (2)}$$

$$\therefore \text{By (1)}, \quad P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i) \cdot P(A/E_i)}{P(A)} \quad \text{by (2)} \quad \text{——— (3)}$$

Now, $A \subseteq \bigcup_{i=1}^{n} E_i$

$$\therefore \quad A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = A \cap (E_1 \cup E_2 \cup \dots \cup E_n)$$

$$\therefore \quad A = (A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)$$

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)$$

(by addition theorem, as $(A \cap E_1) \cap (A \cap E_2) \cap \dots \cap (A \cap E_n) = \phi$ because $E_1 \cap E_2 \cap \dots \cap E_n = \phi$)

$$\therefore \quad P(A) = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i) \cdot P(A/E_i) \quad \text{by (2)} \quad \text{——— (4)}$$

From (3) and (4),

$$P(E_i/A) = \frac{P(E_i).P(A/E_i)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)}$$

Que There are three bags:

I^{st} containing 1 white, 2 red, 3 green balls

II^{nd} containing 2 white, 3 red, 1 green balls

III^{rd} containing 3 white, 1 red, 2 green balls.

Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.

Sol Let us define the following events:

E_1 : I^{st} bag is chosen

E_2 : II^{nd} " "

E_3 : III^{rd} " "

E : The two balls are white and red.

Clearly, we have to find $P(E_2/E)$.

Now, there are 3 bags and one of them is chosen at random

$\therefore P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, $P(E/E_1) = \frac{{}^1C_1 \times {}^2C_1}{{}^6C_2} = \frac{2 \times 2}{6 \times 5} = \frac{2}{15}$

$P(E/E_2) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = \frac{2 \times 3 \times 2}{6 \times 5} = \frac{2}{5}$

$P(E/E_3) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3 \times 1 \times 2}{6 \times 5} = \frac{1}{5}$

$\therefore$ By Baye's theorem, $P(E_2/E) = \dfrac{P(E_2).P(E/E_2)}{P(E_1).P(E/E_1)+P(E_2).P(E/E_2)+P(E_3).P(E/E_3)}$

$$= \frac{\frac{1}{3}\cdot\frac{2}{5}}{\frac{1}{3}\cdot\frac{2}{15}+\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{1}{5}} = \frac{\frac{2}{5}}{\frac{2}{15}+\frac{2}{5}+\frac{1}{5}} = \frac{2}{5}\cdot\frac{15}{11} = \frac{6}{11}$$ A

Que In a bolt factory machines A, B and C manufacture 25%, 30% and 45% respectively. Of the total bolts of their output 5%, 4% and 3% are respectively defective. A bolt is drawn at random from the total production. What is the probability that the bolt drawn is defective. What are the chances that it was manufactured by the machine with the highest output?

Sol Let us define the following events:

E_1: The bolt is manufactured by machine A

E_2: The bolt is manufactured by machine B

E_3: The bolt is manufactured by machine C

E: The bolt is defective

Then we have to find $P(E)$ and $P(E_3/E)$ ($\because$ highest output is from machine C)

Now, $P(E_1) = \frac{25}{100}$, $P(E_2) = \frac{30}{100}$, $P(E_3) = \frac{45}{100}$

$P(E/E_1) = \frac{5}{100}$, $P(E/E_2) = \frac{4}{100}$, $P(E/E_3) = \frac{3}{100}$

$\therefore$ By addition theorem,

$$P(E) = P(E\cap E_1) + P(E\cap E_2) + P(E\cap E_3)$$

$$= P(E_1).P(E/E_1) + P(E_2).P(E/E_2) + P(E_3).P(E/E_3)$$

$$= \frac{25}{100}\cdot\frac{5}{100} + \frac{30}{100}\cdot\frac{4}{100} + \frac{45}{100}\cdot\frac{3}{100} = \frac{125+120+135}{100\times100} = \frac{380}{10000} = 0.0380 \quad (1)$$

Now, by Baye's theorem

$$P(E_3/E) = \frac{P(E_3).P(E/E_3)}{\sum_{i=1}^{3} P(E_i).P(E/E_3)} = \frac{\frac{45}{100}\cdot\frac{3}{100}}{\frac{380}{100\times100}} = \frac{135}{380} = 0.3553$$ A

(by (1))