8 RANDOM VARIABLE AND PROBABILITY DISTRIBUTION

8.1 Since 0, 1, 2 or 3 homes are insured against fire, the random variable X assumes the value 0, 1, 2 or 3. Here $p = P(\text{insured}) = 0.8$ and $q = 1 - p = 1 - 0.8 = 0.2$. The probability function $P(X = x) = P(x)$ is given by

$$P(x) = {}^3C_x p^x q^{3-x}, \qquad x = 0, 1, 2, 3.$$

$$= {}^3C_x (0.8)^x (0.2)^{3-x}, \qquad x = 0, 1, 2, 3$$

The probability distribution in tabular form is given by

x	0	1	2	3
$P(x)$	0.008	0.096	0.384	0.512

8.2(a)(i) Since $\Sigma P(X = x) = 0.7 + y = 1$, $y = 1 - 0.7 = \mathbf{0.3}$.

(ii)

$3x + 5$	8	11	14	17	20
$P(3x + 5)$	0.1	0.3	0.3	0.2	0.1

(iii)

x	1	2	3	4	5	
$P(x)$	0.1	0.3	0.3	0.2	0.1	$\Sigma P(x) = 1$
$xP(x)$	0.1	0.6	0.9	0.8	0.5	$\Sigma xP(x) = 2.9$
$x^2P(x)$	0.1	1.2	2.7	3.2	2.5	$\Sigma x^2P(x) = 9.7$

$\mu = \Sigma xP(x) = \mathbf{2.9}$

$\sigma^2 = \Sigma x^2P(x) - \mu^2 = 9.7 - (2.9)^2 = 9.7 - 8.41 = \mathbf{1.29}$

and $\sigma = \sqrt{1.29} = \mathbf{1.136}$.

(b) Since $\Sigma P(x) = 0.7 + k = 1$, $k = 1 - 0.7 = \mathbf{0.3}$.

The probability distributions of X and $Y = 2X - 8$ are given below where computations of means and variances of X and Y are also shown.

x	2	3	4	5	6	
$P(x)$	0.01	0.25	0.40	0.30	0.04	$\Sigma P(x) = 1$
$xP(x)$	0.02	0.75	1.6	1.5	0.24	$\Sigma xP(x) = 4.11$
$x^2P(x)$	0.04	2.25	6.4	7.5	1.44	$\Sigma x^2P(x) = 17.63$

y	-4	-2	0	2	4	
$P(y)$	0.01	0.25	0.40	0.30	0.04	$\sum P(y) = 1$
$yP(y)$	-0.04	-0.50	0	0.60	0.16	$\sum yP(y) = 0.22$
$y^2P(y)$	0.16	1.00	0	1.20	0.64	$\sum y^2P(y) = 3$

$E(X) = \sum xP(x) = \mathbf{4.11}$, $E(X^2) = \sum x^2P(x) = 17.63$

$\text{Var}(X) = E(X^2) - [E(X)]^2 = 17.63 - (4.11)^2 = \mathbf{0.7379.}$

$E(Y) = \sum yP(y) = \mathbf{0.22}$, $E(Y^2) = \sum y^2P(y) = 3$

$\text{Var}(Y) = E(Y^2) - [E(Y)^2] = 3 - (0.22)^2 = 3 - 0.0484 = \mathbf{2.9516.}$

(i) $E(Y) = 2E(X) - 8 = 2(4.11) - 8 = 8.22 - 8 = \mathbf{0.22.}$

(ii) $\text{Var}(Y) = 4\ \text{Var}(X) = \mathbf{4(0.7379) = 2.9516.}$

8.3(a) The probability function of X is given by

$$P(X = x) = {}^4C_x \cdot {}^6C_{4-x}/{}^{10}C_4, \qquad x = 0, 1, 2, 3, 4.$$

$$P(X = 0) = {}^4C_0 \cdot {}^6C_4/{}^{10}C_4 = \frac{6!}{4!\,2!}\,\frac{4!\,6!}{10!} = \frac{15}{210}$$

$$P(X = 1) = {}^4C_1 \cdot {}^6C_3/{}^{10}C_4 = \frac{4 \infty\ 6!}{3!\,3!}\,\frac{4 \infty\ 6!}{10!} = \frac{80}{210}$$

$$P(X = 2) = {}^4C_2 \cdot {}^6C_2/{}^{10}C_4 = \frac{4!}{2!\,2!}\,\frac{6!}{2!\,4!}\,\frac{4!\,6!}{10!} = \frac{90}{210}$$

$$P(X = 3) = {}^4C_3\,{}^6C_1/{}^{10}C_4 = \frac{4! \infty\ 6}{3!\,1!}\,\frac{4!\,6!}{10!} = \frac{24}{210}$$

$$P(X = 4) = {}^4C_4 \cdot {}^6C_0/{}^{10}C_4 = \frac{4!\,6!}{10!} = \frac{1}{210}$$

Thus the probability distribution of X is given by

x	0	1	2	3	4
$P(x)$	15/210	80/210	90/210	24/210	1/210

(c) $P = P(3 \text{ turns}) = 1/6$, then $q = 1 - p = 1 - 1/6 = 5/6$.

Also $n = 5$.

The probability function of X is

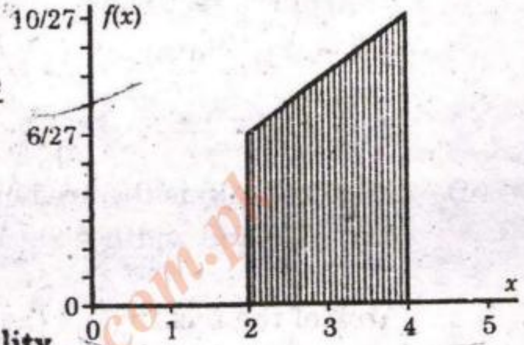
$P(X = x) = {}^5C_x p^x q^{5-x}, \qquad x = 0, 1, 2, 3, 4, 5$

$= {}^5C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{5-x}, \qquad x = 0, 1, 2, 3, 4, 5.$

The probability distribution of X in tabular form is given by

x	0	1	2	3	4	5
$P(x)$	$\frac{3125}{7776}$	$\frac{3125}{7776}$	$\frac{1250}{7776}$	$\frac{250}{7776}$	$\frac{25}{7776}$	$\frac{1}{7776}$

8.4(i) $P(X < 4) = P(2 < X < 4)$ is the area between $x = 2$ and $x = 4$ as shown shaded in the figure, which is a trapezoid. At $x = 2$, $f(2) = 2(1 + 2)/27 = 2/9$ and at $x = 4$, $f(4) = 2(1 + 4)/27 = 10/27$.



Trapezoidal area

$$= \frac{\text{(sum of parallel sides)}\times\text{base}}{2}$$

$$= \left(\frac{[f(2) + f(4)](4 - 2)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{9} + \frac{10}{27}\right)(2) = \frac{16}{27},$$

which is the required probability.

8.4(ii) $P(X > 3.5) = P(3.5 < X < 6)$

$$f(3.5) = \frac{2}{27}(1 + 3.5)$$

$$= \frac{9}{27}$$

$$f(6) = \frac{2}{27}(1 + 6) = \frac{14}{27}$$

$$\text{Trapezoidal Area} = \frac{\text{(Sum of parallel sides)}}{2} \times \text{Base}$$

$$= \frac{f(3.5) + f(6)}{2} \times (6 - 3.5)^2$$

$$= \frac{\frac{9}{27} + \frac{14}{27}}{2} \times 2.5 = 1.06$$

Probability cannot exceed 1.

Alternative Solution

i) $P(X < 4) = P(2 < X < 4) = \int_2^4 f(x)dx = \frac{2}{27}\int_2^4 (1 + x)dx$

$$= \frac{2}{27}\left(x + \frac{x^2}{2}\right)\Big|_2^4 = \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = \frac{16}{27}.$$

i) $P(3 < X < 4) = \int_3^4 f(x)dx = \frac{2}{27}\int_3^4 (1 + x)dx = \frac{2}{27}\left(x + \frac{x^2}{2}\right)\Big|_3^4$

$$= \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(3 + \frac{9}{2}\right)\right] = \frac{2}{27}\left(\frac{9}{2}\right) = \frac{1}{3}$$

8.5(a) (i) The required area is the area between $x = 1$ and $x = 4$ as shown shaded in the figure, which is a rectangle.

