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Bs. 02

9 BINOMIAL AND HYPERGEOMETRIC DISTRIBUTIONS

9.1 Introduction Let us consider an experiment of tossing a coin or rolling a die repeatedly, drawing a card from a pack of playing cards repeatedly, etc. Each toss or drawing is called a *trial.* The possible outcome of each trial is called a *success* if the event occurs and a *failure* if it fails to occur. For example, in tossing a coin the outcome may be called a success if a head occurs and a failure if the head does not occur. Similarly, in drawing cards, the outcome may be called a success if the card drawn is an ace and a failure if it is not an ace. Let the probability of occurrence of an event A or the probability of success be denoted by p and the probability of failure by $q = 1 - p$, so that $p + q = 1$. Suppose the experiment is repeated n times. The number of successes that will be obtained in n trials of the experiment is denoted by x. Then the number of failures is $n - x$.

In some of the above examples, e.g. tossing a coin or rolling a die, the probability of success (or failure) will not change from one trial to the next. In case of drawing cards, the probability will not change from one trial to the other if the card drawn is replaced before drawing the next card. Such trials are said to be *independent* which means that no matter how many times the experiment is repeated the probabilities of success or failure remain the same. Repeated independent trials in which there are only two possible outcomes and probabilities of the outcomes remain the same for all trials are called *Bernoulli trials* after James Bernoulli (1654-1705) who investigated them. An experiment in which the outcome can always be classified as either a success or a failure and in which probability of success remains constant from trial to trial is called a *binomial experiment.* The random variable X, which represents the number of successes, is called a *binomial variable* or *binomial random variable.* The binomial variable is a discrete variable which can assume any of the values $x = 0, 1, 2, \ldots, n$. A binomial experiment is one that possesses the following properties:

(i) Each trial of the experiment results in an outcome that can be classified into one of the two categories: success or failure.
(ii) The probability of a success remains constant from one trial of the experiment to the next.
(iii) Each trial of the experiment is independent of all other trials.
(iv) The experiment is repeated a fixed number of times.

9.2 Binomial Probability Distribution If p is the probability of success in a single trial and q is the probability of failure, then the probability of exactly x successes in n trials of a binomial experiment is given by

$$P(X = x) = P(x) = b(x; n, p) = {}^{n}C_{x}\, p^{x} q^{n-x}, \quad x = 0, 1, 2, \ldots, n \qquad (9.1)$$

The function (9.1) is also called the *binomial probability function.* The notation $b(x; n, p)$ is read 'the binomial probability of x given n and p.' The quantities n and p are called the *parameters* of the binomial distribution because they determine the probabilities for all values of X.

The discrete probability distribution (9.1) is often called the *binomial distribution* because for $x = 0, 1, 2, \ldots, n$, it corresponds to the successive terms in the binomial expansion:

$$(q+p)^n = q^n + {}^nC_1\, q^{n-1}\, p + {}^nC_2\, q^{n-2}\, p^2 + \cdots + p^n$$

where ${}^nC_1$, ${}^nC_2$, ..., are called the *binomial coefficients.* Thus the binomial distribution is that in which the random variable X assumes the values $0, 1, 2, \ldots, n$ with corresponding probabilities q^n, ${}^nC_1\, q^{n-1}\, p$, ${}^nC_2\, q^{n-2}\, p^2$, ..., p^n. The binomial distribution is also called the *Bernoulli distribution* after James Bernoulli who discovered it towards the end of the 17th century.

The binomial distribution is an example of a discrete distribution because the random variable X can take on only discrete values $0, 1, 2, \ldots, n$.

If $p = q = 1/2$, the binomial distribution is a symmetrical distribution and when $p \neq q$, it is a skewed distribution. If $p < q$, the distribution is positively skewed and if $p > q$, the distribution is negatively skewed.

The binomial distribution is applicable to a wide variety of practical problems as well as to problems of an artificial nature, such as those dealing with cards, dice, coins and coloured balls in boxes. For application of the binomial distribution, we must ensure that the trials are independent and the probabilities are the same from trial to trial.

Example 9.1 An event has the probability $p = 3/5$. Find the complete binomial distribution for $n = 5$. (B.I.S.E., Rawalpindi 2007; Gujranwala 2013; Multan 2014)

Solution The probability of x successes in a series of n trials is given by

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$$

Here $n = 5$, $p = 3/5$ and hence $q = 1 - p = 1 - 3/5 = 2/5$.

$$P(X=0) = {}^5C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^5 = \frac{32}{3125} = 0.01024$$

$$P(X=1) = {}^5C_1 \left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^4 = \frac{240}{3125} = 0.0768$$

$$P(X=2) = {}^5C_2 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^3 = \frac{720}{3125} = 0.2304$$

$$P(X=3) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 = \frac{1080}{3125} = 0.3456$$

$$P(X=4) = {}^5C_4 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right) = \frac{810}{3125} = 0.2592$$

$$P(X=5) = {}^5C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^0 = \frac{243}{3125} = 0.07776$$

rnative Solution The required probabilities are the successive terms in the xpansion of $(q+p)^n = \left(\frac{2}{5}+\frac{3}{5}\right)^5$. The expansion is given by

$$\left(\frac{2}{5}+\frac{3}{5}\right)^5 = \left[{}^5C_0\left(\frac{2}{5}\right)^5 + {}^5C_1\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right) + {}^5C_2\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^2 + {}^5C_3\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^3 + {}^5C_4\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 + {}^5C_5\left(\frac{3}{5}\right)^5\right]$$

$$= \frac{1}{2125}[(2)^5 + 5(2)^4(3) + 10(2)^3(3)^2 + 10(2)^3(3)^3 + 5(2)(3)^4 + (3)^5]$$

$$= \frac{1}{2125}[32 + 240 + 720 + 1080 + 810 + 243]$$

$$= [0.01024 + 0.0768 + 0.2304 + 0.3456 + 0.2592 + 0.07776]$$

The required binomial distribution is given in the following table.

x	0	1	2	3	4	5
$P(x)$	0.01024	0.0768	0.2304	0.3456	0.2592	0.07776

Example 9.2 A fair coin is tossed four times. What is the probability of getting (i) exactly two heads (ii) at least two heads (iii) at most 3 heads (iv) between 1 and 3 heads inclusive. (B.I.S.E., Faisalabad 2015 Practical)

