Discrete Structures

Syed Faisal Bukhari, PhD
Associate Professor
Department of Data Science (DDS), Faculty of Computing and Information Technology (FCIT), University of the Punjab (PU)

Text book

Discrete Mathematics and Its Application, 7^{th} Edition

Kenneth H. Rosen

References

Discrete Mathematics and Its Application, 7h Edition

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http://mathworld.wolfram.com/CircularPermutation.html

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http://mathworld.wolfram.com/CircularPermutation.html

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Counting Sample Points

- In many cases, we shall be able to solve a probability problem by counting the number of points in the **sample space without actually listing each element**.

- The **fundamental principle of counting**, often referred to as the **multiplication rule**, is stated in Rule 2.1.

Rule 2.1: If an operation can be performed in ***n_1* ways**, and if for **each of these ways a second operation** can be performed in ***n_2* ways**, then **the two operations** can be performed together in ***$n_1 n_2$* ways**.

- **Example :** How many **sample points** are there in the **sample space** when a **pair of dice** is thrown once?

Solution :

- The first die gives us, **n_1 = 6 ways**.
- For each of these **6 ways**, the **second die** gives us, **n_2 = 6 ways**.
- Therefore, the pair of dice give us **n_1n_2 = (6)(6) = 36 possible ways**.

Example : If a **22-member club** needs to elect a **chair** and a **treasurer**, how many **different ways** can these **two** to be elected?

Solution :

- For the **chair position**, we have n_1 = **22 ways**

- For the **treasurer** position, for each of those **21 possibilities,** we have n_2 = **21ways**

- Total number of ways = $n_1 \times n_2 = 22 \times 21 = 462$

- **Rule 2.2:** If an operation can be performed in n_1 ways, and **if for each of these a second operation** can be performed in n_2 **ways**, and for each of the **first two a third operation** can be performed in n_3 **ways**, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ **ways**.

- **Example:** Sam is going to assemble a computer by himself. He has the choice of chips from **two brands**, a **hard drive** from **four**, **memory** from **three**, and an **accessory bundle** from **five** local stores. How many **different ways** can Sam order the parts?

Solution :

$n_1 = 2$ **(No of brands)**

$n_2 = 4$ **(No of hard drives)**

$n_3 = 3$ **(No of memory sticks)**

$n_4 = 5$ **(No of accessory bundles)**

- Total number of ways = $\mathbf{n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5} = 120$

- **Rule 2.2:** If an operation can be performed in n_1 ways, and **if for each of these a second operation** can be performed in n_2 **ways**, and for each of the **first two a third operation** can be performed in n_3 **ways**, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ **ways**.

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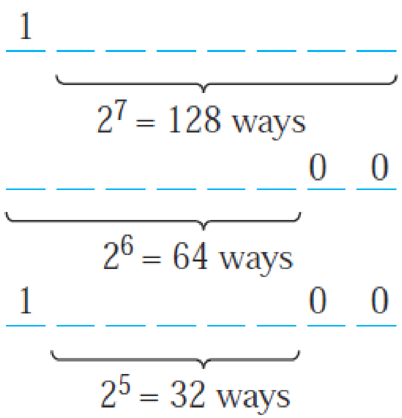
The Subtraction Rule If a task can be done in either n_1 **ways** or n_2 **ways**, then the number of ways to do the task is $n_1 + n_2$ **minus** the number of ways to do the task that **are common to the two different ways**.

The subtraction rule is also known as the **principle of inclusion–exclusion**, especially when it is used to **count the number of elements in the union of two sets**.

Suppose that A_1 and A_2 **are sets**. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2. The number of ways to select an element from **A_1 or from A_2**, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element **from A_1** and the number of ways to select an element **from A_2**, **minus** the number of ways to select an element that is in both A_1 and A_2. Because there are $|A_1 \cup A_2|$ **ways** to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ **ways** to select an element common to both sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Example How many bit strings of **length eight** either start with a **1 bit** or end with the two **bits 00**?



