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CHAPTER 1: SYSTEMS OF LINEAR EQUATIONS AND MATRICES

1.1 Introduction to Systems of Linear Equations

1. **(a)** This is a linear equation in x_1, x_2, and x_3.

(b) This is not a linear equation in x_1, x_2, and x_3 because of the term x_1x_3.

(c) We can rewrite this equation in the form $x_1 + 7x_2 - 3x_3 = 0$ therefore it is a linear equation in x_1, x_2, and x_3.

(d) This is not a linear equation in x_1, x_2, and x_3 because of the term x_1^{-2}.

(e) This is not a linear equation in x_1, x_2, and x_3 because of the term $x_1^{3/5}$.

(f) This is a linear equation in x_1, x_2, and x_3.

2. **(a)** This is a linear equation in x and y.

(b) This is not a linear equation in x and y because of the terms $2x^{1/3}$ and $3\sqrt{y}$.

(c) This is a linear equation in x and y.

(d) This is not a linear equation in x and y because of the term $\frac{\pi}{7}\cos x$.

(e) This is not a linear equation in x and y because of the term xy.

(f) We can rewrite this equation in the form $-x + y = -7$ thus it is a linear equation in x and y.

3. **(a)**
$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

(b)
$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$

(c)
$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & = & b_2 \end{array}$$

4. **(a)**
$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

(c)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \end{bmatrix}$$

5. **(a)**

$$\begin{array}{rcrcl} 2x_1 & & & = & 0 \\ 3x_1 & - & 4x_2 & = & 0 \\ & & x_2 & = & 1 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} 3x_1 & & & - & 2x_3 & = & 5 \\ 7x_1 & + & x_2 & + & 4x_3 & = & -3 \\ & - & 2x_2 & + & x_3 & = & 7 \end{array}$$

6. **(a)**

$$\begin{array}{rcrcrcrcr} & & 3x_2 & - & x_3 & - & x_4 & = & -1 \\ 5x_1 & + & 2x_2 & & & - & 3x_4 & = & -6 \end{array}$$

(b)

$$\begin{array}{rcrcrcrcr} 3x_1 & & & + & x_3 & - & 4x_4 & = & 3 \\ -4x_1 & & & + & 4x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & & & - & 2x_4 & = & -9 \\ & & & & & - & x_4 & = & -2 \end{array}$$

7. **(a)** $\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$

8. **(a)** $\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

9. The values in (a), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system. The 3-tuples in (b) and (c) are not solutions of the system.

10. The values in (b), (d), and (e) satisfy all three equations – these 3-tuples are solutions of the system. The 3-tuples in (a) and (c) are not solutions of the system.

11. **(a)** We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} 3x & - & 2y & = & 4 \\ & & 0 & = & 1 \end{array}$$

The second equation is contradictory, so the original system has no solutions. The lines represented by the equations in that system have no points of intersection (the lines are parallel and distinct).

(b) We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} 2x & - & 4y & = & 1 \\ & & 0 & = & 0 \end{array}$$

The second equation does not impose any restriction on x and y therefore we can omit it. The lines represented by the original system have infinitely many points of intersection. Solving the first equation for x we obtain $x = \frac{1}{2} + 2y$. This allows us to represent the solution using parametric equations

$$x = \frac{1}{2} + 2t, \qquad y = t$$

where the parameter t is an arbitrary real number.

(c) We can eliminate x from the second equation by adding -1 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} x & - & 2y & = & 0 \\ & - & 2y & = & 8 \end{array}$$

From the second equation we obtain $y = -4$. Substituting -4 for y into the first equation results in $x = -8$. Therefore, the original system has the unique solution

$$x = -8, \quad y = -4$$

The represented by the equations in that system have one point of intersection: $(-8, -4)$.

12. We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the system

$$\begin{array}{rcrcl} 2x & - & 3y & = & a \\ & & 0 & = & b - 2a \end{array}$$

If $b - 2a = 0$ (i.e., $b = 2a$) then the second equation imposes no restriction on x and y; consequently, the system has infinitely many solutions.

If $b - 2a \neq 0$ (i.e., $b \neq 2a$) then the second equation becomes contradictory thus the system has no solutions.

There are no values of a and b for which the system has one solution.

13. (a) Solving the equation for x we obtain $x = \frac{3}{7} + \frac{5}{7}y$ therefore the solution set of the original equation can be described by the parametric equations

$$x = \frac{3}{7} + \frac{5}{7}t, \qquad y = t$$

where the parameter t is an arbitrary real number.

(b) Solving the equation for x_1 we obtain $x_1 = \frac{7}{3} + \frac{5}{3}x_2 - \frac{4}{3}x_3$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = \frac{7}{3} + \frac{5}{3}r - \frac{4}{3}s, \qquad x_2 = r, \qquad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

(c) Solving the equation for x_1 we obtain $x_1 = -\frac{1}{8} + \frac{1}{4}x_2 - \frac{5}{8}x_3 + \frac{3}{4}x_4$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = -\frac{1}{8} + \frac{1}{4}r - \frac{5}{8}s + \frac{3}{4}t, \quad x_2 = r, \quad x_3 = s, \quad x_4 = t$$

where the parameters r, s, and t are arbitrary real numbers.

(d) Solving the equation for v we obtain $v = \frac{8}{3}w - \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3}z$ therefore the solution set of the original equation can be described by the parametric equations

$$v = \frac{8}{3}t_1 - \frac{2}{3}t_2 + \frac{1}{3}t_3 - \frac{4}{3}t_4, \quad w = t_1, \quad x = t_2, \quad y = t_3, \quad z = t_4$$

where the parameters t_1, t_2, t_3, and t_4 are arbitrary real numbers.

14. **(a)** Solving the equation for x we obtain $x = 2 - 10y$ therefore the solution set of the original equation can be described by the parametric equations

$$x = 2 - 10t, \quad y = t$$

where the parameter t is an arbitrary real number.

(b) Solving the equation for x_1 we obtain $x_1 = 3 - 3x_2 + 12x_3$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = 3 - 3r + 12s, \quad x_2 = r, \quad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

(c) Solving the equation for x_1 we obtain $x_1 = 5 - \frac{1}{2}x_2 - \frac{3}{4}x_3 - \frac{1}{4}x_4$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = 5 - \frac{1}{2}r - \frac{3}{4}s - \frac{1}{4}t, \quad x_2 = r, \quad y = s, \quad z = t$$

where the parameters r, s, and t are arbitrary real numbers.

(d) Solving the equation for v we obtain $v = -w - x + 5y - 7z$ therefore the solution set of the original equation can be described by the parametric equations

$$v = -t_1 - t_2 + 5t_3 - 7t_4, \quad w = t_1, \quad x = t_2, \quad y = t_3, \quad z = t_4$$

where the parameters t_1, t_2, t_3, and t_4 are arbitrary real numbers.

15. **(a)** We can eliminate x from the second equation by adding -3 times the first equation to the second. This yields the system

$$\begin{array}{rcrcr} 2x & - & 3y & = & 1 \\ & & 0 & = & 0 \end{array}$$

The second equation does not impose any restriction on x and y therefore we can omit it. Solving the first equation for x we obtain $x = \frac{1}{2} + \frac{3}{2}y$. This allows us to represent the solution using parametric equations

$$x = \frac{1}{2} + \frac{3}{2}t, \quad y = t$$

where the parameter t is an arbitrary real number.

(b) We can see that the second and the third equation are multiples of the first: adding -3 times the first equation to the second, then adding the first equation to the third yields the system

$$\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for x_1 we obtain $x_1 = -4 - 3x_2 + x_3$. This allows us to represent the solution using parametric equations

$$x_1 = -4 - 3r + s, \quad x_2 = r, \quad x_3 = s$$

where the parameters r and s are arbitrary real numbers.

16. **(a)** We can eliminate x_1 from the first equation by adding -2 times the second equation to the first. This yields the system

$$\begin{aligned} 0 &= 0 \\ 3x_1 + x_2 &= -4 \end{aligned}$$

The first equation does not impose any restriction on x_1 and x_2 therefore we can omit it. Solving the second equation for x_1 we obtain $x_1 = -\frac{4}{3} - \frac{1}{3}x_2$. This allows us to represent the solution using parametric equations

$$x_1 = -\frac{4}{3} - \frac{1}{3}t, \quad x_2 = t$$

where the parameter t is an arbitrary real number.

(b) We can see that the second and the third equation are multiples of the first: adding -3 times the first equation to the second, then adding 2 times the first equation to the third yields the system

$$\begin{aligned} 2x - y + 2z &= -4 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

The last two equations do not impose any restriction on the unknowns therefore we can omit them. Solving the first equation for x we obtain $x = -2 + \frac{1}{2}y - z$. This allows us to represent the solution using parametric equations

$$x = -2 + \frac{1}{2}r - s, \quad y = r, \quad z = s$$

where the parameters r and s are arbitrary real numbers.

17. **(a)** Add 2 times the second row to the first to obtain $\begin{bmatrix} 1 & -7 & 8 & 8 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$.

(b) Add the third row to the first to obtain $\begin{bmatrix} 1 & 3 & -8 & 3 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

(another solution: interchange the first row and the third row to obtain $\begin{bmatrix} 1 & 4 & -3 & 3 \\ 2 & -9 & 3 & 2 \\ 0 & -1 & -5 & 0 \end{bmatrix}$).

18. **(a)** Multiply the first row by $\frac{1}{2}$ to obtain $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$.

(b) Add the third row to the first to obtain $\begin{bmatrix} 1 & -1 & -3 & 6 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

(another solution: add -2 times the second row to the first to obtain $\begin{bmatrix} 1 & -2 & -18 & 0 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$).

19. **(a)** Add -4 times the first row to the second to obtain $\begin{bmatrix} 1 & k & -4 \\ 0 & 8-4k & 18 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} x + \qquad ky &= -4 \\ (8-4k)y &= 18 \end{aligned}$$

If $k = 2$ then the second equation becomes $0 = 18$, which is contradictory thus the system becomes inconsistent.

If $k \neq 2$ then we can solve the second equation for y and proceed to substitute this value into the first equation and solve for x.

Consequently, for all values of $k \neq 2$ the given augmented matrix corresponds to a consistent linear system.

(b) Add -4 times the first row to the second to obtain $\begin{bmatrix} 1 & k & -1 \\ 0 & 8-4k & 0 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} x + \qquad ky &= -1 \\ (8-4k)y &= 0 \end{aligned}$$

If $k = 2$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq 2$ then the second equation yields $y = 0$ and the first equation becomes $x = -1$.

Consequently, for all values of k the given augmented matrix corresponds to a consistent linear system.

20. **(a)** Add 2 times the first row to the second to obtain $\begin{bmatrix} 3 & -4 & k \\ 0 & 0 & 2k+5 \end{bmatrix}$ which corresponds to the system

$$\begin{aligned} 3x - 4y &= k \\ 0 &= 2k + 5 \end{aligned}$$

If $k = -\frac{5}{2}$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq -\frac{5}{2}$ then the second equation is contradictory thus the system becomes inconsistent.

Consequently, the given augmented matrix corresponds to a consistent linear system only when $k = -\frac{5}{2}$.

(b) Add the first row to the second to obtain $\begin{bmatrix} k & 1 & -2 \\ 4+k & 0 & 0 \end{bmatrix}$ which corresponds to the system

$$\begin{array}{rcrcr} kx & + & y & = & -2 \\ (4+k)x & & & = & 0 \end{array}$$

If $k = -4$ then the second equation becomes $0 = 0$, which does not impose any restriction on x and y therefore we can omit it and proceed to determine the solution set using the first equation. There are infinitely many solutions in this set.

If $k \neq -4$ then the second equation yields $x = 0$ and the first equation becomes $y = -2$.

Consequently, for all values of k the given augmented matrix corresponds to a consistent linear system.

21. Substituting the coordinates of the first point into the equation of the curve we obtain

$$y_1 = ax_1^2 + bx_1 + c$$

Repeating this for the other two points and rearranging the three equations yields

$$\begin{aligned} x_1^2 a + x_1 b + c &= y_1 \\ x_2^2 a + x_2 b + c &= y_2 \\ x_3^2 a + x_3 b + c &= y_3 \end{aligned}$$

This is a linear system in the unknowns a, b, and c. Its augmented matrix is $\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$.

23. Solving the first equation for x_1 we obtain $x_1 = c - kx_2$ therefore the solution set of the original equation can be described by the parametric equations

$$x_1 = c - kt, \qquad x_2 = t$$

where the parameter t is an arbitrary real number.

Substituting these into the second equation yields

$$c - kt + lt = d$$

which can be rewritten as

$$c - kt = d - lt$$

This equation must hold true for all real values t, which requires that the coefficients associated with the same power of t on both sides must be equal. Consequently, $c = d$ and $k = l$.

24. **(a)** The system has no solutions if either

- at least two of the three lines are parallel and distinct or
- each pair of lines intersects at a different point (without any lines being parallel)

(b) The system has exactly one solution if either

- two lines coincide and the third one intersects them or
- all three lines intersect at a single point (without any lines being parallel)

(c) The system has infinitely many solutions if all three lines coincide.

25.
$$\begin{array}{rcrcrcr} 2x & + & 3y & + & z & = & 7 \\ 2x & + & y & + & 3z & = & 9 \\ 4x & + & 2y & + & 5z & = & 16 \end{array}$$

26. We set up the linear system as discussed in Exercise 21:

$$\begin{array}{rcrcrcr} 1^2a & + & 1b & + & c & = & 1 \\ 2^2a & + & 2b & + & c & = & 4 \\ (-1)^2a & - & 1b & + & c & = & 1 \end{array} \quad \text{i.e.} \quad \begin{array}{rcrcrcr} a & + & b & + & c & = & 1 \\ 4a & + & 2b & + & c & = & 4 \\ a & - & b & + & c & = & 1 \end{array}$$

One solution is expected, since exactly one parabola passes through any three given points (x_1, y_1), (x_2, y_2), (x_3, y_3) if x_1, x_2, and x_3 are distinct.

27.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 12 \\ 2x & + & y & + & 2z & = & 5 \\ -x & & & + & z & = & 1 \end{array}$$

True-False Exercises

(a) True. $(0,0, \ldots ,0)$ is a solution.

(b) False. Only multiplication by a **non**zero constant is a valid elementary row operation.

(c) True. If $k = 6$ then the system has infinitely many solutions; otherwise the system is inconsistent.

(d) True. According to the definition, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a linear equation if the a's are not all zero. Let us assume $a_j \neq 0$. The values of all x's except for x_j can be set to be arbitrary parameters, and the equation can be used to express x_j in terms of those parameters.

(e) False. E.g. if the equations are all homogeneous then the system must be consistent. (See True-False Exercise (a) above.)

(f) False. If $c \neq 0$ then the new system has the same solution set as the original one.

(g) True. Adding -1 times one row to another amounts to the same thing as subtracting one row from another.

(h) False. The second row corresponds to the equation $0 = -1$, which is contradictory.

1.2 Gaussian Elimination

1. **(a)** This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(b) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(c) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(d) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(e) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(f) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

(g) This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (-7) above it. The matrix is in row echelon form but not reduced row echelon form.

2. **(a)** This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (2) above it. The matrix is in row echelon form but not reduced row echelon form.

(b) This matrix does not have property 1 since its first nonzero number in the third row (2) is not a 1. The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(c) This matrix has properties 1-3 but does not have property 4: the third column contains a leading 1 and a nonzero number (4) above it. The matrix is in row echelon form but not reduced row echelon form.

(d) This matrix has properties 1-3 but does not have property 4: the second column contains a leading 1 and a nonzero number (5) above it. The matrix is in row echelon form but not reduced row echelon form.

(e) This matrix does not have property 2 since the row that consists entirely of zeros is not at the bottom of the matrix. The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(f) This matrix does not have property 3 since the leading 1 in the second row is directly below the leading 1 in the first (instead of being farther to the right). The matrix is not in row echelon form, therefore it is not in reduced row echelon form either.

(g) This matrix has properties 1-4. It is in reduced row echelon form, therefore it is also in row echelon form.

3. **(a)** The linear system

$$\begin{array}{rcrcrcl} x & - & 3y & + & 4z & = & 7 \\ & & y & + & 2z & = & 2 \\ & & & & z & = & 5 \end{array} \qquad \text{can be rewritten as} \qquad \begin{array}{rcl} x & = & 7 + 3y - 4z \\ y & = & 2 - 2z \\ z & = & 5 \end{array}$$

and solved by back-substitution:

$$\begin{aligned} &z = 5 \\ &y = 2 - 2(5) = -8 \\ &x = 7 + 3(-8) - 4(5) = -37 \end{aligned}$$

therefore the original linear system has a unique solution: $x = -37,\ y = -8,\ z = 5$.

(b) The linear system

$$\begin{array}{rcrcrcrcl} w & & & + & 8y & - & 5z & = & 6 \\ & & x & + & 4y & - & 9z & = & 3 \\ & & & & y & + & z & = & 2 \end{array} \qquad \text{can be rewritten as} \qquad \begin{array}{rcl} w & = & 6 - 8y + 5z \\ x & = & 3 - 4y + 9z \\ y & = & 2 - z \end{array}$$

Let $z = t$. Then

$$\begin{aligned} &y = 2 - t \\ &x = 3 - 4(2 - t) + 9t = -5 + 13t \\ &w = 6 - 8(2 - t) + 5t = -10 + 13t \end{aligned}$$

therefore the original linear system has infinitely many solutions:

$$w = -10 + 13t, x = -5 + 13t, y = 2 - t, z = t$$

where t is an arbitrary value.

(c) The linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 7x_2 & - & 2x_3 & & & - & 8x_5 & = & -3 \\ & & & & x_3 & + & x_4 & + & 6x_5 & = & 5 \\ & & & & & & x_4 & + & 3x_5 & = & 9 \\ & & & & & & & & 0 & = & 0 \end{array}$$

can be rewritten: $x_1 = -3 - 7x_2 + 2x_3 + 8x_5,\ x_3 = 5 - x_4 - 6x_5,\ x_4 = 9 - 3x_5$.

Let $x_2 = s$ and $x_5 = t$. Then

$$\begin{aligned} &x_4 = 9 - 3t \\ &x_3 = 5 - (9 - 3t) - 6t = -4 - 3t \\ &x_1 = -3 - 7s + 2(-4 - 3t) + 8t = -11 - 7s + 2t \end{aligned}$$

therefore the original linear system has infinitely many solutions:

$$x_1 = -11 - 7s + 2t,\ \ x_2 = s,\ \ x_3 = -4 - 3t,\ \ x_4 = 9 - 3t,\ \ x_5 = t$$

where s and t are arbitrary values.

(d) The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

4. **(a)** A unique solution: $x = -3,\ y = 0,\ z = 7$.

(b) Infinitely many solutions: $w = 8 + 7t,\ x = 2 - 3t,\ y = -5 - t, z = t$ where t is an arbitrary value.

(c) Infinitely many solutions: $v = -2 + 6s - 3t,\ w = s,\ x = 7 - 4t,\ y = 8 - 5t,\ z = t$ where s and t are arbitrary values.

(d) The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

5.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} \longleftarrow \text{The first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \longleftarrow -3 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -1\text{ .}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \longleftarrow 10 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{52}\text{ .}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ & & x_2 & - & 5x_3 & = & -9 \\ & & & & x_3 & = & 2 \end{array} \qquad \text{and can be rewritten as} \qquad \begin{array}{rcl} x_1 & = & 8 - x_2 - 2x_3 \\ x_2 & = & -9 + 5x_3 \\ x_3 & = & 2 \end{array}$$

Back-substitution yields

$$\begin{aligned} x_3 &= 2 \\ x_2 &= -9 + 5(2) = 1 \\ x_1 &= 8 - 1 - 2(2) = 3 \end{aligned}$$

The linear system has a unique solution: $x_1 = 3,\ x_2 = 1,\ x_3 = 2.$

6.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}\text{ .}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{2 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow \text{-8 times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow \text{The second row was multiplied by $\frac{1}{7}$.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{7 times the second row was added to the third row.}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + \frac{4}{7}x_3 &= \frac{1}{7} \\ 0 &= 0 \end{aligned}$$

Solve the equations for the leading variables

$$\begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= \frac{1}{7} - \frac{4}{7}x_3 \end{aligned}$$

then substitute the second equation into the first

$$\begin{aligned} x_1 &= -\frac{1}{7} - \frac{3}{7}x_3 \\ x_2 &= \frac{1}{7} - \frac{4}{7}x_3 \end{aligned}$$

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$$x_1 = -\frac{1}{7} - \frac{3}{7}t, \qquad x_2 = \frac{1}{7} - \frac{4}{7}t, \qquad x_3 = t$$

7.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{-2 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{The first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow \text{-3 times the first row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by $\tfrac{1}{3}$.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow \text{-1 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{-3 times the second row was added to the fourth row.}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{array}{rcrcrcrcr} x & - & y & + & 2z & - & w & = & -1 \\ & & y & - & 2z & & & = & 0 \\ & & & & & & 0 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Solve the equations for the leading variables

$$\begin{aligned} x &= -1 + y - 2z + w \\ y &= 2z \end{aligned}$$

then substitute the second equation into the first

$$\begin{aligned} x &= -1 + 2z - 2z + w = -1 + w \\ y &= 2z \end{aligned}$$

If we assign z and w the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x = -1 + t, \qquad y = 2s, \qquad z = s, \qquad w = t$$

8.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The first row was multiplied by $\tfrac{1}{3}$.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix} \longleftarrow \quad -6 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix} \longleftarrow \quad \text{The second row was multiplied by } -\tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix} \longleftarrow \quad 6 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow \quad \text{The third row was multiplied by } \tfrac{1}{6}.$$

The system of equations corresponding to this augmented matrix in row echelon form

$$\begin{array}{rcrcrcr} a & + & 2b & - & c & = & -\frac{2}{3} \\ & & b & - & \frac{3}{2}c & = & -\frac{1}{2} \\ & & & & 0 & = & 1 \end{array}$$

is clearly inconsistent.

9.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \longleftarrow \quad \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} \longleftarrow \quad \text{The first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \longleftarrow \quad -3 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \longleftarrow \quad \text{The second row was multiplied by } -1.$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \longleftarrow \quad 10 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \quad \text{The third row was multiplied by } -\tfrac{1}{52}.$$

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \quad 5 \text{ times the third row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \text{-2 times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \text{-1 times the second row was added to the first row.}$$

The linear system has a unique solution: $x_1 = 3,\ x_2 = 1,\ x_3 = 2$.

10.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The first row was multiplied by $\frac{1}{2}$.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{2 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow \text{-8 times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow \text{The second row was multiplied by $\frac{1}{7}$.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{7 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{-1 times the second row was added to the first row.}$$

Infinitely many solutions: $x_1 = -\frac{1}{7} - \frac{3}{7}t,\ x_2 = \frac{1}{7} - \frac{4}{7}t,\ x_3 = t$ where t is an arbitrary value.

11.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{-2 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longleftarrow \text{the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the first row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the second row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{the second row was added to the first row.}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{array}{rcrcrcrcr} x & & & & & - & w & = & -1 \\ & & y & - & 2z & & & = & 0 \\ & & & & & & 0 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Solve the equations for the leading variables

$$x = -1 + w$$
$$y = 2z$$

If we assign z and w the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x = -1 + t, \qquad y = 2s, \qquad z = s, \qquad w = t$$

12.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix} \longleftarrow -6 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix} \longleftarrow 6 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{6}.$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow \tfrac{1}{2} \text{ times the third row was added to the second row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow \tfrac{2}{3} \text{ times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow -2 \text{ times the second row was added to the first row.}$$

The last row corresponds to the equation

$$0a + 0b + 0c = 1$$

therefore the system is inconsistent.

(Note: this was already evident after the fifth elementary row operation.)

13. Since the number of unknowns (4) exceeds the number of equations (3), it follows from Theorem 1.2.2 that this system has infinitely many solutions. Those include the trivial solution and infinitely many nontrivial solutions.

14. The system does not have nontrivial solutions.
(The third equation requires $x_3 = 0$, which substituted into the second equation yields $x_2 = 0$. Both of these substituted into the first equation result in $x_1 = 0$.)

15. We present two different solutions.

Solution I uses Gauss-Jordan elimination

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{2}{3}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \begin{array}{l}\text{The third row was added to the second row} \\ \text{and } -\tfrac{3}{2} \text{ times the third row was added to the first row}\end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow -\tfrac{1}{2} \text{ times the second row was added to the first row.}$$

Unique solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 0.$

Solution II. This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exact sequence of elementary row operations being used does not matter – see part 1 of the discussion "Some Facts About Echelon Forms" on p. 21)

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longleftarrow \begin{array}{l}\text{The first and second rows were interchanged} \\ \text{(to avoid introducing fractions into the first row).}\end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{-2 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The second row was multiplied by } -\tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{-1 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The third row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The third row was added to the second row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{-2 times the second row was added to the first row.}$$

Unique solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 0.$

16. We present two different solutions.
<u>Solution I</u> uses Gauss-Jordan elimination

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix} \quad \longleftarrow \quad \text{-1 times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix} \quad \longleftarrow \quad \text{The second row was multiplied by } \tfrac{2}{3}.$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

$-\frac{3}{2}$ times the second row was added to the third row.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The third row was multiplied by $\frac{1}{10}$.

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3 times the third row was added to the second row and $\frac{3}{2}$ times the third row was added to the first row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1}{2}$ times the second row was added to the first row.

Unique solution: $x = 0,\ y = 0,\ z = 0.$

<u>Solution II.</u> This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exact sequence of elementary row operations being used does not matter – see part 1 of the discussion "Some Facts About Echelon Forms" on p. 21)

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix}$$

The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ -1 & 2 & -3 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$$

The first and third rows were interchanged
(to avoid introducing fractions into the first row).

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix}$$

The first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & -11 & 0 \end{bmatrix}$$

-2 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{bmatrix}$$

The second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The third row was multiplied by $-\frac{1}{10}$.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

-1 times the third row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{-4 times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by $\tfrac{1}{3}$.}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{-1 times the second row was added to the first row.}$$

Unique solution: $x = 0,\ y = 0,\ z = 0.$

17. $$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \longleftarrow \text{The first row was multiplied by $\tfrac{1}{3}$.}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{bmatrix} \longleftarrow \text{-5 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by $-\tfrac{3}{8}$.}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix} \longleftarrow \text{$-\tfrac{1}{3}$ times the second row was added to the first row.}$$

If we assign x_3 and x_4 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x_1 = -\frac{1}{4}s, \quad x_2 = -\frac{1}{4}s - t, \quad x_3 = s, \quad x_4 = t.$$

(Note that fractions in the solution could be avoided if we assigned $x_3 = 4s$ instead, which along with $x_4 = t$ would yield $x_1 = -s,\ x_2 = -s - t,\ x_3 = 4s,\ x_4 = t$.)

18. $$\begin{bmatrix} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{bmatrix} \longleftarrow$$

-2 times the first row was added to the third row and 4 times the first row was added to the fourth row.

$$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$

-2 times the second row was added to the third row and the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$

$-\frac{1}{2}$ times the second row was added to the first row.

If we assign w and x the arbitrary values s and t, respectively, the general solution is given by the formulas

$$u = \tfrac{7}{2}s - \tfrac{5}{2}t, \quad v = -3s + 2t, \quad w = s, \quad x = t.$$

19.

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \longleftarrow$$

-2 times the first row was added to the third row and 2 times the first row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \longleftarrow$$ -3 times the second row was added to the third and -1 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$ 10 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow$$ -2 times the third row was added to the second and 3 times the third row was added to the first row.

If we assign y an arbitrary value t the general solution is given by the formulas

$$w = t, \quad x = -t, \quad y = t, \quad z = 0.$$

20.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix} \longleftarrow$$ -1 times the first row was added to the second row, -2 times the first row was added to the fourth row, and -1 times the first row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix} \longleftarrow$$ 2 times the second row was added to the third row, 10 times the second row was added to the fourth row, and 5 times the second row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix} \longleftarrow$$ The third row was multiplied by $\frac{1}{2}$.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{41}{2} & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{bmatrix}$$ ← -21 times the third row was added to the fourth row and -9 times the third row was added to the fifth row.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{17}{2} & 0 \end{bmatrix}$$ ← The fourth row was multiplied by $\frac{2}{41}$.

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ ← $-\frac{17}{2}$ times the fourth row was added to the fifth row.

The augmented matrix in row echelon form corresponds to the system

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + x_4 &= 0 \\ x_2 + 2x_3 - x_4 &= 0 \\ x_3 - \frac{3}{2}x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

Using back-substitution, we obtain the unique solution of this system

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0.$$

21. $$\begin{bmatrix} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix}$$ ← The augmented matrix for the system.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 2 & -1 & 3 & 4 & 9 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix}$$ ← The first and second rows were interchanged (to avoid introducing fractions into the first row).

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & -1 & 7 & -10 & -13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix}$$ ⟵ -2 times the first row was added to the second row, -3 times the first row was added to the third row, and -2 times the first row was added to the fourth.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & -3 & 7 & -16 & -25 \\ 0 & 1 & 8 & -10 & -12 \end{bmatrix}$$ ⟵ The second row was multiplied by -1.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & -14 & 14 & 14 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix}$$ ⟵ 3 times the second row was added to the third row and -1 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix}$$ ⟵ The third row was multiplied by $-\frac{1}{14}$.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix}$$ ⟵ -15 times the third row was added to the fourth row.

$$\begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ⟵ The fourth row was multiplied by $-\frac{1}{5}$.

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & -7 & 0 & -7 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ⟵ The fourth row was added to the third row, -10 times the fourth row was added to the second, and -7 times the fourth row was added to the first.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ ⟵ 7 times the third row was added to the second row, and 2 times the third row was added to the first row.

Unique solution: $I_1 = -1,\ I_2 = 0,\ I_3 = 1,\ I_4 = 2.$

22. $$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$ ⟵ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix}$$ ⟵ The first and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \longleftarrow \text{The first row was added to the second row and } -2 \text{ times the first row was added to the last row.}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \longleftarrow \text{The second and third rows were interchanged.}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \longleftarrow -3 \text{ times the second row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow 3 \text{ times the third row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the third row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow 2 \text{ times the second row was added to the first row.}$$

If we assign Z_2 and Z_5 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$Z_1 = -s - t, \quad Z_2 = s, \quad Z_3 = -t, \quad Z_4 = 0, \quad Z_5 = t.$$

23. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has infinitely many solutions (the third unknown can be assigned an arbitrary value t, then back-substitution can be used to solve for the first two unknowns).

(c) The system is inconsistent since the third equation $0 = 1$ is contradictory.

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$ the system is consistent with infinitely many solutions.

- For $\begin{bmatrix}1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1\end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix}1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}$).

24. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has a unique solution (solve the first equation for the first unknown, then proceed to solve the second equation for the second unknown and solve the third equation last.)

(c) The system is inconsistent (adding -1 times the first row to the second yields $\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & * & * & *\end{bmatrix}$; the second equation $0 = 1$ is contradictory).

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix}1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1\end{bmatrix}$ the system is consistent with infinitely many solutions.
- For $\begin{bmatrix}1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1\end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix}1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}$).

25.

$$\begin{bmatrix}1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2\end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix}1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14\end{bmatrix} \longleftarrow \begin{array}{l}-3 \text{ times the first row was added to the second row} \\ \text{and } -4 \text{ times the first row was added to the third row.}\end{array}$$

$$\begin{bmatrix}1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4\end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix}1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4\end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{7}.$$

The system has no solutions when $a = -4$ (since the third row of our last matrix would then correspond to a contradictory equation $0 = -8$).

The system has infinitely many solutions when $a = 4$ (since the third row of our last matrix would then correspond to the equation $0 = 0$).

For all remaining values of a (i.e., $a \neq -4$ and $a \neq 4$) the system has exactly one solution.

26.

$$\begin{bmatrix}1 & 2 & 1 & 2 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & -(a^2-3) & a\end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & -a^2+2 & a-2 \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{2} \\ 0 & 0 & -a^2+2 & a-2 \end{bmatrix} \longleftarrow$$ The second row was multiplied by $-\frac{1}{6}$.

The system has no solutions when $a = \sqrt{2}$ or $a = -\sqrt{2}$ (since the third row of our last matrix would then correspond to a contradictory equation).

For all remaining values of a (i.e., $a \neq \sqrt{2}$ and $a \neq -\sqrt{2}$) the system has exactly one solution.

There is no value of a for which this system has infinitely many solutions.

27.

$$\begin{bmatrix} 1 & 3 & -1 & a \\ 1 & 1 & 2 & b \\ 0 & 2 & -3 & c \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 3 & -1 & a \\ 0 & -2 & 3 & -a+b \\ 0 & 2 & -3 & c \end{bmatrix} \longleftarrow$$ -1 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 3 & -1 & a \\ 0 & -2 & 3 & -a+b \\ 0 & 0 & 0 & -a+b+c \end{bmatrix} \longleftarrow$$ The second row was added to the third row.

$$\begin{bmatrix} 1 & 3 & -1 & a \\ 0 & 1 & -\frac{3}{2} & \frac{a}{2}-\frac{b}{2} \\ 0 & 0 & 0 & -a+b+c \end{bmatrix} \longleftarrow$$ The second row was multiplied by $-\frac{1}{2}$.

If $-a+b+c=0$ then the linear system is consistent. Otherwise (if $-a+b+c \neq 0$) it is inconsistent.

28.

$$\begin{bmatrix} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & -3a+c \end{bmatrix} \longleftarrow$$ The first row was added to the second row and -3 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & -a+2b+c \end{bmatrix} \longleftarrow$$ 2 times the second row was added to the third row.

If $-a+2b+c=0$ then the linear system is consistent. Otherwise (if $-a+2b+c \neq 0$) it is inconsistent.

29.

$$\begin{bmatrix} 2 & 1 & a \\ 3 & 6 & b \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 3 & 6 & b \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 0 & \frac{9}{2} & -\frac{3}{2}a + b \end{bmatrix} \longleftarrow -3 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{9}b \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{2}{9}.$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{3}a - \frac{1}{9}b \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{9}b \end{bmatrix} \longleftarrow -\tfrac{1}{2} \text{ times the second row was added to the first row.}$$

The system has exactly one solution: $x = \frac{2}{3}a - \frac{1}{9}b$ and $y = -\frac{1}{3}a + \frac{2}{9}b$.

30.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 0 & -2a + b \\ 0 & 3 & 3 & c \end{bmatrix} \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a - \frac{b}{2} \\ 0 & 3 & 3 & c \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a - \frac{b}{2} \\ 0 & 0 & 3 & -3a + \frac{3}{2}b + c \end{bmatrix} \longleftarrow -3 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a - \frac{b}{2} \\ 0 & 0 & 1 & -a + \frac{b}{2} + \frac{c}{3} \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 1 & 0 & 2a - \frac{b}{2} - \frac{c}{3} \\ 0 & 1 & 0 & a - \frac{b}{2} \\ 0 & 0 & 1 & -a + \frac{b}{2} + \frac{c}{3} \end{bmatrix} \longleftarrow -1 \text{ times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & a - \frac{c}{3} \\ 0 & 1 & 0 & a - \frac{b}{2} \\ 0 & 0 & 1 & -a + \frac{b}{2} + \frac{c}{3} \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the first row.}$$

The system has exactly one solution: $x_1 = a - \frac{c}{3}$, $x_2 = a - \frac{b}{2}$, and $x_3 = -a + \frac{b}{2} + \frac{c}{3}$.

31. Adding -2 times the first row to the second yields a matrix in row echelon form $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

Adding -3 times its second row to the first results in $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is also in row echelon form.

32. $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix}$ ← -1 times the first row was added to the third row.

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 2 & 1 & 3 \end{bmatrix}$ ← The first and third rows were interchanged.

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & -5 & -1 \end{bmatrix}$ ← -2 times the first row was added to the third row.

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & 1 & 86 \end{bmatrix}$ ← -3 times the second row was added to the third row.

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 86 \\ 0 & -2 & -29 \end{bmatrix}$ ← The second and third rows were interchanged.

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 86 \\ 0 & 0 & 143 \end{bmatrix}$ ← 2 times the second row was added to the third row.

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 86 \\ 0 & 0 & 1 \end{bmatrix}$ ← The third row was multiplied by $\frac{1}{143}$.

$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← -86 times the third row was added to the second row and -2 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← -3 times the second row was added to the first row.

33. We begin by substituting $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$ so that the system becomes

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ 2x & + & 5y & + & 3z & = & 0 \\ -x & - & 5y & + & 5z & = & 0 \end{array}$$

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{bmatrix}$ ⟵ -2 times the first row was added to the second row and the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ ⟵ 3 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was multiplied by -1.

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ 3 times the third row was added to the second row and -3 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ -2 times the second row was added to the first row.

This system has exactly one solution $x = 0, \ \ y = 0, \ \ z = 0$.

On the interval $0 \le \alpha \le 2\pi$, the equation $\sin\alpha = 0$ has three solutions: $\alpha = 0, \alpha = \pi$, and $\alpha = 2\pi$.

On the interval $0 \le \beta \le 2\pi$, the equation $\cos\beta = 0$ has two solutions: $\beta = \frac{\pi}{2}$ and $\beta = \frac{3\pi}{2}$.

On the interval $0 \le \gamma \le 2\pi$, the equation $\tan\gamma = 0$ has three solutions: $\gamma = 0, \gamma = \pi$, and $\gamma = 2\pi$.

Overall, $3 \cdot 2 \cdot 3 = 18$ solutions (α, β, γ) can be obtained by combining the values of α, β, and γ listed above: $\left(0, \frac{\pi}{2}, 0\right), \left(\pi, \frac{\pi}{2}, 0\right)$, etc.

34. We begin by substituting $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$ so that the system becomes

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & 3 \\ 4x & + & 2y & - & 2z & = & 2 \\ 6x & - & 3y & + & z & = & 9 \end{array}$$

$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{bmatrix}$ ⟵ -2 times the first row was added to the second row and -3 times the first row was added to the third row.

$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The third row was multiplied by $-\frac{1}{8}$.

$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ 8 times the third row was added to the second row and -3 times the third row was added to the first row.

$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ⟵ The second row was multiplied by $\frac{1}{4}$.

$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The second row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ← The first row was multiplied by $\frac{1}{2}$.

This system has exactly one solution $x = 1,\ \ y = -1,\ \ z = 0$.

The only angles α, β, and γ that satisfy the inequalities $0 \leq \alpha \leq 2\pi,\ 0 \leq \beta \leq 2\pi, 0 \leq \gamma < \pi$ and the equations

$$\sin \alpha = 1, \qquad \cos \beta = -1, \qquad \tan \gamma = 0$$

are $\alpha = \frac{\pi}{2}$, $\beta = \pi$, and $\gamma = 0$.

35. We begin by substituting $X = x^2$, $Y = y^2$, and $Z = z^2$ so that the system becomes

$$\begin{array}{rcrcrcr} X & + & Y & + & Z & = & 6 \\ X & - & Y & + & 2Z & = & 2 \\ 2X & + & Y & - & Z & = & 3 \end{array}$$

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -4 \\ 0 & -1 & -3 & -9 \end{bmatrix}$ ← -1 times the first row was added to the second row and -2 times the first row was added to the third row.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -9 \\ 0 & -2 & 1 & -4 \end{bmatrix}$ ← The second and third rows were interchanged (to avoid introducing fractions into the second row).

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & -2 & 1 & -4 \end{bmatrix}$ ← The second row was multiplied by -1.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 7 & 14 \end{bmatrix}$ ← 2 times the second row was added to the third row.

$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← The third row was multiplied by $\frac{1}{7}$.

$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -3 times the third row was added to the second row and -1 times the third row was added to the first row.

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ ← -1 times the second row was added to the first row.

We obtain

$$\begin{aligned} X = 1 &\quad \Rightarrow \quad x = \pm 1 \\ Y = 3 &\quad \Rightarrow \quad y = \pm\sqrt{3} \\ Z = 2 &\quad \Rightarrow \quad z = \pm\sqrt{2} \end{aligned}$$

36. We begin by substituting $a = \frac{1}{x}$, $b = \frac{1}{y}$, and $c = \frac{1}{z}$ so that the system becomes

$$\begin{array}{rcrcrcr} a & + & 2b & - & 4c & = & 1 \\ 2a & + & 3b & + & 8c & = & 0 \\ -a & + & 9b & + & 10c & = & 5 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ -1 & 9 & 10 & 5 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 11 & 6 & 6 \end{bmatrix} \longleftarrow -2 \text{ times the first row was added to the second row and the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 11 & 6 & 6 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -1.$$

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 182 & -16 \end{bmatrix} \longleftarrow -11 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & -\frac{8}{91} \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{182}.$$

Using back-substitution, we obtain

$$\begin{aligned} c &= -\frac{8}{91} && \Rightarrow \quad z = \frac{1}{c} = -\frac{91}{8} \\ b &= 2 + 16c = \frac{54}{91} && \Rightarrow \quad y = \frac{1}{b} = \frac{91}{54} \\ a &= 1 - 2b + 4c = -\frac{7}{13} && \Rightarrow \quad x = \frac{1}{a} = -\frac{13}{7} \end{aligned}$$

37. Each point on the curve yields an equation, therefore we have a system of four equations

$$\begin{array}{lrcrcrcrcr} \text{equation corresponding to } (1,7): & a & + & b & + & c & + & d & = & 7 \\ \text{equation corresponding to } (3,-11): & 27a & + & 9b & + & 3c & + & d & = & -11 \\ \text{equation corresponding to } (4,-14): & 64a & + & 16b & + & 4c & + & d & = & -14 \\ \text{equation corresponding to } (0,10): & & & & & & & d & = & 10 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 27 & 9 & 3 & 1 & -11 \\ 64 & 16 & 4 & 1 & -14 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & -18 & -24 & -26 & -200 \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← -27 times the first row was added to the second row and -64 times the first row was added to the third.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & -48 & -60 & -63 & -462 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← The second row was multiplied by $-\frac{1}{18}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 4 & \frac{19}{3} & \frac{214}{3} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← 48 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & \frac{4}{3} & \frac{13}{9} & \frac{100}{9} \\ 0 & 0 & 1 & \frac{19}{12} & \frac{107}{6} \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← The third row was multiplied by $\frac{1}{4}$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -3 \\ 0 & 1 & \frac{4}{3} & 0 & -\frac{10}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← $-\frac{19}{12}$ times the fourth row was added to the third row, $-\frac{13}{9}$ times the fourth row was added to the second row, and -1 times the fourth row was added to the first.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← $-\frac{4}{3}$ times the third row was added to the second row and -1 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$ ← -1 times the second row was added to the first row.

The linear system has a unique solution: $a = 1,\ b = -6,\ c = 2,\ d = 10$. These are the coefficient values required for the curve $y = ax^3 + bx^2 + cx + d$ to pass through the four given points.

38. Each point on the curve yields an equation, therefore we have a system of three equations

equation corresponding to $(-2,7)$:	$53a - 2b + 7c + d = 0$
equation corresponding to $(-4,5)$:	$41a - 4b + 5c + d = 0$
equation corresponding to $(4,-3)$:	$25a + 4b - 3c + d = 0$

The augmented matrix of this system $\begin{bmatrix} 53 & -2 & 7 & 1 & 0 \\ 41 & -4 & 5 & 1 & 0 \\ 25 & 4 & -3 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{29} & 0 \\ 0 & 1 & 0 & -\frac{2}{29} & 0 \\ 0 & 0 & 1 & -\frac{4}{29} & 0 \end{bmatrix}$$

If we assign d an arbitrary value t, the general solution is given by the formulas

$$a = -\frac{1}{29}t, \quad b = \frac{2}{29}t, \quad c = \frac{4}{29}t, \quad d = t$$

(For instance, letting the free variable d have the value -29 yields $a = 1$, $b = -2$, and $c = -4$.)

39. Since the homogeneous system has only the trivial solution, its augmented matrix must be possible to reduce via a sequence of elementary row operations to the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Applying the **same** sequence of elementary row operations to the augmented matrix of the nonhomogeneous system yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{bmatrix}$ where r, s, and t are some real numbers. Therefore, the nonhomogeneous system has one solution.

40. **(a)** 3 (this will be the number of leading 1's if the matrix has no rows of zeros)

(b) 5 (if all entries in B are 0)

(c) 2 (this will be the number of rows of zeros if each column contains a leading 1)

41. **(a)** There are eight possible reduced row echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where r and s can be any real numbers.

(b) There are sixteen possible reduced row echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & r & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & r & t \\ 0 & 1 & s & u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & 0 & s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & r & s & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & r & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & r \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & r & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & r & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{and} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where r, s, t, and u can be any real numbers.

42. **(a)** Either the three lines properly intersect at the origin, or two of them completely overlap and the other one intersects them at the origin.

(b) All three lines completely overlap one another.

43. **(a)** We consider two possible cases: (i) $a = 0$, and (ii) $a \neq 0$.

(i) If $a = 0$ then the assumption $ad - bc \neq 0$ implies that $b \neq 0$ and $c \neq 0$. Gauss-Jordan elimination yields

$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \longleftarrow$ We assumed $a = 0$

$\begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \longleftarrow$ The rows were interchanged.

$\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \longleftarrow$ The first row was multiplied by $\frac{1}{c}$ and the second row was multiplied by $\frac{1}{b}$. (Note that $b, c \neq 0$.)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow$ $-\frac{d}{c}$ times the second row was added to the first row.

(ii) If $a \neq 0$ then we perform Gauss-Jordan elimination as follows:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \longleftarrow$ The first row was multiplied by $\frac{1}{a}$.

$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \longleftarrow$ $-c$ times the first row was added to the second row.

$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \longleftarrow$ The second row was multiplied by $\frac{a}{ad-bc}$. (Note that both a and $ad - bc$ are nonzero.)

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow$ $-\frac{b}{a}$ times the second row was added to the first row.

In both cases ($a = 0$ as well as $a \neq 0$) we established that the reduced row echelon form of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ provided that $ad - bc \neq 0$.

(b) Applying the **same** elementary row operation steps as in part (a) the augmented matrix $\begin{bmatrix} a & b & k \\ c & d & l \end{bmatrix}$ will be transformed to a matrix in reduced row echelon form $\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}$ where p

and q are some real numbers. We conclude that the given linear system has exactly one solution: $x = p,\ y = q$.

True-False Exercises

(a) True. A matrix in reduced row echelon form has all properties required for the row echelon form.

(b) False. For instance, interchanging the rows of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yields a matrix that is not in row echelon form.

(c) False. See Exercise 31.

(d) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. The result follows from Theorem 1.2.1.

(e) True. This is implied by the third property of a row echelon form (see p. 11).

(f) False. Nonzero entries are permitted above the leading 1's in a row echelon form.

(g) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. From Theorem 1.2.1 we conclude that the system has $n - n = 0$ free variables, i.e. it has only the trivial solution.

(h) False. The row of zeros imposes no restriction on the unknowns and can be omitted. Whether the system has infinitely many, one, or no solution(s) depends *solely* on the nonzero rows of the reduced row echelon form.

(i) False. For example, the following system is clearly inconsistent:

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

1.3 Matrices and Matrix Operations

1. **(a)** Undefined (the number of columns in B does not match the number of rows in A)

(b) Defined; 4×4 matrix

(c) Defined; 4×2 matrix

(d) Defined; 5×2 matrix

(e) Defined; 4×5 matrix

(f) Defined; 5×5 matrix

2. **(a)** Defined; 5×4 matrix

(b) Undefined (the number of columns in D does not match the number of rows in C)

(c) Defined; 4×2 matrix

(d) Defined; 2×4 matrix

(e) Defined; 5×2 matrix

(f) Undefined (BA^T is a 4×4 matrix, which cannot be added to a 4×2 matrix D)

3. **(a)** $\begin{bmatrix} 1+6 & 5+1 & 2+3 \\ -1+(-1) & 0+1 & 1+2 \\ 3+4 & 2+1 & 4+3 \end{bmatrix} = \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1-6 & 5-1 & 2-3 \\ -1-(-1) & 0-1 & 1-2 \\ 3-4 & 2-1 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 5 \cdot 3 & 5 \cdot 0 \\ 5 \cdot (-1) & 5 \cdot 2 \\ 5 \cdot 1 & 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -7 \cdot 1 & -7 \cdot 4 & -7 \cdot 2 \\ -7 \cdot 3 & -7 \cdot 1 & -7 \cdot 5 \end{bmatrix} = \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$

(e) Undefined (a 2×3 matrix C cannot be subtracted from a 2×2 matrix $2B$)

(f) $\begin{bmatrix} 4 \cdot 6 & 4 \cdot 1 & 4 \cdot 3 \\ 4 \cdot (-1) & 4 \cdot 1 & 4 \cdot 2 \\ 4 \cdot 4 & 4 \cdot 1 & 4 \cdot 3 \end{bmatrix} - \begin{bmatrix} 2 \cdot 1 & 2 \cdot 5 & 2 \cdot 2 \\ 2 \cdot (-1) & 2 \cdot 0 & 2 \cdot 1 \\ 2 \cdot 3 & 2 \cdot 2 & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 24-2 & 4-10 & 12-4 \\ -4-(-2) & 4-0 & 8-2 \\ 16-6 & 4-4 & 12-8 \end{bmatrix}$
$= \begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$

(g) $-3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 \cdot 6 & 2 \cdot 1 & 2 \cdot 3 \\ 2 \cdot (-1) & 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 4 & 2 \cdot 1 & 2 \cdot 3 \end{bmatrix}\right) = -3\begin{bmatrix} 1+12 & 5+2 & 2+6 \\ -1+(-2) & 0+2 & 1+4 \\ 3+8 & 2+2 & 4+6 \end{bmatrix}$
$= \begin{bmatrix} -3 \cdot 13 & -3 \cdot 7 & -3 \cdot 8 \\ -3 \cdot (-3) & -3 \cdot 2 & -3 \cdot 5 \\ -3 \cdot 11 & -3 \cdot 4 & -3 \cdot 10 \end{bmatrix} = \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$

(h) $\begin{bmatrix} 3-3 & 0-0 \\ -1-(-1) & 2-2 \\ 1-1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) $1+0+4=5$

(j) $\operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 \cdot 6 & 3 \cdot 1 & 3 \cdot 3 \\ 3 \cdot (-1) & 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 1 & 3 \cdot 3 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 1-18 & 5-3 & 2-9 \\ -1-(-3) & 0-3 & 1-6 \\ 3-12 & 2-3 & 4-9 \end{bmatrix}\right)$
$= \operatorname{tr}\left(\begin{bmatrix} -17 & 2 & -7 \\ 2 & -3 & -5 \\ -9 & -1 & -5 \end{bmatrix}\right) = -17-3-5 = -25$

(k) $4\operatorname{tr}\left(\begin{bmatrix} 7 \cdot 4 & 7 \cdot (-1) \\ 7 \cdot 0 & 7 \cdot 2 \end{bmatrix}\right) = 4\operatorname{tr}\left(\begin{bmatrix} 28 & -7 \\ 0 & 14 \end{bmatrix}\right) = 4(28+14) = 4 \cdot 42 = 168$

(l) Undefined (trace is only defined for square matrices)

4. **(a)** $2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3+1 & 2 \cdot (-1)+4 & 2 \cdot 1+2 \\ 2 \cdot 0+3 & 2 \cdot 2+1 & 2 \cdot 1+5 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1-6 & -1-(-1) & 3-4 \\ 5-1 & 0-1 & 2-1 \\ 2-3 & 1-2 & 4-3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(c) $\left(\begin{bmatrix} 1-6 & 5-1 & 2-3 \\ -1-(-1) & 0-1 & 1-2 \\ 3-4 & 2-1 & 4-3 \end{bmatrix}\right)^T = \left(\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(d) Undefined (a 2×2 matrix B^T cannot be added to a 3×2 matrix $5C^T$)

(e) $\begin{bmatrix} \frac{1}{2}\cdot 1 & \frac{1}{2}\cdot 3 \\ \frac{1}{2}\cdot 4 & \frac{1}{2}\cdot 1 \\ \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot 5 \end{bmatrix} - \begin{bmatrix} \frac{1}{4}\cdot 3 & \frac{1}{4}\cdot 0 \\ \frac{1}{4}\cdot(-1) & \frac{1}{4}\cdot 2 \\ \frac{1}{4}\cdot 1 & \frac{1}{4}\cdot 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}-\frac{3}{4} & \frac{3}{2}-0 \\ 2+\frac{1}{4} & \frac{1}{2}-\frac{1}{2} \\ 1-\frac{1}{4} & \frac{5}{2}-\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{9}{4} \end{bmatrix}$

(f) $\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-4 & -1-0 \\ 0-(-1) & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(g) $2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 6 & 2\cdot(-1) & 2\cdot 4 \\ 2\cdot 1 & 2\cdot 1 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 3 \end{bmatrix} - \begin{bmatrix} 3\cdot 1 & 3\cdot(-1) & 3\cdot 3 \\ 3\cdot 5 & 3\cdot 0 & 3\cdot 2 \\ 3\cdot 2 & 3\cdot 1 & 3\cdot 4 \end{bmatrix}$
$= \begin{bmatrix} 12-3 & -2-(-3) & 8-9 \\ 2-15 & 2-0 & 2-6 \\ 6-6 & 4-3 & 6-12 \end{bmatrix} = \begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$

(h) $\left(2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}\right)^T = \left(\begin{bmatrix} 2\cdot 6 & 2\cdot(-1) & 2\cdot 4 \\ 2\cdot 1 & 2\cdot 1 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 3 \end{bmatrix} - \begin{bmatrix} 3\cdot 1 & 3\cdot(-1) & 3\cdot 3 \\ 3\cdot 5 & 3\cdot 0 & 3\cdot 2 \\ 3\cdot 2 & 3\cdot 1 & 3\cdot 4 \end{bmatrix}\right)^T$
$= \left(\begin{bmatrix} 12-3 & -2-(-3) & 8-9 \\ 2-15 & 2-0 & 2-6 \\ 6-6 & 4-3 & 6-12 \end{bmatrix}\right)^T = \left(\begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}\right)^T = \begin{bmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{bmatrix}$

(i) $\begin{bmatrix} (1\cdot 1)-(4\cdot 1)+(2\cdot 3) & (1\cdot 5)+(4\cdot 0)+(2\cdot 2) & (1\cdot 2)+(4\cdot 1)+(2\cdot 4) \\ (3\cdot 1)-(1\cdot 1)+(5\cdot 3) & (3\cdot 5)+(1\cdot 0)+(5\cdot 2) & (3\cdot 2)+(1\cdot 1)+(5\cdot 4) \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$
$= \begin{bmatrix} 3 & 9 & 14 \\ 17 & 25 & 27 \end{bmatrix}\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$
$= \begin{bmatrix} (3\cdot 6)-(9\cdot 1)+(14\cdot 4) & (3\cdot 1)+(9\cdot 1)+(14\cdot 1) & (3\cdot 3)+(9\cdot 2)+(14\cdot 3) \\ (17\cdot 6)-(25\cdot 1)+(27\cdot 4) & (17\cdot 1)+(25\cdot 1)+(27\cdot 1) & (17\cdot 3)+(25\cdot 2)+(27\cdot 3) \end{bmatrix}$
$= \begin{bmatrix} 65 & 26 & 69 \\ 185 & 69 & 182 \end{bmatrix}$

(j) Undefined (a 2×2 matrix B cannot be multiplied by a 3×2 matrix A)

(k) $\operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}\right) =$
$\operatorname{tr}\left(\begin{bmatrix} (1\cdot 6)+(5\cdot 1)+(2\cdot 3) & -(1\cdot 1)+(5\cdot 1)+(2\cdot 2) & (1\cdot 4)+(5\cdot 1)+(2\cdot 3) \\ -(1\cdot 6)+(0\cdot 1)+(1\cdot 3) & (1\cdot 1)+(0\cdot 1)+(1\cdot 2) & -(1\cdot 4)+(0\cdot 1)+(1\cdot 3) \\ (3\cdot 6)+(2\cdot 1)+(4\cdot 3) & -(3\cdot 1)+(2\cdot 1)+(4\cdot 2) & (3\cdot 4)+(2\cdot 1)+(4\cdot 3) \end{bmatrix}\right)$
$= \operatorname{tr}\left(\begin{bmatrix} 17 & 8 & 15 \\ -3 & 3 & -1 \\ 32 & 7 & 26 \end{bmatrix}\right) = 17+3+26 = 46$

(l) Undefined (BC is a 2×3 matrix; trace is only defined for square matrices)

5. **(a)** $\begin{bmatrix} (3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2) \\ -(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2) \\ (1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2) \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$

(b) Undefined (the number of columns of B does not match the number of rows in A)

(c) $\begin{bmatrix} 3\cdot 6 & 3\cdot 1 & 3\cdot 3 \\ 3\cdot(-1) & 3\cdot 1 & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 1 & 3\cdot 3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} (18\cdot 1)-(3\cdot 1)+(9\cdot 3) & (18\cdot 5)+(3\cdot 0)+(9\cdot 2) & (18\cdot 2)+(3\cdot 1)+(9\cdot 4) \\ -(3\cdot 1)-(3\cdot 1)+(6\cdot 3) & -(3\cdot 5)+(3\cdot 0)+(6\cdot 2) & -(3\cdot 2)+(3\cdot 1)+(6\cdot 4) \\ (12\cdot 1)-(3\cdot 1)+(9\cdot 3) & (12\cdot 5)+(3\cdot 0)+(9\cdot 2) & (12\cdot 2)+(3\cdot 1)+(9\cdot 4) \end{bmatrix}$

$= \begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$

(d) $\begin{bmatrix} (3\cdot 4)+(0\cdot 0) & -(3\cdot 1)+(0\cdot 2) \\ -(1\cdot 4)+(2\cdot 0) & (1\cdot 1)+(2\cdot 2) \\ (1\cdot 4)+(1\cdot 0) & -(1\cdot 1)+(1\cdot 2) \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} (12\cdot 1)-(3\cdot 3) & (12\cdot 4)-(3\cdot 1) & (12\cdot 2)-(3\cdot 5) \\ -(4\cdot 1)+(5\cdot 3) & -(4\cdot 4)+(5\cdot 1) & -(4\cdot 2)+(5\cdot 5) \\ (4\cdot 1)+(1\cdot 3) & (4\cdot 4)+(1\cdot 1) & (4\cdot 2)+(1\cdot 5) \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (4\cdot 1)-(1\cdot 3) & (4\cdot 4)-(1\cdot 1) & (4\cdot 2)-(1\cdot 5) \\ (0\cdot 1)+(2\cdot 3) & (0\cdot 4)+(2\cdot 1) & (0\cdot 2)+(2\cdot 5) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$

$= \begin{bmatrix} (3\cdot 1)+(0\cdot 6) & (3\cdot 15)+(0\cdot 2) & (3\cdot 3)+(0\cdot 10) \\ -(1\cdot 1)+(2\cdot 6) & -(1\cdot 15)+(2\cdot 2) & -(1\cdot 3)+(2\cdot 10) \\ (1\cdot 1)+(1\cdot 6) & (1\cdot 15)+(1\cdot 2) & (1\cdot 3)+(1\cdot 10) \end{bmatrix} = \begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} (1\cdot 1)+(4\cdot 4)+(2\cdot 2) & (1\cdot 3)+(4\cdot 1)+(2\cdot 5) \\ (3\cdot 1)+(1\cdot 4)+(5\cdot 2) & (3\cdot 3)+(1\cdot 1)+(5\cdot 5) \end{bmatrix} = \begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$