Solution Considering the probability of getting a head as success, we have $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$ and $n = 4$. Using the formula $P(X = x) = {}^nC_x\, p^x q^{n-x}$, we obtain the required probabilities as computed below:

(i) Probability of getting exactly two heads is

$$P(X=2) = {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{6}{16} = \frac{3}{8}.$$

(ii) Probability of getting at least two heads (two, three or four heads) is

$$P(X \geq 2) = \left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= \left(\frac{1}{2}\right)^4[{}^4C_2 + {}^4C_3 + {}^4C_4] = \frac{1}{16}[6 + 4 + 1] = \frac{11}{16}.$$

(iii) Probability of getting at most 3 heads (0, 1, 2 or 3 heads) is

$$P(X \leq 3) = \left[{}^4C_0\left(\frac{1}{2}\right)^4 + {}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3\right]$$

$$= \left(\frac{1}{2}\right)^4[{}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3] = \frac{1}{16}[1 + 4 + 6 + 4] = \frac{15}{16}.$$

Alternatively $P(X \le 3) = 1 - P(X=4) = 1 - \left[{}^4C_4\left(\frac{1}{2}\right)^4\right] = 1 - \frac{1}{16} = \frac{15}{16}$.

(iv) Probability of getting between 1 and 3 heads inclusive (1, 2, or 3 heads) is

$$P(1 \le X \le 3) = \left[{}^4C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)\right]$$

$$= \left(\frac{1}{2}\right)^4 [{}^4C_1 + {}^4C_2 + {}^4C_3] = \frac{1}{16}[4 + 6 + 4] = \frac{7}{8}.$$

Example 9.3 What is the probability of obtaining (i) 3 sixes (ii) at least 4 sixes and (iii) at most 3 sixes when a perfect cubical die is thrown five times?

Solution Here $n = 5$, $p = P(6) = \frac{1}{6}$ and $q = 1 - \frac{1}{6} = \frac{5}{6}$. Using the formula $P(X = x) = {}^nC_x p^x q^{n-x}$, we obtain the required probabilities as follows.

(i) Probability of obtaining 3 sixes is

$$P(X = 3) = {}^5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = 10\left(\frac{1}{216}\right)\left(\frac{25}{36}\right) = \frac{250}{7776} = 0.03215.$$

(ii) Probability of obtaining at least 4 sixes (4 or 5 sixes) is

$$P(X \ge 4) = \left[{}^5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) + {}^5C_5\left(\frac{1}{6}\right)^5\right] = \frac{1}{(6)^5}[5(5) + 1] = 26/7776 = 0.0033436.$$

(iii) Probability of at most 3 sixes (0, 1, 2 or 3 sixes) is

$$P(X \le 3) = \left[{}^5C_0\left(\frac{5}{6}\right)^5 + {}^5C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4 + {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + {}^5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2\right]$$

$$= \frac{1}{(6)^5}[(5)^5 + (5)(5)^4 + (10)(5)^3 + (10)(5)^2]$$

$$= \frac{1}{7776}[3125 + 3125 + 1250 + 250] = \frac{7750}{7776} = 0.99666.$$

Alternatively $P(X \le 3) = 1 - P(X > 3)$

$$= 1 - \left[{}^5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) + {}^5C_5\left(\frac{1}{6}\right)^5\right] = 1 - \frac{1}{(6)^5}[25 + 1] = 1 - \frac{26}{7776} = \frac{7750}{7776}$$

$= 0.99666.$

Example 9.4 What is the probability of getting a 'total of 9' (i) twice (ii) at least twice in 6 tosses of a pair of dice?

Solution Tossing a pair of dice results in $6 \times 6 = 36$ sample points. Let A denote the event 'total of 9'. Then A contains four sample points: {(3 6), (4 5), (5 4), (6 3)}. $P(A) = 4/36 = 1/9$. Thus $p = 1/9$ and $q = 1 - p = 8/9$. Also $n = 6$. Using the formula $P(X = x) = {}^nC_x p^x q^{n-x}$, we obtain the required probabilities:

Probability of getting a 'total of 9' twice is

$$P(X=2) = {}^6C_2\left(\frac{1}{9}\right)^2\left(\frac{8}{9}\right)^4 = \frac{61440}{531441} = 0.1156.$$

(ii) Probability of getting a 'total of 9' at least twice (2, 3, 4, 5 or 6) is

$$P(X \geq 2) = \left[{}^6C_2\left(\frac{1}{9}\right)^2\left(\frac{8}{9}\right)^4 + {}^6C_3\left(\frac{1}{9}\right)^3\left(\frac{8}{9}\right)^3 + {}^6C_4\left(\frac{1}{9}\right)^4\left(\frac{8}{9}\right)^2\right.$$

$$\left. + {}^6C_5\left(\frac{1}{9}\right)^5\left(\frac{8}{9}\right) + {}^6C_6\left(\frac{1}{9}\right)^6\right]$$

$$= \frac{1}{(9)^6}\left[(15)(8)^4 + (20)(8)^3 + (15)(8)^2 + (6)(8) + 1\right]$$

$$= \frac{1}{521441}\left[61440 + 10240 + 960 + 48 + 1\right] = \frac{72689}{531441} = 0.1368.$$

Alternatively $P(X \geq 2) = 1 - P(X \leq 1)$

$$= 1 - \left[{}^6C_0\left(\frac{8}{9}\right)^6 + {}^6C_1\left(\frac{1}{9}\right)\left(\frac{8}{9}\right)^5\right] = 1 - \left(\frac{262144}{531441} + \frac{196608}{531441}\right) = 1 - \left(\frac{458752}{531441}\right)$$

$$= \frac{72689}{531441} = 0.1368.$$

Example 9.5 An insurance salesman sells policies to 5 men, all of identical age. According to the actuarial tables, the probability that a man of this particular age will be alive 30 years hence is 2/3. Find the probability that in 30 years (a) all 5 men (b) at least 3 men (c) only 2 men (d) at least 1 man will be alive.

(B.I.S.E., Lahore 1978)

Solution The probability that a man will be alive 30 years hence is $p = 2/3$, so that the probability that he will not be alive 30 years hence is $q = 1 - p = 1 - 2/3 = 1/3$. Here $n = 5$. Using the formula $P(X = x) = {}^nC_x\, p^x\, q^{n-x}$, we obtain the required probabilities.

(a) P (All 5 men will be alive) $= P(X=5) = {}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0 = \frac{32}{243}$.