Height of rectangle is

$f(x) = 1/3$

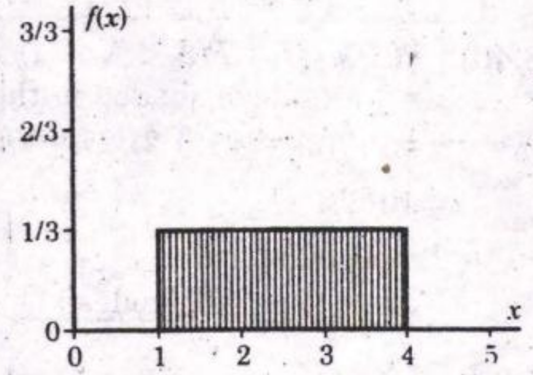
Width or base of rectangle

$= 4 - 1 = 3$

Area of the rectangle

= (height)(width)

$= \left(\frac{1}{3}\right)(3) = \mathbf{1.}$



(ii) $P(1.5 < X < 3)$ is the area of the rectangle whose height $f(x) = 1/3$ and width $= 3 - 1.5 = 1.5$ Thus

Area of the rectangle = (height)(width) $= \left(\frac{1}{3}\right)(1.5) = \mathbf{0.5},$

which is the required probability.

(iii) $P(X \leq 2.2) = P(1 \leq X \leq 2.2)$ is the area of the rectangle whose height is $f(x) = 1/3$ and width $= 2.2 - 1 = 1.2$. Thus

Area of the rectangle = (height)(width) $= \left(\frac{1}{3}\right)(1.2) = \mathbf{0.4},$

which is the required probability.

Alternative Solution

(i) Required area $= \int_1^4 f(x)\,dx = \int_1^4 \frac{1}{3}\,dx = \frac{x}{3}\Big|_1^4 = \frac{1}{3}(4 - 1) = \mathbf{1.}$

(ii) $P(1.5. < X < 3) = \int_{1.5}^3 f(x)\,dx = \int_{1.5}^3 \frac{1}{3}\,dx = \frac{x}{3}\Big|_{1.5}^3$

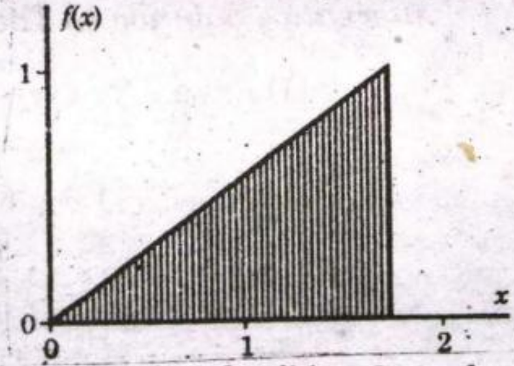
$$= \frac{1}{3}(3 - 1.5) = \mathbf{0.5.}$$

(iii) $P(X \leq 2.2) = P(1 \leq X \leq 2.2) = \int_1^{2.2} f(x)\,dx = \int_1^{2.2} \frac{1}{3}\,dx = \frac{x}{3}\Big|_1^{2.2}$

$$= \frac{1}{3}(2.2 - 1) = \mathbf{0.4.}$$

8.5(b)(i) The density function of X is $f(x) = cx, \quad 0 < x < 2$.

The graph of $f(x) = cx$ is a straight line as shown in the figure. We know that the total area under the line between $x = 0$ and $x = 2$ and above the X-axis must be 1. At $x = 0$, $f(0) = 0$ and at $x = 2$, $f(2) = 2c$. We must choose c so that the shaded area (which a right-angle triangle with height $f(x) = 2c$ and width or base $= 2 - 0 = 2$) is 1. Thus area = (height)(base)/2



$= 2c(2)/2 = 2c = 1$ or $c = 1/2$.

The density function of X is $f(x) = x/2, \quad 0 < x < 2$.

(ii) $P(1 < X < 1.5)$ is the area between $x = 1$ and $x = 1.5$ as shown shaded in the figure, which in the trapezoid. At $x = 1$, $f(1) = 1/2 = 0.5$ and at $x = 1.5$, $f(1.5) = 0.75$.

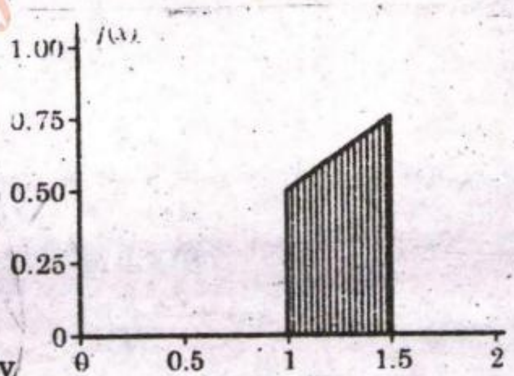
Area

$$= \frac{\text{(sum of parallet sides)}\times\text{base}}{2}$$

$$= \frac{[f(1) + f(1.5)](1.5 - 1)}{2}$$

$$= \frac{1}{2}(0.5 + 0.75)(0.5) = \mathbf{0.3125},$$

which is the required probability

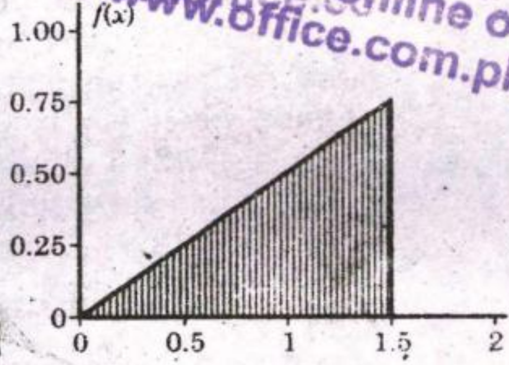


(iii) $P(X < 1.5) = P(0 < X < 1.5)$ is the area between $x = 0$ and $x = 1.5$ as shown shaded in the figure, which is a right-angle triangle whose height is $f(1.5) = 1.5/2 = 0.75$ and base $= 1.5 - 0 = 1.5$.

Area = (height)(base)/2

$= (0.75)(1.5)/2 = \mathbf{0.5625},$

which is the required probability



(iv) $P(X>3) = 0$, since the range of the density function is $0 < x < 2$.

Alternative Solution (i) The function $f(x)$ will be a density function if (1) $f(x) \geq 0$ for every value of x and (2) $\int_{\circledast}^{\circledast} f(x)\,dx = 1$.

For the second condition we must have $\int_0^2 f(x)\,dx = 1$, i.e.

$$\int_0^2 cx\,dx = c\,\frac{x^2}{2}\Big|_0^2 = c\left(\frac{4}{2} - 0\right) = 2c = 1 \text{ or } c = 1/2$$

The density function of X is $f(x) = x/2, \quad 0 < x < 2$.

(ii) $$P(1 < X < 1.5) = \int_1^{1.5} f(x)\,dx = \frac{1}{2}\int_1^{1.5} x\,dx = \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^{1.5}$$

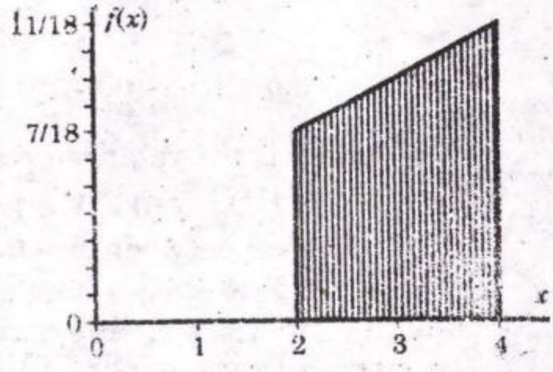
$$= \frac{1}{4}(2.25 - 1) = \mathbf{0.3125}.$$

(iii) $$P(X < 1.5) = P(0 < X < 1.5) = \int_0^{1.5} f(x)\,dx = \frac{1}{2}\int_0^{1.5} x\,dx$$

$$= \frac{1}{2}\cdot\frac{x^2}{2}\Big|_0^{1.5} = \frac{1}{4}(2.25 - 0) = \mathbf{0.5625}.$$

(iv) $$P(X > 3) = \int_3^{\infty} f(x)\,dx = \mathbf{0}.$$

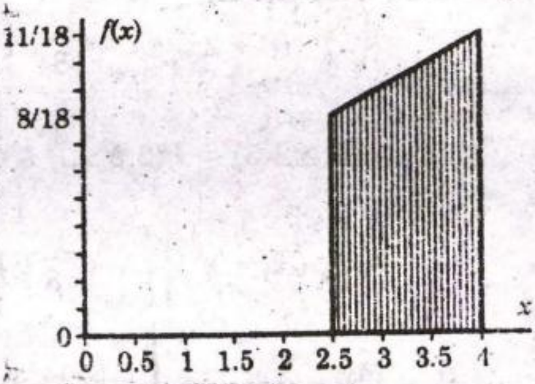
8.6 The density function of X is $f(x) = \dfrac{3 + 2x}{18}$, $2 \leq x \leq 4$.



