

Some Important Derivative Formulas

• $\frac{d}{dx}c = 0$ where c is constant	• $\frac{d}{dx}x^n = nx^{n-1}$
• $\frac{d}{dx}\sin x = \cos x$	• $\frac{d}{dx}\tan x = \sec^2 x$
• $\frac{d}{dx}\cos x = -\sin x$	• $\frac{d}{dx}\cot x = -\csc^2 x$
• $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$	• $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$
• $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$	• $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$
	• $\frac{d}{dx}\csc x = -\csc x \cot x$
	• $\frac{d}{dx}\sec x = \sec x \tan x$
	• $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$
	• $\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

Question # 1

Difference the following trigonometric functions from the first principles.

- | | | | |
|----------------|----------------------|---------------------------|-----------------|
| (i) $\sin 2x$ | (ii) $\tan 3x$ | (iii) $\sin 2x + \cos 2x$ | (iv) $\cos x^2$ |
| (v) $\tan^2 x$ | (vi) $\sqrt{\tan x}$ | (vii) $\cos \sqrt{x}$ | |

Solution

(i) Suppose $y = \sin 2x$

$$\begin{aligned}\Rightarrow y + \delta y &= \sin 2(x + \delta x) \\ \Rightarrow \delta y &= \sin 2(x + \delta x) - y \\ &= \sin 2(x + \delta x) - \sin 2x\end{aligned}$$

Dividing both sides by δx

$$\begin{aligned}\frac{\delta y}{\delta x} &= \frac{\sin(2x + 2\delta x) - \sin 2x}{\delta x} \\ &= \frac{2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right)\sin\left(\frac{2x + 2\delta x - 2x}{2}\right)}{\delta x} \\ &= \frac{2\cos(2x + \delta x)\sin(\delta x)}{\delta x}\end{aligned}$$

Taking limit as $\delta x \rightarrow 0$

$$\begin{aligned}\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \frac{2\cos(2x + \delta x)\sin(\delta x)}{\delta x} \\ \frac{dy}{dx} &= 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \cdot \frac{\sin(\delta x)}{\delta x} \\ &= 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin(\delta x)}{\delta x} \\ &= 2 \cos(2x + 0) \cdot (1) \\ &\quad \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1\end{aligned}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x}$$

(ii) Let $y = \tan 3x$

$$\Rightarrow y + \delta y = \tan 3(x + \delta x)$$

$$\Rightarrow \delta y = \tan(3x + 3\delta x) - \tan 3x$$

$$= \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x} = \frac{\sin(3x + 3\delta x)\cos 3x - \cos(3x + 3\delta x)\sin 3x}{\cos(3x + 3\delta x)\cos 3x}$$

$$= \frac{\sin(3x + 3\delta x - 3x)}{\cos(3x + 3\delta x)\cos 3x} = \frac{\sin(3\delta x)}{\cos(3x + 3\delta x)\cos 3x}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \cdot \frac{\sin(3\delta x)}{\cos(3x + 3\delta x)\cos 3x}$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{\sin(3\delta x)}{\delta x \cos(3x + 3\delta x)\cos 3x}$$

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\sin(3\delta x)}{\delta x} \cdot \frac{1}{\cos(3x + 3\delta x)\cos 3x} \cdot \frac{3}{3} \quad \text{× ing and ÷ ing 3 on R.H.S}$$

$$= 3 \lim_{\delta x \rightarrow 0} \frac{\sin(3\delta x)}{3\delta x} \cdot \lim_{\delta x \rightarrow 0} \frac{1}{\cos(3x + 3\delta x)\cos 3x}$$

$$= 3(1) \cdot \frac{1}{\cos(3x + 3(0))\cos 3x}$$

$$= \frac{3}{\cos 3x \cos 3x} = \frac{3}{\cos^2 3x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 3\sec^2 3x}$$

