Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7^{th} Edition

Kenneth H. Rosen

References

Chapter 3

1. Discrete Mathematics and Its Application, 7^{h} Edition
by
Kenneth H. Rose

2. Discrete Mathematics with Applications
by
Thomas Koshy

3. Discrete Mathematical Structures, CS 173
by
Cinda Heeren, Siebel Center

4. https://www.cs.cornell.edu/courses/JavaAndDS/files/constantTime.pdf

5. Discrete Mathematics for Computer Science by Gary Haggard

These slides contain material from the above resources.

Constant time

Constant time: An operation or method takes constant time if the time it takes to carry it out does not depend on the **size of its operands**.

For example, an **array element reference b[i]** takes **constant time**, but **printing out all elements of array b** is **not constant time** but instead takes time proportional to the **size of b.**

Also, this assignment takes **constant time**:

b[i]= b[i] + 2

Example Give **big-O estimates** for the **factorial function** and the **logarithm of the factorial function**, where the factorial function

$f(n) = n!$ is defined by

$n! = 1 \times 2 \times 3 \ldots n$, whenever n is a positive integer, and

$0! = 1$.

For example,

$1! = 1$, $2! = 1 \times 2 = 2$, $3! = 1 \times 2 \times 3 = 6$, $4! = 1 \times 2 \times 3 \times 4 = 24$.

Note that the function **n! grows rapidly**.

For instance, **$20! = 2{,}432{,}902{,}008{,}176{,}640{,}000$**

Solution:

We have to show $f(n) = n!$ is $O(n^n)$

$|f(n)| \leq C|g(n)|$ whenever $n > k$.

$n! = 1 \times 2 \times \ldots n$

$n! \leq n \times n \times n \ldots n$, where $k > 1$

$0 \leq n! \leq n \times n \times n \ldots n = n^n$

$\Rightarrow n! \leq n^n$

$\Rightarrow n! \leq 1 \times n^n$

$= Cg(n)$

n	$n! \leq n^n$
1	$1! \leq 1^2$ (true)
2	$2! \leq 2^2$ (True)
3	$3! \leq 3^2$ (True)
⋮	⋮

Here $f(n) = n!$ and $g(n) = n^n$

Consequently, we can take C = 1and k = 1 as witnesses to show that f(n) is $O(n^n)$

$\because n! \leq n^n$

Taking log on both sides

$\Rightarrow \log n! \leq \log n^n$

$\Rightarrow \log n! \leq n \log n$ $\because \log m^n = n \times \log m$

This implies that **log n!** is **O(n log n),** again taking C = 1 and k = 1 as witnesses.

