7 PROBABILITY

7.1 Introduction The idea of probability is familiar to everyone. In our day-to-day life we make statements such as the following: "It is likely to rain today." "I will probably go abroad this year." "I am almost sure that I will win this game." "The probability that nuclear war will start is very low." "The odds are four to one that this horse will win." In all these statements there is lack of certainty. In the theory of probability, we introduce a common measure of expression by measuring probability between zero and one. If an event has never happened or is impossible to occur, it is assigned a probability of 0. If an event is sure to occur, it is assigned a probability of 1. For events which may or may not happen, the probability ranges between 0 and 1.

The theory of probability is an interesting branch of mathematics. It has its origin in the games of chance. It plays an important role in making decisions in situations where there is lack of certainty. During the seventeenth century, the gambles called upon the mathematicians to provide them optimum strategies for various games of chance so as to increase their chances of winning. The modern theory of probability has wide applications in fields of insurance, business and biological and physical sciences. We start our study of the theory of probability by introducing some concepts such as experiment, sample space, events, etc.

7.2 Random Experiment In Statistics, the term *experiment* is used in a much wider sense than in Physics or Chemistry. Here we use this term to describe any process which generates raw data. For example, the tossing of a coin is considered an experiment or sometimes a *random experiment*. Other examples of experiments are: rolling a die, selecting a bolt from a lot and observing whether it is defective or not, etc.

All such experiments have two properties in common. One is that each experiment has several possible outcomes which can be specified in advance. For example, in tossing a coin, the possible outcomes are: head and tail. In rolling a die, the possible outcomes are: 1, 2, 3, 4, 5, 6. In selecting a bolt the possible outcomes are: defective and non-defective. The second property is that we are not certain about the outcome or result of the experiment. For example, in tossing a coin, although the possible outcomes are a head and a tail, but we are not certain whether it will be a head or a tail. In selecting a bolt, we are not certain whether the bolt will be defective or non-defective.

7.3 Sample Space The set or collection of all possible outcomes of an experiment is called the *sample space*. It is denoted by S. Each element of a sample space is called a *sample point*. For example, when we toss a coin, the sample space is $S = \{H, T\}$ and each of the elements H (head) and T (tail) is a sample point. When we roll a die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. and each of the elements 1, 2, 3, 4, 5, 6 (spot on the die) is a sample point. In drawing a card from a pack of 52 cards, there are 52 possible outcomes. Thus the sample space of this experiment consists of 52 sample points.

Let us consider some more examples. Suppose a coin is tossed twice or two distinct coins are tossed simultaneously. There will be $2 \times 2 = 4$ possible outcomes as shown below. Thus the sample space is $S = \{HH, HT, TH, TT\}$.

Coin 1 \ Coin 2	H	T
H	HH	HT
T	TH	TT

Suppose a coin is tossed three times or three distinct coins are tossed simultaneously. There will be $2 \times 2 \times 2 = 8$ possible outcomes: *HHH, HHT, HTH, THH, HTT, THT, TTH, TTT*. Thus the sample space of this experiment will be

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

If a die is rolled twice or two distinct dice are rolled simultaneously, there will be $6 \times 6 = 36$ possible outcomes as shown in Table 7.1.

Table 7.1

Outcome of first die	Outcome of second die					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Thus the sample space of this experiment is

$$S = \{ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$$
$$(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$$
$$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),$$
$$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),$$
$$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),$$
$$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \}.$$

7.4 Events The possible outcome of an experiment is called an *event*. Thus an event is a subset of the sample space S. Events are usually denoted by first few capital letters $A, B, C, \ldots$. For example, when a coin is tossed, the sample space is $S = \{H, T\}$ and the subset $A = \{H\}$ is the event that a head occurs. When two coins are tossed, the sample space is $S = \{HH, HT, TH, TT\}$ and the subset $B = \{HH, HT, TH\}$ is the event that at least one head appears when two coins are tossed. We say that an event A occurs when any one of the outcomes in A occurs.

Since events are sets, hence we have an *algebra of events* corresponding to the algebra of sets considered before. By using set operations on events in sample space S we can obtain other events in S as defined below.

7.4.1 The Certain or Sure Event An event consisting of the sample space S itself is the *sure* or *certain event* since an element of S must occur.

7.4.2 The Impossible Event An event consisting of the null set ϕ is called the *impossible event* because an element of ϕ cannot occur.

7.4.3 Mutually Exclusive Events If the sets corresponding to events A and B are disjoints, i.e. $A \cap B = \phi$, then the events A and B are said to be *mutually exclusive*. In other words, two events are mutually exclusive if they cannot both occur at the same time.

7.4.4 Complementary Events The set of possible outcomes in the sample space S which are not in the event A is the *complement* of A relative to S and is denoted by A'.

7.4.5 Union of Events The *union* of two events A and B, denoted by $A \cup B$, consists of those outcomes which belong to either A or B or both A and B.

7.4.6 Intersection of Events The *intersection* of two events A and B, denoted by $A \cap B$, consists of those outcomes which are common to both A and B.

7.4.7 Difference of Events The *differencc* of two events A and B is the event consisting of those outcomes included in event A but not included in event B.

7.4.8 Equal Events When every outcome contained in an event A is also contained in another event B and conversely every outcome of B is also contained in A, the events are said to be *equal*.

7.4.9 Simple and Compound Events If an event consists of only one sample point, it is called a *simple or elementary event*. On the other hand, if the event consists of more than one sample points, it is called a *compound event*. For example, when two coins are tossed, the event $A = \{HH\}$ that two heads appear is simple but the event $B = \{HH, HT, TH\}$ that at least one head appears is a compound event. When two dice are rolled, the event $A = \{1, 1\}$ that an ace appears on both dice is a simple event while the event $B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$ that the sum of the two faces is 7 is a compound event.

Example 7.1 Suppose three distinct coins are tossed. The sample space of this experiment is:

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

The subset $A = \{HHT, HTH, THH\}$ is the event that exactly two heads appear when three coins are tossed. The subset $B = \{HHH\}$ is the event that exactly three heads appear when three coins are tossed.

Example 7.2 The sample space for rolling two dice is given in Table 7.1. The event that the sum of the two faces is 7 is the subset $A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2),$

(6, 1)} and the event that the sum of the two faces is 11 is the subset $B = \{(5, 6), (6, 5)\}$.