Solution

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$|A_1|$ = A bit string of length eight that begins with a **1** = $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ = $2^7 = 128$ **ways**.

$|A_2|$ = A bit string of length eight that end with the two **bits 00** = $2 \times 2 \times 2 \times 2 \times 2 \times 2$ = $2^6 = 64$ **ways**.

$|A_1 \cap A_2|$ = **A** bit strings of **length eight** start with a **1 bit** and end with the two **bits 00** = $2 \times 2 \times 2 \times 2 \times 2$ = $2^5 = 32$ **ways**

$\therefore$ $|A_1 \cup A_2|$ = The number of bit strings of length eight that begin with a 1 or end with a 00 = **128 + 64 – 32 = 160**

Example A computer company receives **350 applications** from computer graduates for a job planning a line of new Web servers. Suppose that **220** of these applicants majored in **computer science**, **147** majored in **business**, and **51** majored both in **computer science and in business**. How many of

these applicants majored **neither in computer science nor in business**?

Solution

$|A_1 \cup A_2| = |A_1| + |A_1| - |A_1 \cap A_2|$

Let A_1 be the set of students who majored in computer science

A_2 the set of students who majored in business

$A_1 \cup A_2$ is the set of students who majored in computer science or business (or both)

$A_1 \cap A_2$ is the set of students who majored both in **computer science and in business**

By the subtraction rule the number of students who majored either in computer science or in business (or both) equals

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$

We conclude that **350 − 316 = 34** of the applicants **majored neither in computer science nor in business**.

The Pigeonhole Principle

- Suppose that a flock of **20 pigeons** flies into a set of **19 pigeonholes** to roost. Because there are 20 pigeons but only 19 pigeonholes, a **least one of these 19 pigeonholes must have at least two pigeons** in it.

- To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated.

- This illustrates a general principle called the **pigeonhole principle**, which states that **if there are more pigeons than pigeonholes**, then there must **be at least one pigeonhole with at least two pigeons** in it (see Figure 1). Of course, this principle applies to other objects besides pigeons and pigeonholes.