(iii) Let $y = \sin 2x + \cos 2x$

$$\Rightarrow y + \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x)$$

$$\Rightarrow \delta y = \sin 2(x + \delta x) + \cos 2(x + \delta x) - y$$

$$= \sin 2(x + \delta x) + \cos 2(x + \delta x) - \sin 2x - \cos 2x$$

$$= [\sin(2x + 2\delta x) - \sin 2x] + [\cos(2x + 2\delta x) - \cos 2x]$$

$$= \left[2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right) \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) \right]$$

$$+ \left[-2\sin\left(\frac{2x + 2\delta x - 2x}{2}\right) \sin\left(\frac{2x + 2\delta x - 2x}{2}\right) \right]$$

$$= 2\cos(2x + \delta x)\sin(\delta x) - 2\sin(2x + \delta x)\sin(\delta x)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} [2\cos(2x + \delta x)\sin(\delta x) - 2\sin(2x + \delta x)\sin(\delta x)]$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} [2\cos(2x + \delta x)\sin(\delta x) - 2\sin(2x + \delta x)\sin(\delta x)]$$

$$\begin{aligned} \frac{dy}{dx} &= 2 \lim_{\delta x \rightarrow 0} \cos(2x + \delta x) \lim_{\delta x \rightarrow 0} \frac{\sin(\delta x)}{\delta x} - 2 \lim_{\delta x \rightarrow 0} \sin(2x + \delta x) \lim_{\delta x \rightarrow 0} \frac{\sin(\delta x)}{\delta x} \\ &= 2\cos(2x + 0) \cdot (1) - 2\sin(2x + 0) \cdot (1) \quad \text{Since } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \end{aligned}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2\cos 2x - 2\sin 2x}$$

$$(iv) \quad \text{Let} \quad y = \cos x^2$$

$$\Rightarrow y + \delta y = \cos(x + \delta x)^2$$

$$\Rightarrow \delta y = \cos(x + \delta x)^2 - \cos x^2$$

$$\begin{aligned} &= -2 \sin\left(\frac{(x + \delta x)^2 + x^2}{2}\right) \sin\left(\frac{(x + \delta x)^2 - x^2}{2}\right) \\ &= -2 \sin\left(\frac{x^2 + 2x\delta x + \delta x^2 + x^2}{2}\right) \sin\left(\frac{x^2 + 2x\delta x + \delta x^2 - x^2}{2}\right) \\ &= -2 \sin\left(\frac{2x^2 + 2x\delta x + \delta x^2}{2}\right) \cdot \sin\left(\frac{2x\delta x + \delta x^2}{2}\right) \\ &= -2 \sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right) \cdot \sin\left(x + \frac{\delta x}{2}\right) \delta x \end{aligned}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{1}{\delta x} \cdot 2 \sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right) \cdot \sin\left(x + \frac{\delta x}{2}\right) \delta x$$

\times ing and \div ing $\left(x + \frac{\delta x}{2}\right)$ on R.H.S

$$\Rightarrow \frac{\delta y}{\delta x} = -\left[\frac{2}{\delta x} \sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right) \cdot \sin\left(x + \frac{\delta x}{2}\right) \delta x \right] \cdot \frac{\left(x + \frac{\delta x}{2}\right)}{\left(x + \frac{\delta x}{2}\right)}$$

$$= -\left[2 \sin\left(x^2 + x\delta x + \frac{\delta x^2}{2}\right) \cdot \frac{\sin\left(x + \frac{\delta x}{2}\right) \delta x}{\left(x + \frac{\delta x}{2}\right) \delta x} \right] \cdot \left(x + \frac{\delta x}{2}\right)$$

Taking limit as $\delta x \rightarrow 0$

$$\begin{aligned}
 \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= - \lim_{\delta x \rightarrow 0} \left[2 \sin \left(x^2 + x\delta x + \frac{\delta x^2}{2} \right) \cdot \frac{\sin \left(x + \frac{\delta x}{2} \right) \delta x}{\left(x + \frac{\delta x}{2} \right) \delta x} \right] \cdot \left(x + \frac{\delta x}{2} \right) \\
 \Rightarrow \frac{dy}{dx} &= - 2 \lim_{\delta x \rightarrow 0} \sin \left(x^2 + x\delta x + \frac{\delta x^2}{2} \right) \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \left(x + \frac{\delta x}{2} \right) \delta x}{\left(x + \frac{\delta x}{2} \right) \delta x} \cdot \lim_{\delta x \rightarrow 0} \left(x + \frac{\delta x}{2} \right) \\
 &= - 2 \sin(x^2 + (0) + (0)) \cdot (1) \cdot (x + (0)) \\
 \Rightarrow \boxed{\frac{dy}{dx} = - 2x \sin x^2}
 \end{aligned}$$