7.5 Equally Likely, Mutually Exclusive and Exhaustive Outcomes When each outcome of a sample space is as likely to occur as any other, the outcomes are said to be *equally likely*. For example, if we toss a fair coin, the head is as likely to occur as the tail. These are, therefore, equally likely outcomes. Similarly, if we roll a fair die, each of the six faces is as likely to occur as the other. Hence all the six faces of a fair die are equally likely outcomes.

If the occurrence of an outcome prevents the occurrence of other outcomes, i.e. if one outcome occurs, others cannot occur, they are called *mutually exclusive outcomes*. Suppose we toss a coin. If the head occurs, the tail cannot occur. Thus they are mutually exclusive. Similarly, if we roll a die and the face 4 occurs, the other faces cannot come up. Thus all the six faces of a die are mutually exclusive.

Outcomes are said to be *exhaustive* if they constitute the entire sample space. For example, if we toss a coin, the possible outcomes are a 'head' and a 'tail'. There is no other possibility; the coin will not stand on the edge. In rolling a cubical die, there are six exhaustive outcomes, namely, the spots 1, 2, 3, 4, 5, 6.

7.6 Definitions of Probability There are two approaches to the interpretation of probability – the *objective approach* and the *subjective approach*. We give some definitions of probability based on the objective approach.

1. Classical or A Priori Definition[1] This definition may be stated as follows:

If an experiment can result in n equally likely, mutually exclusive and exhaustive outcomes and m of which are favourable to the occurrence of an event A, the probability that the event A will occur is given by the ratio m/n. Symbolically, the probability of the occurrence of the event A, denoted by $P(A)$, is given by $P(A) = m/n$.

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Number of all possible outcomes}} = \frac{m}{n} \qquad (7.1)$$

Example 7.3 (a) A fair coin is tossed. What is the probability of getting (i) a head (ii) a tail?

Solution (i) When we toss a coin, there are two possible outcomes: head and tail. Since the coin is fair and only one face (head or tail) can come up, the number of equally likely, mutually exclusive and exhaustive outcomes is 2, i.e. $n = 2$. There is only one outcome favourable to the occurrence of a head, i.e. $m = 1$. Thus the probability of getting a head is $P(\text{head}) = 1/2$.

(ii) Since there is one outcome favourable to the occurrence of a tail, the probability of getting a tail is $P(\text{tail}) = 1/2$.

Example 7.3 (b) A perfect cubical die is rolled. What is the probability of getting (i) a '5' (ii) an even number?

Solution (i) Here the number of total possible outcomes is 6, viz. 1, 2, 3, 4, 5, 6 which are equally likely because the die is perfect. Since only one face can come up at

1. This definition was given by *P.S. Laplace* (1749 – 1897) in his book "*Philosophical Essay on Probabilities*" to deal with problems suggested by games of chance.

a time, they are mutually exclusive. There is only one outcome favourabe to the occurrence of a '5'. Hence the probability of getting a '5' is $P(5) = 1/6$.

(ii) The number of outcomes favourable to the occurrence of an even number is 3, viz. 2, 4, 6. Hence the probability of getting an even number is $P(\text{even}) = 3/6 = 1/2$.

Example 7.4 (a) From a well shuffled pack of 52 cards, a card is drawn at random. What is the probability that it is (i) a black card (ii) an ace (iii) a card of spades (iv) a jack of clubs (v) a pictured card (vi) jack of clubs or queen of diamonds? (B.I.S.E., Multan 2002)

Solution (i) There are 52 playing cards in the pack; hence the number of possible outcomes is 52. Since the pack is well shuffled, we reason that each of these outcomes is equally likely. Since only one card can be drawn at a time, they are mutually exclusive. There are 26 black cards in the pack, so the number of outcomes favourable to the occurrence of a 'black card' is 26. Hence the probability of getting a black card is $P(\text{black card}) = 26/52 = 1/2$.

(ii) There are only four aces in the pack and thus the number of outcomes favourable to the occurrence of an ace is 4. Hence the probability of getting an ace is $P(\text{ace}) = 4/52 = 1/13$.

(iii) There are 13 cards of spades and thus the number of outcomes favourable to the occurrence of a spade is 13. Hence the probability of getting a spade, $P(\text{spade}) = 13/52 = 1/4$.

(iv) There is only one jack of clubs and thus the number of outcomes favourable to the occurrence of jack of clubs is 1. Hence the probability of getting a jack of clubs is $P(\text{jack of clubs}) = 1/52$.

(v) There are 12 pictured cards (3 cards in each suit: hearts, spades, clubs and diamonds). The number of outcomes favourable to the occurrence of a pictured card is 12. Hence the probability of getting a pictured card is $P(\text{pictured card}) = 12/52 = 3/13$.

(vi) There is only one jack of clubs and one queen of diamonds. The number of outcomes favourable to each of these is 1. Hence the probability of getting a jack of clubs or a queen of diamonds is P(jack of clubs or queen of diamonds) = $(1/52) + (1/52) = 1/26$.

Example 7.4 (b) A bag contains 6 red and 4 white balls. Two balls are drawn at random from the bag. Find the probability that (i) both are white (ii) one is red and one is white.

Solution There are 10 balls in the bag. Two balls out of 10 balls can be drawn in ${}^{10}C_2 = 45$ ways.

(i) Two white balls out of 4 white balls can be drawn in ${}^{4}C_2 = 6$ ways. Hence $P(\text{two white balls}) = 6/45 = 2/15$.

(ii) One red ball out of 6 red balls and one white ball out of 4 white balls can be drawn in ${}^{6}C_1 \times {}^{4}C_1 = 6 \times 4 = 24$ ways. Hence $P(\text{one red and one white balls}) = 24/45 = 8/15$.

Example 7.5 A box contains 14 identical balls of which 6 are white, 5 red and 3 black. Four balls are drawn. What is the probability that (i) two are red (ii) one is white?

Solution There are 14 balls in the bag. Four balls out of 14 balls can be drawn in ${}^{14}C_4 = 1001$ ways.

(i) Two red balls out of 5 red balls and two balls out of the remaining 9 balls can be drawn in ${}^5C_2 \times {}^9C_2 = 10 \times 36 = 360$ ways. Hence P(two red balls) = 360/1001.

(ii) One white ball out of 6 white balls and 3 balls out of the remaining 8 balls can be drawn in ${}^6C_1 \times {}^8C_3 = 6 \times 56 = 336$ ways. Hence P(one white ball) = 336/1001.

Example 7.6 From a well shuffled pack of 52 cards, two cards are drawn at random. What is the probability that (i) one is king and other a queen (ii) both are aces?

