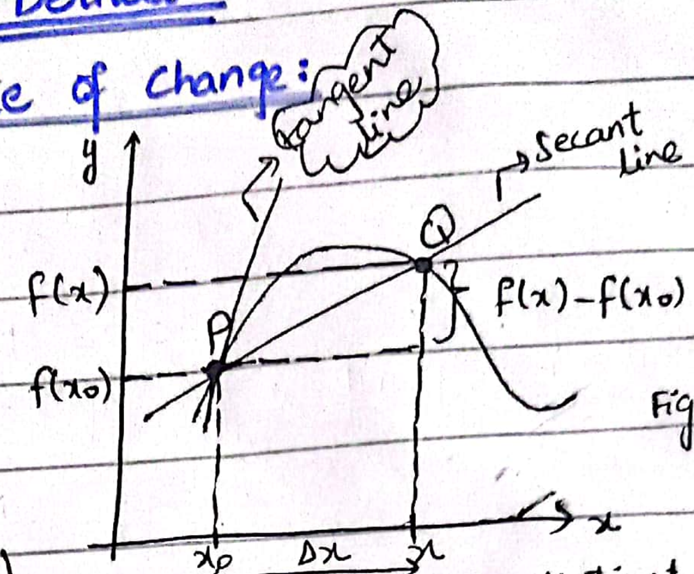
Maths Deficiency

Chapter no. 2: "The Derivative"

Tangent lines & Rate of Change:

↳ Tangent lines:



Fig(1)

$y = f(x)$

In fig (1), consider a point "Q" where "Q" is a point at $x, f(x) \Rightarrow Q = (x, f(x))$

Consider a point Q on the curve that is distinct from P. $\Rightarrow P(x_0, f(x_0))$, and compute the slope m_{pq}.

$$\text{Slope} = \Delta y / \Delta x = \frac{\text{Change in direction of } y}{\text{change in direction of } x}$$

$$\boxed{m_{pq} = \frac{f(x) - f(x_0)}{x - x_0}}$$

If we take $x \to x_0$ then $m_{pq} \to m_{tan}$ (secant)

(secant approaches to tangent)

Thus we make a following definition:

Suppose that "x_0" is in the domain of $f(x)$, the tangent line to the curve $y = f(x)$ at point "p" is the line with equation: $\boxed{y - f(x_0) = m_{tan}(x - x_0)}$

where: $\boxed{m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$ → formula no. 1.

Example: Parabolic equation: $y = x^2$ at the point $P(1,1)$ ($\Rightarrow x_0 = 1$). Find eq. of tangent by using:

$$m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$P(1,1) \Rightarrow x_0 = 1$, $f(x_0) = 1$ $\therefore P(x_0, f(x_0))$

Putting values: $m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$f(x) = y = x^2$

$$= \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$x_0 = 1$

$\therefore \boxed{\begin{array}{c} f(x) = x^2 \\ f(x_0) = 1 \\ x_0 = 1 \end{array}}$

$$= \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$= x + 1$

$= 1 + 1$

$= 2$

$\therefore x \to x_0$
$\Rightarrow x \cong x_0$
$x_0 = 1$
$\Rightarrow x = 1$

There is an alternative way to express formula 1:

$$h = x - x_0$$

then the formula can be written as:

$$\boxed{m_{tan} = \lim_{h \to 0} \frac{f(x) - f(x_0)}{h}}$$

(change at any instant).

$x \to x_0$
$h = x - x_0$
$x \cong x_0$
$\Rightarrow h \to 0$

$m = \frac{\Delta y}{\Delta x}$ when $x \to x_0$ $m = \frac{dy}{dx}$

$V = \frac{\Delta S}{\Delta t}$ [velocity at a particular instant: $V = ds/dt$]

Derivative defines the change i.e., the change at any instant.

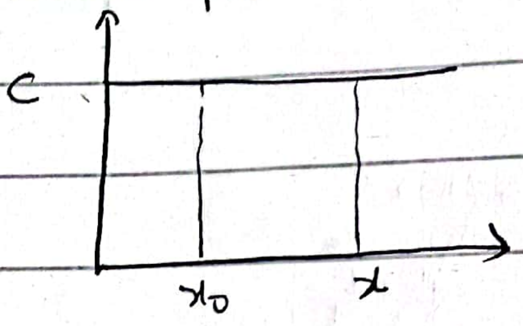
$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d}{dt}\frac{ds}{dt}$$

$$\left[\frac{d}{dt}\left(\frac{ds}{dt}\right)\right]$$

* Introduction to Techniques of Differentiation:

- Derivative is a term.
- Differentiation is a process involving derivative.

→ **Derivative of a Constant:** Consider a constant function:



⇒ (Slope of a constant function = 0)

$f(x) = c$

$$f'(x) = \frac{d}{dx} f(x)$$
↳ derivative

* For a constant:
$$f(x) = f'(x) = \frac{df(x)}{dx} = 0$$

$$f'(x) = \lim_{x \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\boxed{h = x - x_0 \Rightarrow x = x_0 + h}$$

$$= \lim_{h \to 0} \frac{c-c}{h} = 0$$

Example: $\frac{d}{dx}[1] = ?$, $\frac{d}{dx}[-3] = ?$

$\frac{d}{dx}[1] = 0$ [Reason: Derivative of a const. = 0]

$\frac{d}{dx}[-3] = 0$ ["]

1, -3 are constants.

$$m_{tan} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ → Def. of derivative, 1st principle

+ Derivatives Of Power Functions:

The simplest power function is:

$f(x) = x$

$$\frac{df(x)}{dx} = \frac{d}{dx} x = \frac{dx}{dx} = 1$$

$\Rightarrow [1 x^{1-1}]$

$[1 x^0]$ $x^0 = 1$

$(1(1)) = 1$

$f(x) = x^3$

$$\frac{d f(x)}{dx} = f'(x) = 3x^{3-1} = 3x^2$$

The Power Rule: "If 'n' is a positive integer then:

$$\frac{d}{dx} x^n = nx^{n-1}"$$

Proof: Let $f(x) = x^n$

$$\frac{d}{dx} x^n = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

If $f(x) = x^n$

$\Rightarrow f(x+h) = (x+h)^n$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Let $f(x) = x^n$ from the first principle of derivative & the binomial formula for expanding the expression $(x+h)^n$ we obtain:

$$\lim_{h\to 0} \frac{\left([\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)x^{n-2}h^2}{2!} + \dots \dots + nxh^{n-1} + h^n] - \cancel{x^n}\right)}{h}$$

$$= \lim_{h\to 0} \left(\frac{nx^{n-1}h}{h} + \frac{n(n-1)x^{n-2}}{2!}\frac{h^2}{h} + \dots + \frac{nxh^{n-1}}{h} + h^{n-1} \right)$$