(b) P (At least 3 men will be alive)

$= P$(3, 4 or 5 men will be alive) $= P(X \geq 3)$

$$= {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + {}^5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) + {}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0 = \frac{80}{243} + \frac{80}{243} + \frac{32}{243} = \frac{192}{243}$$

$$= \frac{64}{81}.$$

(c) P (Only 2 men will be alive) $= P(X=2) = {}^5C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 = \frac{40}{243}$.

(d) P(At least 1 man will be alive) = 1 − P(no man will be alive) = 1 − $P(X = 0)$

$$= 1 - {}^5C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5 = 1 - \frac{1}{243} = \frac{242}{243}.$$

9.3 Binomial Frequency Distribution Let the n independent trials constitute a binomial experiment and let this experiment be repeated N times. Then we expect x successes in N sets of n trials to occur $N \cdot {}^nC_x\, p^x q^{n-x}$ times. This is called the expected frequency of x successes in N experiments and the possible number of successes together with the expected frequencies is called the *binomial frequency distribution*. In practice, the observed frequencies will differ from the expected frequencies due to chance causes. For N sets, each of n trials, the expected frequencies of 0, 1, 2, ..., n successes are given by the successive terms in the expansion of $N(q + p)^n$.

Example 9.6 Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys (ii) 5 girls (iii) either 2 or 3 boys? Assume equal probabilities for boys and girls. (B.I.S.E., Lahore 2013 Practical)

Solution Here $p = P(\text{boy}) = 1/2$, $q = P(\text{girl}) = 1/2$, $n = 5$ and $N = 800$. Using the formula $P(X = x) = {}^nC_x\, p^x\, q^{n-x}$, we first compute the probability and by multiplying it by N we obtain the expected frequency.

(i) $P(3 \text{ boys}) = P(X = 3) = {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16}$.

Expected number of families with 3 boys = $NP(X = 3) = 800(5/16) = 250$.

(ii) P(5 girls) = P(no boy) = $P(X = 0) = {}^5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

Expected number of families with 5 girls = $NP(X = 0) = 800(1/32) = 25$.

(iii) P(either 2 or 3 boys) = $P(X = 2 \text{ or } X = 3) = {}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

$$= \frac{10}{32} + \frac{10}{32} = \frac{5}{8}.$$

Expected number of families with either 2 or 3 boys
$= NP(X = 2 \text{ or } X = 3) = 800(5/8) = 500$.

Example 9.7 Four dice are thrown 162 times and a throw of '5 or 6' is regarded as success. Find the expected frequencies for 0, 1, 2, 3, 4 successes.

Solution Considering the throw of a '5 or 6' on a die to be a success, we have $p = 2/6 = 1/3$ and $q = 1 - p = 1 - 1/3 = 2/3$. Also $n = 4$ and $N = 162$. If x denotes the number of successes, then the probability $P(x)$ and the expected frequency $NP(x)$ are obtained in the following table.

x	$P(x)$	$NP(x)$
0	${}^4C_0\,p^0q^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$	$162\left(\frac{16}{81}\right) = 32$
1	${}^4C_1\,pq^3 = 4\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{32}{81}$	$162\left(\frac{32}{81}\right) = 64$
2	${}^4C_2\,p^2q^2 = 6\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24}{81}$	$162\left(\frac{24}{81}\right) = 48$
3	${}^4C_3\,p^3q = 4\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) = \frac{8}{81}$	$162\left(\frac{8}{81}\right) = 16$
4	${}^4C_4\,p^4q^0 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$	$162\left(\frac{1}{81}\right) = 2$

Alternative Solution The expected frequencies for 0, 1, 2, ..., n successes are the successive terms in the expansion of $N(q+p)^n$.

$$162\left(\frac{2}{3}+\frac{1}{3}\right)^4 = 162\left\{{}^4C_0\left(\frac{2}{3}\right)^4 + {}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + {}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + {}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + {}^4C_4\left(\frac{1}{3}\right)^4\right\}$$

$$= \frac{162}{(3)^4}\{(2)^4 + (4)(2)^3 + (6)(2)^2 + (4)(2) + 1\} = \frac{162}{81}\{16 + 32 + 24 + 8 + 1\}$$

$$= 32 + 64 + 48 + 16 + 2$$

Thus the expected frequencies of 0, 1, 2, 3, 4 successes are 32, 64, 48, 16, 2 respectively.

Example 9.8 Six dice are thrown 729 times. How many times do you expect at least three dice to show a '5 or 6'?

Solution The probability of getting a '5 or 6' with one dice, is 2/6, i.e. $p = 2/6 = 1/3$ and $q = 1 - p = 2/3$. Also $n = 6$ and $N = 729$.

The expected number of times at least three dice (3, 4, 5 or 6 dice) will show a '5 or 6' is

$$= 729\left[{}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + {}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + {}^6C_6\left(\frac{1}{3}\right)^6\right]$$

$$= \frac{729}{(3)^6}[20(2)^3 + 15(2)^2 + 6(2) + 1] = 160 + 60 + 12 + 1 = 233.$$

9.4 Mean and Variance of the Binomial Distribution For a probability distribution of a discrete random variable X, the mean (μ) and variance (σ^2) are defined as

$$\mu = E(X) = \sum_{i=1}^{n} x_i\, P(x_i) = \Sigma x\, P(x) \tag{9.2}$$

and $$\sigma^2 = E(X^2) - \mu^2 = \sum_{i=1}^{n} x_i^2\, P(x_i) - \mu^2 = \Sigma x^2\, P(x) - \mu^2 \tag{9.3}$$

If X is a binomial random variable, then X can take on any of the values 0, 1, 2, ..., n. The probability that X takes on a particular value x is given by

$$P(X = x) = P(x) = {}^nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

The mean (μ) and variance (σ^2) of the binomial distribution are

$$\mu = E(X) = \sum_{x=0}^{n} x \cdot {}^nC_x\, p^x\, q^{n-x}$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2 = \sum_{x=0}^{n} x^2 \cdot {}^nC_x\, p^x\, q^{n-x} - \mu^2$$

First we find the mean and variance of the binomial distribution for $n = 2$ and $n = 3$.