Solution Two cards out of 52 cards can be drawn in ${}^{52}C_2 = 1326$ ways.

(i) One king (out of 4) and one queen (out of 4) can be drawn in ${}^4C_1 \times {}^4C_1 = 4 \times 4 =$ 16 ways. Hence P(one king and one queen) = 16/1326 = 8/663.

(ii) Two aces (out of 4) can be drawn in ${}^4C_2 = 6$ ways. Hence P(both aces) = 6/1326 = 1/221.

The classical definition is criticized on the following grounds:

(i) The term "equally likely" used in the definition means "equally probable". Thus the definition is circular, i.e. it defines probability in terms of itself.

(ii) The definition fails when the assumption of "equally likely" outcomes is not fulfilled, e.g. when a coin or a die is loaded (not fair).

(iii) In some problems, it is not possible to enumerate the number of possible outcomes because they are infinite.

2. Relative Frequency or A Posteriori Definition This definition may be stated as follows:

If an experiment is repeated n times, where n is very large, under uniform conditions and if an event A occurs m times, then the probability of occurrence of A is defined as the relative frequency m/n.

This is called the *estimated* or *empirical probability* because it is based on the past records of event under consideration instead of pure reasoning. For example, what is the probability that a T.B. patient under thirty years of age will recover? No amount of reasoning will tell us the answer. Such a probability is estimated from the past records. If hospital records show that 80 percent of T.B. patients under thirty years of age recover and 20 percent die, the probability of recovery is 80/100 or 0.80.

The above definition is criticized on the following grounds:

(i) It is difficult to ensure that the experiment is repeated under uniform conditions.

(ii) In the real world, the experiment can be repeated only a finite number of times not an infinite number of times.

3. **Axiomatic Approach** In this approach, the probability that an event A will occur is a number $P(A)$ assigned to this event. This number satisfies the following axioms:

(i) For every event A, $P(A) \geq 0$, which means that probability of an event cannot be negative.

(ii) $P(S) = 1$, which means that the probability of an event sure to occur is 1. Thus for every event A, the probability of A lies between 0 and 1, i.e. $0 \leq P(A) \leq 1$.

(iii) If events A and B are mutually exclusive, then $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$.

4. **Subjective Approach** In this approach, probability is based on the *degree of rational belief* in the occurrence of the event, e.g. the belief that it will rain tomorrow. Such an approach is applicable to real world situations like business, economics, etc.

7.7 **Assignment of Probabilities** To evaluate the likely chance of occurrence of events resulting from a *statistical experiment*, each sample point or each possible outcome of an experiment is assigned a weight ranging from 0 to 1. This weight, which measures the likelihood of this occurrence, is called the *probability* of the sample point. Weights or probabilities are assigned to each sample point following the axioms listed below:

(i) The probability assigned to any sample point ranges from 0 to 1.

(ii) The sum of probabilities assigned to all sample points in the sample space must be equal to one.

A weight or probability close to 1 is assigned to an outcome which is likely to occur. On the other hand, weight or probability close to 0 is assigned to an outcome which is not likely to occur.

7.7.1 **Probability of an Event** The probability of an event A, denoted by $P(A)$, is the sum of the probabilities or weights of all the sample points in A. To find the probability of an event A, we sum all the weights assigned to sample points in A. Thus if the sample space consists of n points and the event A consists of m points $(m \leq n)$, the probability of occurrence of A is

$$P(A) = \frac{\text{Number of sample points in the event } A}{\text{Number of sample points in the sample space } S} = \frac{n(A)}{n(S)}$$

which is equivalent to the classical definition.

Example 7.7 A coin is tossed twice. Find the probability that at least one head turns up.

Solution The sample space for this experiment is $S = \{HH, HT, TH, TT\}$. If A denotes the event "at least one head occurs" then $A = \{HH, HT, TH\}$. Since there are thee sample points in A and each sample point has a probability of 1/4, the required probability is $P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$.

Example 7.8 Three coins are tossed, find the probability that (a) exactly 3 heads appear (b) at the most 2 heads appear. (B.I.S.E., Gujranwala 20 2,

Ans. S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(a) Let A be the event that the exactly 3 heads appear

$$A = \{HHH\}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{8} \text{ Ans.}$$

(b) Let B be the event that at the most 2 heads appear

$$B = \{HHT, THH, HTH, HTT, THT, TTH, TTT\}$$

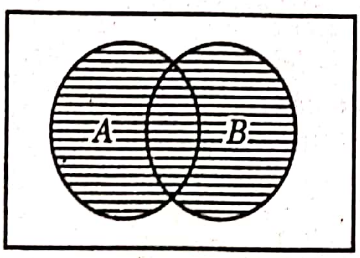
$$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8} \text{ Ans.}$$

Example 7.9 A pair of fair dice is rolled once. Find the probability that the sum of the faces is 9.

Solution Two dice can be thrown in $6 \times 6 = 36$ ways. Thus the sample space for this experiment consists of 36 sample points as listed in Table 7.1. Let A denote the event "sum is 9". Then $A = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$. Since there are four sample points in A, the required probability is $P(A) = 4/36 = 1/9$.

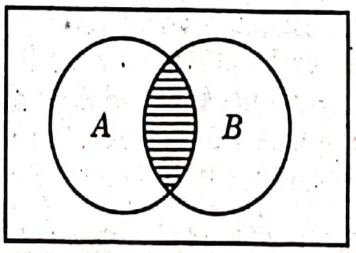
7.8 Union and Intersection of Events The *union* of two events A and B, denoted by $A \cup B$, consists of those outcomes which belong to either A or B or both A and B. The *intersection* of two events A and B, denoted by $A \cap B$, consists of those outcomes which are common to both A and B.

It is helpful to represent the sample space and the combination of events by Venn diagrams. In these diagrams, the sample space S is represented by a rectangle and events are represented by circles inside the rectangle. The events $A \cup B$ and $A \cap B$ are shown by shaded areas in Figures 7.1(a) and 7.1(b) respectively.