Applying limit : $h \to 0$, we get :

$$\boxed{f'(x) = nx^{n-1}}$$

Examples: $f(x) = x^4$

$f'(x) = ?$

$f'(x) = 4x^{4-1} = 4x^3$

$f(t) = t^4 \longrightarrow = 4t^3$

Theorem: "Extended Power Rule"

if 'r' is any real numbers, then:

$$\boxed{\frac{d}{dx}[x^r] = rx^{r-1}}$$

Examples: • $f(x) = x^{\pi} \Rightarrow f'(x) = \pi x^{\pi - 1}$

• $f(x) = \sqrt[3]{x}$ $(x^{1/3})$ $\Rightarrow f'(x) = \frac{1}{3} x^{1/3 - 1}$

$= \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

Derivative of a Constant times a function:

* Constant Multiple Rule: "If 'f' is differentiable at x (derivative of a function exists) and 'c' is any real numbers, then cf is also differentiable at x."

$$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}x}$$

Proof: $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}x$

L.H.S: $\frac{d}{dx}[cf(x)]$

$= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$

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$$= \lim_{h \to 0} \frac{c\,[f(x+h) - f(x)]}{h}$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \text{ or } c\frac{d}{dx}x}$$

Examples: $f(x) = \pi / x$

$$f(x) = \pi \cdot \frac{1}{x}$$

$$= \pi \frac{d}{dx} \frac{1}{x} = \pi \frac{d}{dx} x^{-1}$$

$$= \pi (-1)\, x^{-1-1}$$

$$= -\pi\, x^{-2} = \frac{-\pi}{x^2} \Rightarrow \boxed{f'(x) = \frac{-\pi}{x^2}}$$

Derivatives of Sums & Differences:

"If 'f' and 'g' are differentiable at x, then: $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

Proof: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

Taking L.H.S:

$$\frac{d}{dx}[f(x) + g(x)]$$

$$= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

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Rearranging:

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = R.H.S$$

$$\Rightarrow L.H.S = R.H.S$$

Example:

(i). $f(x) = 2x^6 + x^{-9}$

(ii). $f(x) = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

$$\frac{\sqrt{x}}{\sqrt{x}} - \frac{2x}{\sqrt{x}}$$

$$1 - \frac{2x}{\sqrt{x}} = 1 - 2x \cdot x^{-1/2}$$

$$f(x) = 1 - 2x^{1/2} \qquad 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

$$f'(x) = \frac{d}{dx} 1 - \frac{d}{dx} 2x^{1/2} \qquad \frac{d}{dx} 1 = 0$$

Derivative of a constant = 0

$$= 0 - 2\frac{d}{dx} x^{1/2}$$

$$= -\cancel{2} \times \frac{1}{\cancel{2}} x^{1/2 - 1}$$

$$= -1\, x^{-1/2}$$

$$\boxed{f'(x) = -x^{-1/2}}$$

Example: Find $\frac{dy}{dx}$ if $y = 3x^8 - \frac{2x^5}{\sqrt[3]{x}} + 6x + 6$

$$y' = \frac{dy}{dx} = \frac{d}{dx} 3x^8 - \frac{d}{dx} \frac{2x^5}{\sqrt[3]{x}} + \frac{d}{dx} 6x + 0$$

$$y' = 3 \frac{d}{dx} x^8 - 2 \frac{d}{dx} \frac{x^5}{\sqrt[3]{x}} + 6 \frac{d}{dx} x$$

$$= 3(8x^{8-1}) - 2 \frac{d}{dx} x^{5-1/3} + 6(1x^{1-1})$$

$$= 3(8x^7) - 2 \frac{d}{dx} x^{14/3} + 6(1)$$

$$= 24x^7 - 2\left(\frac{14}{3} x^{14/3 - 1}\right) + 6$$

$$= 24x^7 - 2\left(\frac{14}{3} x^{11/3}\right) + 6$$

$$= \boxed{24x^7 - \frac{28\sqrt[3]{x^{11}}}{3} + 6}$$

Higher Derivatives:

- f': first derivative
- f'': second derivative

$f', f'', f''', \ldots, (f')^n$

$$y' = 24x^7 - \frac{28\sqrt[3]{x^{11}}}{3} + 6$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} \cdot \frac{d}{dx} = \frac{d^2}{dx^2} y$$

$$= 24 \frac{d^2}{dx^2} x^7 - \frac{28}{3} \frac{d^2}{dx^2} x^{1/3} + 0$$

Exercise no. 2.1:

(Question no. 1).

(i). $2x^2+1$

Let $y = f(x) = 2x^2+1$

$$f'(x) = \frac{d}{dx} 2x^2 + \frac{d}{dx} 1$$

$$= 2 \frac{d}{dx} x^2 + 0$$ ∴ derivative of a constant = 0

$$= 2(2x^{2-1}) + 0 \Rightarrow nx^{n-1}$$

$$\boxed{f'(x) = 4x}$$

ii). $2-\sqrt{x}$

Let $y = f(x) = 2-\sqrt{x} = 2 - x^{1/2}$

$$f'(x) = \frac{d}{dx} 2 - \frac{d}{dx} x^{1/2}$$

$$= 0 - \frac{1}{2} x^{1/2-1}$$

$$= -\frac{1}{2} x^{-1/2} = -\frac{1}{2x^{1/2}}$$

$$\boxed{f'(x) = -\frac{1}{2\sqrt{x}}}$$

(iii). $\frac{1}{\sqrt{x}}$

Let $y = f(x) = \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}$

$f'(x) = \frac{d}{dx} x^{-1/2}$ $\qquad$: nx^{n-1} → Power Rule

$= -\frac{1}{2} x^{-1/2-1}$

$f'(x) = -\frac{1}{2} x^{-3/2} = -\frac{1}{2\sqrt{x^3}}$

(iv). $1/x^3$

Let $y = f(x) = 1/x^3$

$f'(x) = x^{-3}$

$= \frac{d}{dx} x^{-3}$

$= -3\, x^{-3-1}$

$f'(x) = -3x^{-4} = -3/x^4$

(v). $1/x-a$

Let $y = f(x) = 1/x-a$

$y = (x-a)^{-1}$

$f'(x) = \frac{d}{dx}(x-a)^{-1}$

$= (-1)(x-a)^{-1-1}$

$= -1(x-a)^{-2}$

$$\boxed{f'(x) = -1/(x-a)^2}$$

Exercise no. 2.2:

(Question no. 1)-

(i). $(ax+b)^3$

Let $y = (ax+b)^3$

$= a^3x^3 + 3a^2x^2b + 3axb^2 + b^3$

$f'(x) = \frac{d}{dx} a^3x^3 + \frac{d}{dx} 3a^2x^2b + \frac{d}{dx} 3axb^2 + \frac{d}{dx} b^3$

$= a^3 \frac{d}{dx} x^3 + 3a^2b \frac{d}{dx} x^2 + 3ab^2 \frac{d}{dx} x + 0$

$= a^3(3x^{3-1}) + 3a^2b(2x^{2-1}) + 3ab^2(x^{1-1})$

$= a^3(3x^2) + 3a^2b(2x) + 3ab^2$ $\quad \therefore x^0 = 1$

$= 3a^3x^2 + 6a^2bx + 3ab^2$

$= 3a(a^2x^2 + 2abx + b^2)$

$$\boxed{f'(x) = 3a(ax+b)^2}$$

Example 9: $y = 3x^4 - 2x^3 + x^2 - 4x + 2$

Find $\frac{dy}{dx}$ & higher order derivatives.