(v) Let $y = \tan^2 x$

$$\Rightarrow y + \delta y = \tan^2(x + \delta x)$$

$$\Rightarrow \delta y = \tan^2(x + \delta x) - \tan^2 x$$

$$= (\tan(x + \delta x) + \tan x) \cdot (\tan(x + \delta x) - \tan x)$$

$$= (\tan(x + \delta x) + \tan x) \cdot \left(\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right)$$

$$= (\tan(x + \delta x) + \tan x) \cdot \left(\frac{\sin(x + \delta x) \cos x - \sin x \cos(x + \delta x)}{\cos(x + \delta x) \cos x} \right)$$

$$= (\tan(x + \delta x) + \tan x) \cdot \left(\frac{\sin(x + \delta x - x)}{\cos(x + \delta x) \cos x} \right)$$

$$= (\tan(x + \delta x) + \tan x) \cdot \left(\frac{\sin \delta x}{\cos(x + \delta x) \cos x} \right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} (\tan(x + \delta x) + \tan x) \cdot \left(\frac{\sin \delta x}{\cos(x + \delta x) \cos x} \right)$$

Taking limit when $\delta x \rightarrow 0$

$$\begin{aligned}
 \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \frac{1}{\delta x} (\tan(x + \delta x) + \tan x) \cdot \left(\frac{\sin \delta x}{\cos(x + \delta x) \cos x} \right) \\
 \Rightarrow \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\tan(x + \delta x) + \tan x}{\cos(x + \delta x) \cos x} \right) \cdot \lim_{\delta x \rightarrow 0} \left(\frac{\sin \delta x}{\delta x} \right) \\
 &= \left(\frac{\tan(x + 0) + \tan x}{\cos(x + 0) \cos x} \right) \cdot (1) = \frac{\tan x + \tan x}{\cos x \cdot \cos x} = \frac{2 \tan x}{\cos^2 x}
 \end{aligned}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2 \tan x \sec^2 x}$$

(vi) Let $y = \sqrt{\tan x}$

$$\Rightarrow y + \delta y = \sqrt{\tan(x + \delta x)}$$

$$\Rightarrow \delta y = \sqrt{\tan(x + \delta x)} - \sqrt{\tan x}$$

$$= (\sqrt{\tan(x + \delta x)} - \sqrt{\tan x}) \cdot \left(\frac{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \right)$$

$$= \frac{\tan(x + \delta x) - \tan x}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}}$$

$$= \frac{1}{\sqrt{\tan(x + \delta x)} + \sqrt{\tan x}} \cdot \left(\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x} \right)$$

Now do yourself as above.

(vii) Let $y = \cos \sqrt{x}$

$$\Rightarrow y + \delta y = \cos \sqrt{x + \delta x}$$

$$\Rightarrow \delta y = \cos \sqrt{x + \delta x} - \cos \sqrt{x}$$

$$= -2 \sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)$$

Dividing by δx

$$\frac{\delta y}{\delta x} = -\frac{2 \sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{\delta x}$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = -2 \lim_{\delta x \rightarrow 0} \frac{\sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{\delta x}$$

As $\delta x = (\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x})$, putting in above

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= -2 \lim_{\delta x \rightarrow 0} \frac{\sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right) \sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{(\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x})} \\ &= -\lim_{\delta x \rightarrow 0} \frac{\sin \left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2} \right)}{(\sqrt{x + \delta x} + \sqrt{x})} \cdot \lim_{\delta x \rightarrow 0} \frac{\sin \left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)}{\left(\frac{\sqrt{x + \delta x} - \sqrt{x}}{2} \right)} \end{aligned}$$

$$= -\frac{\sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right)}{(\sqrt{x+0}+\sqrt{x})} \cdot (1) \Rightarrow \boxed{\frac{dy}{dx} = -\frac{\sin(\sqrt{x})}{2\sqrt{x}}}$$

Question # 2

Differentiate the following w.r.t. the variable involved.