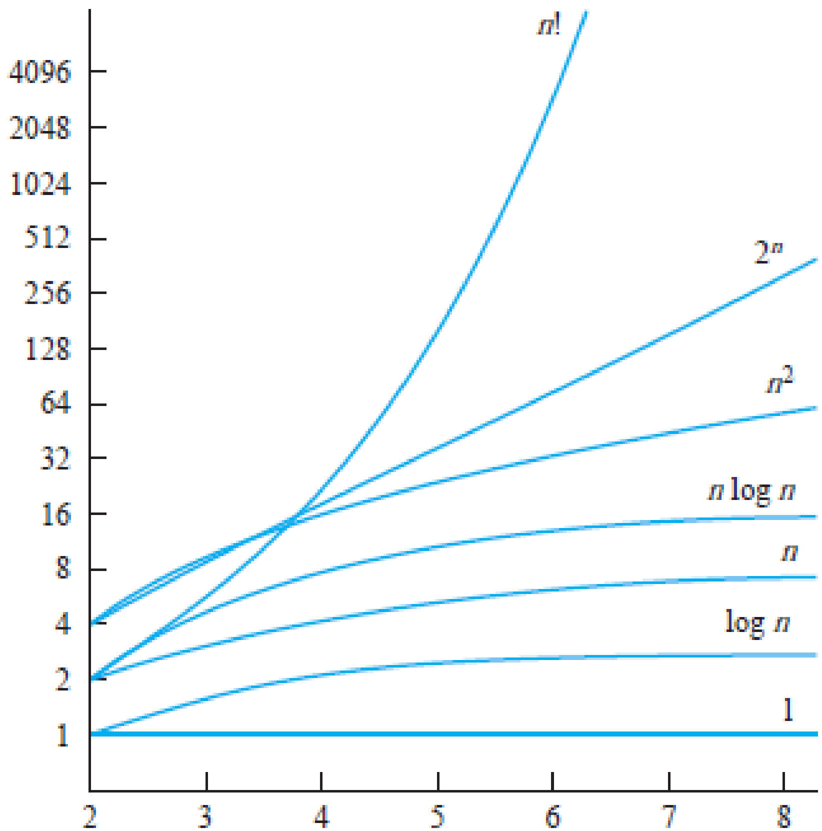
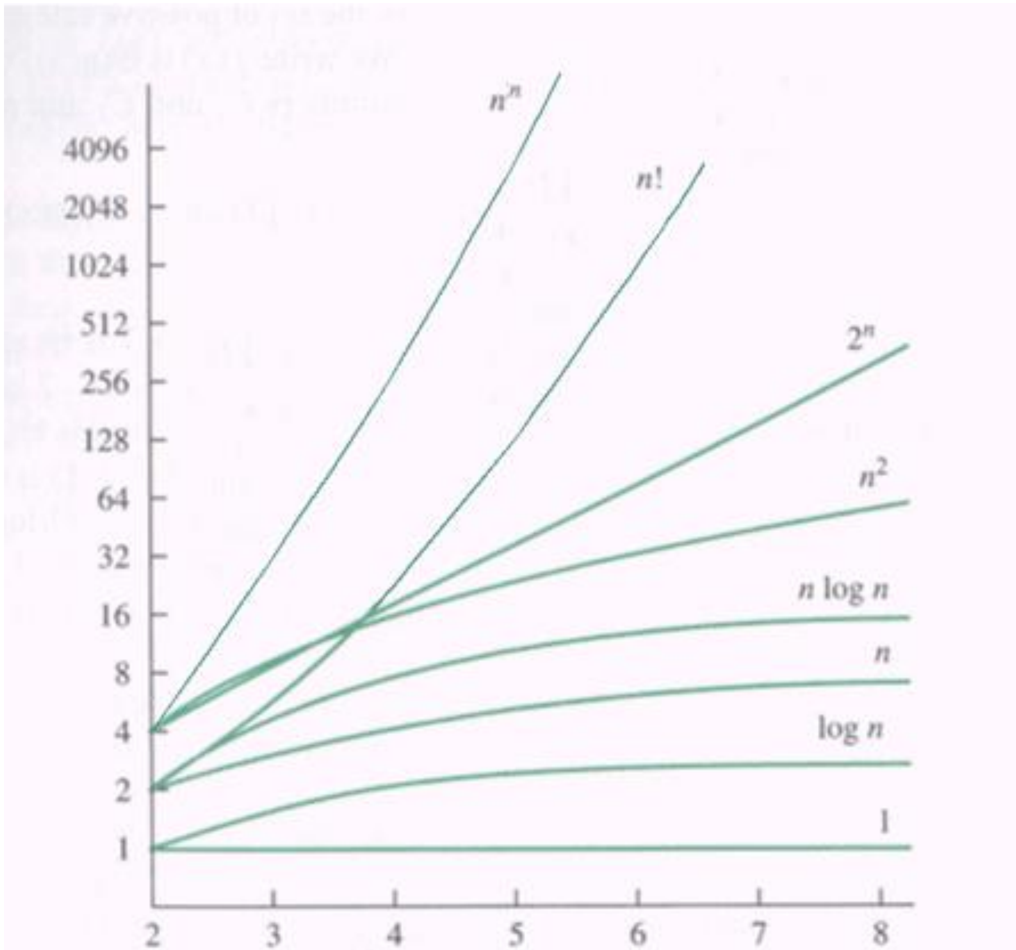


FIGURE 3 **A Display of the Growth of Functions Commonly Used in Big-O Estimates.**



n^n
$n!$
2^n
n^2
$n \log n$
n
$\log n$
1
4096
2048
1024
512
256
128
64
32
16
8
4
2
1
2
3
4
5
6
7
8

Complexity Comparisons for Various Functions

- To see the difference in the **time requirement** for processing **data sets of arbitrary size**, we will assume **a single machine cycle** will require **10^{-6} seconds** to be completed.