When $n = 2$, the number of successes x will be 0, 1 and 2 and corresponding probabilities $P(x)$ are q^2, $2pq$, and p^2. Necessary calculations are shown in the following table.

x	$P(x)$	$xP(x)$	$x^2P(x)$
0	q^2	0	0
1	$2pq$	$2pq$	$2pq$
2	p^2	$2p^2$	$4p^2$

$$\mu = E(X) = \Sigma xP(x) = 2pq + 2p^2 = 2p(q + p) = 2p. \quad (q + p = 1)$$

$$E(X^2) = \Sigma x^2 P(x) = 2pq + 4p^2$$

$$\sigma^2 = E(X^2) - \mu^2 = 2pq + 4p^2 - (2p)^2 = 2pq.$$

When $n = 3$, the number of successes x will be 0, 1, 2 and 3 and the corresponding probabilities $P(x)$ are q^3, $3q^2p$, $3qp^2$ and p^3. Necessary computations are shown in the following table.

x	$P(x)$	$xP(x)$	$x^2P(x)$	$x(x-1)$	$x(x-1)P(x)$
0	q^3	0	0	$0(0-1) = 0$	0
1	$3q^2p$	$3q^2p$	$3q^2p$	$1(1-1) = 0$	0
2	$3qp^2$	$6qp^2$	$12qp^2$	$2(2-1) = 2$	$6qp^2$
3	p^3	$3p^3$	$9p^3$	$3(3-1) = 6$	$6p^3$

Table 9.1

x	$P(x)$	$xP(x)$	$x^2P(x)$	$x(x-1)$	$x(x-1)\,P(x)$
0	q^n	0	0	0	0
1	$nq^{n-1}\,p$	$nq^{n-1}\,p$	$nq^{n-1}\,p$	0	0
2	$\frac{n(n-1)}{2!}\,q^{n-2}\,p^2$	$n(n-1)\,q^{n-2}\,p^2$	$2n(n-1)\,q^{n-2}\,p^2$	2	$\frac{2n(n-1)}{2!}\,q^{n-2}\,p^2$
3	$\frac{n(n-1)(n-2)}{3!}\,q^{n-3}p^3$	$\frac{n(n-1)(n-2)}{2!}\,q^{n-3}p^3$	$\frac{3n(n-1)(n-2)}{2!}\,q^{n-3}p^3$	6	$\frac{6n(n-1)(n-2)}{3!}\,q^{n-3}p^3$
4	$\frac{n(n-1)(n-2)(n-3)}{4!}q^{n-4}p^4$	$\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4$	$\frac{4n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4$	12	$\frac{12n(n-1)(n-2)(n-3)}{4!}q^{n-4}p^4$
⋮	⋮	⋮	⋮	⋮	⋮
n	p^n	np^n	n^2p^n	$n(n-1)$	$n(n-1)p^n$

$$\mu = E(X) = \Sigma xP(x) = 3q^2p + 6qp^2 + 3p^3 = 3p[q^2 + 2qp + p^2] = 3p(q + p)^2 = 3p. \qquad (q + p = 1)$$

$$E(X^2) = \Sigma x^2P(x) = 3q^2p + 12qp^2 + 9p^3 = 3p[q^2 + 4qp + 3p^2]$$

$$= 3p[(q^2 + 2qp + p^2) + (2qp + 2p^2)] = 3p[(q + p)^2 + 2p(q + p)] = 3p(1 + 2p)$$

$$= 3p + 6p^2$$

Alternatively using the relation $x^2 = x(x - 1) + x$

$$E(X^2) = \Sigma x^2P(x) = \Sigma[x(x - 1) + x]P(x) = \Sigma x(x - 1)\, P(x) + \Sigma xP(x) = 6qp^2 + 6p^3 + 3p$$

$$= 6p^2(q + p) + 3p = 3p + 6p^2$$

$$\sigma^2 = E(X^2) - \mu^2 = 3p + 6p^2 - (3p)^2 = 3p - 3p^2 = 3p(1 - p) = 3pq.$$

Now we find the mean and variance of the binomial distribution (9.1). Necessary computations are shown in Table 9.1.

$$\mu = E(X) = \Sigma xP(x)$$

$$= \left[nq^{n-1}p + n(n - 1)q^{n-2}p^2 + \frac{n(n - 1)(n - 2)}{2!} q^{n-3}p^3 + \frac{n(n - 1)(n - 2)(n - 3)}{3!} q^{n-4}p^4 + \cdots + np^n \right]$$

$$= np\left[q^{n-1} + (n - 1)q^{n-2}p + \frac{(n - 1)(n - 2)}{2!} q^{n-3}p^2 + \frac{(n - 1)(n - 2)(n - 3)}{3!} q^{n-4}p^3 + \cdots + p^{n-1} \right]$$

The expression in the square brackets is the expansion of the binomial $(q + p)^{n-1}$. Thus

$$\mu = np(q + p)^{n-1} = np. \qquad (q + p = 1)$$

$$E(X^2) = \Sigma x^2P(x)$$

$$= \left[nq^{n-1}p + 2n(n - 1)q^{n-2}p^2 + \frac{3n(n - 1)(n - 2)}{2!} q^{n-3}p^3 + \frac{4n(n - 1)(n - 2)(n - 3)}{3!} q^{n-4}p^4 + \cdots + n^2p^n \right]$$

$$= np\left[q^{n-1} + 2(n - 1)q^{n-2}p + \frac{3(n - 1)(n - 2)}{2!} q^{n-3}p^2 + \frac{4(n - 1)(n - 2)(n - 3)}{3!} q^{n-4}p^3 + \cdots + np^{n-1} \right]$$

$$= np\Big[\Big\{q^{n-1} + (n-1)\,q^{n-2}p + \frac{(n-1)(n-2)}{2!}q^{n-3}p^2 + \frac{(n-1)(n-2)(n-3)}{3!}q^{n-4}p^3 + \dots + p^{n-1}\Big\}$$

$$+ \Big\{(n-1)\,q^{n-2}p + (n-1)(n-2)\,q^{n-3}p^2 + \frac{(n-1)(n-2)(n-3)}{2!}q^{n-4}p^3 + \dots + (n-1)p^{n-1}\Big\}\Big]$$

The expression in the first curly brackets is the expansion of the binomial $(q+p)^{n-1}$.

$$E(X^2) = np\Big[(q+p)^{n-1} + (n-1)p\Big\{q^{n-2} + (n-2)q^{n-3}p + \frac{(n-2)(n-3)}{2!}q^{n-4}p^2 + \dots + p^{n-2}\Big\}\Big]$$