- (i)
- $x^2 \sec 4x$
- (ii)
- $\tan^3 \theta \ sec^2 \theta$
- (iii)
- $(\sin 2\theta - \cos 3\theta)^2$
- (iv)
- $\cos \sqrt{x} + \sqrt{\sin x}$

Solution

- (i) Assume
- $y = x^2 \sec 4x$

Differentiating w.r.t x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} x^2 \sec 4x \\ &= x^2 \frac{d}{dx} \sec 4x + \sec 4x \frac{d}{dx} x^2 \\ &= x^2 \sec 4x \tan 4x \frac{d}{dx}(4x) + \sec 4x (2x) \\ &= x^2 \sec 4x \tan 4x (4) + 2x \sec 4x \\ &= 2x \sec 4x (2x \tan 4x + 1) \end{aligned}$$

- (ii) Let
- $y = \tan^3 \theta \ sec^2 \theta$

Diff. w.r.t θ

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d}{d\theta} \tan^3 \theta \ sec^2 \theta \\ &= \tan^3 \theta \frac{d}{d\theta} \sec^2 \theta + \sec^2 \theta \frac{d}{d\theta} \tan^3 \theta \\ &= \tan^3 \theta \left(2 \sec \theta \frac{d}{d\theta} \sec \theta \right) + \sec^2 \theta \left(3 \tan^2 \theta \frac{d}{d\theta} \tan \theta \right) \\ &= \tan^3 \theta (2 \sec \theta \cdot \sec \theta \tan \theta) + \sec^2 \theta (3 \tan^2 \theta \cdot \sec^2 \theta) \\ &= \sec^2 \theta \tan^2 \theta (2 \tan^2 \theta + 3 \sec^2 \theta) \end{aligned}$$

- (iii) Let
- $y = (\sin 2\theta - \cos 3\theta)^2$

Diff. w.r.t θ

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta)^2 \\ &= 2(\sin 2\theta - \cos 3\theta) \frac{d}{d\theta} (\sin 2\theta - \cos 3\theta) \\ &= 2(\sin 2\theta - \cos 3\theta) \left(\cos 2\theta \cdot \frac{d}{d\theta}(2\theta) + \sin 3\theta \cdot \frac{d}{d\theta}(3\theta) \right) \\ &= 2(\sin 2\theta - \cos 3\theta) (\cos 2\theta \cdot (2) + \sin 3\theta \cdot (3)) \end{aligned}$$

$$= 2(\sin 2\theta - \cos 3\theta)(2\cos 2\theta + 3\sin 3\theta)$$

(iv) Let $y = \cos \sqrt{x} + \sqrt{\sin x}$

$$= \cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}}$$

Diff. w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\cos(x)^{\frac{1}{2}} + (\sin x)^{\frac{1}{2}} \right) \\ &= -\sin(x)^{\frac{1}{2}} \frac{d}{dx} x^{\frac{1}{2}} + \frac{1}{2} (\sin x)^{-\frac{1}{2}} \frac{d}{dx} (\sin x) \\ &= -\sin(x)^{\frac{1}{2}} \left(\frac{1}{2} x^{-\frac{1}{2}} \right) + \frac{1}{2} (\sin x)^{-\frac{1}{2}} (\cos x) \\ &= \frac{1}{2} \left(\frac{\cos x}{\sqrt{\sin x}} - \frac{\sin \sqrt{x}}{\sqrt{x}} \right)\end{aligned}$$