(g) $\left(\begin{bmatrix} (1\cdot 3)-(5\cdot 1)+(2\cdot 1) & (1\cdot 0)+(5\cdot 2)+(2\cdot 1) \\ -(1\cdot 3)-(0\cdot 1)+(1\cdot 1) & -(1\cdot 0)+(0\cdot 2)+(1\cdot 1) \\ (3\cdot 3)-(2\cdot 1)+(4\cdot 1) & (3\cdot 0)+(2\cdot 2)+(4\cdot 1) \end{bmatrix}\right)^T = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$

(h) $\left(\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (1\cdot 4)+(3\cdot 0) & -(1\cdot 1)+(3\cdot 2) \\ (4\cdot 4)+(1\cdot 0) & -(4\cdot 1)+(1\cdot 2) \\ (2\cdot 4)+(5\cdot 0) & -(2\cdot 1)+(5\cdot 2) \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (4\cdot 3)+(5\cdot 0) & -(4\cdot 1)+(5\cdot 2) & (4\cdot 1)+(5\cdot 1) \\ (16\cdot 3)-(2\cdot 0) & -(16\cdot 1)-(2\cdot 2) & (16\cdot 1)-(2\cdot 1) \\ (8\cdot 3)+(8\cdot 0) & -(8\cdot 1)+(8\cdot 2) & (8\cdot 1)+(8\cdot 1) \end{bmatrix}$

$= \begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$

(i) $$\operatorname{tr}\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} (1\cdot 1)+(5\cdot 5)+(2\cdot 2) & -(1\cdot 1)+(5\cdot 0)+(2\cdot 1) & (1\cdot 3)+(5\cdot 2)+(2\cdot 4) \\ -(1\cdot 1)+(0\cdot 5)+(1\cdot 2) & (1\cdot 1)+(0\cdot 0)+(1\cdot 1) & -(1\cdot 3)+(0\cdot 2)+(1\cdot 4) \\ (3\cdot 1)+(2\cdot 5)+(4\cdot 2) & -(3\cdot 1)+(2\cdot 0)+(4\cdot 1) & (3\cdot 3)+(2\cdot 2)+(4\cdot 4) \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} 30 & 1 & 21 \\ 1 & 2 & 1 \\ 21 & 1 & 29 \end{bmatrix}\right) = 30 + 2 + 29 = 61$$

(j) $$\operatorname{tr}\left(4\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 4\cdot 6-1 & 4\cdot(-1)-5 & 4\cdot 4-2 \\ 4\cdot 1-(-1) & 4\cdot 1-0 & 4\cdot 1-1 \\ 4\cdot 3-3 & 4\cdot 2-2 & 4\cdot 3-4 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} 23 & -9 & 14 \\ 5 & 4 & 3 \\ 9 & 6 & 8 \end{bmatrix}\right) = 23 + 4 + 8 = 35$$

(k) $$\operatorname{tr}\left(\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}\right)$$
$$\operatorname{tr}\left(\begin{bmatrix} (1\cdot 3)+(3\cdot 0) & -(1\cdot 1)+(3\cdot 2) & (1\cdot 1)+(3\cdot 1) \\ (4\cdot 3)+(1\cdot 0) & -(4\cdot 1)+(1\cdot 2) & (4\cdot 1)+(1\cdot 1) \\ (2\cdot 3)+(5\cdot 0) & -(2\cdot 1)+(5\cdot 2) & (2\cdot 1)+(5\cdot 1) \end{bmatrix} + \begin{bmatrix} 2\cdot 6 & 2\cdot(-1) & 2\cdot 4 \\ 2\cdot 1 & 2\cdot 1 & 2\cdot 1 \\ 2\cdot 3 & 2\cdot 2 & 2\cdot 3 \end{bmatrix}\right)$$
$$\operatorname{tr}\left(\begin{bmatrix} 3 & 5 & 4 \\ 12 & -2 & 5 \\ 6 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 12 & -2 & 8 \\ 2 & 2 & 2 \\ 6 & 4 & 6 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 15 & 3 & 12 \\ 14 & 0 & 7 \\ 12 & 12 & 13 \end{bmatrix}\right) = 15 + 0 + 13 = 28$$

(l) $$\operatorname{tr}\left(\left(\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\left(\begin{bmatrix} (6\cdot 1)+(1\cdot 4)+(3\cdot 2) & (6\cdot 3)+(1\cdot 1)+(3\cdot 5) \\ -(1\cdot 1)+(1\cdot 4)+(2\cdot 2) & -(1\cdot 3)+(1\cdot 1)+(2\cdot 5) \\ (4\cdot 1)+(1\cdot 4)+(3\cdot 2) & (4\cdot 3)+(1\cdot 1)+(3\cdot 5) \end{bmatrix}\right)^{T}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\left(\begin{bmatrix} 16 & 34 \\ 7 & 8 \\ 14 & 28 \end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 16 & 7 & 14 \\ 34 & 8 & 28 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} (16\cdot 3)-(7\cdot 1)+(14\cdot 1) & (16\cdot 0)+(7\cdot 2)+(14\cdot 1) \\ (34\cdot 3)-(8\cdot 1)+(28\cdot 1) & (34\cdot 0)+(8\cdot 2)+(28\cdot 1) \end{bmatrix}\right)$$
$$= \operatorname{tr}\left(\begin{bmatrix} 55 & 28 \\ 122 & 44 \end{bmatrix}\right) = 55 + 44 = 99$$

6. **(a)** $$\left(2\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1-6 & 2\cdot(-1)-1 & 2\cdot 3-3 \\ 2\cdot 5-(-1) & 2\cdot 0-1 & 2\cdot 2-2 \\ 2\cdot 2-4 & 2\cdot 1-1 & 2\cdot 4-3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -(4\cdot 3)+(3\cdot 1)+(3\cdot 1) & -(4\cdot 0)-(3\cdot 2)+(3\cdot 1) \\ (11\cdot 3)+(1\cdot 1)+(2\cdot 1) & (11\cdot 0)-(1\cdot 2)+(2\cdot 1) \\ (0\cdot 3)-(1\cdot 1)+(5\cdot 1) & (0\cdot 0)+(1\cdot 2)+(5\cdot 1) \end{bmatrix}$$
$$= \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

(b) Undefined (a 2×3 matrix $(4B)C$ cannot be added to a 2×2 matrix $2B$)

(c) $$\left(-\begin{bmatrix} (3\cdot1)+(0\cdot3) & (3\cdot4)+(0\cdot1) & (3\cdot2)+(0\cdot5)\\ -(1\cdot1)+(2\cdot3) & -(1\cdot4)+(2\cdot1) & -(1\cdot2)+(2\cdot5)\\ (1\cdot1)+(1\cdot3) & (1\cdot4)+(1\cdot1) & (1\cdot2)+(1\cdot5)\end{bmatrix}\right)^T + 5\begin{bmatrix}1&-1&3\\5&0&2\\2&1&4\end{bmatrix}$$

$$=\left(-\begin{bmatrix}3&12&6\\5&-2&8\\4&5&7\end{bmatrix}\right)^T+\begin{bmatrix}5\cdot1&5\cdot(-1)&5\cdot3\\5\cdot5&5\cdot0&5\cdot2\\5\cdot2&5\cdot1&5\cdot4\end{bmatrix}=\begin{bmatrix}-3&-5&-4\\-12&2&-5\\-6&-8&-7\end{bmatrix}+\begin{bmatrix}5&-5&15\\25&0&10\\10&5&20\end{bmatrix}$$

$$=\begin{bmatrix}-3+5&-5+(-5)&-4+15\\-12+25&2+0&-5+10\\-6+10&-8+5&-7+20\end{bmatrix}=\begin{bmatrix}2&-10&11\\13&2&5\\4&-3&13\end{bmatrix}$$

(d) $$\left(\begin{bmatrix}4&-1\\0&2\end{bmatrix}\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}-\begin{bmatrix}2\cdot1&2\cdot4&2\cdot2\\2\cdot3&2\cdot1&2\cdot5\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}(4\cdot3)-(1\cdot0)&-(4\cdot1)-(1\cdot2)&(4\cdot1)-(1\cdot1)\\(0\cdot3)+(2\cdot0)&-(0\cdot1)+(2\cdot2)&(0\cdot1)+(2\cdot1)\end{bmatrix}-\begin{bmatrix}2&8&4\\6&2&10\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}12&-6&3\\0&4&2\end{bmatrix}-\begin{bmatrix}2&8&4\\6&2&10\end{bmatrix}\right)^T=\left(\begin{bmatrix}12-2&-6-8&3-4\\0-6&4-2&2-10\end{bmatrix}\right)^T$$

$$=\left(\begin{bmatrix}10&-14&-1\\-6&2&-8\end{bmatrix}\right)^T=\begin{bmatrix}10&-6\\-14&2\\-1&-8\end{bmatrix}$$

(e) $$\begin{bmatrix}4&0\\-1&2\end{bmatrix}\left(\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}\begin{bmatrix}1&3\\4&1\\2&5\end{bmatrix}-\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}\right)$$

$$=\begin{bmatrix}4&0\\-1&2\end{bmatrix}\left(\begin{bmatrix}(1\cdot1)+(4\cdot4)+(2\cdot2)&(1\cdot3)+(4\cdot1)+(2\cdot5)\\(3\cdot1)+(1\cdot4)+(5\cdot2)&(3\cdot3)+(1\cdot1)+(5\cdot5)\end{bmatrix}\right.$$
$$\left.-\begin{bmatrix}(3\cdot3)+(1\cdot1)+(1\cdot1)&(3\cdot0)-(1\cdot2)+(1\cdot1)\\(0\cdot3)-(2\cdot1)+(1\cdot1)&(0\cdot0)+(2\cdot2)+(1\cdot1)\end{bmatrix}\right)$$

$$=\begin{bmatrix}4&0\\-1&2\end{bmatrix}\left(\begin{bmatrix}21&17\\17&35\end{bmatrix}-\begin{bmatrix}11&-1\\-1&5\end{bmatrix}\right)=\begin{bmatrix}4&0\\-1&2\end{bmatrix}\begin{bmatrix}21-11&17-(-1)\\17-(-1)&35-5\end{bmatrix}=\begin{bmatrix}4&0\\-1&2\end{bmatrix}\begin{bmatrix}10&18\\18&30\end{bmatrix}$$

$$=\begin{bmatrix}(4\cdot10)+(0\cdot18)&(4\cdot18)+(0\cdot30)\\-(1\cdot10)+(2\cdot18)&-(1\cdot18)+(2\cdot30)\end{bmatrix}=\begin{bmatrix}40&72\\26&42\end{bmatrix}$$

(f) $$\begin{bmatrix}1&-1&3\\5&0&2\\2&1&4\end{bmatrix}\begin{bmatrix}6&-1&4\\1&1&1\\3&2&3\end{bmatrix}-\left(\begin{bmatrix}6&1&3\\-1&1&2\\4&1&3\end{bmatrix}\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}\right)^T$$

$$=\begin{bmatrix}(1\cdot6)-(1\cdot1)+(3\cdot3)&-(1\cdot1)-(1\cdot1)+(3\cdot2)&(1\cdot4)-(1\cdot1)+(3\cdot3)\\(5\cdot6)+(0\cdot1)+(2\cdot3)&-(5\cdot1)+(0\cdot1)+(2\cdot2)&(5\cdot4)+(0\cdot1)+(2\cdot3)\\(2\cdot6)+(1\cdot1)+(4\cdot3)&-(2\cdot1)+(1\cdot1)+(4\cdot2)&(2\cdot4)+(1\cdot1)+(4\cdot3)\end{bmatrix}$$
$$-\left(\begin{bmatrix}(6\cdot1)-(1\cdot1)+(3\cdot3)&(6\cdot5)+(1\cdot0)+(3\cdot2)&(6\cdot2)+(1\cdot1)+(3\cdot4)\\-(1\cdot1)-(1\cdot1)+(2\cdot3)&-(1\cdot5)+(1\cdot0)+(2\cdot2)&-(1\cdot2)+(1\cdot1)+(2\cdot4)\\(4\cdot1)-(1\cdot1)+(3\cdot3)&(4\cdot5)+(1\cdot0)+(3\cdot2)&(4\cdot2)+(1\cdot1)+(3\cdot4)\end{bmatrix}\right)^T$$

$$=\begin{bmatrix}14&4&12\\36&-1&26\\25&7&21\end{bmatrix}-\left(\begin{bmatrix}14&36&25\\4&-1&7\\12&26&21\end{bmatrix}\right)^T=\begin{bmatrix}14&4&12\\36&-1&26\\25&7&21\end{bmatrix}-\begin{bmatrix}14&4&12\\36&-1&26\\25&7&21\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

7. **(a)** first row of AB = [first row of A] B $= \begin{bmatrix} 3 & -2 & 7 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$

$= [(3 \cdot 6) - (2 \cdot 0) + (7 \cdot 7) \quad -(3 \cdot 2) - (2 \cdot 1) + (7 \cdot 7) \quad (3 \cdot 4) - (2 \cdot 3) + (7 \cdot 5)]$
$= \begin{bmatrix} 67 & 41 & 41 \end{bmatrix}$

(b) third row of AB = [third row of A] B $= \begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$

$= [(0 \cdot 6) + (4 \cdot 0) + (9 \cdot 7) \quad -(0 \cdot 2) + (4 \cdot 1) + (9 \cdot 7) \quad (0 \cdot 4) + (4 \cdot 3) + (9 \cdot 5)]$
$= \begin{bmatrix} 63 & 67 & 57 \end{bmatrix}$

(c) second column of AB = A [second column of B]

$$= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -(3 \cdot 2) - (2 \cdot 1) + (7 \cdot 7) \\ -(6 \cdot 2) + (5 \cdot 1) + (4 \cdot 7) \\ -(0 \cdot 2) + (4 \cdot 1) + (9 \cdot 7) \end{bmatrix} = \begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix}$$

(d) first column of BA = B [first column of A]

$$= \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} (6 \cdot 3) - (2 \cdot 6) + (4 \cdot 0) \\ (0 \cdot 3) + (1 \cdot 6) + (3 \cdot 0) \\ (7 \cdot 3) + (7 \cdot 6) + (5 \cdot 0) \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$$

(e) third row of AA = [third row of A] A $= \begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$

$= [(0 \cdot 3) + (4 \cdot 6) + (9 \cdot 0) \quad -(0 \cdot 2) + (4 \cdot 5) + (9 \cdot 4) \quad (0 \cdot 7) + (4 \cdot 4) + (9 \cdot 9)]$
$= \begin{bmatrix} 24 & 56 & 97 \end{bmatrix}$

(f) third column of AA = A [third column of A]

$$= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} (3 \cdot 7) - (2 \cdot 4) + (7 \cdot 9) \\ (6 \cdot 7) + (5 \cdot 4) + (4 \cdot 9) \\ (0 \cdot 7) + (4 \cdot 4) + (9 \cdot 9) \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$$

8. **(a)** first column of AB = A [first column of B]

$$= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} (3 \cdot 6) - (2 \cdot 0) + (7 \cdot 7) \\ (6 \cdot 6) + (5 \cdot 0) + (4 \cdot 7) \\ (0 \cdot 6) + (4 \cdot 0) + (9 \cdot 7) \end{bmatrix} = \begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix}$$

(b) third column of BB = B [third column of B]

$$= \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} (6 \cdot 4) - (2 \cdot 3) + (4 \cdot 5) \\ (0 \cdot 4) + (1 \cdot 3) + (3 \cdot 5) \\ (7 \cdot 4) + (7 \cdot 3) + (5 \cdot 5) \end{bmatrix} = \begin{bmatrix} 38 \\ 18 \\ 74 \end{bmatrix}$$

(c) second row of BB = [second row of B] B $= \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$

$= [(0 \cdot 6) + (1 \cdot 0) + (3 \cdot 7) \quad -(0 \cdot 2) + (1 \cdot 1) + (3 \cdot 7) \quad (0 \cdot 4) + (1 \cdot 3) + (3 \cdot 5)]$
$= \begin{bmatrix} 21 & 22 & 18 \end{bmatrix}$

(d) first column of $AA = A$ [first column of A]

$$= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} (3 \cdot 3) - (2 \cdot 6) + (7 \cdot 0) \\ (6 \cdot 3) + (5 \cdot 6) + (4 \cdot 0) \\ (0 \cdot 3) + (4 \cdot 6) + (9 \cdot 0) \end{bmatrix} = \begin{bmatrix} -3 \\ 48 \\ 24 \end{bmatrix}$$

(e) third column of $AB = A$ [third column of B]

$$= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} (3 \cdot 4) - (2 \cdot 3) + (7 \cdot 5) \\ (6 \cdot 4) + (5 \cdot 3) + (4 \cdot 5) \\ (0 \cdot 4) + (4 \cdot 3) + (9 \cdot 5) \end{bmatrix} = \begin{bmatrix} 41 \\ 59 \\ 57 \end{bmatrix}$$

(f) first row of BA = [first row of B] $A = \begin{bmatrix} 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix}$

$$= \begin{bmatrix} (6 \cdot 3) - (2 \cdot 6) + (4 \cdot 0) & -(6 \cdot 2) - (2 \cdot 5) + (4 \cdot 4) & (6 \cdot 7) - (2 \cdot 4) + (4 \cdot 9) \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -6 & 70 \end{bmatrix}$$

9. **(a)** first column of $AA = 3 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 48 \\ 24 \end{bmatrix}$

second column of $AA = -2 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 4 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 29 \\ 56 \end{bmatrix}$

third column of $AA = 7 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 9 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$

(b) first column of $BB = 6 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 64 \\ 21 \\ 77 \end{bmatrix}$

second column of $BB = -2 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 28 \end{bmatrix}$

third column of $BB = 4 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 38 \\ 18 \\ 74 \end{bmatrix}$

10. **(a)** first column of $AB = 6 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix}$

second column of $AB = -2 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix}$

third column of $AB = 4 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 41 \\ 59 \\ 57 \end{bmatrix}$

(b) first column of $BA = 3 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$

$$\text{second column of } BA = -2\begin{bmatrix}6\\0\\7\end{bmatrix}+5\begin{bmatrix}-2\\1\\7\end{bmatrix}+4\begin{bmatrix}4\\3\\5\end{bmatrix}=\begin{bmatrix}-6\\17\\41\end{bmatrix}$$

$$\text{third column of } BA = 7\begin{bmatrix}6\\0\\7\end{bmatrix}+4\begin{bmatrix}-2\\1\\7\end{bmatrix}+9\begin{bmatrix}4\\3\\5\end{bmatrix}=\begin{bmatrix}70\\31\\122\end{bmatrix}$$

11. **(a)** $A=\begin{bmatrix}2&-3&5\\9&-1&1\\1&5&4\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}7\\-1\\0\end{bmatrix}$; the matrix equation: $\begin{bmatrix}2&-3&5\\9&-1&1\\1&5&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7\\-1\\0\end{bmatrix}$

(b) $A=\begin{bmatrix}4&0&-3&1\\5&1&0&-8\\2&-5&9&-1\\0&3&-1&7\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}1\\3\\0\\2\end{bmatrix}$; the matrix equation: $\begin{bmatrix}4&0&-3&1\\5&1&0&-8\\2&-5&9&-1\\0&3&-1&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\3\\0\\2\end{bmatrix}$

12. **(a)** $A=\begin{bmatrix}1&-2&3\\2&1&0\\0&-3&4\\1&0&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-3\\0\\1\\5\end{bmatrix}$; the matrix equation: $\begin{bmatrix}1&-2&3\\2&1&0\\0&-3&4\\1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-3\\0\\1\\5\end{bmatrix}$

(b) $A=\begin{bmatrix}3&3&3\\-1&-5&-2\\0&-4&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\mathbf{b}=\begin{bmatrix}-3\\3\\0\end{bmatrix}$; the matrix equation: $\begin{bmatrix}3&3&3\\-1&-5&-2\\0&-4&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-3\\3\\0\end{bmatrix}$

13. **(a)**
$$\begin{array}{rcrcrcr}5x_1&+&6x_2&-&7x_3&=&2\\-x_1&-&2x_2&+&3x_3&=&0\\&&4x_2&-&x_3&=&3\end{array}$$

(b)
$$\begin{array}{rcrcrcr}x&+&y&+&z&=&2\\2x&+&3y&&&=&2\\5x&-&3y&-&6z&=&-9\end{array}$$

14. **(a)**
$$\begin{array}{rcrcrcr}3x_1&-&x_2&+&2x_3&=&2\\4x_1&+&3x_2&+&7x_3&=&-1\\-2x_1&+&x_2&+&5x_3&=&4\end{array}$$

(b)
$$\begin{array}{rcrcrcrcr}3w&-&2x&&&+&z&=&0\\5w&&&+&2y&-&2z&=&0\\3w&+&x&+&4y&+&7z&=&0\\-2w&+&5x&+&y&+&6z&=&0\end{array}$$

15. $\begin{bmatrix}k&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&2\\0&2&-3\end{bmatrix}\begin{bmatrix}k\\1\\1\end{bmatrix}=\begin{bmatrix}k&1&1\end{bmatrix}\begin{bmatrix}k+1\\k+2\\-1\end{bmatrix}=k^2+k+k+2-1=k^2+2k+1=(k+1)^2$

The only value of k that satisfies the equation is $k=-1$.

16. $\begin{bmatrix}2&2&k\end{bmatrix}\begin{bmatrix}1&2&0\\2&0&3\\0&3&1\end{bmatrix}\begin{bmatrix}2\\2\\k\end{bmatrix}=\begin{bmatrix}2&2&k\end{bmatrix}\begin{bmatrix}6\\3k+4\\k+6\end{bmatrix}=k^2+12k+20=(k+10)(k+2)$

The values of k that satisfy the equation are $k=-10$ and $k=-2$.

17. $\begin{bmatrix}4\\2\end{bmatrix}\begin{bmatrix}0&1&2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}\begin{bmatrix}-2&3&1\end{bmatrix}=\begin{bmatrix}0&4&8\\0&2&4\end{bmatrix}+\begin{bmatrix}6&-9&-3\\2&-3&-1\end{bmatrix}=\begin{bmatrix}6&-5&5\\2&-1&3\end{bmatrix}$

18. $\begin{bmatrix}0\\4\end{bmatrix}\begin{bmatrix}1&4&1\end{bmatrix}+\begin{bmatrix}-2\\-3\end{bmatrix}\begin{bmatrix}-3&0&2\end{bmatrix}=\begin{bmatrix}0&0&0\\4&16&4\end{bmatrix}+\begin{bmatrix}6&0&-4\\9&0&-6\end{bmatrix}=\begin{bmatrix}6&0&-4\\13&16&-2\end{bmatrix}$

19. $\begin{bmatrix}1\\4\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix}+\begin{bmatrix}2\\5\end{bmatrix}\begin{bmatrix}3&4\end{bmatrix}+\begin{bmatrix}3\\6\end{bmatrix}\begin{bmatrix}5&6\end{bmatrix}=\begin{bmatrix}1&2\\4&8\end{bmatrix}+\begin{bmatrix}6&8\\15&20\end{bmatrix}+\begin{bmatrix}15&18\\30&36\end{bmatrix}=\begin{bmatrix}22&28\\49&64\end{bmatrix}$

20. $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}2&-1\end{bmatrix}+\begin{bmatrix}4\\-2\end{bmatrix}\begin{bmatrix}4&0\end{bmatrix}+\begin{bmatrix}2\\5\end{bmatrix}\begin{bmatrix}1&-1\end{bmatrix}=\begin{bmatrix}0&0\\2&-1\end{bmatrix}+\begin{bmatrix}16&0\\-8&0\end{bmatrix}+\begin{bmatrix}2&-2\\5&-5\end{bmatrix}=\begin{bmatrix}18&-2\\-1&-6\end{bmatrix}$

21. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} -3r \\ r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4s \\ 0 \\ -2s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

22. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -3r \\ r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4s \\ 0 \\ -2s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

23. The given matrix equation is equivalent to the linear system

$$\begin{aligned} a &= 4 \\ 3 &= d - 2c \\ -1 &= d + 2c \\ a + b &= -2 \end{aligned}$$

After subtracting first equation from the fourth, adding the second to the third, and back-substituting, we obtain the solution: $a = 4$, $b = -6$, $c = -1$, and $d = 1$.

24. The given matrix equation is equivalent to the linear system

$$\begin{array}{rcrcrcrcl} a & - & b & & & & & = & 8 \\ a & + & b & & & & & = & 1 \\ & & & & c & + & 3d & = & 7 \\ & & & - & c & + & 2d & = & 6 \end{array}$$

After subtracting first equation from the second, adding the third to the fourth, and back-substituting, we obtain the solution: $a = \frac{9}{2}$, $b = -\frac{7}{2}$, $c = -\frac{4}{5}$, and $d = \frac{13}{5}$.

25. **(a)** If the ith row vector of A is $[0 \quad \cdots \quad 0]$ then it follows from Formula (9) in Section 1.3 that

ith row vector of $AB = [0 \quad \cdots \quad 0]\, B = [0 \quad \cdots \quad 0]$

(b) If the jth column vector of B is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the jth column vector of $AB = A\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

26. **(a)** $$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$$

(b) $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$$

(c) $\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$ **(d)** $\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix}$

27. Setting the left hand side $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}$ equal to $\begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$ yields

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= x + y \\ a_{21}x + a_{22}y + a_{23}z &= x - y \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned}$$

Assuming the entries of A are real numbers that do not depend on x, y, and z, this requires that the coefficients corresponding to the same variable on both sides of each equation must match.

Therefore, the only matrix satisfying the given condition is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

28. Setting the left hand side $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}$ equal to $\begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$ yields

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= xy \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned}$$

Assuming the entries of A are real numbers that do not depend on x, y, and z, it follows that no real numbers a_{11}, a_{12}, and a_{13} exist for which the first equation is satisfied for all x, y, and z. Therefore no matrix A with real number entries can satisfy the given condition.

(Note that if A were permitted to depend on x, y, and z, then solutions do exist e.g., $A = \begin{bmatrix} y & 0 & 0 \\ z & 0 & -x \\ 0 & z & -y \end{bmatrix}$.)

29. **(a)** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

(b) Four square roots can be found: $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$, and $\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$.

32. **(a)** $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

33. The given matrix product represents $\begin{bmatrix} \text{the total cost of items purchased in January} \\ \text{the total cost of items purchased in February} \\ \text{the total cost of items purchased in March} \\ \text{the total cost of items purchased in April} \end{bmatrix}$.

34. **(a)** The 4×3 matrix $M + J$ represents sales over the two month period.

(b) The 4×3 matrix $M - J$ represents the decrease in sales of each item from May to June.

(c) $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\mathbf{y} = [1 \quad 1 \quad 1 \quad 1]$

(e) The entry in the 1×1 matrix $\mathbf{y}M\mathbf{x}$ represents the total number of items sold in May.

True-False Exercises

(a) True. The main diagonal is only defined for square matrices.

(b) False. An $m \times n$ matrix has m row vectors and n column vectors.

(c) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not equal $BA = B$.

(d) False. The ith row vector of AB can be computed by multiplying the ith row vector of A by B.

(e) True. Using Formula (14), $((A^T)^T)_{ij} = (A^T)_{ji} = (A)_{ij}$.

(f) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then the trace of $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is 0, which does not equal $\mathrm{tr}(A)\mathrm{tr}(B) = 1$.

(g) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $(AB)^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not equal $A^T B^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(h) True. The main diagonal entries in a square matrix A are the same as those in A^T.

(i) True. Since A^T is a 4×6 matrix, it follows from $B^T A^T$ being a 2×6 matrix that B^T must be a 2×4 matrix. Consequently, B is a 4×2 matrix.

(j) True.

$$\mathrm{tr}\left(c \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \right) = \mathrm{tr}\left(\begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nn} \end{bmatrix} \right)$$

$$= ca_{11} + \cdots + ca_{nn} = c(a_{11} + \cdots + a_{nn}) = c\,\mathrm{tr}\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \right)$$

(k) True. The equality of the matrices $A - C$ and $B - C$ implies that $a_{ij} - c_{ij} = b_{ij} - c_{ij}$ for all i and j. Adding c_{ij} to both sides yields $a_{ij} = b_{ij}$ for all i and j. Consequently, the matrices A and B are equal.

(l) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AC = BC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ even though $A \neq B$.

(m) True. If A is a $p \times q$ matrix and B is an $r \times s$ matrix then AB being defined requires $q = r$ and BA being defined requires $s = p$. For the $p \times p$ matrix AB to be possible to add to the $q \times q$ matrix BA, we must have $p = q$.

(n) True. If the jth column vector of B is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that the jth column vector of $AB = A\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$.

(o) False. E.g., if $A = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ then $BA = A$ does not have a column of zeros even though B does.

1.4 Inverses; Algebraic Properties of Matrices

1. **(a)** $A + (B + C) = (A + B) + C = \begin{bmatrix}7 & 2\\ 0 & -2\end{bmatrix}$ **(b)** $A(BC) = (AB)C = \begin{bmatrix}-34 & -21\\ 52 & 28\end{bmatrix}$

(c) $A(B + C) = AB + AC = \begin{bmatrix}14 & 15\\ 0 & -18\end{bmatrix}$ **(d)** $(a + b)C = aC + bC = \begin{bmatrix}-12 & -3\\ 9 & 6\end{bmatrix}$

2. **(a)** $a(BC) = (aB)C = B(aC) = \begin{bmatrix}-24 & -16\\ 64 & 36\end{bmatrix}$

(b) $A(B - C) = AB - AC = \begin{bmatrix}-16 & 5\\ 8 & -6\end{bmatrix}$

(c) $(B + C)A = BA + CA = \begin{bmatrix}18 & 8\\ -18 & -22\end{bmatrix}$

(d) $a(bC) = (ab)C = \begin{bmatrix}-112 & -28\\ 84 & 56\end{bmatrix}$

3. **(a)** $(A^T)^T = A = \begin{bmatrix}3 & -1\\ 2 & 4\end{bmatrix}$ **(b)** $(AB)^T = B^TA^T = \begin{bmatrix}-1 & 4\\ 10 & -12\end{bmatrix}$

4. **(a)** $(A + B)^T = A^T + B^T = \begin{bmatrix}3 & 3\\ 1 & 0\end{bmatrix}$ **(b)** $(aC)^T = aC^T = \begin{bmatrix}16 & -12\\ 4 & -8\end{bmatrix}$

5. The determinant of A, $\det(A) = (2)(4) - (-3)(4) = 20$, is nonzero. Therefore A is invertible and its inverse is $A^{-1} = \frac{1}{20}\begin{bmatrix}4 & 3\\ -4 & 2\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{3}{20}\\ -\frac{1}{5} & \frac{1}{10}\end{bmatrix}$.

6. The determinant of B, $\det(B) = (3)(2) - (1)(5) = 1$, is nonzero. Therefore B is invertible and its inverse is $B^{-1} = \begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}$.

7. The determinant of C, $\det(C) = (2)(3) - (0)(0) = 6$, is nonzero. Therefore C is invertible and its inverse is $C^{-1} = \frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}$.

8. The determinant of D, $\det(D) = (6)(-1) - (4)(-2) = 2$, is nonzero. Therefore D is invertible and its inverse is $D^{-1} = \frac{1}{2}\begin{bmatrix}-1 & -4\\ 2 & 6\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & -2\\ 1 & 3\end{bmatrix}$.

9. The determinant of $A = \begin{bmatrix} \frac{1}{2}(e^x+e^{-x}) & \frac{1}{2}(e^x-e^{-x}) \\ \frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x}) \end{bmatrix}$,
$\det(A) = \frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x-e^{-x})^2 = \frac{1}{4}(e^{2x}+2+e^{-2x}) - \frac{1}{4}(e^{2x}-2+e^{-2x}) = \frac{1}{4}(2+2) = 1$ is nonzero. Therefore A is invertible and its inverse is $A^{-1} = \begin{bmatrix} \frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x}) \\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x}) \end{bmatrix}$.

10. The determinant of the matrix is $(\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta) = 1 \neq 0$. Therefore the matrix is invertible and its inverse is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

11. $A^T = \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}$; $(A^T)^{-1} = \frac{1}{(2)(4)-(4)(-3)}\begin{bmatrix} 4 & -4 \\ 3 & 2 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix}$

$A^{-1} = \frac{1}{(2)(4)-(-3)(4)}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$; $(A^{-1})^T = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix}$

12. $A^{-1} = \frac{1}{(2)(4)-(-3)(4)}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$;

$(A^{-1})^{-1} = \frac{1}{\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)-\left(\frac{3}{20}\right)\left(-\frac{1}{5}\right)}\begin{bmatrix} \frac{1}{10} & -\frac{3}{20} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \frac{1}{\frac{5}{100}}\begin{bmatrix} \frac{1}{10} & -\frac{3}{20} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = 20\begin{bmatrix} \frac{1}{10} & -\frac{3}{20} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix} = A$

13. $ABC = \begin{bmatrix} -18 & -12 \\ 64 & 36 \end{bmatrix}$; $(ABC)^{-1} = \frac{1}{(-18)(36)-(-12)(64)}\begin{bmatrix} 36 & 12 \\ -64 & -18 \end{bmatrix} = \frac{1}{120}\begin{bmatrix} 36 & 12 \\ -64 & -18 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{8}{15} & -\frac{3}{20} \end{bmatrix}$

$C^{-1}B^{-1}A^{-1} = \left(\frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\left(\frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{8}{15} & -\frac{3}{20} \end{bmatrix}$

14. $ABC = \begin{bmatrix} -18 & -12 \\ 64 & 36 \end{bmatrix}$; $(ABC)^T = \begin{bmatrix} -18 & 64 \\ -12 & 36 \end{bmatrix}$; $C^TB^TA^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -18 & 64 \\ -12 & 36 \end{bmatrix}$

15. From part (a) of Theorem 1.4.7 it follows that the inverse of $(7A)^{-1}$ is $7A$.

Thus $7A = \frac{1}{(-3)(-2)-(-7)(1)}\begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$. Consequently, $A = \frac{1}{7}\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & 1 \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$.

16. From part (a) of Theorem 1.4.7 it follows that the inverse of $(5A^T)^{-1}$ is $5A^T$.

Thus $5A^T = \frac{1}{-1}\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$. Consequently, $A = \begin{bmatrix} -\frac{2}{5} & 1 \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$.

17. From part (a) of Theorem 1.4.7 it follows that the inverse of $(I+2A)^{-1}$ is $I+2A$.

Thus $I+2A=\frac{1}{(-1)(5)-(2)(4)}\begin{bmatrix}5 & -2\\ -4 & -1\end{bmatrix}=\frac{1}{-13}\begin{bmatrix}5 & -2\\ -4 & -1\end{bmatrix}=\begin{bmatrix}-\frac{5}{13} & \frac{2}{13}\\ \frac{4}{13} & \frac{1}{13}\end{bmatrix}$.

Consequently, $A=\frac{1}{2}\left(\begin{bmatrix}-\frac{5}{13} & \frac{2}{13}\\ \frac{4}{13} & \frac{1}{13}\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}-\frac{9}{13} & \frac{1}{13}\\ \frac{2}{13} & -\frac{6}{13}\end{bmatrix}$

18. From part (a) of Theorem 1.4.7 we have $A=(A^{-1})^{-1}$. Therefore $A=\frac{1}{13}\begin{bmatrix}5 & 1\\ -3 & 2\end{bmatrix}=\begin{bmatrix}\frac{5}{13} & \frac{1}{13}\\ -\frac{3}{13} & \frac{2}{13}\end{bmatrix}$.

19. **(a)** $A^3=AAA=\begin{bmatrix}41 & 15\\ 30 & 11\end{bmatrix}$

(b) $(A^3)^{-1}=\frac{1}{(41)(11)-(15)(30)}\begin{bmatrix}11 & -15\\ -30 & 41\end{bmatrix}=\begin{bmatrix}11 & -15\\ -30 & 41\end{bmatrix}$

(c) $A^2-2A+I=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}-2\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 4\\ 8 & 3\end{bmatrix}-\begin{bmatrix}6 & 2\\ 4 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}6 & 2\\ 4 & 2\end{bmatrix}$

20. **(a)** $A^3=AAA=\begin{bmatrix}8 & 0\\ 28 & 1\end{bmatrix}$

(b) $(A^3)^{-1}=\frac{1}{(8)(1)-(0)(28)}\begin{bmatrix}1 & 0\\ -28 & 8\end{bmatrix}=\frac{1}{8}\begin{bmatrix}1 & 0\\ -28 & 8\end{bmatrix}=\begin{bmatrix}\frac{1}{8} & 0\\ -\frac{7}{2} & 1\end{bmatrix}$

(c) $A^2-2A+I=\begin{bmatrix}2 & 0\\ 4 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 4 & 1\end{bmatrix}-2\begin{bmatrix}2 & 0\\ 4 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}4 & 0\\ 12 & 1\end{bmatrix}-\begin{bmatrix}4 & 0\\ 8 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 4 & 0\end{bmatrix}$

21. **(a)** $A-2I=\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$ **(b)** $2A^2-A+I=\begin{bmatrix}20 & 7\\ 14 & 6\end{bmatrix}$ **(c)** $A^3-2A+I=\begin{bmatrix}36 & 13\\ 26 & 10\end{bmatrix}$

22. **(a)** $A-2I=\begin{bmatrix}0 & 0\\ 4 & -1\end{bmatrix}$ **(b)** $2A^2-A+I=\begin{bmatrix}7 & 0\\ 20 & 2\end{bmatrix}$ **(c)** $A^3-2A+I=\begin{bmatrix}5 & 0\\ 20 & 0\end{bmatrix}$

23. $AB=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & a\\ 0 & c\end{bmatrix}$; $BA=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}c & d\\ 0 & 0\end{bmatrix}$.

The matrices A and B commute if $\begin{bmatrix}0 & a\\ 0 & c\end{bmatrix}=\begin{bmatrix}c & d\\ 0 & 0\end{bmatrix}$, i.e.

$$\begin{aligned}0&=c\\ a&=d\\ 0&=0\\ c&=0\end{aligned}$$

Therefore, $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ commute if $c=0$ and $a=d$.

If we assign b and d the arbitrary values s and t, respectively, the general solution is given by the formulas

$$a=t,\quad b=s,\quad c=0,\quad d=t$$

24. $AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$; $CA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$.

The matrices A and C commute if $\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$, i.e.

$$\begin{aligned} b &= 0 \\ 0 &= 0 \\ d &= a \\ 0 &= b \end{aligned}$$

Therefore, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ commute if $b = 0$ and $a = d$.

If we assign c and d the arbitrary values s and t, respectively, the general solution is given by the formulas

$$a = t, \quad b = 0, \quad c = s, \quad d = t$$

25. $x_1 = \frac{(5)(-1)-(-2)(3)}{(3)(5)-(-2)(4)} = \frac{1}{23}$, $x_2 = \frac{(3)(3)-(4)(-1)}{(3)(5)-(-2)(4)} = \frac{13}{23}$

26. $x_1 = \frac{(-3)(4)-(5)(1)}{(-1)(-3)-(5)(-1)} = -\frac{17}{8}$, $x_2 = \frac{(-1)(1)-(-1)(4)}{(-1)(-3)-(5)(-1)} = \frac{3}{8}$

27. $x_1 = \frac{(-3)(0)-(1)(-2)}{(6)(-3)-(1)(4)} = \frac{2}{-22} = -\frac{1}{11}$, $x_2 = \frac{(6)(-2)-(4)(0)}{(6)(-3)-(1)(4)} = \frac{-12}{-22} = \frac{6}{11}$

28. $x_1 = \frac{(4)(4)-(-2)(4)}{(2)(4)-(-2)(1)} = \frac{24}{10} = \frac{12}{5}$, $x_2 = \frac{(2)(4)-(1)(4)}{(2)(4)-(-2)(1)} = \frac{4}{10} = \frac{2}{5}$

29. $p(A) = A^2 - 9I = \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix}$,

$p_1(A) = A + 3I = \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix}$, $p_2(A) = A - 3I = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$, $p_1(A)p_2(A) = \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix}$

30.
$$\begin{aligned} p_1(A)p_2(A) &= (A+3I)(A-3I) \\ &= A(A-3I) + (3I)(A-3I) && \longleftarrow \text{Theorem 1.4.1(e)} \\ &= (A^2 - A(3I)) + ((3I)A - (3I)(3I)) && \longleftarrow \text{Theorem 1.4.1(i)} \\ &= (A^2 - 3(AI)) + (3(IA) - 9II) && \longleftarrow \text{Theorem 1.4.1(m)} \\ &= (A^2 - 3A) + (3A - 9I) && \longleftarrow \text{Property } AI = IA = A \text{ on p. 43} \\ &= A^2 - 9I = p(A) && \longleftarrow \text{Theorem 1.4.1(b)} \end{aligned}$$

31. **(a)** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $(A+B)(A-B) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ does not equal $A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Using the properties in Theorem 1.4.1 we can write

$$(A+B)(A-B) = A(A-B) + B(A-B) = A^2 - AB + BA - B^2$$

(c) If the matrices A and B commute (i.e., $AB = BA$) then $(A+B)(A-B) = A^2 - B^2$.

32. We can let A be one of the following eight matrices: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$.

Note that these eight are not the only solutions - e.g., A can be $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, etc.

33. **(a)** We can rewrite the equation

$$\begin{aligned} A^2+2A+I&=O\\ A^2+2A&=-I\\ -A^2-2A&=I\\ A(-A-2I)&=I \end{aligned}$$

which shows that A is invertible and $A^{-1}=-A-2I$.

(b) Let $p(x)=c_nx^n+\cdots+c_2x^2+c_1x+c_0$ with $c_0\neq 0$. The equation $p(A)=O$ can be rewritten as

$$\begin{aligned} c_nA^n+\cdots+c_2A^2+c_1A+c_0I&=O\\ c_nA^n+\cdots+c_2A^2+c_1A&=-c_0I\\ -\frac{c_n}{c_0}A^n-\cdots-\frac{c_2}{c_0}A^2-\frac{c_1}{c_0}A&=I\\ A\left(-\frac{c_n}{c_0}A^{n-1}-\cdots-\frac{c_2}{c_0}A-\frac{c_1}{c_0}I\right)&=I \end{aligned}$$

which shows that A is invertible and $A^{-1}=-\frac{c_n}{c_0}A^{n-1}-\cdots-\frac{c_2}{c_0}A-\frac{c_1}{c_0}I$.

34. If $A^3=I$ then it follows that $AA^2=I$ therefore A must be invertible $(A^{-1}=A^2)$.

35. If the ith row vector of A is $[0 \ \ \cdots \ \ 0]$ then it follows from Formula (9) in Section 1.3 that
ith row vector of $AB = [0 \ \ \cdots \ \ 0]\,B = [0 \ \ \cdots \ \ 0]$.
Consequently no matrix B can be found to make the product $AB=I$ thus A does not have an inverse.

If the jth column vector of A is $\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that
the jth column vector of $BA = B\begin{bmatrix}0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}0\\\vdots\\0\end{bmatrix}$.
Consequently no matrix B can be found to make the product $BA=I$ thus A does not have an inverse.

36. If the ith and jth row vectors of A are equal then it follows from Formula (9) in Section 1.3 that
ith row vector of AB = jth row vector of AB.
Consequently no matrix B can be found to make the product $AB=I$ thus A does not have an inverse.

If the ith and jth column vectors of A are equal then it follows from Formula (8) in Section 1.3 that
the ith column vector of BA = the jth column vector of BA

Consequently no matrix B can be found to make the product $BA = I$ thus A does not have an inverse.

37. Letting $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$, the matrix equation $AX = I$ becomes

$$\begin{bmatrix} x_{11}+x_{31} & x_{12}+x_{32} & x_{13}+x_{33} \\ x_{11}+x_{21} & x_{12}+x_{22} & x_{13}+x_{23} \\ x_{21}+x_{31} & x_{22}+x_{32} & x_{23}+x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Setting the first columns on both sides equal yields the system

$$\begin{aligned} x_{11} + x_{31} &= 1 \\ x_{11} + x_{21} &= 0 \\ x_{21} + x_{31} &= 0 \end{aligned}$$

Subtracting the second and third equations from the first leads to $-2x_{21} = 1$. Therefore $x_{21} = -\frac{1}{2}$ and (after substituting this into the remaining equations) $x_{11} = x_{31} = \frac{1}{2}$.

The second and the third columns can be treated in a similar manner to result in

$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. We conclude that A invertible and its inverse is $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

38. Letting $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$, the matrix equation $AX = I$ becomes

$$\begin{bmatrix} x_{11}+x_{21}+x_{31} & x_{12}+x_{22}+x_{32} & x_{13}+x_{23}+x_{33} \\ x_{11} & x_{12} & x_{13} \\ x_{21}+x_{31} & x_{22}+x_{32} & x_{23}+x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Although this corresponds to a system of nine equations, it is sufficient to examine just the three equations corresponding to the first column

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 1 \\ x_{11} &= 0 \\ x_{21} + x_{31} &= 0 \end{aligned}$$

to see that subtracting the second and third equations from the first leads to a contradiction $0 = 1$. We conclude that A is not invertible.

39. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

$= (B^{-1}A^{-1})(AC^{-1})((C^{-1})^{-1}(D^{-1})^{-1})D^{-1}$ ⟵ Theorem 1.4.6

$= (B^{-1}A^{-1})(AC^{-1})(CD)D^{-1}$ ⟵ Theorem 1.4.7(a)

$= B^{-1}(A^{-1}A)(C^{-1}C)(DD^{-1})$ ⟵ Theorem 1.4.1(c)

$= B^{-1}III$ ⟵ Formula (1) in Section 1.4

$= B^{-1}$ ⟵ Property $AI = IA = A$ on p. 43

40. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

$= ((C^{-1})^{-1}A^{-1})(AC^{-1})((C^{-1})^{-1}A^{-1})AD^{-1}$ ⟵ Theorem 1.4.6

$= (CA^{-1})(AC^{-1})(CA^{-1})AD^{-1}$ ⟵ Theorem 1.4.7(a)

$= C(A^{-1}A)(C^{-1}C)(A^{-1}A)D^{-1}$ ⟵ Theorem 1.4.1(c)

$= CIIID^{-1}$ ⟵ Formula (1) in Section 1.4

$= CD^{-1}$ ⟵ Property $AI = IA = A$ on p. 43

41. If $R = [r_1 \quad \cdots \quad r_n]$ and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ then $CR = \begin{bmatrix} c_1r_1 & \cdots & c_1r_n \\ \vdots & \ddots & \vdots \\ c_nr_1 & \cdots & c_nr_n \end{bmatrix}$ and $RC = [r_1c_1 + \cdots + r_nc_n] = [\mathrm{tr}(CR)]$.

42. Yes, it is true. From part (e) of Theorem 1.4.8, it follows that $(A^2)^T = (AA)^T = A^TA^T = (A^T)^2$. This statement can be extended to n factors (see p. 49) so that

$$(A^n)^T = (\underbrace{AA \cdots A}_{n \text{ factors}})^T = \underbrace{A^TA^T \cdots A^T}_{n \text{ factors}} = (A^T)^n$$

43. **(a)** Assuming A is invertible, we can multiply (on the left) each side of the equation by A^{-1}:

$AB = AC$

$A^{-1}(AB) = A^{-1}(AC)$ ⟵ Multiply (on the left) each side by A^{-1}

$(A^{-1}A)B = (A^{-1}A)C$ ⟵ Theorem 1.4.1(c)

$IB = IC$ ⟵ Formula (1) in Section 1.4

$B = C$ ⟵ Property $AI = IA = A$ on p. 43

(b) If A is not an invertible matrix then $AB = AC$ does not generally imply $B = C$ as evidenced by Example 3.

44. Invertibility of A implies that A is a square matrix, which is all that is required.
By repeated application of Theorem 1.4.1(m) and (l), we have

$$(kA)^n = \underbrace{(kA) \cdots (kA)(kA)(kA)(kA)}_{n \text{ factors}} = \underbrace{(kA) \cdots (kA)(kA)}_{n-2 \text{ factors}} k^2A^2 = \underbrace{(kA) \cdots (kA)}_{n-3 \text{ factors}} k^3A^3 = \cdots = k^nA^n$$

45. **(a)** $A(A^{-1} + B^{-1})B(A + B)^{-1}$

$= (AA^{-1}B + AB^{-1}B)(A + B)^{-1}$ ⟵ Theorem 1.4.1(d) and (e)

$= (IB + AI)(A + B)^{-1}$ ⟵ Formula (1) in Section 1.4

$= (B + A)(A + B)^{-1}$ ⟵ Property $AI = IA = A$ on p. 43

$= (A+B)(A+B)^{-1}$ ← Theorem 1.4.1(a)

$= I$ ← Formula (1) in Section 1.4

(b) We can multiply each side of the equality from part (a) on the left by A^{-1}, then on the right by A to obtain

$$(A^{-1}+B^{-1})B(A+B)^{-1}A = I$$

which shows that if A, B, and $A+B$ are invertible then so is $A^{-1}+B^{-1}$.
Furthermore, $(A^{-1}+B^{-1})^{-1} = B(A+B)^{-1}A$.

46. **(a)** $(I-A)^2$

$= (I-A)(I-A)$

$= II - IA - AI + AA$ ← Theorem 1.4.1(f) and (g)

$= I - A - A + A^2$ ← Property $AI = IA = A$ on p. 43

$= I - A - A + A$ ← A is idempotent so $A^2 = A$

$= I - A$

(b) $(2A-I)(2A-I)$

$= (2A)(2A) - 2AI - I(2A) + II$ ← Theorem 1.4.1(f) and (g)

$= 4A^2 - 2A - 2A + I$ ← Theorem 1.4.1(l) and (m); Property $AI = IA = A$ on p. 43

$= 4A - 4A + I$ ← A is idempotent so $A^2 = A$

$= I$

47. Applying Theorem 1.4.1(d) and (g), property $AI = IA = A$, and the assumption $A^k = O$ we can write

$$\begin{aligned}&(I-A)\left(I + A + A^2 + \cdots + A^{k-2} + A^{k-1}\right)\\ &= I - A + A - A^2 + A^2 - A^3 + \cdots + A^{k-2} - A^{k-1} + A^{k-1} - A^k\\ &= I - A^k\\ &= I - O\\ &= I\end{aligned}$$

48. $A^2 - (a+d)A + (ad-bc)I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+da & ab+bd \\ ac+dc & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

True-False Exercises

(a) False. A and B are inverses of one another if and only if $AB = BA = I$.

(b) False. $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$ does not generally equal $A^2 + 2AB + B^2$ since AB may not equal BA.

(c) False. $(A-B)(A+B) = A^2 + AB - BA - B^2$ does not generally equal $A^2 - B^2$ since AB may not equal BA.

(d) False. $(AB)^{-1} = B^{-1}A^{-1}$ does not generally equal $A^{-1}B^{-1}$.

(e) False. $(AB)^T = B^T A^T$ does not generally equal $A^T B^T$.

(f) True. This follows from Theorem 1.4.5.

(g) True. This follows from Theorem 1.4.8.

(h) True. This follows from Theorem 1.4.9. (The inverse of A^T is the transpose of A^{-1}.)

(i) False. $p(I) = (a_0 + a_1 + a_2 + \cdots + a_m)I$.

(j) True.
If the ith row vector of A is $[0 \quad \cdots \quad 0]$ then it follows from Formula (9) in Section 1.3 that ith row vector of $AB = [0 \quad \cdots \quad 0]\,B = [0 \quad \cdots \quad 0]$.
Consequently no matrix B can be found to make the product $AB = I$ thus A does not have an inverse.
If the jth column vector of A is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that the jth column vector of $BA = B\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.
Consequently no matrix B can be found to make the product $BA = I$ thus A does not have an inverse.

(k) False. E.g. I and $-I$ are both invertible but $I + (-I) = O$ is not.

1.5 Elementary Matrices and a Method for Finding A^{-1}

1. **(a)** Elementary matrix (corresponds to adding -5 times the first row to the second row)

(b) Not an elementary matrix

(c) Not an elementary matrix

(d) Not an elementary matrix

2. **(a)** Elementary matrix (corresponds to multiplying the second row by $\sqrt{3}$)

(b) Elementary matrix (corresponds to interchanging the first row and the third row)

(c) Elementary matrix (corresponds to adding 9 times the third row to the second row)

(d) Not an elementary matrix

3. **(a)** Add 3 times the second row to the first row: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

(b) Multiply the first row by $-\frac{1}{7}$: $\begin{bmatrix} -\frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Add 5 times the first row to the third row: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

(d) Interchange the first and third rows: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. **(a)** Add 3 times the first row to the second row: $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

(b) Multiply the third row by $\frac{1}{3}$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

(c) Interchange the first and fourth rows: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(d) Add $\frac{1}{7}$ times the third row to the first row: $\begin{bmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. **(a)** Interchange the first and second rows: $EA = \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix}$

(b) Add -3 times the second row to the third row: $EA = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix}$

(c) Add 4 times the third row to the first row: $EA = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

6. **(a)** Multiply the first row by -6 : $EA = \begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

(b) Add -4 times the first row to the second row: $EA = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

(c) Multiply the second row by 5 : $EA = \begin{bmatrix} 1 & 4 \\ 10 & 25 \\ 3 & 6 \end{bmatrix}$

7. **(a)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (B was obtained from A by interchanging the first row and the third row)

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (A was obtained from B by interchanging the first row and the third row)

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (C was obtained from A by adding -2 times the first row to the third row)

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (A was obtained from C by adding 2 times the first row to the third row)

8. **(a)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D was obtained from B by multiplying the second row by -3)

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B was obtained from D by multiplying the second row by $-\frac{1}{3}$)

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (F was obtained from B by adding 2 times the third row to the second row)

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ (B was obtained from F by adding -2 times the third row to the second row)

9. **(a)** (Method I: using Theorem 1.4.5)
The determinant of A, $\det(A) = (1)(7) - (4)(2) = -1$, is nonzero. Therefore A is invertible and its inverse is $A^{-1} = \frac{1}{-1}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$.

(Method II: using the inversion algorithm)

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \longleftarrow \text{The second row was multiplied by } -1.$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right] \longleftarrow -4 \text{ times the second row was added to the first row.}$$

The inverse is $\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$.

(b) (Method I: using Theorem 1.4.5)
The determinant of A, $\det(A) = (2)(8) - (-4)(-4) = 0$. Therefore A is not invertible.
(Method II: using the inversion algorithm)

$\left[\begin{array}{rr|rr} 2 & -4 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{rr|rr} 2 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$ ← 2 times the first row was added to the second row.

A row of zeros was obtained on the left side, therefore A is not invertible.

10. (a) (Method I: using Theorem 1.4.5)
The determinant of A, $\det(A) = (1)(-16) - (-5)(3) = -1$, is nonzero. Therefore A is invertible and its inverse is $A^{-1} = \frac{1}{-1}\begin{bmatrix} -16 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -5 \\ 3 & -1 \end{bmatrix}$.

(Method II: using the inversion algorithm)

$\left[\begin{array}{rr|rr} 1 & -5 & 1 & 0 \\ 3 & -16 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{rr|rr} 1 & -5 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$ ← -3 times the first row was added to the second row.

$\left[\begin{array}{rr|rr} 1 & -5 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{rr|rr} 1 & 0 & 16 & -5 \\ 0 & 1 & 3 & -1 \end{array}\right]$ ← 5 times the second row was added to the first row.

The inverse is $\begin{bmatrix} 16 & -5 \\ 3 & -1 \end{bmatrix}$.

(b) (Method I: using Theorem 1.4.5)
The determinant of A, $\det(A) = (6)(-2) - (4)(-3) = 0$. Therefore A is not invertible.

(Method II: using the inversion algorithm)

$\left[\begin{array}{rr|rr} 6 & 4 & 1 & 0 \\ -3 & -2 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{rr|rr} 0 & 0 & 1 & 2 \\ -3 & -2 & 0 & 1 \end{array}\right]$ ← 2 times the second row was added to the first row.

A row of zeros was obtained on the left side, therefore the matrix is not invertible.

11. (a) $\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$ ← -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$ ← 2 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← The third row was multiplied by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← 3 times the third row was added to the second row and -3 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← -2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

(b) $$\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$$ ← The first row was multiplied by -1.

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$$ ← -2 times the first row was added to the second row and 4 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$ ← The second row was added to the third row.

A row of zeros was obtained on the left side, therefore the matrix is not invertible.

12. (a) $$\left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & 5 & 0 \\ 1 & -4 & \frac{1}{2} & 0 & 0 & 5 \end{array}\right]$$ ← Each row was multiplied by 5.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & -5 & 5 & 0 \\ 0 & -5 & \frac{5}{2} & -5 & 0 & 5 \end{array}\right] \longleftarrow$$ -1 times the first row was added to the second and -1 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -5 & \frac{5}{2} & -5 & 0 & 5 \\ 0 & 0 & \frac{5}{2} & -5 & 5 & 0 \end{array}\right] \longleftarrow$$ The second and third rows were interchanged.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \longleftarrow$$ The second row was multiplied by $-\frac{1}{5}$ and the third row was multiplied by $\frac{2}{5}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \longleftarrow$$ $\frac{1}{2}$ times the third row was added to the second row and 2 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \longleftarrow$$ -1 times the second row was added to the first row.

The inverse is $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$.

(b) $$\left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & -3 & -\frac{3}{2} & 0 & 5 & 0 \\ 1 & -4 & \frac{1}{2} & 0 & 0 & 5 \end{array}\right] \longleftarrow$$ Each row was multiplied by 5.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -5 & \frac{5}{2} & -10 & 5 & 0 \\ 0 & -5 & \frac{5}{2} & -5 & 0 & 5 \end{array}\right] \longleftarrow$$ -2 times the first row was added to the second and -1 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -5 & \frac{5}{2} & -10 & 5 & 0 \\ 0 & 0 & 0 & 5 & -5 & 5 \end{array}\right] \longleftarrow$$ -1 times the second row was added to the third row.

A row of zeros was obtained on the left side, therefore the matrix is not invertible.

13.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \longleftarrow -1 \text{ times the first row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{2}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow \begin{array}{l} -1 \text{ times the third row was added to the second and} \\ -1 \text{ times the third row was added to the first row} \end{array}$$

The inverse is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

14.

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -4 & 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{Each of the first two rows was multiplied by } \tfrac{1}{\sqrt{2}}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 13 & 0 & 2\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow 4 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 & \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The second row was multiplied by } \tfrac{1}{13}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ 0 & 1 & 0 & \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \longleftarrow \quad -3 \text{ times the second row was added to the first row.}$$

The inverse is $\begin{bmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

15.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \quad \longleftarrow \quad \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \quad \longleftarrow \quad \begin{array}{l} -1 \text{ times the first row was added to the second and} \\ -1 \text{ times the first row was added to the third row} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \longleftarrow \quad -1 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \longleftarrow \quad -6 \text{ times the third row was added to the first row}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \longleftarrow \quad -6 \text{ times the second row was added to the first row}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \longleftarrow \quad \text{The first row was multiplied by } \tfrac{1}{2}.$$

The inverse is $\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

16.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \quad \longleftarrow \quad \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{array}\right] \quad \longleftarrow \quad \begin{array}{l} -1 \text{ times the first row was added to each of the} \\ \text{remaining rows.} \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{array}\right]$$ ← -1 times the second row was added to the third row and to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{array}\right]$$ ← -1 times the third row was added to the fourth row

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{array}\right]$$ ← The second row was multiplied by $\frac{1}{3}$, the third row was multiplied by $\frac{1}{5}$, and the fourth row was multiplied by $\frac{1}{7}$.

The inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}$.

17. $$\left[\begin{array}{cccc|cccc} 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← The first and second rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$ ← -2 times the first row was added to the second.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$ ← The second and fourth rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$ ← The second row was multiplied by -1.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right]$$

← 8 times the second row was added to the fourth.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right]$$

← The third row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 40 & 1 & -2 & -4 & -8 \end{array}\right]$$

← -8 times the third row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

← The fourth row was multiplied by $\frac{1}{40}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

← -5 times the fourth row was added to the second row.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & -6 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

← -4 times the third row was added to the second row and -12 times the third row was added to the first row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

← -2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{bmatrix}$.

18.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$
← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$
← The first and second rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$
← -2 times the first row was added to the fourth row and to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$
← The second and third rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -5 & 0 & -2 & 0 & 1 \end{array}\right]$$
← The second row was multiplied by -1.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 8 & -5 & 0 & -2 & 1 & 1 \end{array}\right]$$
← -1 times the second row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -4 & -2 & 1 & 1 \end{array}\right]$$
← -4 times the third row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array}\right]$$
← The third row was multiplied by $\frac{1}{2}$ and the fourth row was multiplied by $-\frac{1}{5}$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

-1 times the fourth row was added to the first row and
3 times the third row was added to the second.

The inverse is $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{3}{2} & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}$.

19. **(a)** $\left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$

The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4} \end{array}\right]$$

The first row was multiplied by $1/k_1$,
the second row was multiplied by $1/k_2$,
the third row was multiplied by $1/k_3$, and
the fourth row was multiplied by $1/k_4$.

The inverse is $\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$.

(b) $\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

First row and third row were both multiplied by $1/k$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$

$-\frac{1}{k}$ times the fourth row was added to the third row and $-\frac{1}{k}$ times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

20. (a) $$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$

The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right] \longleftarrow$$

The first and fourth rows were interchanged; the second and third rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1} & 0 & 0 & 0 \end{array}\right] \longleftarrow$$

The first row was multiplied by $1/k_4$, the second row was multiplied by $1/k_3$, the third row was multiplied by $1/k_2$, and the fourth row was multiplied by $1/k_1$.

The inverse is $\begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}$.

(b) $$\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$

The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$ ← Each row was multiplied by $1/k$.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$ ← $-\frac{1}{k}$ times the first row was added to the second row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right]$$ ← $-\frac{1}{k}$ times the second row was added to the third row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{array}\right]$$ ← $-\frac{1}{k}$ times the third row was added to the fourth row.

The inverse is $\begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 \\ -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{bmatrix}$.

21. It follows from parts (a) and (d) of Theorem 1.5.3 that a square matrix is invertible if and only if its reduced row echelon form is identity.

$$\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & c \\ 1 & c & c \\ c & c & c \end{bmatrix}$$ ← The first and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & c \\ 0 & -1+c & 0 \\ 0 & 0 & c-c^2 \end{bmatrix} \longleftarrow$$ -1 times the first row was added to the second row and $-c$ times the first row was added to the third row.

If $c - c^2 = c(1-c) = 0$ or $-1 + c = 0$, i.e. if $c = 0$ or $c = 1$ the last matrix contains at least one row of zeros, therefore it cannot be reduced to I by elementary row operations.

Otherwise (if $c \neq 0$ and $c \neq 1$), multiplying the second row by $\frac{1}{-1+c}$ and multiplying the third row by $\frac{1}{c-c^2}$ would result in a row echelon form with 1's on the main diagonal. Subsequent elementary row operations would then lead to the identity matrix.

We conclude that for any value of c other than 0 and 1 the matrix is invertible.

22. It follows from parts (a) and (d) of Theorem 1.5.3 that a square matrix is invertible if and only if its reduced row echelon form is identity.

$$\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & c & 1 \\ c & 1 & 0 \\ 0 & 1 & c \end{bmatrix} \longleftarrow$$ The first and second rows were interchanged.

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ c & 1 & 0 \end{bmatrix} \longleftarrow$$ The second and third rows were interchanged.

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 1-c^2 & -c \end{bmatrix} \longleftarrow$$ $-c$ times the first row was added to the third row.

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & c^3-2c \end{bmatrix} \longleftarrow$$ $c^2 - 1$ times the second row was added to the third.

If $c^3 - 2c = c(c^2 - 2) = 0$, i.e. if $c = 0$, $c = \sqrt{2}$ or $c = -\sqrt{2}$ the last matrix contains a row of zeros, therefore it cannot be reduced to I by elementary row operations.

Otherwise (if $c^3 - 2c \neq 0$), multiplying the last row by $\frac{1}{c^3-2c}$ would result in a row echelon form with 1's on the main diagonal. Subsequent elementary row operations would then lead to the identity matrix.

We conclude that for any value of c other than $0, \sqrt{2}$ and $-\sqrt{2}$ the matrix is invertible.

23. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$$A = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \longleftarrow$$ 2 times the second row was added to the first. $E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix} \longleftarrow$ -2 times the first row was added to the second. $E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \longleftarrow$ The second row was multiplied by $-\frac{1}{8}$. $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow$ -5 times the second row was added to the first. $E_4 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

Since $E_4E_3E_2E_1A = I$, then
$A = (E_4E_3E_2E_1)^{-1}I = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ and
$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

24. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \longleftarrow$ 5 times the first row was added to the second row. $E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longleftarrow$ The second row was multiplied by $\frac{1}{2}$. $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Since $E_2E_1A = I$, $A = (E_2E_1)^{-1}I = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $A^{-1} = E_2E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$.

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

25. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow$ The second row was multiplied by $\frac{1}{4}$. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow -\frac{3}{4}$ times the third row was added to the second. $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow 2$ times the third row was added to the first row. $E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $E_3E_2E_1A = I$, we have $A = (E_3E_2E_1)^{-1}I = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

26. We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longleftarrow -1$ times the first row was added to the second row. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow$ The second and third rows were interchanged $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow -1$ times the third row was added to the second. $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow -1$ times the second row was added to the first row. $E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $E_4E_3E_2E_1A = I$, we have $A = (E_4E_3E_2E_1)^{-1}I = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} =$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

27. Let us perform a sequence of elementary row operations to produce B from A. As we do so, we keep track of each corresponding elementary matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \longleftarrow$ -1 times the first row was added to the second row. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \longleftarrow$ -1 times the second row was added to the first row. $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} \longleftarrow$ -1 times the first row was added to the third row. $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Since $E_3E_2E_1A = B$, the equality $CA = B$ is satisfied by the matrix

$$C = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}.$$

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

28. Let us perform a sequence of elementary row operations to produce B from A. As we do so, we keep track of each corresponding elementary matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$\begin{bmatrix} 2 & 1 & 0 \\ -5 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \longleftarrow$ -2 times the first row was added to the second. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 0 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \longleftarrow$ -2 times the first row was added to the third row. $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \longleftarrow$ -4 times the third row was added to the first row. $E_3 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $E_3E_2E_1A = B$, the equality $CA = B$ is satisfied by the matrix

$$C = E_3E_2E_1 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & -4 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

Note that a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead. (However, since both A and B in this exercise are invertible, C is uniquely determined by the formula $C = BA^{-1}$.)

29. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$ cannot result from interchanging two rows of I_3 (since that would create a nonzero entry above the main diagonal).

A can result from multiplying the third row of I_3 by a nonzero number c (in this case, $a = b = 0$, $c \neq 0$).

The other possibilities are that A can be obtained by adding a times the first row to the third ($b = 0, c = 1$) or by adding b times the second row to the third ($a = 0, c = 1$).

In all three cases, at least one entry in the third row must be zero.

30. Consider three cases:

- If $a = 0$ then A has a row of zeros (first row).
- If $a \neq 0$ and $h = 0$ then A has a row of zeros (fifth row).
- If $a \neq 0$ and $h \neq 0$ then adding $-\frac{d}{a}$ times the first row to the third, and adding $-\frac{e}{h}$ times the fifth row to the third results in the third row becoming a row of zeros.

In all three cases, the reduced row echelon form of A is not I_5. By Theorem 1.5.3, A is not invertible.

True-False Exercises

(a) False. An elementary matrix results from performing a *single* elementary row operation on an identity matrix; a product of two elementary matrices would correspond to a sequence of two such operations instead, which generally is not equivalent to a single elementary operation.

(b) True. This follows from Theorem 1.5.2.

(c) True. If A and B are row equivalent then there exist elementary matrices $E_1, \dots, E_p$ such that $B = E_p \cdots E_1 A$. Likewise, if B and C are row equivalent then there exist elementary matrices $E_1^*, \dots, E_q^*$ such that $C = E_q^* \cdots E_1^* B$. Combining the two equalities yields $C = E_q^* \cdots E_1^* E_p \cdots E_1 A$ therefore A and C are row equivalent.

(d) True. A homogeneous system $A\mathbf{x} = \mathbf{0}$ has either one solution (the trivial solution) or infinitely many solutions. If A is not invertible, then by Theorem 1.5.3 the system cannot have just one solution. Consequently, it must have infinitely many solutions.

(e) True. If the matrix A is not invertible then by Theorem 1.5.3 its reduced row echelon form is not I_n. However, the matrix resulting from interchanging two rows of A (an elementary row operation) must have the same reduced row echelon form as A does, so by Theorem 1.5.3 that matrix is not invertible either.

(f) True. Adding a multiple of the first row of a matrix to its second row is an elementary row operation. Denoting by E be the corresponding elementary matrix we can write $(EA)^{-1} = A^{-1}E^{-1}$ so the resulting matrix EA is invertible if A is.

(g) False. For instance, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

1.6 More on Linear Systems and Invertible Matrices

1. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 1 \end{array}\right] \longleftarrow -5 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, i.e., $x_1 = 3$, $x_2 = -1$.

2. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 2 & -5 & 0 & 1 \\ 4 & -3 & 1 & 0 \end{array}\right] \longleftarrow \text{The first and second rows were interchanged.}$$

$$\left[\begin{array}{cc|cc} 2 & -5 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{array}\right] \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{5}{2} & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right] \longleftarrow \begin{array}{l}\text{The first row was multiplied by } \frac{1}{2} \text{ and} \\ \text{the second row was multiplied by } \frac{1}{7}.\end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{14} & -\frac{3}{14} \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} \end{array}\right] \longleftarrow \frac{5}{2} \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix} \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, i.e., $x_1 = x_2 = -3$.

3. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right] \longleftarrow \begin{array}{l} -2 \text{ times the first row was added to the second and} \\ -2 \text{ times the first row was added to the third row.} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & -4 & -1 & -2 & 1 & 0 \end{array}\right] \longleftarrow \text{The second and third rows were interchanged.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right] \longleftarrow 4 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right] \longleftarrow \text{The third row was multiplied by } -1 \text{.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right] \longleftarrow -1 \text{ times the third row was added to the first row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right] \longleftarrow -3 \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$, i.e., $x_1 = -1, x_2 = 4$, and $x_3 = -7$.

4. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ -1 times the second row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ The first row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & 2 & -\frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ -3 times the first row was added to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 3 & 2 & -\frac{3}{2} & \frac{5}{2} & 0 \end{array}\right] \longleftarrow$$ The second and third rows were interchanged.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -\frac{3}{2} & \frac{5}{2} & -3 \end{array}\right] \longleftarrow$$ -3 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right] \longleftarrow$$ The third row was multiplied by -1 .

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right] \longleftarrow$$ -1 times the third row was added to the second row.

Since $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution

$\mathbf{x} = A^{-1}\mathbf{b}$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$, i.e., $x_1 = 1, x_2 = -11$, and $x_3 = 16$.

5. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the coefficient matrix.

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$ ← -1 times the first row was added to the second row and 4 times the first row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$ ← The second and third rows were interchanged.

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$ ← The second row was multiplied by $\frac{1}{5}$ and the third row was multiplied by $-\frac{1}{5}$.

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$ ← -1 times the third row was added to the second row and to the first row.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & 0 \end{array}\right]$ ← -1 times the second row was added to the first row.

Since $A^{-1}=\begin{bmatrix}\frac{1}{5} & 0 & -\frac{1}{5}\\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{1}{5} & -\frac{1}{5} & 0\end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x}=A^{-1}\mathbf{b}$:

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{5} & 0 & -\frac{1}{5}\\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{1}{5} & -\frac{1}{5} & 0\end{bmatrix}\begin{bmatrix}5\\10\\0\end{bmatrix}=\begin{bmatrix}1\\5\\-1\end{bmatrix}$, i.e., $x=1, y=5$, and $z=-1$.

6. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}0 & -1 & -2 & -3\\ 1 & 1 & 4 & 4\\ 1 & 3 & 7 & 9\\ -1 & -2 & -4 & -6\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}w\\x\\y\\z\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}0\\7\\4\\6\end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{rrrr|rrrr}0 & -1 & -2 & -3 & 1 & 0 & 0 & 0\\ 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0\\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1\end{array}\right] \longleftarrow$$ The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{rrrr|rrrr}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0\\ 1 & 3 & 7 & 9 & 0 & 0 & 1 & 0\\ -1 & -2 & -4 & -6 & 0 & 0 & 0 & 1\end{array}\right] \longleftarrow$$ The first and second rows were interchanged.

$$\left[\begin{array}{rrrr|rrrr}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & -1 & -2 & -3 & 1 & 0 & 0 & 0\\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0\\ 0 & -1 & 0 & -2 & 0 & 1 & 0 & 1\end{array}\right] \longleftarrow$$ -1 times the first row was added to the third row and the first row was added to the fourth row.

$$\left[\begin{array}{rrrr|rrrr}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0\\ 0 & 2 & 3 & 5 & 0 & -1 & 1 & 0\\ 0 & -1 & 0 & -2 & 0 & 1 & 0 & 1\end{array}\right] \longleftarrow$$ The second row was multiplied by -1 .

$$\left[\begin{array}{rrrr|rrrr}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & -1 & 2 & -1 & 1 & 0\\ 0 & 0 & 2 & 1 & -1 & 1 & 0 & 1\end{array}\right] \longleftarrow$$ -2 times the second row was added to the third row and the second row was added to the fourth.

$$\left[\begin{array}{rrrr|rrrr}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0\\ 0 & 0 & 2 & 1 & -1 & 1 & 0 & 1\end{array}\right] \longleftarrow$$ The third row was multiplied by -1 .

$$\left[\begin{array}{rrrr|rrrr}1 & 1 & 4 & 4 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0\\ 0 & 0 & 0 & -1 & 3 & -1 & 2 & 1\end{array}\right] \longleftarrow$$ -2 times the third row was added to the fourth.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right] \longleftarrow \text{The fourth row was multiplied by } -1.$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 4 & 0 & 12 & -3 & 8 & 4 \\ 0 & 1 & 2 & 0 & 8 & -3 & 6 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right] \longleftarrow$$

-1 times the last row was added to the third row, -3 times the last row was added to the second row and -4 times the last row was added to the first.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 8 & -3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right] \longleftarrow$$

-2 times the third row was added to the second row and
-4 times the third row was added to the first row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 6 & -3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 1 & -2 & -1 \end{array}\right] \longleftarrow \text{-1 times the second row was added to the first.}$$

Since $A^{-1} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution

$\mathbf{x} = A^{-1}\mathbf{b}$: $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix}$,

i.e., $w = -6$, $x = 1$, $y = 10$, and $z = -7$.

7. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right] \longleftarrow \text{The first and second rows were interchanged.}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right] \longleftarrow \text{-3 times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right] \longleftarrow \text{The second row was multiplied by } -1.$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] \longleftarrow \text{-2 times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$:
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix}$, i.e., $x_1 = 2b_1 - 5b_2,\ \ x_2 = -b_1 + 3b_2$.

8. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right] \longleftarrow \begin{array}{l} -2 \text{ times the first row was added to the second row and} \\ -3 \text{ times the first row was added to the third row.} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 1 & 1 \end{array}\right] \longleftarrow \text{The second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{2}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{13}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow \begin{array}{l} \text{The third row was added to the second row and} \\ -3 \text{ times the third row was added to the first row.} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{15}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \longleftarrow -2 \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} =$ $A^{-1}\mathbf{b}$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3 \\ \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3 \\ \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3 \end{bmatrix}$, i.e.,

$x_1 = -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3$, $x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3$, and $x_3 = \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3$.

9.

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right] \longleftarrow$$ We augmented the coefficient matrix with two columns of constants on the right hand sides of the systems (i) and (ii) – refer to Example 2.

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right] \longleftarrow$$ -3 times the first row was added to the second row.

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right] \longleftarrow$$ The second row was multiplied by $\frac{1}{17}$.

$$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{22}{17} & \frac{21}{17} \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right] \longleftarrow$$ 5 times the second row was added to the first row.

We conclude that the solutions of the two systems are:

(i) $x_1 = \frac{22}{17}$, $x_2 = \frac{1}{17}$ **(ii)** $x_1 = \frac{21}{17}$, $x_2 = \frac{11}{17}$

10.

$$\left[\begin{array}{ccc|c|c} -1 & 4 & 1 & 0 & -3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right] \longleftarrow$$ We augmented the coefficient matrix with two columns of constants on the right hand sides of the systems (i) and (ii) – refer to Example 2.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right] \longleftarrow$$ The first row was multiplied by -1.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 13 & -1 & 1 & 1 \\ 0 & 28 & -2 & 0 & -23 \end{array}\right] \longleftarrow$$ -1 times the first row was added to the second row and -6 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{13} & \frac{1}{13} & \frac{1}{13} \\ 0 & 28 & -2 & 0 & -23 \end{array}\right] \longleftarrow$$ The second row was multiplied by $\frac{1}{13}$.

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{13} & \frac{1}{13} & \frac{1}{13} \\ 0 & 0 & \frac{2}{13} & -\frac{28}{13} & -\frac{327}{13} \end{array}\right] \longleftarrow \text{ -28 times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -\frac{1}{13} & \frac{1}{13} & \frac{1}{13} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right] \longleftarrow \text{ The third row was multiplied by $\frac{13}{2}$.}$$

$$\left[\begin{array}{ccc|c|c} 1 & -4 & 0 & -14 & -\frac{321}{2} \\ 0 & 1 & 0 & -1 & -\frac{25}{2} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right] \longleftarrow \text{ $\frac{1}{13}$ times the third row was added to the second row and the third row was added to the first row.}$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -18 & -\frac{421}{2} \\ 0 & 1 & 0 & -1 & -\frac{25}{2} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right] \longleftarrow \text{ 4 times the second row was added to the first row.}$$

We conclude that the solutions of the two systems are:

(i) $x_1 = -18,\ x_2 = -1,\ x_3 = -14$ **(ii)** $x_1 = -\frac{421}{2},\ x_2 = -\frac{25}{2},\ x_3 = -\frac{327}{2}$.

11. $$\left[\begin{array}{cc|c|c|c|c} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right] \longleftarrow \text{ We augmented the coefficient matrix with four columns of constants on the right hand sides of the systems (i), (ii), (iii), and (iv) – refer to Example 2.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 4 & -7 & 0 & -4 & -1 & -5 \end{array}\right] \longleftarrow \text{ The first and second rows were interchanged.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & -15 & -4 & -28 & -13 & -9 \end{array}\right] \longleftarrow \text{ -4 times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right] \longleftarrow \text{ The second row was multiplied by $-\frac{1}{15}$.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 0 & \frac{7}{15} & \frac{34}{15} & \frac{19}{15} & -\frac{1}{5} \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right] \longleftarrow \text{ -2 times the second row was added to the first row.}$$

We conclude that the solutions of the four systems are:

(i) $x_1 = \frac{7}{15},\ x_2 = \frac{4}{15}$ **(ii)** $x_1 = \frac{34}{15},\ x_2 = \frac{28}{15}$

(iii) $x_1 = \frac{19}{15},\ x_2 = \frac{13}{15}$ **(iv)** $x_1 = -\frac{1}{5},\ x_2 = \frac{3}{5}$

12. $\left[\begin{array}{rrr|r|r|r} 1 & 3 & 5 & 1 & 0 & -1 \\ -1 & -2 & 0 & 0 & 1 & -1 \\ 2 & 5 & 4 & -1 & 1 & 0 \end{array}\right]$ ⟵ We augmented the coefficient matrix with three columns of constants on the right hand sides of the systems (i), (ii) and (iii) – refer to Example 2.

$\left[\begin{array}{rrr|r|r|r} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & -1 & -6 & -3 & 1 & 2 \end{array}\right]$ ⟵ The first row was added to the second row and -2 times the first row was added to the third row.

$\left[\begin{array}{rrr|r|r|r} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & -1 & -2 & 2 & 0 \end{array}\right]$ ⟵ The second row was added to the third row.

$\left[\begin{array}{rrr|r|r|r} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$ ⟵ The third row was multiplied by -1.

$\left[\begin{array}{rrr|r|r|r} 1 & 3 & 0 & -9 & 10 & -1 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$ ⟵ -5 times the third row was added to the first row and to the second row.

$\left[\begin{array}{rrr|r|r|r} 1 & 0 & 0 & 18 & -23 & 5 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$ ⟵ -3 times the second row was added to the first row.

We conclude that the solutions of the three systems are:

(i) $x_1 = 18,\ x_2 = -9,\ x_3 = 2$

(ii) $x_1 = -23,\ x_2 = 11\ ,\ x_3 = -2$

(iii) $x_1 = 5,\ x_2 = -2,\ x_3 = 0$

13. $\left[\begin{array}{rr|r} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right]$ ⟵ The augmented matrix for the system.

$\left[\begin{array}{rr|c} 1 & 3 & b_1 \\ 0 & 7 & 2b_1 + b_2 \end{array}\right]$ ⟵ 2 times the first row was added to the second row.

$\left[\begin{array}{rr|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right]$ ⟵ The second row was multiplied by $\frac{1}{7}$.

The system is consistent for all values of b_1 and b_2.

14. $\left[\begin{array}{rr|r} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right]$ ⟵ The augmented matrix for the system.

$\left[\begin{array}{rr|r} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 3 & -2 & b_2 \end{array}\right]$ ⟵ The first row was multiplied by $\frac{1}{6}$.

$\left[\begin{array}{rr|c} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 0 & 0 & -\frac{1}{2}b_1 + b_2 \end{array}\right]$ ⟵ -3 times the first row was added to the second row.

The system is consistent if and only if $-\frac{1}{2}b_1 + b_2 = 0$, i.e. $b_1 = 2b_2$.

15. $\left[\begin{array}{rrr|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right]$ ← The augmented matrix for the system.

$\left[\begin{array}{rrr|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{array}\right]$ ← -4 times the first row was added to the second row and 3 times the first row was added to the third row.

$\left[\begin{array}{rrr|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$ ← The second row was added to the third row.

$\left[\begin{array}{rrr|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$ ← The second row was multiplied by $\frac{1}{3}$.

The system is consistent if and only if $-b_1 + b_2 + b_3 = 0$, i.e. $b_1 = b_2 + b_3$.

16. $\left[\begin{array}{rrr|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right]$ ← The augmented matrix for the system.

$\left[\begin{array}{rrr|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & 4b_1 + b_2 \\ 0 & -1 & 0 & 4b_1 + b_3 \end{array}\right]$ ← 4 times the first row was added to the second row and to the third row.

$\left[\begin{array}{rrr|c} 1 & -2 & -1 & b_1 \\ 0 & -1 & 0 & 4b_1 + b_3 \\ 0 & -3 & -2 & 4b_1 + b_2 \end{array}\right]$ ← The second and third rows were interchanged.

$\left[\begin{array}{rrr|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & -3 & -2 & 4b_1 + b_2 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{rrr|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & -2 & -8b_1 + b_2 - 3b_3 \end{array}\right]$ ← 3 times the second row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1 - b_3 \\ 0 & 0 & 1 & 4b_1 - \frac{1}{2}b_2 + \frac{3}{2}b_3 \end{array}\right]$$ ← The third row was multiplied by $-\frac{1}{2}$.

The system is consistent for all values of b_1, b_2, and b_3.

17. $$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$$ ← The augmented matrix for the system.

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1 + b_2 \\ 0 & -1 & 11 & 5 & 3b_1 + b_3 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right]$$ ← 2 times the first row was added to the second row, 3 times the first row was added to the third row, and -4 times the first row was added to the fourth row.

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1 - b_2 \\ 0 & -1 & 11 & 5 & 3b_1 + b_3 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right]$$ ← The second row was multiplied by -1.

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1 - b_2 \\ 0 & 0 & 0 & 0 & b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & -2b_1 + b_2 + b_4 \end{array}\right]$$ ← The second row was added to the third row and -1 times the second row was added to the fourth row.

The system is consistent for all values of b_1, b_2, b_3, and b_4 that satisfy the equations $b_1 - b_2 + b_3 = 0$ and $-2b_1 + b_2 + b_4 = 0$.

These equations form a linear system in the variables b_1, b_2, b_3, and b_4 whose augmented matrix $\begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{bmatrix}$. Therefore the system is consistent if $b_1 = b_3 + b_4$ and $b_2 = 2b_3 + b_4$.

18. **(a)** The equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $A\mathbf{x} = I\mathbf{x}$, which yields $A\mathbf{x} - I\mathbf{x} = \mathbf{0}$ and $(A - I)\mathbf{x} = \mathbf{0}$.

This is a matrix form of a homogeneous linear system - to solve it, we reduce its augmented matrix to a row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right]$$ ← The augmented matrix for the homogeneous system $(A - I)\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \longleftarrow \text{-2 times the first row was added to the second row and -3 times the first row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \longleftarrow \text{The second row was multiplied by -1.}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \longleftarrow \text{2 times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longleftarrow \text{The third row was multiplied by $\tfrac{1}{6}$.}$$

Using back-substitution, we obtain the unique solution: $x_1 = x_2 = x_3 = 0$.

(b) As was done in part (a), the equation $A\mathbf{x} = 4\mathbf{x}$ can be rewritten as $(A - 4I)\mathbf{x} = \mathbf{0}$. We solve the latter system by Gauss-Jordan elimination

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right] \longleftarrow \text{The augmented matrix for the homogeneous system $(A - 4I)\mathbf{x} = \mathbf{0}$.}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right] \longleftarrow \text{The first and second rows were interchanged.}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right] \longleftarrow \text{The first row was multiplied by $\tfrac{1}{2}$.}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \longleftarrow \text{2 times the first row was added to the second row and -3 times the first row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right] \longleftarrow \text{The second row was multiplied by -1.}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longleftarrow \text{-4 times the second row was added to the third row and the second row was added to the first row.}$$

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$x_1 = t$, $x_2 = 0$, and $x_3 = t$.

19. $X = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$. Let us find $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the matrix.}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{-2 times the first row was added to the second row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{-2 times the third row was added to the second row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & -1 & 4 & -2 & 5 \end{array}\right] \longleftarrow \text{-2 times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \text{The third row was multiplied by -1.}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 5 & -2 & 5 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \text{-1 times the third row was added to the first row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \text{The second row was added to the first row.}$$

Using $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}$ we obtain

$$X = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$$

20. $X = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}$. Let us find $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix}^{-1}$:

$$\left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -4 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the matrix.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \longleftarrow \text{The first and third rows were interchanged.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -7 & 1 & 0 & 2 \end{array}\right] \longleftarrow \text{2 times the first row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & -7 & 1 & 0 & 2 \end{array}\right] \longleftarrow \text{The second row was multiplied by -1 .}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -9 & 1 & 2 & 2 \end{array}\right] \longleftarrow \text{-2 times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{array}\right] \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{9}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{4}{9} & -\frac{8}{9} & \frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ 0 & 0 & 1 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{array}\right] \longleftarrow$$

-1 times the third row was added to the second row and 4 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ 0 & 0 & 1 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{array}\right] \longleftarrow \text{-1 times the second row was added to the first row.}$$

Using $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{bmatrix}$ we obtain

$$X = \begin{bmatrix} -\frac{5}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \frac{1}{9} & -\frac{7}{9} & \frac{2}{9} \\ -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix} = \begin{bmatrix} -3 & -\frac{25}{9} & -\frac{25}{9} & -\frac{23}{9} \\ -4 & -\frac{40}{9} & -\frac{40}{9} & -\frac{44}{9} \\ -2 & -\frac{23}{9} & -\frac{32}{9} & -\frac{37}{9} \end{bmatrix}$$

True-False Exercises

(a) True. By Theorem 1.6.1, if a system of linear equation has more than one solution then it must have infinitely many.

(b) True. If A is a square matrix such that $A\mathbf{x} = \mathbf{b}$ has a unique solution then the reduced row echelon form of A must be I. Consequently, $A\mathbf{x} = \mathbf{c}$ must have a unique solution as well.

(c) True. Since B is a square matrix then by Theorem 1.6.3(b) $AB = I_n$ implies $B = A^{-1}$. Therefore, $BA = A^{-1}A = I_n$.

(d) True. Since A and B are row equivalent matrices, it must be possible to perform a sequence of elementary row operations on A resulting in B. Let E be the product of the corresponding elementary matrices, i.e., $EA = B$. Note that E must be an invertible matrix thus $A = E^{-1}B$.
Any solution of $A\mathbf{x} = \mathbf{0}$ is also a solution of $B\mathbf{x} = \mathbf{0}$ since $B\mathbf{x} = EA\mathbf{x} = E\mathbf{0} = \mathbf{0}$.
Likewise, any solution of $B\mathbf{x} = \mathbf{0}$ is also a solution of $A\mathbf{x} = \mathbf{0}$ since $A\mathbf{x} = E^{-1}B\mathbf{x} = E^{-1}\mathbf{0} = \mathbf{0}$.

(e) True. If $(S^{-1}AS)\mathbf{x} = \mathbf{b}$ then $SS^{-1}AS\mathbf{x} = A(S\mathbf{x}) = S\mathbf{b}$. Consequently, $\mathbf{y} = S\mathbf{x}$ is a solution of $A\mathbf{y} = S\mathbf{b}$.

(f) True. $A\mathbf{x} = 4\mathbf{x}$ is equivalent to $A\mathbf{x} = 4I_n\mathbf{x}$, which can be rewritten as $(A - 4I_n)\mathbf{x} = \mathbf{0}$. By Theorem 1.6.4, this homogeneous system has a unique solution (the trivial solution) if and only if its coefficient matrix $A - 4I_n$ is invertible.

(g) True. If AB were invertible, then by Theorem 1.6.5 both A and B would be invertible.

1.7 Diagonal, Triangular, and Symmetric Matrices

1. **(a)** The matrix is upper triangular. It is invertible (its diagonal entries are both nonzero).

(b) The matrix is lower triangular. It is not invertible (its diagonal entries are zero).

(c) This is a diagonal matrix, therefore it is also both upper and lower triangular. It is invertible (its diagonal entries are all nonzero).

(d) The matrix is upper triangular. It is not invertible (its diagonal entries include a zero).

2. **(a)** The matrix is lower triangular. It is invertible (its diagonal entries are both nonzero).

(b) The matrix is upper triangular. It is not invertible (its diagonal entries are zero).

(c) This is a diagonal matrix, therefore it is also both upper and lower triangular. It is invertible (its diagonal entries are all nonzero).

(d) The matrix is lower triangular. It is not invertible (its diagonal entries include a zero).

3. $$\begin{bmatrix}3&0&0\\0&-1&0\\0&0&2\end{bmatrix}\begin{bmatrix}2&1\\-4&1\\2&5\end{bmatrix}=\begin{bmatrix}(3)(2)&(3)(1)\\(-1)(-4)&(-1)(1)\\(2)(2)&(2)(5)\end{bmatrix}=\begin{bmatrix}6&3\\4&-1\\4&10\end{bmatrix}$$

4. $$\begin{bmatrix}1&2&-5\\-3&-1&0\end{bmatrix}\begin{bmatrix}-4&0&0\\0&3&0\\0&0&2\end{bmatrix}=\begin{bmatrix}(1)(-4)&(2)(3)&(-5)(2)\\(-3)(-4)&(-1)(3)&(0)(2)\end{bmatrix}=\begin{bmatrix}-4&6&-10\\12&-3&0\end{bmatrix}$$

5. $$\begin{bmatrix}5&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}-3&2&0&4&-4\\1&-5&3&0&3\\-6&2&2&2&2\end{bmatrix}=\begin{bmatrix}(5)(-3)&(5)(2)&(5)(0)&(5)(4)&(5)(-4)\\(2)(1)&(2)(-5)&(2)(3)&(2)(0)&(2)(3)\\(-3)(-6)&(-3)(2)&(-3)(2)&(-3)(2)&(-3)(2)\end{bmatrix}$$
$$=\begin{bmatrix}-15&10&0&20&-20\\2&-10&6&0&6\\18&-6&-6&-6&-6\end{bmatrix}$$

6. $$\begin{bmatrix}2&0&0\\0&-1&0\\0&0&4\end{bmatrix}\begin{bmatrix}4&-1&3\\1&2&0\\-5&1&-2\end{bmatrix}\begin{bmatrix}-3&0&0\\0&5&0\\0&0&2\end{bmatrix}=\begin{bmatrix}(2)(4)(-3)&(2)(-1)(5)&(2)(3)(2)\\(-1)(1)(-3)&(-1)(2)(5)&(-1)(0)(2)\\(4)(-5)(-3)&(4)(1)(5)&(4)(-2)(2)\end{bmatrix}$$
$$=\begin{bmatrix}-24&-10&12\\3&-10&0\\60&20&-16\end{bmatrix}$$

7. $$A^2=\begin{bmatrix}1^2&0\\0&(-2)^2\end{bmatrix}=\begin{bmatrix}1&0\\0&4\end{bmatrix},\quad A^{-2}=\begin{bmatrix}1^{-2}&0\\0&(-2)^{-2}\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{1}{4}\end{bmatrix},\quad A^{-k}=\begin{bmatrix}1^{-k}&0\\0&(-2)^{-k}\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{1}{(-2)^k}\end{bmatrix}$$

8. $A^2 = \begin{bmatrix} (-6)^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 5^2 \end{bmatrix} = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$, $A^{-2} = \begin{bmatrix} (-6)^{-2} & 0 & 0 \\ 0 & 3^{-2} & 0 \\ 0 & 0 & 5^{-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{25} \end{bmatrix}$,

$$A^{-k} = \begin{bmatrix} (-6)^{-k} & 0 & 0 \\ 0 & 3^{-k} & 0 \\ 0 & 0 & 5^{-k} \end{bmatrix} = \begin{bmatrix} \frac{1}{(-6)^k} & 0 & 0 \\ 0 & \frac{1}{3^k} & 0 \\ 0 & 0 & \frac{1}{5^k} \end{bmatrix}$$

9. $A^2 = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^2 & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$, $A^{-2} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-2} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-2} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$,

$$A^{-k} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-k} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-k} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-k} \end{bmatrix} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

10. $A^2 = \begin{bmatrix} (-2)^2 & 0 & 0 & 0 \\ 0 & (-4)^2 & 0 & 0 \\ 0 & 0 & (-3)^2 & 0 \\ 0 & 0 & 0 & 2^2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$,

$$A^{-2} = \begin{bmatrix} (-2)^{-2} & 0 & 0 & 0 \\ 0 & (-4)^{-2} & 0 & 0 \\ 0 & 0 & (-3)^{-2} & 0 \\ 0 & 0 & 0 & 2^{-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix},$$

$$A^{-k} = \begin{bmatrix} (-2)^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & 2^{-k} \end{bmatrix} = \begin{bmatrix} \frac{1}{(-2)^k} & 0 & 0 & 0 \\ 0 & \frac{1}{(-4)^k} & 0 & 0 \\ 0 & 0 & \frac{1}{(-3)^k} & 0 \\ 0 & 0 & 0 & \frac{1}{2^k} \end{bmatrix}$$

11. $\begin{bmatrix} (1)(2)(0) & 0 & 0 \\ 0 & (0)(5)(2) & 0 \\ 0 & 0 & (3)(0)(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} (-1)(3)(5) & 0 & 0 \\ 0 & (2)(5)(-2) & 0 \\ 0 & 0 & (4)(7)(3) \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 84 \end{bmatrix}$

13. $\begin{bmatrix} 1^{39} & 0 \\ 0 & (-1)^{39} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 1^{1000} & 0 \\ 0 & (-1)^{1000} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15. **(a)** $\begin{bmatrix} au & av \\ bw & bx \\ cy & cz \end{bmatrix}$ **(b)** $\begin{bmatrix} ra & sb & tc \\ ua & vb & wc \\ xa & yb & zc \end{bmatrix}$

16. **(a)** $\begin{bmatrix} ua & vb \\ wa & xb \\ ya & zb \end{bmatrix}$ **(b)** $\begin{bmatrix} ar & as & at \\ bu & bv & bw \\ cx & cy & cz \end{bmatrix}$

17. **(a)** $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 3 & 7 & 2 \\ 3 & 1 & -8 & -3 \\ 7 & -8 & 0 & 9 \\ 2 & -3 & 9 & 0 \end{bmatrix}$

18. **(a)** $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 7 & -3 & 2 \\ 7 & 4 & 5 & -7 \\ -3 & 5 & 1 & -6 \\ 2 & -7 & -6 & 3 \end{bmatrix}$

19. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this upper triangular matrix has a 0 on its diagonal, it is not invertible.

20. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this upper triangular matrix has all three diagonal entries nonzero, it is invertible.

21. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this lower triangular matrix has all four diagonal entries nonzero, it is invertible.

22. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this lower triangular matrix has a 0 on its diagonal, it is not invertible.

23. $AB = \begin{bmatrix} (3)(-1) & \times & \times \\ 0 & (1)(5) & \times \\ 0 & 0 & (-1)(6) \end{bmatrix}$. The diagonal entries of AB are: $-3, 5, -6$.

24. $AB = \begin{bmatrix} (4)(6) & 0 & 0 \\ \times & (0)(5) & 0 \\ \times & \times & (7)(6) \end{bmatrix}$. The diagonal entries of AB are: $24, 0, 42$.

25. The matrix is symmetric if and only if $a + 5 = -3$. In order for A to be symmetric, we must have $a = -8$.

26. The matrix is symmetric if and only if the following equations must be satisfied

$$\begin{array}{rcrcrcr} a & - & 2b & + & 2c & = & 3 \\ 2a & + & b & + & c & = & 0 \\ a & & & + & c & = & -2 \end{array}$$

We solve this system by Gauss-Jordan elimination

$$\left[\begin{array}{rrr|r} 1 & -2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -2 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 2 & 1 & 1 & 0 \\ 1 & -2 & 2 & 3 \end{array}\right] \longleftarrow \text{The first and third rows were interchanged.}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & -2 & 1 & 5 \end{array}\right] \longleftarrow \begin{array}{l} -2 \text{ times the first row was added to the second row} \\ \text{and } -1 \text{ times the first row was added to the third.} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & 13 \end{array}\right] \longleftarrow 2 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -13 \end{array}\right] \longleftarrow \text{The third row was multiplied by } -1.$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -13 \end{array}\right] \longleftarrow \begin{array}{l} \text{The third row was added to the second row} \\ \text{and } -1 \text{ times the third row was added to the first.} \end{array}$$

In order for A to be symmetric, we must have $a = 11$, $b = -9$, and $c = -13$.

27. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Therefore, the given upper triangular matrix is invertible for any real number x such that $x \neq 1$, $x \neq -2$, and $x \neq 4$.

28. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Therefore, the given lower triangular matrix is invertible for any real number x such that $x \neq \frac{1}{2}$, $x \neq \frac{1}{3}$, and $x \neq -\frac{1}{4}$.

29. By Theorem 1.7.1, A^{-1} is also an upper triangular or lower triangular invertible matrix. Its diagonal entries must all be nonzero - they are reciprocals of the corresponding diagonal entries of the matrix A.

30. By Theorem 1.4.8(e), $(AB)^T = B^T A^T$. Therefore we have:

$(B^T B)^T = B^T (B^T)^T = B^T B$,
$(BB^T)^T = (B^T)^T B^T = BB^T$, and
$(B^T AB)^T = \left(B^T (AB)\right)^T = (AB)^T (B^T)^T = B^T A^T B = B^T AB$ since A is symmetric.

31. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

32. For example $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (there are seven other possible answers, e.g., $\begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$, etc.)

33. $AB = \begin{bmatrix} (-1)(2)+(2)(0)+(5)(0) & (-1)(-8)+(2)(2)+(5)(0) & (-1)(0)+(2)(1)+(5)(3) \\ (0)(2)+(1)(0)+(3)(0) & (0)(-8)+(1)(2)+(3)(0) & (0)(0)+(1)(1)+(3)(3) \\ (0)(2)+(0)(0)+(-4)(0) & (0)(-8)+(0)(2)+(-4)(0) & (0)(0)+(0)(1)+(-4)(3) \end{bmatrix}$

$= \begin{bmatrix} -2 & 12 & 17 \\ 0 & 2 & 10 \\ 0 & 0 & -12 \end{bmatrix}$. Since this is an upper triangular matrix, we have verified Theorem 1.7.1(b).

34. **(a)** Theorem 1.4.8(e) states that $(AB)^T = B^T A^T$ (if the multiplication can be performed). Therefore,

$$(A^2)^T = (AA)^T = A^T A^T = (A^T)^2 \underset{\substack{A\text{ is}\\ \text{symmetric}}}{=} A^2$$

which shows that A^2 is symmetric.

(b) $(2A^2 - 3A + I)^T \underset{\substack{\text{Th.}\\ 1.4.8\\ \text{(b-d)}}}{=} 2(A^2)^T - 3A^T + I^T \underset{\substack{\text{Th.}\\ 1.4.8\\ \text{(e)}}}{=} 2(A^T)^2 - 3A^T + I^T \underset{\substack{A\text{ and }I\\ \text{are}\\ \text{symmetric}}}{=} 2A^2 - 3A + I$

which shows that $2A^2 - 3A + I$ is symmetric.

35. **(a)** $A^{-1} = \frac{1}{(2)(3)-(-1)(-1)}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$ is symmetric, therefore we verified Theorem 1.7.4.

(b)

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ -2 & 1 & -7 & 0 & 1 & 0 \\ 3 & -7 & 4 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the matrix A.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -1 & 2 & 1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \end{array}\right]$ ← 2 times the first row was added to the second row and -3 times the first row was added to the third row.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \\ 0 & -3 & -1 & 2 & 1 & 0 \end{array}\right]$ ← The second and third rows were interchanged.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & -3 & -1 & 2 & 1 & 0 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & 0 & 14 & 11 & 1 & -3 \end{array}\right]$ ← 3 times the second row was added to the third row.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & 0 & 1 & \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{array}\right]$ ← The third row was multiplied by $\frac{1}{14}$.

$\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & -\frac{19}{14} & -\frac{3}{14} & \frac{9}{14} \\ 0 & 1 & 0 & -\frac{13}{14} & -\frac{5}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{array}\right]$ ← -5 times the third row was added to the second row and -3 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -\frac{45}{14} & -\frac{13}{14} & \frac{11}{14} \\ 0 & 1 & 0 & -\frac{13}{14} & -\frac{5}{14} & \frac{1}{14} \\ 0 & 0 & 1 & \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{array}\right]$$ ← 2 times the second row was added to the first row.

Since $A^{-1} = \begin{bmatrix} -\frac{45}{14} & -\frac{13}{14} & \frac{11}{14} \\ -\frac{13}{14} & -\frac{5}{14} & \frac{1}{14} \\ \frac{11}{14} & \frac{1}{14} & -\frac{3}{14} \end{bmatrix}$ is symmetric, we have verified Theorem 1.7.4

36. All 3×3 diagonal matrices have a form $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

$$\begin{aligned} A^2 - 3A - 4I &= \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} - 3\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} - \begin{bmatrix} 3a & 0 & 0 \\ 0 & 3b & 0 \\ 0 & 0 & 3c \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} a^2 - 3a - 4 & 0 & 0 \\ 0 & b^2 - 3b - 4 & 0 \\ 0 & 0 & c^2 - 3c - 4 \end{bmatrix} \\ &= \begin{bmatrix} (a-4)(a+1) & 0 & 0 \\ 0 & (b-4)(b+1) & 0 \\ 0 & 0 & (c-4)(c+1) \end{bmatrix} \end{aligned}$$

This is a zero matrix whenever the value of a, b, and c is either 4 or -1. We conclude that the following are all 3×3 diagonal matrices that satisfy the equation:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$
$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

37. **(a)** $a_{ji} = j^2 + i^2 = i^2 + j^2 = a_{ij}$ for all i and j therefore A is symmetric.

(b) $a_{ji} = j^2 - i^2$ does not generally equal $a_{ij} = i^2 - j^2$ for $i \neq j$ therefore A is not symmetric (unless $n = 1$).

(c) $a_{ji} = 2j + 2i = 2i + 2j = a_{ij}$ for all i and j therefore A is symmetric.

(d) $a_{ji} = 2j^2 + 2i^3$ does not generally equal $a_{ij} = 2i^2 + 2j^3$ for $i \neq j$ therefore A is not symmetric (unless $n = 1$).

38. If $a_{ij} = f(i, j)$ then A is symmetric if and only if $f(i, j) = f(j, i)$ for all values of i and j.

39. For a general upper triangular 2×2 matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ we have

$$A^3 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a^3 & a^2b+(ab+bc)c \\ 0 & c^3 \end{bmatrix} = \begin{bmatrix} a^3 & (a^2+ac+c^2)b \\ 0 & c^3 \end{bmatrix}$$

Setting $A^3 = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$ we obtain the equations $a^3 = 1$, $(a^2+ac+c^2)b = 30$, $c^3 = -8$.

The first and the third equations yield $a = 1, c = -2$.

Substituting these into the second equation leads to $(1-2+4)b = 30$, i.e., $b = 10$.

We conclude that the only upper triangular matrix A such that $A^3 = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 10 \\ 0 & -2 \end{bmatrix}$.

40. **(a)** Step 1. Solve $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

The first equation is $y_1 = 1$.

The second equation $(-2)(1) + 3y_2 = -2$ yields $y_2 = 0$.

The third equation $(2)(1) + (4)(0) + 1y_3 = 0$ yields $y_3 = -2$.

Step 2. Solve $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ using back-substitution:

The third equation $4x_3 = -2$ yields $x_3 = -\frac{1}{2}$.

The second equation $1x_2 + (2)\left(-\frac{1}{2}\right) = 0$ yields $x_2 = 1$.

The first equation $2x_1 + (-1)(1) + (3)\left(-\frac{1}{2}\right) = 1$ yields $x_1 = \frac{7}{4}$.

(b) Step 1. Solve $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$

The first equation $2y_1 = 4$ yields $y_1 = 2$.

The second equation $(4)(2) + 1y_2 = -5$ yields $y_2 = -13$.

The third equation $(-3)(2) + (-2)(-13) + 3y_3 = 2$ yields $y_3 = -6$.

Step 2. Solve $\begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -13 \\ -6 \end{bmatrix}$ using back-substitution:

The third equation $2x_3 = -6$ yields $x_3 = -3$.

The second equation $4x_2 + (1)(-3) = -13$ yields $x_2 = -\frac{5}{2}$.

The first equation $3x_1 + (-5)\left(-\frac{5}{2}\right) + (2)(-3) = 2$ yields $x_1 = -\frac{3}{2}$.

41. **(a)** $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ -4 & -1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 & -8 \\ 0 & 0 & -4 \\ 8 & 4 & 0 \end{bmatrix}$

42. The condition $A^T = -A$ is equivalent to the linear system

$$\begin{aligned} 2a - 3b + c \quad &= 2 \\ 3a - 5b + 5c \quad &= 3 \\ 5a - 8b + 6c \quad &= 5 \\ d &= 0 \end{aligned}$$

The augmented matrix $\begin{bmatrix} 2 & -3 & 1 & 0 & 2 \\ 3 & -5 & 5 & 0 & 3 \\ 5 & -8 & 6 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -10 & 0 & 1 \\ 0 & 1 & -7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

If we assign c the arbitrary value t, the general solution is given by the formulas

$a = 1 + 10t,\ b = 7t,\ c = t,\ d = 0.$

43. No. If $AB = BA$, $A^T = -A$, and $B^T = -B$ then $(AB)^T = B^T A^T = (-B)(-A) = BA = AB$ which does not generally equal $-AB$. (The product of skew-symmetric matrices that commute is symmetric.)

44. $\frac{1}{2}(A + A^T)$ is symmetric since $\left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}A^T + \frac{1}{2}(A^T)^T = \frac{1}{2}(A + A^T)$ and $\frac{1}{2}(A - A^T)$ is skew-symmetric since $\left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}A^T - \frac{1}{2}(A^T)^T = \frac{1}{2}(A^T - A) = -\left(\frac{1}{2}(A - A^T)\right)$ therefore the result follows from the identity $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A$.

45. **(a)** $(A^{-1})^T$

$= (A^T)^{-1}$ ← Theorem 1.4.9(d)

$= (-A)^{-1}$ ← The assumption: A is skew-symmetric

$= -A^{-1}$ ← Theorem 1.4.7(c)

(b) $(A^T)^T$

$= A$ ← Theorem 1.4.8(a)

$= -A^T$ ← The assumption: A is skew-symmetric

$(A + B)^T$

$= A^T + B^T$ ← Theorem 1.4.8(b)

$= -A - B$ ← The assumption: A and B are skew-symmetric

$= -(A + B)$ ← Theorem 1.4.1(h)

$(A - B)^T$

$= A^T - B^T$ ← Theorem 1.4.8(c)

$= -A - (-B)$ ← The assumption: A and B are skew-symmetric

$= -(A - B)$ ← Theorem 1.4.1(i)

$(kA)^T$

$= kA^T$ ← Theorem 1.4.8(d)

$= k(-A)$ ← The assumption: A is skew-symmetric

$= -kA$ ← Theorem 1.4.1(l)

47. $A^T = (A^TA)^T = A^T(A^T)^T = A^TA = A$ therefore A is symmetric; thus we have $A^2 = AA = A^TA = A$.

True-False Exercises

(a) True. Every diagonal matrix is symmetric: its transpose equals to the original matrix.

(b) False. The transpose of an upper triangular matrix is a *lower* triangular matrix.

(c) False. E.g., $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is not a diagonal matrix.

(d) True. Mirror images of entries across the main diagonal must be equal - see the margin note next to Example 4.

(e) True. All entries below the main diagonal must be zero.

(f) False. By Theorem 1.7.1(d), the inverse of an invertible lower triangular matrix is a lower triangular matrix.

(g) False. A diagonal matrix is invertible if and only if all or its diagonal entries are nonzero (positive or negative).

(h) True. The entries above the main diagonal are zero.

(i) True. If A is upper triangular then A^T is lower triangular. However, if A is also symmetric then it follows that $A^T = A$ must be both upper triangular and lower triangular. This requires A to be a diagonal matrix.

(j) False. For instance, neither $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ nor $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is symmetric even though $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is.

(k) False. For instance, neither $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ nor $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is upper triangular even though $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is.

(l) False. For instance, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not symmetric even though $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is.

(m) True. By Theorem 1.4.8(d), $(kA)^T = kA^T$. Since kA is symmetric, we also have $(kA)^T = kA$. For nonzero k the equality of the right hand sides $kA^T = kA$ implies $A^T = A$.

1.8 Matrix Transformations

1. **(a)** $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w} = A\mathbf{x}$ in R^3.
The domain of T_A is R^2; the codomain is R^3.

(b) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w} = A\mathbf{x}$ in R^2.
The domain of T_A is R^3; the codomain is R^2.

(c) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w} = A\mathbf{x}$ in R^3.
The domain of T_A is R^3; the codomain is R^3.

(d) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^6 into a vector $\mathbf{w} = A\mathbf{x}$ in $R^1 = R$.
The domain of T_A is R^6; the codomain is R.

2. **(a)** $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^5 into a vector $\mathbf{w} = A\mathbf{x}$ in R^4.
The domain of T_A is R^5; the codomain is R^4.

(b) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^4 into a vector $\mathbf{w} = A\mathbf{x}$ in R^5.
The domain of T_A is R^4; the codomain is R^5.

(c) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^4 into a vector $\mathbf{w} = A\mathbf{x}$ in R^4.
The domain of T_A is R^4; the codomain is R^4.

(d) $T_A(\mathbf{x}) = A\mathbf{x}$ maps any vector $\mathbf{x}$ in $R^1 = R$ into a vector $\mathbf{w} = A\mathbf{x}$ in R^3.
The domain of T_A is R; the codomain is R^3.

3. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^3.
Its domain is R^2; the codomain is R^3.

4. **(a)** The transformation maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w}$ in R^3.
Its domain is R^3; the codomain is R^3.

(b) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w}$ in R^2.
Its domain is R^3; the codomain is R^2.

5. **(a)** The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^2.
Its domain is R^3; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^3.
Its domain is R^2; the codomain is R^3.

6. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^3.
Its domain is R^3; the codomain is R^3.

7. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^2.
Its domain is R^3; the codomain is R^2.

8. **(a)** The transformation maps any vector $\mathbf{x}$ in R^4 into a vector in R^2.
Its domain is R^4; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^3.
Its domain is R^3; the codomain is R^3.

9. The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^3. Its domain is R^2; the codomain is R^3.

10. The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^4. Its domain is R^3; the codomain is R^4.

11. **(a)** The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix}$

(b) The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$.

12. **(a)** The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$.

(b) The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

13. **(a)** $T(x_1, x_2) = \begin{bmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$

(b) $T(x_1,x_2,x_3,x_4) = \begin{bmatrix} 7x_1+2x_2-x_3+x_4 \\ x_2+x_3 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$;

the standard matrix is $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

(c) $T(x_1,x_2,x_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $T(x_1,x_2,x_3,x_4) = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1-x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

14. **(a)** $T(x_1,x_2) = \begin{bmatrix} 2x_1-x_2 \\ x_1+x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

(b) $T(x_1,x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $T(x_1,x_2,x_3) = \begin{bmatrix} x_1+2x_2+x_3 \\ x_1+5x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $T(x_1,x_2,x_3) = \begin{bmatrix} 4x_1 \\ 7x_2 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

15. The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ therefore the standard matrix for this operator is $\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$.

By directly substituting $(-1,2,4)$ for (x_1,x_2,x_3) into the given equation we obtain

$$w_1 = -(3)(1) + (5)(2) - (1)(4) = 3$$
$$w_2 = -(4)(1) - (1)(2) + (1)(4) = -2$$
$$w_3 = -(3)(1) + (2)(2) - (1)(4) = -3$$

By matrix multiplication, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -(3)(1)+(5)(2)-(1)(4) \\ -(4)(1)-(1)(2)+(1)(4) \\ -(3)(1)+(2)(2)-(1)(4) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$.

16. The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ therefore the standard matrix for this transformation is $\begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix}$.

By directly substituting $(1,-1,2,4)$ for (x_1,x_2,x_3,x_4) into the given equation we obtain

$$w_1 = (2)(1) - (3)(1) - (5)(2) - (1)(4) = -15$$
$$w_2 = (1)(1) + (5)(1) + (2)(2) - (3)(4) = -2$$

By matrix multiplication,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} (2)(1) - (3)(1) - (5)(2) - (1)(4) \\ (1)(1) + (5)(1) + (2)(2) - (3)(4) \end{bmatrix} = \begin{bmatrix} -15 \\ -2 \end{bmatrix}.$$

17. **(a)** $T(x_1, x_2) = \begin{bmatrix} -x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} (1)(1) + (1)(4) \\ -(0)(1) + (1)(4) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ matches $T(-1,4) = (1 + 4,4) = (5,4)$.

(b) $T(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ x_2 + x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} (2)(2) - (1)(1) - (1)(3) \\ (0)(2) + (1)(1) - (1)(3) \\ (0)(2) + (0)(1) - (0)(3) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

matches $T(2,1,-3) = (4 - 1 - 3, 1 - 3, 0) = (0, -2, 0)$.

18. **(a)** $T(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -(2)(2) - (1)(2) \\ -(1)(2) + (1)(2) \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ matches

$T(-2,2) = (-4 - 2, -2 + 2) = (-6, 0)$.

(b) $T(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} (1)(1) + (0)(0) + (0)(5) \\ (0)(1) + (1)(0) - (1)(5) \\ (0)(1) + (1)(0) + (0)(5) \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$ matches $T(1,0,5) = (1, -5, 0)$.

19. **(a)** $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$

20. **(a)** $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 + 4x_3 \\ 3x_1 + 5x_2 + 7x_3 \\ 6x_1 - x_3 \end{bmatrix}$

(b) $T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ 2x_1 + 4x_2 \\ 7x_1 + 8x_2 \end{bmatrix}$

21. **(a)** If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= \big(2(u_1 + v_1) + (u_2 + v_2), (u_1 + v_1) - (u_2 + v_2)\big) \\ &= (2u_1 + u_2, u_1 - u_2) + (2v_1 + v_2, v_1 - v_2) \end{aligned}$$

$$= T(\mathbf{u}) + T(\mathbf{v})$$

and $T(k\mathbf{u}) = T(ku_1, ku_2) = (2ku_1 + ku_2, ku_1 - ku_2) = k(2u_1 + u_2, u_1 - u_2) = kT(\mathbf{u})$.

(b) If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2, u_3+v_3) \\ &= (u_1+v_1, u_3+v_3, u_1+v_1+u_2+v_2) \\ &= (u_1, u_3, u_1+u_2) + (v_1, v_3, v_1+v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and $T(k\mathbf{u}) = T(ku_1, ku_2, ku_3) = (ku_1, ku_3, ku_1 + ku_2) = k(u_1, u_3, u_1 + u_2) = kT(\mathbf{u})$.

22. (a) If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2, u_3+v_3) \\ &= (u_1+v_1+u_2+v_2, u_2+v_2+u_3+v_3, u_1+v_1) \\ &= (u_1+u_2, u_2+u_3, u_1) + (v_1+v_2, v_2+v_3, v_1) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and $T(k\mathbf{u}) = T(ku_1, ku_2, ku_3) = (ku_1 + ku_2, ku_2 + ku_3, ku_1) = k(u_1 + u_2, u_2 + u_3, u_1) = kT(\mathbf{u})$.

(b) If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2) \\ &= (u_2+v_2, u_1+v_1) \\ &= (u_2, u_1) + (v_2, v_1) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and $T(k\mathbf{u}) = T(ku_1, ku_2) = (ku_2, ku_1) = k(u_2, u_1) = kT(\mathbf{u})$.

23. (a) The homogeneity property fails to hold since $T(kx, ky) = ((kx)^2, ky) = (k^2x^2, ky)$ does not generally equal $kT(x, y) = k(x^2, y) = (kx^2, ky)$. (It can be shown that the additivity property fails to hold as well.)

(b) The homogeneity property fails to hold since $T(kx, ky, kz) = (kx, ky, kxkz) = (kx, ky, k^2xz)$ does not generally equal $kT(x, y, z) = k(x, y, xz) = (kx, ky, kxz)$. (It can be shown that the additivity property fails to hold as well.)

24. (a) The homogeneity property fails to hold since $T(kx, ky) = (kx, ky + 1)$ does not generally equal $kT(x, y) = k(x, y + 1) = (kx, ky + k)$. (It can be shown that the additivity property fails to hold as well.)

(b) The homogeneity property fails to hold since $T(kx_1, kx_2, kx_3) = \left(kx_1, kx_2, \sqrt{kx_3}\right)$ does not generally equal $kT(x_1, x_2, x_3) = k\left(x_1, x_2, \sqrt{x_3}\right) = \left(kx_1, kx_2, k\sqrt{x_3}\right)$. (It can be shown that the additivity property fails to hold as well.)

