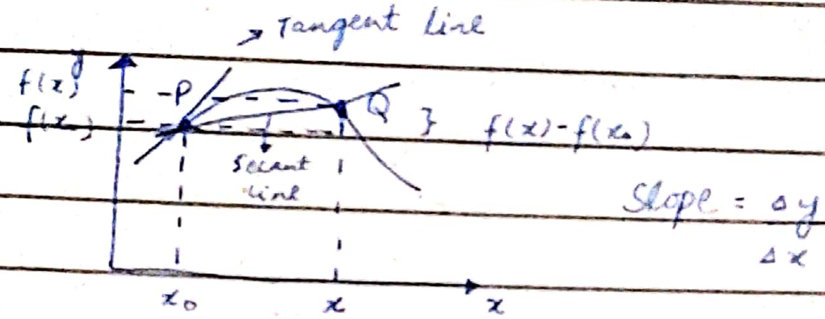
Math-Deficiency II:

Chapter 2: (The Derivative)

Tangent line and rate of change:

TANGENT LINES:



In figure 1, consider a point Q on the curve that is distinct from P.

$$Q = (x, f(x))$$

$$P = (x_0, f(x_0))$$

and compute the slope as:

$$m_{pq} = \frac{f(x) - f(x_0)}{x - x_0}$$

If we take, $x \to x_0$, then:

$$m_{pq} \to m_{tan}$$

→ Suppose that x_0 is in the domain of function f. The tangent line to the curve $y = f(x)$ at point P is the line with equation:

$$y - f(x_0) = m_{tan}(x - x_0)$$

where $m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

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Example:

$\underset{f(x)}{y} = x^2$ at point $P(1,1)$.

$\downarrow x_0$; $\searrow f(x_0)$

Using:

$$\boxed{m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \rightarrow (i)$$

$$= \lim_{x \to x_0} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to x_0} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x \to x_0} (x+1)$$

$$m_{tan} = \lim_{x \to 1} (1+1) = 2$$

$$\boxed{m_{tan} = 2}$$

There is an alternative way to express (i), i.e., if we take:

$$h = x - x_0$$

Then:

$$m_{tan} = \lim_{\substack{h \to 0 \\ \text{(approaches to 0)}}} \frac{f(x) - f(x_0)}{h}$$

$$m = \frac{\Delta y}{\Delta x} \Rightarrow x \to x_0,$$

$$m = \frac{dy}{dx}$$

e.g.: $v = \dfrac{\Delta S}{\Delta t}$, $v = \dfrac{dS}{dt}$

$$a = \frac{dv}{dt}$$

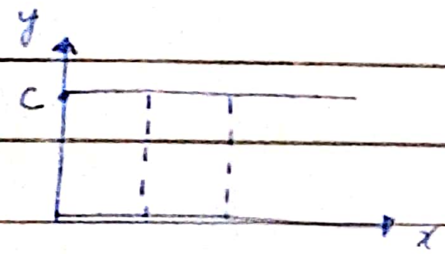
$$a = \frac{d^2 s}{dt^2}$$

$$\boxed{a = \frac{d}{dt} \cdot \frac{ds}{dt}}$$

INTRODUCTION TO TECHNIQUES OF DIFFERENTIATION:

i. Derivative of a constant:

Consider a constant function:



$f(x) = c$

$$\boxed{f_x \,/\, f'(x) = \frac{d}{dt} f(x) = 0}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0$$

e.g. $\frac{d}{dx}[1] = 0$

$\frac{d}{dx}[-3] = 0$

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Derivatives of Power Functions:

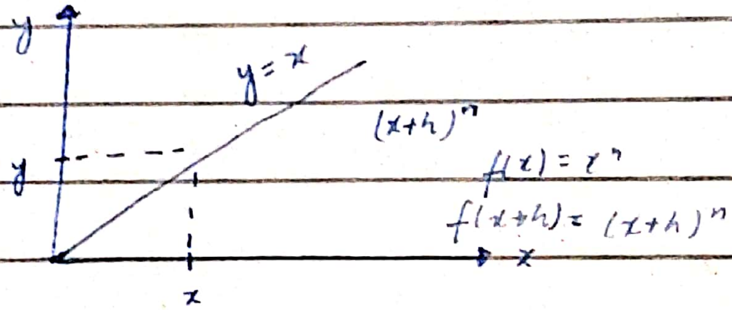
$$\cancel{f(x) = x}$$

$$\cancel{\frac{d\, f(x)}{dx}}$$

The simplest power function is: (power is 1)

$$f(x) = x$$

$$y = f(x) = x$$



Derivative: part of total

$$f'(x) \; \boxed{\frac{d\, f(x)}{dx}} = \frac{d}{dx} \cdot x = \frac{dx}{dx} = 1$$

$$f(x) = x^3$$

$$\frac{d\, f(x)}{dx} = f'(x) = 3x^{3-1} = 3x^2$$

Theorem : THE POWER RULE

"If n is a positive integer

then :

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$$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$$

PROOF: Let $f(x) = x^n$

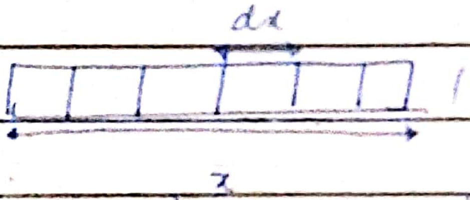
From the first principle and the binomial formula for expanding the expression:

$$(x+h)^n$$

We obtain:

$$\frac{d}{dx} x^n = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$



$$= \lim_{h \to 0} \frac{\left[\cancel{x^n} + n x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2} h^2 + \ldots\ldots n x h^{n-1} + h^n\right] - \cancel{x^n}}{h}$$

Separating h

$$= \lim_{h \to 0} \left(\frac{n x^{n-1} \cancel{h}}{\cancel{h}} + \frac{n(n-1)}{2!} \frac{x^{n-2} h^2}{h} + n x h^{n-2} + h^{n-1} \right)$$

Apply limit $h \to 0$ we get:

$$f'(x) = n x^{n-1}$$

Example:

$$f(x) = x^4$$

$$f'(x) = 4x^{4-1} = 4x^3$$

$$f(t) = t^4 \rightarrow = 4t^3$$

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Theorem : Extended Power Rule

"If r is any real number, then:

$$\frac{d}{dx}[x^r] = r x^{r-1}$$

e.g :

$$f(x) = x^{\pi}$$

$$f'(x) = \pi x^{\pi - 1}$$

- $f(x) = \sqrt[3]{x}$

$$f(x) = (x)^{1/3}$$

$$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1}$$

$$\boxed{f'(x) = \frac{1}{3} x^{-2/3}} \Rightarrow \boxed{f'(x) = \frac{1}{3\sqrt[3]{x^2}}}$$

Derivative of a constant times a function:

Theorem: Constant Multiple Rule

"If f is differentiable at x (derivative of function exists) and c is any real number, then cf is also differentiable at x.

