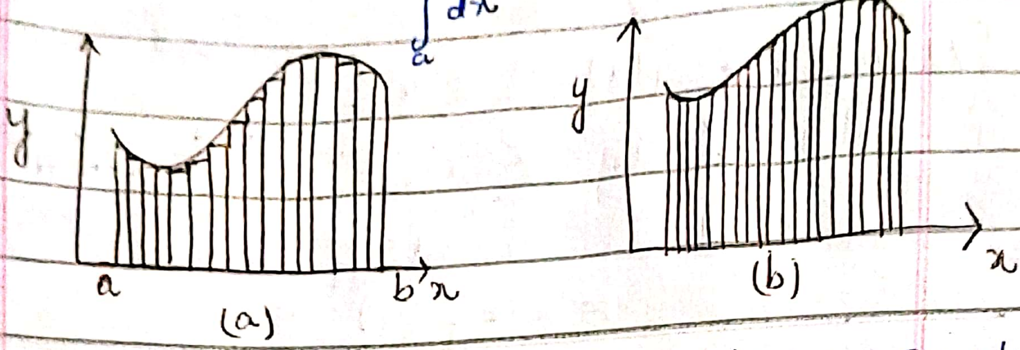
Integration:

An overview of the area problem:

* The rectangular method:

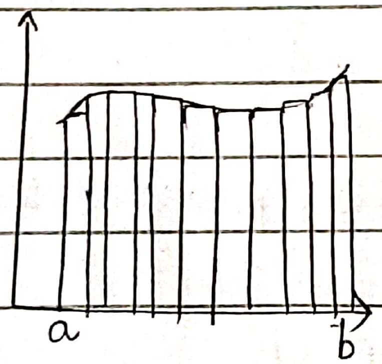
$\int_a dx$



Divide interval $[a,b]$ in 'n' equal sub-intervals for each n, the total area of the rectangles can be viewed as an approximation to the exact area under the curve.

$A = \lim_{n \to \infty} A_n$

↳ exact area under curve



$$\boxed{\sum_{i=1}^{n} A_i = A_1 + A_2 + \cdots + A_n}$$

* Definite integral: $\int_a^b (\ )\,dx$
* Indefinite integral: $\int (\ )\,dx$

→ Indefinite integrals

* Anti-derivative: "A function F is called an anti-derivative of a function f on a given open interval if: $F'(x) = f(x)$ holds for all x in the interval."

Examples: $F(x) = \frac{1}{3}x^3$ is an anti-derivative of $f(x) = x^2$.

$$F'(x) = \frac{3}{3}x^2 = x^2 = f(x)$$

$$\frac{d}{dx}\left[\frac{1}{3}x^3\right] = f(x)$$

Examples: • $G(x) = \frac{1}{3}x^3 + \underbrace{C}_{\text{constant}}$ ∴ derivative of a constant = 0

$$G'(x) = 3\frac{x^2}{\underbrace{3}_{\text{Power rule}}} + 0$$

$$= x^2 = f(x)$$

• $H(x) = \frac{1}{3}x^3 - 5$

Thus $F(x)$, $G(x)$, $H(x)$ are all anti-derivatives of x^2.

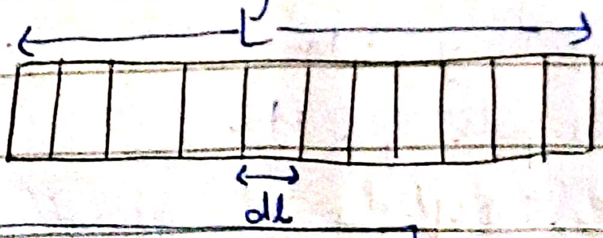
• $\boxed{\text{Indefinite integral:}}$ "The process of finding anti-derivatives is called anti-differentiation or integration".