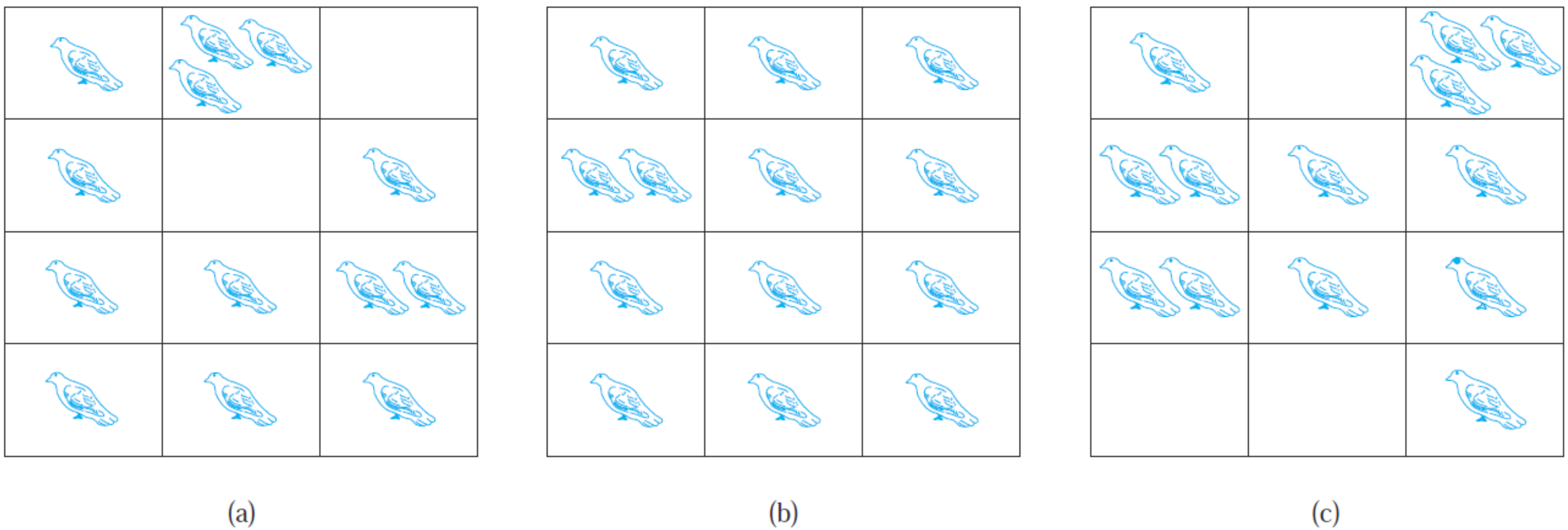


FIGURE 1 **There Are More Pigeons Than Pigeonholes.**

THE PIGEONHOLE PRINCIPLE

Theorem 1 THE PIGEONHOLE PRINCIPLE If k is a **positive integer** and **$k + 1$ or more objects** are placed into **k boxes**, then there is at **least one box** containing **two or more of the objects**.

Example Among any group of **367 people**, there must be at least **two with the same birthday**, because there are only **366 possible birthdays**.

Solution:

Number of days in a leap year = 366 days

Total number of people in a group = 367

By the principle of Pigeonhole, there must be at least **two with the same birthday**, because there are only **366 possible birthdays**.

Note: There are two calendars--one for **normal years** with **365 days**, and one for **leap years** with **366 days**

Example In any group of **27 English words**, there must be at **least two that begin with the same letter**, because there are **26 letters in the English alphabet**.

Example How many students must be in a class to guarantee that **at least two students receive** the same score on the final exam, if the exam is graded on a scale from **0 to 100 points**?

Solution:

Total number of possible scores = 101

The pigeonhole principle shows that among any **102 students** there must be **at least 2 students** with the same score.

Permutation

Permutation: A **permutation** is an arrangement of all or part of a set of objects.

OR

An arrangement of a set of ***n* objects** in a **given order** is called a ***permutation*** of the objects (taken all at **a time**).

Example: Consider the three **letters *a, b,*** and ***c***.

The possible **permutations** are ***abc, acb, bac, bca, cab,*** and ***cba***.

Permutation

- **Definition** For any **non-negative integer** n, $n!$, called **"n factorial,"** is defined as $n! = n(n-1)\cdots(2)(1)$, with special case $0! = 1$.

- **Theorem 2.1:** The number of permutations of **n objects** is $n!$.

- **Example** The number of permutations of the **four letters** a, b, c, and d will be $4! = 24$.

Why 0! one?

The idea of the factorial (in simple terms) is used to compute the number of permutations (combinations) of arranging a set of **n numbers**.

n	Number of permutations (n!)	Visual examples
1	1	{1}
2	2	{1, 2}, {2, 1}
3	6	{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1}
⋮	⋮	⋮
0	1	{}

Why 0! one?

$$n! = n \times (n - 1)!$$

$$\Rightarrow (n - 1)! = \frac{n!}{n}$$

Substitute 1, we get

$$\Rightarrow (1 - 1)! = \frac{1!}{1}$$

$$\Rightarrow 0! = 1$$

Permutations Rule (When items are all different)

- **Theorem 2.2:** The number of permutations of **n distinct objects** taken **r** at a **time** is ${}_nP_r = \frac{n!}{(n-r)!}$, where $r \leq n$

Permutations Rule (When items are all different)

1. There are ***n different* items available**. (This rule **does not apply** if **some of the items are identical** to others.)

2. We **select *r*** of the ***n* items (without replacement).**

3. We consider **rearrangements** of the **same items** to be **different sequences**. (The permutation of ***ABC*** is different from ***CBA*** and is counted separately)

Permutations Rule (When items are all different)

- If the preceding **requirements are satisfied**, the number of **permutations (or sequences)** of ***r* items** selected from ***n* different** available items **(without replacement)** is $_nP_r = \dfrac{n!}{(n-r)!}$, where $r \le n$.