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}x$$

PROOF:

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}x$$

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$$\frac{d}{dx}[c\,f(x)] = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$$

$$= \lim_{h \to 0} c\,\frac{[f(x+h) - f(x)]}{h}$$

$$= c\,\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}[f(x)] \quad \text{or} \quad c\,\frac{d}{dx}\,x$$

Example:

$$f(x) = \pi/x$$

$$f(x) = \pi \cdot \frac{1}{x}$$

$$= \pi \frac{d}{dx} \frac{1}{x}$$

$$= \cancel{\pi \frac{d}{dx} (-1) x^{-1-1}} \; \pi \frac{d}{dx}\, x^{-1}$$

$$= \cancel{\pi}\; \pi \frac{d}{dx}\, (-1)\, x^{-1-1}$$

$$f'(x) = \frac{-\pi}{x^2}$$

Derivatives of Sums and Differences:

"If f and g are differentiable at x, then: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

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Proof: $\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$

$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

$= \frac{d}{dx}(f(x)) + \frac{d}{dx}[g(x)]$

Hence,

$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

Example:

(i) $f(x) = 2x^6 + x^{-9}$

(ii) $f(x) = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

$= \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} - \frac{2x}{\sqrt{x}}$

$= 1 - 2x \cdot x^{-1/2}$

$f(x) = 1 - 2x^{1/2}$

$f'(x) = \frac{d}{dx} 1 - \frac{d}{dx} 2x^{1/2}$

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$$f'(x) = 0 - 2\frac{d}{dx} x^{1/2}$$

$$= -2\frac{d}{dx} \frac{1}{2} x^{\frac{1}{2}-1}$$

$$= -2\frac{d}{dx} \frac{1}{2} x^{-1/2}$$

$$= -\cancel{2} \times \frac{1}{\cancel{2}} x^{-1/2}$$

$$f'(x) = -x^{-1/2}$$

Example:

Find $\frac{dy}{dx}$ if $y = 3x^8 - \frac{2x^5}{\sqrt[3]{x}} + 6x + 6$

$\frac{5-1}{3}$

$$y' = \frac{dy}{dx} = \frac{dy}{dx}3x^8 - \frac{dy}{dx}\frac{2x^5}{\sqrt[3]{x}} + \frac{dy}{dx}6x + 0$$

$\frac{14-1}{3}$

$$y' = 3\frac{dy}{dx}x^8 - 2\frac{dy}{dx}\frac{x^5}{\sqrt[3]{x}} + 6\frac{dy}{dx}x$$

$\frac{11}{3}$

$$= 3\left(8x^7\right) - 2\frac{dy}{dx}x^{5-\frac{1}{3}} + 6\left(1\, x^{1-1}\right)$$

$$= 24x^7 - 2\frac{dy}{dx}x^{14/3} + 6(1)$$

$$= 24x^7 - 2\left(\frac{14}{3}x^{14/3-1}\right) + 6$$

$$= 24x^7 - 2\left(\frac{14}{3}x^{11/3}\right) + 6$$

$$= 24x^7 - \frac{28\sqrt[3]{x^{11}}}{3} + 6$$

Date: Ch. 2 → Ex 2.1 Q(i)–(v)
2.2 Q3 (i, ii)
$y' = \int y$

Higher derivatives:

f' = first derivative

f'' = second derivative

$f', f'', f''', \ldots\ldots (f')^n$

$$y' = \frac{24x^7 - 28\sqrt[3]{x^{11}} + 6}{3}$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} \cdot \frac{d}{dx} = \frac{d^2}{dx^2} \cdot y$$

$$= 24\frac{d^2}{dx^2} x^7 - \frac{28}{3}\frac{d^2}{dx^2} x^{11/3} + 0$$

EXERCISE 2.1:

Q No. 1:

(i) $2x^2+1$

Let $y = f(x) = 2x^2+1$

$$f'(x) = 2 \cdot \frac{d}{dx} x^2 + \frac{d}{dx} 1$$

$$f'(x) = 2\left(2x^{2-1}\right) + 0$$ ∵ derivative of constant = 0

$$f'(x) = 2(2x) = 4x$$

(ii) $2 - \sqrt{x}$

Let $y = f(x) = 2 - \sqrt{x}$

$$f'(x) = 2 - 1x^{1/2}$$

$$= \frac{d}{dx} 2 - \frac{d}{dx} x^{1/2}$$

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$$f'(x) = 0 - \left(\frac{1}{2} x^{\frac{1}{2}-1} \right)$$

$$= \frac{-1}{2} x^{-1/2}$$

$$= \frac{-1}{2x^{1/2}}$$

$$= \frac{-1}{2\sqrt{x}}$$

(iii) $\frac{1}{\sqrt{x}}$

Let $y = f(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \frac{1}{x^{1/2}}$$

$$= x^{-1/2}$$

$$f'(x) = \frac{d}{dx} x^{-1/2}$$

$$= \frac{-1}{2} x^{-1/2-1}$$

$$= \frac{-1}{2} x^{-3/2}$$

$$= \frac{-1}{2x^{3/2}}$$

$$= \frac{-1}{2\sqrt{x^3}}$$

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(iv) $1/x^3$

Let $y = f(x) = \dfrac{1}{x^3}$

$f'(x) = x^{-3}$

$= \dfrac{d}{dx} x^{-3}$

$= -3x^{-3-1}$

$= -3x^{-4} = -3/x^4$

(v) $\dfrac{1}{x-a}$

Let $y = f(x) = \dfrac{1}{x-a}$

$y = (x-a)^{-1}$

$f'(x) = \dfrac{d}{dx}(x-a)^{-1}$

$= (-1)(x-a)^{-1-1}$

$= -1(x-a)^{-2}$

$= \dfrac{-1}{(x-a)^2}$

EXERCISE 2.2:

Q NO. 1:

(i) $(ax+b)^3$

Let $y = (ax+b)^3$

$= a^3x^3 + 3a^2x^2b + 3axb^2 + b^3$

$f'(x) = a^3 \dfrac{d}{dx} x^3 + 3a^2b \dfrac{d}{dx} x^2 + 3ab^2 \dfrac{d}{dx} x + 0$

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Example 9:

$$y = 3x^4 - 2x^3 + x^2 - 4x + 2$$

Find $\frac{dy}{dx}$ and higher order derivatives.

