Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 2

1. Discrete Mathematics and Its Application, 7th Edition

by Kenneth H. Rose

2. Discrete Mathematics with Applications

by Thomas Koshy

These slides contain material from the above resources.

Sequences

Sequence is a **discrete structure** used to represent an **ordered list**.

For example, **1, 2, 3, 5, 8** is a sequence with five terms and **1, 3, 9, 27, 81 , . . . , 3*n*, . . .** is an **infinite sequence**.

Definition: A **sequence is a function** from a subset of the set of integers (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to a set S. We use the **notation a_n** to denote the image of the integer n. We call a_n a term of the sequence.

Example 1 Consider the **sequence $\{a_n\}$,** where

$$\boldsymbol{a_n = \frac{1}{n}}$$

The list of the terms of this sequence, beginning with a_1, namely, $a_1, a_2, a_3, a_4, \ldots,$ starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Geometric progression

Definition A geometric progression is a sequence of the form $a, ar, ar^2, \ldots, ar^n, \ldots,$ where the initial term a and the common ratio r are real numbers.

Example The sequences $\{b_n\}$ with $b_n = (-1)^n$

$\{c_n\}$ with $c_n = 2 \times 5^n$

$\{d_n\}$ with $d_n = 6 \times (1/3)^n$

are geometric progressions with initial term and common ratio equal to **1** and **−1**; **2** and **5**; and **6** and **1/3**, respectively.

Solution:

$b_n = (-1)^n$

The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with

$1, -1, 1, -1, 1, \ldots;$

$c_n = 2 \times 5^n$

the list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with

$2, 10, 50, 250, 1250, \ldots;$

$d_n = 6 \times (1/3)^n$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

Arithmetic progression

Definition An arithmetic progression is a sequence of the form

$a, a + d, a + 2d, \ldots, a + nd, \ldots$

where the initial term **a** and the common difference **d** are real numbers.

Example The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with **$t_n = 7 - 3n$** are both arithmetic progressions with a = −1 and d = 4 , and a = 7 and d = −3, respectively,

$s_n = -1 + 4n$

if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with

−1, 3, 7, 11, . . . ,

$t_n = 7 - 3n$

and the list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with

7, 4, 1,−2,

Strings

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science. These finite sequences are also called **strings**. This string is also denoted by $a_1a_2 \ldots a_n$.

The **length** of a string is the **number of terms in this string**. The **empty string**, denoted by **λ**, is the string that has no terms. The empty string has length zero.

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses an in terms **of one or more of the previous terms** of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with **$n \geq n_0$** where **n_0 is a nonnegative integer**.

A **sequence** is called a **solution of a recurrence relation** if its terms **satisfy the recurrence relation**.

Example Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$\boldsymbol{a_n = a_{n-1} + 3}$ for $n = 1, 2, 3, \ldots,$

and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution

$a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \dots,$

$a_1 = a_0 + 3$

$a_1 = 2 + 3 = 5$

$a_2 = a_1 + 3 = 5 + 3 = 8$

$a_3 = a_2 + 3 = 8 + 3 = 11$

Example Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $\mathbf{a_n = a_{n-1} - a_{n-2}}$ for n = 2, 3, 4, . . . , and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

$\mathbf{a_n = a_{n-1} - a_{n-2}}$ **for n = 2, 3, 4, . . .**

$a_2 = a_1 - a_0$

$a_2 = 5 - 3 = 2$

$\mathbf{a_n = a_{n-1} - a_{n-2}}$ **for n = 2, 3, 4, . . .**

$a_3 = a_2 - a_1$

$a_3 = 2 - 5$

$a_3 = -3$

Recurrence relation of Fibonacci sequence

The **Fibonacci sequence**, $f_0, f_1, f_2, \ldots,$ is defined by the initial conditions

$f_0 = 0$, $f_1 = 1$, and the recurrence relation

$\mathbf{f_n = f_{n-1} + f_{n-2}}$, for $n = 2, 3, 4, \ldots.$

Recurrence relation of Fibonacci sequence

Example Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

$f_n = f_{n-1} + f_{n-2}$

$f_0 = 0$, $f_1 = 1$

$f_2 = f_1 + f_0 = 1 + 0 = 1,$

$f_3 = f_2 + f_1 = 1 + 1 = 2,$

$f_4 = f_3 + f_2 = 2 + 1 = 3,$

$f_5 = f_4 + f_3 = 3 + 2 = 5,$

$f_6 = f_5 + f_4 = 5 + 3 = 8.$

Recurrence relation of factorial sequence

Example Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \ldots.$

$\mathbf{a_n = n!}$

$n! = n((n - 1)(n - 2) \ldots 2 \cdot 1)$

$\quad = n\mathbf{(n - 1)!}$

$\quad = n\mathbf{a_{n-1}}$

$\because a_n = n!$

$\Longrightarrow a_n = na_{n-1}$ is the recurrence relation of factorials with the initial condition

$a_1=1$

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Summations

We use the notation

$$\sum_{j=m}^{n} a_j, \sum\nolimits_{j=m}^{n} a_j, \text{ or } \sum\nolimits_{m\le j\le n} a_j$$

(read as the sum from j = m to j = n of a_j) to represent

$a_m + a_{m+1} + \cdot \quad \cdot \quad \cdot + a_n.$

Here, the variable *j* is called the **index of summation**, and the choice of the letter ***j* as the variable is arbitrary**; that is, we could have used any other letter, such as *i* or *k*. Or, in notation,

$$\sum_{j=m}^{n} a_j = \sum\nolimits_{i=m}^{n} a_i = \sum\nolimits_{k=m}^{n} a_k$$