Example : In one year, **three awards (research, teaching, and service)** will be given to a class of **25** graduate students in a statistics department. If each student can receive at **most one award**, how many possible selections are there?

Solution : Since the awards are **distinguishable**, it is a permutation problem. The total number of sample points is

$$\mathbf{{}_{25}P_3 = \frac{25!}{(25-3)!}} = \mathbf{\frac{25!}{22!}} = (25)(24)(23) = 13,\ 800.$$

Or

$n_1 = 25$ **(research)**

$n_2 = 24$ **(teaching)**

$n_3 = 23$ **(service)**

Total number of ways = $\mathbf{n_1 \times n_2 \times n_3 = 25 \times 24 \times 23}$ = 13800

Example Find the number of ways of forming **four-digit codes** in which no digit is repeated.

Solution

To form a four-digit code with no repeating digits, you need to select 4 digits from a group of 10,

so **n = 10** and **r = 4**.

$$_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$$

So, there are **5040** possible four-digit codes that do not have repeating digits.

Or

$n_1 = 10$ **(number of digits available for the first digit)**

$n_2 = 9$ **(number of digits for the second digit)**

$n_3 = 8$ **(number of digits for the third digit)**

$n_4 = 7$ **(number of digits for the fourth digit)**

Total number of ways = $n_1 \times n_2 \times n_3 \times n_4 = 10 \times 9 \times 8 \times 7 = 5040$

Example How many 4-digit numbers are there with no digit repeated?

Solution

$n_1 = 9$ **(total number available for the first digit of a number)**

$n_2 = 9$ **(total number available for the second digit of a number)**

$n_3 = 8$ **(total number available for the third digit of a number)**

$n_4 = 7$ **(total number available for the fourth digit of a number)**

Total number of ways = $\mathbf{n_1 \times n_2 \times n_3 \times n_4 = 9 \times 9 \times 8 \times 7} = 4536$

Example Forty-three race cars started the **2013 Daytona 500**. How many ways can the cars finish first, second, and third?

Solution

You need to select three race cars from a group of 43, so n = 43 and r = 3. Because the order is important, the number of ways the cars can finish first, second, and third is

$$\mathbf{{}_{43}P_3 = \frac{43!}{(43-3)!} = \frac{43!}{40!} = \frac{43 \times 42 \times 41 \times 40!}{40!} = 74{,}046.}$$

Or

$n_1 = 43$ **(first)**

$n_2 = 42$ **(second)**

$n_3 = 41$ **(third)**

Total number of ways = $\mathbf{n_1 \times n_2 \times n_3 = 43 \times 42 \times 41} = 74046$

Theorem 2.3: The number of **permutations** of n **objects** arranged in a circle is **$(n - 1)!$.**

Example In how many ways can **6 people** be seated at a **round table**?

Solution

Here n = 6 (total number of people)

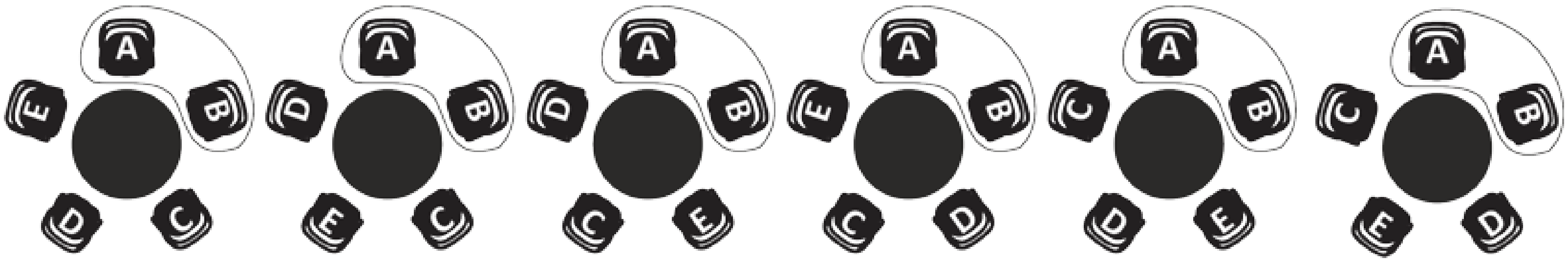
The total number of ways = $(6 - 1)! = 120$