25. The homogeneity property fails to hold since for $b \neq 0$, $f(kx) = m(kx) + b$ does not generally equal $kf(x) = k(mx + b) = kmx + kb$. (It can be shown that the additivity property fails to hold as well.) On the other hand, both properties hold for $b = 0$: $f(x + y) = m(x + y) = mx + my = f(x) + f(y)$ and $f(kx) = m(kx) = k(mx) = kf(x)$.
Consequently, f is not a matrix transformation on R unless $b = 0$

26. Both properties of Theorem 1.8.2 hold for $T(x, y) = (0,0)$:
$T\big((x, y) + (x', y')\big) = T(x + x', y + y') = (0,0) = (0,0) + (0,0) = T(x, y) + T(x', y')$
$T\big(k(x, y)\big) = T(kx, ky) = (0,0) = k(0,0) = kT(x, y)$

On the other hand, neither property holds in general for $T(x, y) = (1,1)$, e.g.,
$T\big((x, y) + (x', y')\big) = T(x + \mathrm{x}', y + y') = (1,1)$ does not equal
$T(x, y) + T(x', y') = (1,1) + (1,1) = (2,2)$

27. By Formula (13), the standard matrix for T is $A = [\ T(\mathbf{e}_1)\ \ |\ T(\mathbf{e}_2)\ |\ \ T(\mathbf{e}_3)\ ]$. Therefore
$A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix}$ and $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} (1)(2) + (0)(1) + (4)(0) \\ (3)(2) + (0)(1) - (3)(0) \\ (0)(2) + (1)(1) - (1)(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$.

28. By Formula (13), the standard matrix for T is $A = [\ T(\mathbf{e}_1)\ \ |\ T(\mathbf{e}_2)\ |\ \ T(\mathbf{e}_3)\ ]$. Therefore
$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$ and $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} (2)(3) - (3)(2) + (1)(1) \\ (1)(3) - (1)(2) + (0)(1) \\ (3)(3) + (0)(2) + (2)(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 11 \end{bmatrix}$.

29. By Formula (13), the standard matrix for T is $A = [\ T(\mathbf{e}_1)\ \ |\ T(\mathbf{e}_2)\ ]$. Therefore
$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $T(1,1) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.

30. For instance, $T(x, y) = (xy, 0)$ satisfies the property $T(0,0) = (0,0)$, but the homogeneity property fails to hold since $T(kx, ky) = (kxky, 0) = (k^2xy, 0)$ does not generally equal $kT(x, y) = k(xy, 0) = (kxy, 0)$.

31. **(a)** $T_A(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $T_A(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$, $T_A(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$.

(b) Since T_A is a matrix transformation,

$$T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T_A(\mathbf{e}_1) + T_A(\mathbf{e}_2) + T_A(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}.$$

(c) Since T_A is a matrix transformation, $T_A(7\boldsymbol{e}_3) = 7T_A(\mathbf{e}_3) = 7\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ -21 \end{bmatrix}$.

True-False Exercises

(a) False. The domain of T_A is R^3.

(b) False. The codomain of T_A is R^m.

(c) True. Since the statement requires the given equality to hold for <u>*some*</u> vector $\mathbf{x}$ in R^n, we can let $\mathbf{x} = \mathbf{0}$.

(d) False. (Refer to Theorem 1.8.3.)

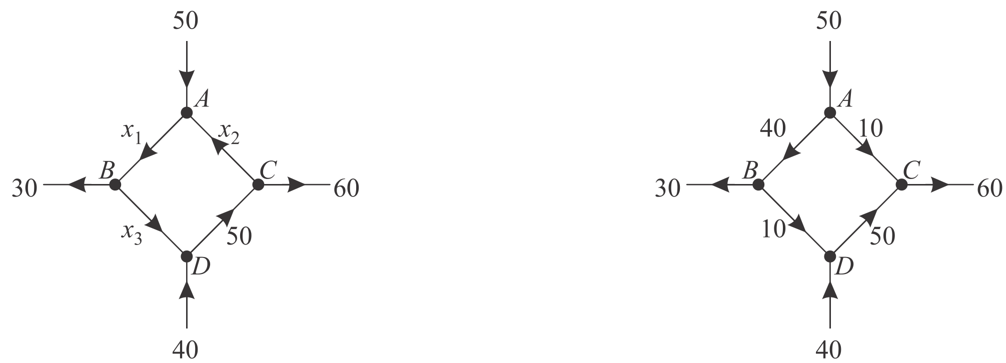
(e) True. The columns of A are $T(\mathbf{e}_i) = \mathbf{0}$.

(f) False. The given equality must hold for every matrix transformation since it follows from the homogeneity property.

(g) False. The homogeneity property fails to hold since $T(k\mathbf{x}) = k\mathbf{x} + \mathbf{b}$ does not generally equal $kT(\mathbf{x}) = k(\mathbf{x} + \mathbf{b}) = k\mathbf{x} + k\mathbf{b}$.

1.9 Applications of Linear Systems

1. There are four nodes, which we denote by A, B, C, and D (see the figure on the left).
We determine the unknown flow rates x_1, x_2, and x_3 assuming the counterclockwise direction (if any of these quantities are found to be negative then the flow direction along the corresponding branch will be reversed).



Network node	Flow In		Flow Out
A	$x_2 + 50$	$=$	x_1
B	x_1	$=$	$x_3 + 30$
C	50	$=$	$x_2 + 60$
D	$x_3 + 40$	$=$	50

This system can be rearranged as follows

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & & & = & -50 \\ x_1 & & & - & x_3 & = & 30 \\ & - & x_2 & & & = & 10 \\ & & & & x_3 & = & 10 \end{array}$$

By inspection, this system has a unique solution $x_1 = 40, x_2 = -10, x_3 = 10$. This yields the flow rates and directions shown in the figure on the right.

2. **(a)** There are five nodes – each of them corresponds to an equation.

Network node	Flow In		Flow Out
top left	200	$=$	$x_1 + x_3$
top right	$x_3 + 150$	$=$	$x_4 + x_5$
bottom left	$x_1 + 25$	$=$	x_2
bottom middle	$x_2 + x_4$	$=$	$x_6 + 175$
bottom right	$x_5 + x_6$	$=$	200

This system can be rearranged as follows

$$\begin{array}{rcrcrcrcrcrcl} x_1 & & & + & x_3 & & & & & & & = & 200 \\ & & & - & x_3 & + & x_4 & + & x_5 & & & = & 150 \\ -x_1 & + & x_2 & & & & & & & & & = & 25 \\ & & x_2 & & & + & x_4 & & & - & x_6 & = & 175 \\ & & & & & & & & x_5 & + & x_6 & = & 200 \end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 1 & -1 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_4 and x_6 the arbitrary values s and t, respectively, the general solution is given by the formulas

$x_1 = 150 - s + t$, $x_2 = 175 - s + t$, $x_3 = 50 + s - t$, $x_4 = s$, $x_5 = 200 - t$, $x_6 = t$

(c) When $x_4 = 50$ and $x_6 = 0$, the remaining flow rates become $x_1 = 100$, $x_2 = 125$, $x_3 = 100$, and $x_5 = 200$. The directions of the flow agree with the arrow orientations in the diagram.

3. **(a)** There are four nodes – each of them corresponds to an equation.

Network node	Flow In		Flow Out
top left	$x_2 + 300$	$=$	$x_3 + 400$
top right (A)	$x_3 + 750$	$=$	$x_4 + 250$
bottom left	$x_1 + 100$	$=$	$x_2 + 400$
bottom right (B)	$x_4 + 200$	$=$	$x_1 + 300$

This system can be rearranged as follows

$$\begin{array}{rcrcrcrcr} & & x_2 & - & x_3 & & & = & 100 \\ & & & & x_3 & - & x_4 & = & -500 \\ x_1 & - & x_2 & & & & & = & 300 \\ -x_1 & & & & & + & x_4 & = & 100 \end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) $\begin{bmatrix} 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -500 \\ 1 & -1 & 0 & 0 & 300 \\ -1 & 0 & 0 & 1 & 100 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & -1 & -400 \\ 0 & 0 & 1 & -1 & -500 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_4 the arbitrary value s, the general solution is given by the formulas

$x_1 = -100 + s$, $x_2 = -400 + s$, $x_3 = -500 + s$, $x_4 = s$

(c) In order for all x_i values to remain positive, we must have $s > 500$. Therefore, to keep the traffic flowing on all roads, the flow from A to B must exceed 500 vehicles per hour.

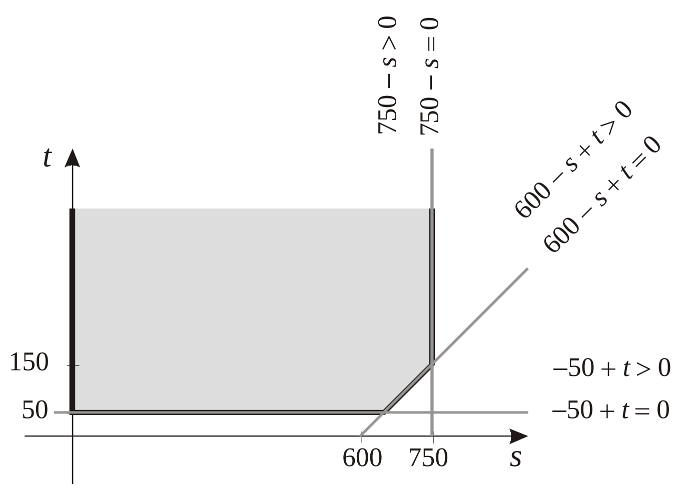
4. **(a)** There are six intersections – each of them corresponds to an equation.

Intersection	Flow In		Flow Out
top left	$500+300$	$=$	x_1+x_3
top middle	x_1+x_4	$=$	x_2+200
top right	x_2+100	$=$	x_5+600
bottom left	x_3+x_6	$=$	$400+350$
bottom middle	x_7+600	$=$	x_4+x_6
bottom right	x_5+450	$=$	x_7+400

We rewrite the system as follows

$$\begin{array}{rrrrrrrrrrrrrrr}
x_1 & & & + & x_3 & & & & & & & & & = & 800 \\
-x_1 & + & x_2 & & & - & x_4 & & & & & & & = & -200 \\
& - & x_2 & & & & & + & x_5 & & & & & = & -500 \\
& & & & x_3 & & & & & + & x_6 & & & = & 750 \\
& & & & & & x_4 & & & + & x_6 & - & x_7 & = & 600 \\
& & & & & & & - & x_5 & & & + & x_7 & = & 50
\end{array}$$

(b) The augmented matrix of the linear system obtained in part (a) has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we assign x_6 and x_7 the arbitrary values s and t, respectively, the general solution is given by the formulas $x_1 = 50 + s$, $x_2 = 450 + t$, $x_3 = 750 - s$, $x_4 = 600 - s + t$, $x_5 = -50 + t$, $x_6 = s$, $x_7 = t$ subject to the restriction that all seven values must be nonnegative. Obviously, we need both $s = x_6 \geq 0$ and $t = x_7 \geq 0$, which in turn imply $x_1 \geq 0$ and $x_2 \geq 0$. Additionally imposing the three inequalities $x_3 = 750 - s \geq 0$, $x_4 = 600 - s + t \geq 0$, and $x_5 = -50 + t \geq 0$ results in the set of allowable s and t values depicted in the grey region on the graph.



(c) Setting $x_1 = 0$ in the general solution obtained in part (b) would result in the negative value $s = x_6 = -50$ which is not allowed (the traffic would flow in a wrong way along the street marked as x_6.)

5. From Kirchhoff's current law at each node, we have $I_1 + I_2 - I_3 = 0$. Kirchhoff's voltage law yields

	Voltage Rises		Voltage Drops
Left Loop (clockwise)	$2I_1$	$=$	$2I_2+6$
Right Loop (clockwise)	$2I_2+4I_3$	$=$	8

(An equation corresponding to the outer loop is a combination of these two equations.)
The linear system can be rewritten as

$$\begin{array}{rcrcrcl} I_1 & + & I_2 & - & I_3 & = & 0 \\ 2I_1 & - & 2I_2 & & & = & 6 \\ & & 2I_2 & + & 4I_3 & = & 8 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{13}{5} \\ 0 & 1 & 0 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{11}{5} \end{bmatrix}$.

The solution is $I_1 = 2.6\text{A}$, $I_2 = -0.4\text{A}$, and $I_3 = 2.2\text{A}$.
Since I_2 is negative, this current is opposite to the direction shown in the diagram.

6. From Kirchhoff's current law at each node, we have $I_1 - I_2 + I_3 = 0$. Kirchhoff's voltage law yields

	Voltage Rises		Voltage Drops
Left Inside Loop (clockwise)	$4I_1 + 6I_2$	=	1
Right Inside Loop (clockwise)	$2I_3$	=	$2 + 4I_1$

(An equation corresponding to the outer loop is a combination of these two equations.)
The linear system can be rewritten as

$$\begin{array}{rcrcrcl} I_1 & - & I_2 & + & I_3 & = & 0 \\ 4I_1 & + & 6I_2 & & & = & 1 \\ -4I_1 & & & + & 2I_3 & = & 2 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{22} \\ 0 & 1 & 0 & \frac{7}{22} \\ 0 & 0 & 1 & \frac{6}{11} \end{bmatrix}$.

The solution is $I_1 = -\frac{5}{22}\text{A}$, $I_2 = \frac{7}{22}\text{A}$, and $I_3 = \frac{6}{11}\text{A}$.
Since I_1 is negative, this current is opposite to the direction shown in the diagram.

7. From Kirchhoff's current law, we have

	Current In		Currrent Out
Top Left Node	I_1	=	$I_2 + I_4$
Top Right Node	I_4	=	$I_3 + I_5$
Bottom Left Node	$I_2 + I_6$	=	I_1
Bottom Right Node	$I_3 + I_5$	=	I_6

Kirchhoff's voltage law yields

	Voltage Rises		Voltage Drops
Left Loop (clockwise)	10	=	$20I_1 + 20I_2$
Middle Loop (clockwise)	$20I_2$	=	$20I_3$
Right Loop (clockwise)	$20I_3 + 10$	=	$20I_5$

(Equations corresponding to the other loops are combinations of these three equations.)

The linear system can be rewritten as

$$\begin{array}{rcrcrcrcrcrcr} I_1 & - & I_2 & & & - & I_4 & & & & & = & 0 \\ & & & - & I_3 & + & I_4 & - & I_5 & & & = & 0 \\ -I_1 & + & I_2 & & & & & & & + & I_6 & = & 0 \\ & & & & I_3 & & & + & I_5 & - & I_6 & = & 0 \\ -20I_1 & - & 20I_2 & & & & & & & & & = & -10 \\ & & 20I_2 & - & 20I_3 & & & & & & & = & 0 \\ & & & & 20I_3 & & & - & 20I_5 & & & = & -10 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The solution is $I_1 = I_4 = I_5 = I_6 = 0.5\text{A}$, $I_2 = I_3 = 0\text{A}$.

8. From Kirchhoff's current law at each node, we have $I_1 - I_2 - I_3 = 0$. Kirchhoff's voltage law yields

	Voltage Rises		Voltage Drops
Top Inside Loop (clockwise)	$3I_1 + 4I_2$	$=$	$5 + 4$
Bottom Inside Loop (clockwise)	$4 + 5I_3$	$=$	$3 + 4I_2$

The corresponding linear system can be rewritten as

$$\begin{array}{rcrcrcr} I_1 & - & I_2 & - & I_3 & = & 0 \\ 3I_1 & + & 4I_2 & & & = & 9 \\ & - & 4I_2 & + & 5I_3 & = & -1 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{77}{47} \\ 0 & 1 & 0 & \frac{48}{47} \\ 0 & 0 & 1 & \frac{29}{47} \end{bmatrix}$.

The solution is $I_1 = \frac{77}{47}\text{A}$, $I_2 = \frac{48}{47}\text{A}$, and $I_3 = \frac{29}{47}\text{A}$.

9. We are looking for positive integers x_1, x_2, x_3, and x_4 such that

$$x_1(C_3H_8) + x_2(O_2) \to x_3(CO_2) + x_4(H_2O)$$

The number of atoms of carbon, hydrogen, and oxygen on both sides must equal:

	Left Side		Right Side
Carbon	$3x_1$	$=$	x_3
Hydrogen	$8x_1$	$=$	$2x_4$
Oxygen	$2x_2$	$=$	$2x_3 + x_4$

The linear system

$$\begin{array}{rcrcrcrcl} 3x_1 & & & - & x_3 & & & = & 0 \\ 8x_1 & & & & & - & 2x_4 & = & 0 \\ & & 2x_2 & - & 2x_3 & - & x_4 & = & 0 \end{array}$$

has the augmented matrix whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{bmatrix}$.

The general solution is $x_1 = \frac{1}{4}t,\ x_2 = \frac{5}{4}t,\ x_3 = \frac{3}{4}t,\ x_4 = t$ where t is arbitrary. The smallest positive integer values for the unknowns occur when $t = 4$, which yields the solution $x_1 = 1,\ x_2 = 5,\ x_3 = 3,\ x_4 = 4.$ The balanced equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

10. We are looking for positive integers x_1, x_2, and x_3 such that

$$x_1(C_6H_{12}O_6) \rightarrow x_2(CO_2) + x_3(C_2H_5OH)$$

The number of atoms of carbon, hydrogen, and oxygen on both sides must equal:

	Left Side		Right Side
Carbon	$6x_1$	$=$	$x_2 + 2x_3$
Hydrogen	$12x_1$	$=$	$6x_3$
Oxygen	$6x_1$	$=$	$2x_2 + x_3$

The linear system

$$\begin{array}{rcrcrcl} 6x_1 & - & x_2 & - & 2x_3 & = & 0 \\ 12x_1 & & & - & 6x_3 & = & 0 \\ 6x_1 & - & 2x_2 & - & x_3 & = & 0 \end{array}$$

has the augmented matrix whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_1 = \frac{1}{2}t,\ x_2 = t,\ x_3 = t$ where t is arbitrary. The smallest positive integer values for the unknowns occur when $t = 2$, which yields the solution $x_1 = 1,\ x_2 = 2,\ x_3 = 2$. The balanced equation is

$$C_6H_{12}O_6 \rightarrow 2CO_2 + 2C_2H_5OH$$

11. We are looking for positive integers x_1, x_2, x_3, and x_4 such that

$$x_1(CH_3COF) + x_2(H_2O) \rightarrow x_3(CH_3COOH) + x_4(HF)$$

The number of atoms of carbon, hydrogen, oxygen, and fluorine on both sides must equal:

	Left Side		Right Side
Carbon	$2x_1$	$=$	$2x_3$
Hydrogen	$3x_1 + 2x_2$	$=$	$4x_3 + x_4$
Oxygen	$x_1 + x_2$	$=$	$2x_3$
Fluorine	x_1	$=$	x_4

The linear system

$$\begin{array}{rcrcrcrcl} 2x_1 & & & - & 2x_3 & & & = & 0 \\ 3x_1 & + & 2x_2 & - & 4x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & = & 0 \\ x_1 & & & & & - & x_4 & = & 0 \end{array}$$

has the augmented matrix whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_1 = t,\ x_2 = t,\ x_3 = t,\ x_4 = t$ where t is arbitrary. The smallest positive integer values for the unknowns occur when $t = 1$, which yields the solution $x_1 = 1,\ x_2 = 1\ x_3 = 1,$ $x_4 = 1$. The balanced equation is

$$\mathrm{CH_3COF} + \mathrm{H_2O} \rightarrow \mathrm{CH_3COOH} + \mathrm{HF}$$

12. We are looking for positive integers x_1, x_2, x_3, and x_4 such that

$$x_1(\mathrm{CO_2}) + x_2(\mathrm{H_2O}) \rightarrow x_3(\mathrm{C_6H_{12}O_6}) + x_4(\mathrm{O_2})$$

The number of atoms of carbon, hydrogen, and oxygen on both sides must equal:

	Left Side		Right Side
Carbon	x_1	$=$	$6x_3$
Hydrogen	$2x_2$	$=$	$12x_3$
Oxygen	$2x_1 + x_2$	$=$	$6x_3 + 2x_4$

The linear system

$$\begin{array}{rcrcrcrcl} x_1 & & & - & 6x_3 & & & = & 0 \\ & & 2x_2 & - & 12x_3 & & & = & 0 \\ 2x_1 & + & x_2 & - & 6x_3 & - & 2x_4 & = & 0 \end{array}$$

has the augmented matrix whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & 0 \end{bmatrix}$.

The general solution is $x_1 = t,\ x_2 = t,\ x_3 = \frac{1}{6}t,\ x_4 = t$ where t is arbitrary. The smallest positive integer values for the unknowns occur when $t = 6$, which yields the solution $x_1 = 6,\ x_2 = 6,\ x_3 = 1,\ x_4 = 6$. The balanced equation is

$$6\mathrm{CO_2} + 6\mathrm{H_2O} \rightarrow \mathrm{C_6H_{12}O_6} + 6\mathrm{O_2}$$

13. We are looking for a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2$ such that $p(1) = 1,\ p(2) = 2,$ and $p(3) = 5$. We obtain a linear system

$$\begin{array}{rcrcrcl} a_0 & + & a_1 & + & a_2 & = & 1 \\ a_0 & + & 2a_1 & + & 4a_2 & = & 2 \\ a_0 & + & 3a_1 & + & 9a_2 & = & 5 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

There is a unique solution $a_0 = 2,\ a_1 = -2,\ a_2 = 1$.
The quadratic polynomial is $p(x) = 2 - 2x + x^2$.

14. We are looking for a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2$ such that $p(0) = 0,\ p(-1) = 1$, and $p(1) = 1$. We obtain a linear system

$$\begin{array}{ccccccc} a_0 & & & & & = & 0 \\ a_0 & - & a_1 & + & a_2 & = & 1 \\ a_0 & + & a_1 & + & a_2 & = & 1 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

There is a unique solution $a_0 = 0,\ a_1 = 0,\ a_2 = 1$. The quadratic polynomial is $p(x) = x^2$.

15. We are looking for a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $p(-1) = -1$, $p(0) = 1,\ p(1) = 3$ and $p(4) = -1$. We obtain a linear system

$$\begin{array}{ccccccccc} a_0 & - & a_1 & + & a_2 & - & a_3 & = & -1 \\ a_0 & & & & & & & = & 1 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = & 3 \\ a_0 & + & 4a_1 & + & 16a_2 & + & 64a_3 & = & -1 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{13}{6} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$.

There is a unique solution $a_0 = 1,\ a_1 = \frac{13}{6},\ a_2 = 0\ ,\ a_3 = -\frac{1}{6}$.

The cubic polynomial is $p(x) = 1 + \frac{13}{6}x - \frac{1}{6}x^3$.

16. We are looking for a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $p(0) = 0$, $p(2) = 5,\ p(4) = 8$ and $p(6) = 3$. We obtain a linear system

$$\begin{array}{ccccccccc} a_0 & & & & & & & = & 0 \\ a_0 & + & 2a_1 & + & 4a_2 & + & 8a_3 & = & 5 \\ a_0 & + & 4a_1 & + & 16a_2 & + & 64a_3 & = & 8 \\ a_0 & + & 6a_1 & + & 36a_2 & + & 216a_3 & = & 3 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{8} \end{bmatrix}$.

There is a unique solution $a_0 = 0,\ a_1 = 2,\ a_2 = \frac{1}{2},\ a_3 = -\frac{1}{8}$.

The cubic polynomial is $p(x) = 2x + \frac{1}{2}x^2 - \frac{1}{8}x^3$.

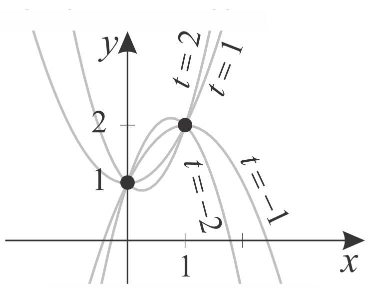
17. **(a)** We are looking for a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2$ such that $p(0) = 1$ and $p(1) = 2$. We obtain a linear system

$$\begin{array}{rcrcrcl} a_0 & & & & & = & 1 \\ a_0 & + & a_1 & + & a_2 & = & 2 \end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

The general solution of the linear system is $a_0 = 1,\ a_1 = 1 - t,\ a_2 = t$ where t is arbitrary. Consequently, the family of all second-degree polynomials that pass through $(0,1)$ and $(1,2)$ can be represented by $p(x) = 1 + (1 - t)x + tx^2$ where t is an arbitrary real number.

(b)



True-False Exercises

(a) False. In general, networks may or may not satisfy the property of flow conservation at each node (although the ones discussed in this section do).

(b) False. When a current passes through a resistor, there is a drop in the electrical potential in a circuit.

(c) True.

(d) False. A chemical equation is said to be balanced if *for each type of atom in the reaction*, the same number of atoms appears on each side of the equation.

(e) False. By Theorem 1.9.1, this is true if the points have distinct x-coordinates.

1.10 Leontief Input-Output Models

1. **(a)** $C = \begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.10 \end{bmatrix}$

(b) The Leontief matrix is $I - C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.10 \end{bmatrix} = \begin{bmatrix} 0.50 & -0.25 \\ -0.25 & 0.90 \end{bmatrix}$;

the outside demand vector is $\mathbf{d} = \begin{bmatrix} 7{,}000 \\ 14{,}000 \end{bmatrix}$.

The Leontief equation $(I-C)\mathbf{x}=\mathbf{d}$ leads to the linear system with the augmented matrix $\begin{bmatrix} 0.50 & -0.25 & 7{,}000 \\ -0.25 & 0.90 & 14{,}000 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & \frac{784{,}000}{31} \\ 0 & 1 & \frac{700{,}000}{31} \end{bmatrix} \approx$ $\begin{bmatrix} 1 & 0 & 25{,}290.32 \\ 0 & 1 & 22{,}580.65 \end{bmatrix}$.

To meet the consumer demand, M must produce approximately \$25,290.32 worth of mechanical work and B must produce approximately \$22,580.65 worth of body work.

2. **(a)** $C=\begin{bmatrix} 0.30 & 0.20 \\ 0.10 & 0.60 \end{bmatrix}$

(b) The Leontief matrix is $I-C=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.30 & 0.20 \\ 0.10 & 0.60 \end{bmatrix}=\begin{bmatrix} 0.70 & -0.20 \\ -0.10 & 0.40 \end{bmatrix}$;

the outside demand vector is $\mathbf{d}=\begin{bmatrix} 130{,}000 \\ 130{,}000 \end{bmatrix}$.

The Leontief equation $(I-C)\mathbf{x}=\mathbf{d}$ leads to the linear system with the augmented matrix $\begin{bmatrix} 0.70 & -0.20 & 130{,}000 \\ -0.10 & 0.40 & 130{,}000 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 300{,}000 \\ 0 & 1 & 400{,}000 \end{bmatrix}$.

To meet the consumer demand, the economy must produce \$300,000 worth of food and \$400,000 worth of housing.

3. **(a)** $C=\begin{bmatrix} 0.10 & 0.60 & 0.40 \\ 0.30 & 0.20 & 0.30 \\ 0.40 & 0.10 & 0.20 \end{bmatrix}$

(b) The Leontief matrix is $I-C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.10 & 0.60 & 0.40 \\ 0.30 & 0.20 & 0.30 \\ 0.40 & 0.10 & 0.20 \end{bmatrix}=\begin{bmatrix} 0.90 & -0.60 & -0.40 \\ -0.30 & 0.80 & -0.30 \\ -0.40 & -0.10 & 0.80 \end{bmatrix}$;

the outside demand vector is $\mathbf{d}=\begin{bmatrix} 1930 \\ 3860 \\ 5790 \end{bmatrix}$.

The Leontief equation $(I-C)\mathbf{x}=\mathbf{d}$ leads to the linear system with the augmented matrix $\begin{bmatrix} 0.90 & -0.60 & -0.40 & 1930 \\ -0.30 & 0.80 & -0.30 & 3860 \\ -0.40 & -0.10 & 0.80 & 5790 \end{bmatrix}$.

Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 31{,}500 \\ 0 & 1 & 0 & 26{,}500 \\ 0 & 0 & 1 & 26{,}300 \end{bmatrix}$.

The production vector that will meet the given demand is $\mathbf{x}=\begin{bmatrix} \$31{,}500 \\ \$26{,}500 \\ \$26{,}300 \end{bmatrix}$.

4. **(a)** $C=\begin{bmatrix} 0.40 & 0.20 & 0.45 \\ 0.30 & 0.35 & 0.30 \\ 0.15 & 0.10 & 0.20 \end{bmatrix}$

(b) The Leontief matrix is $I-C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.40 & 0.20 & 0.45 \\ 0.30 & 0.35 & 0.30 \\ 0.15 & 0.10 & 0.20 \end{bmatrix}=\begin{bmatrix} 0.60 & -0.20 & -0.45 \\ -0.30 & 0.65 & -0.30 \\ -0.15 & -0.10 & 0.80 \end{bmatrix}$;

the outside demand vector is $\mathbf{d} = \begin{bmatrix} 5400 \\ 2700 \\ 900 \end{bmatrix}$.

The Leontief equation $(I - C)\mathbf{x} = \mathbf{d}$ leads to the linear system with the augmented matrix $\begin{bmatrix} 0.60 & -0.20 & -0.45 & 5400 \\ -0.30 & 0.65 & -0.30 & 2700 \\ -0.15 & -0.10 & 0.80 & 900 \end{bmatrix}$.

Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & \frac{9378000}{479} \\ 0 & 1 & 0 & \frac{7830000}{479} \\ 0 & 0 & 1 & \frac{3276000}{479} \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 19578.29 \\ 0 & 1 & 0 & 16346.56 \\ 0 & 0 & 1 & 6839.25 \end{bmatrix}$.

The production vector that will meet the given demand is $\mathbf{x} \approx \begin{bmatrix} \$19578.29 \\ \$16346.56 \\ \$6839.25 \end{bmatrix}$.

5. $I - C = \begin{bmatrix} 0.9 & -0.3 \\ -0.5 & 0.6 \end{bmatrix}$; $(I - C)^{-1} = \frac{100}{39}\begin{bmatrix} 0.6 & 0.3 \\ 0.5 & 0.9 \end{bmatrix} = \begin{bmatrix} \frac{20}{13} & \frac{10}{13} \\ \frac{50}{39} & \frac{30}{13} \end{bmatrix}$

$\mathbf{x} = (I - C)^{-1}\mathbf{d} = \begin{bmatrix} \frac{20}{13} & \frac{10}{13} \\ \frac{50}{39} & \frac{30}{13} \end{bmatrix}\begin{bmatrix} 50 \\ 60 \end{bmatrix} = \begin{bmatrix} \frac{1600}{13} \\ \frac{7900}{39} \end{bmatrix} \approx \begin{bmatrix} 123.08 \\ 202.56 \end{bmatrix}$

6. $I - C = \begin{bmatrix} 0.7 & -0.1 \\ -0.3 & 0.3 \end{bmatrix}$; $(I - C)^{-1} = \frac{100}{18}\begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{5}{9} \\ \frac{5}{3} & \frac{35}{9} \end{bmatrix}$

$\mathbf{x} = (I - C)^{-1}\mathbf{d} = \begin{bmatrix} \frac{5}{3} & \frac{5}{9} \\ \frac{5}{3} & \frac{35}{9} \end{bmatrix}\begin{bmatrix} 22 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{400}{9} \\ \frac{820}{9} \end{bmatrix} \approx \begin{bmatrix} 44.44 \\ 91.11 \end{bmatrix}$

7. **(a)** The Leontief matrix is $I - C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$.

The Leontief equation $(I - C)\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ leads to the linear system with the augmented matrix $\begin{bmatrix} \frac{1}{2} & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ therefore a production vector can be found (namely, $\begin{bmatrix} 4 \\ t \end{bmatrix}$ for an arbitrary nonnegative t) to meet the demand.

On the other hand, the Leontief equation $(I - C)\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ leads to the linear system with the augmented matrix $\begin{bmatrix} \frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; the system is inconsistent, therefore a production vector cannot be found to meet the demand.

(b) Mathematically, the linear system represented by $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ can be rewritten as $\begin{bmatrix} \frac{1}{2} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

Clearly, if $d_2 = 0$ the system has infinitely many solutions: $x_1 = 2d_1$; $x_2 = t$ where t is an arbitrary nonnegative number.
If $d_2 \neq 0$ the system is inconsistent. (Note that the Leontief matrix is not invertible.)

An economic explanation of the result in part (a) is that $\mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ therefore the second sector consumes all of its own output, making it impossible to meet any outside demand for its products.

8. $I - C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{7}{8} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{7}{8} \end{bmatrix}$

If the open sector demands k dollars worth from each product-producing sector, i.e. the outside demand vector is $\mathbf{d} = \begin{bmatrix} k \\ k \\ k \end{bmatrix}$. The Leontief equation $(I - C)\mathbf{x} = \mathbf{d}$ leads to the linear system with the augmented matrix $\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & k \\ -\frac{1}{2} & \frac{7}{8} & -\frac{1}{4} & k \\ -\frac{1}{2} & -\frac{1}{4} & \frac{7}{8} & k \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 18k \\ 0 & 1 & 0 & 16k \\ 0 & 0 & 1 & 16k \end{bmatrix}$.

We conclude that the first sector must produce the greatest dollar value to meet the specified open sector demand.

9. From the assumption $c_{21}c_{12} < 1 - c_{11}$, it follows that the determinant of $\det(I - C) = \det\left(\begin{bmatrix} 1 - c_{11} & -c_{12} \\ -c_{21} & 1 \end{bmatrix}\right) = 1 - c_{11} - c_{12}c_{21}$ is nonzero. Consequently, the Leontief matrix is invertible; its inverse is $(I - C)^{-1} = \frac{1}{1 - c_{11} - c_{12}c_{21}} \begin{bmatrix} 1 & c_{12} \\ c_{21} & 1 - c_{11} \end{bmatrix}$. Since the consumption matrix C has nonnegative entries and $1 - c_{11} > c_{21}c_{12} \geq 0$, we conclude that all entries of $(I - C)^{-1}$ are nonnegative as well. This economy is productive (see the discussion above Theorem 1.10.1) - the equation $\mathbf{x} - C\mathbf{x} = \mathbf{d}$ has a unique solution $\mathbf{x} = (I - C)^{-1}\mathbf{d}$ for every demand vector $\mathbf{d}$.

True-False Exercises

(a) False. Sectors that do *not* produce outputs are called open sectors.

(b) True.

(c) False. The ith row vector of a consumption matrix contains the monetary values required of the ith sector by the other sectors for each of them to produce one monetary unit of output.

(d) True. This follows from Theorem 1.10.1.

(e) True.

Chapter 1 Supplementary Exercises

1. The corresponding system of linear equations is

$$\begin{array}{rcrcrcrcr} 3x_1 & - & x_2 & & & + & 4x_4 & = & 1 \\ 2x_1 & & & + & 3x_3 & + & 3x_4 & = & -1 \end{array}$$

$\begin{bmatrix} 3 & -1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$ ← The original augmented matrix.

$\begin{bmatrix} 1 & -1 & -3 & 1 & 2 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$ ← -1 times the second row was added to the first row.

$\begin{bmatrix} 1 & -1 & -3 & 1 & 2 \\ 0 & 2 & 9 & 1 & -5 \end{bmatrix}$ ← -2 times the first row was added to the second row.

$\begin{bmatrix} 1 & -1 & -3 & 1 & 2 \\ 0 & 1 & \frac{9}{2} & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$ ← The second row was multiplied by $\frac{1}{2}$.

This matrix is in row echelon form. It corresponds to the system of equations

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & 3x_3 & + & x_4 & = & 2 \\ & & x_2 & + & \frac{9}{2}x_3 & + & \frac{1}{2}x_4 & = & -\frac{5}{2} \end{array}$$

Solve the equations for the leading variables

$$x_1 = x_2 + 3x_3 - x_4 + 2$$
$$x_2 = -\frac{9}{2}x_3 - \frac{1}{2}x_4 - \frac{5}{2}$$

then substitute the second equation into the first

$$x_1 = -\frac{3}{2}x_3 - \frac{3}{2}x_4 - \frac{1}{2}$$
$$x_2 = -\frac{9}{2}x_3 - \frac{1}{2}x_4 - \frac{5}{2}$$

If we assign x_3 and x_4 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$x_1 = -\frac{3}{2}s - \frac{3}{2}t - \frac{1}{2}, \quad x_2 = -\frac{9}{2}s - \frac{1}{2}t - \frac{5}{2}, \quad x_3 = s, \quad x_4 = t$$

2. The corresponding system of linear equations is

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -1 \\ -2x_1 & - & 8x_2 & = & 2 \\ 3x_1 & + & 12x_2 & = & -3 \\ & & 0 & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & 4 & -1 \\ -2 & -8 & 2 \\ 3 & 12 & -3 \\ 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{The original augmented matrix.}$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longleftarrow \begin{array}{l} 2 \text{ times the first row was added to the second row and} \\ -3 \text{ times the first row was added to the third row.} \end{array}$$

This matrix is both in row echelon form and in reduced row echelon form. It corresponds to the system of equations

$$\begin{array}{rcrcr} x_1 & + & 4x_2 & = & -1 \\ & & 0 & = & 0 \\ & & 0 & = & 0 \\ & & 0 & = & 0 \end{array}$$

If we assign x_2 an arbitrary value t, the general solution is given by the formulas

$$x_1 = -1 - 4t, \qquad x_2 = t$$

3. The corresponding system of linear equations is

$$\begin{array}{rcrcrcr} 2x_1 & - & 4x_2 & + & x_3 & = & 6 \\ -4x_1 & & & + & 3x_3 & = & -1 \\ & & x_2 & - & x_3 & = & 3 \end{array}$$

$$\begin{bmatrix} 2 & -4 & 1 & 6 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix} \longleftarrow \text{The original augmented matrix.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & -8 & 5 & 11 \\ 0 & 1 & -1 & 3 \end{bmatrix} \longleftarrow 4 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & -8 & 5 & 11 \end{bmatrix} \longleftarrow \text{The second and third rows were interchanged.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -3 & 35 \end{bmatrix} \longleftarrow 8 \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} & 3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -\frac{35}{3} \end{bmatrix} \longleftarrow \text{The third row was multiplied by } -\tfrac{1}{3}.$$

This matrix is in row echelon form. It corresponds to the system of equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & \frac{1}{2}x_3 & = & 3 \\ & & x_2 & - & x_3 & = & 3 \\ & & & & x_3 & = & -\frac{35}{3} \end{array}$$

Solve the equations for the leading variables

$$\begin{aligned} x_1 &= 2x_2 - \frac{1}{2}x_3 + 3 \\ x_2 &= x_3 + 3 \\ x_3 &= -\frac{35}{3} \end{aligned}$$

then finish back-substituting to obtain the unique solution

$$x_1 = -\frac{17}{2}, \qquad x_2 = -\frac{26}{3}, \qquad x_3 = -\frac{35}{3}$$

4. The corresponding system of linear equations is

$$\begin{array}{rcrcr} 3x_1 & + & x_2 & = & -2 \\ -9x_1 & - & 3x_2 & = & 6 \\ 6x_1 & + & 2x_2 & = & 1 \end{array}$$

$$\begin{bmatrix} 3 & 1 & -2 \\ -9 & -3 & 6 \\ 6 & 2 & 1 \end{bmatrix} \longleftarrow \text{The original augmented matrix.}$$

$$\begin{bmatrix} 3 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \longleftarrow \begin{array}{l} 3 \text{ times the first row was added to the second row and} \\ -2 \text{ times the first row was added to the third row.} \end{array}$$

Although this matrix is not in row echelon form yet, clearly it corresponds to an inconsistent linear system

$$\begin{array}{rcrcr} 3x_1 & + & x_2 & = & -2 \\ & & 0 & = & 0 \\ & & 0 & = & 5 \end{array}$$

since the third equation is contradictory. (We could have performed additional elementary row operations to obtain a matrix in row echelon form $\begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.)

5.

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & x \\ \frac{4}{5} & \frac{3}{5} & y \end{bmatrix}$$

The augmented matrix corresponding to the system.

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ \frac{4}{5} & \frac{3}{5} & y \end{bmatrix}$$

The first row was multiplied by $\frac{5}{3}$.

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ 0 & \frac{5}{3} & -\frac{4}{3}x + y \end{bmatrix}$$

$-\frac{4}{5}$ times the first row was added to the second row.

$$\begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3}x \\ 0 & 1 & -\frac{4}{5}x + \frac{3}{5}y \end{bmatrix}$$

The second row was multiplied by $\frac{3}{5}$.

$$\begin{bmatrix} 1 & 0 & \frac{3}{5}x + \frac{4}{5}y \\ 0 & 1 & -\frac{4}{5}x + \frac{3}{5}y \end{bmatrix}$$

$\frac{4}{3}$ times the second row was added to the first row.

The system has exactly one solution: $x' = \frac{3}{5}x + \frac{4}{5}y$ and $y' = -\frac{4}{5}x + \frac{3}{5}y$.

6. We break up the solution into three cases:

Case I: $\cos\theta \neq 0$ and $\sin\theta \neq 0$

$$\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \end{bmatrix}$$

The augmented matrix corresponding to the system.

$$\begin{bmatrix} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{x}{\cos\theta} \\ \sin\theta & \cos\theta & y \end{bmatrix}$$

The first row was multiplied by $\frac{1}{\cos\theta}$.

$$\begin{bmatrix} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{x}{\cos\theta} \\ 0 & \frac{1}{\cos\theta} & y - x\frac{\sin\theta}{\cos\theta} \end{bmatrix}$$

$-\sin\theta$ times the first row was added to the second $\left(\frac{\sin^2\theta}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta}\right)$.

$$\begin{bmatrix} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{x}{\cos\theta} \\ 0 & 1 & y\cos\theta - x\sin\theta \end{bmatrix}$$

The second row was multiplied by $\cos\theta$.

$$\begin{bmatrix} 1 & 0 & x\cos\theta + y\sin\theta \\ 0 & 1 & y\cos\theta - x\sin\theta \end{bmatrix}$$

$\frac{\sin\theta}{\cos\theta}$ times the second row was added to the first row $\left(-\frac{x\sin^2\theta}{\cos\theta} + \frac{x}{\cos\theta} = \frac{x\cos^2\theta}{\cos\theta} = x\cos\theta\right)$.

The system has exactly one solution: $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$.

Case II: $\cos\theta = 0$ which implies $\sin^2\theta = 1$. The original system becomes $x = -y'\sin\theta,\ y = x'\sin\theta$. Multiplying both sides of the each equation by $\sin\theta$ yields $x' = y\sin\theta,\ \ y' = -x\sin\theta$.

Case III: $\sin\theta = 0$, which implies $\cos^2\theta = 1$. The original system becomes $x = x'\cos\theta,\ y = y'\cos\theta$. Multiplying both sides of each equation by $\cos\theta$ yields $x' = x\cos\theta,\ y' = y\cos\theta$.

Notice that the solution found in case I

$$x' = x\cos\theta + y\sin\theta \text{ and } y' = -x\sin\theta + y\cos\theta.$$

actually applies to all three cases.

7. $\begin{bmatrix} 1 & 1 & 1 & 9 \\ 1 & 5 & 10 & 44 \end{bmatrix} \longleftarrow$ The original augmented matrix.

$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 4 & 9 & 35 \end{bmatrix} \longleftarrow$ -1 times the first row was added to the second row.

$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{bmatrix} \longleftarrow$ The second row was multiplied by $\frac{1}{4}$.

$\begin{bmatrix} 1 & 0 & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & \frac{9}{4} & \frac{35}{4} \end{bmatrix} \longleftarrow$ -1 times the second row was added to the first row.

If we assign z an arbitrary value t, the general solution is given by the formulas

$$x = \frac{1}{4} + \frac{5}{4}t, \qquad y = \frac{35}{4} - \frac{9}{4}t, \qquad z = t$$

The positivity of the three variables requires that $\frac{1}{4} + \frac{5}{4}t > 0$, $\frac{35}{4} - \frac{9}{4}t > 0$, and $t > 0$. The first inequality can be rewritten as $t > -\frac{1}{4}$, while the second inequality is equivalent to $t < \frac{35}{9}$. All three unknowns are positive whenever $0 < t < \frac{35}{9}$. There are three integer values of $t = z$ in this interval: 1, 2, and 3. Of those, only $z = t = 3$ yields integer values for the remaining variables: $x = 4, y = 2$.

8. Let x, y, and z denote the number of pennies, nickels, and dimes, respectively. Since there are 13 coins, we must have

$$x + y + z = 13.$$

On the other hand, the total value of the coins is 83 cents so that

$$x + 5y + 10z = 83.$$

The resulting system of equations has the augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 13 \\ 1 & 5 & 10 & 83 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & -\frac{5}{4} & -\frac{9}{2} \\ 0 & 1 & \frac{9}{4} & \frac{35}{2} \end{bmatrix}$

If we assign z an arbitrary value t, the general solution is given by the formulas

$$x = -\frac{9}{2} + \frac{5}{4}t, \qquad y = \frac{35}{2} - \frac{9}{4}t, \qquad z = t$$

However, all three unknowns must be nonnegative integers.

The nonnegativity of x requires the inequality $-\frac{9}{2}+\frac{5}{4}t \geq 0$, i.e., $t \geq \frac{18}{5}$.

Likewise for y, $\frac{35}{2}-\frac{9}{4}t \geq 0$ yields $t \leq \frac{70}{9}$.

When $\frac{18}{5} \leq t \leq \frac{70}{9}$, all three variables are nonnegative. Of the four integer $t = z$ values inside this interval (4, 5, 6, and 7), only $t = z = 6$ yields integer values for x and y.

We conclude that the box has to contain 3 pennies, 4 nickels, and 6 dimes.

9. $\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{bmatrix}$ ← -1 times the first row was added to the second row.

$\begin{bmatrix} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{bmatrix}$ ← -1 times the second row was added to the third row.

(a) the system has a unique solution if $a \neq 0$ and $b \neq 2$ (multiplying the rows by $\frac{1}{a}$, $\frac{1}{a}$, and $\frac{1}{b-2}$, respectively, yields a row echelon form of the augmented matrix $\begin{bmatrix} 1 & 0 & \frac{b}{a} & \frac{2}{a} \\ 0 & 1 & \frac{4-b}{a} & \frac{2}{a} \\ 0 & 0 & 1 & 1 \end{bmatrix}$).

(b) the system has a one-parameter solution if $a \neq 0$ and $b = 2$ (multiplying the first two rows by $\frac{1}{a}$ yields a reduced row echelon form of the augmented matrix $\begin{bmatrix} 1 & 0 & \frac{2}{a} & \frac{2}{a} \\ 0 & 1 & \frac{2}{a} & \frac{2}{a} \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

(c) the system has a two-parameter solution if $a = 0$ and $b = 2$ (the reduced row echelon form of the augmented matrix is $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

(d) the system has no solution if $a = 0$ and $b \neq 2$ (the reduced row echelon form of the augmented matrix is $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

10. $\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & a^2-4 & a-2 \end{bmatrix}$ ← The augmented matrix for the system.

$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2a^2+a+6 \end{bmatrix}$ ← $-a^2+4$ times the second row was added to the third .

From quadratic formula we have $-2a^2 + a + 6 = -2\left(a + \frac{3}{2}\right)(a-2)$.

The system has no solutions when $a \neq 2$ and $a \neq -\frac{3}{2}$ (since the third row of our last matrix would then correspond to a contradictory equation).

The system has infinitely many solutions when $a = 2$ or $a = -\frac{3}{2}$.

No values of a result in a system with exactly one solution.

11. For the product AKB to be defined, K must be a 2×2 matrix. Letting $K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$ABC = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2a & b & -b \\ 2c & d & -d \end{bmatrix} =$$
$$\begin{bmatrix} 2a + 8c & b + 4d & -b - 4d \\ -4a + 6c & -2b + 3d & 2b - 3d \\ 2a - 4c & b - 2d & -b + 2d \end{bmatrix}.$$

The matrix equation $AKB = C$ can be rewritten as a system of nine linear equations

$$\begin{array}{rcrcrcrcr} 2a & & & + & 8c & & & = & 8 \\ & & b & & & + & 4d & = & 6 \\ & - & b & & & - & 4d & = & -6 \\ -4a & & & + & 6c & & & = & 6 \\ & - & 2b & & & + & 3d & = & -1 \\ & & 2b & & & - & 3d & = & 1 \\ 2a & & & - & 4c & & & = & -4 \\ & & b & & & - & 2d & = & 0 \\ & - & b & & & + & 2d & = & 0 \end{array}$$

which has a unique solution $a = 0, b = 2, c = 1, d = 1$. (An easy way to solve this system is to first split it into two smaller systems. The system $2a + 8c = 8, -4a + 6c = 6, 2a - 4c = -4$ involves a and c only, whereas the remaining six equations involve just b and d.) We conclude that $K = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

12. Substituting the values $x = 1$, $y = -1$, and $z = 2$ into the original system yields a system of three equations in the unknowns a, b, and c:

$$\begin{array}{rcrcrcr} a & - & b & - & (3)(2) & = & -3 \\ (-2)(1) & + & b & + & 2c & = & -1 \\ a & + & (3)(-1) & - & 2c & = & -3 \end{array}$$

that can be rewritten as

$$\begin{array}{rcrcrcr} a & - & b & & & = & 3 \\ & & b & + & 2c & = & 1 \\ a & & & - & 2c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. We conclude that for the original system to have $x = 1, y = -1$, and $z = 2$ as its solution, we must let $a = 2$, $b = -1$, and $c = 1$.

(Note that it can also be shown that the system with $a = 2$, $b = -1$, and $c = 1$ has $x = 1, y = -1$, and $z = 2$ as its **only** solution. One way to do that would be to verify that the reduced row echelon form of the coefficient matrix of the original system with these specific values of a, b and c is the identity matrix.)

13. (a) X must be a 2×3 matrix. Letting $X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ we can write

$$X\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -a+b+3c & b+c & a-c \\ -d+e+3f & e+f & d-f \end{bmatrix}$$

therefore the given matrix equation can be rewritten as a system of linear equations:

$$\begin{array}{rcrcrcrcrcrcr} -a & + & b & + & 3c & & & & & & & = & 1 \\ & & b & + & c & & & & & & & = & 2 \\ a & & & - & c & & & & & & & = & 0 \\ & & & & & - & d & + & e & + & 3f & = & -3 \\ & & & & & & & & e & + & f & = & 1 \\ & & & & & & d & & & - & f & = & 5 \end{array}$$

The augmented matrix of this system has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ so the system has a unique solution

$a = -1, b = 3, c = -1, d = 6, e = 0, f = 1$ and $X = \begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$.

(An alternative to dealing with this large system is to split it into two smaller systems instead: the first three equations involve a, b, and c only, whereas the remaining three equations involve just d, e, and f. Since the coefficient matrix for both systems is the same, we can follow the procedure of Example 2 in Section 1.6; the reduced row echelon form of the matrix

$\left[\begin{array}{rrr|r|r} -1 & 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 & 5 \end{array}\right]$ is $\left[\begin{array}{rrr|r|r} 1 & 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$.)

Yet another way of solving this problem would be to determine the inverse

$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ using the method introduced in Section 1.5, then multiply both sides of the given matrix equation on the right by this inverse to determine X:

$$X = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$$

(b) X must be a 2×2 matrix. Letting $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$X\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3b & -a & 2a+b \\ c+3d & -c & 2c+d \end{bmatrix}$$

therefore the given matrix equation can be rewritten as a system of linear equations:

$$\begin{array}{rcrcrcrcrcr} a & + & 3b & & & & & = & -5 \\ -a & & & & & & & = & -1 \\ 2a & + & b & & & & & = & 0 \\ & & & & c & + & 3d & = & 6 \\ & & & - & c & & & = & -3 \\ & & & & 2c & + & d & = & 7 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so the system has a unique solution $a = 1, b = -2, c = 3, d = 1$. We conclude that $X = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$.

(An alternative to dealing with this large system is to split it into two smaller systems instead: the first three equations involve a and b only, whereas the remaining three equations involve just c and d. Since the coefficient matrix for both systems is the same, we can follow the procedure of Example 2 in Section 1.6; the reduced row echelon form of the matrix $\left[\begin{array}{rr|r|r} 1 & 3 & -5 & 6 \\ -1 & 0 & -1 & -3 \\ 2 & 1 & 0 & 7 \end{array}\right]$ is $\left[\begin{array}{rr|r|r} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.)

(c) X must be a 2×2 matrix. Letting $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we can write

$$\begin{aligned} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} X - X \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3a + c & 3b + d \\ -a + 2c & -b + 2d \end{bmatrix} - \begin{bmatrix} a + 2b & 4a \\ c + 2d & 4c \end{bmatrix} \\ &= \begin{bmatrix} 2a - 2b + c & -4a + 3b + d \\ -a + c - 2d & -b - 4c + 2d \end{bmatrix} \end{aligned}$$

therefore the given matrix equation can be rewritten as a system of linear equations:

$$\begin{array}{rcrcrcrcr} 2a & - & 2b & + & c & & & = & 2 \\ -4a & + & 3b & & & + & d & = & -2 \\ -a & & & + & c & - & 2d & = & 5 \\ & - & b & - & 4c & + & 2d & = & 4 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{113}{37} \\ 0 & 1 & 0 & 0 & -\frac{160}{37} \\ 0 & 0 & 1 & 0 & -\frac{20}{37} \\ 0 & 0 & 0 & 1 & -\frac{46}{37} \end{bmatrix}$

so the system has a unique solution $a = -\frac{113}{37}, b = -\frac{160}{37}, c = -\frac{20}{37}, d = -\frac{46}{37}$.

We conclude that $X = \begin{bmatrix} -\frac{113}{37} & -\frac{160}{37} \\ -\frac{20}{37} & -\frac{46}{37} \end{bmatrix}$.

14. (a) From Theorem 1.4.1, the properties $AI = IA = A$ (page 43) and the assumption $A^4 = 0$, we have

$$\begin{aligned}(I-A)(I+A+A^2+A^3) &= II+IA+IA^2+IA^3-AI-AA-AA^2-AA^3\\ &= I+A+A^2+A^3-A-A^2-A^3-A^4\\ &= I\end{aligned}$$

This shows that $(I-A)^{-1} = I+A+A^2+A^3$.

(b) From Theorem 1.4.1, the properties $AI = IA = A$ (page 43) and the assumption $A^{n+1} = 0$, we have

$$\begin{aligned}&(I-A)(I+A+A^2+\cdots+A^{n-1}+A^n)\\ &= II+IA+IA^2+\cdots+IA^{n-1}+IA^n-AI-AA-AA^2-\cdots-AA^{n-1}-AA^n\\ &= I+A+A^2+\cdots+A^{n-1}+A^n-A-A^2-A^3-\cdots-A^n-A^{n+1}\\ &= I\end{aligned}$$

15. We are looking for a polynomial of the form

$$p(x) = ax^2+bx+c$$

such that $p(1) = 2$, $p(-1) = 6$, and $p(2) = 3$. We obtain a linear system

$$\begin{array}{rcrcrcr} a & + & b & + & c & = & 2\\ a & - & b & + & c & = & 6\\ 4a & + & 2b & + & c & = & 3\end{array}$$

Its augmented matrix has the reduced row echelon form $\begin{bmatrix}1&0&0&1\\0&1&0&-2\\0&0&1&3\end{bmatrix}$.

There is a unique solution $a = 1$, $b = -2$, $c = 3$.

16. Since $p(-1) = 0$ and $p(2) = -9$ we have the equations $a-b+c = 0$ and $4a+2b+c = -9$.

From calculus, the derivative of $p(x) = ax^2+bx+c$ is $p'(x) = 2ax+b$.

For the tangent to be horizontal, the derivative $p'(2) = 4a+b$ must equal zero. This leads to the equation $4a+b = 0$.

We proceed to solve the resulting system of two equations:

$$\begin{array}{rcrcrcr} a & - & b & + & c & = & 0\\ 4a & + & 2b & + & c & = & -9\\ 4a & + & b & & & = & 0\end{array}$$

The reduced row echelon form of the augmented matrix of this system is $\begin{bmatrix}1&0&0&1\\0&1&0&-4\\0&0&1&-5\end{bmatrix}$. Therefore, the values $a = 1$, $b = -4$, and $c = -5$ result in a polynomial that satisfies the conditions specified.

17. When multiplying the matrix J_n by itself, each entry in the product equals n. Therefore, $J_nJ_n = nJ_n$.

$(I-J_n)\left(I-\frac{1}{n-1}J_n\right)$

$= I^2 - I\frac{1}{n-1}J_n - J_nI + J_n\frac{1}{n-1}J_n$ ← Theorem 1.4.1(f) and (g)

$= I - \frac{1}{n-1}J_n - J_n + J_n\frac{1}{n-1}J_n$ ← Property $AI = IA = A$ on p. 43

$$= I - \tfrac{1}{n-1}J_n - J_n + \tfrac{1}{n-1}J_nJ_n \qquad \longleftarrow \text{Theorem 1.4.1(m)}$$

$$= I - \tfrac{1}{n-1}J_n - J_n + \tfrac{n}{n-1}J_n \qquad \longleftarrow J_nJ_n = nJ_n$$

$$= I + \left(\tfrac{-1}{n-1} - 1 + \tfrac{n}{n-1}\right)J_n \qquad \longleftarrow \text{Theorem 1.4.1(j) and (k)}$$

$$= I + \left(\tfrac{-1}{n-1} - \tfrac{n-1}{n-1} + \tfrac{n}{n-1}\right)J_n$$

$$= I$$

CHAPTER 2: DETERMINANTS

2.1 Determinants by Cofactor Expansion

1.

$$M_{11} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 29 \qquad C_{11} = (-1)^{1+1}M_{11} = M_{11} = 29$$

$$M_{12} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} = 21 \qquad C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -21$$

$$M_{13} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} = 27 \qquad C_{13} = (-1)^{1+3}M_{13} = M_{13} = 27$$

$$M_{21} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -11 \qquad C_{21} = (-1)^{2+1}M_{21} = -M_{21} = 11$$

$$M_{22} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 13 \qquad C_{22} = (-1)^{2+2}M_{22} = M_{22} = 13$$

$$M_{23} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} = -5 \qquad C_{23} = (-1)^{2+3}M_{23} = -M_{23} = 5$$

$$M_{31} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix} = -19 \qquad C_{31} = (-1)^{3+1}M_{31} = M_{31} = -19$$

$$M_{32} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix} = -19 \qquad C_{32} = (-1)^{3+2}M_{32} = -M_{32} = 19$$

$$M_{33} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix} = 19 \qquad C_{33} = (-1)^{3+3}M_{33} = M_{33} = 19$$

2.

$$M_{11} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & 4 \end{vmatrix} = 6 \qquad C_{11} = (-1)^{1+1}M_{11} = M_{11} = 6$$

$$M_{12} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 0 & 4 \end{vmatrix} = 12 \qquad C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -12$$

$$M_{13} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 0 & 1 \end{vmatrix} = 3 \qquad C_{13} = (-1)^{1+3}M_{13} = M_{13} = 3$$

$$M_{21} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \qquad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -2$$

$$M_{22} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4 \qquad C_{22} = (-1)^{2+2} M_{22} = M_{22} = 4$$

$$M_{23} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -1$$

$$M_{31} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \qquad C_{31} = (-1)^{3+1} M_{31} = M_{31} = 0$$

$$M_{32} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \qquad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 0$$

$$M_{33} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0 \qquad C_{33} = (-1)^{3+3} M_{33} = M_{33} = 0$$

3. **(a)**

$$M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 0\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

cofactor expansion along the first row

$$= 0 - 0 + 3(0) = 0$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13} = 0$$

(b)

$$M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$$

cofactor expansion along the first row

$$= 4(-12) + 1(-48) + 6(0) = -96$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 96$$

(c)

$$M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix} = -4\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 4 & 2 \end{vmatrix} - 14\begin{vmatrix} 4 & 1 \\ 4 & 3 \end{vmatrix}$$

cofactor expansion along the second row

$$= -4(-16) + 0 - 14(8) = -48$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22} = -48$$

(d)

$$M_{21} = \begin{vmatrix} -1 & 1 & 6 \\ 1 & 0 & 14 \\ 1 & 3 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix} - 14\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}$$

cofactor expansion along the second row

$= -1(-16) + 0 - 14(-4) = 72$

$C_{21} = (-1)^{2+1}M_{21} = -M_{21} = -72$

4. **(a)** $M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(-3) + 1(-21) + 1(-3) = -30$

$C_{32} = (-1)^{3+2}M_{32} = -M_{32} = 30$

(b) $M_{44} = \begin{vmatrix} 2 & 3 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 2 \\ 3 & -2 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(2) - 3(-3) - 1(0) = 13$

$C_{44} = (-1)^{4+4}M_{44} = M_{44} = 13$

(c) $M_{41} = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 3 \\ -2 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 3(-3) + 1(6) + 1(2) = -1$

$C_{41} = (-1)^{4+1}M_{41} = -M_{41} = 1$

(d) $M_{24} = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(0) - 3(0) - 1(0) = 0$

$C_{24} = (-1)^{2+4}M_{24} = M_{24} = 0$

5. $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = (3)(4) - (5)(-2) = 12 + 10 = 22 \neq 0.$ Inverse: $\frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{11} & \frac{-5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$

6. $\begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = (4)(2) - (1)(8) = 0;$ The matrix is not invertible.