$$\frac{df(x)}{dx} = f'$$

$$y = f(x) = x^2 + 1$$

$$y(x) = x^2 + 1$$

$$\frac{dy}{dx}\cancel{(x)} =$$

$$y' = \frac{dy}{dx}$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} \cdot \frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$y''' = \frac{d^3y}{dy^3}$$

$$\boxed{y^n = \frac{d^n y}{dy^n}}$$

$$y' = 3\frac{d}{dx}x^4 - 2\frac{d}{dx}x^3 + \frac{d}{dx}x^2 - 4\frac{d}{dx}x + 0$$

$$y' = 3(3x^3) - 2(2x^2) + 2x + (-4)$$

$$y' = 9x^3 - 4x^2 + 2x - 4$$

$y'' =$

$y''' =$

$y'''' =$

$y''''' =$

$\vdots$

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The Product Rule:

Consider the function:

$f(x) = x$

$g(x) = x^2$

$f(x) \cdot g(x) = x^3 \Rightarrow (f(x) \cdot g(x))' = 3x^2$

$f'(x) \cdot g'(x) = 2x$

$(f(x) \cdot g(x))' \neq f'(x) \cdot g'(x)$

Theorem:

"If f and g are differentiable at x and so their product is:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$$

Example 1:

$$y = (\underbrace{4x^2 - 1}_{f})(\underbrace{7x^3 + x}_{g})$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(4x^2-1)\right] \cdot (7x^3+x) + (4x^2-1) \cdot \frac{d}{dx}(7x^3+x)$$

$$= 8x(7x^3+x) + (4x^2-1)(21x^2+1)$$

$$= 56x^4 + 8x^2 + 84x^4 + 4x^2 - 21x^2 - 1$$

$$= 140x^4 - 9x^2 - 1$$

Find $\frac{dS}{dt}$, if $S = (1+t)\sqrt{t}$

$$\frac{dS}{dt} = \left[\frac{d}{dt}(1+t)\right] \cdot \sqrt{t} + \left[\frac{d}{dt}\sqrt{t}\right] \cdot (1+t)$$

$$= (1 \cdot t^{1/2}) + \frac{1}{2} t^{1/2 - 1} \cdot (1+t)$$

$$= \sqrt{t} + \frac{1}{2\sqrt{t}}(1+t) = \sqrt{t} + \frac{1}{2\sqrt{t}} + \frac{t}{2\sqrt{t}}$$

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Quotient rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot \frac{d}{dx} f(x) - f(x)\cdot \frac{d}{dx} g(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$$

Example:

$$y'(x) = ?$$

$$y = \frac{x^3 + 2x^2 - 1}{x+5}$$

$$y' = \frac{(x+5)\frac{d}{dx}(x^3+2x^2-1) - (x^3+2x^2-1)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)(1)}{(x+5)^2}$$

$$= \frac{(3x^3+4x^2+15x^2+20x) - (x^3+2x^2-1)}{(x+5)^2}$$

$$= \frac{3x^3 - x^3 + 19x^2 - 2x^2 + 20x + 1}{(x+5)^2}$$

$$= \frac{2x^3 + 17x^2 + 20x + 1}{(x+5)^2}$$

Derivatives of some trigonometric functions:

1. $\frac{d}{dx} \sin x = \cos x$

2. $\frac{d}{dx} \cos x = -\sin x$

Date:

$$\frac{dy}{dx} = ?, \quad y = x \sin x$$

$$\frac{dy}{dx} = x \cdot \frac{d}{dx} \sin x + \left(\frac{d}{dx} x \cdot\right) \sin x$$

$$= x \cos x + (1) \sin x$$

3. $\frac{d}{dx} \tan x = \sec^2 x$

4. $\frac{d}{dx} \sec x = \sec x \cdot \tan x$

5. $\frac{d}{dx} \cot x = - \sec^2 x$

6. $\frac{d}{dx} \operatorname{cosec} x = - \operatorname{cosec} x \cdot \cot x$

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Example:

$$y = 2x^2 \cos x$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2)\cos x + 2x^2 \frac{d}{dx} \cos x$$

$$= 4x \cos x - 2x^2 \sin x$$

Example 2:

$$y = \frac{\sin x}{1+\cos x}$$

Apply quotient rule:

$$y' = \frac{(1+\cos x)\, d/dx \sin x - \sin x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$y' = \frac{(1+\cos x)(\cos x) - \sin x\,(0 - \sin x)}{(1+\cos x)^2}$$

$$y' = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$y' = \frac{(\cancel{\cos}\, x + 1)}{(1+\cos x)^{\cancel{2}}}$$

$$y' = \frac{1}{1+\cos x}$$

Example 3:

$$f''(x) = ?$$

$$f''(\pi/4) = ?$$

- $f(x) = \sec x$

Applying identities:

$$f'(x) = \frac{d}{dx} \sec x$$

Date:

$f'(x) = \sec x . \tan x$

$f''(x) = \left(\frac{d^2}{dx^2}(\sec x)\right) \tan x + \left(\frac{d^2}{dx^2} \tan x\right) \sec x$

$f''(x) = (\sec x \cdot \tan x) \tan x + \sec^2 x \cdot \sec x$

$f''(x) = \sec x \cdot \tan^2 x + \sec^3 x$

Put $x = 45 \, (\pi/4)$

$f''(x) = \sec \frac{\pi}{4} \cdot \tan^2 \frac{\pi}{4} + \sec^3 \frac{\pi}{4}$

$= \sqrt{2} +$ $=$

THE CHAIN RULE:

"If g is differentiable at x and f is differentiable at $g(x)$, then:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

$g \to x$

$y \to f(x)$

$y = f(\underbrace{g(x)}_{u})$

$u = g(x)$

Find $\frac{dy}{dx}$, if $y = \cos(x^3)$

Let $u = x^3$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Date:

$$\frac{dy}{dx} = \frac{d}{du}\cos u \cdot \frac{d}{dx} x^3$$

$$= -\sin u \cdot 3x^2$$

$$= (-\sin(x^3).)\,3x^2$$

$$= -3x^2 \sin x^3$$

Example 2:

Find $\frac{dw}{dt} = ?$

$w = \tan x$ and $x = 4t^3 + t$

Sol:

$$w = \tan(4t^3 + t)$$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dw}{dt} = \frac{d}{dx}\tan x \cdot \frac{d}{dt}(4t^3 + t)$$

$$= (\sec^2(4t^3 + t)) \cdot (12t^2 + 1)$$

$$= (12t^2 + 1)\sec^2(4t^3 + t)$$

AN ALTERNATIVE VERSION OF CHAIN RULE:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$f(g(x))' = f'\,g'$$

e.g.:

$$y = \sqrt{x^3 + 1}$$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{x^3 + 1}}\right) \cdot 3x^2$$