Example Find the number of ways in which **5 people A, B, C, D, E** can be seated at **a round table**, such that

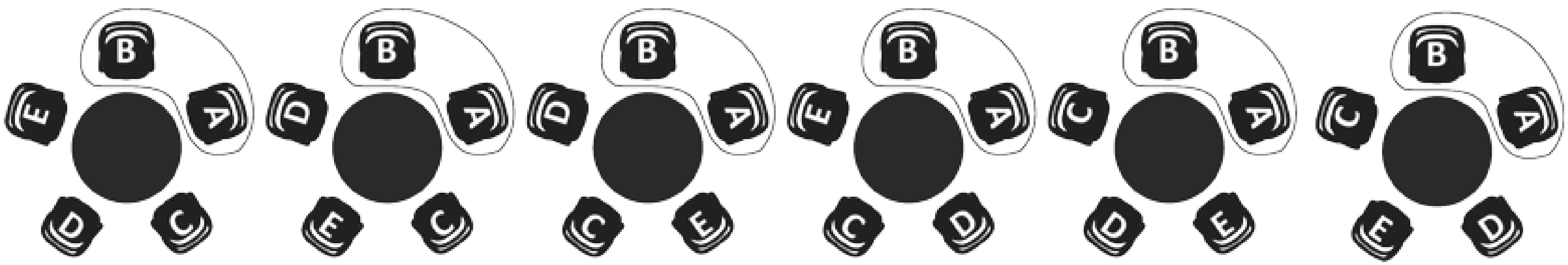
a. **A and B** must always sit together.

b. **C and D** must not sit together.

- **Solution a.** If we wish to seat **A and B** together in all arrangements, we can consider these **two as one unit**, along with 3 **others**. So effectively we've to arrange **4 people** in a circle. The number of ways = **(4 – 1)! = 6**



But in each of these arrangements, **A** and **B** can themselves interchange places in **2 ways**.



Therefore, the total number of ways will be $\mathbf{6} \times \mathbf{2} = \mathbf{12}$.

5 people A, B, C, D, E

- **b.** The total number of ways will be **(5 – 1)!** or **24**.
- Similar to **a.** above, the number of cases in which **C** and **D** are seated together, will be **12**.
- Therefore the required number of ways = **24 – 12 = 12.**

Permutations when repetition is allowed

- So far we have considered **permutations** of **distinct objects**. That is, all the objects were completely different or distinguishable.

- Obviously, if the letters ***b* and *c*** are both equal to ***x***, then the **6 permutations** of the **letters *a*, *b*, and *c*** become ***axx*, *axx*, *xax*, *xax*, *xxa*,** and ***xxa***, of which only **3** are distinct.

- Therefore, with **3 letters**, **2 being the same**, we have **3!/2! = 3** distinct permutations.

Permutations Rule (When some items are identical to others)

- **Theorem 2.4:** The number of **distinct permutations** of n **things** of which n_1 are of one kind, n_2 of a second kind, . . . , n_k of a k^{th} kind is

$$\frac{n!}{n_1!n_2!\cdots nk!}$$

where $n_1 + n_2 + n_3 + ... + n_k = n$.

