the fourth toss. The negative binomial distribution then reduces to the form $b^*(x; 1, p) = pq^{x-1}$, $x = 1, 2, 3, \ldots$. Since the successive terms constitute a geometric progression, it is customary to refer to this special case as the **geometric distribution** and denote its values by $g(x; p)$.

DEFINITION

Geometric Distribution. If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is given by

$$g(x; p) = pq^{x-1}, \qquad \text{for } x = 1, 2, 3, \ldots.$$

Example 16. Find the probability that a person flipping a balanced coin requires 4 tosses to get a head.

Solution. Using the geometric distribution with $x = 4$ and $p = \frac{1}{2}$, we have

$$g(4; \tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{1}{2})^3 = \tfrac{1}{16}.$$

6.5 Poisson Distribution

Experiments yielding numerical values of a random variable X, the number of outcomes occurring during a given time interval or in a specified region, are often called **Poisson experiments.** The given time interval may be of any length, such as a minute, a day, a week, a month, or even a year. Hence a Poisson experiment might generate observations for the random variable X representing the number of telephone calls per hour received by an office, the number of days school is closed due to snow during the winter, or the number of postponed games due to rain during a baseball season. The specified region could be a line segment, an area, a volume, or perhaps a piece of material. In this case X might represent the number of field mice per acre, the number of bacteria in a given culture, or the number of typing errors per page.

A Poisson experiment is one that possesses the following properties:

1. The number of outcomes occurring in one time interval or specified region is independent of the number that occur in any other disjoint time interval or region of space.
2. The probability that a single outcome will occur during a very short time interval or in a small region is proportional to the length of the time interval

or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.

3. The probability that more than one outcome will occur in such a short time interval or fall in such a small region is negligible.

The number X of outcomes occurring in a Poisson experiment is called a **Poisson random variable** and its probability distribution is called the **Poisson distribution.** Since its probabilities depend only on μ, the average number of outcomes occurring in the given time interval or specified region, we shall denote them by the symbol $p(x; \mu)$. The derivation of the formula for $p(x; \mu)$, based on the properties for a Poisson experiment listed above, is beyond the scope of this text. We list the result in the following definition.

DEFINITION

Poisson Distribution. The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region, is

$$p(x; \mu) = \frac{e^{-\mu}\mu^x}{x!}, \qquad \text{for } x = 0, 1, 2, \ldots,$$

where μ is the average number of outcomes occurring in the given time interval or specified region and $e = 2.71828 \cdots$.

Table A.3 contains Poisson probability sums $\sum_{x=0}^{r} p(x; \mu)$ for a few selected values of μ ranging from 0.1 to 18. We illustrate the use of this table with the following two examples.

Example 17. The average number of days school is closed due to snow during the winter in a certain city in the eastern part of United States is 4. What is the probability that the schools in this city will close for 6 days during a winter?

Solution. Using the Poisson distribution with $x = 6$ and $\mu = 4$, we find from Table A.3 that

$$p(6; 4) = \frac{e^{-4}4^6}{6!} = \sum_{x=0}^{6} p(x; 4) - \sum_{x=0}^{5} p(x; 4)$$
$$= 0.8893 - 0.7851 = 0.1042.$$

Example 18. The average number of field mice per acre in a 5-acre wheat field is estimated to be 10. Find the probability that a given acre contains more than 15 mice.

Solution. Let X be the number of field mice per acre. Then using Table A.3, we have

$$\begin{aligned} P(X > 15) &= 1 - P(X \leq 15) \\ &= 1 - \sum_{x=0}^{15} p(x; 10) \\ &= 1 - 0.9513 \\ &= 0.0487. \end{aligned}$$

The variance of the Poisson distribution can be shown to be equal to the mean. Thus in Example 17, where $\mu = 4$, we also have $\sigma^2 = 4$ and hence $\sigma = 2$. Using Chebyshev's theorem, we can state that our random variable has a probability of at least $\frac{3}{4}$ of falling in the interval $\mu \pm 2\sigma = 4 \pm (2)(2)$, or from 0 to 8. Therefore, we conclude that at least $\frac{3}{4}$ of the time the schools of the given city will be closed anywhere from 0 to 8 days during the winter season.

The Poisson and binomial distributions have histograms with approximately the same shape when n is large and p is close to zero. Hence, if these two conditions hold, the Poisson distribution, with $\mu = np$, can be used to approximate binomial probabilities. If p is close to 1, we can interchange what we have defined to be a success and a failure, thereby changing p to a value close to zero.

Example 19. Suppose that on the average 1 person in every 1000 is an alcoholic. Find the probability that a random sample of 8000 people will yield fewer than 7 alcoholics.

Solution. This is essentially a binomial experiment where $n = 8000$ and $p = 0.001$. Since p is very close to zero and n is quite large, we shall approximate with the Poisson distribution using $\mu = (8000)(0.001) = 8$. Hence, if X represents the number of alcoholics, we have

$$\begin{aligned} P(X < 7) &= \sum_{x=0}^{6} b(x; 8000, 0.001) \\ &\simeq \sum_{x=0}^{6} p(x; 8) \\ &= 0.3134. \end{aligned}$$

9. A certain area of the eastern United States is, on the average, hit by 6 hurricanes a year. Find the probability that in a given year this area will be hit by
(a) fewer than 4 hurricanes;
(b) anywhere from 6 to 8 hurricanes.

10. The average number of oil tankers arriving each day at a certain port city is known to be 10. The facilities at the port can handle at most 15 tankers per day. What is the probability that the port is unable to handle all the tankers that arrive
(a) on a given day?
(b) on one of the next 3 days?

11. A restaurant prepares a tossed salad containing on the average 5 vegetables. Find the probability that the salad contains more than 5 vegetables
    (a) on a given day;
    (b) on 3 of the next 4 days;
    (c) for the first time in April on April 5.

12. The probability that a person dies from a certain respiratory infection is 0.002. Find the probability that fewer than 5 of the next 2000 so infected will die.

13. Suppose that on the average 1 person in 1000 makes a numerical error in preparing his income tax return. If 10,000 forms are selected at random and examined, find the probability that 6, 7, or 8 of the forms will be in error.

14. The probability that a student fails the screening test for scoliosis (curvature of the spine) at a local high school is known to be 0.004. Of the next 1875 students who are screened for scoliosis, find the probability that
    (a) fewer than 5 fail the test;
    (b) 8, 9, or 10 fail the test.

15. Using Chebyshev's theorem, find and interpret the interval $\mu \pm 2\sigma$ for Exercise 12.

16. Using Chebyshev's theorem, find and interpret the interval $\mu \pm 3\sigma$ for Exercise 13.