$A \cup B$ is shaded

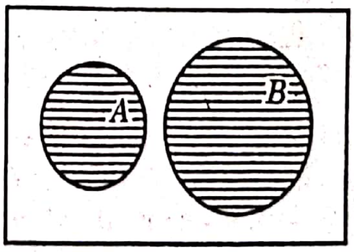
Fig. 7.1(a)



$A \cap B$ is shaded

Fig. 7.1(b)

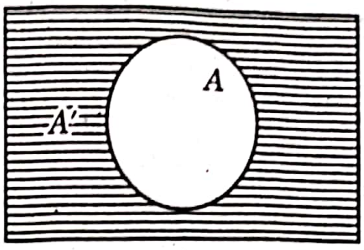
7.9 Mutually Exclusive Events Two events A and B are said to be *mutually exclusive* if the occurrence of one excludes the occurrence of the other. In other words, two events A and B are mutually exclusive if they cannot occur together, i.e. if A occurs, B can not occur and if B occurs, A cannot occur. For mutually exclusive events there are no outcomes or sample points that are common to both A and B in the sample space i.e. $A \cap B = \phi$ as shown by shaded areas in Fig. 7.2.



$A \cup B$ is shaded

Fig. 7.2

7.10 Complementary Events The set of possible outcomes in the sample space S which are not in the event A is the complement of A relative to S and is denoted A' or $\bar{A}$. This is shown by shaded area in Fig.7.3. It is obvious that the events A and A' are mutually exclusive because they do not have any points in common in the sample space.



A' is shaded

Fig. 7.3

7.11 Addition Law of Probability If A and B are any two events belonging to the sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{7.2}$$

This is known as the *addition theorem of probability* which states that the probability of event A or B or both occurring is equal to the probability that A occurs plus the probability that B occurs minus the probability that both events A and B occur together.

Proof From the Venn diagram of Fig.7.4, we note that the set $B-(A \cap B)$ consists of the set of points in B but not in A. Also the set $A \cup B$ is the union of the two mutually exclusive sets A and $B - (A \cap B)$. Thus

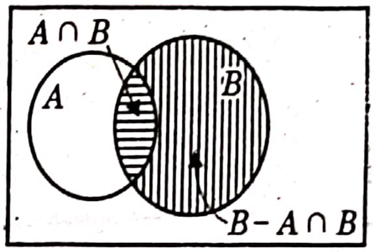


Fig. 7.4

$$A \cup B = A \cup [B - (A \cap B)]$$

or $\quad P(A \cup B) = P\{A \cup [B - (A \cap B)]\}$

Since the sets A and $B - (A \cap B)$ are mutually exclusive, we have

$$P(A \cup B) = P(A) + P[B - (A \cap B)] \tag{i}$$

Again the set B is the union of two mutually exclusive sets $A \cap B$ and $B - (A \cap B)$. Thus

$$B = (A \cap B) \cup [B - (A \cap B)]$$

or $\quad P(B) = P\{(A \cap B) \cup [B - A \cap B]\}$

Since the sets $A \cap B$ and $B - (A \cap B)$ are mutually exclusive, we have

$$P(B) = P(A \cap B) + P[B - (A \cap B)]$$

or $\quad P[B - (A \cap B)] = P(B) - P(A \cap B) \qquad$ (ii)

Making substitution from (ii) into (i) we get the result (7.2):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 7.10 A can solve 80% of the problems in a book while B can solve 60% of the problems. What is the probability that A or B will solve a problem chosen at random. (B.I.S.E., Lahore 1977)

Solution $P(A) = 80/100 = 0.8$, $P(B) = 60/100 = 0.6$ and $P(A \cap B) = (0.8)(0.6) = 0.48$. The probability that A or B will solve the problem is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.48 = 0.92.$$

Example 7.11 A fair die is rolled once. You win the game if the outcome is either even or divisible by 3. What is the probability of winning the game?

Solution Let A be the event that the outcome is even and let B the event that the outcome is divisible by 3. The sample space S and the events A and B consist of the outcomes as

$$S = \{1, 2, 3, 4, 5, 6\},\ A = \{2, 4, 6\},\ B = \{3, 6\} \text{ and } A \cap B = \{6\}.$$

The game is won if either A or B or both A and B occur. Thus according to the addition theorem, the probability of winning the game is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 7.12 A customer enters a food store. The probabilities that the customer buys bread, milk and both bread and milk are respectively 0.60, 0.50 and 0.30. What is the probability that the customer would buy either bread or milk or both?

Solution Let B be the event that customer buys bread and M be the event that the customer buys milk. Thus $(B \cap M)$ is the event that the customer buys both bread and milk. $P(B) = 0.60$, $P(M) = 0.50$ and $P(B \cap M) = 0.30$. Hence according to addition theorem, the probability of the event $B \cup M$ is

$$P(B \cup M) = P(B) + P(M) - P(B \cap M) = 0.60 + 0.50 - 0.30 = 0.80.$$

7.11.1 Addition Theorem for Mutually Exclusive Events We have described events A and B as mutually exclusive when $A \cap B = \phi$. Consequently, for two mutually exclusive events A and B, $P(A \cap B) = P(\phi) = 0$. Thus if events A and B are mutually exclusive (in the sense that they cannot occur at the same time), then the addition formula (7.2) becomes (B.I.S.E., Lahore 2018)

$$P(A \cup B) = P(A) + P(B) \qquad (7.3)$$

This is known as addition theorem of probability for mutually exclusive events. This theorem states that if A and B are two mutually exclusive events, then the probability that the event A or B occur is equal to the probability that A occurs plus the probability that B occurs.

Alternative Proof Let an experiment consist of n sample points and let m_1 and m_2 be respectively the number of sample points in A and B so that $P(A) = m_1/n$ and $P(B) = m_2/n$. Since the events A and B are mutually exclusive, they have no sample points in common, i.e. the sets A and B are distinct and non-overlapping. Thus the number of sample points belonging to $A \cup B$ is $m_1 + m_2$. By definition

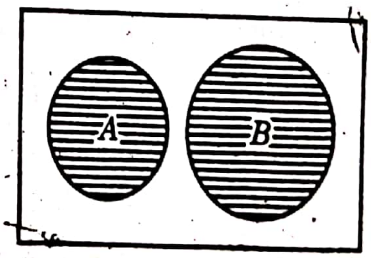


Fig. 7.5

$$P(A \cup B) = \frac{\text{Number of sample points in } A \cup B}{\text{Number of sample points in the sample space } S} = \frac{m_1 + m_2}{n}$$

$$= \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

The addition theorem for mutually exclusive events also applies to more than two events.

In general, if $A_1, A_2, ..., A_n$ are n mutually exclusive events, then the probability of occurrence of either A_1 or A_2 or ... or A_n is

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) \qquad (7.4)$$

Example 7.13 A pair of dice is rolled. What is the probability of getting a total of 7 or 11?