(a) $\frac{d}{dx} = \sin 2x$

(b) $\frac{d}{dx}(\tan x^2+1)$

→ (a) $\cancel{\frac{d}{dx} = 2 \sin x \cdot \cos x}$

Let $2x = u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du} \sin u \cdot \frac{d}{dx} 2x$$

$$= (+\cos u) \cdot 2$$

$$= +2 \cos 2x.$$

(b) $\tan(x^2+1)$

Let $u = x^2+1$

$$\cancel{dy}\ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du} \tan u \cdot \frac{d}{dx}(x^2+1)$$

$$= (\sec^2 x) \cdot 2x$$

$$= 2x \cdot \sec^2 x.$$

(c) $\sqrt{x^3 + \operatorname{cosec} x}$

Let $u = x^3 + \operatorname{cosec} x$

$$\frac{d}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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$$\frac{d}{du} = \frac{d}{du}(tt)^{1/2} \cdot \frac{d}{dx}\ x^3 + \operatorname{cosec} x$$

$$= \left(\frac{1}{2}\ u^{1/2-1}\right) \cdot \left[2x^2 + \ -\operatorname{cosec} x \cdot \cot x\right]$$

$$= \frac{1}{2\sqrt{u}} \cdot 3x^2 - \operatorname{cosec} x \cdot \cot x$$

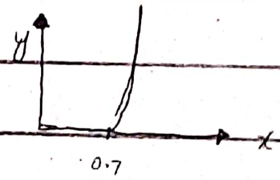
$$= \left(\frac{1}{2\left(\sqrt{x^3 + \operatorname{cosec} x}\right)}\right) \cdot 3x^2 - \operatorname{cosec} x \cdot \cot x$$

Next Lecture;

Derivative of Exponential Function:

$$\frac{d}{dx} e^x = e^x$$

e.g.: PN- junction



Example:

$$y = e^{x^2+1}$$

$$\frac{dy}{dx} = ?$$

Let $u = x^2+1$

$$y = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du} e^u \cdot \frac{d}{dx}(x^2+1)$$

$$\frac{dy}{dx} = e^u \cdot 2x = 2x\, e^{x^2+1}$$

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Example 2:

$$f(x) = x^3 e^{1/x}$$

Let $u = 1/x$

$$\frac{dy}{dx} = x^3 e^{1/x}$$

$$= e^{1/x} \frac{d}{dx} x^3 + x^3 \frac{d}{dx} e^{1/x}$$

$$= \cancel{e^{1/x}(3x^2) + x^3}$$

$$= e^{1/x}(3x^2) + x^3 e^{1/x} \cdot \frac{d}{dx}(-1)x^{-1-1}$$

$$= e^{1/x} 3x^2 + x^3 e^{1/x} \cdot x^{-2}(-1)$$

$$= 3x^2 e^{1/x} + (-1)e^{1/x} \cdot x$$

$$= (-x + 3x^2) e^{1/x}$$

Rule:

$$\boxed{\frac{d}{dx} a^x = a^x \ln a}$$

$$y = a^{\sqrt{x}}$$

Find $dy/dx = ?$

Let $u = \sqrt{x}$

$$y = a^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du} a^u + \frac{d}{dx}(\sqrt{x})$$

$$= a^u \ln a + \frac{1}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx} = a^{\sqrt{x}} \ln a + \frac{1}{2\sqrt{x}}}$$

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Example:

$$y = a^x$$

$$y = e^{\boxed{x \ln a}}$$

$$= e^{\boxed{\ln a^x}}$$

$$y = a^x$$

$$\frac{dy}{dx} = e^{x \ln a} \cdot \frac{d}{dx} x \ln a$$

$$\frac{dy}{dx} = e^{x \ln a} \cdot \ln a$$

Derivative of logarithmic functions:

1. $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

2. $\boxed{\frac{d}{dx}[\log_a x] = \frac{1}{x} \cdot \frac{1}{\ln a}}$

Example:

$$y = \ln(x^2+2x)$$

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(x^2+2x)]$$

$$= \frac{1}{(x^2+2x)} \cdot \frac{d}{dx}(x^2+2x)$$

$$= \frac{1}{(x^2+2x)} \times (2x+2)$$

Example:

$$y = \log_{10}(ax^2+bx+c)$$

Let $u = ax^2+bx+c$

$$y = \log_{10} u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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Ex: 2.6 Q1 (i-v)
Date: Q2 (i-iv)

$$\frac{dy}{dx} = \frac{d}{du}\log_{10} u + \frac{d}{dx}(ax^2+bx+c)$$

$$= \frac{1}{u}\frac{1}{\ln 10} + (2ax+b)$$

$$= \frac{1}{ax^2+bx+c} + (2ax+b)$$

Derivative of Inverse Trigonometric Functions:

1. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x}}$

2. $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x}}$

3. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

4. $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

5. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2+1}}$

6. $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x}$

PROOF:

Let $y = \sin^{-1}x$

$\sin y = x \Rightarrow \boxed{x = \sin y}$

Derivative w.r.t x:

$$\frac{dx}{dx} = \frac{d}{dx}\sin y$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \rightarrow (i)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} \Rightarrow \frac{dy}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

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PROOF 2: Let $y = \cos^{-1}x$

$x = \cos y \to (i)$

Derivative wrt x: $\frac{dx}{dx} = \frac{d}{dx} \cos y$

$1 = -\sin y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{1}{\sin y}$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$

$\frac{d}{dx}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}}$

Example 1:

Find dy/dx when $y = x \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{a}\right) \frac{d}{dx} x + x \frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]$

$= \sin^{-1} \frac{x(1)}{a} + x \frac{1}{\sqrt{1-(x/a)^2}}$ $\quad \frac{1-x^2}{a^2}$

$= \sin^{-1} \frac{x}{a} + \frac{x}{\sqrt{\frac{a^2-x^2}{a^2}}}$

$= \sin^{-1} \frac{x}{a} + \frac{ax}{\sqrt{a^2-x^2}}$ $\quad \frac{\sqrt{a^2-x^2}}{\sqrt{a^2}}$

2.5 =

2. $dy/dx = ?$ $y = \cos^{-1} x/a$

$y = 1/a \cdot \sin^{-1}(a/x)$

$y = \sin^{-1}(\sqrt{1-x^2})$

$y = \cot^{-1} \frac{2x^2}{\sqrt{1-x}}$

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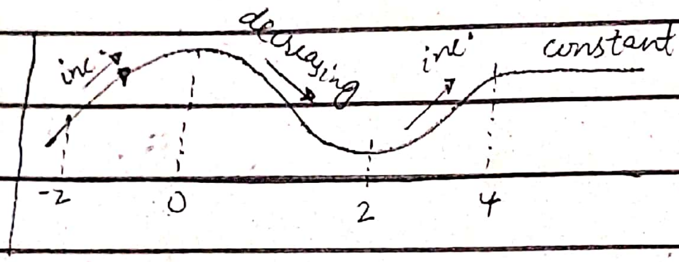
The Derivative in Graphics and Applications:

Analysis of functions I: Increase/Decrease

Function value test:

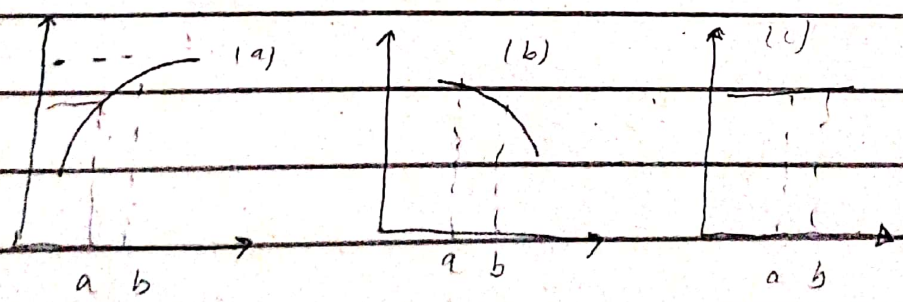


(i) f is increasing in the interval of $f(x_1) < f(x_2)$ when $x_1 < x_2$.

(ii) f is decreasing in the interval of $f(x_1) > f(x_2)$ ~~when~~ if $f(x_1) > f(x_2)$ when $x_1 < x_2$.

(iii) f is constant in the interval of $f(x_1) = f(x_2)$ for all $x_1 = x_2$.

First Derivative Test:



(a) If $f'(x) > 0$ for every value of

$(\, , \,) \rightarrow$ open interval
$[\, , \,] \rightarrow$ closed interval

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x in (a,b), then f is increasing on $[a,b] \rightarrow$ closed interval

(b) If $f'(x) < 0$ for every value of x in (a,b) then f is decreasing on $[a,b]$.

(c) If $f'(x) = 0$ for every value of x in (a,b), then f is constant on $[a,b]$.

Example :

Find intervals on which :

$$f = x^2 - 4x + 3$$

(i) decreasing (ii) increasing

$$f'(x) = 2x - 4$$
$$= 2(x-2)$$

$\rightarrow f'(x) < 0$ if $x < 2$

$\rightarrow f'(x) > 0$ if $x > 2$

- f is decreasing on $[-\infty, 2]$
- f is increasing on $[2, +\infty]$

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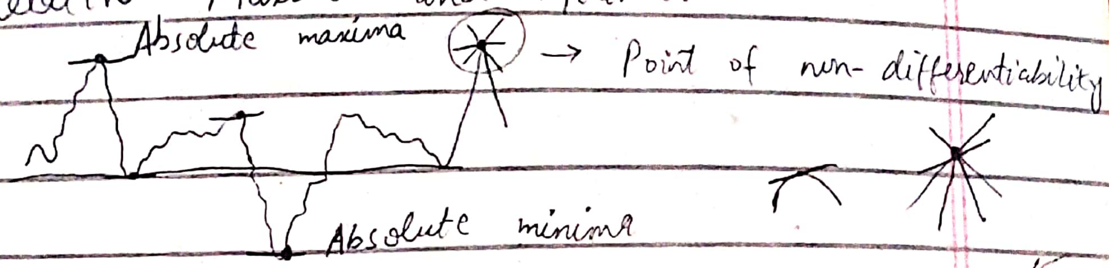
Analysis of function I: Relative Extrema:

Relative Maxima and Minima:



A function 'f' is said to have a relative maxima at x_0 if there is an open interval containing x_0 for which $f(x_0) \geq f(x)$ for all x in that interval.

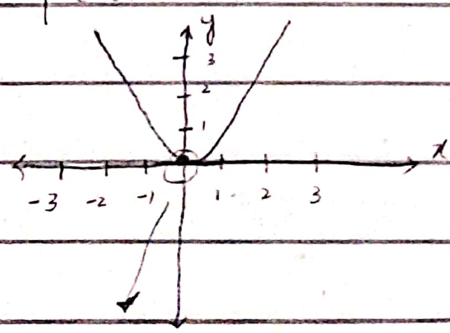
f is either a relative maxima or minima at x_0, then f is said to have relative extrema.

Maxima : $f(x_0) \geq f(x)$

Minima : $f(x_0) \leq f(x)$

Example 1:

$$f(x) = x^2$$



Relative minima

Function has a relative minima at (0,0) but no relative maxima.

$$f(x) = x^3$$

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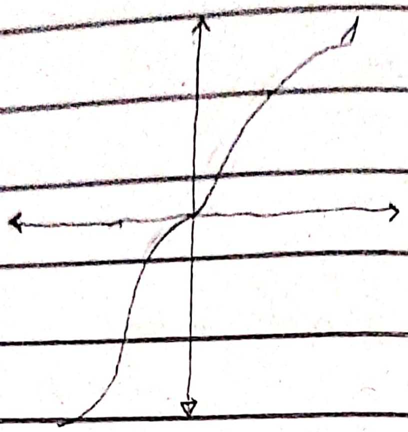
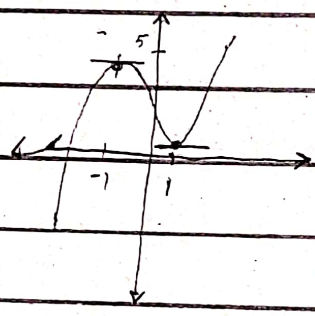
$f(x) = x^3$



There is no relative minima or maxima.

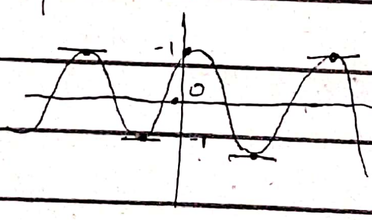
$f(x) = x^3 - 3x + 3$



Relative maxima: $(-1, 5)$ at $x = -1$

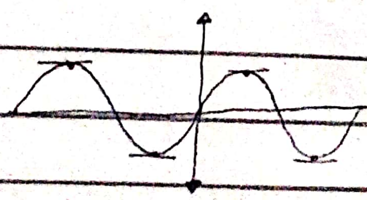
Relative minima at $x = 1$

• $y = f(x) = \cos x$



For all $n\pi$ values, function is a minima.