$y = f(x) = x^2 + 1$ $\quad \frac{df(x)}{d(x)} = f'$

$y(x) = x^2 + 1$

$$y' = \frac{dy}{dx}$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} \cdot \frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$y''' = \frac{d^3y}{dx^3}$$

$\vdots$

$$y^n = \frac{d^ny}{dx^n}$$

$$y' = 3\frac{d}{dx}x^4 - 2\frac{d}{dx}x^3 + \frac{d}{dx}x^2 - 4\frac{d}{dx}x + 0$$

$$y' = 3(3x^3) - 2(2x^2) + 2x - 4$$

$$y' = 9x^3 - 4x^2 + 2x - 4$$

$y'' =$

$y''' =$

$\vdots$

$y^n = 0$

The Product Rule: Consider the functions:

$f(x) = x$, $g(x) = x^2$

$f(x).g(x) = x^3 \Rightarrow (f(x).g(x))' = 3x^2$

But $f'(x).g'(x) = 2x$

$\Rightarrow (f(x).g(x))' \neq f'(x).g'(x)$

Theorem: "If 'f' & 'g' are differentiable at 'x' & so their product is: $\frac{d}{dx}[f(x).\&g(x)] = f(x).\frac{d}{dx}g(x) + g(x).\frac{d}{dx}f(x)$

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Example 1:

$$y = (\underbrace{4x^2 - 1}_{f})(\underbrace{7x^3 + x}_{g})$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(4x^2-1)\right] \cdot (7x^3+x) + (4x^2-1)\frac{d}{dx}(7x^3+x)$$

$$= 8x(7x^3+x) + (4x^2-1)(21x^2+1)$$

$$= 56x^4 + 8x^2 + 84x^4 + 4x^2 - 21x^2 - 1$$

$$= 140x^4 - 9x^2 - 1$$

Find $\frac{ds}{dt}$ if $s = (1+t)\sqrt{t}$

$$\frac{ds}{dt} = \left[\frac{d}{dt}(1+t)\right] \cdot \sqrt{t} + (1+t)\left[\frac{d}{dt}\sqrt{t}\right]$$

$$= (1 \cdot t^{1/2}) + \left(\frac{1}{2} t^{1/2 - 1}\right)(1+t)$$

$$= \sqrt{t} + \frac{1}{2\sqrt{t}}(1+t) = \boxed{\sqrt{t} + \frac{1}{2\sqrt{t}} + \frac{t}{2\sqrt{t}}}$$

The Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x)\frac{d}{dx} g(x)}{[g(x)]^2}$$

$$\boxed{\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}}$$

Examples: $y'(x) = ?$

$$y = \frac{x^3 + 2x^2 - 1}{x+5}$$

$$y' = \frac{(x+5)\frac{d}{dx}(x^3+2x^2-1) - (x^3+2x^2-1)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)(1)}{(x+5)^2}$$

$$= \frac{(3x^3+4x^2+15x^2+20x) - (x^3+2x^2-1)}{(x+5)^2}$$

$$= \frac{3x^3 - x^3 + 19x^2 - 2x^2 + 20x + 1}{(x+5)^2}$$

$$\boxed{y' = \frac{2x^3 + 17x^2 + 20x + 1}{(x+5)^2}}$$

Derivatives of some trigonometric functions:

(1). $\frac{d}{dx} \sin x = \cos x$

(2). $\frac{d}{dx} \cos x = -\sin x$

$\frac{dy}{dx} = ?$, $y = x \sin x$

$$\frac{dy}{dx} = x \cdot \frac{d}{dx} \sin x + \left[\frac{d}{dx} x\right] \sin x$$

$$= x \cos x + (1) \sin x$$

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(3). $\frac{d}{dx} \tan x = \sec^2 x$

(4). $\frac{d}{dx} \sec x = \sec x \tan x$

(5). $\frac{d}{dx} \cot x = -\sec^2 x$

(6). $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x . \cot x$

Example: $y = 2x^2 \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2)\cos x + 2x^2 \frac{d}{dx} \cos x$$

$$= 4x \cos x - 2x^2 \sin x$$

Example 2: $y = \frac{\sin x}{1+\cos x}$

Applying Quotient Rule:

$$y' = \frac{(1+\cos x)\, d/dx \sin x - \sin x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - \sin x(0 - \sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cancel{\cos x + 1}}{(1+\cos x)^{\cancel{2}}} \Rightarrow \boxed{y' = \frac{1}{1+\cos x}}$$

Example 3:

$f''(x) = ?$

$f''(\pi/4) = ?$

- $f(x) = \sec x$

Applying identities:

$$f'(x) = \frac{d}{dx} \sec x$$

$$f'(x) = \sec x \tan x$$

$$f''(x) = \left[\frac{d^2}{dx^2}(\sec x)\right] \tan x + \sec x \left[\frac{d^2}{dx^2} \tan x\right]$$

$$= (\sec x . \tan x) \tan x + \sec^2 x . \sec x$$

$$= \sec x . \tan^2 x + \sec^3 x$$

Putting, $x = 45 \Rightarrow x = \pi/4$

$$f''(x) = \sec \frac{\pi}{4} . \tan^2 \frac{\pi}{4} + \sec^3 \frac{\pi}{4}$$

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"The Chain Rule"

If 'g' is differentiable at 'x' and 'f' is differentiable at g(x), then:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