The function $f(x)$ will be a density function if (1) $f(x) \geq 0$ and (2) The area between $x = 2$ and $x = 4$ is equal to one as shown shaded in the figure, which is a trapezoid. At $x = 2$, $f(2) = (3 + 4)/18 = 7/18$ and at $x = 4$, $f(4) = (3 + 8)/18 = 11/18$.

$$\text{Area} = \frac{[f(2) + f(4)](4 - 2)}{2} = \frac{1}{2}\left(\frac{7}{18} + \frac{11}{18}\right)(2) = \mathbf{1}.$$

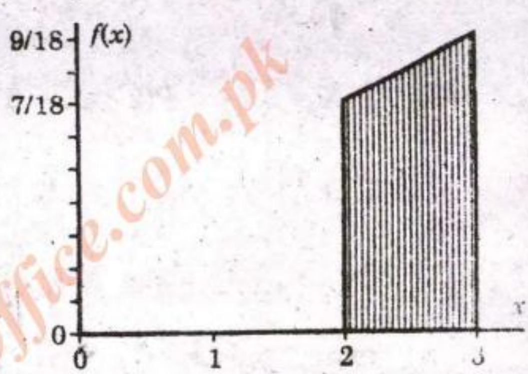
(i) $P(X \geq 2.5) = P(2.5 \leq X \leq 4)$ is the area between $x = 2.5$ and $x = 4$ as shown shaded in the figure. At $x = 2.5$, $f(2.5) = (3 + 5)/18 = 8/18$ and at $x=4$, $f(4)=(3+8)/18=11/18$.



Area = $\dfrac{[f(2.5) + f(4)](4 - 2.5)}{2}$

$= \frac{1}{2}\left(\frac{8}{18} + \frac{11}{18}\right)(1.5) = \mathbf{0.7917}$, which is the required probability.

(ii) $P(2 \leq X \leq 3)$ is the area between $x = 2$ and $x = 3$ as shown shaded in the figure. At $x = 2$, $f(2) = (3 + 4)/18 = 7/18$ and at $x = 3$, $f(3) = (3 + 6)/18 = 9/18$.

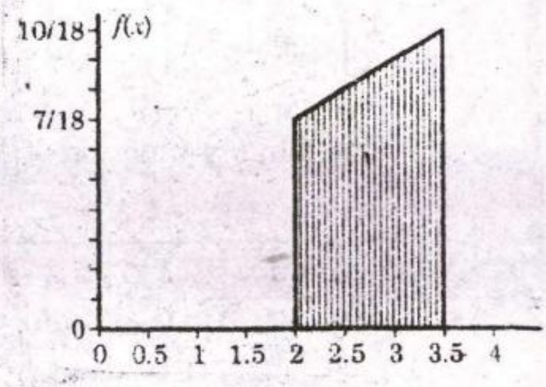


Area = $\dfrac{[f(2) + f(3)](3 - 2)}{2}$

$= \frac{1}{2}\left(\frac{7}{18} + \frac{9}{18}\right) = \frac{4}{9} = \mathbf{0.4444}$,

which is the required probability.

(iii) $P(X < 3.5) = P(2 < X < 3.5)$ is the area between $x = 2$ and $x = 3.5$ as shown shaded in the figure. At $x = 2$, $f(2) = (3 + 4)/18 = 7/18$ and at $x = 3.5$, $f(3.5) = (3 + 7)/18 = 10/18$.



Area = $\dfrac{[f(2) + f(3.5)](3.5 - 2)}{2} = \dfrac{1}{2}$

$\left(\frac{7}{18} + \frac{10}{18}\right)(1.5) = \mathbf{0.7083}$, which is the required probability.

Alternative Solution The function $f(x)$ will be a density function if

(1) $f(x) \geq 0$ and (2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$. For the second condition,

we must have $\int_{2}^{4} f(x)\, dx = 1$.

$$\frac{1}{18}\int_2^4 (3+2x)\,dx = \frac{1}{18}(3x+x^2)\Big|_2^4 = \frac{1}{18}[(12+16)-(6+4)] = \mathbf{1.}$$

(i) $P(X \geq 2.5) = P(2.5 \leq P \leq 4) = \int_{2.5}^4 f(x)\,dx = \frac{1}{18}\int_{2.5}^4 (3+2x)$

$$= \frac{1}{18}(3x+x^2)\Big|_{2.5}^4 = \frac{1}{18}[(12+16)-(7.5+6.25)] = \mathbf{0.7917}$$

(ii) $P(2 \leq X \leq 3) = \int_2^3 f(x)\,dx = \frac{1}{18}\int_2^3 (3+2x)dx = \frac{1}{18}(3x+x^2)\Big|_2^3$

$$= \frac{1}{18}[(9+9)-(6+4)] = \frac{4}{9} = \mathbf{0.4444.}$$

(iii) $P(X < 3.5) = P(2 < X < 3.5) = \int_2^{3.5} f(x)\,dx = \frac{1}{18}\int_2^{3.5}(3+2x)dx$

$$= \frac{1}{18}(3x+x^2)\Big|_2^{3.5} = \frac{1}{18}[(10.5+12.25)-(6+4)] = \mathbf{0.7063.}$$

(b) The function $f(x)$ will be a density function if (1) $f(x) > 0$ for every value of x and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For the second condition, we must have $\frac{1}{k}\int_a^b dx = 1$ i.e.

$$\frac{1}{k}\int_a^b dx = \frac{1}{k}x\Big|_a^b = \frac{1}{k}(b-a) = 1 \text{ or } k = b-a.$$

The density function of X is $f(x) = \frac{1}{b-a}$, $a < x < b$.

8.7

x	0	1	2	3	4	
$P(x)$	$\frac{15}{210}$	$\frac{80}{210}$	$\frac{90}{210}$	$\frac{24}{210}$	$\frac{1}{210}$	$\sum P(x) = 1$
$xP(x)$	0	$\frac{80}{210}$	$\frac{180}{210}$	$\frac{72}{210}$	$\frac{4}{210}$	$\sum xP(x) = 336/210$

$\mu = \sum xP(x) = 336/210 = \mathbf{8/5.}$

8.8(a)

x	-1	0	1	2	3	
$P(x)$	0.125	0.50	0.20	0.05	0.125	
$xP(x)$	-0.125	0	0.20	0.10	0.375	$\sum xP(x) = 0.55$
$x^2P(x)$	0.125	0	0.20	0.20	1.125	$\sum x^2P(x) = 1.65$

(i) $E(X) = \sum xP(x) = \mathbf{0.55}$, (ii) $E(X)^2 = \sum x^2P(x) = \mathbf{1.65}$

(b)(i) $(2x - 3)$ assumes the values $-5, -3, -1, 1, 3$ corresponding to $x = -1, 0, 1, 2, 3$. The probability distribution of $(2X - 3)$ is

$2x - 3$	-5	-3	-1	1	3
$P(2x - 3)$	0.125	0.50	0.20	0.05	0.125

(ii) $(x^2 - 1)$ assumes the values $0, -1, 0, 3, 8$ corresponding to $x = -1, 0, 1, 2, 3$ with probabilities 0.125, 0.50, 0.20, 0.05, 0.125 respectively. The probability distribution of $(X^2 - 1)$ is