Again the expression in the curly brackets is the expansion of the binomial $(q+p)^{n-2}$. Thus

$$E(X^2) = np[1 + (n-1)p\,(q+p)^{n-2}] = np[1 + (n-1)p] = np[1 + np - p]$$

$$= np + n^2p^2 - np^2.$$

Alternatively using the relation $x^2 = x(x-1) + x$, we have from Table 9.1.

$$E(X^2) = \Sigma x^2P(x) = \Sigma[x(x-1) + x]P(x) = \Sigma x(x-1)P(x) + \Sigma xP(x)$$

$$= \frac{2n(n-1)}{2!}q^{n-2}p^2 + \frac{6n(n-1)(n-2)}{3!}q^{n-3}p^3 + \frac{12n(n-1)(n-2)(n-3)}{4!}q^{n-4}p^4 + \dots + n(n-1)p^n + np$$

$$= n(n-1)q^{n-2}p^2 + n(n-1)(n-2)q^{n-3}p^3 + \frac{n(n-1)(n-2)(n-3)}{2!}q^{n-4}p^4 + \dots + n(n-1)p^n + np$$

$$= n(n-1)p^2\Big[q^{n-2} + (n-2)q^{n-3}p + \frac{(n-2)(n-3)}{2!}q^{n-4}p^2 + \dots + p^{n-2}\Big] + np$$

The expression in the square brackets is the expansion of the binomial $(q+p)^{n-2}$. Thus

$$E(X^2) = n(n-1)p^2\,(1+p)^{n-2} + np = n(n-1)\,p^2 + np$$

$$\sigma^2 = E(X^2) - \mu^2 = n(n-1)p^2 + np - (np)^2 = np + n^2p^2 - np^2 - n^2p^2 = np - np^2$$

$$= np(1-p) = npq$$

and the standard deviation $\sigma = \sqrt{npq}$.

Example 9.9 In a binomial distribution $n = 10$ and $p = 3/5$. Find the mean and variance of the distribution.

Solution The mean μ and variance σ^2 are $\boxed{\mu = np} = 10\left(\frac{3}{5}\right) = 6.$

$$\boxed{\sigma^2 = npq} = 10\left(\frac{3}{5}\right)\left(1-\frac{3}{5}\right) = 10\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) = \frac{12}{5} = 2.4.$$

Example 9.10 In a binomial distribution, the mean and the standard deviation were found to be 4.5 and 1.5 respectively. Find n and p.

Solution We have $\mu = np = 4.5$ and $\sigma^2 = npq = (1.5)^2 = 2.25.$

$$\frac{\sigma^2}{\mu} = \frac{npq}{np} = \frac{2.25}{4.5} \text{ or } q = 0.5 \text{ and } p = 1 - q = 1 - 0.5 = 0.5.$$

$$\mu = np = 4.5 \text{ or } n = 4.5/p = 4.5/0.5 = 9.$$

Since p or q cannot be greater than 1 or negative, so it is not possible to have a binomial distribution with mean 10 and standard deviation 4.

Example 9.11 If $n = 4$, $p = \frac{1}{2}$ find $P(X = 3)$. (B.I.S.E., Lahore 2015)

Solution $n = 4, \quad p = \frac{1}{2}, \; q = 1 - \frac{1}{2} = \frac{1}{2}$

$$P(X = 3) = {}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{4-3} = 4\left(\frac{1}{8}\right) b\left(\frac{1}{2}\right) = \frac{4}{16} = \frac{1}{4}$$

Example 9.12 Three dice are thrown and the number of aces in each throw are recorded. This is repeated 216 times. Write down theoretical (or expected) frequencies of 0,1,2,3 aces. Also find the mean and variance.

Solution Probability of throwing an ace with one die is 1/6, i.e. $p = 1/6$ and $q = 1 - 1/6 = 5/6$. Also $n = 3$ and $N = 216$.

$$N(q+p)^n = 216\left(\frac{5}{6}+\frac{1}{6}\right)^3$$

$$= 216\left[{}^3C_0\left(\frac{5}{6}\right)^3 + {}^3C_1\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + {}^3C_2\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2 + {}^3C_3\left(\frac{1}{6}\right)^3\right]$$

$$= \frac{216}{(6)^3}\left[(5)^3 + 3(5)^2 + 3(5) + 1\right] = 125 + 75 + 15 + 1.$$

The theoretical frequencies are shown in the table below, where computation of the mean and variance is also shown.

No. of aces (x)	Theoretical Frequency (f)	fx	fx^2
0	125	0	0
1	75	75	75
2	15	30	60
3	1	3	9
	$\Sigma f = 216$	$\Sigma fx = 108$	$\Sigma fx^2 = 144$

$$\mu = \frac{\Sigma fx}{\Sigma f} = \frac{108}{216} = \frac{1}{2} = 0.5.$$

$$\sigma^2 = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2 = \frac{144}{216} - \left(\frac{108}{216}\right)^2 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12} = 0.42.$$

9.5 Hypergeometric Distribution The binomial distribution is based on the assumption that the successive trials are independent and the probability of success remains unchanged from trial to trial. This assumption holds if the successive trials or drawings are with replacement. For example, if we draw balls from a bag containing 10 white, 8 red and 5 blue balls with replacement, the probability of drawing white balls will remain the same from one trial to the other. In sampling without replacement, the probability will change from trial to trial. In drawing balls from the bag, the probability of a white ball on first draw will be 10/23, on the second draw 9/22 and so on. Thus when the successive trials or drawings are without replacement, then they are not independent and the probability of success does not remain the same from one trial to the other. Such an experiment in which a random sample is selected without replacement from a finite population is called *hypergeometric experiment.*

A hypergeometric experiment has the following properties:

1. Each trial of an experiment results in an outcome that can be classified into one of the two categories: *success* or *failure*
2. The successive trials are dependent
3. The probability of success changes from trial to trial
4. The experiment is repeated a fixed number of times

The random variable X representing the number of successes in a hypergeometric experiment is called a *hypergeometric random variable*. The probability distribution of the hypergeometric variable is called the *hypergeometric distribution.*

9.5.1 Hypergeometric Probability Function Suppose a population of N items consists of k items of one kind (successes) and $N - k$ items of another kind (failures), and we are interested in the probability of getting x successes among n items selected at random from the population of N items. The formula for this probability function is

$$P(X = x) = h(x; N, n, k) = \frac{{}^{k}C_x \, {}^{N-k}C_{n-x}}{{}^{N}C_n} \qquad (9.4)$$

n = items
N = population , K = Success
n − x = Failures

$$x = 0, 1, 2, \ldots, n \text{ if } n \leq k$$
$$= 0, 1, 2, \ldots, k \text{ if } n > k$$

Formula (9.4) can be proved as follows:

The total number of ways in which n items can be selected from the population of N items is ${}^{N}C_n$. In a similar manner, the x successes can be selected from among the k successes in ${}^{k}C_x$ ways. The $n - x$ failures can be selected from among the $N - k$ failures in ${}^{N-k}C_{n-x}$ ways. Thus the x successes and $n - x$ failures can be selected in ${}^{k}C_x \, {}^{N-k}C_{n-x}$ ways in accordance with the fundamental principle of counting.