$g \to x$

$y \to f(x)$

$y = f(\underbrace{g(x)}_{u})$

$\Rightarrow u = g(x)$

* Find $\frac{dy}{du}$, if $y = \cos(x^3)$

Let $u = x^3$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du} \cos u \cdot \frac{d}{dx} x^3$$ ∴ derivative of cos = -sin

$$= -\sin u \cdot 3x^2$$

$$= (-\sin(x^3))\, 3x^2$$ ∴ $u = x^3$

$$= -3x^2 \sin x^3$$

Example 2 : $\frac{dw}{dt} = ?$

$w = \tan x$ & $x = 4t^3 + t$

Solution: $w = \tan(4t^3 + t)$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dw}{dt} = \frac{d}{dx} \tan x \cdot \frac{d}{dt} (4t^3 + t)$$

$$= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1)$$

$$= (12t^2 + 1)\sec^2 (4t^3 + t)$$

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An Alternative version of Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$f(g(x))' = f'g'$$

e.g.:

$$y = \sqrt{x^3+1}$$

$$\frac{dy}{dx} = \left(\frac{1}{2\sqrt{x^3+1}}\right) \cdot 3x^2$$ ∴ Applying Power rule

(a). $\frac{d}{dx} = \sin 2x$

Solution: Let $2x = u$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du} \sin u \cdot \frac{d}{dx} 2x$$

$= (\cos u) \cdot 2$ → power rule

↳ derivative of sin = cos

$= \boxed{2\cos 2x}$ ∴ $\boxed{u = 2x}$

(b). $\frac{d}{dx} (\tan x^2 + 1)$

Let $u = x^2 + 1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du} \tan u \cdot \frac{d}{dx} (x^2+1)$$

$= (\sec^2 x).2x \quad = \boxed{2x.\sec^2 x}$

(c). $\sqrt{x^3 + \operatorname{cosec} x}$

Let $u = x^3 + \operatorname{cosec} x$

$$\frac{d}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{du} = \frac{d}{du}(u)^{1/2} \cdot \frac{d}{dx} x^3 + \operatorname{cosec} x$$

$$= \left[\frac{1}{2} u^{1/2 - 1}\right] \cdot \left[2x^2 - \operatorname{cosec} x . \cot x\right]$$

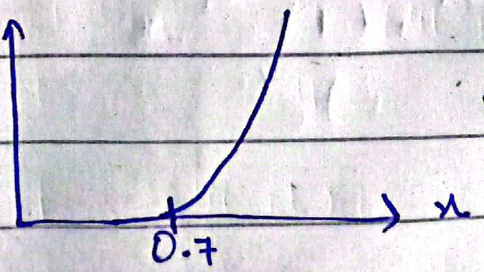
$$= \frac{1}{2\sqrt{u}} \cdot \overset{2x^2 - \operatorname{cosec} x . \cot x}{\cancel{3x}\ \cancel{2x^2}}$$

$$= \frac{1}{2\sqrt{u}} \cdot 2x^2 - \operatorname{cosec} x . \cot x$$

$$= \left(\frac{1}{2(\sqrt{x^3 + \operatorname{cosec} x})}\right) 2x^2 - \operatorname{cosec} x . \cot x$$

Derivative of Exponential Functions:

$$\frac{d}{dx} e^x = e^x$$



$\therefore$ PN-Junction

Example: $y = e^{x^2+1}$

$\frac{dy}{dx} = ?$

Let $u = x^2+1$, $y = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du} e^u \cdot \frac{d}{dx} (x^2+1)$$

$$= e^u \cdot 2x$$ ∴ Applying power rule (nx^{n-1})

$$= 2x e^{x^2+1}$$ ∴ $u = x^2+1$

Example 2:

$$f(x) = x^3 e^{1/x}$$

Let $u = 1/x$

$$\frac{dy}{dx} = x^3 e^u$$

$$= e^u \frac{d}{dx} x^3 + x^3 \frac{d}{dx} e^u$$

$$= e^u (3x^2) + x^3 e^u \cdot \frac{d}{dx} (-1) x^{-1-1}$$

$$= e^{1/x} (3x^2) + x^3 e^{1/x} \cdot x^{-2} (-1)$$

$$= 3x^2 e^{1/x} + (-1) e^{1/x} \cdot x$$

$$= \boxed{(-x + 3x^2) e^{1/x}}$$

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Rule: $\boxed{\frac{d}{dx} a^x = a^x \ln a}$

Example: $y = a^{\sqrt{x}}$, find $\frac{dy}{dx}$

Sol. Let $u = \sqrt{x}$

$y = a^u$

Now: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{da^u}{du} + \frac{d}{dx}(\sqrt{x})$

$= a^u \ln a + \underbrace{\frac{1}{2\sqrt{x}}}_{\text{power rule}}$

$$\boxed{\frac{dy}{dx} = a^{\sqrt{x}} \ln a + \frac{1}{2\sqrt{x}}}$$

Example:

$y = a^x$

$y = e^{x \ln a}$

$= e^{\ln a^x}$

$y = a^x$

$\frac{dy}{dx} = e^{x \ln a} \cdot \frac{d}{dx} x \ln a$

$$\boxed{\frac{dy}{dx} = e^{x \ln a} \cdot \ln a}$$

Derivative of logarithmic functions:

(1). $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

(2). $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{x} \cdot \dfrac{1}{\ln a}$

Example: $y = \ln(x^2 + 2x)$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(x^2+2x)\right]$$

$$= \frac{1}{(x^2+2x)} \cdot \frac{d}{dx}(x^2+2x)$$

$$= \boxed{\frac{1}{x^2+2x} \cdot (2x+2)}$$ Power Rule

Example: $y = \log_{10}(ax^2+bx+c)$

Let $u = ax^2+bx+c$

$y = \log_{10} u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}\log_{10} u + \frac{d}{dx}(ax^2+bx+c)$$

$$= \frac{1}{u}\,\frac{1}{\ln 10} + (2ax+b)$$

$$= \boxed{\frac{1}{ax^2+bx+c} + (2ax+b)}$$

Derivative of Inverse Trigonometric Functions:

(1). $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x}}$

(2). $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x}}$

(3). $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

(4). $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

(5). $\frac{d}{dx} \sec^{-1} x = \frac{1}{x^2\sqrt{x^2+1}}$

(6). $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x}$

Proof:

Let $y = \sin^{-1} x$

$\sin y = x \Rightarrow \boxed{x = \sin y}$

Derivative w.r.t x:

$$\frac{d}{dx} = \frac{d}{dx} \sin y$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad \text{—(i)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\Rightarrow \boxed{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}}$$

Proof 2: let $y = \cos^{-1}x$

$x = \cos y \quad -(i)$

Derivative w.r.t x:

$$\frac{dx}{dx} = \frac{d}{dx}\cos y$$

$$1 = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$\boxed{\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}}$$

Example:

$$\frac{dy}{dx} \, ? \qquad y = x \sin^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \sin^{-1}\left(\frac{x}{a}\right)\frac{d}{dx}x + x\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]$$