$x^2 - 1$	-1	0	3	8
$P(x^2 - 1)$	0.50	0.325	0.05	0.125

8.9(a)

x	$P(x)$	$xP(x)$	$x^2P(x)$	$5x + 10$	$(5x+10)P(x)$	$(5x + 10)^2P(x)$
-3	1/5	$-3/5$	9/5	-5	-1	5
-2	1/10	$-2/10$	4/10	0	0	0
2	1/10	2/10	4/10	20	2	40
3	1/5	3/5	9/5	25	5	125
4	2/5	8/5	32/5	30	12	360
	$\sum xP(x) = 8/5$		$\sum x^2P(x) = 108/10$	$\sum(5x+10)P(x)=18$		$\sum(5x+10)^2P(x) = 530$

$E(X) = \sum xP(x) = \mathbf{8/5} = 1.6$, $E(5x + 10) = \sum(5X + 10)P(x) = 18.$

$E(5X + 10) = 5E(X) + 10 = 5(1.6) + 10 = \mathbf{18}$

(b) $E(X^2) = \sum x^2P(x) = 108/10 = 10.8$

$E(5X + 10)^2 = \sum(5x + 10)^2P(x) = 530$

$\sigma_x^{\ 2} = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 10.8 - (1.6)^2 = \mathbf{8.24}.$

$\sigma_x = \sqrt{8.24} = \mathbf{2.87}.$

$\sigma^2_{(5x+10)} = \text{Var}(5X+10) = E(5X+10)^2 - [E(5X+10)]^2$

$= 530 - (18)^2 = \mathbf{206}.$

$\sigma_{(5x+10)} = \sqrt{206} = \mathbf{14.35}.$

$$\frac{\sigma_x}{\sigma_{(5x+10)}} = \frac{2.87}{14.35} = \mathbf{0.2}.$$

(c) There will be no change in $E(X)$ and σ_x because the relative frequencies 2/10, 1/10, 1/10, 2/10, 4/10 are the same as the probabilities $P(x)$.

8.10(b) Since A and B throw the die alternately and A takes the first turn, A can win if he throws a 6 in either the first, third or fifth, ... throw while B can win if he throws a 6 in second, fourth or sixth, ... throw. Let A_1, A_3, A_5, ... denote the events of getting a 6 on the first, third, fifth..., throw by A and A_1', A_3', A_5', ... denote the event not getting a 6 on the first, third, fifth, ... throw by A. Let B_2, B_4, B_6, ... denote the events of getting a 6 on the second, fourth, six, ... throw by B and B_2', B_4', B_6', ... denote the events of not getting a 6 on the second, fourth, sixth, ... throw by B. Then

$P(A_1) = P(B_2) = \cdots = 1/6$, $P(A_1') = P(B_2') = \cdots = 5/6$

A's chance of success, denoted by $P(A)$, is

$$\begin{aligned} P(A) &= P(A_1) + P(A_1')\,P(B_2')\,P(A_3) \\ &\quad + P(A_1')\,P(B_2')\,P(A_3')\,P(B_4')\,P(A_5) + \cdots \\ &= \frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \cdots \\ &= \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right] \end{aligned}$$

The series in the square brackets is an infinite geometric series whose sum is obtained as

$$S_\infty = \frac{1}{1-(5/6)^2} = \frac{1}{1-25/36} = \frac{36}{11}.$$

Hence $P(A) = \frac{1}{6}\left(\frac{36}{11}\right) = \frac{6}{11}$ and $E(A) = 11\left(\frac{6}{11}\right) = \textbf{Rs.6}.$

Now B's chance of success, denoted by $P(B)$, is

$$P(B) = P(A_1')\,P(B_2) + P(A_1')\,P(B_2')\,P(A_3')\,P(B_4) + P(A_1')\,P(B_2')\,P(A_3')\,P(B_4')\,P(A_5')\,P(B_6) + \cdots$$

$$= \frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \cdots$$

$$= \frac{5}{36}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right] = \frac{5}{36}\left(\frac{36}{11}\right) = \frac{5}{11}.$$

Alternatively, $P(B) = 1 - P(A) = 1 - \frac{6}{4} = \frac{5}{11}$.

Hence $E(B) = 11\left(\frac{5}{11}\right) =$ **Rs. 5.**

8.11 $P(\text{both white balls}) = {}^3C_2/{}^5C_2 = \frac{3}{10}$

$P(\text{both black balls}) = {}^2C_2/{}^5C_2 = \frac{1}{10}$

$P(\text{one white and one black ball}) = {}^3C_1\,{}^2C_1/{}^5C_2 = \frac{6}{10}$

The required expectation is given by

$$E(X) = 140\left(\frac{3}{10}\right) + 14\left(\frac{1}{10}\right) + 77\left(\frac{6}{10}\right)$$

$$= 42 + 1.4 + 46.2 = \text{Rs. } 89.60.$$

8.12 Let A, B, C denote hot, fair and cloudy respectively.

$P(A) = 0.8$, $P(C) = 0.05$, $P(B) = 1 - (0.8 + 0.05) = 0.15$

Then $E(A) = 500(0.8) = \text{Rs. } 400$, $E(B) = 200(0.15) = \text{Rs. } 30$ and $E(C) = 60(0.05) = \text{Rs. } 3$.

The required expectation is given by

$E(X) = E(A) + E(B) - E(C) = 400 + 30 - 3 =$ **Rs. 427.**

8.13(a) Let A denote the event 'sum less than 5'. The sample space contains 36 sample points and the event A contains 6 sample points: {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)}.

$P(A) = 6/36 = 1/6$. The number of times 'sum will be less than 5' is $E(A) = 900(1/6) =$ **150.**

(b) Let X denote the number of heads in tossing 3 coins.

Probability of getting 3 heads in a single toss of 3 coins is

$$P(X = 3) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

The required expectation is $E(X) = 80(1/8) =$ **Rs.10.**

8.14 The probability function of X(number of black balls) is

$$P(X=x) = {}^7C_x\, {}^4C_{5-x} / {}^{11}C_5, \quad x = 0, 1, 2, 3, 4, 5$$

$$P(X=0) = {}^7C_0\, {}^4C_5 / {}^{11}C_5 = 0, \qquad ({}^4C_5 = 0)$$

$$P(X=1) = {}^7C_1\, {}^4C_4 / {}^{11}C_5 = \frac{7}{462},\ P(X=2) = {}^7C_2\, {}^4C_3 / {}^{11}C_5 = \frac{84}{462}$$

$$P(X=3) = {}^7C_3\, {}^4C_2 / {}^{11}C_5 = \frac{210}{462},\ P(X=4) = {}^7C_4\, {}^4C_1 / {}^{11}C_5 = \frac{140}{462}$$

$$P(X=5) = {}^7C_5\, {}^4C_0 / {}^{11}C_5 = \frac{21}{462}$$

The probability distribution of X is given in the following table where computation of its mean and variance is shown.

x	1	2	3	4	5	
$P(x)$	$\frac{7}{462}$	$\frac{84}{462}$	$\frac{210}{462}$	$\frac{140}{462}$	$\frac{21}{462}$	$\sum P(x) = 1$
$xP(x)$	$\frac{7}{462}$	$\frac{168}{462}$	$\frac{630}{462}$	$\frac{560}{462}$	$\frac{105}{462}$	$\sum x\, P(x) = \frac{1470}{462}$
$x^2P(x)$	$\frac{7}{462}$	$\frac{336}{462}$	$\frac{1890}{462}$	$\frac{2240}{462}$	$\frac{525}{462}$	$\sum x^2P(x) = \frac{4998}{462}$

$$\mu = \sum xP(x) = \frac{1470}{462} = \frac{35}{11} = 3.182.$$

$$\sigma^2 = \sum x^2P(x) - \mu^2 = \frac{4998}{462} - \left(\frac{35}{11}\right)^2 = \frac{119}{11} - \frac{1225}{121} = \frac{84}{121}$$

$$= \mathbf{0.694.}$$

8.15(a) Let p denote the probability of a tail. Thus $p = 1/2$, $q = 1 - p = 1/2$, $n = 4$. The probability function of X is

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}, \qquad x = 0, 1, 2, 3, 4$$

In tabular form, the probability distribution of X is given in the first two rows of the following table. Computation of $E(Y)$ and $\text{Var}(Y)$ is also shown in the table.

x	0	1	2	3	4	
$P(x)$	1/16	4/16	6/16	4/16	1/16	
$y = 2x$	0	2	4	6	8	
$P(y)$	1/16	4/16	6/16	4/16	1/16	
$yP(y)$	0	8/16	24/16	24/16	8/16	$\Sigma yP(y) = 64/16$
$y^2P(y)$	0	16/16	96/16	144/16	64/16	$\Sigma y^2P(y) = 320/16$

(b) $E(Y) = \Sigma yP(y) = 64/16 = 4$, $E(Y^2) = \Sigma y^2 P(y) = 320/16 = 20$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 20 - (4)^2 = 4.$

8.16 Let p and q denote respectively the probability of a head and the probability of a tail. Since the head is twice as likely to occur as the tail, we have $p = 2q$. Now $p + q = 1$ or $2q + q = 1$ or $3q = 1$ or $q = 1/3$ and $p = 2/3$.