Hence the desired probability that n items randomly selected from the population of N items consisting of x successes and $n - x$ failures is given by (9.4).

Example 9.13 A box contains 5 red and 10 white marbles. If 8 marbles are chosen at random (without replacement) determine the probability that (a) 4 are red (b) all are white (c) at least one is red. (B.I.S.E., Gujranwala 2013)

Solution (a) We have $N = 15$, $n = 8$, $k = 5$, $x = 4$. Using the formula

$P(X = x) = h(x; N, n, k) = {}^{k}C_x \, {}^{N-k}C_{n-x} \Big/ {}^{N}C_n$, the required probability is

$$P(X = 4) = {}^{5}C_4 \, {}^{15-5}C_{8-4} \Big/ {}^{15}C_8 = {}^{5}C_4 \, {}^{10}C_4 \Big/ {}^{15}C_8 = \frac{70}{429}.$$

Alternative Solution The total number of different ways of selecting 8 marbles out of $(5 + 10) = 15$ marbles in the box is ${}^{15}C_8$. The number of different ways of selecting 4 red marbles out of 5 red marbles is ${}^{5}C_4$. The number of different ways of selecting the remaining 4 marbles out of 10 white marbles is ${}^{10}C_4$. Thus the number of different samples containing 4 red marbles and 4 white marbles is ${}^{5}C_4 \, {}^{10}C_4$. Hence the required probability is given by

$${}^{5}C_4 \, {}^{10}C_4 \Big/ {}^{15}C_8 = \frac{70}{429}.$$

(b) We have $N = 15$, $n = 8$, $k = 10$, $x = 8$. Using the formula, the required probability is

$$P(X = 8) = {}^{10}C_8 \, {}^{15-10}C_{8-8} \Big/ {}^{15}C_8 = {}^{10}C_8 \, {}^{5}C_0 \Big/ {}^{15}C_8 = \frac{1}{143}.$$

Alternative Solution The total number of different ways of selecting 8 marbles out of $(5 + 10) = 15$ marbles in the box is ${}^{15}C_8$. The number of different ways of selecting 8 white marbles out of 10 white marbles is ${}^{10}C_8$. The number of different ways of selecting 0 red marble out of 5 red marbles is ${}^{5}C_0$. Thus the number of different

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...mples containing 8 white marbles and 0 red marble is ${}^{10}C_8\,{}^{5}C_0$. Hence the required probability is given by ${}^{10}C_8\,{}^{5}C_0 \Big/ {}^{15}C_8 = \frac{1}{143}$.

(c) $P(\text{at least one red}) = 1 - P(0 \text{ red}) = 1 - P(\text{all white}) = 1 - P(X = 8) = 1 - \frac{1}{143} = \frac{142}{143}$.

Example 9.14 Out of 60 applicants for a job 40 are from Lahore. If 20 applicants are selected at random find the probability that (a) 10 (b) not more than 2 are from Lahore.

Solution (a) We have $N = 60$, $k = 40$, $n = 20$, $x = 10$. Using the formula, the required probability is

$$P(X = 10) = h(x; N, n, k) = \frac{{}^{40}C_{10}\,{}^{60-40}C_{20-10}}{{}^{60}C_{20}} = \frac{{}^{40}C_{10}\,{}^{20}C_{10}}{{}^{60}C_{20}}.$$

$$= \frac{{}^{k}C_{x}\,{}^{N-k}C_{n-x}}{{}^{N}C_{n}}$$

(b) We have $N = 60$, $k = 40$, $n = 20$, $x = 0, 1, 2$. Using the formula, the required probability is

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{{}^{40}C_0\,{}^{60-40}C_{20-0}}{{}^{60}C_{20}} + \frac{{}^{40}C_1\,{}^{60-40}C_{20-1}}{{}^{60}C_{20}} + \frac{{}^{40}C_2\,{}^{60-40}C_{20-2}}{{}^{60}C_{20}}$$

$$= \left[1 + {}^{40}C_1\,{}^{20}C_1 + {}^{40}C_2\,{}^{20}C_2\right] \Big/ {}^{60}C_{20}.$$

Example 9.15 If 13 cards are chosen at random (without replacement) from an ordinary deck of 52 cards find the probability that 6 are picture cards.

(B.I.S.E., Lahore 2009)

Solution We have $N = 52$, $n = 13$, $k = 12$, $x = 6$. Using the formula,

$$P(X = 6) = h(x; N,n,k) = \frac{{}^{12}C_6\,{}^{55-12}C_{13-6}}{{}^{52}C_{13}} = \frac{{}^{12}C_6\,{}^{40}C_7}{{}^{52}C_{13}}.$$

Alternative Solution Thirteen cards out of 52 cards can be chosen in ${}^{52}C_{13}$ ways. Six picture cards out of 12 picture cards can be chosen in ${}^{12}C_6$ ways. The remaining seven cards out of 40 cards can be chosen in ${}^{40}C_7$ ways. Thus the number of different samples containing 6 picture cards and 7 other cards is ${}^{12}C_6\,{}^{40}C_7$. Hence the required probability is ${}^{12}C_6\,{}^{40}C_7 \Big/ {}^{52}C_{13}$.