$y = f(x) = \sin x$



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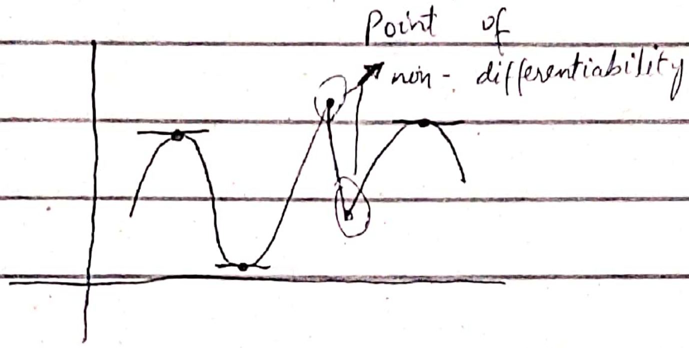
cos:

$f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π.

NOTE:

At extrema points tangent is a horizontal line.



Critical Point : Horizontal tangent line

↳ Stationary point (here derivative is zero).

→ Find all critical points of:

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3$$

$$= 3(x^2 - 1)$$

$$= 3(x+1)(x-1)$$

$\Rightarrow$ $x + 1 = 0$, $x - 1 = 0$

$x = -1$, $x = 1$

→ Which is consistent with this graph.

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First derivative test:

The function f has a relative ~~max~~ extreme at those critical points where f′ changes its signs.

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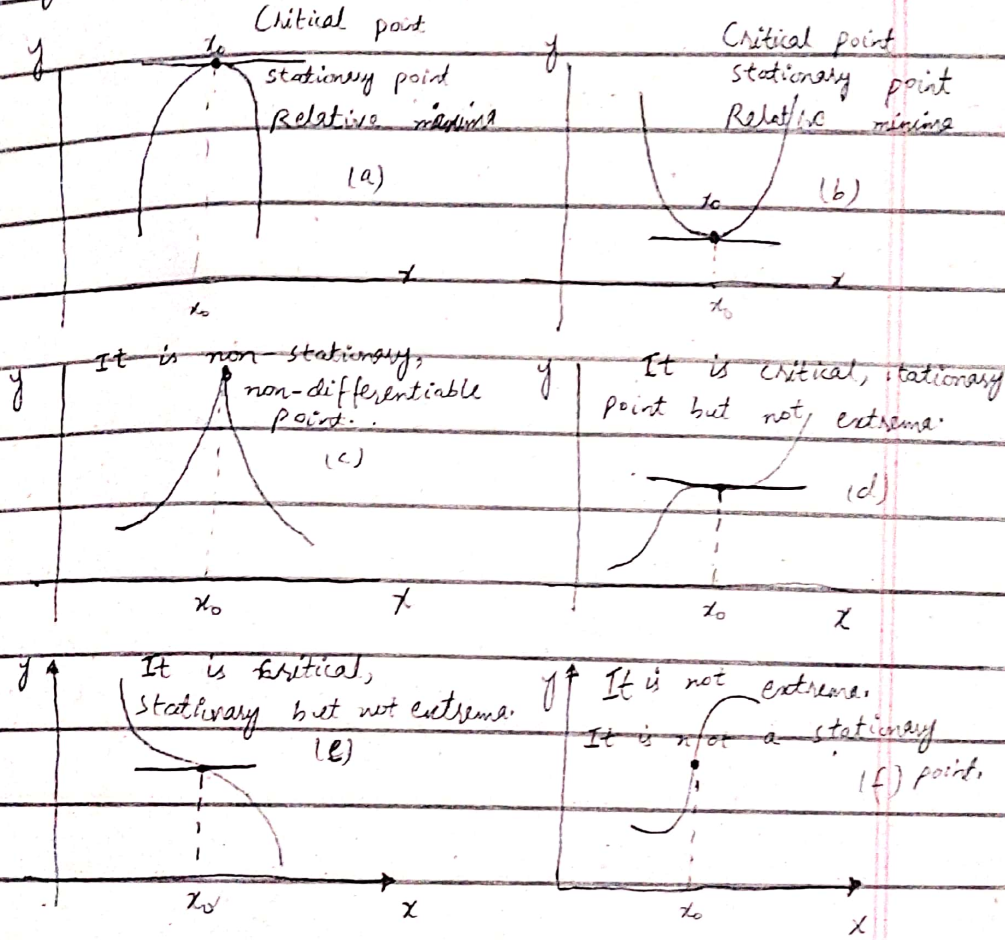
Analysis of function II :

First derivative test (for extremes)

"A function f has a relative extrema at those critical points where f' changes sign."



Theorem:

(a) If $f'(x) > 0$ on an interval extending left from x_0 and $f'(x) < 0$ on an interval extending right from x_0 then x has a relative maxima at x_0.

(b) If $f'(x) < 0$ on an interval extending left from x_0 and $f'(x) > 0$ on an

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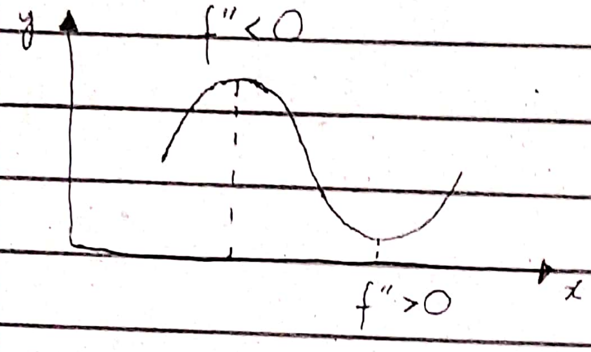
interval extending right from x_0, then x has a relative minima at x_0.

(a.c.f) If f' has the same sign extending left and right from x_0, then f does not have a relative extrema at x_0.

Second Derivative Test:



(a) If $f'(x_0)=0$ and $f''(x_0)>0$, f has relative minima at x_0.

(b) If $f'(x_0)=0$ and $f''(x_0)<0$, f has relative maxima at x_0.

(c) If $f'(x_0)=0$ and $f''(x_0)=0$, f is ineffective.