The probability function of X(number of heads when the coin is tossed $n = 4$ times) is given by

$$P(X = x) = {}^4C_x p^x q^{n-x}, \qquad x = 0, 1, 2, 3, 4$$

$$= {}^4C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{4-x}, \qquad x = 0,1,2,3,4$$

$$P(X = 0) = {}^4C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^4 = \frac{1}{81}, P(X = 1) = {}^4C_1 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 = \frac{8}{81}$$

$$P(X = 2) = {}^4C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^2 = \frac{24}{81}, P(X = 3) = {}^4C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right) = \frac{32}{81}$$

$$P(X = 4) = {}^4C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^0 = \frac{16}{81}$$

The probability distribution of X is given in the following table where computation of its mean and variance is shown.

x	0	1	2	3	4	
$P(x)$	1/81	8/81	24/81	32/81	16/81	$\Sigma P(x) = 1$
$xP(x)$	0	8/81	48/81	96/81	64/81	$\Sigma xP(x) = 216/81$
$x^2P(x)$	0	8/81	96/81	288/81	256/81	$\Sigma x^2P(x) = 648/81$

$$\mu = \sum xP(x) = \frac{216}{81} = \frac{8}{3}.$$

$$\sigma^2 = \sum x^2P(x) - \mu^2 = \frac{648}{81} - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \frac{8}{9}.$$

8.17(a) The probability distribution of X in tabular form is given in the following table where the computation of the mean and variance is also shown. (For example, For $x = 9$, $f(x) = \frac{6 - |7-9|}{36} = \frac{6 - |-2|}{36} = \frac{6-2}{36} = \frac{4}{36}$)

x	2	3	4	5	6	7
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36
$xf(x)$	2/36	6/36	12/36	20/36	30/36	42/36
$x^2f(x)$	4/36	18/36	48/36	100/36	180/36	294/36
x	8	9	10	11	12	
$f(x)$	5/36	4/36	3/36	2/36	1/36	$\sum f(x) = 1$
$xf(x)$	40/36	36/36	30/36	22/36	12/36	$\sum xf(x) = 252/36$
$x^2f(x)$	320/36	324/36	300/36	242/36	144/36	$\sum x^2f(x) = 1974/36$

$\mu = E(X) = \sum xf(x) = \frac{252}{36} = 7$. $E(X^2) = \sum x^2f(x) = \frac{1974}{36} = \mathbf{54.8333.}$

$\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = \frac{1974}{36} - (7)^2 = 54.8333 - 49 = \mathbf{5.8333.}$

(b)(i) For the probability distribution of sum of dots on the two dice, see Example 8.7 in the Text.

(ii) The difference of dots on the dice X and Y, i.e. $(X - Y)$ assumes the values –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5 with frequency of occurrence 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1 respectively. The probability distribution of $Z = (X - Y)$ is obtained in the following table.

z	–5	–4	–3	–2	–1	0	1	2	3	4	5
$P(z)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

8.18

x	P(x)	x P(x)	x^2 P(x)
1	6/9	6/9	6/9
2	2/9	4/9	8/9
3	1/9	3/9	9/9
		13/9	23/9

$\mu = \Sigma x\, P(x) = 13/9$

$$\sigma = \sqrt{E(X^2) - [E(X)]^2} = \sqrt{\frac{23}{9} - \left(\frac{13}{9}\right)^2} = \sqrt{\frac{23}{9} - \frac{169}{81}}$$

$$= \sqrt{2.5556 - 2.0864} = \sqrt{0.4692} = 0.6850$$

8.19

x	0	1	2	3	4	
$P(x)$	1/126	20/126	60/126	40/126	5/126	$\Sigma P(x) = 1$
$xP(x)$	0	20/126	120/126	120/126	20/126	$\Sigma xP(x) = 280/126$
$2x+3$	3	5	7	9	11	
$(2x+3)P(x)$	3/126	100/126	420/126	360/126	55/126	$\Sigma(2x+3)P(x) = 938/126$

$$E(X) = \Sigma xP(x) = \frac{280}{126} = \frac{20}{9} \cdot E(2x+3) = \Sigma(2x+3)P(x) = \frac{938}{126} = \frac{67}{9}.$$

$$E(2x+3) = 2E(X) + 3 = 2\left(\frac{20}{9}\right) + 3 = \frac{67}{9}.$$

8.20 There are 36 outcomes when two dice are thrown simultaneously. The variable X (sum of outcomes on both dice) assumes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

(i) Range of X = max. value – min. value = 12 – 2 = 10.

(ii) 'Sum 2' consists of 1 outcome: (1, 1)

'Sum 3' consists of 2 outcomes: {(1, 2), (2, 1)}

'Sum 4' consists of 3 outcomes: {(1, 3), (2, 2), (3, 1)}

'Sum 5' consists of 4 outcomes: {(1, 4), (2, 3), (3, 2), (4, 1)}

'Sum 6' consists of 5 outcomes:

$\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

'Sum 7' consists of 6 outcomes:

$\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

'Sum 8' consists of 5 outcomes:

$\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

'Sum 9' consists of 4 outcomes: $\{(3, 6), (4, 5), (5, 4), (6, 3)\}$

'Sum 10' consists of 3 outcomes: $\{(4, 6), (5, 5), (6, 4)\}$

'Sum 11' consists of 2 outcomes: $\{(5, 6), (6, 5)\}$

'Sum 12' consists of 1 outcome: $\{(6, 6)\}$

The probability distribution of X is obtained as

x	2	3	4	5	6	7
$P(x)$	1/36	2/36	3/36	4/36	5/36	6/36
x	8	9	10	11	12	
$P(x)$	5/36	4/36	3/36	2/36	1/36	

(iii) The mean and variance of X is computed in the solution of Exercise 8.22(b): $\mu = 7$ and $\sigma^2 = 5.8333$.

8.21(a) (i) The function $f(x)$ will be a density function if

(1) $f(x) \geq 0$ for every value of x and (2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

For the second condition we must have $\int_0^2 f(x)\, dx = 1$, i.e.

$$\int_0^2 A(2x^3 + 1)dx = A\left(\frac{2x^4}{4} + x\right)\Big|_0^2 = A[(8 + 2) - 0] = 10A = 1 \text{ or}$$

$A = 1/10$.

The density function of X is $f(x) = (2x^3 + 1)/10, \quad 0 \leq x \leq 2$.