Solution The sample space of this experiment consists of 36 sample points. Let A denote the event "the sum is 7" and B the event "the sum is 11". Then $A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$ and $B = \{(5, 6), (6, 5)\}$. Since A and B consist of six and two sample points respectively, $P(A) = 6/36$ and $P(B) = 2/36$. Thus the probability of getting a total of 7 or 11 is $P(A \cup B) = P(A) + P(B) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$.

Example 7.14 Two cards are drawn from a well-shuffled pack of 52 cards. What is the probability that both cards are of the same colour?

Solution Two cards out of 52 cards can be drawn in ${}^{52}C_2 = 1326$ ways. There are 26 black cards and 26 red cards. Cards of the same colour, i.e. either both black or both red cards can be drawn in ${}^{26}C_2 + {}^{26}C_2 = 325 + 325 = 650$ ways. Thus

$$P(\text{both cards of same colour}) = \frac{650}{1326} = \frac{25}{51}.$$

Example 7.15 A box contains 6 white, 5 red and 4 black balls. Find the probability of getting at least two red balls if four balls are drawn.

Solution There are 15 balls. Four balls out of 15 balls can be drawn in ${}^{15}C_4 = 1365$ ways. At least two red balls mean 2, 3 or 4 red balls. Two red balls (out of 5) and two other balls (out of 10) can be drawn in ${}^{5}C_2 \times {}^{10}C_2 = 10 \times 45 = 450$ ways. Three red balls (out of 5) and one other ball (our of 10) can be drawn in ${}^{5}C_3 \times {}^{10}C_1 = 10 \times 10 = 100$ ways. Four red balls (our of 5) can be drawn in ${}^{5}C_4 = 5$ ways. Thus the number of ways of drawing at least 2 red balls is $450 + 100 + 5 = 555$. Hence

$$P(\text{at least two red balls}) = \frac{555}{1365} = \frac{37}{91}.$$

7.12 Complementary Events If A and A' are complementary events, then $P(A') = 1 - P(A)$. (7.5)

Proof From Fig. 7.6, we note that

$$A + A' = S \text{ or } P(A + A') = P(S)$$

Since A and A' are mutually exclusive and $P(S) = 1$, we have

$$P(A) + P(A') = 1 \text{ or } P(A') = 1 - P(A).$$

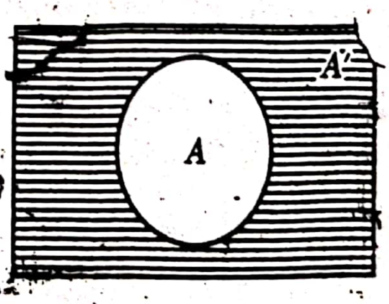


Fig. 7.6

The relationship (7.5) can be used to compute $P(A)$

when A' is simple in nature and its probability $P(A')$ is easy to compute directly.

Example 7.16 A coin is tossed three times. What is the probability of getting at least one head?

Solution A coin can be tossed three times in $2 \times 2 \times 2 = 8$ ways. Thus the sample space consists of 8 sample points:

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

Let A be the event that at least one head appears. Then A' denotes that no head appears i.e. $A' = \{TTT\}$ and $P(A') = 1/8$. Thus

$$P(A) = 1 - P(A') = 1 - (1/8) = 7/8.$$

7.13 Independent and Dependent Events The events A and B are said to be *independent* if the occurrence or non-occurrence of event A does not affect the probability of occurrence of B. This means that irrespective whether event A has occurred or not, the probability of B is going to be the same. If the events A and B are not independent, they are said to be *dependent*.

7.14 Conditional Probability The probability of the occurrence of an event A when it is known that some other event B has already occurred is called the *conditional probability* of A given that the event B has already occurred and is denoted by $P(A|B)$. The symbol $P(A|B)$ is usually read as "the probability that A occurs given that B has already occurred" or simply "the probability of A given B."

To illustrate the concept of conditional probability let us consider an example. Suppose a die is rolled. The sample space of this experiment is $S = \{1, 2, 3, 4, 5, 6\}$. The probability of getting a '6' (event A) is 1/6, i.e. $P(A) = 1/6$. Suppose we are told that on a particular throw of the die, the outcome is an even number (event B). $B = \{2, 4, 6\}$ and the probability of getting a '6' in this reduced sample space is 1/3. This is the conditional probability of the occurrence of the event A (getting a '6') given that B (getting an even number) has occurred, viz. $P(A|B)$. Thus $P(A|B) = 1/3$.

Definition If events A and B belong to the sample space S and if $P(B) \neq 0$, then the conditional probability of A given that B has occurred, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (7.6a)$$

If we interchange the roles of A and B and if $P(A) \neq 0$, then the conditional probability of B given that A has occurred, denoted by $P(B|A)$, is given by

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad (7.6b)$$

Example 7.17 A fair die is rolled once. Given that the outcome is even, what is the probability of getting a number greater than 3?

Solution Let us first define the following sets:

$S = \{1, 2, 3, 4, 5, 6\}$ (sample space)

$A = \{2, 4, 6\}$ (set of even numbers)

given that A has occurred is the same as the unconditional probability of B, that is, $P(B|A) = P(B)$. As it can be shown that the event A is independent of event B whenever event B is independent of event A, i.e. $P(A|B) = P(A)$ whenever $P(B|A) = P(B)$, it is customary to say simply that A and B are independent whenever one is independent of the other. Also when two events A and B are not independent, they are said to be *dependent*. In this case the multiplication formula (7.7) for independent events becomes

$$P(A \cap B) = P(A)\, P(B) \tag{7.8}$$

This theorem states that "If A and B are two independent events, then the probability that both of them occur is equal to the probability that A occurs multiplied by the probability that B occurs."

Alternative Proof Suppose that an event A can occur in m_1 ways out of m possible ways and an event B can occur in n_1 ways out of n possible ways. Then

$$P(A) = m_1/m \text{ and } P(B) = n_1/n$$

Since the events A and B are independent, any one of the m possible outcomes for A can occur with any one of the n possible outcomes for B. Therefore the number of possible outcomes for the event $A \cap B$ is $m \times n$. Similarly, the number of favourable outcomes for the joint occurrence of the event $A \cap B$ is $m_1 \times n_1$. Thus

$$P(A \cap B) = \frac{m_1 \times n_1}{m \times n} = \frac{m_1}{m} \cdot \frac{n_1}{n} = P(A)\, P(B)$$

which proves the theorem.

In view of (7.8), we can say that two events are independent if and only if the probability of their joint occurrence is equal to the product of their individual probabilities.