Question # 3

Find $\frac{dy}{dx}$ if

$$(i) y = x \cos y$$

$$(ii) x = y \cos y$$

Solution

(i) Since $y = x \cos y$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} x \cos y \\ &= x \frac{d}{dx} \cos y + \cos y \frac{dx}{dx} \\ &= x(-\sin y) \frac{dy}{dx} + \cos y(1) \\ \Rightarrow \frac{dy}{dx} + x \sin y \frac{dy}{dx} &= \cos y \quad \Rightarrow (1+x \sin y) \frac{dy}{dx} = \cos y \\ \Rightarrow \frac{dy}{dx} &= \frac{\cos y}{1+x \sin y}\end{aligned}$$

(ii)

Do yourself as above

Question # 4

Find the derivative w.r.t. "x"

$$(i) \cos \sqrt{\frac{1+x}{1+2x}}$$

$$(ii) \sin \sqrt{\frac{1+2x}{1+x}}$$

Solution

$$(i) \text{ Since } y = \cos \sqrt{\frac{1+x}{1+2x}}$$

Diff. w.r.t x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \cos \sqrt{\frac{1+x}{1+2x}} \\
&= -\sin \sqrt{\frac{1+x}{1+2x}} \frac{d}{dx} \left(\sqrt{\frac{1+x}{1+2x}} \right) = -\sin \sqrt{\frac{1+x}{1+2x}} \frac{d}{dx} \left(\frac{1+x}{1+2x} \right)^{\frac{1}{2}} \\
&= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left(\frac{1+x}{1+2x} \right)^{-\frac{1}{2}} \frac{d}{dx} \left(\frac{1+x}{1+2x} \right) \\
&= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{1}{2} \left(\frac{1+2x}{1+x} \right)^{\frac{1}{2}} \left(\frac{(1+2x) \frac{d}{dx}(1+x) - (1+x) \frac{d}{dx}(1+2x)}{(1+2x)^2} \right) \\
&= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}} \left(\frac{(1+2x)(1) - (1+x)(2)}{(1+2x)^2} \right) \\
&= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}} \left(\frac{1+2x-2-2x}{(1+2x)^2} \right) \\
&= -\sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}} \left(\frac{-1}{(1+2x)^2} \right) \\
&= \frac{1}{2} \sin \sqrt{\frac{1+x}{1+2x}} \cdot \frac{(1+2x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}(1+2x)^{\frac{2-1}{2}}} \\
\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{1+x}(1+2x)^{\frac{3}{2}}} \sin \sqrt{\frac{1+x}{1+2x}}}
\end{aligned}$$

(ii)

Do yourself as above.

Question # 5

Differentiate

(i) $\sin x$ w.r.t. $\cot x$ (ii) $\sin^2 x$ w.r.t. $\cos^4 x$ **Solution**(i) Let $y = \sin x$ and $u = \cot x$ Diff. y w.r.t x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \sin x \\
&= \cos x
\end{aligned}$$

Now diff. u w.r.t x

$$\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx} \cot x \\
&= -\csc^2 x
\end{aligned}$$

$$\Rightarrow \frac{dx}{du} = -\frac{1}{\csc^2 x}$$

$$= -\sin^2 x$$

Now by chain rule

$$\begin{aligned}\frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ &= (\cos x)(-\sin^2 x) = -\sin^2 x \cos x\end{aligned}$$

(ii) Let $y = \sin^2 x$ and $u = \cos^4 x$

Diff. y w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \sin^2 x \\ &= 2\sin x \frac{d}{dx}(\sin x) = 2\sin x \cos x\end{aligned}$$

Now diff. u w.r.t x

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx} \cos^4 x \\ &= 4\cos^3 x \frac{d}{dx}(\cos x) = 4\cos^3 x(-\sin x) \\ &= -4\sin x \cos^3 x \\ \Rightarrow \frac{dx}{du} &= -\frac{1}{4\sin x \cos^3 x}\end{aligned}$$

Now by chain rule

$$\begin{aligned}\frac{dy}{du} &= \frac{dy}{dx} \cdot \frac{dx}{du} \\ &= (2\sin x \cos x) \left(-\frac{1}{4\sin x \cos^3 x} \right) \\ &= -\frac{1}{2} \sec^2 x\end{aligned}$$