(ii) $P(X > 1.5) = P(1.5 < X < 2) = \frac{1}{10}\int_{1.5}^{2}(2x^3 + 1)dx$

$$= \frac{1}{10}\left(\frac{x^4}{2} + x\right)\Big|_{1.5}^{2} = \frac{1}{10}\left[(8 + 2) - \left(\frac{5.0625}{2} + 1.5\right)\right]$$

$$= \frac{1}{10}(10 - 4.03125) = \mathbf{0.5969}$$

(iii) $P(X < 1.2) = P(0 < X < 1.2) = \frac{1}{10}\int_0^{1.2}(2x^3 + 1)dx$

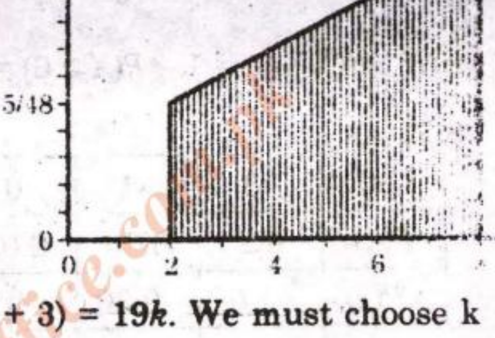
$$= \frac{1}{10}\left(\frac{x^4}{2} + x\right)\Bigg|_0^{1.2} = \frac{1}{10}\left[\frac{2.0736}{2} + 1.2 - 0\right] = \mathbf{0.2237.}$$

(iv) $P(1 < X < 2) = \frac{1}{10}\int_1^2 (2x^3 + 1)dx = \frac{1}{10}\left(\frac{x^4}{2} + x\right)\Bigg|_1^2$

$$= \frac{1}{10}\left[(8 + 2) - \left(\frac{1}{2} + 1\right)\right] = \frac{1}{10}(10 - 1.5) = \mathbf{0.85.}$$

(b)(i) The density function of X is $f(x) = k(2x + 3)$, $\quad 2 \le x \le 8$

The graph of $f(x) = k(2x + 3)$ is a straight line as shown in the figure. We know that the total area under the line between $x = 2$ and $x = 8$ and above the X-axis must be 1. At $x = 2$, $f(2) = k(4 + 3) = 7k$ and at $x = 8$, $f(8) = k(16 + 3) = 19k$. We must choose k so that the trapezoidal area is 1.

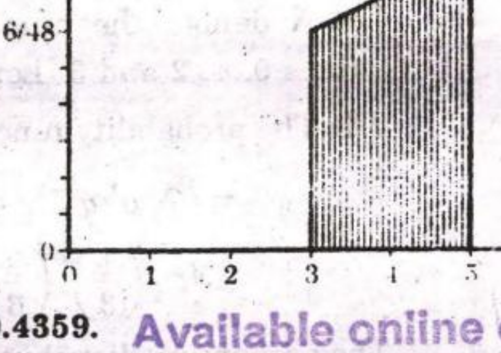


Area $= \frac{[f(2) + f(8)]\text{base}}{2} = 1$ or $\frac{1}{2}(7k + 19k)(8 - 2) = 78k = 1$ or

$k = \mathbf{1/78.}$

(ii) The density function of X is $f(x) = (2x + 3)/78$, $2 \le x \le 8$.

$P(X \ge 6) = P(6 \le X \le 8)$ is the area between $x = 6$ and $x = 8$ as shown shaded in the figure. At $x = 6$, $f(2) = (12 + 3)/78 = 15/78$ and at $x = 8$, $f(8) = (16 + 3)/78 = 19/78$.



Area $= \frac{[f(2) + f(8)]\text{base}}{2}$

$= \frac{1}{2}\left(\frac{15}{78} + \frac{19}{78}\right)(8 - 6) = \frac{34}{78} = \mathbf{0.4359.}$

(iii) $P(X \le 6) = 1 - P(X \ge 6) = 1 - \frac{34}{78} = \frac{44}{78} = \mathbf{0.5641.}$

Alternative Solution (i) The function will be a density function if $f(x) \ge 0$ for every value of x and (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

For the second condition, we must have $\int_2^8 f(x)\, dx = 1$ i.e.

$$\int_2^8 k(2x+3)\, dx = k(x^2+3x)\Big|_2^8 = k[(64+24)-(4+6)]^2$$

$= 78k = 1$ or $k = \mathbf{1/78}$.

(ii) $P(X \ge 6) = P(6 \le X \le 8) = \int_6^8 f(x)\, dx = \frac{1}{78}\int_6^8 (2x+3)\, dx$

$$= \frac{1}{78}(x^2+3x)\Big|_6^8 = \frac{1}{78}[(64+24)-(36+18)] = \frac{34}{78} = \mathbf{0.4359.}$$

(iii) $P(X \le 6) = 1 - P(X \ge 6) = 1 - \frac{34}{78} = \frac{44}{78} = \mathbf{0.5641.}$

8.22(a)

x	-5	-1	0	1	5	
$P(x)$	0.2	0.30	0.05	0.15	0.30	$\sum P(x) = 1$
$xP(x)$	-1.0	-0.30	0	0.15	1.5	$\sum xP(x) = 0.35$
$x^2P(x)$	5.0	0.30	0	0.15	7.5	$\sum x^2P(x) = 12.95$

$\mu = E(X) = \sum xP(x) = 0.35.$

$\sigma^2 = E(X^2) - \mu^2 = \sum x^2P(x) - \mu^2 = 12.95 - (0.35)^2 = 12.8275$

$\sigma = \sqrt{12.8275} = 3.58.$

C.V. $= \frac{\sigma}{\mu} \times 100 = \left(\frac{3.58}{0.35}\right) 100 = \mathbf{1022.86\%}$

(b) Let X denote the number of girls. Then X assumes the values 0, 1, 2 and 3. Let $p = P(\text{girl}) = 1/3$ and $q = P(\text{boy}) = 2/3$. The probability function of X is

$$P(X=x) = {}^3C_x\, p^x q^{3-x}, \qquad x = 0, 1, 2, 3$$

$$= {}^3C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x}, \qquad x = 0, 1, 2, 3$$

The probability distribution of X in tabular form is

x	0	1	2	3
$P(x)$	8/27	12/27	6/27	1/27

8.23. Sample space

S = {HHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

Let X denote the no. of tail

X	0	1	2	3	4	
f(X)	1/16	4/16	6/16	4/16	1/16	
x p(x)	0	4/16	12/16	12/16	4/16	$\Sigma x\, p(x) = 32/16$
x^2 p(x)	0	4/16	24/16	36/16	16/16	$\Sigma x^2\, p(x) = 80/16$

$$\mu = \Sigma x\, p(x)$$
$$= 32/16 = 2$$
$$\sigma^2 = E(X)^2 - [E(X)]^2$$
$$= \frac{80}{16} - (2)^2$$
$$= 5 - 4$$
$$= 1$$

8.24. $f(x) = \frac{4-x}{4} \quad 1 \le x \le 3$

Sol.(1) Area $= \frac{\text{Sum of the parallel sides}}{2} \times \text{Base}$

$$f(1) = \frac{4-1}{4} = 3/4$$

$$f(3) = \frac{4-3}{4} = 1/4$$

$$\text{Area} = \frac{f(1) + f(3)] \times (3-1)}{2}$$

$$= \frac{\frac{3}{4} + \frac{1}{4}}{2} \times 2$$

$$= \frac{4}{4} = 1$$

Yes this is a valid probability function.

(2) $f(x) = 1/4 \qquad -2 \le x \le 3$

$$\text{Area} = \frac{\text{Sum of the parallel sides}}{2} \times \text{Base}$$

$$f(-2) = \frac{1}{4}$$

$$f(3) = 1/4$$

$$\text{Area} = \frac{\left(\frac{1}{4} + \frac{1}{4}\right) \times (3 - (-2))}{2}$$

$$= \frac{\frac{2}{4} \times 5}{2}$$

$$= \frac{10}{8} = 1.25$$

No this is not a valid probability function because probability cannot exceed 1.

(3) $f(x) = \frac{x+1}{8} \qquad 2 \le x \le 4$

$$\text{Area} = \frac{\text{Sum of the parallel sides}}{2} \times \text{Base}$$

$$f(2) = \frac{2+1}{8} = 3/8$$

$$f(4) = \frac{4+1}{8} = 5/8$$

$$\text{Area} = \frac{[f(2) + f(4)]}{2} \times (4 - 2)$$

$$= \frac{\frac{3}{8} + 5/8}{2} = 8/8$$

$$= 1$$

Yes this is a valid probability function.

8.25. $f(x) = \frac{2(5-x)}{25} \qquad 0 \le x < 5$

(1) $\text{Area} = \frac{\text{Sum of the parallel sides}}{2} \times \text{Base}$

$$f(0) = \frac{2(5-0)}{25} = \frac{10}{25}$$

$$f(5) = \frac{2(5-5)}{25} = 0$$

$$\text{Area} = \frac{[f(0) + f(5)]}{2} \times (5-0)$$

$$= \frac{\frac{10}{25} + 0}{2} \times 5$$

$$= \frac{\frac{10}{25} \times 5}{2}$$

$$= 1$$

Yes this is a p.d.f.