- **Table** on the next slide gives the **time required to process a data set of size n**, for six different values of n, if it takes $\log_2(n)$ *(n, n^2, n^5 ,* and 2^n, respectively) machine cycles to make the computation.

- For example, in the column labeled n^2 for the row labeled n = 100, 100^2 operations are needed to complete execution. The time is **$(10^2)^2 10^{-6}$ seconds = 10^{-2} seconds.**

Complexity Comparisons for Various Functions

$F(n)$	$log_2(n)$	n	n^2	n^5	2^n
$n = 10$	3×10^{-6} sec	10^{-5} sec	10^{-4} sec	0.1 sec	10^{-3} sec
$n = 20$	4×10^{-6} sec	2×10^{-5} sec	4×10^{-4} sec	3 sec	1 sec
$n = 50$	6×10^{-6} sec	5×10^{-5} sec	3×10^{-3} sec	5 min	36 yrs
$n = 100$	7×10^{-6} sec	10^{-4} sec	10^{-2} sec	3 hrs	4×10^{16} yrs
$n = 1000$	1×10^{-5} sec	10^{-3} sec	1 sec	32 yrs	3.9×10^{287} yrs
$n = 100,000$	2×10^{-5} sec	0.1 sec	2.7 hrs	3×10^{11} yrs	$> 10^{30,089}$ yrs

What Does Machine Cycle Mean?

- A machine cycle consists of the steps that a computer's **processor executes whenever it receives a machine language** instruction.

- It is the most basic CPU operation, and modern CPUs are able to perform millions of machine cycles per second.

- The cycle consists of three standard steps: fetch, decode and execute. In some cases, store is also incorporated into the cycle.

Show that $7x^2$ is $O(x^3)$.

Solution:

$7x^2$ is $O(x^3)$.

$|f(x)| \leq C|g(x)|$ whenever $x > k$.

$f(x) = 7x^2$

$g(x) = x^3$

We observe that we can readily estimate the size of f (x) when **$x > 7$**

$\because 7x^2 < x^3$. when $x > 7$.

$0 \leq 7x^2 \leq x^3$

$\Rightarrow 7x^2 \leq \mathbf{1} \times x^3$

$\Rightarrow f(x) \leq Cg(x)$

Consequently, we can take ***C* = 1** and ***k* = 7** as witnesses to establish

Show that $7x^2$ is $O(x^3)$.

Alternative solution:

$7x^2$ is $O(x^3)$.

$|f(x)| \leq C|g(x)|$ whenever $x > k$.

$f(x) = 7x^2$

$g(x) = x^3$

We observe that we can readily estimate the size of f (x) when x > 1

$\because 7x^2 < 7x^3$. **when x > 1**.

$0 \leq 7x^2 \leq 7x^3$

$\Rightarrow 7x^2 \leq \mathbf{7 \times x^3}$

$\Rightarrow f(x) \leq Cg(x)$

Consequently, we can take ***C* = 7** and ***k* = 1** as witnesses to establish

Show that n^2 is $O(n)$.

Solution:

n^2 is O(n).

$|f(n)| \leq C|g(n)|$ whenever $n > k$.

$f(n) = n^2$

$g(n) = n$

We have to show that

$n^2 \leq Cn$

Dividing both sides by n

$\Rightarrow n \leq C$

We cannot find any *C* and *k* as witnesses

Note: C and k are constants, whereas k is a positive real number and C is a real number

Example: Find Big-oh notation of $1 + 2 + 3 + \ldots + n$

Let $f(n) = 1 + 2 + 3 + \ldots + n$

We have to show that $f(n) = 1 + 2 + 3 + \ldots + n$ is **$O(n^2)$**

$\because |f(n)| \leq C|g(n)|$ whenever $n > k$.