Permutations Rule (When some items are identical to others)

Requirements

1. There are ***n* items available**, and some items are **identical** to others.

2. We select all of the ***n* items (without replacement).**

3. We consider **rearrangements of distinct items** to be **different sequences**.

If the preceding requirements are satisfied, and if there are n_1 **alike**, n_2 **alike**, . . . , n_k **alike**, the number

$$\frac{n!}{n_1!n_2! \cdots nk!}$$

Example A building contractor is planning to develop a subdivision. The subdivision is to consist of **6 one-story houses**, **4 two-story houses**, and **2 split-level houses**. In how many **distinguishable** ways can the houses be arranged?

Solution

The total number of arrangements = $\dfrac{12!}{6!\,4!\,2!}$

= 13,860

distinguishable ways

Example Calculate the number of distinguishable permutations of the letters **AAAABBC**.

Solution

The total number of arrangements = $\dfrac{7!}{4!\,2!\,1!}$

= 105

distinguishable ways

- **Example** In a college football training session, the defensive coordinator needs to have **10 players** standing in a row. Among these **10 players**, there are **1 freshman**, **2 sophomores**, **4 juniors**, and **3 seniors**. How many **different ways** can they be arranged in a row if only their **class level** will be distinguished?

❑ Solution

n = 10 (Total number of players)

$\mathbf{n_1 = 1}$ (Total number of freshman)

$\mathbf{n_2 = 2}$ (Total number of sophomores)

$\mathbf{n_3 = 4}$ (Total number of juniors)

$\mathbf{n_4 = 3}$ (Total number of seniors)

The total number of arrangements = $\dfrac{10!}{1!\,2!\,4!\,3!}$

= 12, 600.

Theorem 2.5: The number of ways of **partitioning** a set of ***n* objects** into ***r* cells** with **n_1 elements** in the first cell, **n_2 elements** in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$

Example : In how many ways can **7 graduate** students be assigned to **1 triple** and **2 double** hotel rooms during a conference?

Solution :

$$\frac{n!}{n_1!n_2! \cdots nk_!}$$

Here n = 7

$n_1 = 3$

$n_2 = 2$

$n_3 = 2$

The total number of possible partitions would be

$$\frac{7!}{3!\ 2!\ 2!} = 210$$

Theorem 2.6: The number of combinations of ***n* distinct objects** taken ***r*** at a time is

$$ {}_nC_r = \frac{n!}{r!(n-r)!}, \text{ where } r \leq n. $$

Combinations Rule

Requirements

1. There are ***n different*** items available.

2. We select ***r*** of the ***n* items** (**without replacement**).

3. We consider rearrangements of the **same items** to **be the same**. (The combination ***ABC*** is the same as ***CBA***.)

If the preceding requirements are satisfied, the number of **combinations** of ***r* items** selected from ***n* different items** is

$$_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Combination vs. Permutations

- The **2-permutations** of the letters **A, B, C, and D** are:

  AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC.

- The combinations of two out of these four letters are:

  AB, AC, AD, BC, BD, CD.

(Since the elements of a combination are unordered, **BA** is not viewed as being distinct from **AB**.)

- **Example:** In a lottery, each ticket has **5 one-digit numbers 0-9** on it.

a) You win if your ticket has the **digits in any order.** What are your changes of winning?

b) You would win only if your ticket has the digits in **the required order.** What are your chances of winning?

Solution:

There are 10 digits to be taken 5 at a time.

a) The number of ways of **selecting 5** tickets **from 10** is

$$ {}_{10}C_{5} = \frac{10!}{5!(10-5)!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5!} = 252 $$

∴The chances of winning are **1 out of 252 (0.0040 or 0.3968 %**

- b) Since the order matters, we should use permutation instead of combination.

$$ {}_{10}P_5 = \frac{10!}{(10-5)!} = \frac{10!}{5!} = 10 \times 9 \times 8 \times 7 \times 6 = \mathbf{30240} $$

$\therefore$ The chances of winning are 1 out of 30240 (or **0.0033 %)**

Suggested Readings

6.1 The Basics of Counting

6.2 The Pigeonhole Principle

6.3 Permutations and Combinations