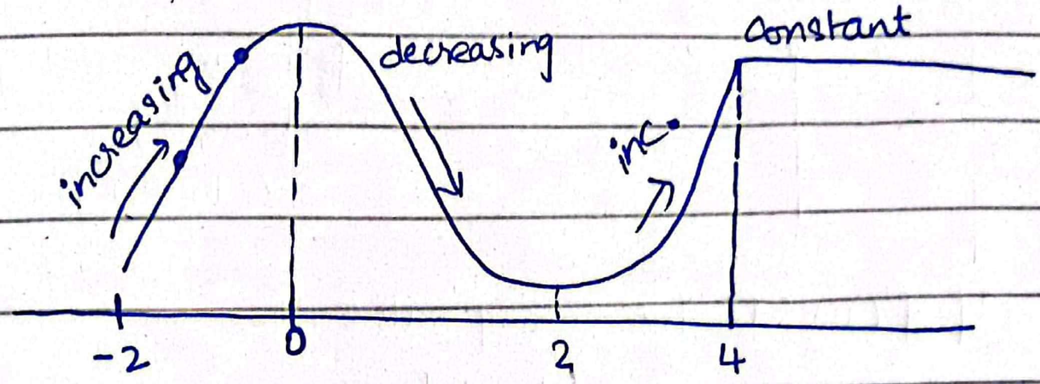
$$= \sin^{-1}\left(\frac{x}{a}\right)(1) + x\frac{1}{\sqrt{1-(x/a)^2}}$$

$$= \boxed{\sin^{-1}\frac{x}{a} + \frac{ax}{\sqrt{a^2-x^2}}}$$

The Derivative in Graphing & Applications

∘ Analysis of Functions I: Increase, Decrease

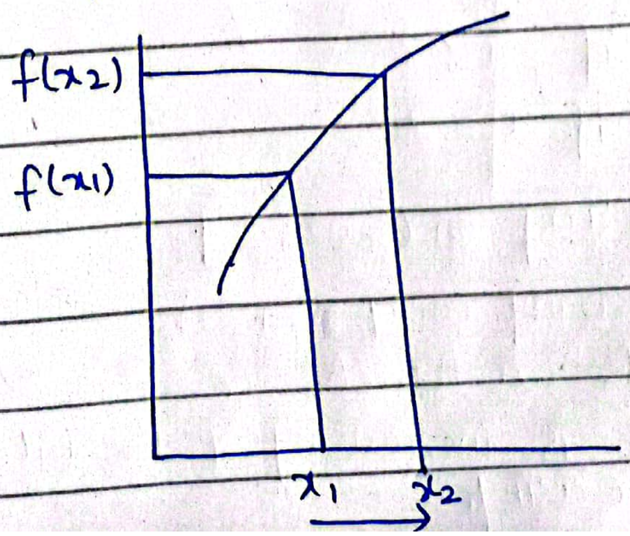
* Function Value Test:



(a). f is increasing on the interval if $f(x_1) < f(x_2)$ where $x_1 < x_2$.

(b). f is decreasing on the interval if $f(x_1) > f(x_2)$ where $x_1 < x_2$.

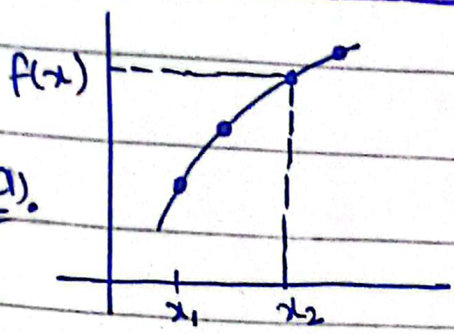
(c). f is constant on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



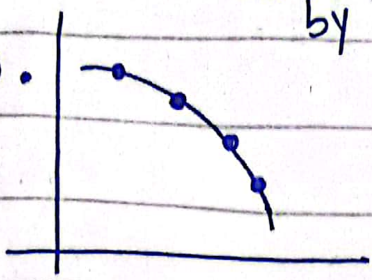
First Derivative Test:

- open interval is denoted by: $(\, , \,)$
- closed interval is denoted by: $[\, , \,]$

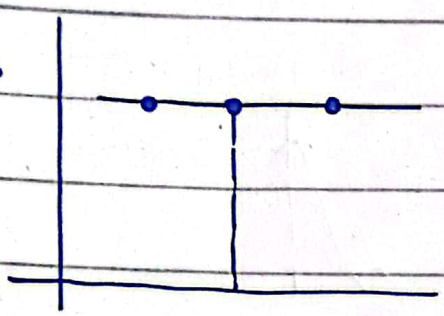
(a).



(b).



(c).



(a). If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.
(open interval) (closed interval)

(b). If $f'(x) < 0$ for every value ....
f is decreasing on $[a, b]$.

(c). If $f'(x) = 0$ for every value ....
f is constant on $[a, b]$.

Exp #1: $f(x) = x^2 - 4x + 3$

Solution: $f'(x) = \frac{d}{dx} x^2 - 4 \frac{d}{dx} x + 0$

$= 2x - 4$

$= 2(x - 2)$

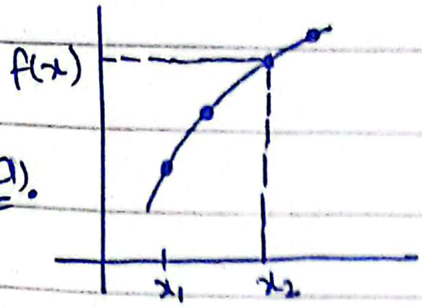
- $f'(x) < 0$ if $x < 2$
- $f'(x) > 0$ if $x > 2$
- f is decreasing on $(-\infty, 2]$
- 'f' is increasing on $[2, +\infty)$

- Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing & the intervals on which it is decreasing.

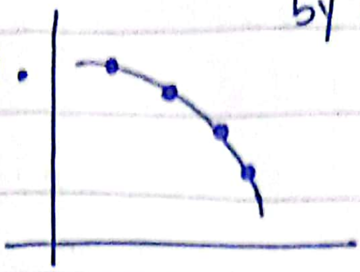
First Derivative Test:

- open interval is denoted by: (,)
- closed interval is denpted by: [,]

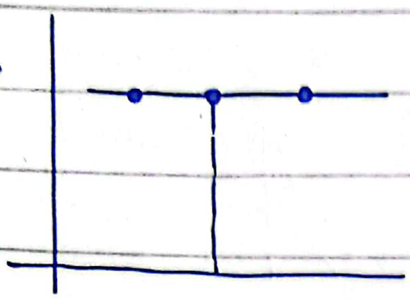
(a).



(b).



(c).



(a). If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

open interval

closed interval

(b). If $f'(x) < 0$ for every value ....

f is decreasing on $[a, b]$.