The **usual laws for arithmetic apply to summations**. For example

$$\sum_{j=1}^{n}(ax_j + by_j) = \sum_{j=1}^{n} ax_j + b\sum_{j=1}^{n} y_j$$

Example Use summation notation to express the **sum of the first 100 terms of the sequence** $\{a_j\}$, where $\mathbf{a_j = 1/n}$ for $j = 1, 2, 3, \ldots$.

$$\sum_{j=1}^{100} \frac{1}{j}$$

How do you know this is true?

$$\sum_{i=1}^{k}(ca_i + b_i) = c\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$$

1. Use associative law to separate the bs from the as.

2. Use distributive law to factor the cs.

Example What is the value of $\boldsymbol{\sum_{j=1}^{5} j^2}$?

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= 55$$

Example What is the value of $\sum_{k=4}^{8}(-1)^k$?

$\sum_{k=4}^{8}(-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$

$= 1 + (-1) + 1 + (-1) + 1$

$= 1$

Example Suppose we have the sum $\sum_{j=1}^{5} j^2$ but want the index of summation to run **between 0 and 4** rather than **from 1 to 5**

Let **j = k +1**

When **j = 1**, then 1 = k +1 ⇒ $\mathbf{k = 0}$

When **j = 5**, then 5 = k + 1 ⇒ $k = 5 - 1$ ⇒ $\mathbf{k = 4}$

$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k + 1)2$

It is easily checked that both sums are 1 + 4 + 9 + 16 + 25 = 55

What is **S = 1 + 2 + 3 + … + n?**

S	=	1	+	2	+	…	+	n	Write the sum.
S	=	n	+	n -1	+	…	+	1	Write it again in reverse order.
2s	=	n+1	+	n+1	+	…	+	n +1	Add together.

2s = n(n+1)

s = n(n+1)/2

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Some important summations

What is S = 1 + r + r^2 + … + r^n

$$\sum_{k=0}^{n} r^k = 1 + r + \ldots + r^n$$

Geometric Series

$$r\sum_{k=0}^{n} r^k = r + r^2 + \ldots + r^{n+1}$$

Multiply by r

$$\sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k = 1 - r^{n+1}$$

Subtract 2nd from 1st

$$(1-r)\sum_{k=0}^{n} r^k = 1 - r^{n+1}$$

Factor

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{(1-r)}$$

Divide

Some important summations

What is S = 1 + 3 + 5 + … + (2n - 1)?

Sum of first n odds

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1$$

$$= 2\left(\frac{n(n+1)}{2}\right) - n$$

$$= n^2$$

Some important summations

What is $S = 1 + 3 + 5 + \ldots + (2n - 1)$?

Sum of first n odds

$= n^2$

Double summations arise in many contexts (as in the analysis of nested loops in computer programs).

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} i(1 + 2 + 3)$$

$$= \sum_{i=1}^{4} (i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24 = 60.$$

We can also use summation notation to **add all values of a function**, or **terms of an indexed set**, where the index of summation runs over all values in a set. We write,

$$\sum_{s \in S} f(s)$$

to represent the sum of the values f(s), for all s of S.

What is the value of $\sum_{s \in \{0,\ 2,\ 4\}} s$?

$\sum_{s \in \{0,\ 2,\ 4\}} s$ = 0 + 2 + 4 = 6.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

$\sum_{k=50}^{100} k^2 = ?$

$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$

$\because \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$

$$\sum_{k=50}^{100} k^2 = \frac{100\times101\times201}{6} - \frac{49\times50\times99}{6}$$

$= 33850 - 40425$

$= 297925$

Cardinality

Definition: The sets A and B have the same cardinality if and only if there is a **one-to-one correspondence** from A to B. When A and B have the **same cardinality**, we write **|A| = |B|**

Definition: If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Countable Sets

We will now split **infinite sets** into two groups, those with the **same cardinality** as the set of natural numbers and those with a different cardinality.

Definition: A set that is either **finite** or has the **same cardinality** as the set of **positive integers** is called ***countable***. *A* set that is not countable is called ***uncountable***. *When* an **infinite set *S*** is **countable**, we denote the cardinality of *S* by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that *S* has cardinality "aleph null."

Countable Sets

Example Show that the set of odd positive integers is a countable set. To show that the set of odd positive integers is countable, we will exhibit **a one-to-one correspondence between this set and the set of positive integers**.

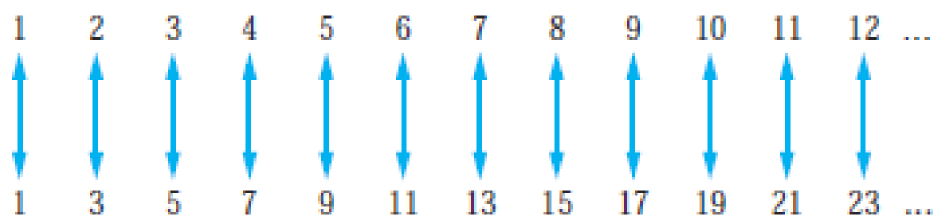

FIGURE 1 A One-to-One Correspondence Between Z^+ and the Set of Odd Positive Integers.

Suggested Readings

Chapter 2

2.4 Sequences and Summations

2.5 Cardinality of Sets