The multiplication theorem for independent events can be extended to cover the case of more than two events.

In general, if $A_1, A_2, \ldots, A_n$ are n independent events, then the probability of occurrence of A_1 and A_2 ... and A_n is

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2) \ldots P(A_n). \tag{7.9}$$

It is important to remember that mutually exclusive and independent events are different concepts and should not be confused. Mutually exclusive events are such that the occurrence of one event rules out the possibility of occurrence of the other. Thus a coin can fall a head or a tail, but it cannot fall both. Clearly mutually exclusive events are very much dependent and were never concerned with the simultaneous occurrence of two or more of them although we might well be concerned with unions of such events.

Example 7.23 A fair die is tossed twice. Find the probability of getting a 4, 5 or 6 on the first toss and a 1, 2, 3 or 4 on the second.

Solution Let A denote the event "4, 5 or 6 occurs on the first throw of a die" and B the event "1, 2, 3 or 4 occurs on the second throw of a die". Then $P(A) = 3/6 = 1/2$ and $P(B) = 4/6 = 2/3$. Since A and B are independent events,

$$P(A \cap B) = P(A)\, P(B) = (1/2) \cdot (2/3) = 1/3.$$

common to A) and m_3 be the number of sample points common to both A and B, viz. $A \cap B$. Then assuming $m_1 > 0$ and $m_2 > 0$, $P(A) = m_1/n$, $P(B) = m_2/n$ and $P(A \cap B) = m_3/n$. By definition

$$P(A \cap B) = \frac{\text{Number of sample points in } A \cap B}{\text{Number of sample points in } S} = \frac{m_3}{n}$$

Multiplying and dividing on R.H.S. by m_1, we have

$$P(A \cap B) = \frac{m_3}{n} \cdot \frac{m_1}{m_1} = \frac{m_1}{n} \cdot \frac{m_3}{m_1}$$

If event A has occurred, the sample space S is reduced to m_1 sample points, the number of sample points belonging to event A only. Then the ratio m_3/m_1 is the probability of the occurrence of the event B in this reduced sample space, called the *conditional probability* of B given A, i.e. $P(B \mid A)$. Thus

$$P(A \cap B) = P(A)\, P(B \mid A)$$

As it does not matter which event is referred to as A and which event is referred to as B, the above formula can also be written as

$$P(A \cap B) = P(B)\, P(A \mid B)$$

Example 7.21 From a well-shuffled pack of 52 cards two cards are drawn in succession without replacement. Find the probability that the first card is a king and the second is a queen.

Solution Let A be the event that the first card drawn is a king and B be the event that the second card drawn is a queen. The probability of obtaining a king on the first draw is $P(A) = 4/52$. Since the first card drawn is not replaced, we are left with 51 cards in the pack. The probability of obtaining a queen on the second draw, given that the first card drawn is not replaced is $P(B \mid A) = 4/51$. Thus

$$P(A \cap B) = P(A)\, P(B \mid A) = (4/52)\,(4/51) = 4/663.$$

Example 7.22 Two balls are drawn in succession from a bag containing 5 white and 8 black balls. What is the probability that both balls are (i) black (ii) white.

Solution Let W_1 = white ball on the first draw, W_2 = white ball on the second draw, B_1 = black ball on the first draw, B_2 = black ball on the second draw. $P(W_1) = 5/13$ and $P(W_2 \mid W_1) = 4/12$.

$$P(B_1) = 8/13 \text{ and } P(B_2 \mid B_1) = 7/12.$$

(i) $P(W_1 \cap W_2) = P(W_1)\, P(W_2 \mid W_1) = (5/13)\,(4/12) = 5/39.$

(ii) $P(B_1 \cap B_2) = P(B_1)\, P(B_2 \mid B_1) = (8/13)\,(7/12) = 14/39.$

7.15.1 Multiplication Theorem for Independent Events Events A and B are said to be *independent* when the occurrence or non-occurrence of one does not influence the probability of the occurrence of the other. This means that regardless of whether event A has or has not occurred, the probability of B will remain the same. Stated differently, when A and B are independent, the conditional probability of B

Example 7.20 A man tosses two fair dice. What is the conditional probability that the sum of the two dice will be 7 given that (i) the sum is odd (ii) the sum is greater than 6 (iii) the two dice have the same outcome.

Solution The sample space consists of 36 sample points. Let A = sum is 7, B = sum is odd, C = sum is greater than 6 and D = two dice show the same outcome.

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$B = \{(1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (3, 6), (4, 1), (4, 3), (4, 5), (5, 2), (5, 4), (5, 6), (6, 1), (6, 3), (6, 5)\}$$

$$C = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)\}$$

$$D = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$A \cap B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$A \cap C = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

$$A \cap D = \{\ \} = \phi$$

$$P(A) = 6/36,\ P(B) = 18/36,\ P(C) = 21/36,\ P(D) = 6/36.$$

$$P(A \cap B) = 6/36,\ P(A \cap C) = 6/36,\ P(A \cap D) = 0/36 = 0.$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{6/36}{18/36} = \frac{1}{3}.$$

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{6/36}{21/36} = \frac{2}{7}.$$

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{0}{6/36} = 0.$$

7.15 Multiplication Theorem of Probability If A and B are any two events belonging to the sample space S, then

$$P(A \cap B) = P(A)\, P(B \mid A) \tag{7.7}$$

This is known as the *multiplication theorem of probability* which states that the probability that both events A and B occur is equal to the probability that A occurs multiplied by the conditional probability that B occurs given that A has already occurred.

Proof This theorem can be proved from the definition of conditional probability. According to (7.6b), conditional probability of B given A, denoted by $P(B \mid A)$, is defined as

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad (P(A) \neq 0)$$

Multiplying both sides by $P(A)$, we get the result (7.7), i.e. $P(A \cap B) = P(A)\, P(B \mid A)$.

Alternative Proof Let the sample space of an experiment consists of n equally likely outcomes or sample points. Let m_1 be the number of sample points in A (including those common to B), m_2 be the number of sample points in B (including those

$B = \{4, 5, 6\}$ (set of numbers greater than 3)

$A \cap B = \{4, 6\}$ (set of even numbers which are greater than 3)

Since the die is fair, we assign a probability of 1/6 to each sample point. Thus $P(A) = 3/6$ and $P(A \cap B) = 2/6$. Using the definition of conditional probability, we can now determine the probability of getting a number greater than 3 given that the outcome is even as

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{3/6} = \frac{2}{3}.$$

This result can be verified by the fact that out of three even outcomes {2, 4, 6} only two are greater than 3.