Question # 6

If $\tan y(1 + \tan x) = 1 - \tan x$, show that $\frac{dy}{dx} = -1$

Solution

Since $\tan y(1 + \tan x) = 1 - \tan x$

$$\begin{aligned}\Rightarrow \tan y &= \frac{1 - \tan x}{1 + \tan x} \\ &= \frac{1 - \tan x}{1 + 1 \cdot \tan x} = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \cdot \tan x} = \tan \left(\frac{\pi}{4} - x \right) \\ \Rightarrow y &= \frac{\pi}{4} - x\end{aligned}$$

Diff. w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\pi}{4} - x\right) \\ &= 0 - 1 \Rightarrow \frac{dy}{dx} = -1\end{aligned}$$

Question # 7If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$, Prove that $(2y-1)\frac{dy}{dx} = \sec^2 x$.**Solution**

Since $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}$

Taking square on both sides

$$\begin{aligned}y^2 &= \tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}} \\ &= \tan x + \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots \infty}}}\end{aligned}$$

$$\Rightarrow y^2 = \tan x + y$$

Diff. w.r.t x

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}(\tan x + y) \\ \Rightarrow 2y\frac{dy}{dx} &= \sec^2 x + \frac{dy}{dx} \Rightarrow 2y\frac{dy}{dx} - \frac{dy}{dx} &= \sec^2 x \\ \Rightarrow (2y-1)\frac{dy}{dx} &= \sec^2 x\end{aligned}$$

Question # 8If $x = a\cos^3 \theta$, $y = b\sin^3 \theta$, Show that $a\frac{dy}{dx} + b\tan \theta = 0$ **Solution**

$x = a\cos^3 \theta$, $y = b\sin^3 \theta$

Diff. x w.r.t θ

$$\begin{aligned}\frac{dx}{d\theta} &= \frac{d}{d\theta}(a\cos^3 \theta) \\ &= a \cdot 3\cos^2 \theta \frac{d}{d\theta}(\cos \theta) = 3a\cos^2 \theta(-\sin \theta) \\ \Rightarrow \frac{dx}{d\theta} &= -3a\sin \theta \cos^2 \theta \Rightarrow \frac{d\theta}{dx} = \frac{-1}{3a\sin \theta \cos^2 \theta}\end{aligned}$$

Now diff. y w.r.t θ

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta}(b\sin^3 \theta) \\ &= b \cdot 3\sin^2 \theta \frac{d}{d\theta}(\sin \theta) = 3b\sin^2 \theta \cos \theta\end{aligned}$$

Now by chain rule

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \\
 &= 3b\sin^2\theta\cos\theta \cdot -\frac{1}{3a\sin\theta\cos^2\theta} \\
 &= -\frac{b}{a}\tan\theta \\
 \Rightarrow a\frac{dy}{dx} &= -b\tan\theta \quad \Rightarrow a\frac{dy}{dx} + b\tan\theta = 0
 \end{aligned}$$

Question # 9

Find $\frac{dy}{dx}$ if $x = a(\cos t + \sin t)$ and $y = a(\sin t - t \cos t)$

Solution

$$x = a(\cos t + \sin t) \text{ and } y = a(\sin t - t \cos t)$$

Do yourself

Derivative of inverse trigonometric formulas

$$(i) \quad \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

See proof on book page 76

$$(ii) \quad \frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$$

Proof

$$\begin{aligned} \text{Let } y &= \cos^{-1} x && \text{where } x \in [0, \pi] \\ \Rightarrow \cos y &= x \end{aligned}$$

Diff. w.r.t x

$$\begin{aligned}
 \frac{d}{dx} \cos y &= \frac{dx}{dx} \quad \Rightarrow -\sin y \frac{dy}{dx} = 1 \\
 \frac{dy}{dx} &= -\frac{1}{\sin y} \\
 &= -\frac{1}{\sqrt{1-\cos^2 y}} \quad \text{Since } \sin y \text{ is positive for } x \in [0, \pi] \\
 &= -\frac{1}{\sqrt{1-x^2}}
 \end{aligned}$$

$$(iii) \quad \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

See proof on book at page 77

$$(iv) \quad \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$$

Proof

$$\begin{aligned} \text{Let } y &= \cot^{-1} x \\ \Rightarrow \cot y &= x \end{aligned}$$