(2) $P(X < 4) = P(0 < X < 4)$

$$\text{Area} = \frac{[f(0) + f(4)] \times 4 - 0}{2}$$

$$f(0) = \frac{2(5-0)}{25} = \frac{10}{25}$$

$$f(4) = \frac{2(5-4)}{25} = \frac{2}{25}$$

$$\text{Area} = \frac{\left(\frac{10}{25} + \frac{2}{25}\right) \times 4}{2}$$

$$= \frac{12}{25} \times 2$$

$$= \frac{24}{25}$$

(3) $P(X > 6) = 0$

Becaus continuous function lies in this range $0 < X < 5$.

(4) $P(2 < X < 3)$

$$\text{Area} = \frac{[f(2) + f(3)] \times (3-2)}{2}$$

$$f(2) = \frac{2(5-2)}{25} = \frac{6}{25}$$

$$f(3) = \frac{2(5-3)}{25} = \frac{4}{25}$$

$$\text{Area} = \frac{\left(\frac{6}{25}+\frac{4}{25}\right)}{2} \times 1$$

$$= \frac{10}{25} \times \frac{1}{2}$$

$$= \frac{1}{5}$$

(5) $P(1 < X < 4)$

$$\text{Area} = \frac{[\text{Sum of the parallel sides}] \times \text{Base}}{2}$$

$$f(1) = \frac{2(5-1)}{25} = 8/25$$

$$f(4) = \frac{2(5-4)}{25} = \frac{2}{25}$$

$$\text{Area} = \frac{\left[\frac{8}{25}+\frac{2}{25}\right] \times (4-1)}{2}$$

$$= \frac{\frac{10}{25} \times 3}{2}$$

$$= \frac{30}{25} \times \frac{1}{2}$$

$$= \frac{30}{50}$$

$$= 3/5$$

8.26.

X	P(x)
1	a
5	b
9	c

$P(X > 4) = P(X > 4)$

$a = b + c$ (1)

$P(X \leq 5) = 2P(X < 5)$

$b + c = 2C$ (2)

From eq. (1) and (2)

$= 2C$

$a + b + c = 1$

$a + a = 1$

$2a = 1$

$a = 1/2$

$a = 2C$

$\frac{1}{2} = 2C$

$c = \frac{1}{4}$

$a = b + C$

$\frac{1}{2} = b + \frac{1}{4}$

$\frac{1}{2} - \frac{1}{4} = b$

$\frac{2-1}{4} = b$

$b = 1/4$

8.27.

x	0	1	3	4	6	
$P(x)$	1/16	1/4	3/8	1/4	1/16	$\Sigma P(x) = 1$
$x\,P(x)$	0	1/4	9/8	1	6/16	$\Sigma xP(x) = 11/4$
$x^2\,P(x)$	0	1/4	27/8	4	36/16	$\Sigma x^2P(x) = 158/16$

(i) $\mu = E(X) = x\,\Sigma P(X) = 11/4$

$$\sigma^2 = E(X)^2 - \mu^2 = \frac{158}{16} - \left(\frac{11}{4}\right)^2 = \frac{37}{16}$$

(ii) $Y = 2X - 5$

$$\bar{Y} = 2\bar{X} - 5 = 2\left(\frac{11}{4}\right) - 5 = \frac{1}{2}$$

$$\text{Var}(Y) = \text{Var}(2X - 5) = 4\,\text{Var}(X) + \text{Var}(5)$$

$$= 4\left(\frac{37}{16}\right) + 0 = \frac{37}{4}$$

8.28.

x	0	1	2	3	
$P(x)$	0.1	0.2	0.3	0.4	$\Sigma P(x) = 1$
$x\,P(x)$	0	0.2	0.6	1.2	$\Sigma x\,P(x) = 2.0$
$x^2\,P(x)$	0	0.2	1.2	3.6	$\Sigma x^2 P(x) = 5.0$
$Y=5X+8$	8	13	18	23	
$P(y)$	0.1	0.2	0.3	0.4	$\Sigma P(y) = 1$
$y\,P(y)$	0.8	2.6	5.4	9.2	$\Sigma y\,P(y) = 18.0$

$\mu = E(X) = \Sigma x\,P(x) = 2.0$

$\sigma^2 = E(X^2) - \mu^2 = 5.0 - (2.0)^2 = 1.0$

$E(5X + 8) = E(Y) = \Sigma y\,P(y) = 18.0$

$5E(X) + 8 = 5(2.0) + 8 = 18.0$

Thus $E(5X + 8) = 5E(X) + 8$

8.29. $f(x) = cx, \quad 0 \le x \le 2.$

(i) $\int f(x)\,dx = 1$ or $\int_0^2 cx\,dx = 1$ or $c\dfrac{x^2}{2}\Big|_0^2 = 1$

or $\dfrac{c}{2}(4 - 0) = 1$ or $c = \dfrac{1}{2}$.

$f(x) = \dfrac{x}{2}, \quad 0 \le x \le 1$

$$P(X < 1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^1 = \frac{1}{4}(1 - 0) = \frac{1}{4}$$

$$P\left(\frac{1}{2} \le X \le \frac{3}{2}\right) = \int_{1/2}^{3/2} \frac{x}{2}\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_{1/2}^{3/2} = \frac{1}{4}\left(\frac{9}{4} - \frac{1}{4}\right) = \frac{1}{2}$$

$P(X > 3) = 0$, since the function outside the limits $0 \le x \le 2$ is zero.

8.30. $f(x) = \dfrac{1}{18}(3 + 2x), \quad 2 \le x \le 4.$

The function $f(x)$ is a density function if $\int f(x)\,dx = 1$

i.e. $\displaystyle\int_2^4 \frac{1}{18}(3 + 2x)\,dx = 1$

Now $\frac{1}{18}\int_2^4 (3+2x)dx = \frac{1}{8}(3x+x^2)\Big|_2^4$

$= \frac{1}{18}[(12+16)-(6+4)] = 1$

(i) $P(X>2.5) = \int_{2.5}^4 f(x)\,dx = \int_{2.5}^4 \frac{1}{18}(3+2x)\,dx$

$= \frac{1}{18}(3x+x^2)\Big|_{2.5}^4 = \frac{1}{18}[(12+16)-(7.5+6.25)]$

$= \frac{1}{18}(14.25) = 0.7917$

(ii) $P(2 \le X \le 3) = \int_2^3 \frac{1}{18}(3+2x)dx = \frac{1}{18}(3x+x^2)\Big|_2^3$

$= \frac{1}{18}[(9+9)-(6+4)] = \frac{4}{9}$

$P(X<3.5) = \int_2^{3.5} \frac{1}{18}(3+2x)dx = \frac{1}{18}(3x+x^2)\Big|_2^{3.5}$

$= \frac{1}{18}[(10.5+12.25)-(6+4)] = 0.7083$

8.31 $f(x) = a(x+3), \quad 2<x<8.$

(i) $\int f(x)\,dx = 1$ or $\int_2^8 a(x+3)\,dx = 1$ or $a\left(\frac{x^2}{2}+3x\right)\Big|_2^8 = 1$

or $a[32+24)-(2+6) = 1$ or $48a = 1$ or $a = 1/48$

$f(x) = \frac{1}{48}(x+3), \quad 2<x<8$

$P(X<6) = \int_2^6 \frac{1}{48}(x+3)\,dx = \frac{1}{48}\left(\frac{x^2}{2}+3x\right)\Big|_2^6$

$= \frac{1}{48}[(18+18)-(2+6)] = \frac{28}{48} = \frac{7}{12}$

8.32. If $P(\text{tail}) = a$, then $P(\text{head}) = 2a$, so that
$P(\text{tail}) + P(\text{head}) = 1$ or $a+2a = 1$ or $a = 1/3$.
Let X denote the number of heads. Then X takes the values 0, 1, 2, 3, 4.