7. $\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = (-5)(-2) - (7)(-7) = 10 + 49 = 59 \neq 0.$ Inverse: $\frac{1}{59}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} \frac{-2}{59} & \frac{-7}{59} \\ \frac{7}{59} & \frac{-5}{59} \end{bmatrix}$

8. $\begin{vmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{vmatrix} = (\sqrt{2})(\sqrt{3}) - (\sqrt{6})(4) = \sqrt{6} - 4\sqrt{6} = -3\sqrt{6} \neq 0.$ Inverse: $\frac{1}{-3\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ -4 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} & \frac{1}{3} \\ \frac{4}{3\sqrt{6}} & \frac{-1}{3\sqrt{3}} \end{bmatrix}$

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = \begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2) - 5(-3) = a^2 - 5a + 6 + 15 = a^2 - 5a + 21$

10. $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix} = \left|\begin{array}{ccc|cc} -2 & 7 & 6 & -2 & 7 \\ 5 & 1 & -2 & 5 & 1 \\ 3 & 8 & 4 & 3 & 8 \end{array}\right. = [-8 - 42 + 240] - [18 + 32 + 140] = 0$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \left|\begin{array}{ccc|cc} -2 & 1 & 4 & -2 & 1 \\ 3 & 5 & -7 & 3 & 5 \\ 1 & 6 & 2 & 1 & 6 \end{array}\right. = [-20 - 7 + 72] - [20 + 84 + 6] = -65$

12. $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = \left|\begin{array}{ccc|cc} -1 & 1 & 2 & -1 & 1 \\ 3 & 0 & -5 & 3 & 0 \\ 1 & 7 & 2 & 1 & 7 \end{array}\right. = [0 - 5 + 42] - [0 + 35 + 6] = -4$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = \left|\begin{array}{ccc|cc} 3 & 0 & 0 & 3 & 0 \\ 2 & -1 & 5 & 2 & -1 \\ 1 & 9 & -4 & 1 & 9 \end{array}\right. = [12 + 0 + 0] - [0 + 135 + 0] = -123$

14. $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix} = \left|\begin{array}{ccc|cc} c & -4 & 3 & c & -4 \\ 2 & 1 & c^2 & 2 & 1 \\ 4 & c-1 & 2 & 4 & c-1 \end{array}\right. = [2c - 16c^2 + 6(c-1)] - [12 + (c-1)c^3 - 16]$

$$= 2c - 16c^2 + 6c - 6 - 12 - c^4 + c^3 + 16 = -c^4 + c^3 - 16c^2 + 8c - 2$$

15. $\det(A) = \begin{vmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{vmatrix} = (\lambda-2)(\lambda+4) - (1)(-5) = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1)$

The determinant is zero if $\lambda = -3$ or $\lambda = 1$.

16. Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = \begin{vmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{vmatrix} = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix} - 0 + 0$$

$$= (\lambda-4)[\lambda(\lambda-1) - 6] = (\lambda-4)[\lambda^2 - \lambda - 6] = (\lambda-4)(\lambda-3)(\lambda+2)$$

The determinant is zero if $\lambda = -2$, $\lambda = 3$, or $\lambda = 4$.

17. $\det(A) = \begin{vmatrix} \lambda - 1 & 0 \\ 2 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1)$

The determinant is zero if $\lambda = 1$ or $\lambda = -1$.

18. Calculate the determinant by a cofactor expansion along the third row:

$$\det(A) = \begin{vmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = 0 - 0 + (\lambda - 5)\begin{vmatrix} \lambda - 4 & 4 \\ -1 & \lambda \end{vmatrix}$$

$$= (\lambda - 5)[(\lambda - 4)\lambda + 4] = (\lambda - 5)[\lambda^2 - 4\lambda + 4] = (\lambda - 5)(\lambda - 2)^2$$

The determinant is zero if $\lambda = 2$ or $\lambda = 5$.

19. **(a)** $3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 + 0 = 3(-41) = -123$

(b) $3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} = 3(-41) - 2(0) + 1(0) = -123$

(c) $-2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} = -2(0) - 1(-12) - 5(27) = -123$

(d) $-0 + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} = -1(-12) - 9(15) = -123$

(e) $1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = 1(0) - 9(15) - 4(-3) = -123$

(f) $0 - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -5(27) - 4(-3) = -123$

20. **(a)** $(-1)\begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} = (-1)(35) - 1(11) + 2(21) = -4$

(b) $(-1)\begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 7 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} = (-1)(35) - 3(-12) + 1(-5) = -4$

(c) $-3\begin{vmatrix} 1 & 2 \\ 7 & 2 \end{vmatrix} + 0 - (-5)\begin{vmatrix} -1 & 1 \\ 1 & 7 \end{vmatrix} = -3(-12) + 0 + 5(-8) = -4$

(d) $-1\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} + 0 - 7\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} = -1(11) + 0 - 7(-1) = -4$

(e) $1\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} - 7\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = 1(-5) - 7(-1) + 2(-3) = -4$

(f) $2\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} - (-5)\begin{vmatrix} -1 & 1 \\ 1 & 7 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = 2(21) + 5(-8) + 2(-3) = -4$

21. Calculate the determinant by a cofactor expansion along the second column:

$$-0 + 5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 0 = 5(-8) = -40$$

22. Calculate the determinant by a cofactor expansion along the second row:

$$-1\begin{vmatrix} 3 & 1 \\ -3 & 5 \end{vmatrix} + 0 - (-4)\begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix} = -1(18) + 0 + 4(-12) = -66$$

23. Calculate the determinant by a cofactor expansion along the first column:

$$1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} = 1(0) - 1(0) + 1(0) = 0$$

24. Calculate the determinant by a cofactor expansion along the second column:

$$\begin{aligned} &-(k-1)\begin{vmatrix} 2 & 4 \\ 5 & k \end{vmatrix} + (k-3)\begin{vmatrix} k+1 & 7 \\ 5 & k \end{vmatrix} - (k+1)\begin{vmatrix} k+1 & 7 \\ 2 & 4 \end{vmatrix} \\ &= -(k-1)(2k-20) + (k-3)\big((k+1)k - 35\big) - (k+1)(4(k+1) - 14) \\ &= k^3 - 8k^2 - 10k + 95 \end{aligned}$$

25. Calculate the determinant by a cofactor expansion along the third column:

$$\det(A) = 0 - 0 + (-3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix}$$

Calculate the determinants in the third and fourth terms by a cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix} = 3(24) - 3(8) + 5(16) = 128$$

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 3(2) - 3(8) + 5(-6) = -48$$

Therefore $\det(A) = 0 - 0 - 3(128) - 3(-48) = -240$.

26. Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = 4\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} - 0 + 0 - 1\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} + 0$$

Calculate each of the two determinants by a cofactor expansion along its first row:

$$\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 & 3 \\ 6 & 2 & 3 \\ 4 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 & 3 \\ 4 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 0 = 3(0) - 3(0) - 1(0) - 0 = 0$$

$$\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 & 3 \\ 9 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 9 & 4 & 3 \\ 2 & 2 & 3 \end{vmatrix} - 0 = 3(0) - 3(-6) + 3(-6) - 0 = 0$$

Therefore $\det(A) = 4(0) - 0 + 0 - 1(0) = 0$.

27. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (1)(-1)(1) = -1$.

28. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (2)(2)(2) = 8$.

29. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (0)(2)(3)(8) = 0$.

30. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(2)(3)(4) = 24$.

31. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(1)(2)(3) = 6$.

32. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (-3)(2)(-1)(3) = 18$.

33. **(a)** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = (\sin\theta)(\sin\theta) - (\cos\theta)(-\cos\theta) = \sin^2\theta + \cos^2\theta = 1$

(b) Calculate the determinant by a cofactor expansion along the third column:
$0 - 0 + 1\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 0 - 0 + (1)(1) = 1$ (we used the result of part (a))

35. The minor M_{11} in both determinants is $\begin{vmatrix} 1 & f \\ 0 & 1 \end{vmatrix} = 1$. Expanding both determinants along the first row yields $d_1 + \lambda = d_2$.

37. If $n = 1$ then the determinant is 1.
If $n = 2$ then $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.
If $n = 3$ then a cofactor expansion will involves minors $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$. Therefore the determinant is 0.
By induction, we can show that the determinant will be 0 for all $n > 3$ as well.

True-False Exercises

(a) False. The determinant is $ad - bc$.

(b) False. E.g., $\det(I_2) = \det(I_3) = 1$.

(c) True. If $i + j$ is even then $(-1)^{i+j} = 1$ therefore $C_{ij} = (-1)^{i+j}M_{ij} = M_{ij}$.

(d) True. Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.
Then $C_{12} = (-1)^{1+2}\begin{vmatrix} b & e \\ c & f \end{vmatrix} = -(bf - ec)$ and $C_{21} = (-1)^{2+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix} = -(bf - ce)$ therefore $C_{12} = C_{21}$. In the same way, one can show $C_{13} = C_{31}$ and $C_{23} = C_{32}$.

(e) True. This follows from Theorem 2.1.1.

(f) True. In formulas (7) and (8), each cofactor C_{ij} is zero.

(g) False. The determinant of a lower triangular matrix is the *product* of the entries along the main diagonal.

(h) False. E.g. $\det(2I_2) = 4 \neq 2 = 2\det(I_2)$.

(i) False. E.g., $\det(I_2 + I_2) = 4 \neq 2 = \det(I_2) + \det(I_2)$.

(j) True. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right) = \begin{vmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{vmatrix} = (a^2+bc)(bc+d^2) - (ab+bd)(ac+cd)$
$= a^2bc + a^2d^2 + b^2c^2 + bcd^2 - a^2bc - abcd - abcd - bcd^2 = a^2d^2 + b^2c^2 - 2abcd.$
$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 = (ad-bc)^2 = a^2d^2 - 2adbc + b^2c^2$ therefore $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right) = \left(\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)^2.$

2.2 Evaluating Determinants by Row Reduction

1. $\det(A) = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = (-2)(4) - (3)(1) = -11$; $\det(A^T) = \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = (-2)(4) - (1)(3) = -11$

2. $\det(A) = \begin{vmatrix} -6 & 1 \\ 2 & -2 \end{vmatrix} = (-6)(-2) - (1)(2) = 10$; $\det(A^T) = \begin{vmatrix} -6 & 2 \\ 1 & -2 \end{vmatrix} = (-6)(-2) - (2)(1) = 10$

3. $\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{vmatrix} = [24 - 20 - 9] - [30 - 24 - 6] = -5;$

$\det(A^T) = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 2 & -3 \\ 3 & 4 & 6 \end{vmatrix} = [24 - 9 - 20] - [30 - 24 - 6] = -5$ (we used the arrow technique)

4. $\det(A) = \begin{vmatrix} 4 & 2 & -1 \\ 0 & 2 & -3 \\ -1 & 1 & 5 \end{vmatrix} = [40 + 6 - 0] - [2 - 12 + 0] = 56;$

$\det(A^T) = \begin{vmatrix} 4 & 0 & -1 \\ 2 & 2 & 1 \\ -1 & -3 & 5 \end{vmatrix} = [40 - 0 + 6] - [2 - 12 + 0] = 56$ (we used the arrow technique)

5. The third row of I_4 was multiplied by -5. By Theorem 2.2.4, the determinant equals -5.

6. -5 times the first row of I_3 was added to the third row. By Theorem 2.2.4, the determinant equals 1.

7. The second and the third rows of I_4 were interchanged. By Theorem 2.2.4, the determinant equals -1.

8. The second row of I_4 was multiplied by $-\frac{1}{3}$. By Theorem 2.2.4, the determinant equals $-\frac{1}{3}$.

9.

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$$
← A common factor of 3 from the first row was taken through the determinant sign.

$$= 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix}$$
← 2 times the first row was added to the second row.

$$= 3(-1)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{vmatrix}$$
← The second and third rows were interchanged.

$$= (3)(-1)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix}$$
← -3 times the second row was added to the third row.

$$= (3)(-1)(-11)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -11 from the last row was taken through the determinant sign.

$$= (3)(-1)(-11)(1) = 33$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = 3\left[1\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} - 0 + 0\right] = 3[(1)(11)] = 33.$$

10.

$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ 0 & 5 & -1 \end{vmatrix}$$

2 times the first row was added to the third row.

$$= 3(-1)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 5 & -1 \\ 0 & 0 & -2 \end{vmatrix}$$

The second and third rows were interchanged.

$$= (3)(-1)(5)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & -2 \end{vmatrix}$$

A common factor of 5 from the second row was taken through the determinant sign.

$$= (3)(-1)(5)(-2)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -2 from the last row was taken through the determinant sign.

$$= (3)(-1)(5)(-2)(1) = 30$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ 0 & 5 & -1 \end{vmatrix} = 3\left[1\begin{vmatrix} 0 & -2 \\ 5 & -1 \end{vmatrix} - 0 + 0\right] = 3[(1)(10)] = 30.$$

11.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

The first and second rows were interchanged.

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

-2 times the first row was added to the second row.

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix} \longleftarrow$$

-2 times the second row was added to the third row.

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix} \longleftarrow$$

-1 times the second row was added to the fourth row.

$$= (-1)(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix} \longleftarrow$$

A common factor of -1 from the third row was taken through the determinant sign.

$$= (-1)(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{vmatrix} \longleftarrow$$

-1 times the third row was added to the fourth row.

$$= (-1)(-1)(6)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \longleftarrow$$

A common factor of 6 from the third row was taken through the determinant sign.

$$= (-1)(-1)(6)(1) = 6$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the fourth step above:

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix} = (-1)(1)\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix} = (-1)(1)(1)\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= (-1)(1)(1)(-6) = 6\,.$$

12.

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 5 & -2 & 2 \end{vmatrix} \longleftarrow$$

2 times the first row was added to the second row.

$$= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} \longleftarrow$$

-5 times the first row was added to the third row.

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 13 & 2 \end{vmatrix} \longleftarrow$$

A common factor of -2 from the second row was taken through the determinant sign.

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{17}{2} \end{vmatrix} \longleftarrow$$

-13 times the second row was added to the third row.

$$= (-2)\left(\frac{17}{2}\right)\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of $\frac{17}{2}$ from the last row was taken through the determinant sign.

$$= (-2)\left(\frac{17}{2}\right)(1) = -17$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = (1)\begin{vmatrix} -2 & 1 \\ 13 & 2 \end{vmatrix} = (1)(-17) = -17.$$

13.

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

2 times the first row was added to the second row.

$$= (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

A common factor of -1 from the second row was taken through the determinant sign.

$$= (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

-2 times the third row was added to the fourth row.

$$= (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

-1 times the fourth row was added to the fifth row.

$$= (-1)(2)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

A common factor of 2 from the fifth row was taken through the determinant sign.

$$= (-1)(2)(1) = -2$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the third step above:

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = (-1)(1)\begin{vmatrix} 1 & -2 & -6 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= (-1)(1)(1)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (-1)(1)(1)(1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)(1)(1)(1)(2) = -2\,.$$

14. $$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix}$$ ← -5 times the first row was added to the second row.

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix}$$ ← The first row was added to the third row.

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$$ ← -2 times the first row was added to the fourth row.

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix}$$ ← -12 times the second row was added to the fourth row.

$$= -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 108 & 23 \end{vmatrix}$$ ← A common factor of -3 from the third row was taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & -13 \end{vmatrix}$$ ← -108 times the third row was added to the fourth row.

$$= (-3)(-13)\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ ← A common factor of -13 from the third row was taken through the determinant sign.

$$= (-3)(-13)(1) = 39$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the fourth step above:

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix} = (1)\begin{vmatrix} 1 & -9 & -2 \\ 0 & -3 & -1 \\ 0 & 108 & 23 \end{vmatrix} = (1)(1)\begin{vmatrix} -3 & -1 \\ 108 & 23 \end{vmatrix}$$

$$= (1)(1)(1)(39) = 39\,.$$

15.
$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$
The first and third rows were interchanged.

$$= (-1)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
The second and third rows were interchanged.

$$= (-1)(-1)(-6) = -6$$

16. The first and the third rows were interchanged, therefore $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-6) = 6.$

17.
$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$
A common factor of 3 from the first row was taken through the determinant sign.

$$= 3(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix}$$
A common factor of -1 from the second row was taken through the determinant sign.

$$= 3(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
A common factor of 4 from the third row was taken through the determinant sign.

$$= 3(-1)(4)(-6) = 72$$

18.
$$\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{vmatrix}$$
The second row was added to the first row.

$$= -1\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
A common factor of -1 from the second row was taken through the determinant sign.

$$= (-1)(-6) = 6$$

19.
$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
-1 times the third row was added to the first row.

$$= -6$$

20. $$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix}$$ ← -3 times the first row was added to the last row.

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← A common factor of 2 from the second row was taken through the determinant sign.

$$= (2)(-6) = -12$$

21. $$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$ ← A common factor of -3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← 4 times the second row was added to the last row.

$$= (-3)(-6) = 18$$

22. The third row is proportional to the first row, therefore by Theorem 2.2.5 $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix} = 0$.

(This can also be shown by adding -2 times the first row to the third, then performing a cofactor expansion of the resulting determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix}$ along the third row.)

23. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ a^2 & b^2 & c^2 \end{vmatrix}$$ ← $-a$ times the first row was added to the second row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$ ← $-a^2$ times the first row was added to the third row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{vmatrix}$$ ← $-(b+a)$ times the second row was added to the third row.

$$= (1)(b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

24. **(a)** Interchanging the first row and the third row and applying Theorem 2.1.2 yields

$$\det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)\det\begin{bmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{13} \end{bmatrix} = -a_{13}a_{22}a_{31}$$

(b) We interchange the first and the fourth row, as well as the second and the third row. Then we use Theorem 2.1.2 to obtain

$$\det\begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = (-1)(-1)\det\begin{bmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{14} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$$

Generally for any $n \times n$ matrix A such that $a_{ij} = 0$ if $i + j \leq n$ we have

$\det(A) = (-1)^n a_{1n}a_{2,n-1} \cdots a_{n1}$.

25.

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & b_1+c_1 \\ a_2 & b_2 & b_2+c_2 \\ a_3 & b_3 & b_3+c_3 \end{vmatrix}$$

-1 times the first column was added to the third column.

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

-1 times the second column was added to the third column.

26.

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ (1-t^2)b_1 & (1-t^2)b_2 & (1-t^2)b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$-t$ times the first row was added to the second row.

$$= (1-t^2)\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

A common factor of $1-t^2$ from the second row was taken through the determinant sign.

$$= (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$-t$ times the second row was added to the first row.

27.

$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1 & -2b_1 & c_1 \\ a_2+b_2 & -2b_2 & c_2 \\ a_3+b_3 & -2b_3 & c_3 \end{vmatrix}$$ ← -1 times the first column was added to the second column.

$$= -2\begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}$$ ← A common factor of -2 from the second column was taken through the determinant sign.

$$= -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ ← -1 times the second column was added to the first column.

28. $$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1+rb_1+sa_1 \\ a_2 & b_2 & c_2+rb_2+sa_2 \\ a_3 & b_3 & c_3+rb_3+sa_3 \end{vmatrix}$$ ← $-t$ times the first column was added to the second column.

$$= \begin{vmatrix} a_1 & b_1 & c_1+rb_1 \\ a_2 & b_2 & c_2+rb_2 \\ a_3 & b_3 & c_3+rb_3 \end{vmatrix}$$ ← $-s$ times the first column was added to the third column.

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ ← $-r$ times the second column was added to the third column.

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ ← The matrix was transposed. (Theorem 2.2.2)

29. The second column vector is a scalar multiple of the fourth. By Theorem 2.2.5, the determinant is 0.

30. Adding the second, third, fourth, and fifth rows to the first results in the first row made up of zeros.

31. $\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ -1 & 3 & 2 \end{vmatrix}\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 8 & -4 \end{vmatrix} = \left(0-0+2\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right)\left(0-0+(-4)\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}\right) = (2)(-12) = -24$

32. $\det(M) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = \big((1)(1)(1)\big)\big((1)(1)\big) = 1$

33. In order to reverse the order of rows in 2×2 and 3×3 matrix, the first and the last rows can be interchanged, so $\det(B) = -\det(A)$.
For 4×4 and 5×5 matrices, two such interchanges are needed: the first and last rows can be swapped, then the second and the penultimate one can follow.
Thus, $\det(B) = (-1)(-1)\det(A) = \det(A)$ in this case.
Generally, to rows in an $n \times n$ matrix can be reversed by

- interchanging row 1 with row n,

- interchanging row 2 with row $n-1$,
- $\vdots$
- interchanging row $\lfloor n/2 \rfloor$ with row $n - \lfloor n/2 \rfloor$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x (also known as the "floor" of x).

We conclude that $\det(B) = (-1)^{\lfloor n/2 \rfloor}\det(A)$.

34.
$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & b \\ b-a & a-b & 0 & 0 \\ b-a & 0 & a-b & 0 \\ b-a & 0 & 0 & a-b \end{vmatrix}$$
-1 times the first row was added to each of the remaining rows.

$$= \begin{vmatrix} a+b & b & b & b \\ b-a & a-b & 0 & 0 \\ b-a & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}$$
The last column was added to the first column.

$$= \begin{vmatrix} a+2b & b & b & b \\ b-a & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}$$
The third column was added to the first column.

$$= \begin{vmatrix} a+3b & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}$$
The second column was added to the first column.

$$= (a+3b)(a-b)^3$$

True-False Exercises

(a) True. $\det(B) = (-1)(-1)\det(A) = \det(A)$.

(b) True. $\det(B) = (4)\left(\frac{3}{4}\right)\det(A) = 3\det(A)$.

(c) False. $\det(B) = \det(A)$.

(d) False. $\det(B) = n(n-1)\cdots 3 \cdot 2 \cdot 1 \cdot \det(A) = (n!)\det(A)$.

(e) True. This follows from Theorem 2.2.5.

(f) True. Let B be obtained from A by adding the second row to the fourth row, so $\det(A) = \det(B)$. Since the fourth row and the sixth row of B are identical, by Theorem 2.2.5 $\det(B) = 0$.

2.3 Properties of Determinants; Cramer's Rule

1. $\det(2A) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} = (-2)(8) - (4)(6) = -40$

$(2)^2 \det(A) = 4\begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = 4\big((-1)(4) - (2)(3)\big) = (4)(-10) = -40$

2. $\det(-4A) = \begin{vmatrix} -8 & -8 \\ -20 & 8 \end{vmatrix} = (-8)(8) - (-8)(-20) = -224$

$(-4)^2 \det(A) = 16 \begin{vmatrix} 2 & 2 \\ 5 & -2 \end{vmatrix} = 16\big((2)(-2) - (2)(5)\big) = (16)(-14) = -224$

3. We are using the arrow technique to evaluate both determinants.

$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix} = (-160 + 8 - 288) - (-48 - 64 + 120) = -448$

$(-2)^3 \det(A) = -8 \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} = (-8)\big((20 - 1 + 36) - (6 + 8 - 15)\big) = (-8)(56) = -448$

4. We are using the cofactor expansion along the first column to evaluate both determinants.

$\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} = 3 \begin{vmatrix} 6 & 9 \\ 3 & -6 \end{vmatrix} = 3\big((6)(-6) - (9)(3)\big) = (3)(-63) = -189$

$3^3 \det(A) = 27 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} = (27)(1) \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 27\big((2)(-2) - (3)(1)\big) = (27)(-7) = -189$

5. We are using the arrow technique to evaluate the determinants in this problem.

$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} = (18 - 170 + 0) - (80 + 0 - 62) = -170;$

$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix} = (-22 - 120 + 510) - (660 - 20 - 102) = -170;$

$\det(A + B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix} = (45 + 0 + 0) - (75 + 0 + 0) = -30;$

$\det(A) = (16 + 0 + 0) - (0 + 0 + 6) = 10;$
$\det(B) = (1 - 10 + 0) - (15 + 0 - 7) = -17;$
$\det(A + B) \neq \det(A) + \det(B)$

6. We are using the arrow technique to evaluate the determinants in this problem.

$\det(AB) = \begin{vmatrix} 6 & 15 & 26 \\ 2 & -4 & -3 \\ -2 & 10 & 12 \end{vmatrix} = (-288 + 90 + 520) - (208 - 180 + 360) = -66;$

$\det(BA) = \begin{vmatrix} 5 & 8 & -3 \\ -6 & 14 & 7 \\ 5 & -2 & -5 \end{vmatrix} = (-350 + 280 - 36) - (-210 - 70 + 240) = -66;$

$\det(A + B) = \begin{vmatrix} 1 & 7 & -2 \\ 2 & 1 & 2 \\ -2 & 5 & 1 \end{vmatrix} = (1 - 28 - 20) - (4 + 10 + 14) = -75;$

$\det(A) = (0 + 16 + 4) - (0 + 2 + 16) = 2$;
$\det(B) = (-2 + 0 - 12) - (0 + 18 + 1) = -33$;
$\det(A + B) \neq \det(A) + \det(B)$;

7. $\det(A) = (-6 + 0 - 20) - (-10 + 0 - 15) = -1 \neq 0$ therefore A is invertible by Theorem 2.3.3

8. $\det(A) = (-24 + 0 + 0) - (-18 + 0 + 0) = -6 \neq 0$ therefore A is invertible by Theorem 2.3.3

9. $\det(A) = (2)(1)(2) = 4 \neq 0$ therefore A is invertible by Theorem 2.3.3

10. $\det(A) = 0$ (second column contains only zeros) therefore A is not invertible by Theorem 2.3.3

11. $\det(A) = (24 - 24 - 16) - (24 - 16 - 24) = 0$ therefore A is not invertible by Theorem 2.3.3

12. $\det(A) = (1 + 0 - 81) - (8 + 36 + 0) = -124 \neq 0$ therefore A is invertible by Theorem 2.3.3

13. $\det(A) = (2)(1)(6) = 12 \neq 0$ therefore A is invertible by Theorem 2.3.3

14. $\det(A) = 0$ (third column contains only zeros) therefore A is not invertible by Theorem 2.3.3

15. $\det(A) = (k-3)(k-2) - (-2)(-2) = k^2 - 5k + 2 = \left(k - \frac{5-\sqrt{17}}{2}\right)\left(k - \frac{5+\sqrt{17}}{2}\right)$. By Theorem 2.3.3, A is invertible if $k \neq \frac{5-\sqrt{17}}{2}$ and $k \neq \frac{5+\sqrt{17}}{2}$.

16. $\det(A) = k^2 - 4 = (k-2)(k+2)$. By Theorem 2.3.3, A is invertible if $k \neq 2$ and $k \neq -2$.

17. $\det(A) = (2 + 12k + 36) - (4k + 18 + 12) = 8 + 8k = 8(1 + k)$.
By Theorem 2.3.3, A is invertible if $k \neq -1$.

18. $\det(A) = (1 + 0 + 0) - (0 + 2k + 2k) = 1 - 4k$. By Theorem 2.3.3, A is invertible if $k \neq \frac{1}{4}$.

19. $\det(A) = (-6 + 0 - 20) - (-10 + 0 - 15) = -1 \neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} -1 & 0 \\ 4 & 3 \end{vmatrix} = -3 \qquad C_{12} = -\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} = 3 \qquad C_{13} = \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} = -2$$
$$C_{21} = -\begin{vmatrix} 5 & 5 \\ 4 & 3 \end{vmatrix} = 5 \qquad C_{22} = \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} = -4 \qquad C_{23} = -\begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} = 2$$
$$C_{31} = \begin{vmatrix} 5 & 5 \\ -1 & 0 \end{vmatrix} = 5 \qquad C_{32} = -\begin{vmatrix} 2 & 5 \\ -1 & 0 \end{vmatrix} = -5 \qquad C_{33} = \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} = 3$$

The matrix of cofactors is $\begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-1}\begin{bmatrix} -3 & 5 & 5 \\ 3 & -4 & -5 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$.

20. $\det(A) = (-24 + 0 + 0) - (-18 + 0 + 0) = -6 \neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12 \qquad C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4 \qquad C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$
$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & -4 \end{vmatrix} = 0 \qquad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2 \qquad C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0$$
$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9 \qquad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4 \qquad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

The matrix of cofactors is $\begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}$.

21. $\det(A) = (2)(1)(2) = 4 \neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2 \qquad C_{12} = -\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix} = 0 \qquad C_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$
$$C_{21} = -\begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix} = 6 \qquad C_{22} = \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} = 4 \qquad C_{23} = -\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} = 0$$
$$C_{31} = \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} = 4 \qquad C_{32} = -\begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} = 6 \qquad C_{33} = \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = 2$$

The matrix of cofactors is $\begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 4 & 6 & 2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{4}\begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

22. $\det(A) = (2)(1)(6) = 12$ is nonzero, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix} = 6 \qquad C_{12} = -\begin{vmatrix} 8 & 0 \\ -5 & 6 \end{vmatrix} = -48 \qquad C_{13} = \begin{vmatrix} 8 & 1 \\ -5 & 3 \end{vmatrix} = 29$$
$$C_{21} = -\begin{vmatrix} 0 & 0 \\ 3 & 6 \end{vmatrix} = 0 \qquad C_{22} = \begin{vmatrix} 2 & 0 \\ -5 & 6 \end{vmatrix} = 12 \qquad C_{23} = -\begin{vmatrix} 2 & 0 \\ -5 & 3 \end{vmatrix} = -6$$
$$C_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 \qquad C_{32} = -\begin{vmatrix} 2 & 0 \\ 8 & 0 \end{vmatrix} = 0 \qquad C_{33} = \begin{vmatrix} 2 & 0 \\ 8 & 1 \end{vmatrix} = 2$$

The matrix of cofactors is $\begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{12}\begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$.

23. $\begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ ← -2 times the first row was added to the second row; -1 times the first row was added to the third and fourth rows.

$= -\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 8 \end{vmatrix}$ ← The third row and the fourth row were interchanged.

$= -\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ ← -7 times the third row was added to the fourth row

$= -(1)(-1)(1)(1) = 1$

The determinant of A is nonzero therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$C_{11} = \begin{vmatrix} 5 & 2 & 2 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = (80 + 54 + 12) - (48 + 90 + 12) = -4$

$C_{12} = -\begin{vmatrix} 2 & 2 & 2 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -[(32 + 18 + 4) - (16 + 36 + 4)] = 2$

$C_{13} = \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = (12 + 45 + 6) - (6 + 54 + 10) = -7$

$C_{14} = -\begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = -[(12 + 40 + 6) - (6 + 48 + 10)] = 6$

$C_{21} = -\begin{vmatrix} 3 & 1 & 1 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -[(48 + 27 + 6) - (24 + 54 + 6)] = 3$

$C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = (16 + 9 + 2) - (8 + 18 + 2) = -1$

$C_{23} = -\begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = -[(6 + 27 + 3) - (3 + 27 + 6)] = 0$

$C_{24} = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = (6 + 24 + 3) - (3 + 24 + 6) = 0$

$C_{31} = \begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = (12 + 6 + 10) - (6 + 12 + 10) = 0$

$$C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = -[(4+2+4)-(2+4+4)] = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = (10+6+6)-(5+6+12) = -1$$

$$C_{34} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -[(10+6+6)-(5+6+12)] = 1$$

$$C_{41} = -\begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = -[(54+6+40)-(6+48+45)] = -1$$

$$C_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 8 & 9 \end{vmatrix} = (18+2+16)-(2+16+18) = 0$$

$$C_{43} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 9 \end{vmatrix} = -[(45+6+6)-(5+6+54)] = 8$$

$$C_{44} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{vmatrix} = (40+6+6)-(5+6+48) = -7$$

The matrix of cofactors is $\begin{bmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{bmatrix}$ and the adjoint matrix is $\text{adj}(A) =$
$\begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

24. $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$; $x_1 = \frac{\det(A_1)}{\det(A)} = \frac{13}{13} = 1$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{26}{13} = 2$

25. $\det(A) = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = (8+10+0)-(0+40+110) = -132,$

$\det(A_1) = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = (4+10+0)-(0+20+30) = -36,$

$\det(A_2) = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = (24+4+0)-(0+8+44) = -24,$

$\det(A_3) = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = (4+15+110)-(2+60+55) = 12;$

$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11}$, $y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11}$, $z = \frac{\det(A_3)}{\det(A)} = \frac{12}{-132} = -\frac{1}{11}$.

26. $\det(A) = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = (3 - 16 + 8) - (-2 + 4 + 48) = -55,$

$\det(A_1) = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = (18 + 160 - 2) - (20 + 24 - 12) = 144,$

$\det(A_2) = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = (3 + 24 - 80) - (-2 - 40 - 72) = 61,$

$\det(A_3) = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = (20 + 8 + 48) - (-12 - 2 + 320) = -230;$

$x = \frac{\det(A_1)}{\det(A)} = \frac{144}{-55} = -\frac{144}{55},\ \ y = \frac{\det(A_2)}{\det(A)} = \frac{61}{-55} = -\frac{61}{55},\ \ z = \frac{\det(A_3)}{\det(A)} = \frac{-230}{-55} = \frac{46}{11}.$

27. $\det(A) = \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{vmatrix} = (3 + 0 + 0) - (-4 + 0 + 18) = -11,$

$\det(A_1) = \begin{vmatrix} 4 & -3 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -3\begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix} = (-3)(-4 - 6) = 30,$

$\det(A_2) = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{vmatrix} = (6 + 0 + 0) - (-8 + 0 - 24) = 38,$

$\det(A_3) = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{vmatrix} = 4\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = (4)(6 + 4) = 40;$

$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{30}{-11} = -\frac{30}{11},\ \ x_2 = \frac{\det(A_2)}{\det(A)} = \frac{38}{-11} = -\frac{38}{11},\ \ x_3 = \frac{\det(A_3)}{\det(A)} = \frac{40}{-11} = -\frac{40}{11}.$

28. $\det(A) = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{vmatrix}$

$$= -1\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 7 & 9 \\ -1 & 3 & 1 \\ 1 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 & 9 \\ -1 & 1 & 1 \\ 1 & -2 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 & 7 \\ -1 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= -[(12 - 14 + 9) - (-54 - 1 - 28)] + 4[(-24 + 7 - 9) - (27 + 2 + 28)]$$
$$+2[(-8 - 1 + 18) - (9 - 4 - 4)] - [(2 - 3 + 14) - (7 - 12 + 1)]$$
$$= -90 - 332 + 16 - 17 = -423$$

$\det(A_1) = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$

$$= -32\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 14 & 7 & 9 \\ 11 & 3 & 1 \\ -4 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 14 & -1 & 9 \\ 11 & 1 & 1 \\ -4 & -2 & -4 \end{vmatrix} - 1\begin{vmatrix} 14 & -1 & 7 \\ 11 & 1 & 3 \\ -4 & -2 & 1 \end{vmatrix}$$

$$= -32[(12 - 14 + 9) - (-54 - 1 - 28)] + 4[(-168 - 28 + 99) - (-108 + 14 - 308)]$$
$$+2[(-56 + 4 - 198) - (-36 - 28 + 44)] - [(14 + 12 - 154) - (-28 - 84 - 11)]$$
$$= -2880 + 1220 - 460 + 5 = -2115$$

$$\det(A_2)=\begin{vmatrix}-1&-32&2&1\\2&14&7&9\\-1&11&3&1\\1&-4&1&-4\end{vmatrix}$$
$$=-1\begin{vmatrix}14&7&9\\11&3&1\\-4&1&-4\end{vmatrix}+32\begin{vmatrix}2&7&9\\-1&3&1\\1&1&-4\end{vmatrix}+2\begin{vmatrix}2&14&9\\-1&11&1\\1&-4&-4\end{vmatrix}-1\begin{vmatrix}2&14&7\\-1&11&3\\1&-4&1\end{vmatrix}$$
$$=-[(-168-28+99)-(-108+14-308)]+32[(-24+7-9)-(27+2+28)]$$
$$+2[(-88+14+36)-(99-8+56)]-[(22+42+28)-(77-24-14)]$$
$$=-305-2656-370-53=-3384$$
$$\det(A_3)=\begin{vmatrix}-1&-4&-32&1\\2&-1&14&9\\-1&1&11&1\\1&-2&-4&-4\end{vmatrix}$$
$$=-1\begin{vmatrix}-1&14&9\\1&11&1\\-2&-4&-4\end{vmatrix}+4\begin{vmatrix}2&14&9\\-1&11&1\\1&-4&-4\end{vmatrix}-32\begin{vmatrix}2&-1&9\\-1&1&1\\1&-2&-4\end{vmatrix}-1\begin{vmatrix}2&-1&14\\-1&1&11\\1&-2&-4\end{vmatrix}$$
$$=-[(44-28-36)-(-198+4-56)]+4[(-88+14+36)-(99-8+56)]$$
$$-32[(-8-1+18)-(9-4-4)]-[(-8-11+28)-(14-44-4)]$$
$$=-230-740-256-43=-1269$$
$$\det(A_4)=\begin{vmatrix}-1&-4&2&-32\\2&-1&7&14\\-1&1&3&11\\1&-2&1&-4\end{vmatrix}$$
$$=-1\begin{vmatrix}-1&7&14\\1&3&11\\-2&1&-4\end{vmatrix}+4\begin{vmatrix}2&7&14\\-1&3&11\\1&1&-4\end{vmatrix}+2\begin{vmatrix}2&-1&14\\-1&1&11\\1&-2&-4\end{vmatrix}+32\begin{vmatrix}2&-1&7\\-1&1&3\\1&-2&1\end{vmatrix}$$
$$=-[(12-154+14)-(-84-11-28)]+4[(-24+77-14)-(42+22+28)]$$
$$+2[(-8-11+28)-(14-44-4)]+32[(2-3+14)-(7-12+1)]$$
$$=5-212+86+544=423$$

$$x_1=\frac{\det(A_1)}{\det(A)}=\frac{-2115}{-423}=5,\ \ x_2=\frac{\det(A_2)}{\det(A)}=\frac{-3384}{-423}=8,$$
$$x_3=\frac{\det(A_3)}{\det(A)}=\frac{-1269}{-423}=3,\ \ x_4=\frac{\det(A_4)}{\det(A)}=\frac{423}{-423}=-1$$

29. $\det(A)=0$ therefore Cramer's rule does not apply.

30. $\det(A)=\cos^2\theta+\sin^2\theta=1$ is nonzero for all values of θ, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll}C_{11}=\cos\theta & C_{12}=\sin\theta & C_{13}=0\\ C_{21}=-\sin\theta & C_{22}=\cos\theta & C_{23}=0\\ C_{31}=0 & C_{32}=0 & C_{33}=\cos^2\theta+\sin^2\theta=1\end{array}$$

The matrix of cofactors is

$$\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$

and the adjoint matrix is

$$\operatorname{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

31. $\det(A) = \begin{vmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{vmatrix} = -424$; $\det(A_2) = \begin{vmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{vmatrix} = 0$; $y = \frac{\det(A_2)}{\det(A)} = \frac{0}{-424} = 0$

32. **(a)** $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix}$,

$A_1 = \begin{bmatrix} 6 & 1 & 1 & 1 \\ 1 & 7 & -1 & 1 \\ -3 & 3 & -5 & 8 \\ 3 & 1 & 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{bmatrix}$, $A_4 =$
$\begin{bmatrix} 4 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & -3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$;

$x = \frac{\det(A_1)}{\det(A)} = \frac{-424}{-424} = 1,\ \ y = \frac{\det(A_2)}{\det(A)} = \frac{0}{-424} = 0,\ \ z = \frac{\det(A_3)}{\det(A)} = \frac{-848}{-424} = 2,\ \ w = \frac{\det(A_4)}{\det(A)} = \frac{0}{-424} = 0$

(b) The augmented matrix of the system $\begin{bmatrix} 4 & 1 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 & 1 \\ 7 & 3 & -5 & 8 & -3 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only one solution: $x = 1,\ y = 0,\ z = 2,$ and $w = 0.$

(c) The method in part (b) requires fewer computations.

33. **(a)** $\det(3A) = 3^3\det(A) = (27)(-7) = -189$ (using Formula (1))

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-7} = -\frac{1}{7}$ (using Theorem 2.3.5)

(c) $\det(2A^{-1}) = 2^3\det(A^{-1}) = \frac{8}{\det(A)} = \frac{8}{-7} = -\frac{8}{7}$ (using Formula (1) and Theorem 2.3.5)

(d) $\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = \frac{1}{(8)(-7)} = -\frac{1}{56}$ (using Theorem 2.3.5 and Formula (1))

(e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = -\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-7) = 7$ (in the first step we interchanged the last two columns applying Theorem 2.2.3(b); in the second step we transposed the matrix applying Theorem 2.2.2)

34. **(a)** $\det(-A) = \det\big((-1)A\big) = (-1)^4 \det(A) = \det(A) = -2$ (using Formula (1))

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-2} = -\frac{1}{2}$ (using Theorem 2.3.5)

(c) $\det(2A^T) = 2^4 \det(A^T) = 16\det(A) = -32$ (using Formula (1) and Theorem 2.2.2)

(d) $\det(A^3) = \det(AAA) = \det(A)\det(A)\det(A) = (-2)^3 = -8$ (using Theorem 2.3.4)

35. **(a)** $\det(3A) = 3^3\det(A) = (27)(7) = 189$ (using Formula (1))

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{7}$ (using Theorem 2.3.5)

(c) $\det(2A^{-1}) = 2^3\det(A^{-1}) = \frac{8}{\det(A)} = \frac{8}{7}$ (using Formula (1) and Theorem 2.3.5)

(d) $\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = \frac{1}{(8)(7)} = \frac{1}{56}$ (using Theorem 2.3.5 and Formula (1))

True-False Exercises

(a) False. By Formula (1), $\det(2A) = 2^3\det(A) = 8\det(A)$.

(b) False. E.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have $\det(A) = \det(B) = 0$ but $\det(A+B) = 1 \neq 2\det(A)$.

(c) True. By Theorems 2.3.4 and 2.3.5,
$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A) = \frac{1}{\det(A)}\det(B)\det(A) = \det(B)$.

(d) False. A square matrix A is invertible if and only if $\det(A) \neq 0$.

(e) True. This follows from Definition 1.

(f) True. This is Formula (8).

(g) True. If $\det(A) \neq 0$ then by Theorem 2.3.8 $A\mathbf{x} = \mathbf{0}$ must have only the trivial solution, which contradicts our assumption. Consequently, $\det(A) = 0$.

(h) True. If the reduced row echelon form of A is I_n then by Theorem 2.3.8 $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$, which contradicts our assumption. Consequently, the reduced row echelon form of A cannot be I_n.

(i) True. Since the reduced row echelon form of E is I then by Theorem 2.3.8 $E\mathbf{x} = \mathbf{0}$ must have only the trivial solution.

(j) True. If A is invertible, so is A^{-1}. By Theorem 2.3.8, each system has only the trivial solution.

(k) True. From Theorem 2.3.6, $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A) = \det(A)\,A^{-1}$. Consequently,
$\left(\frac{1}{\det(A)}A\right)\operatorname{adj}(A) = \left(\frac{1}{\det(A)}A\right)\left(\det(A)\;A^{-1}\right) = \frac{\det(A)}{\det(A)}(AA^{-1}) = I_n$ so $\left(\operatorname{adj}(A)\right)^{-1} = \frac{1}{\det(A)}A$.

(l) False. If the kth row of A contains only zeros then all cofactors C_{jk} where $j \neq i$ are zero (since each of them involves a determinant of a matrix with a zero row). This means the matrix of cofactors contains at least one zero row, therefore $\operatorname{adj}(A)$ has a *column* of zeros.

Chapter 2 Supplementary Exercises

1. **(a)** Cofactor expansion along the first row: $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = (-4)(3) - (2)(3) = -12 - 6 = -18$

(b) $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -\begin{vmatrix} 3 & 3 \\ -4 & 2 \end{vmatrix}$ ← The first and second rows were interchanged.

$= -(3)\begin{vmatrix} 1 & 1 \\ -4 & 2 \end{vmatrix}$ ← A common factor of 3 from the first row was taken through the determinant sign.

$= -(3)\begin{vmatrix} 1 & 1 \\ 0 & 6 \end{vmatrix}$ ← 4 times the first row was added to the second row

$= -(3)(1)(6) = -18$ ← Use Theorem 2.1.2.

2. **(a)** Cofactor expansion along the first row: $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = (7)(-6) - (-1)(-2) = -42 - 2 = -44$

(b) $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -\begin{vmatrix} -2 & -6 \\ 7 & -1 \end{vmatrix}$ ← The first and second rows were interchanged.

$= -(-2)\begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix}$ ← A common factor of -2 from the first row was taken through the determinant sign.

$= -(-2)\begin{vmatrix} 1 & 3 \\ 0 & -22 \end{vmatrix}$ ← -7 times the first row was added to the second row

$= -(-2)(1)(-22)$ ← Use Theorem 2.1.2.

$= -44$

3. **(a)** Cofactor expansion along the second row:

$$\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = -0 + 2\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 5 \\ -3 & 1 \end{vmatrix}$$

$$= 0 + 2[(-1)(1) - (2)(-3)] - (-1)[(-1)(1) - (5)(-3)]$$

$$= 0 + (2)(5) - (-1)(14) = 0 + 10 + 14 = 24$$

(b) $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix}$ ← A common factor of -1 from the first row was taken through the determinant sign.

$= (-1)\begin{vmatrix} 1 & -5 & -2 \\ 0 & 2 & -1 \\ 0 & -14 & -5 \end{vmatrix}$ ← 3 times the first row was added to the third row.

$= (-1)\begin{vmatrix} 1 & -5 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & -12 \end{vmatrix}$ ← 7 times the second row was added to the third.

$= (-1)(1)(2)(-12) = 24$ ← Use Theorem 2.1.2.

4. **(a)** Cofactor expansion along the first row:

$$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = (-1)\begin{vmatrix} -5 & -6 \\ -8 & -9 \end{vmatrix} - (-2)\begin{vmatrix} -4 & -6 \\ -7 & -9 \end{vmatrix} + (-3)\begin{vmatrix} -4 & -5 \\ -7 & -8 \end{vmatrix}$$
$$= (-1)[(-5)(-9) - (-6)(-8)] - (-2)[(-4)(-9) - (-6)(-7)]$$
$$+(-3)[(-4)(-8) - (-5)(-7)]$$
$$= (-1)(-3) - (-2)(-6) + (-3)(-3) = 3 - 12 + 9 = 0$$

(b) $\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix}$ ← A common factor of -1 from the first row was taken through the determinant sign.

$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{vmatrix}$ ← 4 times the first row was added to the second row and 7 times the first row was added to the third row

$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{vmatrix}$ ← -2 times the second row was added to the third row

$= (-1)(0) = 0$ ← Use Theorem 2.2.1.

5. **(a)** Cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = (3)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}$$
$$= (3)[(1)(2) - (1)(4)] - 0 + (-1)[(1)(4) - (1)(0)]$$
$$= (3)(-2) - 0 + (-1)(4) = -6 + 0 - 4 = -10$$

(b) $\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 2 \end{vmatrix}$ ← The first and second rows were interchanged.

$$= (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 4 & 2 \end{vmatrix}$$ ← -3 times the first row was added to the second.

$$= (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 1 & -2 \end{vmatrix}$$ ← The second row was added to the third row

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & -4 \end{vmatrix}$$ ← The second and third rows were interchanged.

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{vmatrix}$$ ← 3 times the second row was added to the third.

$$= (-1)(-1)(1)(1)(-10) = -10$$ ← Use Theorem 2.1.2.

6. **(a)** Cofactor expansion along the second row:

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -3\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} + 0 - 2\begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= -3[(1)(2) - (4)(-2)] - 2[(-5)(-2) - (1)(1)] = (-3)(10) - 2(9) = -30 - 18 = -48$$

(b) $$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 2 \\ -5 & 1 & 4 \end{vmatrix}$$ ← The first and third rows were interchanged.

$$= -\begin{vmatrix} 1 & -2 & 2 \\ 0 & 6 & -4 \\ 0 & -9 & 14 \end{vmatrix}$$ ← -3 times the first row was added to the second row and 5 times the first row was added to the third row

$$= -6\begin{vmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & -9 & 14 \end{vmatrix}$$ ← A common factor of 6 from the second row was taken through the determinant sign.

$$= -6\begin{vmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 8 \end{vmatrix}$$ ← 9 times the second row was added to the third row.

$$= -6(1)(1)(8) = -48$$ ← Use Theorem 2.1.2.

7. **(a)** We perform cofactor expansions along the first row in the 4x4 determinant. In each of the 3x3 determinants, we expand along the second row:

$$\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} - 6\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} + 0 - 1\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix}$$

$$=3\left(-0+(-1)\begin{vmatrix}3 & 4\\ 2 & 2\end{vmatrix}-1\begin{vmatrix}3 & 1\\ 2 & -2\end{vmatrix}\right)-6\left(-1\begin{vmatrix}1 & 4\\ -2 & 2\end{vmatrix}+(-1)\begin{vmatrix}-2 & 4\\ -9 & 2\end{vmatrix}-1\begin{vmatrix}-2 & 1\\ -9 & -2\end{vmatrix}\right)+0$$
$$-1\left(-1\begin{vmatrix}3 & 1\\ 2 & -2\end{vmatrix}+0-(-1)\begin{vmatrix}-2 & 3\\ -9 & 2\end{vmatrix}\right)$$
$$=3\big(0-1(-2)-1(-8)\big)-6\big(-1(10)-1(32)-1(13)\big)+0-1\big(-1(-8)+0+1(23)\big)$$
$$=3(10)-6(-55)+0-1(31)$$
$$=329$$

(b)

$$\begin{vmatrix}3 & 6 & 0 & 1\\ -2 & 3 & 1 & 4\\ 1 & 0 & -1 & 1\\ -9 & 2 & -2 & 2\end{vmatrix}=(-1)\begin{vmatrix}1 & 0 & -1 & 1\\ -2 & 3 & 1 & 4\\ 3 & 6 & 0 & 1\\ -9 & 2 & -2 & 2\end{vmatrix}$$ ← The first and third rows were interchanged.

$$=(-1)\begin{vmatrix}1 & 0 & -1 & 1\\ 0 & 3 & -1 & 6\\ 0 & 6 & 3 & -2\\ 0 & 2 & -11 & 11\end{vmatrix}$$ ← 2 times the first row was added to the second, -3 times the first row was added to the third and 9 times the first row was added to the fourth.

$$=(-1)\begin{vmatrix}1 & 0 & -1 & 1\\ 0 & 3 & -1 & 6\\ 0 & 0 & 5 & -14\\ 0 & 0 & -\frac{31}{3} & 7\end{vmatrix}$$ ← -2 times the second row was added to the third and $-\frac{2}{3}$ times the second row was added to the fourth.

$$=(-1)\begin{vmatrix}1 & 0 & -1 & 1\\ 0 & 3 & -1 & 6\\ 0 & 0 & 5 & -14\\ 0 & 0 & 0 & -\frac{329}{15}\end{vmatrix}$$ ← $\frac{31}{15}$ times the third row was added to the fourth.

$$=(-1)(1)(3)(5)\left(-\frac{329}{15}\right)=329$$ ← Use Theorem 2.1.2.

8. **(a)** We perform cofactor expansions along the first row in the 4x4 determinant, as well as in each of the 3x3 determinants:

$$\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix}$$
$$= -1\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \\ -3 & -2 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 4 & 2 & 1 \\ 1 & 3 & 4 \\ -4 & -2 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 4 & 3 & 1 \\ 1 & 2 & 4 \\ -4 & -3 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ -4 & -3 & -2 \end{vmatrix}$$
$$= -1\left(3\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix}\right)$$
$$+ 2\left(4\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ -4 & -2 \end{vmatrix}\right)$$
$$- 3\left(4\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ -4 & -3 \end{vmatrix}\right) + 4\left(4\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -4 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -4 & -3 \end{vmatrix}\right)$$
$$= -\big((3)(5) - (2)(10) + 5\big) + (2)((4)(5) - 2\big((4)(5) - (2)(15) + 10\big)$$
$$- 3\big((4)(10) - (3)(15) + 5\big) + 4\big((4)(5) - (3)(10) + (2)(5)\big)$$
$$= 0 + 0 + 0 + 0$$
$$= 0$$

(b)
$$\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ -4 & -3 & -2 & -1 \end{vmatrix}$$ ⟵ The first row was added to the third row.

$$= 0$$ ⟵ Use Theorem 2.2.1.

9.
$$\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = \left|\begin{matrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{matrix}\right| \begin{matrix} -1 & 5 \\ 0 & 2 \\ -3 & 1 \end{matrix} = [-2 + 15 + 0] - [-12 + 1 + 0] = 24$$

$$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = \left|\begin{matrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{matrix}\right| \begin{matrix} -1 & -2 \\ -4 & -5 \\ -7 & -8 \end{matrix} = [-45 - 84 - 96] - [-105 - 48 - 72] = 0$$

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = \left|\begin{matrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{matrix}\right| \begin{matrix} 3 & 0 \\ 1 & 1 \\ 0 & 4 \end{matrix} = [6 + 0 - 4] - [0 + 12 + 0] = -10$$

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = \left|\begin{matrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{matrix}\right| \begin{matrix} -5 & 1 \\ 3 & 0 \\ 1 & -2 \end{matrix} = [0 + 2 - 24] - [0 + 20 + 6] = -48$$

10. **(a)** e.g. $\begin{vmatrix} 4 & 0 & 3 & 6 \\ 8 & 0 & 5 & 0 \\ 7 & 3 & 7 & 10 \\ 13 & 0 & 10 & 0 \end{vmatrix} = -3\begin{vmatrix} 4 & 3 & 6 \\ 8 & 5 & 0 \\ 13 & 10 & 0 \end{vmatrix} = -18\begin{vmatrix} 8 & 5 \\ 13 & 10 \end{vmatrix} = (-18)(15) = -270$ was easy to calculate by cofactor expansions (first, we expanded along the second column, then along the third column), but would be more difficult to calculate using elementary row operations.

(b) e.g., $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix}$ of Exercise 8 was easy to calculate using elementary row operations, but more difficult using cofactor expansion.

11. In Exercise 1: $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -18 \neq 0$ therefore the matrix is invertible.

In Exercise 2: $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -44 \neq 0$ therefore the matrix is invertible.

In Exercise 3: $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = 24 \neq 0$ therefore the matrix is invertible.

In Exercise 4: $\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = 0$ therefore the matrix is not invertible.

12. In Exercise 5: $\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = -10 \neq 0$ therefore the matrix is invertible.

In Exercise 6: $\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -48 \neq 0$ therefore the matrix is invertible.

In Exercise 7: $\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 329 \neq 0$ therefore the matrix is invertible.

In Exercise 8: $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = 0$ therefore the matrix is not invertible.

13. $\begin{vmatrix} 5 & b-3 \\ b-2 & -3 \end{vmatrix} = (5)(-3) - (b-3)(b-2) = -15 - b^2 + 2b + 3b - 6 = -b^2 + 5b - 21$

14. $\begin{vmatrix} 3 & -4 & a \\ a^2 & 1 & 2 \\ 2 & a-1 & 4 \end{vmatrix} = \begin{vmatrix} 4a^2+3 & 0 & 8+a \\ a^2 & 1 & 2 \\ -a^3+a^2+2 & 0 & -2a+6 \end{vmatrix}$ ← 4 times the second row was added to the first row and $1-a$ times the second row was added to the last row.

$= -0 + 1\begin{vmatrix} 4a^2+3 & 8+a \\ -a^3+a^2+2 & -2a+6 \end{vmatrix} - 0$ ← Cofactor expansion along the second column.

$= (4a^2+3)(-2a+6) - (8+a)(-a^3+a^2+2)$

$$= a^4 - a^3 + 16a^2 - 8a + 2$$

15. $\begin{vmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{vmatrix}$

$= (-1)\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix}$ ← The first row and the fifth row were interchanged.

$= (-1)(-1)\begin{vmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix}$ ← The second row and the fourth row were interchanged.

$= (-1)(-1)(5)(2)(-1)(-4)(-3) = -120$

16. $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = x(1-x) - (-1)(3) = -x^2 + x + 3;$

Adding -2 times the first row to the second row, then performing cofactor expansion along the second row yields $\begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 0 & x & 0 \\ 1 & 3 & x-5 \end{vmatrix} = x\begin{vmatrix} 1 & -3 \\ 1 & x-5 \end{vmatrix} = x(x-5+3) = x^2 - 2x$

Solve the equation

$$-x^2 + x + 3 = x^2 - 2x$$
$$-2x^2 + 3x + 3 = 0$$

From quadratic formula $x = \frac{-3+\sqrt{9+24}}{-4} = \frac{3-\sqrt{33}}{4}$ or $x = \frac{-3-\sqrt{9+24}}{-4} = \frac{3+\sqrt{33}}{4}$.

17. It was shown in the solution of Exercise 1 that $\begin{vmatrix} -4 & 2 \\ 3 & 3 \end{vmatrix} = -18$. The determinant is nonzero, therefore by Theorem 2.3.3, the matrix $A = \begin{bmatrix} -4 & 2 \\ 3 & 3 \end{bmatrix}$ is invertible.

The cofactors are:

$$C_{11} = 3 \qquad C_{12} = -3$$
$$C_{21} = -2 \qquad C_{22} = -4$$

The matrix of cofactors is $\begin{bmatrix} 3 & -3 \\ -2 & -4 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-18}\begin{bmatrix} 3 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{9} \\ \frac{1}{6} & \frac{2}{9} \end{bmatrix}$.