Example 7.18 The probability that it will rain on the first day of July is 0.50. The probability that it will rain on both the first and second day of July is 0.40. Given that July 1 is a rainy day, what is the probability of rain on the next day?

Solution Let R_1 denote the event that it will rain on the first day of July and R_2 the event that it will rain on the second day of July. We have $P(R_1) = 0.50$ and $P(R_1 \cap R_2) = 0.40$. Using the definition of conditional probability, we have

$$P(R_2 \mid R_1) = \frac{P(R_1 \cap R_2)}{P(R_1)} = \frac{0.40}{0.50} = 0.80.$$

Example 7.19 A pair of dice is rolled. Let A denote the event "the sum shown is 6" and B the event "the two dice show the same number". Find (i) $P(A \mid B)$ (ii) $P(B \mid A)$ (iii) $P(A \mid B')$.

Solution The sample space contains 36 sample points.

$A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

$B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$B' = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1),$
$(3, 2), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6), (5, 1), (5, 2),$
$(5, 3), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$

$A \cap B = \{(3, 3)\}$ and $A \cap B' = \{(1, 5), (2, 4), (4, 2), (5, 1)\}$

$P(A) = 5/36$, $P(B) = 6/36$, $P(B') = 30/36$, $P(A \cap B) = 1/36$

and $P(A \cap B') = 4/36$.

(i) $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}.$$

(ii) $$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/36}{5/36} = \frac{1}{5}.$$

(iii) $$P(A \mid B') = \frac{P(A \cap B')}{P(B')} = \frac{4/36}{30/36} = \frac{2}{15}.$$

$$4P + 2P + P = 1$$

$$7P = 1, \quad P = \frac{1}{7}$$

$$P(C) = \frac{1}{7};\ P(B) = \frac{2}{7};\ P(A) = \frac{4}{7}$$

2. B and C are mutually exclusive

$$P(B \cup C) = P(B) + P(C) = \frac{2}{7} + \frac{1}{7} = \frac{3}{7}$$

Example 7.35 Find the probability of obtaining a '5' at least once in four throws of a die.

Solution Probability of getting a '5' in a single throw is 1/6, i.e. $p = 1/6$. and $q = 1 - p = 1 - 1/6 = 5/6$. Also $n = 4$. Hence

P(at least one '5' in four throws) = 1 – P(no '5' in four throws)

$$= 1 - {}^5C_0\,(1/6)^0\,(5/6)^4 = 1 - (5/6)^4 = 1 - 625/1296 = 671/1296.$$

7.17 Miscellaneous Examples

Example 7.36 A and B toss a coin alternately on the under-standing that the first to throw a head will win. Find their chances of winning if A takes the fist turn.

Solution Since A and B toss the coin alternately and A takes the first turn, A can win if he throws a head in either the first, third or fifth, ... throw, while B can win if he throws a head in second, fourth, or sixth, ... throw.

Let H_1 and T_1, H_3 and T_3, H_5 and T_5, ... denote the event of getting head and tail respectively on the first, third, fifth, ... throw by A. Let H_2 and T_2, H_4 and T_4, H_6 and T_6, ... denote the event of getting head and tail respectively on the second, fourth, sixth ... throw by B. Then

$$P(H_1) = P(H_2) = \cdots = P(T_1) = P(T_2) = \cdots = 1/2$$

A's chance of winning, denoted by $P(A)$, is

$$P(A) = P(H_1) + P(T_1)P(T_2)P(H_3) + P(T_1)P(T_2)P(T_3)P(T_4)\,P(H_5) + \cdots$$

$$= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \cdots = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots$$

This is an *infinite geometric series* whose sum is given by the formula

$$S_\alpha = \frac{a}{1-r}, \text{ where } a = \text{first term and } r = \text{common ratio.}$$

$$\text{Here } a = \frac{1}{2} \text{ and } r = \frac{(1/2)^3}{(1/2)} = \frac{(1/2)^5}{(1/2)^3} = \cdots = \left(\frac{1}{2}\right)^2$$

$$\text{Therefore} \quad P(A) = \frac{1/2}{1-(1/2)^2} = \frac{1/2}{3/4} = \frac{2}{3}.$$

Example 7.31 A can hit a target four times in 5 shots, B three times in 4 shots and C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit?

Solution $P(A \text{ hits}) = 4/5$, $P(B \text{ hits}) = 3/4$ and $P(C \text{ hits}) = 2/3$. $P(A \text{ fails})$ $1 - 4/5 = 1/5$, $P(B \text{ fails}) = 1 - 3/4 = 1/4$ and $P(C \text{ fails}) = 1 - 2/3 = 1/3$.

P(at least two shots hit)

= P(A hits B hits C fails) or P(A hits B fails C hits) or P(A fails B hits C hits) or P(A hits B hits C hits).

$$= \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3}+\frac{4}{5}\cdot\frac{1}{4}\cdot\frac{2}{3}+\frac{1}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}+\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}=\frac{1}{5}+\frac{2}{15}+\frac{1}{10}+\frac{2}{5}=\frac{25}{30}=\frac{5}{6}.$$

Example 7.32 Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected from each group. Find the probability that the three selected consist of 1 girl and 2 boys.

Solution P(1 girl and 2 boys)

= P(girl from first, boy from second and boy from third group) or P(boy from first, girl from second and boy from third group) or P(boy from first, boy from second and girl from third group)

$$=\frac{3}{4}\cdot\frac{2}{4}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{2}{4}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{2}{4}\cdot\frac{1}{4}=\frac{9}{32}+\frac{3}{32}+\frac{1}{32}=\frac{13}{32}.$$

7.16 Repeated Independent Events If the probability of occurrence of an event (success) is p and the probability of its non-occurrence (failure) is $q = 1 - p$, then the probability of exactly x successes in n independent repetitions is given by ${}^nC_x p^x q^{n-x}$.

Example 7.33 Six dice are rolled once (or one die is rolled six times). Find the probability of getting 3 sixes. (B.I.S.E., Gujranwala 2015)

Solution The probability of getting a six in one throw of a die is $1/6$, i.e. $p = 1/6$ and $q = 1 - p = 1 - 1/6 = 5/6$. Also $n = 6$. Hence

$$P(3 \text{ sixes}) = {}^6C_3\,(1/6)^3\,(5/6)^3 = 625/11664.$$

Example 7.34 Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C.