9.5.2 Properties of the Hypergeometric Distribution The mean (μ) and variance (σ^2) of the hypergeometric distribution are

$$\mu = E(X) = \frac{nk}{N} \qquad (9.5)$$

$$\sigma^2 = \text{Var}(X) = \frac{nk(N-k)(N-n)}{N^2(N-1)} \tag{9.6}$$

If $p = \frac{k}{N}$ (proportion of successes), then

$$q = 1 - p = 1 - \frac{k}{N} = \frac{(N-k)}{N}. \text{ Thus}$$

$$\mu = np \text{ and } \sigma^2 = \frac{nk}{N}\frac{N-k}{N}\left(\frac{N-n}{N-1}\right) = npq\left(\frac{N-n}{N-1}\right)$$

Note that the expected value of the hypergeometric distribution is the same as that of the binomial distribution, whereas the variances differ by a factor $\left(\frac{N-n}{N-1}\right)$. An interesting features of this factor is that for fixed k/N and n, the factor tends to unity as N becomes infinite. Hence it can be shown that the hypergeometric distribution tends towards the binomial distribution; that is $h(x; N, n, k) \rightarrow b(x; n, p)$ for fixed k/N and n as N increases without limit.

Example 9.16 An event has probability $P = \frac{3}{4}$. Find complete binomial distribution for $n = 4$ trials. (B.I.S.E., Lahore 2015)

Solution The probability of x success in a series of n trials is given by

$$P(X = x) = C_x^n p^x q^{n-x}$$

Here $n = 4$, $P = \frac{3}{4}$ and hence $q = 1 - p = 1 - \frac{3}{4} = \frac{1}{4}$.

$$P(X=0) = C_0^4\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{4-0} = \frac{1}{256}$$

$$P(X=1) = C_1^4\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{4-0} = 4\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^3 = \frac{12}{256}$$

$$P(X=2) = C_2^4\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{4-2} = 6\left(\frac{9}{16}\right)\left(\frac{1}{16}\right) = \frac{54}{256}$$

$$P(X=3) = C_3^4\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^{4-3} = 4\left(\frac{27}{64}\right)\left(\frac{1}{4}\right) = \frac{108}{256}$$

$$P(X=4) = C_4^4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^{4-4} = \frac{81}{256}$$

Example 9.17(a) A committee of size 4 is to be selected at random form 3 women and 4 men. Find the probability. (B.I.S.E., Multan Practical 2014; Gujranwala 2014)

1. 2 women
2. At least 2 women will be selected in the committee.

Sol. Here $n = 4$

W	M	T
3	4	7

N = 7

k = 3

Let x be the no. of women.

(i) $$P(X = 2) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} = \frac{\binom{3}{2}\binom{7-3}{4-2}}{\binom{7}{4}} = \frac{\binom{3}{2}\binom{4}{2}}{\binom{7}{4}} = \frac{(3)\,(6)}{35} = \frac{18}{35} = 0.5143$$

2. P(at least 2 women will be selected)

(ii) $$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)] = \left[\frac{{}^3C_0 \times {}^4C_4}{{}^7C_4} + \frac{{}^3C_1 \times {}^4C_3}{{}^7C_4}\right]$$

$$= 1 - \left[\frac{1 \times 1}{35} + \frac{3 \times 4}{35}\right] = 1 - \left[\frac{1}{35} + \frac{12}{35}\right] = 1 - \frac{13}{35} = \frac{22}{35} = 0.6286$$

Example 9.17(b) A bag contains 6 green and 4 red balls. Three are selected at random without replacement. Compute the probability distribution of green balls.
(B.I.S.E. Multan Board, 2013; Faisalabad 2015)

Sol. Here N = 6 + 4 = 10, n = 3. Let X be the No. of green balls k = 6.

x	P(X = x)
0	${}^6C_0 \times {}^4C_3/{}^{10}C_3 = \frac{4}{120}$
1	${}^6C_1 \times {}^4C_2/{}^{10}C_3 = \frac{36}{120}$
2	${}^6C_2 \times {}^4C_1/{}^{10}C_3 = \frac{60}{120}$
3	${}^6C_3 \times {}^4C_0/{}^{10}C_3 = \frac{20}{120}$

OBJECTIVE QUESTIONS

Q9.1 Write short answers to the following questions.

Q.1. Define the binomial probability distribution.

(B.I.S.E., Lahore 2011, 2014; Faisalabad 2015, 2017; Gujranwala 2015, 2018)

Ans. If p is the probability of success in a single trial and q is the probability of failure, then the probability of exactly x successes in n trials of a binomial experiment is given by

$$P(X = x) = P(x) = b(x; n, p) = {}^nC_x p^x q^{n-x}, \; x = 0, 1, 2, \ldots, n$$

Q.2. Why is the discrete probability function $b(x; n, p) = {}^nC_x p^n q^{n-x}$ called the binomial distribution?

Ans. The discrete probability function $b(x; n, p)$ is called the binomial distribution because for $x = 0, 1, 2, \ldots, n$, it corresponds to the successive terms in the binomial expansion

$$(q + p)^n = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \ldots + p^n,$$

where ${}^nC_1$, ${}^nC_2$, ... are called the binomial coefficients.

Q.3. What is a binomial experiment?

(B.I.S.E., Multan 2009, Lahore 2012, 2013, 2015; Gujranwala 2014)

Ans. An experiment in which the outcome can always be classified as either a success or a failure and in which the probability of success remains constant from trial to trial is called a binomial experiment.

Q.4. State the properties of the binomial experiment.

(B.I.S.E., Multan 2008, 2013, 2010, 2014; Bahawalpur 2011; Gujranwala 2011, 2013)

Ans. (i) Each trial of the experiment results in an outcome that can be classified into one of the two categories: success or failure

(ii) The probability of success remains constant from one trial of the experiment to the next

(iii) Each trial of the experiment is independent of all other trials

(iv) The experiment is repeated a fixed number of times.

Q.5. What are Bernoulli trials? (B.I.S.E., Lahore 2009, 2011, 2012, 2014, 2015, 2018)

Ans. Repeated independent trials in which there are only two possible outcomes and probabilities of the outcomes remain the same for all trials are called Bernoulli trials.

Q.6. Describe the applications of the binomial distribution.

Ans. The binomial distribution is applicable to a wide variety of practical problems as well as to problems of an artificial sort, such as those dealing with cards, dice, coins and coloured balls in boxes.

It can be applied to any industrial situation where an outcome of a process is dichotomous and the results of the process are independent with a probability of success being the same from trial to trial.

It is also used in medical and military situations. A drug either cures or does not cure the disease. A missile either hits or does not hit the target.