18. It was shown in the solution of Exercise 2 that $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -44$. The determinant is nonzero, therefore by Theorem 2.3.3, the matrix $A = \begin{bmatrix} 7 & -1 \\ -2 & -6 \end{bmatrix}$ is invertible.

The cofactors are:

$$C_{11} = -6 \qquad C_{12} = 2$$
$$C_{21} = 1 \qquad C_{22} = 7$$

The matrix of cofactors is $\begin{bmatrix} -6 & 2 \\ 1 & 7 \end{bmatrix}$ and the adjoint matrix is $\text{adj}(A) = \begin{bmatrix} -6 & 1 \\ 2 & 7 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-44}\begin{bmatrix} -6 & 1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & -\frac{1}{44} \\ -\frac{1}{22} & -\frac{7}{44} \end{bmatrix}$.

19. It was shown in the solution of Exercise 3 that $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix} = 24$. The determinant is nonzero, therefore by Theorem 2.3.3, $A = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \qquad C_{12} = -\begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} = 3 \qquad C_{13} = \begin{vmatrix} 0 & 2 \\ -3 & 1 \end{vmatrix} = 6$$
$$C_{21} = -\begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = -3 \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = 5 \qquad C_{23} = -\begin{vmatrix} -1 & 5 \\ -3 & 1 \end{vmatrix} = -14$$
$$C_{31} = \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = -9 \qquad C_{32} = -\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = -1 \qquad C_{33} = \begin{vmatrix} -1 & 5 \\ 0 & 2 \end{vmatrix} = -2$$

The matrix of cofactors is $\begin{bmatrix} 3 & 3 & 6 \\ -3 & 5 & -14 \\ -9 & -1 & -2 \end{bmatrix}$ and the adjoint matrix is $\text{adj}(A) = \begin{bmatrix} 3 & -3 & -9 \\ 3 & 5 & -1 \\ 6 & -14 & -2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{24}\begin{bmatrix} 3 & -3 & -9 \\ 3 & 5 & -1 \\ 6 & -14 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{8} & \frac{5}{24} & -\frac{1}{24} \\ \frac{1}{4} & -\frac{7}{12} & -\frac{1}{12} \end{bmatrix}$.

20. It was shown in the solution of Exercise 4 that $\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = 0$ therefore by Theorem 2.3.3, the matrix is not invertible.

21. It was shown in the solution of Exercise 5 that $\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = -10$. The determinant is nonzero, therefore by Theorem 2.3.3, $A = \begin{bmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = -2 \qquad C_{12} = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2 \qquad C_{13} = \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} = 4$$
$$C_{21} = -\begin{vmatrix} 0 & -1 \\ 4 & 2 \end{vmatrix} = -4 \qquad C_{22} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6 \qquad C_{23} = -\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} = -12$$
$$C_{31} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1 \qquad C_{32} = -\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = -4 \qquad C_{33} = \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = 3$$

The matrix of cofactors is $\begin{bmatrix} -2 & -2 & 4 \\ -4 & 6 & -12 \\ 1 & -4 & 3 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -2 & -4 & 1 \\ -2 & 6 & -4 \\ 4 & -12 & 3 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-10}\begin{bmatrix} -2 & -4 & 1 \\ -2 & 6 & -4 \\ 4 & -12 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{1}{10} \\ \frac{1}{5} & -\frac{3}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{6}{5} & -\frac{3}{10} \end{bmatrix}$.

22. It was shown in the solution of Exercise 6 that $\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -48$. The determinant is nonzero, therefore by Theorem 2.3.3, $A = \begin{bmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 0 & 2 \\ -2 & 2 \end{vmatrix} = 4 \qquad C_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = -4 \qquad C_{13} = \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix} = -6$$
$$C_{21} = -\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} = -10 \qquad C_{22} = \begin{vmatrix} -5 & 4 \\ 1 & 2 \end{vmatrix} = -14 \qquad C_{23} = -\begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix} = -9$$
$$C_{31} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} -5 & 4 \\ 3 & 2 \end{vmatrix} = 22 \qquad C_{33} = \begin{vmatrix} -5 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

The matrix of cofactors is $\begin{bmatrix} 4 & -4 & -6 \\ -10 & -14 & -9 \\ 2 & 22 & -3 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} 4 & -10 & 2 \\ -4 & -14 & 22 \\ -6 & -9 & -3 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-48}\begin{bmatrix} 4 & -10 & 2 \\ -4 & -14 & 22 \\ -6 & -9 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{5}{24} & -\frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & -\frac{11}{24} \\ \frac{1}{8} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}$.

23. It was shown in the solution of Exercise 7 that $\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 329$. The determinant of A is nonzero therefore by Theorem 2.3.3, $A = \begin{bmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 3 & 1 & 4 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} = (-6+2+0) - (-8-6+0) = 10$$

$$C_{12} = -\begin{vmatrix} -2 & 1 & 4 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} = -[(4-9-8) - (36+4+2)] = 55$$

$$C_{13} = \begin{vmatrix} -2 & 3 & 4 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{vmatrix} = (0-27+8) - (0-4+6) = -21$$

$$C_{14} = -\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix} = -[(0+27+2) - (0+4-6)] = -31$$

$$C_{21} = -\begin{vmatrix} 6 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & -2 & 2 \end{vmatrix} = -[(-12+0+0) - (-2-12+0)] = -2$$

$$C_{22} = \begin{vmatrix} 3 & 0 & 1 \\ 1 & -1 & 1 \\ -9 & -2 & 2 \end{vmatrix} = (-6+0-2) - (9-6+0) = -11$$

$$C_{23} = -\begin{vmatrix} 3 & 6 & 1 \\ 1 & 0 & 1 \\ -9 & 2 & 2 \end{vmatrix} = -[(0-54+2) - (0+6+12)] = 70$$

$$C_{24} = \begin{vmatrix} 3 & 6 & 0 \\ 1 & 0 & -1 \\ -9 & 2 & -2 \end{vmatrix} = (0+54+0) - (0-6-12) = 72$$

$$C_{31} = \begin{vmatrix} 6 & 0 & 1 \\ 3 & 1 & 4 \\ 2 & -2 & 2 \end{vmatrix} = (12+0-6) - (2-48+0) = 52$$

$$C_{32} = -\begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 4 \\ -9 & -2 & 2 \end{vmatrix} = -[(6+0+4) - (-9-24+0)] = -43$$

$$C_{33} = \begin{vmatrix} 3 & 6 & 1 \\ -2 & 3 & 4 \\ -9 & 2 & 2 \end{vmatrix} = (18-216-4) - (-27+24-24) = -175$$

$$C_{34} = -\begin{vmatrix} 3 & 6 & 0 \\ -2 & 3 & 1 \\ -9 & 2 & -2 \end{vmatrix} = -[(-18-54+0) - (0+6+24)] = 102$$

$$C_{41} = -\begin{vmatrix} 6 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & -1 & 1 \end{vmatrix} = -[(6+0-3) - (0-24+0)] = -27$$

$$C_{42} = \begin{vmatrix} 3 & 0 & 1 \\ -2 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = (3+0+2) - (1-12+0) = 16$$

$C_{43} = -\begin{vmatrix} 3 & 6 & 1 \\ -2 & 3 & 4 \\ 1 & 0 & 1 \end{vmatrix} = -[(9+24+0)-(3+0-12)] = -42$

$C_{44} = \begin{vmatrix} 3 & 6 & 0 \\ -2 & 3 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-9+6+0)-(0+0+12) = -15$

The matrix of cofactors is $\begin{bmatrix} 10 & 55 & -21 & -31 \\ -2 & -11 & 70 & 72 \\ 52 & -43 & -175 & 102 \\ -27 & 16 & -42 & -15 \end{bmatrix}$ and $\operatorname{adj}(A) = \begin{bmatrix} 10 & -2 & 52 & -27 \\ 55 & -11 & -43 & 16 \\ -21 & 70 & -175 & -42 \\ -31 & 72 & 102 & -15 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{329}\begin{bmatrix} 10 & -2 & 52 & -27 \\ 55 & -11 & -43 & 16 \\ -21 & 70 & -175 & -42 \\ -31 & 72 & 102 & -15 \end{bmatrix} =$

$$\begin{bmatrix} \frac{10}{329} & -\frac{2}{329} & \frac{52}{329} & -\frac{27}{329} \\ \frac{55}{329} & -\frac{11}{329} & -\frac{43}{329} & \frac{16}{329} \\ -\frac{3}{47} & \frac{10}{47} & -\frac{25}{47} & -\frac{6}{47} \\ -\frac{31}{329} & \frac{72}{329} & \frac{102}{329} & -\frac{15}{329} \end{bmatrix}.$$

24. It was shown in the solution of Exercise 8 that $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = 0$ therefore by Theorem 2.3.3, the matrix is not invertible.

25. $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, $\det(A) = \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) - \left(-\frac{4}{5}\right)\left(\frac{4}{5}\right) = \frac{9}{25} + \frac{16}{25} = 1$; $A_1 = \begin{bmatrix} x & -\frac{4}{5} \\ y & \frac{3}{5} \end{bmatrix}$, $A_2 = \begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & y \end{bmatrix}$;

$$x' = \frac{\det(A_1)}{\det(A)} = \frac{3}{5}x + \frac{4}{5}y, \ y' = \frac{\det(A_2)}{\det(A)} = \frac{3}{5}y - \frac{4}{5}x$$

26. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $A_1 = \begin{bmatrix} x & -\sin\theta \\ y & \cos\theta \end{bmatrix}$, $A_2 = \begin{bmatrix} \cos\theta & x \\ \sin\theta & y \end{bmatrix}$;

$$x' = \frac{\det(A_1)}{\det(A)} = \frac{x\cos\theta + y\sin\theta}{\cos^2\theta + \sin^2\theta} = x\cos\theta + y\sin\theta, \ y' = \frac{\det(A_2)}{\det(A)} = \frac{y\cos\theta - x\sin\theta}{\cos^2\theta + \sin^2\theta} = y\cos\theta - x\sin\theta$$

27. The coefficient matrix of the given system is $A = \begin{bmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{bmatrix}$. Coefficient expansion along the first row yields

$$\begin{aligned} \det(A) &= 1\begin{vmatrix} 1 & \beta \\ \beta & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & \beta \\ \alpha & 1 \end{vmatrix} + \alpha\begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} \\ &= 1 - \beta^2 - (1 - \alpha\beta) + \alpha(\beta - \alpha) = -\alpha^2 + 2\alpha\beta - \beta^2 = -(\alpha - \beta)^2 \end{aligned}$$

By Theorem 2.3.8, the given system has a nontrivial solution if and only if $\det(A) = 0$, i.e., $\alpha = \beta$.

28. According to the arrow technique (see Example 7 in Section 2.1), the determinant of a 3×3 matrix can be expressed as a sum of six terms:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

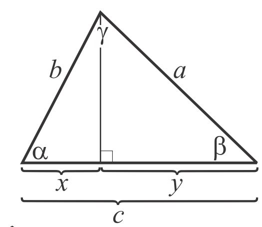
If each entry of A is either 0 or 1, then each of the terms must be either 0 or ± 1. The largest value 3 would result from the terms $1 + 1 + 1 - 0 - 0 - 0$, however, this is not possible since the first three terms all equal 1 would require that all nine matrix entries be equal 1, making the determinant 0.

The largest value of the determinant that is actually attainable is 2, e.g., let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

29. (a) We will justify the third equality, $a\cos\beta + b\cos\alpha = c$ by considering three cases:

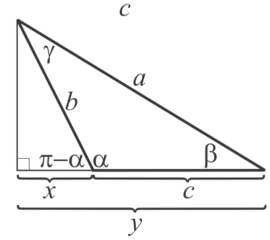
CASE I: $\alpha \le \frac{\pi}{2}$ and $\beta \le \frac{\pi}{2}$

Referring to the figure on the right side, we have
$x = b\cos\alpha$ and $y = a\cos\beta$.
Since $x + y = c$ we obtain , $a\cos\beta + b\cos\alpha = c$.



CASE II: $\alpha > \frac{\pi}{2}$ and $\beta < \frac{\pi}{2}$

Referring to the picture on the right side, we can write
$x = b\cos(\pi - \alpha) = -b\cos\alpha$ and $y = a\cos\beta$
This time we can write $c = y - x = a\cos\beta - (-b\cos\alpha)$
therefore once again $a\cos\beta + b\cos\alpha = c$.



CASE III: $\beta > \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$ (similarly to case II, $c = b\cos\alpha - a\cos(\pi - \beta) = b\cos\alpha + a\cos\beta$)

The first two equations can be justified in the same manner.

Denoting $X = \cos\alpha$, $Y = \cos\beta$, and $Z = \cos\gamma$ we can rewrite the linear system as

$$\begin{array}{rcrcrcl} & & cY & + & bZ & = & a \\ cX & & & + & aZ & = & b \\ bX & + & aY & & & = & c \end{array}$$

We have $\det(A) = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = [0 + abc + abc] - [0 + 0 + 0] = 2abc$ and

$\det(A_1) = \begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = [0 + ac^2 + ab^2] - [0 + a^3 + 0] = a(b^2 + c^2 - a^2)$ therefore by Cramer's rule

$\cos\alpha = X = \frac{\det(A_1)}{\det(A)} = \frac{a(b^2+c^2-a^2)}{2abc} = \frac{b^2+c^2-a^2}{2bc}$.

(b) Using the results obtained in part (a) along with

$\det(A_2) = \begin{vmatrix} 0 & a & b \\ c & b & a \\ b & c & 0 \end{vmatrix} = [0 + a^2b + bc^2] - [b^3 + 0 + 0] = b(a^2 + c^2 - b^2)$ and

$\det(A_3) = \begin{vmatrix} 0 & c & a \\ c & 0 & b \\ b & a & c \end{vmatrix} = [0 + b^2c + a^2c] - [0 + 0 + c^3] = c(a^2 + b^2 - c^2)$ therefore by

Cramer's rule

$\cos\beta = Y = \frac{\det(A_2)}{\det(A)} = \frac{a^2+c^2-b^2}{2ac}$ and $\cos\gamma = Z = \frac{\det(A_3)}{\det(A)} = \frac{a^2+b^2-c^2}{2ab}$.

31. From Theorem 2.3.6, $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A) = \det(A)\,A^{-1}$. Consequently,

$$\left(\frac{1}{\det(A)}A\right)\operatorname{adj}(A) = \left(\frac{1}{\det(A)}A\right)\left(\det(A)\ A^{-1}\right) = \frac{\det(A)}{\det(A)}(AA^{-1}) = I_n \text{ so } \left(\operatorname{adj}(A)\right)^{-1} = \frac{1}{\det(A)}A.$$

Using Theorem 2.3.5, we can also write $\operatorname{adj}(A^{-1}) = \det(A^{-1})\,(A^{-1})^{-1} = \frac{1}{\det(A)}A$.

33. The equality $A\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ means that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x} = \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}$. Consequently, it follows from Theorem 2.3.8 that $\det(A) = 0$.

34. **(b)** $\frac{1}{2}\begin{vmatrix}3 & 3 & 1\\ 4 & 0 & 1\\ -2 & -1 & 1\end{vmatrix} = -\frac{19}{2}$ is the negative of the area of the triangle because it is being traced clockwise; (reversing the order of the points would change the orientation to counterclockwise, and thereby result in the positive area: $\frac{1}{2}\begin{vmatrix}-2 & -1 & 1\\ 4 & 0 & 1\\ 3 & 3 & 1\end{vmatrix} = \frac{19}{2}$

CHAPTER 3: EUCLIDEAN VECTOR SPACES

3.1 Vectors in 2-Space, 3-Space, and n-Space

1. **(a)** $(4-1, 1-5) = (3, -4)$ **(b)** $(0-2, 0-3, 4-0) = (-2, -3, 4)$

2. **(a)** $(-3-2, 3-3) = (-5, 0)$ **(b)** $(0-3,\ 4-0,\ 4-4) = (-3, 4, 0)$

3. **(a)** $\overrightarrow{P_1P_2} = (2-3, 8-5) = (-1, 3)$ **(b)** $\overrightarrow{P_1P_2} = (2-5, 4-(-2), 2-1) = (-3, 6, 1)$

4. **(a)** $\overrightarrow{P_1P_2} = (-4-(-6), -1-2) = (2, -3)$ **(b)** $\overrightarrow{P_1P_2} = (-1-0,\ 6-0,\ 1-0) = (-1, 6, 1)$

5. **(a)** Denote the terminal point by $B(b_1, b_2)$. Since the vector $\overrightarrow{AB} = (b_1 - 1, b_2 - 1)$ is to be equivalent to the vector $\mathbf{u} = (1, 2)$, the coordinates of B must satisfy the equations

$$b_1 - 1 = 1 \quad \text{and} \quad b_2 - 1 = 2$$

therefore $b_1 = 2$ and $b_2 = 3$. The terminal point is $B(2, 3)$.

(b) Denote the initial point by $A(a_1, a_2, a_3)$. Since the vector $\overrightarrow{AB} = (-1 - a_1, -1 - a_2, 2 - a_3)$ is to be equivalent to the vector $\mathrm{u} = (1, 1, 3)$, the coordinates of A must satisfy the equations

$$-1 - a_1 = 1, \qquad -1 - a_2 = 1, \qquad \text{and} \qquad 2 - a_3 = 3$$

therefore $a_1 = -2$, $a_2 = -2$, and $a_3 = -1$. The initial point is $A(-2, -2, -1)$.

6. **(a)** Denote the initial point by $A(a_1, a_2)$. Since the vector $\overrightarrow{AB} = (2 - a_1, 0 - a_2) = (2 - a_1, -a_2)$ is to be equivalent to the vector $\mathbf{u} = (1, 2)$, the coordinates of A must satisfy the equations

$$2 - a_1 = 1 \ \text{ and } \ -a_2 = 2$$

therefore $a_1 = 1$ and $a_2 = -2$. The initial point is $A(1, -2)$.

(b) Denote the terminal point by $B(b_1, b_2, b_3)$. Since the vector $\overrightarrow{AB} = (b_1 - 0, b_2 - 2, b_3 - 0) = (b_1, b_2 - 2, b_3)$ is to be equivalent to the vector $\mathbf{u} = (1, 1, 3)$, the coordinates of B must satisfy the equations

$$b_1 = 1, \quad b_2 - 2 = 1, \ \text{ and } \ b_3 = 3$$

therefore $b_1 = 1$, $b_2 = 3$, and $b_3 = 3$. The terminal point is $B(1, 3, 3)$.

7. **(a)** For any positive real number k, the vector $\mathbf{u} = k\mathbf{v}$ has the same direction as $\mathbf{v}$. For example, letting $k = 1$, we have $\mathbf{u} = (4, -2, -1)$. If the terminal point is $Q(3, 0, -5)$ then the initial point has coordinates $(3-4, 0-(-2), -5-(-1))$, i.e., $(-1, 2, -4)$.

(b) For any negative real number k, the vector $\mathbf{u} = k\mathbf{v}$ is oppositely directed to $\mathbf{v}$. For example, letting $k = -1$, we have $\mathbf{u} = (-4, 2, 1)$. If the terminal point is $Q(3, 0, -5)$ then the initial point has coordinates $(3-(-4), 0-2, -5-1)$, i.e., $(7, -2, -6)$.

8. **(a)** For any positive real number k, the vector $\mathbf{u} = k\mathbf{v}$ has the same direction as $\mathbf{v}$. For example, letting $k = 1$, we have $\mathbf{u} = (6, 7, -3)$. If the initial point is $P(-1,3,-5)$ then the terminal point has coordinates $(-1 + 6, 3 + 7, -5 - 3)$, i.e., $(5,10,-8)$.

(b) For any negative real number k, the vector $\mathbf{u} = k\mathbf{v}$ is oppositely directed to $\mathbf{v}$. For example, letting $k = -1$, we have $\mathbf{u} = (-6, -7, 3)$. If the initial point is $P(-1,3,-5)$ then the terminal point has coordinates $(-1 - 6, 3 - 7, -5 + 3)$, i.e., $(-7, -4, -2)$.

9. **(a)** $\mathbf{u} + \mathbf{w} = \big(4 + (-3), -1 + (-3)\big) = (1, -4)$

(b) $\mathbf{v} - 3\mathbf{u} = (0, 5) - (12, -3) = \big(0 - 12, 5 - (-3)\big) = (-12, 8)$

(c) $2(\mathbf{u} - 5\mathbf{w}) = 2[(4, -1) - (-15, -15)] = 2(19, 14) = (38, 28)$

(d) $3\mathbf{v} - 2(\mathbf{u} + 2\mathbf{w}) = (0, 15) - 2[(4, -1) + (-6, -6)] = (0, 15) - 2(-2, -7)$
$= (0, 15) - (-4, -14) = (4, 29)$

10. **(a)** $\mathbf{v} - \mathbf{w} = \big(4 - 6, 0 - (-1), -8 - (-4)\big) = (-2, 1, -4)$

(b) $6\mathbf{u} + 2\mathbf{v} = (-18, 6, 12) + (8, 0, -16) = (-10, 6, -4)$

(c) $-3(\mathbf{v} - 8\mathbf{w}) = -3[(4, 0, -8) - (48, -8, -32)] = -3(-44, 8, 24) = (132, -24, -72)$

(d) $(2\mathbf{u} - 7\mathbf{w}) - (8\mathbf{v} + \mathbf{u}) = [(-6, 2, 4) - (42, -7, -28)] - [(32, 0, -64) + (-3, 1, 2)]$
$= (-48, 9, 32) - (29, 1, -62) = (-77, 8, 94)$

11. **(a)** $\mathbf{v} - \mathbf{w} = (4 - 5, 7 - (-2), -3 - 8, 2 - 1) = (-1, 9, -11, 1)$

(b) $-\mathbf{u} + (\mathbf{v} - 4\mathbf{w}) = (3, -2, -1,0) + [(4, 7, -3, 2) - (20, -8, 32, 4)]$
$= (3, -2, -1,0) + (-16, 15, -35, -2) = (-13, 13, -36, -2)$

(c) $6(\mathbf{u} - 3\mathbf{v}) = 6[(-3, 2, 1, 0) - (12, 21, -9, 6)] = 6(-15, -19,10, -6) = (-90, -114, 60, -36)$

(d) $(6\mathbf{v} - \mathbf{w}) - (4\mathbf{u} + \mathbf{v}) = [(24, 42, -18, 12) - (5, -2, 8, 1)] - [(-12, 8, 4, 0) + (4, 7, -3, 2)]$
$= (19, 44, -26, 11) - (-8, 15, 1, 2) = (27, 29, -27, 9)$

12. **(a)** $\mathbf{v} + \mathbf{w} = (0 + 7, 4 + 1, -1 - 4, 1 - 2, 2 + 3) = (7, 5, -5, -1, 5)$

(b) $3(2\mathbf{u} - \mathbf{v}) = 3[(2, 4, -6, 10, 0) - (0, 4, -1, 1, 2)] = 3(2, 0, -5, 9, -2) = (6, 0, -15, 27, -6)$

(c) $(3\mathbf{u} - \mathbf{v}) - (2\mathbf{u} + 4\mathbf{w})$
$= [(3, 6, -9, 15, 0) - (0, 4, -1, 1, 2)] - [(2, 4, -6, 10, 0) + (28, 4, -16, -8, 12)]$
$= (3, 2, -8, 14, -2) - (30, 8, -22, 2, 12) = (-27, -6, 14, 12, -14)$

(d) $\frac{1}{2}(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v} = \frac{1}{2}[(7, 1, -4, -2,3) - (0, 20, -5, 5, 10) + (2, 4, -6, 10, 0)] + (0, 4, -1, 1, 2)$
$= \frac{1}{2}(9, -15, -5, 3, -7) + (0, 4, -1, 1, 2) = (\frac{9}{2}, -\frac{7}{2}, -\frac{7}{2}, \frac{5}{2}, -\frac{3}{2})$

13. Solve the vector equation using the properties listed in Theorems 3.1.1 and 3.1.2:

$3\mathbf{u} + \mathbf{v} + (-2)\mathbf{w} = 3\mathbf{x} + 2\mathbf{w}$ [Part (c) of Theorem 3.1.2 and part (g) of Theorem 3.1.1]

$(3\mathbf{u} + \mathbf{v}) + (-4)\mathbf{w} = 3\mathbf{x} + 0\mathbf{w}$ [Add $-2\mathbf{w}$ to both sides, use parts (b) and (d) of Th. 3.1.1]

$(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}=3\mathbf{x}$ [Use part (a) of Theorem 3.1.2]

$\frac{1}{3}[(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}]=\frac{1}{3}(3\mathbf{x})$ [Multiply both sides by $\frac{1}{3}$]

$\frac{1}{3}[(3\mathbf{u}+\mathbf{v})+(-4)\mathbf{w}]=\mathbf{x}$ [Parts (g) and (h) of Theorem 3.1.1]

Therefore $\mathbf{x}=\frac{1}{3}[(-5,13,0,2)+(-20,8,-32,-4)]=\frac{1}{3}(-25,21,-32,-2)=(-\frac{25}{3},7,-\frac{32}{3},-\frac{2}{3})$.

14. Solve the vector equation using the properties listed in Theorems 3.1.1 and 3.1.2:

$2\mathbf{u}+(-1)\mathbf{v}+\mathbf{x}=7\mathbf{x}+\mathbf{w}$ [Part (c) of Theorem 3.1.2]

$2\mathbf{u}+(-1)\mathbf{v}+0\mathbf{x}=6\mathbf{x}+\mathbf{w}$ [Add $-\mathbf{x}$ to both sides, use parts (b) and (d) of Th. 3.1.1]

$2\mathbf{u}+(-1)\mathbf{v}=6\mathbf{x}+\mathbf{w}$ [Use part (a) of Theorem 3.1.2]

$2\mathbf{u}+(-1)\mathbf{v}+(-1)\mathbf{w}=6\mathbf{x}+0\mathbf{w}$ [Add $-\mathbf{w}$ to both sides, use parts (b) and (d) of Th. 3.1.1]

$2\mathbf{u}+(-1)\mathbf{v}+(-1)\mathbf{w}=6\mathbf{x}$ [Use part (a) of Theorem 3.1.2]

$\frac{1}{6}(2\mathbf{u}+(-1)\mathbf{v}+(-1)\mathbf{w})=\frac{1}{6}(6\mathbf{x})$ [Multiply both sides by $\frac{1}{6}$]

$\frac{1}{6}(2\mathbf{u}+(-1)\mathbf{v}+(-1)\mathbf{w})=\mathbf{x}$ [Parts (g) and (h) of Theorem 3.1.1]

Therefore $\mathbf{x}=\frac{1}{6}[(2,4,-6,10,0)+(0,-4,1,-1,-2)+(-7,-1,4,2,-3)]=\left(-\frac{5}{6},-\frac{1}{6},-\frac{1}{6},\frac{11}{6},-\frac{5}{6}\right)$.

15. Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel (collinear) if one of them is a scalar multiple of the other one, i.e. either $\mathbf{u}=a\mathbf{v}$ for some scalar a or $\mathbf{v}=b\mathbf{u}$ for some scalar b or both (the two conditions are not equivalent if one of the vectors is a zero vector, but the other one is not.)

(a) $\mathbf{v}=(4,2,0,6,10,2)$ does not equal $k\mathbf{u}=(-2k,k,0,3k,5k,k)$ for any scalar k; $\mathbf{v}$ is not parallel to $\mathbf{u}$

(b) $\mathbf{v}=(4,-2,0,-6,-10,-2)=-2\mathbf{u}$; $\mathbf{v}$ is parallel to $\mathbf{u}$

(c) $\mathbf{v}=(0,0,0,0,0,0)=0\mathbf{u}$; $\mathbf{v}$ is parallel to $\mathbf{u}$

16. Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel (collinear) if one of them is a scalar multiple of the other one, i.e. either $\mathbf{u}=a\mathbf{v}$ for some scalar a or $\mathbf{v}=b\mathbf{u}$ for some scalar b or both (the two conditions are not equivalent if one of the vectors is a zero vector, but the other one is not.)

(a) Let $\mathbf{v}=(8t,-2)$.

$\mathbf{u}=a\mathbf{v} \quad\Leftrightarrow\quad 4=8at \text{ and } -1=-2a \quad\Leftrightarrow\quad a=\frac{1}{2} \text{ and } t=1$

$\mathbf{v}=b\mathbf{u} \quad\Leftrightarrow\quad 8t=4b \text{ and } -2=-b \quad\Leftrightarrow\quad b=2 \text{ and } t=1$

Therefore the vector $(8t,-2)$ is parallel to $(4,-1)$ if and only if $t=1$.

(b) Let $\mathbf{v}=(8t,2t)$.

$\mathbf{u} = a\mathbf{v} \quad \Leftrightarrow \quad 4 = 8at$ and $-1 = 2at \quad \Leftrightarrow \quad 1 = 2at$ and $-1 = 2at$ - contradiction

$\mathbf{v} = b\mathbf{u} \quad \Leftrightarrow \quad 8t = 4b$ and $2t = -b \quad \Leftrightarrow \quad b = 0$ and $t = 0$

Therefore the vector $(8t, 2t)$ is parallel to $(4, -1)$ if and only if $t = 0$.

(c) Let $\mathbf{v} = (1, t^2)$.

$\mathbf{u} = a\mathbf{v} \quad \Leftrightarrow \quad 4 = a$ and $-1 = at^2$ - contradiction

$\mathbf{v} = b\mathbf{u} \quad \Leftrightarrow \quad 1 = 4b$ and $t^2 = -b$ - contradiction

Therefore the vector $(1, t^2)$ is not parallel to $(4, -1)$ for any real value t.

17. The vector equation $a(1, -1, 3, 5) + b(2, 1, 0, -3) = (1, -4, 9, 18)$ is equivalent to the linear system

$$\begin{array}{rcrcr} 1a & + & 2b & = & 1 \\ -1a & + & 1b & = & -4 \\ 3a & + & 0b & = & 9 \\ 5a & - & 3b & = & 18 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 3 & 0 & 9 \\ 5 & -3 & 18 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, the unique solution is $a = 3$ and $b = -1$.

18. The vector equation $a(2, 1, 0, 1, -1) + b(-2, 3, 1, 0, 2) = (-8, 8, 3, -1, 7)$ is equivalent to the linear system

$$\begin{array}{rcrcr} 2a & - & 2b & = & -8 \\ 1a & + & 3b & = & 8 \\ 0a & + & 1b & = & 3 \\ 1a & + & 0b & = & -1 \\ -1a & + & 2b & = & 7 \end{array}$$

whose augmented matrix $\begin{bmatrix} 2 & -2 & 8 \\ 1 & 3 & 8 \\ 0 & 1 & 3 \\ 1 & 0 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, the unique solution is $a = -1$ and $b = 3$.

19. The vector equation $c_1(1, -1, 0) + c_2(3, 2, 1) + c_3(0, 1, 4) = (-1, 1, 19)$ is equivalent to the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 3c_2 & + & 0c_3 & = & -1 \\ -1c_1 & + & 2c_2 & + & 1c_3 & = & 1 \\ 0c_1 & + & 1c_2 & + & 4c_3 & = & 19 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 3 & 0 & -1 \\ -1 & 2 & 1 & 1 \\ 0 & 1 & 4 & 19 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$.

Therefore, the unique solution is $c_1 = 2$, $c_2 = -1$, and $c_3 = 5$.

20. The vector equation $c_1(-1, 0, 2) + c_2(2, 2, -2) + c_3(1, -2, 1) = (-6, 12, 4)$ is equivalent to the linear system

$$\begin{array}{rcrcrcr} -1c_1 & + & 2c_2 & + & 1c_3 & = & -6 \\ 0c_1 & + & 2c_2 & - & 2c_3 & = & 12 \\ 2c_1 & - & 2c_2 & + & 1c_3 & = & 4 \end{array}$$

whose augmented matrix $\begin{bmatrix} -1 & 2 & 1 & -6 \\ 0 & 2 & -2 & 12 \\ 2 & -2 & 1 & 4 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

Therefore, the unique solution is $c_1 = 6,\ c_2 = 2,$ and $c_3 = -4$.

21. The vector equation $c_1(-2, 9, 6) + c_2(-3, 2, 1) + c_3(1,\ 7, 5) = (0, 5, 4)$ is equivalent to the linear system

$$\begin{array}{rcrcrcr} -2c_1 & - & 3c_2 & + & 1c_3 & = & 0 \\ 9c_1 & + & 2c_2 & + & 7c_3 & = & 5 \\ 6c_1 & + & 1c_2 & + & 5c_3 & = & 4 \end{array}$$

whose augmented matrix $\begin{bmatrix} -2 & -3 & 1 & 0 \\ 9 & 2 & 7 & 5 \\ 6 & 1 & 5 & 4 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The system has no solution.

22. Equating the second components on both sides yields a contradictory equation $0 = -2$.

23. **(a)** The midpoint of the segment is the terminal point of the vector
$\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ} = (2, 3, -2) + \frac{1}{2}(7 - 2, -4 - 3, 1 - (-2)) = \left(\frac{9}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$
therefore the midpoint has coordinates $\left(\frac{9}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

(b) The desired point is the terminal point of the vector
$\overrightarrow{ON} = \overrightarrow{OP} + \frac{3}{4}\overrightarrow{PQ} = (2, 3, -2) + \frac{3}{4}(7 - 2, -4 - 3, 1 - (-2)) = \left(\frac{23}{4}, -\frac{9}{4}, \frac{1}{4}\right)$
therefore this point has coordinates $\left(\frac{23}{4}, -\frac{9}{4}, \frac{1}{4}\right)$.

24. When the vector $\mathbf{u} = \overrightarrow{OP_1} + \frac{1}{2}\left(\overrightarrow{OP_2} - \overrightarrow{OP_1}\right)$ is positioned so its initial point is at the origin, its terminal point is the midpoint of the line segment connecting the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ since

$$\mathbf{u} = (x_1, y_1) + \frac{1}{2}(x_2 - x_1, y_2 - y_1) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

25. **(a)** $\mathbf{u} + \mathbf{v} + \mathbf{w} = (5, -5) + (-10, 2) + (3, 8) = (-2, 5)$

(b) $\mathbf{u} + \mathbf{v} + \mathbf{w} = (10, -7) + (-3, 8) + (-4, -9) = (3, -8)$

26. **(a)** $\mathbf{u} - \mathbf{v} + \mathbf{w} = (5, -5) - (-10, 2) + (3, 8) = (18, 1)$

(b) $\mathbf{u} - \mathbf{v} + \mathbf{w} = (10, -7) - (-3, 8) + (-4, -9) = (9, -24)$

27. The midpoint of the line segment connecting the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Therefore we have

$$\left(\frac{1 + x_2}{2}, \frac{3 + y_2}{2}, \frac{7 + z_2}{2}\right) = (4, 0, -6).$$

This vector equation is equivalent to a system of three linear equations in three unknowns that is easy to solve:

$$\frac{1+x_2}{2}=4 \quad\Leftrightarrow\quad x_2=7$$
$$\frac{3+y_2}{2}=0 \quad\Leftrightarrow\quad y_2=-3$$
$$\frac{7+z_2}{2}=-6 \quad\Leftrightarrow\quad z_2=-19$$

We conclude that the point Q is $(7,-3,-19)$.

28. Yes. Arranging the three vectors "tip-to-tail" we obtain a triangle since the terminal point of the last vector is the same as the initial point of the first one.

29. **(a)** We have $\mathbf{a}+\mathbf{d}=\mathbf{b}+\mathbf{e}=\mathbf{c}+\mathbf{f}=\mathbf{0}$ therefore $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}=\mathbf{0}$.

(b) The sum is $\frac{1}{2}(\mathbf{0})=\mathbf{0}$.

(c) From part (a), $\mathbf{b}+\mathbf{c}+\mathbf{d}+\mathbf{e}+\mathbf{f}=-\mathbf{a}$.

(d) From part (a), the sum of any five vectors remaining after one is removed equals to the negative of the removed vector.

30. The sum of all radial vectors of a regular n-sided polygon is always $\mathbf{0}$. When consecutive vectors are arranged "tip-to-tail", a regular n-sided polygon is obtained. (An argument similar to the one used in 29(a) could also be used when n is even.)

True-False Exercises

(a) False. Equivalent vectors have the same length and direction - they may have different initial points.

(b) False. According to Definition 2, equivalent vectors must have the same number of components.

(c) False. $\mathbf{v}$ and $k\mathbf{v}$ are parallel for any k.

(d) True. This is a consequence of Theorem 3.1.1.

(e) True. This is a consequence of Theorem 3.1.1.

(f) False. At least one of the scalars must be nonzero for the vectors to be parallel.

(g) False. For nonzero vector $\mathbf{u}$, the vectors $\mathbf{u}$ and $-\mathbf{u}$ are collinear and have the same length but are not equal.

(h) True.

(i) False. $(k+m)(\mathbf{u}+\mathbf{v})=(k+m)\mathbf{u}+(\mathrm{k}+m)\mathbf{v}$.

(j) True. $\mathbf{x}=\frac{5}{8}\mathbf{v}+\frac{1}{2}\mathbf{w}$.

(k) False. For instance, if $\mathbf{v}_2=2\mathbf{v}_1$ then $4\mathbf{v}_1+2\mathbf{v}_2=2\mathbf{v}_1+3\mathbf{v}_2$.

3.2 Norm, Dot Product, and Distance in R^n

1. **(a)** $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$;

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{2\sqrt{3}}(2,2,2) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$; $-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{2\sqrt{3}}(2,2,2) = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + 0^2 + 2^2 + 1^2 + 3^2} = \sqrt{15}$;

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{15}}(1,0,2,1,3) = (\frac{1}{\sqrt{15}}, 0, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{3}{\sqrt{15}})$; $-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{15}}(1,0,2,1,3) =$ $(-\frac{1}{\sqrt{15}}, 0, -\frac{2}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{3}{\sqrt{15}})$

2. **(a)** $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$;

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{6}}(1,-1,2) = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$; $-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{6}}(1,-1,2) = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$

(b) $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2 + 3^2 + (-1)^2} = \sqrt{23}$;

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{23}}(-2,3,3,-1) = (-\frac{2}{\sqrt{23}}, \frac{3}{\sqrt{23}}, \frac{3}{\sqrt{23}}, -\frac{1}{\sqrt{23}})$;

$-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{23}}(-2,3,3,-1) = (\frac{2}{\sqrt{23}}, -\frac{3}{\sqrt{23}}, -\frac{3}{\sqrt{23}}, \frac{1}{\sqrt{23}})$

3. **(a)** $\mathbf{u} + \mathbf{v} = (3,-5,7)$; $\|\mathbf{u} + \mathbf{v}\| = \sqrt{3^2 + (-5)^2 + 7^2} = \sqrt{83}$

(b) $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{2^2 + (-2)^2 + 3^2} + \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{17} + \sqrt{26}$

(c) $-2\mathbf{u} + 2\mathbf{v} = (-4,4,-6) + (2,-6,8) = (-2,-2,2)$;

$\|-2\mathbf{u} + 2\mathbf{v}\| = \sqrt{(-2)^2 + (-2)^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$

(d) $3\mathbf{u} - 5\mathbf{v} + \mathbf{w} = (6,-6,9) - (5,-15,20) + (3,6,-4) = (4,15,-15)$;

$\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\| = \sqrt{4^2 + 15^2 + (-15)^2} = \sqrt{466}$

4. **(a)** $\mathbf{u} + \mathbf{v} + \mathbf{w} = (6,1,3)$; $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\| = \sqrt{6^2 + 1^2 + 3^2} = \sqrt{46}$

(b) $\mathbf{u} - \mathbf{v} = (1,1,-1)$; $\|\mathbf{u} - \mathbf{v}\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$

(c) $3\mathbf{v} = (3,-9,12)$;

$\|3\mathbf{v}\| - 3\|\mathbf{v}\| = \sqrt{3^2 + (-9)^2 + 12^2} - 3\sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{234} - 3\sqrt{26} = 0$

(d) $\|\mathbf{u}\| - \|\mathbf{v}\| = \sqrt{2^2 + (-2)^2 + 3^2} - \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{17} - \sqrt{26}$

5. **(a)** $3\mathbf{u} - 5\mathbf{v} + \mathbf{w} = (-6,-3,12,15) - (15,5,-25,35) + (-6,2,1,1) = (-27,-6,38,-19)$;

$\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\| = \sqrt{(-27)^2 + (-6)^2 + 38^2 + (-19)^2} = \sqrt{2570}$

(b) $\|3\mathbf{u}\| - 5\|\mathbf{v}\| + \|\mathbf{w}\|$

$= \sqrt{(-6)^2 + (-3)^2 + 12^2 + 15^2} - 5\sqrt{3^2 + 1^2 + (-5)^2 + 7^2} + \sqrt{(-6)^2 + 2^2 + 1^2 + 1^2}$

$= \sqrt{414} - 5\sqrt{84} + \sqrt{42} = 3\sqrt{46} - 10\sqrt{21} + \sqrt{42}$

(c) $\|\mathbf{u}\| = \sqrt{(-2)^2 + (-1)^2 + 4^2 + 5^2} = \sqrt{46}$;
$\|-\|\mathbf{u}\|\mathbf{v}\| = \|-\sqrt{46}\mathbf{v}\| = \sqrt{46}\sqrt{3^2 + 1^2 + (-5)^2 + 7^2} = \sqrt{46}\sqrt{84} = 2\sqrt{966}$

6. **(a)** $-2\mathbf{v} = (-6, -2, 10, -14)$, $-3\mathbf{w} = (18, -6, -3, -3)$

$\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$
$= \sqrt{(-2)^2 + (-1)^2 + 4^2 + 5^2} + \sqrt{(-6)^2 + (-2)^2 + 10^2 + (-14)^2}$
$+ \sqrt{18^2 + (-6)^2 + (-3)^2 + (-3)^2}$
$= \sqrt{46} + \sqrt{336} + \sqrt{378} = \sqrt{46} + 4\sqrt{21} + 3\sqrt{42}$

(b) $\mathbf{u} - \mathbf{v} = (-5, -2, 9, -2)$, $\|\mathbf{u} - \mathbf{v}\| = \sqrt{(-5)^2 + (-2)^2 + 9^2 + (-2)^2} = \sqrt{114}$

$\|\mathbf{u} - \mathbf{v}\|\mathbf{w} = (-6\sqrt{114}, 2\sqrt{114}, \sqrt{114}, \sqrt{114})$; $\|\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\| = \sqrt{4788} = 6\sqrt{133}$

7. $\|k\mathbf{v}\| = \sqrt{(-2k)^2 + (3k)^2 + 0^2 + (6k)^2} = \sqrt{49k^2} = 7\sqrt{k^2}$; this quantity equals 5 if $k = \frac{5}{7}$ or $k = -\frac{5}{7}$

8. $\|k\mathbf{v}\| = \sqrt{k^2 + k^2 + (2k)^2 + (-3k)^2 + k^2} = \sqrt{16k^2} = 4\sqrt{k^2}$; this quantity equals 4 if $k = 1$ or $k = -1$

9. **(a)** $\mathbf{u} \cdot \mathbf{v} = (3)(2) + (1)(2) + (4)(-4) = -8$
$\mathbf{u} \cdot \mathbf{u} = (3)(3) + (1)(1) + (4)(4) = 26$
$\mathbf{v} \cdot \mathbf{v} = (2)(2) + (2)(2) + (-4)(-4) = 24$

(b) $\mathbf{u} \cdot \mathbf{v} = (1)(2) + (1)(-2) + (4)(3) + (6)(-2) = 0$
$\mathbf{u} \cdot \mathbf{u} = (1)(1) + (1)(1) + (4)(4) + (6)(6) = 54$
$\mathbf{v} \cdot \mathbf{v} = (2)(2) + (-2)(-2) + (3)(3) + (-2)(-2) = 21$

10. **(a)** $\mathbf{u} \cdot \mathbf{v} = (1)(-1) + (1)(0) + (-2)(5) + (3)(1) = -8$
$\mathbf{u} \cdot \mathbf{u} = (1)(1) + (1)(1) + (-2)(-2) + (3)(3) = 15$
$\mathbf{v} \cdot \mathbf{v} = (-1)(-1) + (0)(0) + (5)(5) + (1)(1) = 27$

(b) $\mathbf{u} \cdot \mathbf{v} = (2)(1) + (-1)(2) + (1)(2) + (0)(2) + (-2)(1) = 0$
$\mathbf{u} \cdot \mathbf{u} = (2)(2) + (-1)(-1) + (1)(1) + (0)(0) + (-2)(-2) = 10$
$\mathbf{v} \cdot \mathbf{v} = (1)(1) + (2)(2) + (2)(2) + (2)(2) + (1)(1) = 14$

11. **(a)** $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(3-1)^2 + (3-0)^2 + (3-4)^2} = \sqrt{14}$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(3)(1)+(3)(0)+(3)(4)}{\sqrt{3^2+3^2+3^2}\sqrt{1^2+0^2+4^2}} = \frac{15}{\sqrt{27}\sqrt{17}} = \frac{5}{\sqrt{51}}$; the angle is acute since $\mathbf{u} \cdot \mathbf{v} > 0$

(b) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(0-(-3))^2 + (-2-2)^2 + (-1-4)^2 + (1-4)^2} = \sqrt{59}$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(0)(-3)+(-2)(2)+(-1)(4)+(1)(4)}{\sqrt{0^2+(-2)^2+(-1)^2+1^2}\sqrt{(-3)^2+2^2+4^2+4^2}} = \frac{-4}{\sqrt{6}\sqrt{45}}$; the angle is obtuse since $\mathbf{u} \cdot \mathbf{v} < 0$

12. **(a)** $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(1-5)^2 + (2-1)^2 + (-3-2)^2 + \left(0-(-2)\right)^2} = \sqrt{46}$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(1)(5)+(2)(1)+(-3)(2)+(0)(-2)}{\sqrt{1^2+2^2+(-3)^2+0^2}\sqrt{5^2+1^2+2^2+(-2)^2}} = \frac{1}{\sqrt{14}\sqrt{34}}$; the angle is acute since $\mathbf{u} \cdot \mathbf{v} > 0$

(b) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(0-2)^2+(1-1)^2+(1-0)^2+\left(1-(-1)\right)^2+(2-3)^2} = \sqrt{10}$

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(0)(2)+(1)(1)+(1)(0)+(1)(-1)+(2)(3)}{\sqrt{0^2+1^2+1^2+1^2+2^2}\sqrt{2^2+1^2+0^2+(-1)^2+3^2}} = \frac{6}{\sqrt{7}\sqrt{15}}$; the angle is acute since $\mathbf{u}\cdot\mathbf{v} > 0$

13. The angle between the two vectors is $30°$, so by Formula (1) we have $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos 30° = \frac{45\sqrt{3}}{2}$.

14. $\mathbf{a}\cdot\mathbf{b} = 0$ since the angle between the two vectors is $90°$

15. **(a)** $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ does not make sense; $\mathbf{v}\cdot\mathbf{w}$ is a scalar, whereas the dot product is only defined for vectors

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ makes sense (the result is a scalar)

(c) $\|\mathbf{u}\cdot\mathbf{v}\|$ does not make sense; $\mathbf{u}\cdot\mathbf{v}$ is a scalar, whereas the norm is only defined for vectors

(d) $(\mathbf{u}\cdot\mathbf{v}) - \|\mathbf{u}\|$ makes sense (the result is a scalar)

16. **(a)** $\|\mathbf{u}\|\cdot\|\mathbf{v}\|$ does not make sense: $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are scalars, whereas the dot product is only defined for vectors

(b) $(\mathbf{u}\cdot\mathbf{v}) - \mathbf{w}$ does not make sense: $\mathbf{u}\cdot\mathbf{v}$ is a scalar so the vector $\mathbf{w}$ cannot be subtracted from it

(c) $(\mathbf{u}\cdot\mathbf{v}) - k$ makes sense (the result is a scalar)

(d) $k\cdot\mathbf{u}$ does not make sense: k is a scalar, whereas the dot product is only defined for vectors

17. **(a)** $|\mathbf{u}\cdot\mathbf{v}| = |(-3)(2)+(1)(-1)+(0)(3)| = 7$;
$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{(-3)^2+1^2+0^2}\sqrt{2^2+(-1)^2+3^2} = \sqrt{10}\sqrt{14}$
Since $|\mathbf{u}\cdot\mathbf{v}| = 7 = \sqrt{49} \le \sqrt{140} = \sqrt{10}\sqrt{14} = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

(b) $|\mathbf{u}\cdot\mathbf{v}| = |(0)(1)+(2)(1)+(2)(1)+(1)(1)| = 5$
$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{0^2+2^2+2^2+1^2}\sqrt{1^2+1^2+1^2+1^2} = \sqrt{9}\sqrt{4} = 6$
Since $|\mathbf{u}\cdot\mathbf{v}| = 5 \le 6 = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

18. **(a)** $|\mathbf{u}\cdot\mathbf{v}| = |(4)(1)+(1)(2)+(1)(3)| = 9$; $\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{4^2+1^2+1^2}\sqrt{1^2+2^2+3^2} = \sqrt{18}\sqrt{14}$
Since $|\mathbf{u}\cdot\mathbf{v}| = 9 = \sqrt{81} \le \sqrt{252} = \sqrt{18}\sqrt{14} = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

(b) $|\mathbf{u}\cdot\mathbf{v}| = |(1)(0)+(2)(1)+(1)(1)+(2)(5)+(3)(-2)| = 7$
$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{1^2+2^2+1^2+2^2+3^2}\sqrt{0^2+1^2+1^2+5^2+(-2)^2} = \sqrt{19}\sqrt{31}$
Since $|\mathbf{u}\cdot\mathbf{v}| = 7 = \sqrt{49} \le \sqrt{589} = \sqrt{19}\sqrt{31} = \|\mathbf{u}\|\|\mathbf{v}\|$, the Cauchy-Schwarz inequality holds.

21. We have $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$. Therefore

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \frac{(v_1)(1)+(v_2)(0)+(v_3)(0)}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}$$
$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|} = \frac{(v_1)(0)+(v_2)(1)+(v_3)(0)}{\|\mathbf{v}\|} = \frac{v_2}{\|\mathbf{v}\|}$$
$$\cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|} = \frac{(v_1)(0)+(v_2)(0)+(v_3)(1)}{\|\mathbf{v}\|} = \frac{v_3}{\|\mathbf{v}\|}$$

22. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{v_1}{\|\mathbf{v}\|}\right)^2+\left(\frac{v_2}{\|\mathbf{v}\|}\right)^2+\left(\frac{v_3}{\|\mathbf{v}\|}\right)^2=\frac{v_1^2+v_2^2+v_3^2}{\|\mathbf{v}\|^2}=\frac{v_1^2+v_2^2+v_3^2}{v_1^2+v_2^2+v_3^2}=1$

23. Using the result of Exercise 21, and letting $\mathbf{v}_1=(a_1,b_1,c_1)$ and $\mathbf{v}_2=(a_2,b_2,c_2)$, we can have

$$\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=\frac{a_1}{\|\mathbf{v}_1\|}\frac{a_2}{\|\mathbf{v}_2\|}+\frac{b_1}{\|\mathbf{v}_1\|}\frac{b_2}{\|\mathbf{v}_2\|}+\frac{c_1}{\|\mathbf{v}_1\|}\frac{c_2}{\|\mathbf{v}_2\|}=\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}$$

The left-hand side is zero if and only if the right-hand side is zero; this happens if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero orthogonal vectors.

24. **(a)** We have $\mathbf{d}=(1,1,1)$ and $\mathbf{u}=(1,1,0)$.

$\cos\theta=\frac{\mathbf{d}\cdot\mathbf{u}}{\|\mathbf{d}\|\|\mathbf{u}\|}=\frac{(1)(1)+(1)(1)+(1)(0)}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+0^2}}=\frac{2}{\sqrt{3}\sqrt{2}}=\sqrt{\frac{2}{3}}$ therefore $\theta=\cos^{-1}\sqrt{\frac{2}{3}}\approx 35°$.

(b) The vectors $\mathbf{d}$ and $\mathbf{v}=(-1,0,1)$ form a right angle since

$\cos\varphi=\frac{(1)(-1)+(1)(0)+(1)(1)}{\sqrt{1^2+1^2+1^2}\sqrt{(-1)^2+0^2+1^2}}=0$.

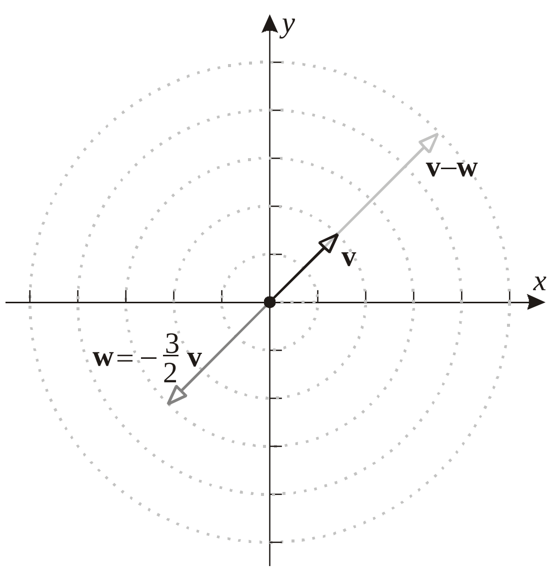
25. Align the edges of the box with the coordinate axes so that the diagonal becomes the vector $\mathbf{v}=(10,15,25)$. The length of this vector is $\|\mathbf{v}\|=\sqrt{10^2+15^2+25^2}=5\sqrt{38}$ therefore

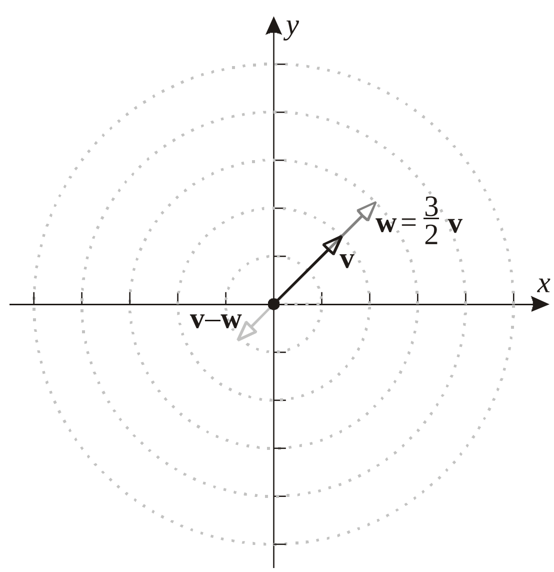
- the angle between $\mathbf{v}$ and the x-axis is $\cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|}\right)=\cos^{-1}\left(\frac{2}{\sqrt{38}}\right)\approx 71°$,
- the angle between $\mathbf{v}$ and the y-axis is $\cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\|\mathbf{j}\|}\right)=\cos^{-1}\left(\frac{3}{\sqrt{38}}\right)\approx 61°$,
- the angle between $\mathbf{v}$ and the z-axis is $\cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\|\mathbf{k}\|}\right)=\cos^{-1}\left(\frac{5}{\sqrt{38}}\right)\approx 36°$.

26. Let us assume both vectors $\mathbf{v}$ and $\mathbf{w}$ have the same number of components (otherwise $\mathbf{v}-\mathbf{w}$ would be undefined).

From Theorem 3.2.5(a), we obtain two inequalities: $\|\mathbf{v}-\mathbf{w}\|\le\|\mathbf{v}\|+\|-\mathbf{w}\|=\|\mathbf{v}\|+\|\mathbf{w}\|=5$.

The norm $\|\mathbf{v}-\mathbf{w}\|$ can actually attain this upper bound if $\mathbf{w}=-\frac{3}{2}\mathbf{v}$ (so that the two vectors have





opposite directions):

$$\|\mathbf{v}-\mathbf{w}\| = \left\|\mathbf{v}-\left(-\frac{3}{2}\mathbf{v}\right)\right\| = \left\|\frac{5}{2}\mathbf{v}\right\| \underbrace{=}_{\substack{\text{Theorem}\\ 3.2.1c}} \left|\frac{5}{2}\right| \|\mathbf{v}\| = 5$$

Applying Theorem 3.2.5(a) to $-\mathbf{w} = (\mathbf{v}-\mathbf{w}) + (-\mathbf{v})$ yields $\|-\mathbf{w}\| \le \|\mathbf{v}-\mathbf{w}\| + \|-\mathbf{v}\|$ thus $\|\mathbf{v}-\mathbf{w}\| \ge \|\mathbf{w}\| - \|\mathbf{v}\| = 1$.

The norm $\|\mathbf{v}-\mathbf{w}\|$ attains this lower bound if $\mathbf{w} = \frac{3}{2}\mathbf{v}$ (so that the two vectors have the same direction):

$$\|\mathbf{v}-\mathbf{w}\| = \left\|\mathbf{v}-\frac{3}{2}\mathbf{v}\right\| = \left\|-\frac{1}{2}\mathbf{v}\right\| \underbrace{=}_{\substack{\text{Theorem}\\ 3.2.1c}} \left|-\frac{1}{2}\right| \|\mathbf{v}\| = 1$$

29. The scalar product of $\frac{m}{\|\mathbf{v}\|}\mathbf{v}$ has the same direction as $\mathbf{v}$ and its length is $\left\|\frac{m}{\|\mathbf{v}\|}\mathbf{v}\right\| = m$.

31. We are looking for the force $\mathbf{F}$ such that $\mathbf{F} + (10\cos 60°, 10\sin 60°) + (-8, 0) = (0, 0)$. This yields $\mathbf{F} = -(5, 5\sqrt{3}) - (-8, 0) = (3, -5\sqrt{3})$. The magnitude of $\mathbf{F}$ is $\sqrt{84}$ lb ≈ 9.17 lb; the vector forms the angle $\approx -70.9°$ with the positive x-axis.

32. We are looking for the force $\mathbf{F}$ such that $\mathbf{F} + (100, 0) + (150\cos 60°, 150\sin 60°) + (120\cos 135°, 120\sin 135°) = (0, 0)$. This yields $\mathbf{F} = -(100, 0) - (75, 75\sqrt{3}) - (-60\sqrt{2}, 60\sqrt{2}) = (-175 + 60\sqrt{2}, -75\sqrt{3} - 60\sqrt{2})$.
The magnitude of $\mathbf{F}$ is ≈ 232.91 lb; the vector forms the angle $\approx -112.8°$ with the positive x-axis.

True-False Exercises

(a) True. By Theorem 3.2.1(b), $\|2\mathbf{v}\| = |2|\|\mathbf{v}\| = 2\|\mathbf{v}\|$.

(b) True.

(c) False. Norm can be zero for the zero vector.

(d) True. The two vectors are $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ and $-\frac{1}{\|\mathbf{v}\|}\mathbf{v}$.

(e) True. This follows from Formula (13).

(f) False. The first expression does not make sense since the scalar $\mathbf{u}\cdot\mathbf{v}$ cannot be added to a vector.

(g) False. For example, let $\mathbf{u} = (1,0)$, $\mathbf{v} = (0,1)$, and $\mathbf{w} = (0,2)$. We have $\mathbf{v} \ne \mathbf{w}$ even though $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$.

(h) False. For example, for $\mathbf{u} = (1,1) \ne (0,0)$ and $\mathbf{v} = (1,-1) \ne (0,0)$ we have $\mathbf{u}\cdot\mathbf{v} = 0$.

(i) True. Cosine of such angle cannot be positive, therefore neither can $\mathbf{u}\cdot\mathbf{v}$.