$$\frac{d}{dx}F(x) = f(x)$$

$$f(x).d(x) = dF(x) \quad \therefore \text{Cross-Multiplication}$$

Taking integral on both sides

$$\int f(x).d(x) = \int d\,F(x)$$



$$\therefore \boxed{\int dL = L \qquad \int dF = F \qquad \int dx = x}$$

$$\int f(x)\, d(x) = F(x) + C$$

integral ; $f(x)$ = integrand ; C = constant of integration

Power Rule:

$$\boxed{\int x^3 . dx = \frac{x^{n+1}}{n+1} + C}$$

Derivative Formula	Equivalent integration Formula
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2 . dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}} . dx = \sqrt{x} + C$
$\frac{d}{dx}(\tan t) = \sec^2 t$	$\int \sec^2 t . dt = \tan(t) + C$

Example:

$$\int \frac{1}{2\sqrt{x}} . dx = ?$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x}} . dx \qquad \therefore \boxed{\frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}}$$

$$= \frac{1}{2} \int x^{-1/2} . dx$$

$$= \frac{1}{2} \frac{x^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{1}{2} . \frac{x^{1/2}}{1/2} + C = \frac{1}{2} . \frac{2}{1} \sqrt{x} + C$$

$$= \boxed{\sqrt{x} + C}$$

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$$\rightarrow \frac{d}{dx}[u^{3/2}] = \frac{3}{2}\sqrt{u}$$

$$\rightarrow \int \frac{3}{2}\sqrt{u}\,.\,du = u^{3/2} + C$$

$$= \int \frac{3}{2}\sqrt{u}\,.\,du$$

$$= \frac{3}{2}\int (u)^{1/2}.\,du \quad = \frac{3}{2}\,\frac{u^{1/2+1}}{1/2+1} + C$$

$$= \frac{\cancel{3}}{\cancel{2}}\,\frac{u^{3/2}}{\cancel{3/2}} + C$$

$$= u^{3/2} + C$$

- **Properties of Integrals:**
  - $\int c\,f(x).\,dx = c\,F(x) + C$
  - $\int [f(x) \pm g(x)].\,dx = F(x) \pm G(x) \pm C$

Examples:

(a). $\int 4\cos x.\,dx$

$= 4\int \cos x.\,dx$ $\quad\therefore \boxed{\int \cos x = \sin x}$

$= 4\,(\sin x) + C$

(b). $\int (x + x^2)\,dx$

$= \int x.\,dx + \int x^2.\,dx$

$= \frac{x^2}{2} + \frac{x^3}{3} + C$

Example 5: $\int \left(3x^6 - 2x^2 + \frac{x^2 - 2x^4}{x^4}\right) dx$

Solution:

$= 3\int x^6.\,dx - 2\int x^2.\,dx + \int \frac{x^2}{x^4}\,dx - 2\int \frac{x^4}{x^4}\,dx$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \int \frac{1}{x^2}\, dx - 2\int dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \int \frac{x^{-2}\, dx}{x^{-1}/-1} - 2x$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} - \frac{1}{x} - 2x + C$$

Example 6: Evaluate: $\int \frac{\cos x}{\sin^2 x}\, dx$

Solution:

$$\int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\, dx = \int \cot x \cdot \operatorname{cosec} x \cdot dx$$

$$= -\operatorname{cosec} x + C$$

① INTEGRATION BY SUBSTITUTION:

Method of u-sub:

Example:

Evaluate $\int (x^2+1)^{50} \cdot 2x\, dx$

Let $u = x^2 + 1$

$$\frac{du}{dx} = 2x \Rightarrow \boxed{du = 2x \cdot dx}$$

$$\int u^{50} \cdot du = \frac{u^{51}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C \qquad \because u = x^2 + 1$$

Example 2:

$\int \sin(2x+9).dx$ —(i)

Solution: Let $u = 2x+9$

$$\frac{du}{dx} = 2$$

$$\frac{1}{2}du = dx \quad -(A)$$

Putting value of dx from (A) in (i):

$$= \int \sin u . \frac{du}{2}$$

$$= -\frac{1}{2}\cos u + C$$

$$= -\frac{1}{2}(x+9) + C$$

Example 3:

Evaluate: $\int \cos 5x . dx$

Solution: Let $u = 5x$ —(2)

$\int \cos 5x . dx$ —(1)

$$\frac{du}{dx} = 5\frac{dx}{dx}$$

$$\Rightarrow \frac{du}{dx} = 5 \quad \text{OR} \quad \boxed{dx = \frac{1}{5}du} \text{ —(3)}$$