(c). If $f'(x) = 0$ for every value ....

f is constant on $[a, b]$.

Exp #1: $f(x) = x^2 - 4x + 3$

Solution: $f'(x) = \frac{d}{dx} x^2 - 4 \frac{d}{dx} x + 0$

$= 2x - 4$

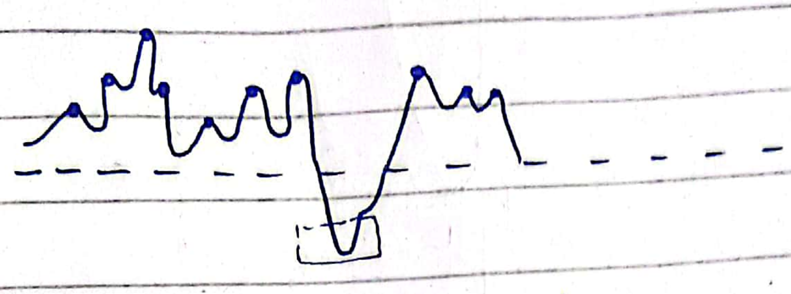
$= 2(x - 2)$

- $f'(x) < 0$ if $x < 2$
- $f'(x) > 0$ if $x > 2$
- f is decreasing on $(-\infty, 2]$
- 'f' is increasing on $[2, +\infty)$

Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing & the intervals on which it is decreasing.

Analysis of Functions II : Relative Extrema

Relative Maxima and Minima :



* A function 'f' is said to have a relative maxima at x_0 if there is an open interval containing x_0 for which

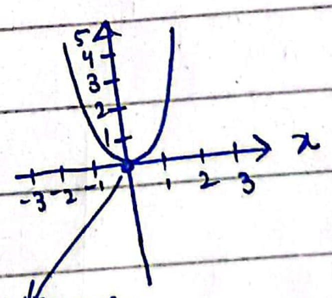
$\boxed{f(x_0) \geq f(x)}$ → relative maxima.

'f' is either a relative maxima or a relative minima at x_0, then 'f' is said to have relative extremas.

$\boxed{f(x_0) \leq f(x)}$ → relative minima

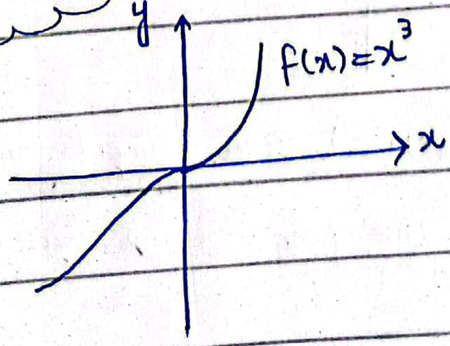
Exp. #1: → $f(x) = x^2$

↳ graph: always +ve



function has relative minima at $x=0$, but has no relative maxima.

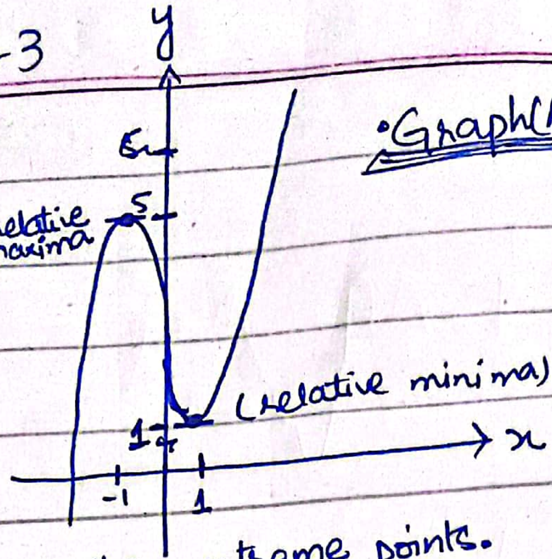
→ $f(x) = x^3$



no relative extremas ↳ (no minima, no maxima)

$$\bullet\ f(x) = x^3 - 3x + 3 \rightarrow \text{polynomial}$$

• Graph(A).

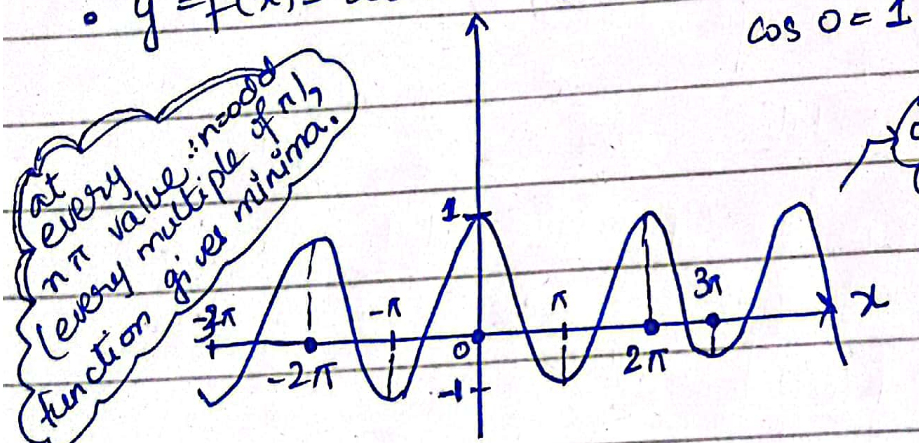


⊛ Relative means all the extreme points.

⊛ Absolute means maximum & minimum points.

• $y = f(x) = \cos x$ → oscillatory function (oscillates b/w 1 and −1)