(j) True. Applying triangle inequality twice, $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\| \le \|\mathbf{u}+\mathbf{v}\| + \|\mathbf{w}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| + \|\mathbf{w}\|$.

3.3 Orthogonality

1. **(a)** $\mathbf{u}\cdot\mathbf{v} = (6)(2) + (1)(0) + (4)(-3) = 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

(b) $\mathbf{u}\cdot\mathbf{v}=(0)(1)+(0)(1)+(-1)(1)=-1\neq 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal vectors

(c) $\mathbf{u}\cdot\mathbf{v}=(3)(-4)+(-2)(1)+(1)(-3)+(3)(7)=4\neq 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal vectors

(d) $\mathbf{u}\cdot\mathbf{v}=(5)(-4)+(-4)(1)+(0)(-3)+(3)(7)=-3\neq 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal vectors

2. **(a)** $\mathbf{u}\cdot\mathbf{v}=(2)(5)+(3)(-7)=-11\neq 0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal vectors

(b) $\mathbf{u}\cdot\mathbf{v}=(1)(0)+(1)(0)+(1)(0)=0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

(c) $\mathbf{u}\cdot\mathbf{v}=(1)(3)+(-5)(3)+(4)(3)=0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

(d) $\mathbf{u}\cdot\mathbf{v}=(4)(-1)+(1)(5)+(-2)(3)+(5)(1)=0$ therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors

3. $-2(x-(-1))+1(y-3)-1(z-(-2))=0$ can be rewritten as $-2(x+1)+(y-3)-(z+2)=0$

4. $1(x-1)+9(y-1)+8(z-4)=0$ can be rewritten as $x-1+9(y-1)+8(z-4)=0$

5. $0(x-2)+0(y-0)+2(z-0)=0$ can be rewritten as $2z=0$

6. $1(x-0)+2(y-0)+3(z-0)=0$ can be rewritten as $x+2y+3z=0$

7. The plane $4x-y+2z=5$ has a normal vector $(4,-1,2)$.
The plane $7x-3y+4z=8$ has a normal vector $(7,-3,4)$.
The two normal vectors are not parallel (neither of them can be expressed as a scalar multiple of the other one) therefore the planes are not parallel either.

8. The plane $x-4y-3z-2=0$ has a normal vector $(1,-4,-3)$.
The plane $3x-12y-9z-7=0$ has a normal vector $(3,-12,-9)$.
The two normal vectors are parallel: $(3,-12,-9)=3(1,-4,-3)$ therefore the planes are parallel as well.

9. Rewriting the first plane equation $2y=8x-4z+5$ as $-8x+2y+4z=5$ yields a normal vector $(-8,2,4)$.
Rewriting the second plane equation $x=\frac{1}{2}z+\frac{1}{4}y$ as $x-\frac{1}{4}y-\frac{1}{2}z=0$ yields a normal vector $\left(1,-\frac{1}{4},-\frac{1}{2}\right)$.
The two normal vectors are parallel: $(-8,2,4)=-8\left(1,-\frac{1}{4},-\frac{1}{2}\right)$ therefore the planes are parallel as well.

10. The normal vectors of the two planes are parallel: $(8,-2,-4)=-2(-4,1,2)$ therefore the planes are parallel as well.

11. Normal vectors of the two planes are not orthogonal:

$$(3,-1,1)\cdot(1,0,2)=(3)(1)+(-1)(0)+(1)(2)=5\neq 0$$

therefore the given planes are not perpendicular.

12. Normal vectors of the two planes are orthogonal:

$$(1,-2,3)\cdot(-2,5,4)=(1)(-2)+(-2)(5)+(3)(4)=0$$

therefore the given planes are perpendicular.

13. **(a)** From Formula (12), $\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(1)(-4)+(-2)(-3)|}{\sqrt{(-4)^2+(-3)^2}} = \frac{2}{\sqrt{25}} = \frac{2}{5}$

(b) From Formula (12) , $\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(3)(2)+(0)(3)+(4)(3)|}{\sqrt{2^2+3^2+3^2}} = \frac{18}{\sqrt{22}}$

14. **(a)** From Formula (12), $\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(5)(2)+(6)(-1)|}{\sqrt{2^2+(-1)^2}} = \frac{4}{\sqrt{5}}$

(b) From Formula (12), $\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(3)(1)+(-2)(2)+(6)(-7)|}{\sqrt{1^2+2^2+(-7)^2}} = \frac{43}{\sqrt{54}} = \frac{43}{3\sqrt{6}}$

15. $\mathbf{u}\cdot\mathbf{a} = (6)(3) + (2)(-9) = 0$, $\|\mathbf{a}\|^2 = (3)^2 + (-9)^2 = 90$,
the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{0}{90}(3,-9) = (\,0,0\,)$,
the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (6,2) - (\,0,0\,) = (6,2)$

16. $\mathbf{u}\cdot\mathbf{a} = (-1)(-2) + (-2)(3) = -4$, $\|\mathbf{a}\|^2 = (-2)^2 + 3^2 = 13$,
the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = -\frac{4}{13}(-2,3) = (\frac{8}{13},-\frac{12}{13})$,
the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (-1,-2) - \left(\frac{8}{13},-\frac{12}{13}\right) = (-\frac{21}{13},-\frac{14}{13})$

17. $\mathbf{u}\cdot\mathbf{a} = (3)(1) + (1)(0) + (-7)(5) = -32$, $\|\mathbf{a}\|^2 = 1^2 + 0^2 + 5^2 = 26$,
the vector component of $\mathbf{u}$ along $\mathbf{a}$ is
$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{-32}{26}(1,0,5) = \left(-\frac{32}{26},0,-\frac{160}{26}\right) = (-\frac{16}{13},0,-\frac{80}{13})$,
the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is
$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (3,1,-7) - (-\frac{16}{13},\ 0,-\frac{80}{13}) = (\frac{55}{13},\ 1,-\frac{11}{13})$

18. $\mathbf{u}\cdot\mathbf{a} = (2)(1) + (0)(2) + (1)(3) = 5$, $\|\mathbf{a}\|^2 = 1^2 + 2^2 + 3^2 = 14$,
the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{5}{14}(1,2,3) = (\frac{5}{14},\frac{5}{7},\frac{15}{14})$,
the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2,0,1) - (\frac{5}{14},\frac{5}{7},\frac{15}{14}) = (\frac{23}{14},-\frac{5}{7},-\frac{1}{14})$

19. $\mathbf{u}\cdot\mathbf{a} = (2)(4) + (1)(-4) + (1)(2) + (2)(-2) = 2$, $\|\mathbf{a}\|^2 = 4^2 + (-4)^2 + 2^2 + (-2)^2 = 40$,
the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{2}{40}(4,-4,2,-2) = (\frac{1}{5},-\frac{1}{5},\frac{1}{10},-\frac{1}{10})$,
the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is
$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2,1,1,2) - (\frac{1}{5},-\frac{1}{5},\frac{1}{10},-\frac{1}{10}) = (\frac{9}{5},\frac{6}{5},\frac{9}{10},\frac{21}{10})$

20. $\mathbf{u}\cdot\mathbf{a} = (5)(2) + (0)(1) + (-3)(-1) + (7)(-1) = 6$, $\|\mathbf{a}\|^2 = 2^2 + 1^2 + (-1)^2 + (-1)^2 = 7$,
the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{6}{7}(2,1,-1,-1) = (\frac{12}{7},\frac{6}{7},-\frac{6}{7},-\frac{6}{7})$,
the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is
$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (5,0,-3,7) - (\frac{12}{7},\frac{6}{7},-\frac{6}{7},-\frac{6}{7}) = (\frac{23}{7},-\frac{6}{7},-\frac{15}{7},\frac{55}{7})$

21. From Theorem 3.3.4(a) the distance between the point and the line is $D = \frac{|(4)(-3)+(3)(1)+4|}{\sqrt{4^2+3^2}} = \frac{5}{\sqrt{25}} = 1$

22. From Theorem 3.3.4(a) the distance between the point and the line is $D = \frac{|(1)(-1)+(-3)(4)+2|}{\sqrt{1^2+(-3)^2}} = \frac{11}{\sqrt{10}}$

23. From Theorem 3.3.4(a) the distance between the point and the line is $D = \frac{|(4)(2)+(1)(-5)-2|}{\sqrt{4^2+1^2}} = \frac{1}{\sqrt{17}}$ (the equation of the line had to be rewritten in the form $ax + by + c = 0$ as $4x + y - 2 = 0$)

24. From Theorem 3.3.4(a) the distance between the point and the line is $D = \frac{|(3)(1)+(1)(8)-5|}{\sqrt{3^2+1^2}} = \frac{6}{\sqrt{10}}$ (the equation of the line had to be rewritten in the form $ax + by + c = 0$ as $3x + y - 5 = 0$)

25. From Theorem 3.3.4(b) the distance between the point and the plane is $D = \frac{|(1)(3)+(2)(1)+(-2)(-2)-4|}{\sqrt{1^2+2^2+(-2)^2}} = \frac{5}{\sqrt{9}} = \frac{5}{3}$ (the equation of the plane had to be rewritten in the form $ax + by + cz + d = 0$ as $x + 2y - 2z - 4 = 0$)

26. From Theorem 3.3.4(b) the distance between the point and the plane is $D = \frac{|(2)(-1)+(5)(-1)+(-6)(2)-4|}{\sqrt{2^2+5^2+(-6)^2}} = \frac{23}{\sqrt{65}}$ (the equation of the plane had to be rewritten in the form $ax + by + cz + d = 0$ as $2x + 5y - 6z - 4 = 0$)

27. First, select an arbitrary point in the plane $2x - y - z = 5$ by setting $x = y = 0$; we obtain $P_0(0,0,-5)$. From Theorem 3.3.4(b) the distance between P_0 and the plane $-4x + 2y + 2z - 12 = 0$ is $D = \frac{|(-4)(0)+(2)(0)+(2)(-5)-12|}{\sqrt{(-4)^2+2^2+2^2}} = \frac{22}{\sqrt{24}} = \frac{11}{\sqrt{6}}$

28. First, select an arbitrary point in the plane $2x - y + z = 1$ by setting $x = y = 0$; we obtain $P_0(0,0,1)$. From Theorem 3.3.4(b) the distance between P_0 and the plane $2x - y + z + 1 = 0$ is $D = \frac{|(2)(0)+(-1)(0)+(1)(1)+1|}{\sqrt{2^2+(-1)^2+1^2}} = \frac{2}{\sqrt{6}}$

29. In order for $\mathbf{w} = (a, b, c)$ to be orthogonal to both $(1,0,1)$ and $(0,1,1)$, we must have $a + c = 0$ and $b + c = 0$. These equations form a linear system whose augmented matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ is already in reduced row echelon form. For arbitrary real number t, the solutions are $a = -t, b = -t, c = t$. Since $\mathbf{w}$ is also required to be a unit vector, we must have $\|\mathbf{w}\| = \sqrt{(-t)^2 + (-t)^2 + t^2} = \sqrt{3t^2} = 1$. This yields $t = \pm\frac{1}{\sqrt{3}}$, consequently there are two possible vectors that satisfy the given conditions: $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

30. **(a)** $\mathbf{v} \cdot \mathbf{w} = (a)(-b) + (b)(a) = 0$ therefore $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors

(b) $(3,2)$ and $(-3,-2)$

(c) $(4,3)$ and $(4,3)$

31. $\overrightarrow{AB} = (-2 - 1, 0 - 1, 3 - 1) = (-3, -1, 2)$, $\overrightarrow{AC} = (-3 - 1, -1 - 1, 1 - 1) = (-4, -2, 0)$,
$\overrightarrow{BC} = (-3 - (-2), -1 - 0, 1 - 3) = (-1, -1, -2)$
$\overrightarrow{AB} \cdot \overrightarrow{BC} = (-3)(-1) + (-1)(-1) + (2)(-2) = 0$
therefore the points A, B, and C form the vertices of a right triangle

32. $\overrightarrow{AB} = (4 - 3, 3 - 0, 0 - 2) = (1, 3, -2)$, $\overrightarrow{AC} = (8 - 3, 1 - 0, -1 - 2) = (5, 1, -3)$,
$\overrightarrow{BC} = (8 - 4, 1 - 3, -1 - 0) = (4, -2, -1)$
$\overrightarrow{AB} \cdot \overrightarrow{BC} = (1)(4) + (3)(-2) + (-2)(-1) = 0$
therefore the points A, B, and C form the vertices of a right triangle

33. Assuming $\mathbf{v}\cdot\mathbf{w}_1 = \mathbf{v}\cdot\mathbf{w}_2 = 0$ and using Theorem 3.2.2, we have
$\mathbf{v}\cdot(k_1\mathbf{w}_1 + k_2\mathbf{w}_2) = \mathbf{v}\cdot(k_1\mathbf{w}_1) + \mathbf{v}\cdot(k_2\mathbf{w}_2) = k_1(\mathbf{v}\cdot\mathbf{w}_1) + k_2(\mathbf{v}\cdot\mathbf{w}_2) = (k_1)(0) + (k_2)(0) = 0$.

34. Yes.
One possible scenario is when $\mathbf{u} = \mathbf{a}$ - in this case, $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a} = \text{proj}_{\mathbf{u}}\mathbf{u} = \mathbf{u}$.
Another possibility is to take $\mathbf{u}$ and $\mathbf{a}$ to be orthogonal vectors, so that $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a} = \mathbf{0}$.

38. $W = \|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos\theta = (10)(50)\cos\frac{\pi}{3} = 125$ ft-lb

39. $W = \|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos\theta = (500)(100)\cos\frac{\pi}{4} = \frac{50{,}000}{\sqrt{2}} \approx 35{,}355$ Nm.

True-False Exercises

(a) True. $(3,-1,2)\cdot(0,0,0) = 0$.

(b) True. By Theorem 3.2.2(c) and Theorem 3.2.3(e), $(k\mathbf{u})\cdot(m\mathbf{v}) = (km)(\mathbf{u}\cdot\mathbf{v}) = (km)(0) = 0$.

(c) True. This follows from Theorem 3.3.2.

(d) True. $\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{b}}(\mathbf{u})) = \frac{\left(\frac{\mathbf{u}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}\right)\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\frac{\mathbf{u}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}(\mathbf{b}\cdot\mathbf{a})}{\|\mathbf{a}\|^2}\mathbf{a} = 0\mathbf{a} = \mathbf{0}$
($\text{proj}_{\mathbf{b}}(\mathbf{u})$ has the same direction as $\mathbf{b}$, so it is also orthogonal to $\mathbf{a}$).

(e) True. $\text{proj}_{\mathbf{a}}(\text{proj}_{\mathbf{a}}(\mathbf{u})) = \frac{\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\|\mathbf{a}\|^2}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \text{proj}_{\mathbf{a}}(\mathbf{u})$
($\text{proj}_{\mathbf{a}}(\mathbf{u}) = k\mathbf{a}$ for some scalar k and then $\text{proj}_{\mathbf{a}}(k\mathbf{a}) = k\mathbf{a}$).

(f) False. For instance, let $\mathbf{u}$ be a nonzero vector orthogonal to $\mathbf{a}$. Then
$\text{proj}_{\mathbf{a}}(\mathbf{u}) = \text{proj}_{\mathbf{a}}(2\mathbf{u}) = \mathbf{0}$ even though $\mathbf{u} \neq 2\mathbf{u}$.

(g) False. By Theorem 3.2.5(a), $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. This becomes an equality only when $\mathbf{u}$ and $\mathbf{v}$ are collinear vectors in the same direction. (For instance, $\|(1,0)+(0,1)\| = \|(1,1)\| = \sqrt{2}$ does not equal $\|(1,0)\| + \|(0,1)\| = 1 + 1 = 2$.)

3.4 The Geometry of Linear Systems

1. The vector equation in Formula (5) can be expressed as $(x,y) = (-4,1) + t(0,-8)$.
This yields the parametric equations $x = -4,\ y = 1 - 8t$.

2. The vector equation in Formula (5) can be expressed as $(x,y) = (2,-1) + t(-4,-2)$.
This yields the parametric equations $x = 2 - 4t,\ y = -1 - 2t$.

3. The vector equation in Formula (5) can be expressed as $(x,y,z) = t(-3,0,1)$.
This yields the parametric equations $x = -3t,\ y = 0,\ z = t$.

4. The vector equation in Formula (5) can be expressed as $(x,y,z) = (-9,3,4) + t(-1,6,0)$.
This yields the parametric equations $x = -9 - t,\ y = 3 + 6t,\ z = 4$.

5. A point on the line: $(3,-6)$; a vector parallel to the line: $(-5,-1)$.

6. A point on the line: $(0,7,4)$; a vector parallel to the line: $(4,0,3)$.

7. Rewriting the vector equation as $(x,y) = (4-6t, 6-6t)$ yields a point on the line: $(4,6)$ and a vector parallel to the line: $(-6,-6)$.

8. A point on the line: $(0,-5,1)$; a vector parallel to the line: $(0,5,-1)$.

9. The vector equation in Formula (6) can be expressed as
$(x,y,z) = (-3,1,0) + t_1(0,-3,6) + t_2(-5,1,2)$.
This yields the parametric equations $x = -3 - 5t_2$, $y = 1 - 3t_1 + t_2$, $z = 6t_1 + 2t_2$.

10. The vector equation in Formula (6) can be expressed as
$(x,y,z) = (0,6,-2) + t_1(0,9,-1) + t_2(0,-3,0)$.
This yields the parametric equations $x = 0$, $y = 6 + 9t_1 - 3t_2$, $z = -2 - t_1$.

11. The vector equation in Formula (6) can be expressed as
$(x,y,z) = (-1,1,4) + t_1(6,-1,0) + t_2(-1,3,1)$.
This yields the parametric equations $x = -1 + 6t_1 - t_2$, $y = 1 - t_1 + 3t_2$, $z = 4 + t_2$.

12. The vector equation in Formula (6) can be expressed as
$(x,y,z) = (0,5,-4) + t_1(0,0,-5) + t_2(1,-3,-2)$.
This yields the parametric equations $x = t_2$, $y = 5 - 3t_2$, $z = -4 - 5t_1 - 2t_2$.

13. We find a nonzero vector orthogonal to $\mathbf{v}$, e.g., $(3,2)$. The vector equation of the line passing through $(0,0)$ and parallel to $(3,2)$ can be expressed as $(x,y) = t(3,2)$. Parametric equations are $x = 3t$ and $y = 2t$.

14. We find a nonzero vector orthogonal to $\mathbf{v}$, e.g., $(4,1)$. The vector equation of the line passing through $(0,0)$ and parallel to $(4,1)$ can be expressed as $(x,y) = t(4,1)$. Parametric equations are $x = 4t$ and $y = t$.

15. We find two nonparallel nonzero vectors orthogonal to $\mathbf{v}$, e.g., $(5,0,4)$ and $(0,1,0)$. The vector equation of the plane that contains the origin and these two vectors can be expressed as
$(x,y,z) = t_1(5,0,4) + t_2(0,1,0)$. Parametric equations are $x = 5t_1$, $y = t_2$, and $z = 4t_1$.

16. We find two nonparallel nonzero vectors orthogonal to $\mathbf{v}$, e.g., $(-1,3,0)$ and $(0,6,1)$. The vector equation of the plane that contains the origin and these two vectors can be expressed as
$(x,y,z) = t_1(-1,3,0) + t_2(0,6,1)$. Parametric equations are $x = -t_1$, $y = 3t_1 + 6t_2$, and $z = t_2$.

17. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system, $x_1 = -s - t$, $x_2 = s$, $x_3 = t$ expressed in vector form as $\mathbf{x} = (-s-t, s, t)$ is orthogonal to the rows of the coefficient matrix of the original system $\mathbf{r}_1 = (1,1,1)$, $\mathbf{r}_2 = (2,2,2)$, and $\mathbf{r}_3 = (3,3,3)$ since $\mathbf{r}_1 \cdot \mathbf{x} = (1)(-s-t) + (1)(s) + (1)(t) = 0$, $\mathbf{r}_2 \cdot \mathbf{x} = (2)(-s-t) + (2)(s) + (2)(t) = 0$, and $\mathbf{r}_3 \cdot \mathbf{x} = (3)(-s-t) + (3)(s) + (3)(t) = 0$.

18. The augmented matrix of the linear system $\begin{bmatrix} 1 & 3 & -4 & 0 \\ 2 & 6 & -8 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system, $x_1 = -3s + 4t$, $x_2 = s$, $x_3 = t$ expressed in

vector form as $\mathbf{x} = (-3s + 4t, s, t)$ is orthogonal to the rows of the coefficient matrix of the original system $\mathbf{r}_1 = (1,3,-4)$ and $\mathbf{r}_2 = (2,6,-8)$ since
$\mathbf{r}_1 \cdot \mathbf{x} = (1)(-3s + 4t) + (3)(s) + (-4)(t) = 0$ and $\mathbf{r}_2 \cdot \mathbf{x} = (2)(-3s + 4t) + (6)(s) + (-8)(t) = 0$.

19. The augmented matrix of the linear system $\begin{bmatrix} 1 & 5 & 1 & 2 & -1 & 0 \\ 1 & -2 & -1 & 3 & 2 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{3}{7} & \frac{19}{7} & \frac{8}{7} & 0 \\ 0 & 1 & \frac{2}{7} & -\frac{1}{7} & -\frac{3}{7} & 0 \end{bmatrix}$. A general solution of the system,
$x_1 = \frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t$, $x_2 = -\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t$, $x_3 = r, x_4 = s, x_5 = t$ expressed in vector form as $\mathbf{x} = \left(\frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t, -\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t, r, s, t\right)$ is orthogonal to the rows of the coefficient matrix of the original system $\mathbf{r}_1 = (1,5,1,2,-1)$ and $\mathbf{r}_2 = (1,-2,-1,3,2)$ since
$\mathbf{r}_1 \cdot \mathbf{x} = (1)\left(\frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t\right) + (5)\left(-\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t\right) + (1)(r) + (2)(s) + (-1)(t) = 0$ and
$\mathbf{r}_2 \cdot \mathbf{x} = (1)\left(\frac{3}{7}r - \frac{19}{7}s - \frac{8}{7}t\right) + (-2)\left(-\frac{2}{7}r + \frac{1}{7}s + \frac{3}{7}t\right) + (-1)(r) + (3)(s) + (2)(t) = 0$.

20. The augmented matrix of the linear system $\begin{bmatrix} 1 & 3 & -4 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 17 & 0 \\ 0 & 1 & -7 & 0 \end{bmatrix}$. A general solution of the system, $x_1 = -17t$, $x_2 = 7t$, $x_3 = t$ expressed in vector form as $\mathbf{x} = (-17t, 7t, t)$ is orthogonal to the rows of the coefficient matrix of the original system $\mathbf{r}_1 = (1,3,-4)$ and $\mathbf{r}_2 = (1,2,3)$ since

$\mathbf{r}_1 \cdot \mathbf{x} = (1)(-17t) + (3)(7t) + (-4)(t) = 0$ and $\mathbf{r}_2 \cdot \mathbf{x} = (1)(-17t) + (2)(7t) + (3)(t) = 0$.

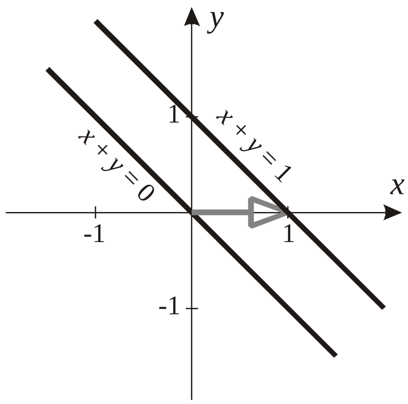
21. **(a)** The associated homogeneous system $x + y + z = 0$ has a general solution $x = -s - t,\ y = s,\ z = t$. The original nonhomogeneous system has a general solution $x = 1 - s - t,\ y = s,\ z = t$, which can be expressed in vector form as

$$(x, y, z) = (1 - s - t, s, t) = \underbrace{(1,0,0)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{(-s - t, s, t)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(b) Geometrically, the points (x, y, z) corresponding to solutions of $x + y + z = 1$ form a plane passing through the point (1,0,0) and parallel to the vectors (−1,1,0) and (−1,0,1).

22. **(a)** The associated homogeneous system $x + y = 0$ has a general solution $x = -t,\ y = t$.
The original nonhomogeneous system has a general solution $x = 1 - t,\ y = t$, which can be expressed in vector form as

$$(x, y) = (1 - t, t) = \underbrace{(1,0)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{(-t, t)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(b) 

Geometrically, the points (x, y, z) corresponding to solutions of $x + y = 1$ form a line passing through the point $(1,0,)$ and parallel to the vector $(-1,1)$.

23. **(a)** Theorem 3.4.3 yields the following homogeneous linear system that satisfies our requirements:

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ -2x & + & 3y & & & = & 0 \end{array}$$

(b) A straight line passing through the origin – this line is parallel to any vector that is orthogonal to both **a** and **b**.

(c) The augmented matrix of the system obtained in part (a) has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{2}{5} & 0 \end{bmatrix}$. A general solution of the system is $x = -\frac{3}{5}t,\ y = -\frac{2}{5}t,\ z = t$. It can also be expressed in vector form as $\mathbf{u} = (x, y, z) = (-\frac{3}{5}t, -\frac{2}{5}t, t)$. To confirm that Theorem 3.4.3 holds, we verify that $\mathbf{u}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:
$\mathbf{u} \cdot \mathbf{a} = \left(-\frac{3}{5}t\right)(1) + \left(-\frac{2}{5}t\right)(1) + (t)(1) = 0,\ \mathbf{u} \cdot \mathbf{b} = \left(-\frac{3}{5}t\right)(-2) + \left(-\frac{2}{5}t\right)(3) + (t)(0) = 0.$

24. **(a)** Theorem 3.4.3 yields the following homogeneous linear system that satisfies our requirements:

$$\begin{array}{rcrcrcl} -3x & + & 2y & - & z & = & 0 \\ & - & 2y & - & 2z & = & 0 \end{array}$$

(b) A straight line passing through the origin – this line is parallel to any vector that is orthogonal to both **a** and **b**.

(c) The augmented matrix of the system obtained in part (a) has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. A general solution of the system is $x = -t,\ y = -t,\ z = t$. It can also be expressed in vector form as $\mathbf{u} = (x, y, z) = (-t, -t, t)$. To confirm that Theorem 3.4.3 holds, we verify that $\mathbf{u}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{u} \cdot \mathbf{a} = (-t)(-3) + (-t)(2) + (t)(-1) = 0,\ \mathbf{u} \cdot \mathbf{b} = (-t)(0) + (-t)(-2) + (t)(-2) = 0.$$

25. **(a)** The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -\frac{2}{3}s + \frac{1}{3}t,\ x_2 = s,\ x_3 = t$.

(b) Multiplying $\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ therefore $x_1 = 1,\ x_2 = 0,\ x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1, x_2, x_3) = \underbrace{(1,0,1)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{(-\tfrac{2}{3}s + \tfrac{1}{3}t, s, t)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = \frac{2}{3} - \frac{2}{3}p + \frac{1}{3}q,\ x_2 = p,\ x_3 = q$.

If we let $p = s$ and $q = t + 1$ then this agrees with the solution we obtained in part (c).

26. **(a)** The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{11}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -\frac{11}{5}t,\ x_2 = \frac{2}{5}t,\ x_3 = t$.

(b) Multiplying $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$ therefore $x_1 = x_2 = x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1, x_2, x_3) = \underbrace{(1,1,1)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{(-\tfrac{11}{5}t, \tfrac{2}{5}t, t)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{11}{5} & \frac{16}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = \frac{16}{5} - \frac{11}{5}s,\ x_2 = \frac{3}{5} + \frac{2}{5}s,\ x_3 = s$.

If we let $s = 1 + t$ then this agrees with the solution we obtained in part (c).

27. The augmented matrix of the nonhomogeneous system $\begin{bmatrix} 3 & 4 & 1 & 2 & 3 \\ 6 & 8 & 2 & 5 & 7 \\ 9 & 12 & 3 & 10 & 13 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system

$x_1 = \frac{1}{3} - \frac{4}{3}r - \frac{1}{3}s, \quad x_2 = r, \quad x_3 = s, \quad x_4 = 1$ can be expressed in vector form as

$$(x_1, x_2, x_3, x_4) = \underbrace{\left(\tfrac{1}{3}, 0, 0, 1\right)}_{\substack{\text{particular}\\\text{solution}\\\text{of the}\\\text{nonhomogeneous}\\\text{system}}} + \underbrace{\left(-\tfrac{4}{3}r - \tfrac{1}{3}s,\ r,\ s,\ 0\right)}_{\substack{\text{general}\\\text{solution}\\\text{of the}\\\text{associated}\\\text{homogeneous}\\\text{system}}}$$

28. The augmented matrix of the nonhomogeneous system $\begin{bmatrix} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{13}{3} & \frac{13}{3} \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system

$x_1 = \frac{13}{3} + \frac{1}{3}s + \frac{13}{3}t, \quad x_2 = s, \quad x_3 = -7 - 9t, \quad x_4 = t$

can be expressed in vector form as

$$(x_1, x_2, x_3, x_4) = \underbrace{\left(\tfrac{13}{3}, 0, -7, 0\right)}_{\substack{\text{particular}\\\text{solution}\\\text{of the}\\\text{nonhomogeneous}\\\text{system}}} + \underbrace{\left(\tfrac{1}{3}s + \tfrac{13}{3}t, s, -9t, t\right)}_{\substack{\text{general}\\\text{solution}\\\text{of the}\\\text{associated}\\\text{homogeneous}\\\text{system}}}$$

29. By Theorem 1.8.2, we can write $T(\mathbf{x}_0 + t\mathbf{v}) = T(\mathbf{x}_0) + T(t\mathbf{v}) = T(\mathbf{x}_0) + tT(\mathbf{v})$.
If $T(\mathbf{v}) = \mathbf{0}$ then the image of the entire line is a single point $T(\mathbf{x}_0)$.
Otherwise, the image is a line through $T(\mathbf{x}_0)$ that is parallel to $T(\mathbf{v})$.

True-False Exercises

(a) True. This follows from Definition 1.

(b) False. We need *two* vectors parallel to the plane that are not collinear.

(c) True. This follows from Theorem 3.4.1.

(d) True.
If $\mathbf{b} = \mathbf{0}$ then by Theorem 3.4.3, all solution vectors of $A\mathbf{x} = \mathbf{b}$ are orthogonal to the row vectors of A.
If all solution vectors of $A\mathbf{x} = \mathbf{b}$ are orthogonal to the row vectors of A, $\mathbf{r}_1, \ldots, \mathbf{r}_m$ then the ith component of the product $A\mathbf{x}$ is $\mathbf{r}_i \cdot \mathbf{x} = 0$, so we must have $\mathbf{b} = \mathbf{0}$.

(e) False. By Theorem 3.4.4, the general solution of $A\mathbf{x} = \mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x} = \mathbf{b}$ to the general solution of $A\mathbf{x} = \mathbf{0}$.

(f) True. Subtracting $A\mathbf{x}_1 = \mathbf{b}$ from $A\mathbf{x}_2 = \mathbf{b}$ yields $A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b}$, i.e., $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$.

3.5 Cross Product

1. **(a)** $\mathbf{v}\times\mathbf{w} = \left(\begin{vmatrix}2 & -3\\6 & 7\end{vmatrix}, -\begin{vmatrix}0 & -3\\2 & 7\end{vmatrix}, \begin{vmatrix}0 & 2\\2 & 6\end{vmatrix}\right) = (32,-6,-4)$

(b) $\mathbf{w}\times\mathbf{v} = \left(\begin{vmatrix}6 & 7\\2 & -3\end{vmatrix}, -\begin{vmatrix}2 & 7\\0 & -3\end{vmatrix}, \begin{vmatrix}2 & 6\\0 & 2\end{vmatrix}\right) = (-32,6,4)$

(c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = (3,4,-4)\times(2,6,7) = \left(\begin{vmatrix}4 & -4\\6 & 7\end{vmatrix}, -\begin{vmatrix}3 & -4\\2 & 7\end{vmatrix}, \begin{vmatrix}3 & 4\\2 & 6\end{vmatrix}\right) = (52,-29,10)$

(d) Using the result of part (a),
$\mathbf{v}\cdot(\mathbf{v}\times\mathbf{w}) = (0,2,-3)\cdot(32,-6,-4) = (0)(32)+(2)(-6)+(-3)(-4) = 0$

(e) $\mathbf{v}\times\mathbf{v} = \left(\begin{vmatrix}2 & -3\\2 & -3\end{vmatrix}, -\begin{vmatrix}0 & -3\\0 & -3\end{vmatrix}, \begin{vmatrix}0 & 2\\0 & 2\end{vmatrix}\right) = (0,0,0)$

(f) $(\mathbf{u}-3\mathbf{w})\times(\mathbf{u}-3\mathbf{w}) = (-3,-16,-22)\times(-3,-16,-22)$
$= \left(\begin{vmatrix}-16 & -22\\-16 & -22\end{vmatrix}, -\begin{vmatrix}-3 & -22\\-3 & -22\end{vmatrix}, \begin{vmatrix}-3 & -16\\-3 & -16\end{vmatrix}\right) = (0,0,0)$

2. **(a)** $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}2 & -1\\2 & -3\end{vmatrix}, -\begin{vmatrix}3 & -1\\0 & -3\end{vmatrix}, \begin{vmatrix}3 & 2\\0 & 2\end{vmatrix}\right) = (-4,9,6)$

(b) Using the result of part (a), $-(\mathbf{u}\times\mathbf{v}) = -(-4,9,6) = (4,-9,-6)$

(c) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (3,2,-1)\times(2,8,4) = \left(\begin{vmatrix}2 & -1\\8 & 4\end{vmatrix}, -\begin{vmatrix}3 & -1\\2 & 4\end{vmatrix}, \begin{vmatrix}3 & 2\\2 & 8\end{vmatrix}\right) = (16,-14,20)$

(d) Using the result of Exercise 1(b),
$\mathbf{w}\cdot(\mathbf{w}\times\mathbf{v}) = (2,6,7)\cdot(-32,6,4) = (2)(-32)+(6)(6)+(7)(4) = 0$

(e) $\mathbf{w}\times\mathbf{w} = \left(\begin{vmatrix}6 & 7\\6 & 7\end{vmatrix}, -\begin{vmatrix}2 & 7\\2 & 7\end{vmatrix}, \begin{vmatrix}2 & 6\\2 & 6\end{vmatrix}\right) = (0,0,0)$

(f) $(7\mathbf{v}-3\mathbf{u})\times(7\mathbf{v}-3\mathbf{u}) = (-9,8,-18)\times(-9,8,-18) =$
$\left(\begin{vmatrix}8 & -18\\8 & -18\end{vmatrix}, -\begin{vmatrix}-9 & -18\\-9 & -18\end{vmatrix}, \begin{vmatrix}-9 & 8\\-9 & 8\end{vmatrix}\right) = (0,0,0)$

3. By Lagrange's identity (Theorem 3.5.1(c)) and Formula (18) in Section 3.2, we have
$\|\mathbf{u}\times\mathbf{w}\|^2 = \|\mathbf{u}\|^2\|\mathbf{w}\|^2 - (\mathbf{u}\cdot\mathbf{w})^2 = (\mathbf{u}\cdot\mathbf{u})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{u}\cdot\mathbf{w})^2$.

$\mathbf{u}\times\mathbf{w} = \left(\begin{vmatrix}2 & -1\\6 & 7\end{vmatrix}, -\begin{vmatrix}3 & -1\\2 & 7\end{vmatrix}, \begin{vmatrix}3 & 2\\2 & 6\end{vmatrix}\right) = (20,-23,14)$

$\|\mathbf{u}\times\mathbf{w}\|^2 = \left(\sqrt{20^2+(-23)^2+14^2}\right)^2 = 1125$

$(\mathbf{u}\cdot\mathbf{u})(\mathbf{w}\cdot\mathbf{w}) - (\mathbf{u}\cdot\mathbf{w})^2 = (3^2+2^2+(-1)^2)(2^2+6^2+7^2) - [(3)(2)+(2)(6)+(-1)(7)]^2$
$= (14)(89) - 11^2 = 1246 - 121 = 1125$

4. By Lagrange's identity (Theorem 3.5.1(c)) and Formula (18) in Section 3.2, we have
$\|\mathbf{v}\times\mathbf{u}\|^2 = \|\mathbf{v}\|^2\|\mathbf{u}\|^2 - (\mathbf{v}\cdot\mathbf{u})^2 = (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{v}\cdot\mathbf{u})^2$.

$\mathbf{v}\times\mathbf{u} = \left(\begin{vmatrix}2 & -3\\2 & -1\end{vmatrix}, -\begin{vmatrix}0 & -3\\3 & -1\end{vmatrix}, \begin{vmatrix}0 & 2\\3 & 2\end{vmatrix}\right) = (4,-9,-6)$

$\|\mathbf{v}\times\mathbf{u}\|^2 = \left(\sqrt{4^2+(-9)^2+(-6)^2}\right)^2 = 133$

$(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{v}\cdot\mathbf{u})^2 = (0^2+2^2+(-3)^2)(3^2+2^2+(-1)^2) - [(0)(3)+(2)(2)+(-3)(-1)]^2$
$= (13)(14) - 7^2 = 182 - 49 = 133$

5. $\mathbf{v}\times\mathbf{w} = \left(\begin{vmatrix}2 & -3\\6 & 7\end{vmatrix}, -\begin{vmatrix}0 & -3\\2 & 7\end{vmatrix}, \begin{vmatrix}0 & 2\\2 & 6\end{vmatrix}\right) = (32,-6,-4)$

$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \left(\begin{vmatrix}2 & -1\\-6 & -4\end{vmatrix}, -\begin{vmatrix}3 & -1\\32 & -4\end{vmatrix}, \begin{vmatrix}3 & 2\\32 & -6\end{vmatrix}\right) = (-14,-20,-82)$

By Theorem 3.5.1(d),
$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$
$= [(3)(2)+(2)(6)+(-1)(7)](0,2,-3) - [(3)(0)+(2)(2)+(-1)(-3)](2,6,7)$
$= 11(0,2,-3) - 7(2,6,7) = (0,22,-33) - (14,42,49) = (-14,-20,-82)$

6. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}2 & -1\\2 & -3\end{vmatrix}, -\begin{vmatrix}3 & -1\\0 & -3\end{vmatrix}, \begin{vmatrix}3 & 2\\0 & 2\end{vmatrix}\right) = (-4,9,6)$

$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \left(\begin{vmatrix}9 & 6\\6 & 7\end{vmatrix}, -\begin{vmatrix}-4 & 6\\2 & 7\end{vmatrix}, \begin{vmatrix}-4 & 9\\2 & 6\end{vmatrix}\right) = (27,40,-42)$

By Theorem 3.5.1(e),
$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$
$= [(3)(2)+(2)(6)+(-1)(7)](0,2,-3) - [(0)(2)+(2)(6)+(-3)(7)](3,2,-1)$
$= 11(0,2,-3) - (-9)(3,2,-1) = (0,22,-33) + (27,18,-9) = (27,40,-42)$

7. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}4 & 2\\1 & 5\end{vmatrix}, -\begin{vmatrix}-6 & 2\\3 & 5\end{vmatrix}, \begin{vmatrix}-6 & 4\\3 & 1\end{vmatrix}\right) = (18,36,-18)$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

8. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}1 & -2\\-1 & 2\end{vmatrix}, -\begin{vmatrix}1 & -2\\2 & 2\end{vmatrix}, \begin{vmatrix}1 & 1\\2 & -1\end{vmatrix}\right) = (0,-6,-3)$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

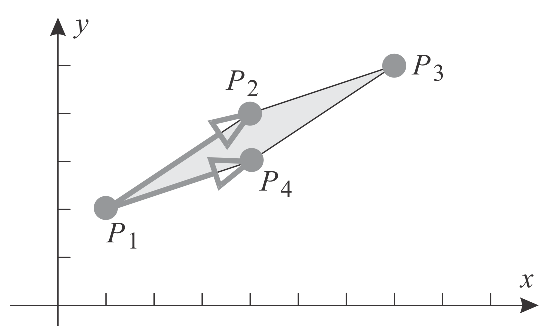
9. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}-1 & 2\\3 & 1\end{vmatrix}, -\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix}, \begin{vmatrix}1 & -1\\0 & 3\end{vmatrix}\right) = (-7,-1,3)$

The area of the parallelogram determined by both $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\times\mathbf{v}\| = \sqrt{(-7)^2+(-1)^2+3} = \sqrt{59}$.

10. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}-1 & 4\\-2 & 8\end{vmatrix}, -\begin{vmatrix}3 & 4\\6 & 8\end{vmatrix}, \begin{vmatrix}3 & -1\\6 & -2\end{vmatrix}\right) = (0,0,0)$

The area of the parallelogram determined by both $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\times\mathbf{v}\| = \sqrt{0^2+0^2+0^2} = 0$.

11. $\overrightarrow{P_1P_2} = (3,2) = \overrightarrow{P_4P_3}$, $\overrightarrow{P_1P_4} = (3,1) = \overrightarrow{P_2P_3}$



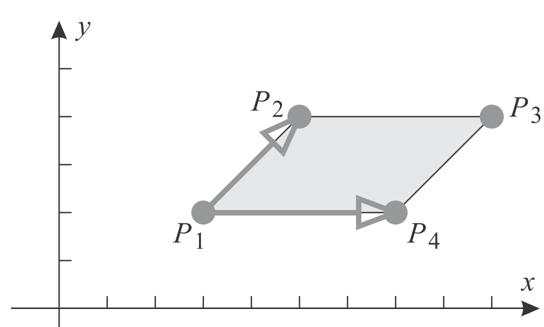
Viewing these as vectors in 3-space, we obtain

$$\begin{aligned}\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_4} &= (3,2,0)\times(3,1,0)\\ &= \left(\begin{vmatrix}2 & 0\\1 & 0\end{vmatrix}, -\begin{vmatrix}3 & 0\\3 & 0\end{vmatrix}, \begin{vmatrix}3 & 2\\3 & 1\end{vmatrix}\right)\\ &= (0,0,-3)\end{aligned}$$

The area of the parallelogram is

$$\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_4}\right\| = \sqrt{0^2+0^2+(-3)^2} = 3.$$

12. $\overrightarrow{P_1P_2} = (2,2) = \overrightarrow{P_4P_3}$, $\overrightarrow{P_1P_4} = (4,0) = \overrightarrow{P_2P_3}$



Viewing these as vectors in 3-space, we obtain

$$\begin{aligned}\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_4} &= (2,2,0)\times(4,0,0)\\ &= \left(\begin{vmatrix}2 & 0\\ 0 & 0\end{vmatrix}, -\begin{vmatrix}2 & 0\\ 4 & 0\end{vmatrix}, \begin{vmatrix}2 & 2\\ 4 & 0\end{vmatrix}\right)\\ &= (0,0,-8)\end{aligned}$$

The area of the parallelogram is

$$\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_4}\right\| = \sqrt{0^2+0^2+(-8)^2} = 8.$$

13. We have $\overrightarrow{AB} = (1,4)$ and $\overrightarrow{AC} = (-3,2)$. Viewing these as vectors in 3-space, we obtain $\overrightarrow{AB}\times\overrightarrow{AC} = (1,4,0)\times(-3,2,0) = \left(\begin{vmatrix}4 & 0\\ 2 & 0\end{vmatrix}, -\begin{vmatrix}1 & 0\\ -3 & 0\end{vmatrix}, \begin{vmatrix}1 & 4\\ -3 & 2\end{vmatrix}\right) = (0,0,14)$.
The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{0^2+0^2+14^2} = 7$.

14. We have $\overrightarrow{AB} = (1,1)$ and $\overrightarrow{AC} = (2,-4)$. Viewing these as vectors in 3-space, we obtain $\overrightarrow{AB}\times\overrightarrow{AC} = (1,1,0)\times(2,-4,0) = \left(\begin{vmatrix}1 & 0\\ -4 & 0\end{vmatrix}, -\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix}, \begin{vmatrix}1 & 1\\ 2 & -4\end{vmatrix}\right) = (0,0,-6)$.
The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{0^2+0^2+(-6)^2} = 3$.

15. $\overrightarrow{P_1P_2} = (-1,-5,2)$, $\overrightarrow{P_1P_3} = (2,0,3)$
$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \left(\begin{vmatrix}-5 & 2\\ 0 & 3\end{vmatrix}, -\begin{vmatrix}-1 & 2\\ 2 & 3\end{vmatrix}, \begin{vmatrix}-1 & -5\\ 2 & 0\end{vmatrix}\right) = (-15,7,10)$.
The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\| = \frac{1}{2}\sqrt{(-15)^2+7^2+10^2} = \frac{\sqrt{374}}{2}$.

16. $\overrightarrow{PQ} = (-1,4,2)$, $\overrightarrow{PR} = (5,2,6)$
$\overrightarrow{PQ}\times\overrightarrow{PR} = \left(\begin{vmatrix}4 & 2\\ 2 & 6\end{vmatrix}, -\begin{vmatrix}-1 & 2\\ 5 & 6\end{vmatrix}, \begin{vmatrix}-1 & 4\\ 5 & 2\end{vmatrix}\right) = (20,16,-22)$.
The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{20^2+16^2+(-22)^2} = \sqrt{285}$.

17. From Theorem 3.5.4(b), the volume of the parallelepiped is equal to $\left|\det\begin{bmatrix}2 & -6 & 2\\ 0 & 4 & -2\\ 2 & 2 & -4\end{bmatrix}\right| = 16$.

18. From Theorem 3.5.4(b), the volume of the parallelepiped is equal to $\left|\det\begin{bmatrix}3 & 1 & 2\\ 4 & 5 & 1\\ 1 & 2 & 4\end{bmatrix}\right| = 45$.

19. $\begin{vmatrix}-1 & -2 & 1\\ 3 & 0 & -2\\ 5 & -4 & 0\end{vmatrix} = 16 \neq 0$ therefore by Theorem 3.5.5 these vectors do not lie in the same plane when they have the same initial point.

20. $\begin{vmatrix}5 & -2 & 1\\ 4 & -1 & 1\\ 1 & -1 & 0\end{vmatrix} = 0$ therefore by Theorem 3.5.5 these vectors lie in the same plane when they have the same initial point.

21. From Formula (7), $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix}-2 & 0 & 6\\ 1 & -3 & 1\\ -5 & -1 & 1\end{vmatrix} = -92$.

22. From Formula (7), $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{vmatrix} = -10$.

23. From Formula (7), $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$.

24. From Formula (7), $\mathbf{i}\cdot(\mathbf{j}\times\mathbf{k}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$.

25. **(a)** $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ can be obtained from $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ by interchanging the second row and the third row. This reverses the sign of the determinant, therefore $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -3$.

(b) By Theorem 3.2.2(a), $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 3$.

(c) $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ can be obtained from $\begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ by interchanging the first row and the second row.

The latter determinant can be obtained from $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ by interchanging the second row and the third row.

Overall, we reversed the sign of the determinant twice, therefore $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = (-1)(-1)3 = 3$.

26. **(a)** $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ can be obtained from $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ by interchanging the first row and the second row. This reverses the sign of the determinant, therefore $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = -3$.

(b) $(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v} = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = -3$ as shown in part (a) above

(c) $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{w}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$ since this determinant has two equal rows (this follows from Theorem 2.2.5).

27. **(a)** From $\overrightarrow{AB} = (-1,2,2)$ and $\overrightarrow{AC} = (1,1,-1)$ we obtain
$\overrightarrow{AB}\times\overrightarrow{AC} = \left(\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}\right) = (-4,1,-3)$.
The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \frac{1}{2}\sqrt{(-4)^2+1^2+(-3)^2} = \frac{\sqrt{26}}{2}$.

(b) Denoting the altitude from C to AB by h, we must have $\frac{1}{2}\left\|\overrightarrow{AB}\right\|h = \frac{\sqrt{26}}{2}$.
Since $\left\|\overrightarrow{AB}\right\| = \sqrt{(-1)^2+2^2+2^2} = 3$, we conclude that $h = \frac{\sqrt{26}}{3}$.

28. $\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}3 & -6\\3 & 6\end{vmatrix}, -\begin{vmatrix}2 & -6\\2 & 6\end{vmatrix}, \begin{vmatrix}2 & 3\\2 & 3\end{vmatrix}\right) = (36,-24,0)$; $\|\mathbf{u}\times\mathbf{v}\| = \sqrt{36^2+(-24)^2+0^2} = 12\sqrt{13}$;
$\|\mathbf{u}\| = \sqrt{2^2+3^2+(-6)^2} = 7$; $\|\mathbf{v}\| = \sqrt{2^2+3^2+6^2} = 7$
From Formula (6), $\sin\theta = \frac{\|\mathbf{u}\times\mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{12}{49}\sqrt{13}$.

29. Using parts (a), (b), (c), and (f) of Theorem 3.5.2, we can write $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$
$= (\mathbf{u}\times\mathbf{u}) - (\mathbf{u}\times\mathbf{v}) + (\mathbf{v}\times\mathbf{u}) + (\mathbf{v}\times\mathbf{v}) = \mathbf{0} - (-(\mathbf{v}\times\mathbf{u})) + (\mathbf{v}\times\mathbf{u}) + \mathbf{0} = 2(\mathbf{v}\times\mathbf{u})$.

30. The result follows directly from part (b) of Theorem 3.2.3 with $\mathbf{u}=\mathbf{a}$, $\mathbf{v}=\mathbf{d}$, and $\mathbf{w}=\mathbf{b}\times\mathbf{c}$.

31. **(a)** Taking $\mathbf{F} = 1000\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = 500\sqrt{2}(-1,0,1)$ and $\mathbf{d} = \overrightarrow{PQ} = (0,2,-1)$ we obtain
$\mathbf{F}\times\mathbf{d} = 500\sqrt{2}[(-1,0,1)\times(0,2,-1)] = 500\sqrt{2}\left(\begin{vmatrix}0 & 1\\2 & -1\end{vmatrix}, -\begin{vmatrix}-1 & 1\\0 & -1\end{vmatrix}, \begin{vmatrix}-1 & 0\\0 & 2\end{vmatrix}\right)$
$= 500\sqrt{2}(-2,-1,-2)$. $\|\mathbf{F}\times\mathbf{d}\| = 500\sqrt{2}\sqrt{(-2)^2+(-1)^2+(-2)^2} = 1500\sqrt{2}$ therefore the scalar moment of $\mathbf{F}$ about the point P is $1500\sqrt{2}$ Nm ≈ 2121.32 Nm.

(b) It was shown in the solution of part (a) that the vector moment of $\mathbf{F}$ about the point P is $\mathbf{F}\times\mathbf{d} = 500\sqrt{2}(-2,-1,-2)$ and its magnitude is $1500\sqrt{2}$. The direction angles are $\cos^{-1}\left(-\frac{1000\sqrt{2}}{1500\sqrt{2}}\right) \approx 132°$, $\cos^{-1}\left(-\frac{500\sqrt{2}}{1500\sqrt{2}}\right) \approx 109°$, and $\cos^{-1}\left(-\frac{1000\sqrt{2}}{1500\sqrt{2}}\right) \approx 132°$.

32. Taking $\mathbf{F} = 200(\cos 72°, \sin 72°, 0)$ and $\mathbf{d} = (0.2, 0.03, 0)$ we obtain $\|\mathbf{F}\times\mathbf{d}\| \approx 36.19$ Nm.

39. **(a)** The volume is $\frac{1}{6}\left|\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})\right| = \frac{1}{6}\left|\det\begin{bmatrix}3 & -1 & -3\\2 & -1 & 1\\4 & -4 & 3\end{bmatrix}\right| = \frac{1}{6}|17| = \frac{17}{6}$.

(b) The volume is $\frac{1}{6}\left|\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})\right| = \frac{1}{6}\left|\det\begin{bmatrix}1 & 2 & -1\\3 & 4 & 0\\-1 & -3 & 4\end{bmatrix}\right| = \frac{1}{6}|-3| = \frac{1}{2}$.

True-False Exercises

(a) True. This follows from Formula (6): for nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ is zero if and only if $\sin\theta = 0$ (i.e., the vectors are parallel).

(b) True. The cross product of two nonzero noncollinear vectors in a plane is a nonzero vector perpendicular to both vectors, and therefore to the entire plane.

(c) False. The scalar triple product is a scalar, rather than a vector.

(d) True. This follows from Theorem 3.5.3 and from the equality $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{v}\times\mathbf{u}\|$.

(e) False. These two triple vector products are generally not the same, as evidenced by parts (d) and (e) of Theorem 3.5.1.

(f) False. For instance, let $\mathbf{u}=\mathbf{v}=\mathbf{i}$ and $\mathbf{w}=2\mathbf{i}$. We have $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w} = \mathbf{0}$ even though $\mathbf{v}\neq\mathbf{w}$.

Chapter 3 Supplementary Exercises

1. **(a)** $3\mathbf{v} - 2\mathbf{u} = (9,-3,18) - (-4,0,8) = (13,-3,10)$

(b) $\mathbf{u}+\mathbf{v}+\mathbf{w}=(3,-6,5)$; $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|=\sqrt{3^2+(-6)^2+5^2}=\sqrt{70}$

(c) $-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})=(6,0,-12)-\big((3,-1,6)+(10,-25,-25)\big)=(-7,26,7)$

$d(-3\mathbf{u},\mathbf{v}+5\mathbf{w})=\|-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})\|=\sqrt{(-7)^2+26^2+7^2}=\sqrt{774}=3\sqrt{86}$

(d) $\mathbf{u}\cdot\mathbf{w}=(-2)(2)+(0)(-5)+(4)(-5)=-24$; $\|\mathbf{w}\|^2=2^2+(-5)^2+(-5)^2=54$;

$\text{proj}_{\mathbf{w}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}=\frac{-24}{54}(2,-5,-5)=(-\frac{8}{9},\frac{20}{9},\frac{20}{9})$

(e) From Formula (7) in Section 3.5, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}-2&0&4\\3&-1&6\\2&-5&-5\end{vmatrix}=-122$

(f) $-5\mathbf{v}+\mathbf{w}=(-15,5,-30)+(2,-5,-5)=(-13,0,-35)$
$(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=[(-2)(3)+(0)(-1)+(4)(6)]\mathbf{w}=18\mathbf{w}=(36,-90,-90)$
$(-5\mathbf{v}+\mathbf{w})\times\big((\mathbf{u}\cdot\mathbf{v})\mathbf{w}\big)=\left(\begin{vmatrix}0&-35\\-90&-90\end{vmatrix},-\begin{vmatrix}-13&-35\\36&-90\end{vmatrix},\begin{vmatrix}-13&0\\36&-90\end{vmatrix}\right)$
$=(-3150,-2430,1170)$

2. Rewrite $\mathbf{u}=(3,-5,1)$, $\mathbf{v}=(-2,0,2)$, and $\mathbf{w}=(0,-1,4)$.

(a) $3\mathbf{v}-2\mathbf{u}=(-6,0,6)-(6,-10,2)=(-12,10,4)$

(b) $\mathbf{u}+\mathbf{v}+\mathbf{w}=(1,-6,7)$; $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|=\sqrt{1^2+(-6)^2+7^2}=\sqrt{86}$

(c) $-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})=(-9,15,-3)-\big((-2,0,2)+(0,-5,20)\big)=(-7,20,-25)$

$d(-3\mathbf{u},\mathbf{v}+5\mathbf{w})=\|-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})\|=\sqrt{(-7)^2+20^2+(-25)^2}=\sqrt{1074}$

(d) $\mathbf{u}\cdot\mathbf{w}=(3)(0)+(-5)(-1)+(1)(4)=9$; $\|\mathbf{w}\|^2=0^2+(-1)^2+4^2=17$;

$\text{proj}_{\mathbf{w}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}=\frac{9}{17}(0,-1,4)=(0,-\frac{9}{17},\frac{36}{17})$

(e) From Formula (7) in Section 3.5, $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix}3&-5&1\\-2&0&2\\0&-1&4\end{vmatrix}=-32$

(f) $-5\mathbf{v}+\mathbf{w}=(10,0,-10)+(0,-1,4)=(10,-1,-6)$
$(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=[(3)(-2)+(-5)(0)+(1)(2)]\mathbf{w}=-4\mathbf{w}=(0,4,-16)$
$(-5\mathbf{v}+\mathbf{w})\times\big((\mathbf{u}\cdot\mathbf{v})\mathbf{w}\big)=\left(\begin{vmatrix}-1&-6\\4&-16\end{vmatrix},-\begin{vmatrix}10&-6\\0&-16\end{vmatrix},\begin{vmatrix}10&-1\\0&4\end{vmatrix}\right)=(40,160,40)$

3. **(a)** $3\mathbf{v}-2\mathbf{u}=(-9,0,24,0)-(-4,12,4,2)=(-5,-12,20,-2)$

(b) $\mathbf{u}+\mathbf{v}+\mathbf{w}=(4,7,4,-5)$; $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|=\sqrt{4^2+7^2+4^2+(-5)^2}=\sqrt{106}$

(c) $-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})=(6,-18,-6,-3)-\big((-3,0,8,0)+(45,5,-30,-30)\big)=(-36,-23,16,27)$
$d(-3\mathbf{u},\mathbf{v}+5\mathbf{w})=\|-3\mathbf{u}-(\mathbf{v}+5\mathbf{w})\|=\sqrt{(-36)^2+(-23)^2+16^2+(-27)^2}=\sqrt{2810}$

(d) $\mathbf{u}\cdot\mathbf{w}=(-2)(9)+(6)(1)+(2)(-6)+(1)(-6)=-30$;
$\|\mathbf{w}\|^2=9^2+1^2+(-6)^2+(-6)^2=154$;

$\text{proj}_{\mathbf{w}}\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}=\frac{-30}{154}(9,1,-6,-6)=\frac{-15}{77}(9,1,-6,-6)=\left(-\frac{135}{77},-\frac{15}{77},\frac{90}{77},\frac{90}{77}\right)$

4. **(a)** A line through the origin perpendicular to the given vector.

(b) A plane through the origin perpendicular to the given vector.

(c) The origin.

(d) A line through the origin perpendicular to the given vectors (and to the plane containing them).

5. By Theorem 3.5.5, this set is the plane containing A, B, and C.

6. By part (b) of Theorem 3.5.2, $\overrightarrow{AB} \times \overrightarrow{AD} = \overrightarrow{AB} \times \left(\overrightarrow{AC} + \overrightarrow{CD}\right) = \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) + \left(\overrightarrow{AB} \times \overrightarrow{CD}\right)$. Therefore $\overrightarrow{AC} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AD}\right) = \overrightarrow{AC} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) + \overrightarrow{AC} \cdot \left(\overrightarrow{AB} \times \overrightarrow{CD}\right) = 0$ (since $\overrightarrow{AB} \times \overrightarrow{AC}$ is orthogonal to $\overrightarrow{AC}$).
By Theorem 3.5.5, this implies that the four points lie on the same plane.
We assumed that $\overrightarrow{AB} \times \overrightarrow{CD} \neq 0$ so the line through A and B cannot be parallel to the line through C and D. We conclude that these two coplanar lines must intersect.

7. Denoting $S(-1, a, b)$ we have have $\overrightarrow{RS} = (-6, a-1, b-1)$. For this vector to be parallel to $\overrightarrow{PQ} = (3, 1, -2)$ there must exist a scalar k such that $\overrightarrow{RS} = k\,\overrightarrow{PQ}$. The equality of the first components immediately leads to $k = -2$. Equating the remaining pairs of components yields the equations:

$$a - 1 = (-2)(1), \quad b - 1 = (-2)(-2)$$

therefore $a = -1$ and $b = 5$. We conclude that the point S has coordinates $(-1, -1, 5)$.