$1 < n^2$, $2 < n^2$, $3 < n^2$, … so on when $n > 1$

$$
\begin{aligned}
f(n) &= 1 + 2 + 3 + \ldots + n \\
&\leq n + n + n + \ldots + n \\
&\leq n \times n \\
&\leq 1 \times n^2
\end{aligned}
$$

$\Rightarrow f(n) \leq Cg(n)$

Consequently, we can take $C = 1$ and $k = 1$ as witnesses to establish

Show that logn is O(n)

We will prove by mathematical induction

$n < 2^n$

$\Rightarrow f(x) \leq Cg(x)$

Consequently, we can take $C = 1$ and $k = 1$ as witnesses to establish

We conclude that **n** is **$O(2^n)$**

$n < 2^n$

Taking log on both sides with base 2

$\lg n < \lg 2^n$

$\lg n < n\lg 2$ $\qquad\qquad$ **$\because \lg 2 = 1$**

$\lg n < n$

$\Rightarrow f(x) \leq Cg(x)$

Consequently, we can take $C = 1$ and $k = 1$ as witnesses to establish. We conclude that **lgn is O(n)**

Cont.

If we have logarithms to a base *b*, where *b* is different from 2, we still have **$\log_b n$** is ***O(n)***

Because

$\because \lg n < n$

$$\log_b n = \frac{\log n}{\log b} < \frac{n}{\log b} \qquad \because \log_b n = \frac{\log_c n}{\log_c b}$$

$$\log_b n = \frac{\log n}{\log b} < \frac{1}{\log b} \times n$$

whenever *n* is a positive integer. We take **$C = 1/\log b$** and $k = 1$ as witnesses.

- $n! = O(n^n)$
- $\log(n!) = O(n \log n)$.
- $\log_b n = O(n)$
- $n = O(2^n)$
- $f(n) = 1 + 2 + 3 + \ldots + n$ is $O(n^2)$

The Growth of Combinations of Functions

Theorem Suppose that $f_1(x)$ is $O(g_1(x))$ and that $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

Corollary Suppose that $f_1(x)$ and $f_2(x)$ are both $O(g(x))$. Then $(f_1 + f_2)(x)$ is $O(g(x))$.

Theorem Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$.

Example Give a **big-O estimate** for $f(n) = 3n \log(n!) + (n^2 + 3) \log n$, where n is a positive integer.

Solution: for $f(n) = 3n \log(n!) + (n^2 + 3) \log n$

First, the product $3n \log(n!)$ will be estimated.

$\log(n!) = O(n \log n)$.

$3n$ is $O(n)$

Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$.

$\Rightarrow 3n \log(n!) = O(n^2 \log n)$.

$(n^2 + 3) \log n$ will be estimated:

Because $(n^2 + 3) < 2n^2$ when $n > 2$, it follows that

$n^2 + 3 = O(n^2)$.

$\Rightarrow (n^2 + 3) \log n = O(n^2 \log n)$.

Cont.

Theorem Suppose that $f_1(x)$ is $O(g_1(x))$ and that $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.

$\therefore f(n) = 3n \log(n!) + (n^2 + 3) \log n$

is $O(\max(n^2 \log n, n^2 \log n))$ is $O(n^2 \log n)$.