$\cos 0 = 1$

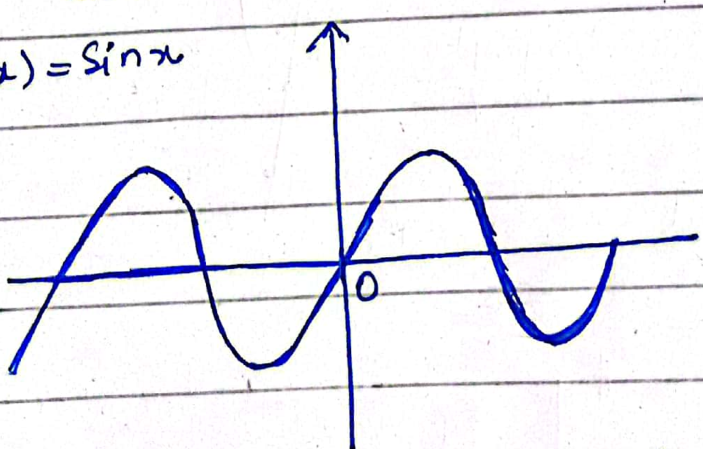


• $f(x) = \cos x$ has relative maxima at all even multiples of π

• and has relative minima at all odd multiples of π

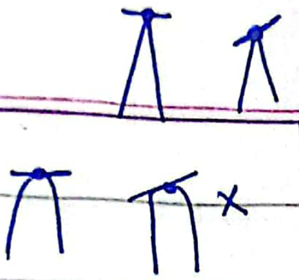
$\sin 0 = 0$

• $f(x) = \sin x$



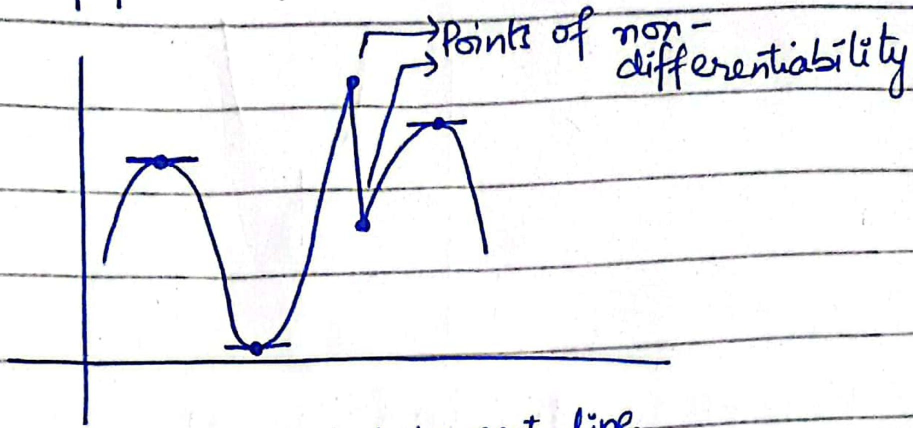
Extremas (minima, maxima) have tangents on ~~straight~~ horizontal line.

⊛ Extremas are the points which has zero derivative.



⇒ non-differentiable point

↳ both are correct.



Critical Points:
- → horizontal tangent line
- → stationary points
- → $f'(x)=0$ (derivative $=0$)

Exp #2. Find all critical points of function $f(x)=x^3-3x+1$.

$$f'(x) = \frac{d}{dx}x^3 - \frac{d}{dx}3x + 0$$

$$= 3x^2 - 3$$

$$= 3(x^2-1)$$

$$= 3(x+1)(x-1)$$

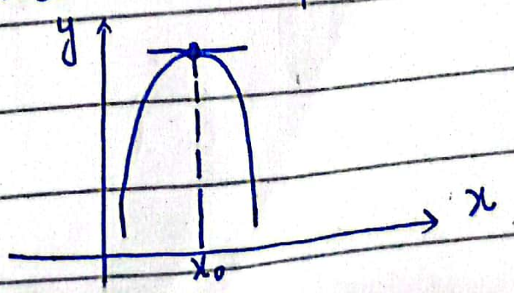
Putting $x = -1$ or $x = 1$

$$= 3(1+1)\cancel{(1-1)}0$$

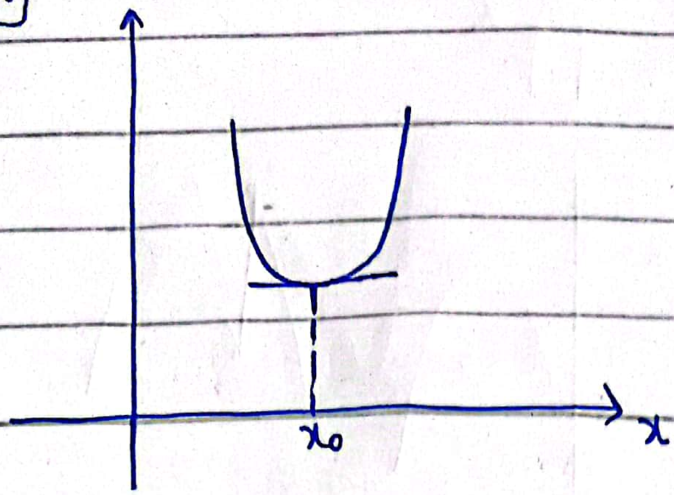
$$= 0$$

It is consistent with Graph(A).

First Derivative Test: "The function 'f' has a relative maxima at those critical points where 'f'' changes its sign



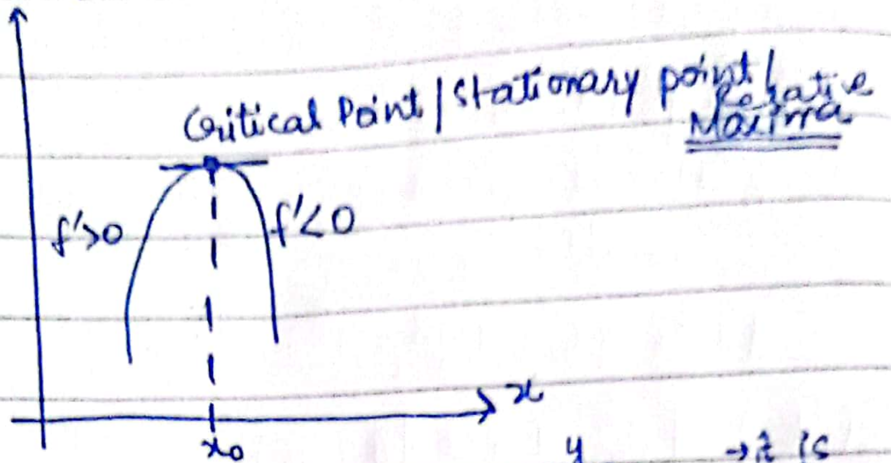
• Sign of a function changes after the critical point.



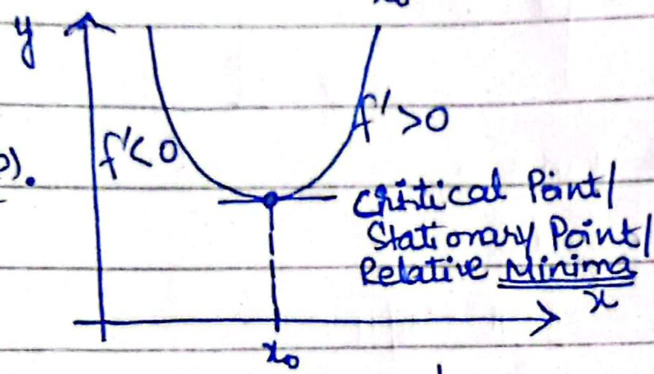
⊛ First Derivative Test for Extremas:

"A function 'f' has a relative extrema at those critical points where 'f′' changes sign."