Eq (1) becomes:

$$\int \cos u . \frac{1}{5} du \quad \because dx = \frac{1}{5}du$$

$$= \frac{1}{5}\int \cos u . du \quad \because \int \cos u = \sin u$$

$$= \frac{1}{5}\sin u + C \quad = \frac{1}{5}\sin 5x + C$$

$\because u = 5x$

Example 4:

$$\int \frac{dx}{(1/3x-8)} + u$$

$$\int \frac{dx}{\left(\frac{1}{3}x-8\right)} + u$$

Let $u = \frac{1}{3}x - 8$

$$\frac{du}{dx} = \frac{1}{3}$$

$dx = 3 \cdot du$

$$\int \frac{dx}{\frac{1}{3}x-8} = \int \frac{dx}{u}$$

$$= \int 3 \cdot \frac{du}{u^5} = 3\int u^{-5} \cdot du$$

$$= 3\,\frac{u^{-5+1}}{-5+1} + C$$

$$= \frac{-3u^{-4}}{4} + C = \frac{-3}{4} \cdot \frac{1}{(1/3x-8)^4} + C$$

$$= -\frac{3}{4} \cdot \frac{1}{\left(\frac{1}{3}x-8\right)^4} + C$$

Example 5: $\int \left(\frac{1}{x^2} + \sec^2 \pi x\right) dx$

$$= \int \frac{dx}{x^2} + \int \sec^2 \pi x \cdot dx$$

Let $\pi x = u$

$$\boxed{\frac{du}{dx} = \pi \frac{dx}{dx}}$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \int \frac{1}{x^2}\, dx - 2\int dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \int \frac{x^{-2}\, dx}{x^{-1}/-1} - 2x$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} - \frac{1}{x} - 2x + C$$

Example 6: Evaluate : $\int \frac{\cos x}{\sin^2 x}\, dx$

Solution:

$$\int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\, dx = \int \cot x \cdot \operatorname{cosec} x \cdot dx$$

$$= -\operatorname{cosec} x + C$$

ⓧ INTEGRATION BY SUBSTITUTION:

Method of u-sub:

Example :

Evaluate $\int (x^2+1)^{50} \cdot 2x\, dx$

Let $u = x^2 + 1$

$$\frac{du}{dx} = 2x \quad \Rightarrow \boxed{du = 2x \cdot dx}$$

$$\int u^{50} \cdot du = \frac{u^{51}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C \qquad \because u = x^2+1$$

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Example 2:

$$\int \sin(2x+9).dx \quad \text{—(i)}$$

Solution: Let $u = 2x+9$

$$\frac{du}{dx} = 2$$

$$\frac{1}{2} du = dx \quad \text{-(A)}$$

Putting value of dx from (A) in (i):

$$= \int \sin u . \frac{du}{2}$$

$$= -\frac{1}{2} \cos u + C$$

$$= -\frac{1}{2} (x+9) + C$$

Example 3:

Evaluate: $\int \cos 5x . dx$

Solution: Let $u = 5x$ —(2)

$$\int \cos 5x . dx \quad \text{—(1)}$$

$$\frac{du}{dx} = 5 \frac{dx}{dx}$$

$$\Rightarrow \frac{du}{dx} = 5 \quad \text{OR} \quad \boxed{dx = \frac{1}{5} du} \text{—(3)}$$

Eq (1) becomes:

$$\int \cos u . \frac{1}{5} du \quad \because dx = \frac{1}{5} du$$

$$= \frac{1}{5} \int \cos u . du \qquad \because \int \cos u = \sin u$$

$$= \frac{1}{5} \sin u + C \qquad = \frac{1}{5} \sin 5x + C$$

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Example 4:

$$\int \frac{dx}{(1/3x-8)+u}$$

$$\int \frac{dx}{\left(\frac{1}{3}x-8\right)} + u$$

Let $u = \frac{1}{3}x-8$

$$\frac{du}{dx} = \frac{1}{3}$$

$dx = 3.du$

$$\int \frac{dx}{\frac{1}{3}x-8} = \int \frac{dx}{u}$$

$$= \int 3.\frac{du}{u^5} = 3\int u^{-5}.du$$

$$= 3\frac{u^{-5+1}}{-5+1} + C$$

$$= \frac{-3u^{-4}}{4} + C = \frac{-3}{4}.\frac{1}{(1/3x-8)^4} + C$$

$$= \frac{-3}{4}.\frac{1}{\left(\frac{1}{3}x-8\right)^4} + C$$

Example 5: $\int \left(\frac{1}{x^2} + \sec^2 \pi x\right) dx$

$$= \int \frac{dx}{x^2} + \int \sec^2 \pi x . dx$$

let $\pi x = u$

$$\boxed{\frac{du}{dx} = \pi \frac{d}{dx} x}$$

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$$\frac{du}{dx} = \pi \Rightarrow \boxed{dx = \frac{du}{\pi}}$$

$$= \frac{x^{-2+1}}{-2+1} + \int \sec^2 u \cdot \frac{du}{\pi}$$

$$= \frac{-1}{x} + \frac{1}{\pi} \tan u + C$$

$$= \frac{-1}{x} + \frac{1}{\pi} \tan(\pi x) + C$$

Evaluate: $\int \sin^2 x \cdot \cos x \cdot dx$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x \Rightarrow du = \cos x \cdot dx$$

$$\int u^2 \cdot du = \frac{u^{2+1}}{2+1} + C$$

$$= \frac{u^3}{3} + C = \boxed{\frac{\sin^3 x}{3} + C}$$

Evaluate:

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx$$

Solution: Let $\sqrt{x} = u$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1}{2u} \quad \therefore \sqrt{x} = u$$

$$dx = du \cdot 2u$$

$$\int \frac{\cos u}{\cancel{u}} \cdot du \cdot 2\cancel{u}$$

$$2 \int \cos u \cdot du \qquad \int \cos u = \sin$$

$$= 2 \sin u = \boxed{2 \sin\sqrt{x} + C} \quad \therefore u = \sqrt{x}$$

Less apparent substitutions:

Example 8: $\int x^2 \sqrt{x-1} \, . \, dx$

Solution:

Let $u = x - 1 \Rightarrow x^2 = (u+1)^2$

$\frac{du}{dx} = 1$

$\Rightarrow \boxed{du = dx}$

$$\int (u+1)^2 \cdot \sqrt{u} \cdot du$$
$$= \int (u^2+2u+1) \cdot u^{1/2} \cdot du$$
$$\int (u^{5/2}+u^{3/2}+u^{1/2}) \cdot du$$
$$= \int u^{5/2}\, du + \int u^{3/2}\, du + \int u^{1/2} \cdot du$$
$$= \boxed{\frac{2u^{7/2}}{7} + \frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3}}$$

Example 10:

Evaluate: $\int \cos^3 x \cdot dx$

Solution:

$$= \int \cos^2 x \cdot \cos x \cdot dx \quad : \sin^2 x + \cos^2 x = 1$$

↳ Trigonometric identity

$$= \int (1-\sin^2 x) \cdot \cos x \cdot dx$$
$$= \int \cos x\, du - \int \sin^2 x \cos x\, dx$$
$$= \boxed{\sin x - \int \frac{\sin^3 x}{3} + C}$$

Integration by Parts:

Example:

Evaluate $\int \underbrace{x}_{①} \underbrace{\sin 2x}_{②}\, dx$

$$= x\left(\int \sin 2x \cdot dx\right) - \int \left(\text{Integration of integrated function}\right) \cdot dx$$
$$= x\left(\frac{-\cos 2x}{2}\right) - \int \left(\frac{-\cos 2x}{2}\right) \cdot dx$$

PTO

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General Formula:

$$\int u \cdot dv = uv - \int v \cdot du$$

$$= \left(\begin{matrix}\text{First function} \\ \times \\ \text{Integration of second function}\end{matrix}\right) - \left(\begin{matrix}\text{Integration of integrated function} \\ \times \\ \text{Derivative of First function}\end{matrix}\right)$$