Example: Let $f(n) = 6n^2 + 5n + 7\lg n!$ Estimate the growth of $f(n)$

Solution:

$f(n) = 6n^2 + 5n + 7\lg n!$

$6 = O(1)$

$6n^2 = O(n^2)$

$5 = O(1)$

$5n = O(n)$

$\because$ $(f_1 + f_2)(x)$ is **$O(\max(|g_1(x)|, |g_2(x)|))$**

$6n^2 + 5n = O(\max(n^2, n)) = O(n^2)$

$7 = O(1)$

$\lg n! = O(n\lg n)$

$\because$ $(f_1f_2)(x)$ is $O(g_1(x)g_2(x))$

$7\lg n! = O(1.n\lg n) = O(n\lg n)$

$\because$ $(f_1 + f_2)(x)$ is **$O(\max(|g_1(x)|, |g_2(x)|))$**

$6n^2 + 5n + 7\lg n!$ is $O($**max**$(n^2, n\lg n)) = O(n^2)$

big-Omega (Ω)

Let **f** and **g** be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is **$\Omega(g(x))$** if there are positive constants C and k such that

$|f(x)| \geq C|g(x)|$ whenever $x > k$. [This is read as "f (x) is big-Omega of g(x)."]

Note: There is a strong connection between big-O and big-Omega notation. In particular, **f(x) is Ω (g(x))** if and only **if g(x) is O(f(x))**

Example The function $\mathbf{f(x) = 8x^3 + 5x^2 + 7}$ is $\Omega(g(x))$, where $g(x)$ is the function $g(x) = x^3$.

$\mathbf{|f(x)| \geq C|g(x)|}$ **whenever** $\mathbf{x > k}$**.**

This is easy to see because $\mathbf{f(x) = 8x^3 + 5x^2 + 7 \geq 8x^3}$ for all positive real numbers x.

This is equivalent to saying that $g(x) = x^3$ is $O(8x^3 + 5x^2 + 7)$, which can be established directly by turning the inequality around.

big-Theta(Θ)

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that f(x) is **Θ(g(x)) if f(x) is O(g(x)) and f(x) is Ω(g(x)).** When f(x) is Θ(g(x)) we say that "f is big-Theta of g(x)'', that f(x) is of order g(x), and that f(x) and g(x) are of the same order.

- When f(x) is Θ(g(x)), it is also the case that g(x) is Θ(f (x)). Also note that f (x) is Θ(g(x)) if and only if f (x) is O(g(x)) and g(x) is O(f (x)).

- **f (x) is Θ(g(x))** if and only if there are real numbers C_1 and C_2 and a positive real number k such that **$C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$** whenever $x > k$. The existence of the constants C_1, C_2, and k tells us that f (x) is Ω(g(x)) and that f (x) is O(g(x)), respectively

Example Show that $3x^2 + 8x \log x$ is $\Theta(x^2)$.

Solution:

f(x) = $3x^2 + 8x \log x$

$C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$ whenever $x > k$

For O:

$|f(x)| \leq C_2|g(x)|$ whenever $x > k$.

$x \log x < x^2$ whenever $x > 1$

$0 \leq 3x^2 + 8x \log x \leq 3x^2 + 8x^2 \leq 11x^2$ whenever $x > 1$

$3x^2 + 8x \log x \leq 11x^2$

$3x^2 + 8x \log x \leq Cg(x)$ whenever $x > 1$

$C_2 = 11, k = 1$

Note: C_2 and k are constants.

We have to show that $3x^2 + 8x \log x$ is $\Theta(x^2)$.

f(x) = $3x^2 + 8x \log x$

For Ω:

$|f(x)| \geq C_1|g(x)|$ **whenever x > k.**

$3x^2 + 8x \log x \geq \mathbf{3}x^2$ whenever $x > 1$.

$3x^2 + 8x \log x \geq C_2|g(x)|$

$C_1 = 3$, $k = 1$

Note: C_1 and k are constants.

$3x^2 + 8x \log x$ is $\Theta(x^2)$.

Theorem Let $\mathbf{f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}$, where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Then ***f(x)*** is of order $\mathbf{x^n}$.

Example The polynomials

$3x^8 + 10x^7 + 221x^2 + 1444$,

$x^{19} - 18x^4 - 10{,}112$,

and

$-x^{99} + 40{,}001x^{98} + 100{,}003x$

are of orders x^8, x^{19}, and x^{99}, respectively.

Suggested Readings

Chapter 3

3.1 Algorithms

3.2 The Growth of Functions