Example 5:

Find relative extrema of function:

$$f(x)= 3x^5 - 5x^3$$

Sol:

(i) Find its first derivative:

$$f'(x)= 15x^4 - 15x^2$$
$$= 15x^2(x^2-1)$$
$$= 15x^2(x+1)(x-1)$$

Substitute $f'(x)=0$, so:

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$15x^2 = 0$, $x+1=0$, $x-1=0$

$x=0$, $x=-1$, $x=+1$

$f'' = 60x^3 - 30x$

At $x=0$, function is inconclusive $f''=0$

At $x=1$, function >0. $f''=30$

At $x=-1$, function <0. $f''=-30$

(i) $x=0$

$f'' = 60(0) - 30(0) = 0$

(ii) $x=+1$

$f'' = 60(1) - 30(1) = 30$

(iii) $x=-1$

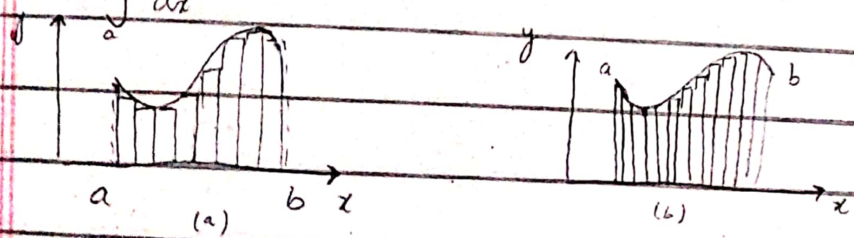
$f'' = 60(-1) - 30(-1) = -30$

Integration:

An overview of the area problem:

The rectangular method for finding area:

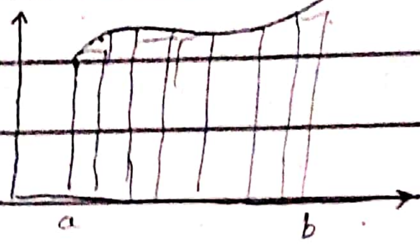
$\int_a^b dx$



Divide interval $[a,b]$ in n equal sub-intervals for each n, the total area of the rectangles can be viewed as an approximation to the exact area under the curve.

exact area under curve $A = \lim_{n\to\infty} A_n$

MD:



$$\sum_{i=1}^{n} A_i = A_1 + A_2 + \cdots + A_n \qquad A = \lim_{n\to\infty} A_n$$

$$\int$$

Definite integral: $\int_a^b (\quad) dx$

Indefinite integral: $\int (\quad) dx$

Indefinite integral:

Anti-derivative:

"A function F is called an anti-derivative of a function f on a given open interval if:
$F'(x) = f(x)$ for all x in the interval.

$$\frac{d}{dx} F = f(x)$$

Example: $F(x) = \frac{1}{3} x^3$ is an anti-derivati of $f(x) = x^2$

Date:

$$F'(x) = \frac{3}{3}x^2 = x^2 = f(x)$$

$$\frac{d}{dx}\left[\frac{1}{3}x^3\right] = f(x)$$

Example:

- $$G(x) = \frac{1}{3}x^3 + c$$

$$G'(x) = \frac{3x^2}{3} + 0$$

$$= x^2 = f(x)$$

- $$H(x) = \frac{1}{3}x^3 - 5$$

Thus are $F(x)$, $G(x)$, $H(x)$ are all anti-derivatives of x^2.

Indefinite integral:

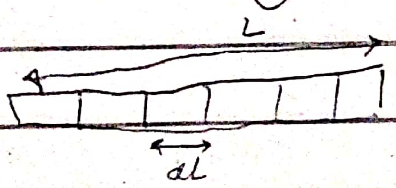
"The process of finding anti-derivatives is called anti-differentiation or integration."

$$\frac{d}{dx}F(x) = f(x)$$

$$f(x) \cdot dx = dF(x)$$

Taking integral on both sides:

$$\int f(x) \cdot d(x) = \int dF(x)$$



$\int dL = L$

$\int dx = x$

$\int dF = F$

$$\underbrace{\int}_{\text{integral}} \underbrace{f(x)\, d(x)}_{\text{integrand}} = F(x) + \underbrace{c}_{\text{constant of integration}}$$

Power rule: $\boxed{\int x^3 . dx = \frac{x^{n+1}}{n+1} + C}$

Derivative formula	Equivalent integration formula
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2 . dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}} . dx = \sqrt{x} + c$
$\frac{d}{dx}(\tan t) = \sec^2 t$	$\int \sec^2 t . dt = \tan(t) + c$

$3\int x^2 . dx$

$\frac{3x^3}{3} + c$

$x^3 + c$

Examples

$$\int \frac{1}{2\sqrt{x}} . dx = ?$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x}} . dx$$

$$= \frac{1}{2} \int x^{-1/2} . dx$$

$$= \frac{1}{2} \frac{x^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{1}{2} . \frac{x^{1/2}}{1/2} + c$$

$$= \frac{1}{\cancel{2}} . \frac{\cancel{2}\sqrt{x}}{1} + C$$

$$= \sqrt{x} + c$$

Date:

$\rightarrow \frac{d}{dx}\left(u^{3/2}\right) = \frac{3}{2}\sqrt{u}$

$\rightarrow \int \frac{3}{2}\sqrt{u} \cdot du = u^{3/2} + c$

$= \int \frac{3}{2}\sqrt{u} \cdot du$

$= \frac{3}{2}\int (u)^{1/2} \cdot du$

$= \frac{3}{2}\frac{u^{1/2+1}}{1/2+1} + c$

$= \frac{\cancel{3}}{2}\frac{u^{3/2}}{3/2} + c$

$= u^{3/2} + c$

Properties of Integrals:

- $\int c\, f(x) \cdot dx = c\,F(x) + C$
- $\int (f(x) \pm g(x)) \cdot dx = F(x) \pm G(x) \pm C$

Example:

(a) $\int 4 \cos x \cdot dx$

$= 4\int \cos x \cdot dx$

$= 4(\sin x) + c$

(b) $\int (x + x^2)\, dx$

$= \int x \cdot dx + \int x^2 \cdot dx$

$= \frac{x^2}{2} + \frac{x^3}{3} + C$

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Example 5:

$$\int \left(\frac{3x^6 - 2x^2 + x^2 - 2x^4}{x^4} \right) dx$$

$$= 3\int x^6 dx - 2\int x^2 dx + \int \frac{x^2}{x^4} \cdot dx - 2\int \frac{x^4}{x^4} \cdot dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \int \frac{1}{x^2} dx - 2\int dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \int x^{-2} dx - 2x$$

$$x^{-1}/-1$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} - \frac{1}{x} - 2x + C$$

Example 6:

Evaluate: $\int \frac{\cos x}{\sin^2 x} dx$

$$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} dx$$

$$= \int \cot x \cdot \operatorname{cosec} x \cdot dx$$

$$= -\operatorname{cosec} x + C$$

INTEGRATION BY SUBSTITUTION:

Method of u-sub:

Example 1:

Evaluate: $\int (x^2+1)^{50} \cdot 2x \, dx$

Let $u = x^2+1$

$$\frac{du}{dx} = 2x$$

 E

$du = 2x.dx$

$$\int u^{50} . du = \frac{u^{51}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C$$

