Question # 1

Find from first principles, the derivatives of the following expensions w.r.t. their respective independent variables:

(i) $(ax+b)^3$ (ii) $(2x+3)^5$ (iii) $(3t+2)^{-2}$

(iv) $(ax+b)^{-5}$ (v) $\dfrac{1}{(az-b)^7}$

Solution

(i) Let $y = (ax+b)^3$

$$\Rightarrow y+\delta y = \left(a(x+\delta x)+b\right)^3$$

$$\Rightarrow \delta y = \left(ax+b+a\delta x\right)^3 - y$$

$$= \left((ax+b)+a\delta x\right)^3 - (ax+b)^3$$

$$= \left[(ax+b)^3+3(ax+b)^2(a\delta x)+3(ax+b)(a\delta x)^2+(a\delta x)^3\right]-(ax+b)^3$$

$$= 3a(ax+b)^2\delta x+3a^2(ax+b)\delta x^2+a^3\delta x^3$$

$$= \delta x\left(3a(ax+b)^2+3a^2(ax+b)\delta x+a^3\delta x^2\right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = 3a(ax+b)^2+3a^2(ax+b)\delta x+a^3\delta x^2$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[3a(ax+b)^2+3a^2(ax+b)\delta x+a^3\delta x^2\right]$$

$$\Rightarrow \frac{dy}{dx} = 3a(ax+b)^2+3a^2(ax+b)(0)+a^3(0)^2$$

$$\Rightarrow \frac{dy}{dx} = 3a(ax+b)^2+0+0 \qquad \Rightarrow \boxed{\frac{dy}{dx} = 3a(ax+b)^2}$$

(ii) Let $y = (2x+3)^5$

$$\Rightarrow y+\delta y = \left(2(x+\delta x)+3\right)^5$$

$$\Rightarrow \delta y = \left(2x+2\delta x+3\right)^5 - y$$

$$= \left((2x+3)+2\delta x\right)^5-(2x+3)^5$$

$$= \left[\binom{5}{0}(2x+3)^5+\binom{5}{1}(2x+3)^4(2\delta x)+\binom{5}{2}(2x+3)^3(2\delta x)^2+\dots\right.$$

$$\left.\dots+\binom{5}{5}(2\delta x)^5\right]-(2x+3)^5$$

$$= \left[(1)(2x+3)^5 + 2\binom{5}{1}(2x+3)^4\delta x + 4\binom{5}{2}(2x+3)^3\delta x^2 + \ldots\right.$$
$$\left.\ldots + 32\binom{5}{5}\delta x^5\right] - (2x+3)^5$$
$$= 2\binom{5}{1}(2x+3)^4\delta x + 4\binom{5}{2}(2x+3)^3\delta x^2 + \ldots + 32\binom{5}{5}\delta x^5$$

Dividing by δx

$$\frac{\delta y}{\delta x} = 2\binom{5}{1}(2x+3)^4 + 4\binom{5}{2}(2x+3)^3\delta x + \ldots + 32\binom{5}{5}\delta x^4$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[2\binom{5}{1}(2x+3)^4 + 4\binom{5}{2}(2x+3)^3\delta x + \ldots + 32\binom{5}{5}\delta x^4\right]$$
$$\Rightarrow \frac{dy}{dx} = \left[2\binom{5}{1}(2x+3)^4 + 0 + 0 + \ldots + 0\right]$$
$$\Rightarrow \frac{dy}{dx} = 2(5)(2x+3)^4 \quad \text{or} \quad \boxed{\frac{dy}{dx} = 10(2x+3)^4}$$

(iii) Let $y = (3t+2)^{-2}$

$$\Rightarrow y + \delta y = (3(t+\delta t)+2)^{-2}$$
$$\Rightarrow \delta y = (3t+3\delta t+2)^{-2} - y$$
$$\Rightarrow \delta y = ((3t+2)+3\delta t)^{-2} - (3t+2)^{-2}$$
$$= (3t+2)^{-2}\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - (3t+2)^{-2} = (3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2} - 1\right]$$
$$= (3t+2)^{-2}\left[\left(1+(-2)\frac{3\delta t}{3t+2} + \frac{-2(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2 + \ldots\right) - 1\right]$$
$$\Rightarrow \delta y = (3t+2)^{-2}\left[1 - \frac{6\delta t}{3t+2} + \frac{-2(-3)}{2}\left(\frac{\delta t}{3t+2}\right)^2 + \ldots - 1\right]$$
$$= (3t+2)^{-2}\left[-\frac{6\delta t}{3t+2} + 3\left(\frac{3\delta t}{3t+2}\right)^2 + \ldots\right]$$
$$= (3t+2)^{-1}\cdot\frac{3\delta t}{3t+2}\left[-2 + 3\left(\frac{3\delta t}{3t+2}\right) + \ldots\right]$$

Dividing by δt

$$\frac{\delta y}{\delta t} = 3(3t+2)^{-2-1}\left[-2 + \left(\frac{3\delta t}{3t+2}\right) + \ldots\right]$$

Taking limit when $\delta t \to 0$, we have

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \lim_{\delta t \to 0} 3(3t+2)^{-3}\left[-2+\left(\frac{3\delta t}{3t+2}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx} = 3(3t+2)^{-3}[-2+0-0+....] \Rightarrow \boxed{\frac{dy}{dx} = -6(3t+2)^{-3}}$$

(iv) Let $y = (ax+b)^{-5}$

Do yourself

(v) Let $y = \dfrac{1}{(az-b)^7} = (az-b)^{-7}$

$$\Rightarrow y+\delta y = \left(a(z+\delta z)-b\right)^{-7}$$

$$\Rightarrow \delta y = \left((az-b)+a\delta z\right)^{-7} - \left(az-b\right)^{-7}$$

$$\Rightarrow \delta y = \left(az-b\right)^{-7}\left[\left(1+\frac{a\delta z}{(az-b)}\right)^{-7} - 1\right]$$