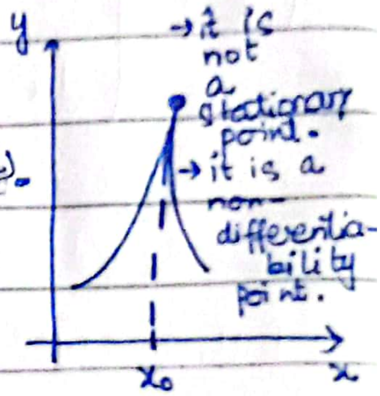
(a).



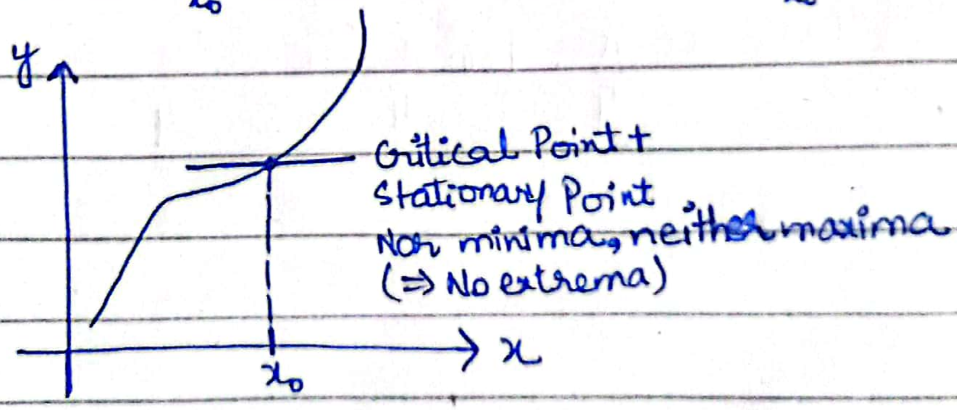
(b).



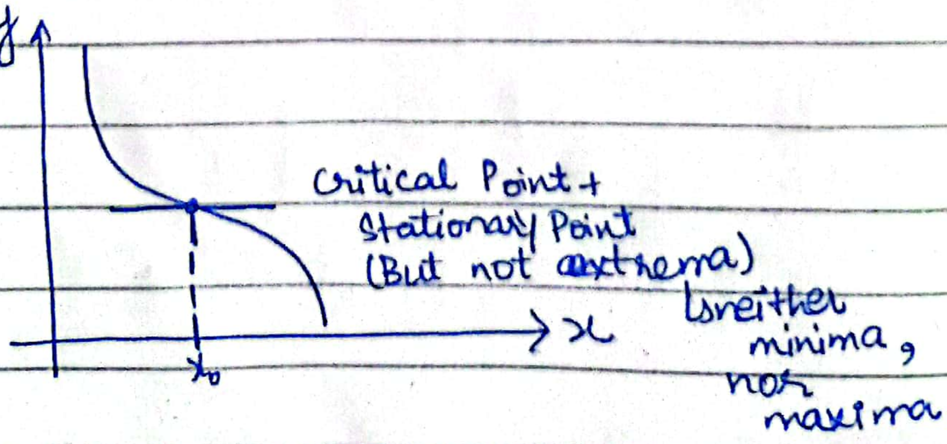
(c).



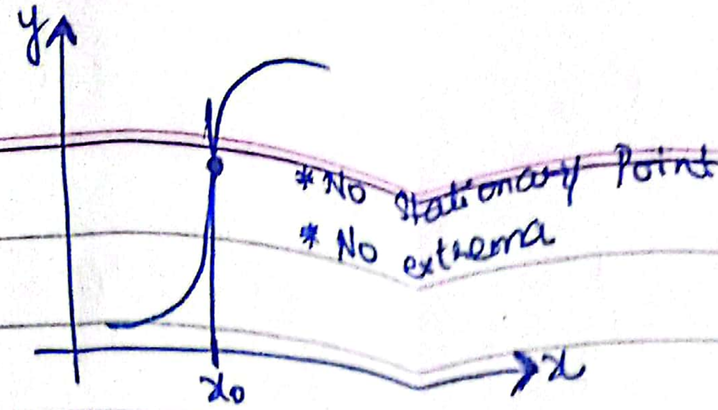
(d).



(e).



(f).



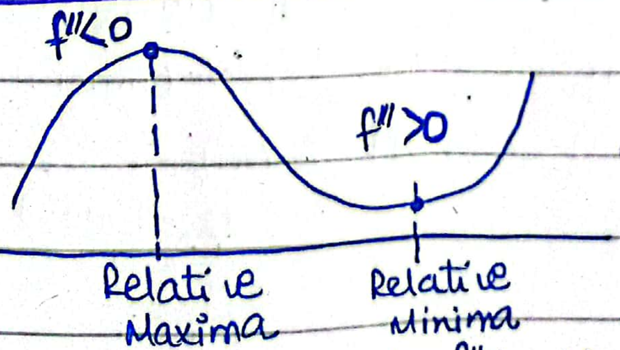
Theorem of First Derivative Test:

(a). If $f'(x) > 0$ on an interval extending left from x_0 and $f'(x) < 0$ on an interval extending right from x_0, then 'f' has a relative maxima at x_0.

(b). If $f'(x) < 0$ on an interval extending left from x_0 and $f'(x) > 0$ on an interval extending right from x_0, then 'f' has a relative minima at x_0.

(c). If $f'(x)$ has the same sign extending left and right from x_0 then $f(x)$ does not have a relative extrema at x_0.

Second Derivative Test:



(a). If $f'(x_0) = 0$ and $f''(x_0) > 0$, then function 'f' has relative minima at x_0.

(b). If $f'(x_0)=0$ and $f''(x_0)<0$, 'f' has relative maxima at x_0.

(c). If $f'(x_0)=0$ and $f''(x_0)=0$, then the test is inconclusive.

Find the relative extrema of $f(x)=3x^5-5x^3$

Solution:

Step 1: Calculate first derivative.

$$f'(x) = 15x^4 - 15x^2$$
$$= 15x^2(x^2-1)$$
$$= 15x^2(x+1)(x-1)$$

~~From here~~ Now substitute $f'(x)=0$, so that we can find the values.

$$0 = 15x^2(x+1)(x-1)$$
$$\Rightarrow x=0,\ x=1,\ x=-1$$

$$f''(x)\ \frac{d}{dx}15x^4 - \frac{d}{dx}15x^2$$

$$= 60x^3 - 30x$$

$x=0 \Rightarrow 60(0)^3-30(0) = 0$ $f''=0$ (inconclusive)

$x=1 \Rightarrow 60(1)^3-30(1) = 30$ $f''=30$ (minima)

$x=-1 \Rightarrow 60(-1)^3-30(-1) = -60+30$

$f'' = -30$ (maxima)