

# Discrete Structures

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# Text book

Discrete Mathematics and Its Application, 7<sup>th</sup> Edition

Kenneth H. Rosen

# References

## Chapter 5

1. Discrete Mathematics and Its Application, 7<sup>h</sup> Edition

By Kenneth H. Rose

2. Discrete Mathematics with Applications

By Thomas Koshy

These slides contain material from the above resources.

# Principle of Mathematical Induction

To prove that  $P(n)$  is true for all positive integers  $n$ , where  $P(n)$  is a propositional function, we complete two steps:

**Basis Step:** We verify that  $P(1)$  is true.

**Inductive Step:** We show that the conditional statement  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ .

Expressed as a **rule of inference**, this proof technique can be stated as

$(P(1) \wedge \forall k (P(k) \rightarrow P(k + 1))) \rightarrow \forall n P(n)$ , when the domain is the set of positive integers.

**Example:** Conjecture a formula for the **sum of the first  $n$  positive odd integers**. Then prove your conjecture using mathematical induction.

## Solution

$$1 + 3 + 5 + \dots + (2n - 1) = n^2$$

Let  $P(n)$  be the proposition that the sum of the first  $n$  positive odd integers is  $n^2$

**Basis Step:**  $P(1)$  is true

$$\because 2(1) - 1 = (1)^2 \Rightarrow 1 = 1$$

**Inductive Step:** Let it will be true for  $k$

$$1 + 3 + 5 + \dots + (2k - 1) = k^2$$

**Under this assumption**, it must be shown that  **$P(k + 1)$  is true**

Adding  **$2(K+1) - 1$**  on both sides

$$1 + 3 + 5 + \dots + (2k - 1) + \mathbf{2(k+1) - 1} = k^2 + \mathbf{2(k+1) - 1}$$

$$1 + 3 + 5 + \dots + (2k - 1) + 2(k + 1) - 1 = k^2 + 2k + 1$$

$$1 + 3 + 5 + \dots + (2k - 1) + 2(\overline{k + 1}) - 1 = (\overline{k + 1})^2$$

Consequently, by the principle of mathematical induction we can conclude that  $P(n)$  is true for all positive integers  $n$ . That is, we know that  $1 + 3 + 5 + \dots + (2n - 1) = n^2$  for all positive integers  $n$ .

**Example** Use mathematical induction to show that  
 $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$  for all **nonnegative**  
integers  $n$ .



## **Solution:**

Let  $P(n)$  be the proposition that  $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$  for the integer  $n$ .

**Basis Step:**  $P(0)$  is true

$$\because 2^0 = 2^{0+1} - 1 \Rightarrow 1 = 1$$

**Inductive Step:** Let it will be true for  $k$

$$1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

Under this assumption, it must be shown that  $P(k + 1)$  is true

Adding  $2^{k+1}$  on both sides

## Cont.

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2 \cdot 2^{k+1} - 1$$

$$1 + 2 + 2^2 + \dots + 2^k + 2^{K+1} = 2^{K+1+1} - 1$$

We have completed the **basis step** and the **inductive step**, by mathematical induction we know that **P(n)** is true for all nonnegative integers n.

That is,  $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$  for all nonnegative integers n.

**Example** Sums of **Geometric Progressions**. Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression:

$$\sum_{j=0}^n ar^j = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r-1} \quad \text{where } r \neq 1,$$

where  $n$  is a nonnegative integer.

## Solution

Let  $P(n)$  be the statement that the sum of the first  $n + 1$  terms of a geometric progression in this formula is correct.

**Basis Step:**  $P(0)$  is true, because

$$ar^0 = \frac{ar^{0+1} - a}{r-1} \Rightarrow a = a$$

**Inductive Step:** Let it will be true for  $k$

$$a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r-1}$$

Under this assumption, it must be shown that  $P(k + 1)$  is true

Adding  $ar^{k+1}$  on both sides

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}$$

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a + (r - 1)ar^{k+1}}{r - 1}$$

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a + ar^{k+1+1} - ar^{k+1}}{r - 1}$$

$$a + ar + ar^2 + \dots + ar^k + ar^{\overline{K+1}} = \frac{ar^{\overline{K+1}+1} - a}{r - 1}$$

We have completed the **basis step** and the **inductive step**, so by mathematical induction  $P(n)$  is true for all nonnegative integers  $n$ . This shows that the formula for the sum of the terms of a geometric series is correct.

**Example** Use mathematical induction to prove the inequality  $n < 2^n$  for all positive integers  $n$ .

## Solution

$$n < 2^n$$

**Basis Step:**  $P(1)$  is true, because  $1 < 2^1 \Rightarrow 1 < 2$

**Inductive Step:** Let it will be true for  $n = k$

$$k < 2^k$$

Under this assumption, it must be shown that  $P(k + 1)$  is true

Adding 1 on both sides

$$k + 1 < 2^k + 1$$

$$\Rightarrow k + 1 < 2^k + 2^k$$

$$\because 1 \leq 2^k$$

$$\Rightarrow k + 1 < 2 \cdot 2^k$$

$$\Rightarrow \overline{k + 1} < \overline{2^{k+1}}$$

We have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that  $n < 2^n$  is true for all positive integers  $n$

**Example** Use mathematical induction to prove that  $2^n < n!$  for every positive integer  $n$  with  $n \geq 4$ . (Note that this inequality is false for  $n = 1, 2$ , and  $3$ .)



Let  $P(n)$  be the proposition that  $2^n < n!$

**Basis Step:** To prove the inequality for  $n \geq 4$  requires that the basis step be

$P(4)$ . Note that  $P(4)$  is true, because

$$2^4 < 4!$$

$$16 < 24$$

**Inductive Step:** For the inductive step, we assume that  $P(k)$  is true for the positive integer  $k$  with  $k \geq 4$ .

$$2^k < k! \text{ -----(1)}$$

We have to show to that  $2^{k+1} < (k+1)!$ . Multiply (1) by 2

$$2 \times 2^k < 2 \times k!$$

$$2^{k+1} < 2 \times k!$$

$$2^{k+1} < (k+1)k!$$

$$2^{k+1} < (k+1)!$$

$$\because 2^{k+1} = 2 \times 2^k$$

$$\because 2 < k+1$$

$$\because (k+1)! = (k+1)k!$$

This shows that  $P(k+1)$  is true when  $P(k)$  is true. This completes the inductive step of the proof

**Example** Use mathematical induction to prove that  $n^3 - n$  is divisible by 3 whenever  $n$  is a positive integer.

## Solution

Let  $P(n) = n^3 - n$

**Basis Step:**  $P(1)$  is true

$\because P(1) = 1^3 - 1 = 0$ , which is divisible by 3

**Inductive Step:** Let it will be true for  $n = k$

$$P(k) = k^3 - k$$

To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that  $P(k + 1)$ , the statement that

$(k + 1)^3 - (k + 1)$  is **divisible by 3**, is also true.

We have to show that  $(k + 1)^3 - (k + 1)$  is divisible by 3

$$P(k + 1) = (k + 1)^3 - (k + 1)$$

$$P(k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$P(k + 1) = k^3 - k + 3k^2 + 3k$$

$$P(k + 1) = k^3 - k + 3k(k + 1)$$

$P(k + 1)$  = first term is divisible by 3 + second term is divisible by 3

$P(k + 1)$  = sum is divisible by 3

# Suggested Readings

## 5.1 Mathematical Induction