Example 2:

$$\int \sin(2x+9) \cdot dx \rightarrow (i)$$

Let $u = 2x+9$

$$\frac{du}{dx} = 2$$

$$\frac{1}{2} du = dx$$

Put in (i):

$$= \int \sin u \cdot \frac{du}{2}$$

$$= \frac{-1}{2} \cos u + C$$

$$= \frac{-1}{2} \cos(x+9) + C$$

Example 3: Let $\boxed{u = 5x}$

Evaluate $\int \cos 5x . dx \rightarrow (i)$

Let $u = .5x \rightarrow (ii)$

$$\frac{du}{dx} = 5 \frac{d}{dx} x \Rightarrow \boxed{\frac{du}{dx} = 5}$$

$$dx = 1/5 \, du \rightarrow (iii)$$

(i) becomes:

$$= \int \cos u . \frac{1}{5} du = \frac{1}{5} \int \cos u \cdot du$$

$$= \frac{1}{5} \sin u + C = \frac{1}{5} \sin 5x + C$$

Example 4:

$$\int \frac{dx}{(1/3x-8)^5} \to u$$

Let $u = \frac{1}{3}x - 8$ $\to$ $\frac{du}{dx} = \frac{1}{3}$

$\boxed{dx = 3 \cdot du}$

$$\int \frac{dx}{1/3x - 8} = \int \frac{dx}{u}$$

$$= \int \frac{3 \cdot du}{u^5} = 3 \int u^{-5} \cdot du$$

$$= \frac{3\, u^{-5+1}}{-5+1} + C = -\frac{3u^{-4}}{4} + C = -\frac{3}{4} \cdot \frac{1}{(1/3x-8)^4} + C$$

Example 5:

$$\int \left(\frac{1}{x^2} + \sec^2 \pi x \right) dx \qquad \text{let } \pi x = u$$

$$= \int \frac{dx}{x^2} + \int \sec^2 \pi x \cdot dx$$

Let $\pi x = u$

$$\frac{du}{dx} = \pi \frac{d}{dx} x$$

$$\frac{du}{dx} = \pi \Rightarrow \boxed{dx = \frac{du}{\pi}}$$

$$= \int dx \cdot x^{-2} + \cancel{\frac{1}{\pi}} \int \cancel{dx}$$

$$= \frac{x^{-2+1}}{-2+1} + \int \sec^2 u \cdot \frac{du}{\pi} = \frac{-1}{x} + \frac{1}{\pi} \tan u + C$$

$$= \frac{-1}{x} + \frac{1}{\pi} \tan(\pi x) + C$$

$\int \sin^2 x \cdot \cos x \cdot dx$

Let $\boxed{u = \sin x}$

$\frac{du}{dx} = \cos x$

$\boxed{du = \cos x \cdot dx}$

$\int u^2 \cdot du$

$= \frac{u^{2+1}}{2+1} + C$

$= \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$

Evaluate:

nx^{n-1}

$\frac{x^{n+1}}{n+1}$

$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Let $\sqrt{x} = u$

$\frac{du}{dx} = \frac{1}{2\sqrt{x} = u}$ $\Rightarrow$ $\frac{du}{dx} = \frac{1}{2u}$

$dx = du \cdot 2u$

$\int \frac{\cos u}{\cancel{u}} \cdot du \cdot 2\cancel{u}$

$2 \int \cos u \cdot du$

$2 \sin u$

$2 \sin \sqrt{x} + C$

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Evaluate:

$$\int t^4 \sqrt[3]{3-5t^5} \cdot dt$$

Let $u = 3-5t^5$ nx^{n-1}

$$\frac{du}{dt} = -10t$$

~~du = -10~~

$$dt = -\frac{du}{10t}$$ $\frac{1}{3}+1$

$$= \int t^{4\,3} \cdot \int \sqrt[3]{u} \cdot \frac{-du}{10\cancel{t}}$$ $1/3+1$ $4/3$

$$= \left(\frac{t^4}{4}\right) + c + \left(\left(\frac{-1}{10}\right) \cdot \frac{3u^{4/3}}{4}\right) + c$$

$$= \frac{t^4}{4}\left(\frac{-1}{10} \cdot \frac{3(3-5t^2)^{4/3}}{4}\right) + C$$

Less apparent substitution:

Example 8:

$$\int x^2 \sqrt{x-1} \cdot dx$$

Let $u = x-1 \Rightarrow x^2 = (u+1)^2$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int (u+1)^2 \cdot \sqrt{u} \cdot du = \int (u^2+2u+1) \cdot u^{1/2} \cdot du$$

$$\int (u^{5/2} + u^{3/2} + u^{1/2}) \cdot du$$

$$\int u^{5/2} \cdot du + \int u^{3/2} \cdot du + \int u^{1/2} \cdot du$$

$$= \frac{2u^{7/2}}{7} + \frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}$$

Example 10:

$$\int \cos^3 x \cdot dx$$

$$= \int \cos^2 x \cdot \cos x \cdot dx$$

$$= \int (1 - \sin^2 x) \cdot \cos x \cdot dx$$

$$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Integration by Parts:

Example 1:

Evaluate : $\int \underbrace{x}_{①} \underbrace{\sin 2x}_{②} \cdot dx$

$$= x \left(\int \sin 2x \cdot dx \right) - \int \left(\begin{matrix} \text{Integration} \\ \text{of integrated} \\ \text{function} \end{matrix} \right) \cdot \frac{d}{dx}$$

Derivative of first function

$$= x \left(\frac{-\cos 2x}{2} \right) - \int \left(\frac{-\cos 2x}{2} \right) \cdot dx$$

General formula:

$$\int u \cdot dv = uv - \int v \cdot du$$

$$= \left(\begin{matrix} \text{First} \\ \text{function} \end{matrix} \times \begin{matrix} \text{Integration} \\ \text{of} \\ \text{second} \\ \text{function} \end{matrix} \right) - \left(\begin{matrix} \text{Integration} \\ \text{of} \\ \text{integrated} \\ \text{function} \end{matrix} \cdot \begin{matrix} \text{Derivative} \\ \text{of} \\ \text{1st} \\ \text{function} \end{matrix} \right)$$

$$= x \left(\frac{-\cos 2x}{2} \right) + \frac{\sin 2x}{2 \times 2} + c$$

$$= \frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} + C$$

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$x \sec^2 x \cdot dx$

Example 2:

$$\int x\sqrt{x+1}\,.dx$$

$$= x\,\frac{(x+1)^{1/2+1}}{1/2+1} - \int \frac{2(x+1)^{3/2}}{3}\,.dx$$

$$= \frac{2}{3}x\cdot(x+1)^{3/2} - \frac{2}{3}\int \frac{2(x+1)^{3/2}}{3}\,.dx$$

$$= \frac{2}{3}x\cdot(x+1)^{3/2} - \frac{2}{3}\left(\frac{(x+1)^{3/2+1}}{3/2+1}\right) + C$$

Example 3:

$$\int x\sec^2 x\cdot dx$$

$$= x\tan x - \int \tan x\cdot dx$$

$$= x\cdot\tan x - \ln|\sec x| + C$$