Q.7. What is the mean and variance of the binomial distribution with parameters n and p? (B.I.S.E., Lahore 2009; Multan 2010)

Ans. The mean $\mu = np$ and the variance $\sigma^2 = npq$.

Q.8. Describe the binomial frequency distribution. (B.I.S.E., Multan 2011)

Ans. Let the n independent trials constitute one experiment and let this experiment be repeated N times. The expected frequency of x successes in N sets of n trials is given by $N \cdot {}^nC_x\, p^x q^{n-x}$. The possible number of successes together with the expected frequencies is called the *binomial expected frequency distribution.*

Q.9. What is a hypergeometric experiment?(B.I.S.E., Rawalpindi 2007; Multan 2008)

Ans. An experiment in which a random sample is selected without replacement from a finite population is called a hypergeometric experiment. Here the successive trials are not independent and the probability of success does not remain the same from one trial to the other.

Q.10. Define the hypergeometric probability function.

(B.I.S.E., Bahawalpur 2011, Lahore 2012, 2018; Gujranwala 2013; Faisalabad 2015, 2017)

Ans. If a population of N items consists of k items of one kind (successes) and $N - k$ items of another kind (failures), then the probability of getting x successes among n items selected at random from the population of N items is given by

$$P(X = x) = h\,(x; N, n, k) = \frac{{}^kC_x \cdot {}^{N-k}C_{n-x}}{{}^NC_n}, \; x = 0, 1, 2, \ldots, n.$$

This is the hypergeometric probability function.

Q.11. State the properties of the hyperogemoetric experiment.

(B.I.S.E., Multan 2009, 2011; Lahore 2014; Gujranwala 2014, 2015, 2017, 2018; Faisalabad 2015)

Ans. (i) Each trial of the hypergeometric experiment results in an outcome that can be classified into one of the two categories: success or failure.

(ii) The successive trials are dependent.

(iii) The probability of success changes from trial to trial.

(iv) The experiment is repeated a fixed number of times.

Q.12. Define binomial random variable.

Ans. The random variable X which represents the number of successes in a binomial experiment is called a *binomial random variable.* The binomial variable is a discrete variable which can assume any of the values $x = 0,1,2,\ldots, n$.

Q.13. Define hypergeometric random variable.

Ans. The random variable X which represents the number of successes in a hypergeometric experiment is called a *hypergeometric random variable.* It is a discrete variable which can assume any of the values $x = 0, 1, 2, \ldots, n$.

Q.14. Identify the parameters of the binomial distribution $b(x;\ n,\ p)$ and the hypergeometric distribution $h(x; N, n, k)$.

(B.I.S.E., Multan 2013, 2014; Gujranwala 2014; Lahore 2017)

Ans. The binomial distribution $b(x; n, p)$ has two parameters n and p, whereas the hypergeometric distribution $h(x; N, n, k)$ has three parameters N, n and k.

Q.15. Establish the relation between the mean and variance of the binomial distribution $b(x; n, p)$ and the hypergeometric distribution $h(x; N, n, k)$.

(B.I.S.E., Gujranwala 2009)

Ans. The mean μ and variance σ^2 of the hypergeometric distribution $h(x; N, n, k)$ are

$$\mu = \frac{nk}{N} = n\left(\frac{k}{N}\right) = np \text{ and}$$

$$\sigma^2 = \frac{nk(N-k)(N-n)}{N^2(N-1)} = n\left(\frac{k}{N}\right)\left(\frac{N-k}{N}\right)\left(\frac{N-n}{N-1}\right) = npq\left(\frac{N-n}{n-1}\right)$$

The mean of $h(x; N, n, k)$ is the same as that of $b(x; n, p)$. However the variances differ by a factor $\frac{N-n}{N-1}$.

Q.16 If $n = 10$, $q = 0.4$ find the mean and variance of the binomial distribution. (B.I.S.E., Gujranwala 2017, 2018)

Ans. $n = 10, q = 0.4,\qquad p = 1 - q = 1 - 0.4 = 0.6$

Mean $= \mu = np = 10\,(0.6) = 6$

σ^2 = variance $= npq = (10)\,(0.6)\,(0.4) = 2.4$

$P(X = 2) = {}^{10}C_2(0.6)^2\,(0.4)^{10-2} = 45 \times 0.35(0.0006) = 0.0106$

Q.17. Is it possible to have binomial distribution with mean 10 and standard deviation 4? (B.I.S.E., Lahore 2009, 2015, 2018; Gujranwala 2013)

Ans. Here $\mu = np = 10$ and $\sigma^2 = npq = (4)^2 = 16$.

$\frac{\sigma^2}{\mu} = \frac{npq}{np} = \frac{16}{10} = 1.6$ or $q = 1.6$ and $p = 1 - q = 1 - 1.6 = -0.6$.

Since p or q cannot be greater than 1 or negative, so it is not possible have a binomial distribution wit μ=10 and σ=4.

Q.18. Find the binomial distribution whose mean is 12 and standard deviation is 3. (B.I.S.E., Multan 2010; Lahore 2014, 2018)

Ans. $\mu = np = 12$ and $\sigma = \sqrt{npq} = 3$ or $npq = 9$. Thus

$\frac{npq}{np} = \frac{9}{12}$ or $q = 3/4$, $p = 1/4$ and $n = 12/p = 12(4) = 48$.

The binomial distribution is $(q+p)^n = \left(\frac{3}{4}+\frac{1}{4}\right)^{48}$.

Q.19. If the probability of getting caught copying in the examination is 0.2, find the probability of not getting caught in 3 attempts.

Ans. Let p denote the probability of getting caught copying. Then $p = 0.2$, $q = 0.8$ and $n = 3$. Let X denote the number of times caught copying.

P(not getting caught copying) $= P(X = 0) = {}^3C_0\,p^0q^3 = (0.8)^3 = 0.512$.

Q.20. Mean = 6, Variance, 2, find out parameters of binomial distribution.

Ans. In binomial distribution

$\mu = np \qquad \sigma^2 = npq$

$6 = np \qquad (1)$

$2 = npq \qquad (2)$

equation (1) dividing by equation (2)

$\frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$

$q = \frac{1}{3} \qquad p = 2/3 \qquad \mu = np$

$6 = n(2/3)$

$n = \frac{6 \times 3}{2} = 9 \qquad$ **So** $P = \frac{2}{3}$.