$P(X=0) = P(\text{all 4 tails}) = P(TTTT) = P(T)P(T)P(T)P(T)$

$$= \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{81}$$

$P(X=1) = P(\text{one head and 3 tails})$

$- P(HTTT \text{ or } THTT \text{ or } TTHT \text{ or } TTTH)$

$= P(H)P(T)P(T)P(T) + P(T)P(H)P(T)P(T)$

$+ P(T)P(T)P(H)P(T) + P(T)P(T)P(T)P(H)$

$$= \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}$$

$$+ \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{8}{81}$$

$P(X=2) = P(\text{2 head 2 tails})$

$= P(HHTT \text{ or } HTHT \text{ or } HTTH \text{ or } TTHH \text{ or } THTH$
$\text{ or } THHT)$

$= P(H)P(H)P(T)P(T) + P(H)P(T)P(H)P(T)$

$+ P(H)P(T)P(T)P(H) + P(T)P(T)P(H)P(H)$

$+ P(T)P(H)P(T)P(H) + P(T)P(H)P(H)P(T)$

$$= \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}$$

$$+ \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{24}{81}$$

$P(X=3) = P(\text{3 head 1 tail})$

$= P(HHHT \text{ or } HHTH \text{ or } HTHH \text{ or } THHH)$

$= P(H)P(H)P(H)P(T) + P(H)P(H)P(T)P(H)$

$+ P(H)P(T)P(H)P(H) + P(T)P(H)P(H)P(H)$

$$= \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}$$

$$= \frac{24}{81}$$

$P(X=4) = P(\text{all 4 heads})$

$= P(HHH) = P(H)P(H)P(H)P(H)$

$$= \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{16}{81}$$

The probability distribution of X is given by

x	0	1	2	3	4	
$P(x)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$	$\Sigma P(x)=1$
$x\,P(x)$	0	$\frac{8}{81}$	$\frac{48}{81}$	$\frac{96}{81}$	$\frac{64}{81}$	$\Sigma x\,P(x) = \frac{216}{81}$
$x^2\,P(x)$	0	$\frac{8}{81}$	$\frac{96}{81}$	$\frac{288}{81}$	$\frac{256}{81}$	$\Sigma x^2\,P(x) = \frac{648}{81}$

$$\mu = E(X) = \frac{216}{81} = \frac{8}{3}$$

$$\sigma^2 = E(X^2) - \mu^2 = \frac{648}{81} - \frac{64}{9} = 8 - \frac{64}{9} = \frac{8}{9}$$

8.32(b)

(i) $P(X = 0) = \dfrac{{}^6C_0 \cdot {}^{48}C_6}{{}^{54}C_6} = \dfrac{12271512}{25827165} = 0.47514$

(ii) P(At least one will be winning no)

$$= 1 - p(\text{no number win}) = 1 - \frac{{}^6C_0 \cdot {}^{48}C_6}{{}^{54}C_6}$$

$$= 1 - 0.4751 = 0.5249$$

8.33 $f(x) = 2x \quad 0 < x < 1$

(i) $P(X \leq 1/2)$ (ii) $P(x > 1/4)$ (iii) $P\left(\frac{1}{4} < x < 1/2\right)$

(i) $P(x \leq 1/2) = \left[\dfrac{f(1/2) + f(\theta)}{2}\right] \times \text{Base}$

$$= \left[\frac{2(1/2) + 2(\theta)}{2}\right] \times (1/2 - 0) = \left[\frac{1}{2}\right][1/2] = \frac{1}{4}$$

(ii) $P(x > 1/4) = \left[\dfrac{f(1) + f(1/4)}{2}\right] \times \text{Base}$

$$= \left[\frac{2(1) + 2(1/4)}{2}\right](1 - 1/4)$$

$$= \left[\frac{2 + 1/2}{2}\right][3/4] = = \left[\frac{5}{4}\right][3/4] = \frac{15}{16}$$

(iii) $P\left(\frac{1}{4} < x < 1/\dot{2}\right)$

$$= \left[\frac{f(1/2) + f(1/4)}{2}\right]\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \left[\frac{\frac{2}{2} + \frac{2}{4}}{2}\right]\left(\frac{1}{4}\right)^2 = \left[\frac{1 + \frac{1}{2}}{2}\right]\left(\frac{1}{4}\right)$$

$$= \left(\frac{1}{4}\right) = \frac{3}{16}$$

8.34

X	P(X)	P(X)
–2	0.1	0.1
–1	K	0.667
0	0.2	0.2
1	2K	0.1333
2	0.3	0.3
3	3K	0.19998
	6K + 0.6	

As Total Area equal to one

$\Sigma P(x) = 1$

(i) $6K + 0.6 = 1$

$6K = 1 - 0.6$

$K = \frac{0.4}{6} = 0.067$

(ii) $P(X = -2) + P(x = -1) + P(x = 0) + P(x = 1)$

$P(X < 2) = 0.1 + 0.067 + 0.2 + 0.133 = 0.5$

(iii) $P(-2 < X < 2) = 0.400$

8.35(a)

Y	P(y)
1	C
2	2C
3	3C
4	4C
	$\Sigma p(y) =$ 10C

$\Sigma p(y) = 1, \quad 10C = 1, \quad C = 1/10$

y	P(y)	P(y)
0	$(1 - C)C^0$	$(1 - C)$
1	$(1 - C)C^1$	$(1 - C)C$
2	$(1 - C)C^2$	$(1 - C)C^2$

$(1 - C) + (1 - C)C + (1 - C)C^2 + \ldots$

$(1 - C)\,[1 + C + C^2 + \ldots]$

Solving this term by geometric series

$1 + C + C^2 + \ldots$

$a = 1, r = C$

$S = \frac{a}{1 - r} = \frac{1}{1 - C}$

$(1 - C)\left[\frac{1}{1 - C}\right] = 1$

8.36

X	P(X)	X P(X)	X^2 P(X)
0	0.1	0	0
1	0.2	0.2	0.2
2	0.3	0.6	1.2
3	0.2	0.6	1.8
4	0.1	0.4	1.6
5	0.1	0.5	2.5
		$\Sigma xP(x) = 2.3$	$\Sigma x^2P(x) = 7.3$

$\mu = E(X) = \Sigma xP(x) = 2.3$

$\sigma^2 = E(X^2) - [E(X)]^2$

$= 7.3 - (2.3)^2$

$= 7.3 - 5.29$

$= 2.01$

8.37

X	*f(x)*	*x f(x)*	$x^2 f(x)$	*3X – 2*	*E(3X–2)*	*f(3X–2)*
3	0.1	0.3	0.9	7	0.1	0.7
5	0.2	1	5	13	0.2	2.6

X	f(x)	x f(x)	x^2 f(x)	3X − 2	E(3X−2)	f(3X−2)
7	0.4	2.8	19.6	19	0.4	7.6
9	0.2	1.8	16.2	25	0.2	5
11	0.1	1.1	12.1	31	0.1	3.1
		7	53.8			19

$E(X) = \Sigma x\, f(x) = 7$

$Var(X) = E(X^2) - [E(X)]^2 = \Sigma x^2 f(x) - [\Sigma x\, f(x)]^2 = 53.8 - (7)^2$

$= 53.8 - 49 = 4.8$

$E(3X - 2) = (3X - 2)\, f(3X - 2) = 19$

8.38

X	P(x)	x P(x)	x^2 P(x)
1	0.05	0.05	0.05
2	0.04	0.08	0.16
3	0.10	0.30	0.90
4	0.24	0.96	3.84
5	0.05	0.25	1.25
		1.64	6.2

$\mu = E(X)$

$= \Sigma x\, P(x) = 1.64$

$E(X^2) = \Sigma x^2 P(x)$

$= 6.2$

8.39 A random variable has the probability distribution.

X	0	1	2	3
P(x)	0.1	0.2	0.3	0.4

Show that $E(5X + 8) = 5\,E(X) + 8$. (B.I.S.E. Lahore, 2017)

Sol.

X	P(x)	x P(x)	5X + 8	P(5X + 8)	(5X + 8) P(5X + 8)
0	0.1	0	8	0.1	0.8
1	0.2	0.2	13	0.2	2.6
2	0.3	0.6	18	0.3	5.4
3	0.4	1.2	23	0.4	9.2
		Σx P(x) = 2			