Diff. w.r.t x

$$\begin{aligned}\frac{d}{dx} \cot y &= \frac{d}{dx} x \Rightarrow -\csc^2 y \frac{dy}{dx} = 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{-1}{\csc^2 y} \\ &= \frac{-1}{1 + \cot^2 y} \quad \because 1 + \cot^2 y = \csc^2 y \\ \Rightarrow \frac{dy}{dx} &= -\frac{1}{1 + x^2}\end{aligned}$$

(v) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$

Proof

Let $y = \sec^{-1} x \Rightarrow \sec y = x$

Diff. w.r.t x

$$\begin{aligned}\frac{d}{dx} \sec y &= \frac{d}{dx} x \Rightarrow \sec y \tan y \frac{dy}{dx} = 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sec y \tan y} \\ &= \frac{1}{\sec y \sqrt{\sec^2 y - 1}} \quad \because 1 + \tan^2 y = \sec^2 y \\ \Rightarrow \frac{d}{dx} \sec^{-1} x &= \frac{1}{x\sqrt{x^2 - 1}} \quad \because \sec y = x\end{aligned}$$

(vi) $\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$

See on book at page 77

Question # 10

Differentiate w.r.t. "x"

- | | | |
|--|--|--|
| (i) $\cos^{-1} \frac{x}{a}$ | (ii) $\cot^{-1} \frac{x}{a}$ | (iii) $\frac{1}{a} \sin^{-1} \frac{a}{x}$ |
| (iv) $\sin^{-1} \sqrt{1-x^2}$ | (v) $\sec^{-1} \left(\frac{x^2+1}{x^2-1} \right)$ | (vi) $\cot^{-1} \left(\frac{2x}{1-x^2} \right)$ |
| (vii) $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$ | | |

Solution

(i) Let $y = \cos^{-1} \frac{x}{a}$

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \cos^{-1} \frac{x}{a}$$

$$\begin{aligned}
&= \frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{a} \right) = \frac{-1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a} \frac{d}{dx} x \\
&= \frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a} (1) = \frac{-a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a} = \frac{-1}{\sqrt{a^2-x^2}} \quad \text{Ans}
\end{aligned}$$

(ii) Let $y = \cot^{-1} \frac{x}{a}$

Diff w.r.t x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \cot^{-1} \frac{x}{a} \\
&= \frac{-1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx} \left(\frac{x}{a} \right) = \frac{-1}{\frac{a^2+x^2}{a^2}} \cdot \frac{1}{a} \frac{d}{dx} (x) \\
&= \frac{-a^2}{a^2+x^2} \cdot \frac{1}{a} (1) = \frac{-a}{a^2+x^2}.
\end{aligned}$$

(iii) Let $y = \frac{1}{a} \sin^{-1} \frac{a}{x}$

Diff. w.r.t x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{a} \frac{d}{dx} \sin^{-1} \frac{a}{x} \\
&= \frac{1}{a} \frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \frac{d}{dx} \left(\frac{a}{x} \right) = \frac{1}{a} \frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}} \cdot a \frac{d}{dx} (x^{-1}) \\
&= \frac{x}{\sqrt{x^2-a^2}} (-x^{-2}) = \frac{x}{\sqrt{x^2-a^2}} \left(-\frac{1}{x^2} \right) = -\frac{1}{x\sqrt{x^2-a^2}} \quad \text{Ans}
\end{aligned}$$

(iv) Let $y = \sin^{-1} \sqrt{1-x^2}$

Diff. w.r.t x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \sin^{-1} \sqrt{1-x^2} \\
&= \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \cdot \frac{d}{dx} \sqrt{1-x^2} = \frac{1}{\sqrt{1-1+x^2}} \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \frac{d}{dx} (1-x^2) \\
&= \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2} \frac{1}{(1-x^2)^{\frac{1}{2}}} (-2x) = -\frac{1}{x} \cdot \frac{x}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