1. What are their respective chances of winning.
2. What is the probability that B are C wins.

Ans. Let P(C) = P

P(B) = 2P(C) = 2P

and P(A) = 2P(B) = 2 × 2P = 4P

1. A, B and C are mutually exclusive and collectively exhaustive.

$\therefore$ $P(A \cup B \cup C) = P(S)$

$\Rightarrow$ $P(A) + P(B) + P(C) = 1$

(ii) If the first card is not replaced, then A and B are dependent events. Then

$$P(A \cap B) = P(A)\, P(B \mid A) = (4/52)\,(4/51) = 4/663.$$

Here $P(B \mid A)$ is the probability of a king on second draw given that first card drawn is a queen.

Example 7.28 A bag contains 8 red, 5 white and 7 black balls. If three balls are drawn, find the probability that they are drawn in the order red, white, black if each ball is (i) replaced (ii) not replaced.

Solution Let R denote the event "red ball on first draw", W the event "white ball on second draw" and B the event "black ball on third draw". The total number of balls in the bag is 20.

(i) If each ball is replaced, then R, W and B are independent events. Then

$$P(R \cap W \cap B) = P(R)\, P(W)\, P(B) = \frac{8}{20} \cdot \frac{5}{20} \cdot \frac{7}{20} = \frac{7}{200}.$$

(ii) If each ball is not replaced, then R, W and B are dependent events. On the second and third draws, we will have 19 and 18 balls respectively. Therefore

$$P(R \cap W \cap B) = P(R)\, P(W \mid R)\, P(B \mid WR) = \frac{8}{20} \cdot \frac{5}{19} \cdot \frac{7}{18} = \frac{7}{171}.$$

Example 7.29 A bag contains 4 white and 3 black balls. Another bag contains 5 black and 3 white balls. A ball is drawn from one of the bags selected at random. What is the probability that it is white?

Solution Since there are two bags, the probability of selecting a bag is 1/2. The first bag contains 7 balls of which 4 are white. Thus the probability of drawing a white ball from the first bag is 4/7. Since the second bag contains 8 balls out of which 3 are white, the probability of drawing a white ball from the second bag is 3/8.

$$P(\text{first bag is selected and a white ball is drawn}) = \frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$$

$$P(\text{second bag is selected and a white ball is drawn}) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$$

$$P(\text{white ball from either bag}) = \frac{2}{7} + \frac{3}{16} = \frac{53}{112}.$$

29

Example 7.30 In Example 7.28, if one ball is drawn from each bag, find the probability that the balls are of the same colour.

Solution Let W_1 denote the event "white ball from the first bag", W_2 the event "white ball from the second bag", B_1 the event "black ball from the first bag" and B_2 the event "black ball from the second bag". By "balls of the same colour", we mean either white from the first bag and white from the second bag or black from the first bag and black from the second bag. Thus

$$P(\text{balls of same colour}) = P(W_1 \cap W_2) + P(B_1 \cap B_2) = P(W_1)\, P(W_2) + P(B_1)\, P(B_2)$$

$$= \frac{4}{7} \cdot \frac{3}{8} + \frac{3}{7} \cdot \frac{5}{8} = \frac{12}{56} + \frac{15}{56} = \frac{27}{56}.$$

Example 7.24 A pair of dice is rolled twice. What is the probability of getting a total of 7 on the first throw and a total of 11 on the second?(B.I.S.E., Gujranwala 2004)

Solution Let A denote the event "getting a total of 7 on the first throw" and B the event "getting a total of 11 on the second throw". From Example 7.14, $P(A) = 1/6$ and $P(B) = 1/18$. Since the events A and B are independent,

$$P(A \cap B) = P(A)\,P(B) = (1/6)\,(1/18) = 1/108.$$

Example 7.25 A pair of dice is thrown twice. Find the probability of getting a total of 7 and 11.

Solution Let A_1 and A_2 denote the events "a 7 appears on the first throw" and "a 7 appears on the second throw" and B_1 and B_2 the events "an 11 appears on the first throw" and "an 11 appear on the second throw". From Example 7.14, $P(A_1) = P(A_2) = 1/6$ and $P(B_1) = P(B_2) = 1/18$.

$$\begin{aligned} P(\text{total of 7 and 11}) &= P(\text{total of 7 on first die and total of 11 on the second die}) \\ &\quad + P(\text{total of 11 on first die and total of 7 on the second die}) \end{aligned}$$

$$P(A_1 \cap B_2) + P(B_1 \cap A_2) = P(A_1)\,P(B_2) + P(B_1)\,P(A_2) = \frac{1}{6}\cdot\frac{1}{18} + \frac{1}{18}\cdot\frac{1}{6}$$

$$= \frac{1}{108} + \frac{1}{108} = \frac{1}{54}.$$

Example 7.26 The probability that A will be alive after 10 years is 5/7 and B will be alive after 10 years is 7/9. Find the probability that (i) both of them will die (ii) A will be alive and B dead (iii) B will be alive and A dead (iv) both will be alive after 10 years. (B.I.S,E., Lahore 1978)

Solution Let A and B denote the events 'A alive' and 'B alive' respectively. Then A' and B' denote 'A dead' and 'B dead' respectively.

$$P(A) = 5/7 \text{ and } P(B) = 7/9,$$

$$P(A') = 1 - P(A) = 1 - 5/7 = 2/7 \text{ and } P(B') = 1 - P(B) = 1 - 7/9 = 2/9$$

(i) $P(\text{both will die}) = P(A' \cap B') = P(A')\,P(B') = (2/7)\,(2/9) = 4/63.$

(ii) $P(A \text{ will alive and } B \text{ dead}) = P(A \cap B') = P(A)\,P(B') = (5/7)\,(2/9) = 10/63.$

(iii) $P(B \text{ will be alive and } A \text{ dead}) = P(A' \cap B) = P(A')\,P(B) = (2/7)\,(7/9) = 2/9.$

(iv) $P(\text{both will be alive}) = P(A \cap B) = P(A)\,P(B) = (5/7)\,(7/9) = 5/9.$

Example 7.27 From a well-shuffled pack of 52 cards, two cards are drawn. Find the probability that the first card is a queen and the second a king if the first card is (i) replaced (ii) not replaced.

Solution Let A denote the event "first card is a queen" and let B denote the event "second card is a king."

(i) If the first card is replaced, A and B are independent events. Then

$$P(A \cap B) = P(A)\,P(B) = (4/52)\,(4/52) = 1/169.$$