Ans $x\left(\frac{-\cos 2x}{2}\right) + \frac{\sin 2x}{2\times 2} + C$

$$= -x\frac{\cos 2x}{2} + \frac{\sin 2x}{4} + C$$

Example 2:

$$\int x\sqrt{x+1} \cdot dx$$

Solution:

$$= x\frac{(x+1)^{1/2+1}}{1/2+1} - \int \frac{2(x+1)^{3/2}}{3} \cdot dx$$

$$= \frac{2}{3}x\cdot(x+1)^{3/2} - \frac{2}{3}\int \frac{2}{3}(x+1)^{3/2} \cdot dx$$

$$= \boxed{\frac{2}{3}x\cdot(x+1)^{3/2} - \frac{2}{3}\left(\frac{(x+1)^{3/2+1}}{3/2+1}\right) + C}$$

Example 3:

$$\int x\sec^2 x \cdot dx$$

Solution:

$$= x\tan x - \int \tan x \cdot dx$$

$$= \boxed{x\tan x - \ln|\sec x| + C}$$

Definite Integration:

Recall indefinite integration

$\int 2x\,dx$

$= 2\int x\,dx$

$= \cancel{2}\frac{x^2}{\cancel{2}} + C$

$= x^2 + C$

Definite integration:

- $\int_1^2 2x\,dx$

$= 2\int_1^2 x\,dx$ → upper limit

$= 2\frac{x^2}{2}\Big|_1^2 + C$

lower limit

$= (2)^2 - (1)^2 + C$

$= 4 - 1 + C$

$= 3 \,|\, 3 + C$

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

let $u = 1-x^2$

$$= -\frac{1}{2}\int_1^0 \sqrt{u}\,du$$

$$= -\frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\,\frac{u^{1/2+1}}{\left(\frac{1}{2}+1\right)} = -\frac{1}{2}\,\frac{u^{3/2}}{3/2}\Bigg|_1^0$$

$$= -\frac{1}{3}\left(1-x^2\right)^{3/2}\Bigg|_0^1$$

$$= -\frac{1}{3}\left[\left(1-1^2\right)^{3/2} - \left(1-0^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}$$

Properties of Definite integration:

① $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ See theorem. $\int_{-1}^{2} f(x)\,d(x) = \int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$

Definite:

* $\int_{-1}^{2} (x + 2\sqrt[3]{x})\, dx$

$= \int_{-1}^{2} x\, dx + 2\int_{-1}^{2} x^{1/3}\, dx$

$= \frac{x^2}{2}\Big|_{-1}^{2} + \frac{2x^{4/3}}{4/3}\Big|_{-1}^{2}$

$= \frac{1}{2}\left[(2)^2 - (-1)^2\right] + \frac{6}{4}\left[(2)^{4/3} - (-1)^{4/3}\right]$

* $\int_{a}^{b} f(x)\, dx = \int_{a}^{b} f(x)\, dx + \int_{b}^{c} f(x)\, dx$

* $\int_{0}^{\pi/2} \frac{\sin x}{5\, dx} = \frac{1}{5}\int_{-\pi/4}^{\pi/4} \sec x \tan x$

$= \int_{0}^{\pi/2} \frac{\sin x\, dx}{5\, \frac{dx}{dx}}$

$= \int_{0}^{\pi/2} \frac{-\cos x}{5} = -\frac{1}{5} \cos x \Big|_{0}^{\pi/2}$

$= -\frac{1}{5}\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= -\frac{1}{5}\left(0 - (+1)\right) = \boxed{\frac{1}{5}}$

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Definite integration by substitutions

- $\sec x \Big|_{-\pi/4}^{+\pi/4}$

Solution:

$$\sec \frac{\pi}{4} - \sec\left(-\frac{\pi}{4}\right)$$

$$= \sqrt{2} - \sqrt{2} = 0$$

Evaluate :

$$\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx$$

derivative of function

let $u = \sin 2x$,

is function $\times$ derivative of function

$$\frac{du}{dx} = \cos 2x \ 2$$

$$\frac{du}{dx} = \cos 2x \, 2$$

$$\frac{du}{2} = \cos 2x \, dx$$

$$= \frac{1}{2} \int_0^{1/\sqrt{2}} u^5 \, du$$

Chain Rule : $\left[\dfrac{df}{dx} = \dfrac{df}{du} \cdot \dfrac{du}{dx} \right]$