(v) Let $y = \operatorname{Sec}^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Diff. w.r.t x

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \operatorname{Sec}^{-1}\left(\frac{x^2+1}{x^2-1}\right) \\
 &= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right) \sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2 - 1}} \cdot \frac{d}{dx} \left(\frac{x^2+1}{x^2-1}\right) \\
 &= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right) \sqrt{\frac{(x^2+1)^2 - (x^2-1)^2}{(x^2-1)^2}}} \cdot \left(\frac{(x^2-1) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-1)}{(x^2-1)^2} \right) \\
 &= \frac{1}{\left(\frac{x^2+1}{x^2-1}\right) \cdot \frac{\sqrt{(x^4+2x^2+1) - (x^4+2x^2-1)}}{(x^2-1)}} \cdot \left(\frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} \right) \\
 &= \frac{(x^2-1)^2}{(x^2+1) \cdot \sqrt{x^4+2x^2+1-x^4-2x^2+1}} \cdot \left(\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \right) \\
 &= \frac{1}{(x^2+1) \cdot \sqrt{4x^2}} \cdot (2x(-2)) = \frac{-4x}{(x^2+1) \cdot 2x} = \frac{-2}{(x^2+1)} \quad \text{Ans}
 \end{aligned}$$

(vi) *Do yourself as above.*

(vii) *Do yourself as above.*

Question # 11

Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \tan^{-1} \frac{x}{y}$

Solution

$$\text{Since } \frac{y}{x} = \tan^{-1} \frac{x}{y} \Rightarrow y = x \tan^{-1} \frac{x}{y}$$

Diff. w.r.t x

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(x \tan^{-1} \frac{x}{y} \right) \\
 &= x \frac{d}{dx} \left(\tan^{-1} \frac{x}{y} \right) + \tan^{-1} \frac{x}{y} \cdot \frac{d}{dx} (x)
 \end{aligned}$$

$$\begin{aligned}
&= x \left(\frac{1}{1 + \left(\frac{x}{y} \right)^2} \frac{d}{dx} \left(\frac{x}{y} \right) \right) + \tan^{-1} \frac{x}{y} \cdot (1) \\
&= x \left(\frac{1}{\frac{y^2 + x^2}{y^2}} \left(\frac{y(1) - x \frac{dy}{dx}}{y^2} \right) \right) + \tan^{-1} \frac{x}{y} = \frac{x}{y^2 + x^2} \left(y - x \frac{dy}{dx} \right) + \frac{y}{x} \\
\Rightarrow \frac{dy}{dx} &= \frac{xy}{y^2 + x^2} - \frac{x^2}{y^2 + x^2} \cdot \frac{dy}{dx} + \frac{y}{x} \\
\Rightarrow \frac{dy}{dx} + \frac{x^2}{y^2 + x^2} \cdot \frac{dy}{dx} &= \frac{xy}{y^2 + x^2} + \frac{y}{x} \quad \Rightarrow \left(1 + \frac{x^2}{y^2 + x^2} \right) \cdot \frac{dy}{dx} = \frac{y}{x} \left(\frac{x^2}{y^2 + x^2} + 1 \right) \\
\Rightarrow \frac{dy}{dx} &= \frac{y}{x} \quad \text{Proved}
\end{aligned}$$

Question # 12

If $y = \tan(p \tan^{-1} x)$, show that $(1+x^2)y_1 - p(1+y^2) = 0$

Solution

$$\text{Since } y = \tan(p \tan^{-1} x) \Rightarrow \tan^{-1} y = p \tan^{-1} x$$

Differentiating w.r.t x

$$\begin{aligned}
\frac{d}{dx} \tan^{-1} y &= p \frac{d}{dx} \tan^{-1} x \\
\Rightarrow \frac{1}{1+y^2} \frac{dy}{dx} &= p \cdot \frac{1}{1+x^2} \quad \Rightarrow (1+x^2) \frac{dy}{dx} = p(1+y^2) \\
\Rightarrow (1+x^2)y_1 - p(1+y^2) &= 0 \quad \text{Since } \frac{dy}{dx} = y_1
\end{aligned}$$