8. Denoting $S(a, b, 6, c)$ we have $\overrightarrow{RS} = (a+4, b-1, 2, c)$. For this vector to be parallel to $\overrightarrow{PQ} = (3,4,1,-8)$ there must exist a scalar k such that $\overrightarrow{RS} = k\,\overrightarrow{PQ}$. The equality of the third components immediately leads to $k = 2$. Equating the remaining pairs of components yields the equations:

$$a + 4 = (2)(3), \quad b - 1 = (2)(4), \quad c = (2)(-8)$$

therefore $a = 2$, $b = 9$, and $c = -16$. We conclude that the point S has coordinates $(2,9,6,-16)$.

9. $\overrightarrow{PQ} = (3, 1, -2)$; $\overrightarrow{PR} = (2, 2, -3)$; $\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \frac{(3)(2)+(1)(2)+(-2)(-3)}{\sqrt{3^2+1^2+(-2)^2}\sqrt{2^2+2^2+(-3)^2}} = \frac{14}{\sqrt{14}\sqrt{17}} = \sqrt{\frac{14}{17}}$

10. $\overrightarrow{PQ} = (3,4,1,-8)$; $\overrightarrow{PR} = (-1,0,4,-6)$;
$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \frac{(3)(-1)+(4)(0)+(1)(4)+(-8)(-6)}{\sqrt{3^2+4^2+1^2+(-8)^2}\sqrt{(-1)^2+0^2+4^2+(-6)^2}} = \frac{49}{\sqrt{90}\sqrt{53}} = \frac{49}{3\sqrt{530}}$

11. From Theorem 3.3.4(b) the distance between the point and the plane is
$D = \frac{|(5)(-3)+(-3)(1)+(1)(3)+4|}{\sqrt{5^2+(-3)^2+1^2}} = \frac{11}{\sqrt{35}}$ (the equation of the plane had to be rewritten in the form $ax + by + cz + d = 0$ as $5x - 3y + z + 4 = 0$)

12. The planes are parallel since their normal vectors, $(3,-1,6)$ and $(-6,2,-12)$, are parallel: $(-6,2,-12) = -2(3,-1,6)$. We select an arbitrary point in the plane $3x - y + 6z = 7$ by setting $x = z = 0$ to obtain $P_0(0,-7,0)$.
From Theorem 3.3.4(b) the distance between P_0 and the plane $-6x + 2y - 12z - 1 = 0$ is

$$D = \frac{|(-6)(0)+(2)(-7)+(-12)(0)-1|}{\sqrt{(-6)^2+2^2+(-12)^2}} = \frac{15}{\sqrt{184}} = \frac{15}{2\sqrt{46}}$$

13. A vector equation of the plane that contains the point P and vectors $\overrightarrow{PQ} = (1,-2,-2)$ and $\overrightarrow{PR} = (5,-1,-5)$ can be expressed as $(x,y,z) = (-2,1,3) + t_1(1,-2,-2) + t_2(5,-1,-5)$.
Parametric equations are $x = -2 + t_1 + 5t_2,\ y = 1 - 2t_1 - t_2$, and $z = 3-2t_1 - 5t_2$.

14. Since the line is to be orthogonal to the plane $4x - z = 5$, it must be parallel to a normal vector to the plane $(4,0,-1)$.
A vector equation of the line can be expressed as $(x,y,z) = (-1,6,0) + t(4,0,-1)$.
This yields parametric equations $x = -1 + 4t,\ y = 6,\ z = -t$.

15. A vector equation of the line can be expressed as $(x,y) = (0,-3) + t(8,-1)$.
This yields parametric equations $x = 8t,\ y = -3 - t$.

16. Since the plane is to be parallel to the plane $-8x + 6y - z = 4$, it must be orthogonal to a normal vector to the given plane $(-8,6,-1)$. To find a vector form and parametric form of the plane equation, we construct two nonzero nonparallel vectors orthogonal to $(-8,6,-1)$, e.g., $(1,0,-8)$ and $(0,1,6)$.
A vector equation of the plane that contains $P(-2,1,0)$ and these two vectors can be expressed as $(x,y,z) = (-2,1,0) + t_1(1,0,-8) + t_2(0,1,6)$.
Parametric equations are $x = -2 + t_1,\ y = 1 + t_2$, and $z = -8t_1 + 6t_2$.

17. Since the line has a slope 3, the vector $(1,3)$ is parallel to the line (any scalar multiple of (1,3) could be used instead).
Substituting an arbitrary number into the line equation for x, we can solve for y to obtain coordinates of a point on the line. For instance, letting $x = 0$ yields $y = -5$ resulting in the point $(0,-5)$.
A vector equation of the line can now be expressed as $(x,y) = (0,-5) + t(1,3)$.
This yields parametric equations $x = t,\ y = -5 + 3t$.

18. To find a vector form and parametric form of the plane equation, we construct two nonzero nonparallel vectors orthogonal to the plane's normal vector $(2,-6,3)$, e.g., $(-3,0,2)$ and $(3,1,0)$.
We also need a point on the plane $2x - 6y + 3z = 5$, e.g., $(1,0,1)$ - note that any one of the infinitely many solutions can be used here.
A vector equation of the plane that contains the point $(1,0,1)$ and the vectors $(-3,0,2)$ and $(3,1,0)$ can be expressed as $(x,y,z) = (1,0,1) + t_1(-3,0,2) + t_2(3,1,0)$.
Parametric equations are $x = 1 - 3t_1 + 3t_2,\ y = t_2$, and $z = 1 + 2t_1$.

19. The given vector equation specifies a point on the plane, $(-1,5,6)$, as well as two vectors parallel to the plane. A normal vector can be obtained as a cross product of these two vectors:

$$(0,-1,3) \times (2,-1,0) = \left(\begin{vmatrix} -1 & 3 \\ -1 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 0 & -1 \\ 2 & -1 \end{vmatrix} \right) = (3,6,2)$$

A point-normal equation for the plane can be written as $3(x+1) + 6(y-5) + 2(z-6) = 0$.

20. Since the plane is to be orthogonal to the line $x = 3 - 5t,\ y = 2t,\ z = 7$, we can use the vector $(-5,2,0)$ as a normal vector for the plane. This yields a point-normal equation $-5(x+5) + 2(y-1) = 0$.

21. Begin by forming two vectors parallel to the plane: $\overrightarrow{PQ} = (-10,4,-1)$ and $\overrightarrow{PR} = (-9,6,-6)$. A normal vector can be obtained as a cross product of these two vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left(\begin{vmatrix} 4 & -1 \\ 6 & -6 \end{vmatrix}, - \begin{vmatrix} -10 & -1 \\ -9 & -6 \end{vmatrix}, \begin{vmatrix} -10 & 4 \\ -9 & 6 \end{vmatrix} \right) = (-18, -51, -24)$$

A point-normal equation for the plane can be written as $-18(x-9) - 51y - 24(z-4) = 0$.

25. The equation represents a plane through the origin perpendicular to the xy-plane. It intersects the xy-plane along the line $Ax + By = 0$.

CHAPTER 4: GENERAL VECTOR SPACES

4.1 Real Vector Spaces

1. **(a)** $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $\quad k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$ is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k\big((u_1,u_2)+(v_1,v_2)\big)=k(u_1+v_1,u_2+v_2)=\big(0,k(u_2+v_2)\big)=(0,ku_2)+(0,kv_2)$ $=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1,u_2,v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$ $=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m,u_1, and u_2;

Axiom 9: $k\big(m(u_1,u_2)\big)=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m,u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

2. **(a)** $\mathbf{u}+\mathbf{v}=(0+1+1,4-3+1)=(2,2)$; $\quad k\mathbf{u}=(2\cdot 0,2\cdot 4)=(0,8)$

(b) $(0,0)+(u_1,u_2)=(0+u_1+1,0+u_2+1)=(u_1+1,u_2+1)\neq(u_1,u_2)$ therefore $(0,0)$ is not the zero vector $\mathbf{0}$ required by Axiom 4

(c) For all real numbers u_1 and u_2, we have
$(-1,-1)+(u_1,u_2)=(-1+u_1+1,-1+u_2+1)=(u_1,u_2)$ and
$(u_1,u_2)+(-1,-1)=(u_1-1+1,u_2-1+1)=(u_1,u_2)$ therefore Axiom 4 holds for $\mathbf{0}=(-1,-1)$

(d) For any pair of real numbers $\mathbf{u}=(u_1,u_2)$, letting $-\mathbf{u}=(-2-u_1,-2-u_2)$ yields
$\mathbf{u}+(-\mathbf{u})=(u_1+(-2-u_1)+1,u_2+(-2-u_2)+1)=(-1,-1)=\mathbf{0}$;
Since $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ holds as well, Axiom 5 holds.

(e) Axiom 7 fails to hold:
$k(\mathbf{u}+\mathbf{v})=k(u_1+v_1+1,u_2+v_2+1)=(ku_1+kv_1+k,ku_2+kv_2+k)$
$k\mathbf{u}+k\mathbf{v}=(ku_1,ku_2)+(kv_1,kv_2)=(ku_1+kv_1+1,ku_2+kv_2+1)$
therefore in general $k(\mathbf{u}+\mathbf{v})\neq k\mathbf{u}+k\mathbf{v}$

Axiom 8 fails to hold:
$(k+m)\mathbf{u}=\big((k+m)u_1,(k+m)u_2\big)=(ku_1+mu_1,ku_2+mu_2)$
$k\mathbf{u}+m\mathbf{u}=(ku_1,ku_2)+(mu_1,mu_2)=(ku_1+mu_1+1,ku_2+mu_2+1)$
therefore in general $(k+m)\mathbf{u}\neq k\mathbf{u}+m\mathbf{u}$

3. Let V denote the set of all real numbers.

Axiom 1: $x + y$ is in V for all real x and y;

Axiom 2: $x + y = y + x$ for all real x and y;

Axiom 3: $x + (y + z) = (x + y) + z$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0} = 0$, we have $0 + x = x + 0 = x$ for all real x;

Axiom 5: for each $\mathbf{u} = x$, let $-\mathbf{u} = -x$; then $x + (-x) = (-x) + x = 0$

Axiom 6: kx is in V for all real k and x;

Axiom 7: $k(x + y) = kx + ky$ for all real k, x, and y;

Axiom 8: $(k + m)x = kx + mx$ for all real k, m, and x;

Axiom 9: $k(mx) = (km)x$ for all real k, m, and x;

Axiom 10: $1x = x$ for all real x.

This is a vector space – all axioms hold.

4. Let V denote the set of all pairs of real numbers of the form $(x, 0)$.

Axiom 1: $(x, 0) + (y, 0) = (x + y, 0)$ is in V for all real x and y;

Axiom 2: $(x, 0) + (y, 0) = (x + y, 0) = (y + x, 0) = (y, 0) + (x, 0)$ for all real x and y;

Axiom 3: $(x, 0) + \big((y, 0) + (z, 0)\big) = (x, 0) + (y + z, 0) = (x + y + z, 0) = (x + y, 0) + (z, 0)$
$= \big((x, 0) + (y, 0)\big) + (z, 0)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0} = (0,0)$, we have $(0,0) + (x, 0) = (x, 0)$ and $(x, 0) + (0,0) = (x, 0)$ for all real x;

Axiom 5: for each $\mathbf{u} = (x, 0)$, let $-\mathbf{u} = (-x, 0)$;
then $(x, 0) + (-x, 0) = (0,0)$ and $(-x, 0) + (x, 0) = (0,0)$;

Axiom 6: $k(x, 0) = (kx, 0)$ is in V for all real k and x;

Axiom 7: $k\big((x, 0) + (y, 0)\big) = k(x + y, 0) = (kx + ky, 0) = k(x, 0) + k(y, 0)$ for all real k, x, and y;

Axiom 8: $(k + m)(x, 0) = \big((k + m)x, 0\big) = (kx + mx, 0) = k(x, 0) + m(x, 0)$ for all real k, m, and x;

Axiom 9: $k\big(m(x, 0)\big) = k(mx, 0) = (kmx, 0) = (km)(x, 0)$ for all real k, m, and x;

Axiom 10: $1(x, 0) = (x, 0)$ for all real x.

This is a vector space – all axioms hold.

5. Axiom 5 fails whenever $x \neq 0$ since it is then impossible to find (x', y') satisfying $x' \geq 0$ for which $(x, y) + (x', y') = (0, 0)$. (The zero vector from axiom 4 must be $\mathbf{0} = (0,0)$.)

Axiom 6 fails whenever $k < 0$ and $x \neq 0$.

This is not a vector space.

6. Let V denote the set of all n-tuples of real numbers of the form $(x, x, \dots, x)$.

Axiom 1: $(x, x, \dots, x) + (y, y, \dots, y) = (x + y, x + y, \dots, x + y)$ is in V for all real x and y;

Axiom 2: $(x, x, \dots, x) + (y, y, \dots, y) = (x + y, x + y, \dots, x + y) = (y + x, y + x, \dots, y + x)$
$= (y, y, \dots, y) + (x, x, \dots, x)$ for all real x and y;

Axiom 3: $(x, x, \dots, x) + \big((y, y, \dots, y) + (z, z, \dots, z)\big) = (x, x, \dots, x) + (y + z, y + z, \dots, y + z)$
$= (x + y + z, x + y + z, \dots, x + y + z) = (x + y, x + y, \dots, x + y) + (z, z, \dots, z)$
$= \big((x, x, \dots, x) + (y, y, \dots, y)\big) + (z, z, \dots, z)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0} = (0,0, \dots,0)$, we have $(0,0, \dots,0) + (x, x, \dots, x) = (x, x, \dots, x)$ and
$(x, x, \dots, x) + (0,0, \dots,0) = (x, x, \dots, x)$ for all real x;

Axiom 5: for each $\mathbf{u} = (x, x, \dots, x)$, let $-\mathbf{u} = (-x, -x, \dots, -x)$;
then $(x, x, \dots, x) + (-x, -x, \dots, -x) = (0,0, \dots,0)$ and
$(-x, -x, \dots, -x) + (x, x, \dots, x) = (0,0, \dots,0)$;

Axiom 6: $k(x, x, \dots, x) = (kx, kx, \dots, kx)$ is in V for all real k and x;

Axiom 7: $k\big((x, x, \dots, x) + (y, y, \dots, y)\big) = k(x + y, x + y, \dots, x + y) = (kx + ky, kx + ky, \dots, kx + ky)$
$= k(x, x, \dots, x) + k(y, y, \dots, y)$ for all real k, x, and y;

Axiom 8: $(k + m)(x, x, \dots, x) = \big((k + m)x, (k + m)x, \dots, (k + m)x\big)$
$= (kx + mx, kx + mx, \dots, kx + mx) = k(x, x, \dots, x) + m(x, x, \dots, x)$
for all real k, m, and x;

Axiom 9: $k\big(m(x, x, \dots, x)\big) = k(mx, mx, \dots, mx) = (kmx, kmx, \dots, kmx) = (km)(x, x, \dots, x)$
for all real k, m, and x;

Axiom 10: $1(x, x, \dots, x) = (x, x, \dots, x)$ for all real x.

This is a vector space – all axioms hold.

7. Axiom 8 fails to hold:
$(k + m)\mathbf{u} = ((k + m)^2 x, (k + m)^2 y, (k + m)^2 z)$
$k\mathbf{u} + m\mathbf{u} = (k^2 x, k^2 y, k^2 z) + (m^2 x, m^2 y, m^2 z) = ((k^2 + m^2)x, (k^2 + m^2)y, (k^2 + m^2)z)$
therefore in general $(k + m)\mathbf{u} \neq k\mathbf{u} + m\mathbf{u}$.

This is not a vector space.

8. Axiom 1 fails since a sum of two 2×2 invertible matrices may or may not be invertible, e.g. both $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are invertible, but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Axiom 6 fails whenever $k = 0$.

9. Let V be the set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (i.e., all diagonal 2×2 matrices)

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$;
follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.

10. Let V be the set of all real-valued functions f defined for all real numbers and such that $f(1) = 0$.

Axiom 1: If f and g are in V then $f + g$ is a function defined for all real numbers and $(f+g)(1) = f(1) + g(1) = 0$ therefore V is closed under the operation of addition defined by Formula (2).

Axiom 6: If k is a scalar and f is in V then kf is a function defined for all real numbers and $(kf)(1) = k\big(f(1)\big) = 0$ therefore V is closed under the operation of scalar multiplication defined by Formula (3).

Verification of the eight remaining axioms proceeds analogously to Example 6.

This is a vector space – all axioms hold.

11. Let V denote the set of all pairs of real numbers of the form $(1, x)$.

Axiom 1: $(1, y) + (1, y') = (1, y + y')$ is in V for all real y and y';

Axiom 2: $(1, y) + (1, y') = (1, y + y') = (1, y' + y) = (1, y') + (1, y)$ for all real y and y';

Axiom 3: $(1, y) + \big((1, y') + (1, y'')\big) = (1, y) + (1, y' + y'') = (1, y + y' + y'') = (1, y + y') + (1, y'')$
$= \big((1, y) + (1, y')\big) + (1, y'')$ for all real y, y', and y'';

Axiom 4: taking $\mathbf{0} = (1,0)$, we have $(1,0) + (1, y) = (1, y)$ and $(1, y) + (1,0) = (1, y)$
for all real y;

Axiom 5: for each $\mathbf{u} = (1, y)$, let $-\mathbf{u} = (1, -y)$;
then $(1, y) + (1, -y) = (1,0)$ and $(1, -y) + (1, y) = (1,0)$;

Axiom 6: $k(1, y) = (1, ky)$ is in V for all real k and y;

Axiom 7: $k\big((1, y) + (1, y')\big) = k(1, y + y') = (1, ky + ky') = (1, ky) + (1, ky') = k(1, y) + k(1, y')$
for all real k, y, and y';

Axiom 8: $(k+m)(1,y) = (1,(k+m)y) = (1,ky+my) = (1,ky)+(1,my) = k(1,y)+m(1,y)$ for all real k, m, and y;

Axiom 9: $k\big(m(1,y)\big) = k(1,my) = (1,kmy) = (km)(1,y)$ for all real k, m, and y;

Axiom 10: $1(1,y) = (1,y)$ for all real y.

This is a vector space – all axioms hold.

12. Let V be the set of polynomials of the form $a+bx$.

Axiom 1: $(a_0+b_0x)+(a_1+b_1x) = (a_0+a_1)+(b_0+b_1)x$ is in V for all real a_0, a_1, b_0, and b_1;

Axiom 2: $(a_0+b_0x)+(a_1+b_1x) = (a_0+a_1)+(b_0+b_1)x = (a_1+a_0)+(b_1+b_0)x$
$= (a_1+b_1x)+(a_0+b_0x)$ for all real a_0, a_1, b_0, and b_1;

Axiom 3: $(a_0+b_0x)+\big((a_1+b_1x)+(a_2+b_2x)\big) = (a_0+a_1+a_2)+(b_0+b_1+b_2)x$
$\big((a_0+b_0x)+(a_1+b_1x)\big)+(a_2+b_2x)$ for all real a_0, a_1, a_2, b_0, b_1, and b_2;

Axiom 4: taking $\mathbf{0} = 0+0x$, we have $(0+0x)+(a+bx) = a+bx$ and
$(a+bx)+(0+0x) = a+bx$ for all real a and b;

Axiom 5: for each $\mathbf{u} = a+bx$, let $-\mathbf{u} = -a-bx$;
then $(a+bx)+(-a-bx) = 0+0x = (-a-bx)+(a+bx)$ for all real a and b;

Axiom 6: $k(a+bx) = ka+(kb)x$ is in V for all real a, b, and k;

Axiom 7: $k\big((a_0+b_0x)+(a_1+b_1x)\big) = k\big((a_0+a_1)+(b_0+b_1)x\big) = k(a_0+b_0x)+k(a_1+b_1x)$
for all real a_0, a_1, b_0, b_1, and k;

Axiom 8: $(k+m)(a+bx) = (k+m)a+(k+m)bx = k(a+bx)+m(a+bx)$
for all real a, b, k, and m;

Axiom 9: $k\big(m(a+bx)\big) = k(ma+mbx) = kma+kmbx = (km)(a+bx)$
for all real a, b, k, and m;

Axiom 10: $1(a+bx) = a+bx$ for all real a and b.

This is a vector space – all axioms hold.

13. Axiom 3: follows from part (b) of Theorem 1.4.1 since

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \left(\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\right) + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = (\mathbf{u}+\mathbf{v})+\mathbf{w}$$

Axiom 7: follows from part (h) of Theorem 1.4.1 since

$$k(\mathbf{u}+\mathbf{v}) = k\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}\right) = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = k\mathbf{u}+k\mathbf{v}$$

Axiom 8: follows from part (j) of Theorem 1.4.1 since

$$(k+m)\mathbf{u} = (k+m)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + m\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = k\mathbf{u}+m\mathbf{u}$$

Axiom 9: follows from part (l) of Theorem 1.4.1 since
$$k(m\mathbf{u}) = k\left(m\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}\right) = (km)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = (km)\mathbf{u}$$

15. Axiom 1: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$ is in V

Axiom 2: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = (v_1, v_2) + (u_1, u_2)$

Axiom 3: $(u_1, u_2) + \big((v_1, v_2) + (w_1, w_2)\big) = (u_1, u_2) + (v_1 + w_1, v_2 + w_2)$
$= (u_1 + v_1 + w_1, u_2 + v_2 + w_2) = (u_1 + v_1, u_2 + v_2) + (w_1, w_2)$
$= \big((u_1, u_2) + (v_1, v_2)\big) + (w_1, w_2)$

Axiom 4: taking $\mathbf{0} = (0,0)$, we have $(0,0) + (u_1, u_2) = (u_1, u_2)$ and $(u_1, u_2) + (0,0) = (u_1, u_2)$

Axiom 5: for each $\mathbf{u} = (u_1, u_2)$, let $-\mathbf{u} = (-u_1, -u_2)$;
then $(u_1, u_2) + (-u_1, -u_2) = (0,0)$ and $(-u_1, -u_2) + (u_1, u_2) = (0,0)$

Axiom 6: $k(u_1, u_2) = (ku_1, 0)$ is in V

Axiom 7: $k\big((u_1, u_2) + (v_1, v_2)\big) = k(u_1 + v_1, u_2 + v_2) = (ku_1 + kv_1, 0)$
$= (ku_1, 0) + (kv_1, 0) = k(u_1, u_2) + k(v_1, v_2)$

Axiom 8: $(k + m)(u_1, u_2) = \big((k + m)u_1, 0\big) = (ku_1 + mu_1, 0) = (ku_1, 0) + (mu_1, 0)$
$= k(u_1, u_2) + m(u_1, u_2)$

Axiom 9: $k\big(m(u_1, u_2)\big) = k(mu_1, 0) = (kmu_1, 0) = (km)(u_1, u_2)$

19. $\frac{1}{u} = u^{-1}$

20. For positive real numbers u, $u^k = 1$ if and only if $k = 0$ or $u = 1$.

21.

$\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$	Hypothesis
$(\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w})$	Add $-\mathbf{w}$ to both sides
$\mathbf{u} + [\mathbf{w} + (-\mathbf{w})] = \mathbf{v} + [\mathbf{w} + (-\mathbf{w})]$	Axiom 3
$\mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0}$	Axiom 5
$\mathbf{u} = \mathbf{v}$	Axiom 4

22. (1) Axiom 7
(2) Axiom 4
(3) Axiom 5
(4) Axiom 1
(5) Axiom 3
(6) Axiom 5
(7) Axiom 4

True-False Exercises

(a) True. This is a part of Definition 1.

(b) False. Example 1 discusses a vector space containing only one vector.

(c) False. By part (d) of Theorem 4.1.1, if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

(d) False. Axiom 6 fails to hold if $k < 0$. (Also, Axiom 4 fails to hold.)

(e) True. This follows from part (c) of Theorem 4.1.1.

(f) False. This function must have a value of zero at *every* point in $(-\infty, \infty)$.

4.2 Subspaces

1. **(a)** Let W be the set of all vectors of the form $(a, 0, 0)$, i.e. all vectors in R^3 with last two components equal to zero.
This set contains at least one vector, e.g. $(0, 0, 0)$.
Adding two vectors in W results in another vector in W: $(a, 0, 0) + (b, 0, 0) = (a + b, 0, 0)$ since the result has zeros as the last two components.
Likewise, a scalar multiple of a vector in W is also in W: $k(a, 0, 0) = (ka, 0, 0)$ - the result also has zeros as the last two components.
According to Theorem 4.2.1, W is a subspace of R^3.

(b) Let W be the set of all vectors of the form $(a, 1, 1)$, i.e. all vectors in R^3 with last two components equal to one. The set W is not closed under the operation of vector addition since $(a, 1,1) + (b, 1,1) = (a + b, 2,2)$ does not have ones as its last two components thus it is outside W.
According to Theorem 4.2.1, W is not a subspace of R^3.

(c) Let W be the set of all vectors of the form (a, b, c), where $b = a + c$.
This set contains at least one vector, e.g. $(0, 0, 0)$. (The condition $b = a + c$ is satisfied when $a = b = c = 0$.)
Adding two vectors in W results in another vector in W
$(a, a + c, c) + (a', a' + c', c') = (a + a',\ a + c + a' + c',\ c + c')$ since in this result, the second component is the sum of the first and the third: $a + c + a' + c' = (a + a') + (c + c')$.
Likewise, a scalar multiple of a vector in W is also in W: $k(a, a + c, c) = (ka, k(a + c), kc)$ since in this result, the second component is once again the sum of the first and the third: $k(a + c) = ka + kc$.
According to Theorem 4.2.1, W is a subspace of R^3.

(d) Let W be the set of all vectors of the form (a, b, c), where $b = a + c + 1$. The set W is not closed under the operation of vector addition, since in the result of the following addition of two vectors from W
$(a, a + c + 1, c) + (a', a' + c' + 1, c') = (a + a',\ a + c + a' + c' + 2,\ c + c')$
the second component does not equal to the sum of the first, the third, and 1:
$a + c + a' + c' + 2 \neq (a + a') + (c + c') + 1$. Consequently, this result is not a vector in W.
According to Theorem 4.2.1, W is not a subspace of R^3.

(e) Let W be the set of all vectors of the form $(a, b, 0)$, i.e. all vectors in R^3 with last component equal to zero.
This set contains at least one vector, e.g. $(0, 0, 0)$.

Adding two vectors in W results in another vector in W
$(a, b, 0) + (a', b', 0) = (a + a', b + b', 0)$ since the result has 0 as the last component.
Likewise, a scalar multiple of a vector in W is also in W: $k(a, b, 0) = (ka, kb, 0)$ - the result also has 0 as the last component.
According to Theorem 4.2.1, W is a subspace of R^3.

2. **(a)** Let W be the set of all $n \times n$ diagonal matrices.
This set contains at least one matrix, e.g. the zero $n \times n$ matrix.
Adding two matrices in W results in another $n \times n$ diagonal matrix, i.e. a matrix in W:

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

Likewise, a scalar multiple of a matrix in W is also in W:

$$k\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ka_{11} & 0 & \cdots & 0 \\ 0 & ka_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ka_{nn} \end{bmatrix}$$

According to Theorem 4.2.1, W is a subspace of M_{nn}.

(b) Let W be the set of all $n \times n$ matrices such whose determinant is zero. We shall show that W is not closed under the operation of matrix addition. For instance, consider the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ - both have determinant equal 0, therefore both matrices are in W.
However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has nonzero determinant, thus it is outside W.
According to Theorem 4.2.1, W is not a subspace of M_{nn}.

(c) Let W be the set of all $n \times n$ matrices with zero trace.
This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.
Let us assume $A = [a_{ij}]$ and $B = [b_{ij}]$ are both in W, i.e. $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and $\operatorname{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$.
Since $\operatorname{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})$
$= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A + B$ is in W.
A scalar multiple of the same matrix A with a scalar k has $tr(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(d) Let W be the set of all symmetric $n \times n$ matrices (i.e., $n \times n$ matrices such that $A^T = A$).
This set contains at least one matrix, e.g., I_n is in W.
Let us assume A and B are both in W, i.e. $A^T = A$ and $B^T = B$. By Theorem 1.4.8(b), their sum satisfies $(A + B)^T = A^T + B^T = A + B$ therefore W is closed under addition.
From Theorem 1.4.8(d), a scalar multiple of a symmetric matrix is also symmetric: $(kA)^T = kA^T = kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(e) Let W be the set of all $n \times n$ matrices such that $A^T = -A$.
This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.
Let us assume A and B are both in W, i.e. $A^T = -A$ and $B^T = -B$. By Theorem 1.4.8(b), their sum satisfies $(A+B)^T = A^T + B^T = -A - B = -(A+B)$ therefore W is closed under addition.
From Theorem 1.4.8(d), we have $(kA)^T = kA^T = k(-A) = -kA$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

(f) Let W be the set of $n \times n$ matrices for which $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. It follows from Theorem 1.5.3 that the set W consists of all $n \times n$ matrices that are invertible. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of M_{nn}.

(g) Let B be some fixed $n \times n$ matrix, and let W be the set of all $n \times n$ matrices A such that $AB = BA$.
This set contains at least one matrix, e.g., I_n is in W.
Let us assume A and C are both in W, i.e. $AB = BA$ and $CB = BC$. By Theorem 1.4.1(d,e), their sum satisfies $(A+C)B = AB + CB = BA + BC = B(A+C)$ therefore W is closed under addition.
From Theorem 1.4.1(m), we have $(kA)B = k(AB) = k(BA) = B(kA)$ which makes W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of M_{nn}.

3. **(a)** Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
This set contains at least one polynomial, $0 + 0x + 0x^2 + 0x^3 = 0$.
Adding two polynomials in W results in another polynomial in W:
$(0 + a_1x + a_2x^2 + a_3x^3) + (0 + b_1x + b_2x^2 + b_3x^3)$
$= 0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$.
Likewise, a scalar multiple of a polynomial in W is also in W:
$k(0 + a_1x + a_2x^2 + a_3x^3) = 0 + (ka_1)x + (ka_2)x^2 + (ka_3)x^3$.
According to Theorem 4.2.1, W is a subspace of P_3.

(b) Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3$.
Adding two polynomials in W results in another polynomial in W
$(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) + (-b_1 - b_2 - b_3 + b_1x + b_2x^2 + b_3x^3)$
$= (-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$
since we have $(-a_1 - a_2 - a_3 - b_1 - b_2 - b_3) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$.
Likewise, a scalar multiple of a polynomial in W is also in W
$k(-a_1 - a_2 - a_3 + a_1x + a_2x^2 + a_3x^3) = -ka_1 - ka_2 - ka_3 + ka_1x + ka_2x^2 + ka_3x^3$
since it meets the condition $(-ka_1 - ka_2 - ka_3) + (ka_1) + (ka_2) + (ka_3) = 0$.
According to Theorem 4.2.1, W is a subspace of P_3.

(c) Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers. The set W is not closed under the operation of scalar multiplication, e.g., the

scalar product of the polynomial x^3 in W by $k = \pi$ is πx^3, which is not in W.
According to Theorem 4.2.1, W is not a subspace of P_3.

(d) The set of all polynomials of degree ≤ 1 is a subset of P_3. It is also a vector space (called P_1) with same operations of addition and scalar multiplication as those defined in P_3. By Definition 1, we conclude that P_1 is a subspace of P_3.

4. (a) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 0$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(0) = g(0) = 0$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(0) = k\big(f(0)\big) = 0$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(b) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(0) = 1$.
We will show that W is not closed under addition. For instance, let $f(x) = 1$ and $g(x) = \cos x$ be two functions in W. Their sum, $f + g$, is not in W since $(f + g)(0) = f(0) + g(0) = 1 + 1 = 2$.
We conclude that W is not a subspace of $F(-\infty,\infty)$.

(c) Let W be the set of all functions f in $F(-\infty,\infty)$ for which $f(-x) = f(x)$.
This set contains at least one function, e.g., the constant function $f(x) = 0$.
Assume we have two functions f and g in W, i.e., $f(-x) = f(x)$ and $g(-x) = g(x)$. Their sum $f + g$ is also a function in $F(-\infty,\infty)$ and satisfies $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$ therefore W is closed under addition.
A scalar multiple of a function f in W, kf, is also a function in $F(-\infty,\infty)$ for which $(kf)(-x) = k\big(f(-x)\big) = k\big(f(x)\big) = (kf)(x)$ making W closed under scalar multiplication.
According to Theorem 4.2.1, W is a subspace of $F(-\infty,\infty)$.

(d) A sum of two polynomials of degree 2 may be a polynomial of lower degree, e.g., $(1 + x^2) + (x - x^2) = 1 + x$ therefore the set is not closed under addition, and consequently is not a subspace of $F(-\infty,\infty)$.

5. (a) Let W be the set of all sequences in R^∞ of the form $(v, 0, v, 0, v, 0, \dots)$.
This set contains at least one sequence, e.g. $(0, 0, 0, \dots)$.
Adding two sequences in W results in another sequence in W:
$(v, 0, v, 0, v, 0, \dots) + (w, 0, w, 0, w, 0, \dots) = (v + w, 0, v + w, 0, v + w, 0, \dots)$.
Likewise, a scalar multiple of a vector in W is also in W: $k(v, 0, v, 0, v, 0, \dots) = (kv, 0, kv, 0, kv, 0, \dots)$.
According to Theorem 4.2.1, W is a subspace of R^∞.

(b) Let W be the set of all sequences in R^∞ of the form $(v, 1, v, 1, v, 1, \dots)$.
This set is not closed under addition since
$(v, 1, v, 1, v, 1, \dots) + (w, 1, w, 1, w, 1, \dots) = (v + w, 2, v + w, 2, v + w, 2, \dots)$ is not in W.
We conclude that W is not a subspace of R^∞.

(c) Let W be the set of all sequences in R^{∞} of the form $(v, 2v, 4v, 8v, 16v, \ldots)$.
This set contains at least one sequence, e.g. $(0, 0, 0, \ldots)$.
Adding two sequences in W results in another sequence in W:
$(v, 2v, 4v, 8v, 16v, \ldots) + (w, 2w, 4w, 8w, 16w, \ldots)$
$= (v + w, 2(v + w), 4(v + w), 8(v + w), 16(v + w), \ldots)$.
Likewise, a scalar multiple of a vector in W is also in W:
$k(v, 2v, 4v, 8v, 16v, \ldots) = (kv, 2kv, 4kv, 8kv, 16kv, \ldots)$.
According to Theorem 4.2.1, W is a subspace of R^{∞}.

(d) Let W be the set of all sequences in R^{∞} whose components are 0 from some point on.
This set contains at least one sequence, e.g. $(0, 0, 0, \ldots)$.
Let a sequence $\mathbf{u}$ in W have 0 components starting from the ith element; also, let a sequence $\mathbf{v}$ in W have 0 components starting from the jth element. It follows that $\mathbf{u} + \mathbf{v}$ must have 0 component starting no later than from the position corresponding to $\max(i, j)$ - the larger of the two numbers. Therefore, $\mathbf{u} + \mathbf{v}$ is in W.
The scalar product $k\mathbf{u}$ must have 0 components starting no later than from the ith element, therefore $k\mathbf{u}$ is also in W.
According to Theorem 4.2.1, W is a subspace of R^{∞}.

6. The line L contains at least one point – e.g., the origin.
If the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are both on L, then there must exist real numbers t_1 and t_2 such that $x_1 = at_1$, $y_1 = bt_1$, $z_1 = ct_1$, $x_2 = at_2$, $y_2 = bt_2$, and $z_2 = ct_2$.
L is closed under addition since $(x_1, y_1, z_1) + (x_2, y_2, z_2) = ((a)(t_1 + t_2), (b)(t_1 + t_2), (c)(t_1 + t_2))$.
It is also closed under scalar multiplication because $k(x_1, y_1, z_1) = ((a)(kt_1), (b)(kt_1), (c)(kt_1))$.
It follows from Theorem 4.2.1 that L is a subspace of R^3.

7. (a) For $(2, 2, 2)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0, -2, 2) + b(1, 3, -1) = (2, 2, 2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 2 \\ -2a & + & 3b & = & 2 \\ 2a & - & 1b & = & 2 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. The linear system is consistent, therefore $(2, 2, 2)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) For $(0, 4, 5)$ to be a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$, there must exist scalars a and b such that

$$a(0, -2, 2) + b(1, 3, -1) = (0, 4, 5)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 0a & + & 1b & = & 0 \\ -2a & + & 3b & = & 4 \\ 2a & - & 1b & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0 = 1$ which is contradictory. We conclude that $(0, 4, 5)$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) By inspection, the zero vector $(0, 0, 0)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ since

$$0(0, -2, 2) + 0(1, 3, -1) = (0, 0, 0)$$

8. **(a)** For $(-9, -7, -15)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (-9,-7,-15)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a = -2$, $b = 1$, $c = -2$, therefore $(-9, -7, -15) = -2\mathbf{u} + 1\mathbf{v} - 2\mathbf{w}$.

(b) For $(6,11,6)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (6,11,6)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a = 4$, $b = -5$, $c = 1$, therefore $(6,11,6) = 4\mathbf{u} - 5\mathbf{v} + 1\mathbf{w}$.

(c) For $(0,0,0)$ to be a linear combination of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, there must exist scalars a, b, and c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (0,0,0)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 0 \\ 1a & - & 1b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = 0,\ c = 0,$ therefore $(0,0,0) = 0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w}$.

9. **(a)** For $\begin{bmatrix}6 & -8\\-1 & -8\end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix}4 & 0\\-2 & -2\end{bmatrix} + b\begin{bmatrix}1 & -1\\2 & 3\end{bmatrix} + c\begin{bmatrix}0 & 2\\1 & 4\end{bmatrix} = \begin{bmatrix}6 & -8\\-1 & -8\end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & 6\\ 0a & - & 1b & + & 2c & = & -8\\ -2a & + & 2b & + & 1c & = & -1\\ -2a & + & 3b & + & 4c & = & -8\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -3\\0 & 0 & 0 & 0\end{bmatrix}$. The linear system is consistent, therefore $\begin{bmatrix}6 & -8\\-1 & -8\end{bmatrix}$ is a linear combination of A, B, and C.

(b) The zero matrix $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ is a linear combination of A, B, and C since $0A + 0B + 0C = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$.

(c) For $\begin{bmatrix}-1 & 5\\7 & 1\end{bmatrix}$ to be a linear combination of A, B, and C, there must exist scalars a, b, and c such that

$$a\begin{bmatrix}4 & 0\\-2 & -2\end{bmatrix} + b\begin{bmatrix}1 & -1\\2 & 3\end{bmatrix} + c\begin{bmatrix}0 & 2\\1 & 4\end{bmatrix} = \begin{bmatrix}-1 & 5\\7 & 1\end{bmatrix}$$

Equating corresponding entries on both sides yields the linear system

$$\begin{array}{rcrcrcr} 4a & + & 1b & + & 0c & = & -1\\ 0a & - & 1b & + & 2c & = & 5\\ -2a & + & 2b & + & 1c & = & 7\\ -2a & + & 3b & + & 4c & = & 1\end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$. The last row corresponds to the equation $0 = 1$ which is contradictory. We conclude that $\begin{bmatrix}-1 & 5\\7 & 1\end{bmatrix}$ is not a linear combination of A, B, and C.

10. **(a)** For $-9 - 7x - 15x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = -9 - 7x - 15x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2 = -9-7x-15x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a=-2,\ b=1,\ c=-2,$ therefore $-9-7x-15x^2 = -2\mathbf{p}_1+1\mathbf{p}_2-2\mathbf{p}_3$.

(b) For $6+11x+6x^2$ to be a linear combination of the vectors $\mathbf{p}_1,\ \mathbf{p}_2,$ and $\mathbf{p}_3,$ there must exist scalars $a,\ b,$ and c such that

$$a(2+x+4x^2)+b(1-x+3x^2)+c(3+2x+5x^2) = 6+11x+6x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2 = 6+11x+6x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a=4,\ b=-5,\ c=1,$ therefore $6+11x+6x^2 = 4\mathbf{p}_1-5\mathbf{p}_2+1\mathbf{p}_3$.

(c) By inspection, $0 = 0\mathbf{p}_1+0\mathbf{p}_2+0\mathbf{p}_3$.

(d) For $7+8x+9x^2$ to be a linear combination of the vectors $\mathbf{p}_1,\ \mathbf{p}_2,$ and $\mathbf{p}_3,$ there must exist scalars $a,\ b,$ and c such that

$$a(2+x+4x^2)+b(1-x+3x^2)+c(3+2x+5x^2) = 7+8x+9x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a+b+3c)+(a-b+2c)x+(4a+3b+5c)x^2 = 7+8x+9x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & 7 \\ 1a & - & 1b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = -2,\ c = 3,$ therefore $7 + 8x + 9x^2 = 0\mathbf{p}_1 - 2\mathbf{p}_2 + 3\mathbf{p}_3$.

11. (a) The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 0k_2 & + & 0k_3 & = & b_1 \\ 2k_1 & + & 0k_2 & + & 1k_3 & = & b_2 \\ 2k_1 & + & 3k_2 & + & 1k_3 & = & b_3 \end{array}$$

By inspection, regardless of the right hand side values b_1, b_2, b_3, the first equation can be solved for k_1, then the second equation can be used to obtain k_3, and the third would yield k_2.
We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3.

(b) The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2,-1,3) + k_2(4,1,2) + k_3(8,-1,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 4k_2 & + & 8k_3 & = & b_1 \\ -1k_1 & + & 1k_2 & - & 1k_3 & = & b_2 \\ 3k_1 & + & 2k_2 & + & 8k_3 & = & b_3 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

12. (a) In order for the vector $(2,3,-7,3)$ to be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (2,3,-7,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 2 \\ 1a & - & 1b & + & 0c & = & 3 \\ 0a & + & 5b & + & 2c & = & -7 \\ 3a & + & 2b & + & 1c & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
This system is consistent (its only solution is $a = 2,\ b = -1,\ c = -1$), therefore $(2,3,-7,3)$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) The vector $(0,0,0,0)$ is obviously in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ since

$$0(2,1,0,3) + 0(3,-1,5,2) + 0(-1,0,2,1) = (0,0,0,0)$$

(c) In order for the vector $(1,1,1,1)$ to be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (1,1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 1 \\ 1a & - & 1b & + & 0c & = & 1 \\ 0a & + & 5b & + & 2c & = & 1 \\ 3a & + & 2b & + & 1c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This system is inconsistent therefore $(1,1,1,1)$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(d) In order for the vector $(-4,6,-13,4)$ to be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (-4,6,-13,4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & -4 \\ 1a & - & 1b & + & 0c & = & 6 \\ 0a & + & 5b & + & 2c & = & -13 \\ 3a & + & 2b & + & 1c & = & 4 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
This system is consistent (its only solution is $a = 3,\ b = -3,\ c = 1$), therefore $(-4,6,-13,4)$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

13. The given polynomials span P_2 if an arbitrary polynomial in P_2, $\mathbf{p} = a_0 + a_1x + a_2x^2$ can be expressed as a linear combination

$$a_0 + a_1x + a_2x^2 = k_1(1 - x + 2x^2) + k_2(3 + x) + k_3(5 - x + 4x^2) + k_4(-2 - 2x + 2x^2)$$

Grouping the terms according to the powers of x yields

$$a_0 + a_1x + a_2x^2 = (k_1 + 3k_2 + 5k_3 - 2k_4) + (-k_1 + k_2 - k_3 - 2k_4)x + (2k_1 + 4k_3 + 2k_4)x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcrcl} 1k_1 & + & 3k_2 & + & 5k_3 & - & 2k_4 & = & a_0 \\ -1k_1 & + & 1k_2 & - & 1k_3 & - & 2k_4 & = & a_1 \\ 2k_1 & + & 0k_2 & + & 4k_3 & + & 2k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{bmatrix}$

therefore the system has no solution if $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$.

Since polynomials $\mathbf{p} = a_0 + a_1x + a_2x^2$ for which $-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$ cannot be expressed as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$, we conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ do not span P_2.

14. **(a)** It follows from the trigonometric identity $\cos 2x = \cos^2 x - \sin^2 x$ that $\cos 2x$ is in $\text{span}\{\mathbf{f}, \mathbf{g}\}$.

(b) In order for $3 + x^2$ to be in $\text{span}\{\mathbf{f}, \mathbf{g}\}$, there must exist scalars a and b such that

$$a\cos^2 x + b\sin^2 x = 3 + x^2$$

holds for all real x values. When $x = 0$ the equation becomes $a = 3$, however if $x = \pi$ then it yields $a = 3 + \pi^2$ - a contradiction. We conclude that $3 + x^2$ is not in $\text{span}\{\mathbf{f}, \mathbf{g}\}$.

(c) It follows from the trigonometric identity $\cos^2 x + \sin^2 x = 1$ that 1 is in $\text{span}\{\mathbf{f}, \mathbf{g}\}$.

(d) In order for $\sin x$ to be in $\text{span}\{\mathbf{f}, \mathbf{g}\}$, there must exist scalars a and b such that

$$a\cos^2 x + b\sin^2 x = \sin x$$

holds for all real x values. When $x = \frac{\pi}{2}$ the equation becomes $b = 1$, however if $x = -\frac{\pi}{2}$ then it yields $b = -1$ - a contradiction. We conclude that $\sin x$ is not in $\text{span}\{\mathbf{f}, \mathbf{g}\}$.

(e) Since $0\cos^2 x + 0\sin^2 x = 0$ holds for all real x values, we conclude that 0 is in $\text{span}\{\mathbf{f}, \mathbf{g}\}$.

15. **(a)** The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solutions are $x = -\frac{1}{2}t,\ y = -\frac{3}{2}t, z = t$. These are parametric equations of a line through the origin.

(b) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ therefore the only solution is $x = y = z = 0$ - the origin.

(c) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which corresponds to an equation of a plane through the origin $x - 3y + z = 0$.

(d) The reduced row echelon form of the coefficient matrix A is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the solutions are $x = -3t,\ y = -2t, z = t$. These are parametric equations of a line through the origin.

17. Let W denote the set of all continuous functions $f = f(x)$ on $[a,b]$ such that $\int_a^b f(x)dx = 0$.
This set contains at least one function $f(x) \equiv 0$.
Let us assume $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are functions in W. From calculus,
$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx = 0$ and $\int_a^b kf(x)dx = k\int_a^b f(x)dx = 0$ therefore both $\mathbf{f}+\mathbf{g}$ and $k\mathbf{f}$ are in W for any scalar k. According to Theorem 4.2.1, W is a subspace of $C[a,b]$.

19. **(a)** The vectors $T_A(1,2) = (-1,4)$ and $T_A(-1,1) = (-2,2)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1,4) + k_2(-2,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & b_1 \\ 4k_1 & + & 2k_2 & = & b_2 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 4 & 2 \end{vmatrix} = 6 \neq 0$, therefore by Theorem 2.3.8, the system is consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ span R^2.

(b) The vectors $T_A(1,2) = (-1,2)$ and $T_A(-1,1) = (-2,4)$ span R^2 if an arbitrary vector $\mathbf{b} = (b_1, b_2)$ can be expressed as a linear combination

$$(b_1, b_2) = k_1(-1,2) + k_2(-2,4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & b_1 \\ 2k_1 & + & 4k_2 & = & b_2 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} = 0$, therefore by Theorem 2.3.8, the system cannot be consistent for all right hand side vectors $\mathbf{b}$.
We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ do not span R^2.

21. Since $T_A: R^3 \to R^m$, it follows from Theorem 4.2.5 that the kernel of T_A must be a subspace of R^3. Hence, according to Table 1 the kernel can be one of the following four geometric objects:

- the origin,
- a line through the origin,
- a plane through the origin,
- R^3.

22. We begin by showing that the vector $\mathbf{w}_1$ is a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, i.e., that there exist scalars a, b, and c such that

$$a(1,6,4) + b(2,4,-1) + c(-1,2,5) = (1,-2,-5)$$

Equating corresponding components on both sides leads to the linear system

$$\begin{array}{rcrcrcr} 1a & + & 2b & - & 1c & = & 1 \\ 6a & + & 4b & + & 2c & = & -2 \\ 4a & - & 1b & + & 5c & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of this system is $a = -1 - t,\ b = 1 + t,\ c = t$. E.g., letting $t = 0$ yields a solution $a = -1,\ b = 1,\ c = 0$.

Applying the same procedure repeatedly to each of the remaining four vectors, we can show that

$$\begin{aligned} \mathbf{w}_1 &= -1\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3 \\ \mathbf{w}_2 &= 2\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3 \\ \mathbf{v}_1 &= 1\mathbf{w}_1 + 1\mathbf{w}_2 \\ \mathbf{v}_2 &= 2\mathbf{w}_1 + 1\mathbf{w}_2 \\ \mathbf{v}_3 &= -1\mathbf{w}_1 + 0\mathbf{w}_2 \end{aligned}$$

It follows from Theorem 4.2.6 that the sets $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ span the same subspace of R^3.

23. Let W be the set of all functions of the form $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ - W is a subset of $C^\infty(-\infty, \infty)$.
This set contains at least one function $x(t) \equiv 0$.
A sum of two functions in W is also in W:
$(c_1 \cos \omega t + c_2 \sin \omega t) + (d_1 \cos \omega t + d_2 \sin \omega t) = (c_1 + d_1) \cos \omega t + (c_2 + d_2) \sin \omega t$.
A scalar product of a function in W by any scalar k is also a function in W:
$k(c_1 \cos \omega t + c_2 \sin \omega t) = (kc_1) \cos \omega t + (kc_2) \sin \omega t$.
According to Theorem 4.2.1, W is a subspace of $C^\infty(-\infty, \infty)$.

True-False Exercises

(a) True. This follows from Definition 1.

(b) True.

(c) False. The set of all nonnegative real numbers is a subset of the vector space R containing 0, but it is not closed under scalar multiplication.

(d) False. By Theorem 4.2.5, the kernel of $T_A: R^n \to R^m$ is a subspace of R^n.

(e) False. The solution set of a nonhomogeneous system is not closed under addition: $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}$ do not imply $A(\mathbf{x} + \mathbf{y}) = \mathbf{b}$.

(f) True. This follows from part (a) of Theorem 4.2.3.

(g) True. This follows from Theorem 4.2.2.

(h) False. Consider $W_1 = \text{span}\{(1,0)\}$ and $W_2 = \text{span}\{(0,1)\}$. The union of these sets is not closed under vector addition, e.g. $(1,0) + (0,1) = (1,1)$ is outside the union.

(i) False. For any nonzero vector $\mathbf{v}$ in a vector space V, both $\{\mathbf{v}\}$ and $\{2\mathbf{v}\}$ span the same subspace of V.

(j) True. This set contains at least one matrix (e.g., I_n). A sum of two upper triangular matrices is also upper triangular, therefore the set is closed under addition. A scalar multiple of an upper triangular matrix is also upper triangular, hence the set is closed under scalar multiplication.

(k) False. The constant polynomial $p(x) = 1$ cannot be represented as a linear combination of these, since at $x = 1$ all three are zero, whereas $p(1) = 1$.

4.3 Linear Independence

1. **(a)** Since $\mathbf{u}_2 = -5\mathbf{u}_1$, linear dependence follows from Definition 1.

(b) A set of 3 vectors in R^2 must be linearly dependent by Theorem 4.3.3.

(c) Since $\mathbf{p}_2 = 2\mathbf{p}_1$, linear dependence follows from Definition 1.

(d) Since $A = (-1)B$, linear dependence follows from Definition 1.

2. **(a)** The vector equation $a(-3,0,4) + b(5,-1,2) + c(1,1,3) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} -3a & + & 5b & + & 1c & = & 0 \\ 0a & - & 1b & + & 1c & = & 0 \\ 4a & + & 2b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given set of vectors is linearly independent.

(b) A set of 4 vectors in R^3 must be linearly dependent by Theorem 4.3.3.

3. **(a)** The vector equation $a(3,8,7,-3) + b(1,5,3,-1) + c(2,-1,2,6) + d(4,2,6,4) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcr} 3a & + & 1b & + & 2c & + & 4d & = & 0 \\ 8a & + & 5b & - & 1c & + & 2d & = & 0 \\ 7a & + & 3b & + & 2c & + & 6d & = & 0 \\ -3a & - & 1b & + & 6c & + & 4d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = -t,\ b = t,\ c = -t,\ d = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) The vector equation $a(3,0,-3,6) + b(0,2,3,1) + c(0,-2,-2,0) + d(-2,1,2,1) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcrcr} 3a & + & 0b & + & 0c & - & 2d & = & 0 \\ 0a & + & 2b & - & 2c & + & 1d & = & 0 \\ -3a & + & 3b & - & 2c & + & 2d & = & 0 \\ 6a & + & 1b & + & 0c & + & 1d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = d = 0$. We conclude that the given set of vectors is linearly independent.

4. (a) The terms in the equation

$$a(2 - x + 4x^2) + b(3 + 6x + 2x^2) + c(2 + 10x - 4x^2) = 0$$

can be grouped according to the powers of x

$$(2a + 3b + 2c) + (-a + 6b + 10c)x + (4a + 2b - 4c)x^2 = 0 + 0x + 0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & + & 2c & = & 0 \\ -a & + & 6b & + & 10c & = & 0 \\ 4a & + & 2b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given set of vectors in P_2 is linearly independent.

(b) The terms in the equation

$$a(1 + 3x + 3x^2) + b(x + 4x^2) + c(5 + 6x + 3x^2) + d(7 + 2x - x^2) = 0$$

can be grouped according to the powers of x

$$(a + 5c + 7d) + (3a + b + 6c + 2d)x + (3a + 4b + 3c - d)x^2 = 0 + 0x + 0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcrcr} a & & & + & 5c & + & 7d & = & 0 \\ 3a & + & b & + & 6c & + & 2d & = & 0 \\ 3a & + & 4b & + & 3c & - & d & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{17}{4} & 0 \\ 0 & 1 & 0 & \frac{5}{4} & 0 \\ 0 & 0 & 1 & \frac{9}{4} & 0 \end{bmatrix}$

therefore a general solution of the system is $a = \frac{17}{4}t,\ b = -\frac{5}{4}t,\ c = -\frac{9}{4}t,\ d = t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

5. (a) The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1a & + & 1b & + & 0c & = & 0 \\ 0a & + & 2b & + & 1c & = & 0 \\ 1a & + & 2b & + & 2c & = & 0 \\ 2a & + & 1b & + & 1c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given matrices are linearly independent.

(b) By inspection, the matrix equation $a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has only the trivial solution $a = b = c = 0$. We conclude that the given matrices are linearly independent.

6. The matrix equation $a\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1a & - & 1b & + & 2c & = & 0 \\ 0a & + & 0b & + & 0c & = & 0 \\ 1a & + & kb & + & 1c & = & 0 \\ ka & + & 1b & + & 3c & = & 0 \end{array}$$

Omitting the second equation (which imposes no restrictions on the unknowns), we obtain the coefficient matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix}$. Performing elementary row operations

- add -1 times the first row to the second row,
- add $-k$ times the first row to the third row, and
- add -1 times the second row to the third row

yields $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1+k & -1 \\ 0 & 0 & 4-2k \end{bmatrix}$. We have $\det(A) = \det(B) = (1+k)(4-2k)$ therefore by Theorem 2.3.8, the system has only the trivial solution, whenever $(1+k)(4-2k) \neq 0$. Consequently, the given matrices are linearly independent for all k values except -1 and 2.

7. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) The vector equation $a(2,-2,0) + b(6,1,4) + c(2,0,-4) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} 2a & + & 6b & + & 2c & = & 0 \\ -2a & + & 1b & + & 0c & = & 0 \\ 0a & + & 4b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a = b = c = 0$. We conclude that the given vectors are linearly independent, hence they do not lie in a plane.

(b) The vector equation $a(-6, 7, 2) + b(3, 2, 4) + c(4, -1, 2) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} -6a & + & 3b & + & 4c & = & 0 \\ 7a & + & 2b & - & 1c & = & 0 \\ 2a & + & 4b & + & 2c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a = \frac{1}{3}t,\ b = -\frac{2}{3}t,\ c = t$.

Since the system has nontrivial solutions, the given vectors are linearly dependent, hence they lie in a plane.

8. **(a)** The set $\{\mathbf{v}_1, \mathbf{v}_3\}$ can be shown to be linearly independent since $a(-1,2,3) + b(-3,6,0) = (0,0,0)$ has only the trivial solution $a = b = 0$. Therefore the three vectors do not lie on the same line (even though the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ *are* collinear).

(b) Any subset of two vectors chosen from these three vectors can be shown to be linearly independent (e.g., $a(2,-1,4) + b(4,2,3) = (0,0,0)$ has only the trivial solution $a = b = 0$). Therefore the three vectors do not lie on the same line.
(An alternate way to show this would be to demonstrate that the three vectors form a linearly independent set, therefore they do not even lie on the same plane, so that they cannot possibly lie on the same line.)

(c) Each subset of two vectors chosen from these three vectors can be shown to be linearly dependent since $-1\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0}$, $1\mathbf{v}_1 + 2\mathbf{v}_3 = \mathbf{0}$, and $1\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{0}$. Therefore all three vectors lie on the same line.

9. **(a)** The vector equation $a(0, 3, 1, -1) + b(6, 0, 5, 1) + c(4, -7, 1, 3) = (0, 0, 0, 0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 0a & + & 6b & + & 4c & = & 0 \\ 3a & + & 0b & - & 7c & = & 0 \\ 1a & + & 5b & + & 1c & = & 0 \\ -1a & + & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

therefore a general solution of the system is $a = \frac{7}{3}t,\ b = -\frac{2}{3}t,\ c = t.$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) From part (a), we have $\frac{7}{3}t\mathbf{v}_1 - \frac{2}{3}t\mathbf{v}_2 + t\mathbf{v}_3 = 0.$
Letting $t = \frac{3}{7}$, we obtain $\mathbf{v}_1 = \frac{2}{7}\mathbf{v}_2 - \frac{3}{7}\mathbf{v}_3.$
Letting $t = -\frac{3}{2}$, we obtain $\mathbf{v}_2 = \frac{7}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_3.$
Letting $t = 1$, we obtain $\mathbf{v}_3 = -\frac{7}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2.$

10. **(a)** The vector equation $a(1,2,3,4) + b(0,1,0,-1) + c(1,3,3,3) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} 1a & + & 0b & + & 1c & = & 0 \\ 2a & + & 1b & + & 3c & = & 0 \\ 3a & + & 0b & + & 3c & = & 0 \\ 4a & - & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

therefore a general solution of the system is

$$a = -t,\ b = -t,\ c = t$$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) In the general solution we obtained in part (a), let the parameter t have a nonzero value, e.g., $t = 1$. Then $a = -1,\ b = -1$, and $c = 1$ so that $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. This can be solved for each of the three vectors: $\mathbf{v}_1 = -\mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_2 = -\mathbf{v}_1 + \mathbf{v}_3$, and $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

11. By inspection, when $\lambda = -\frac{1}{2}$, the vectors become linearly dependent (since they all become equal). We proceed to find the remaining values of λ.

The vector equation $a\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right) + b\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right) + c\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right) = (0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} \lambda a & - & \frac{1}{2}b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & + & \lambda b & - & \frac{1}{2}c & = & 0 \\ -\frac{1}{2}a & - & \frac{1}{2}b & + & \lambda c & = & 0 \end{array}$$

The determinant of the coefficient matrix is $\begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^3 - \frac{3}{4}\lambda - \frac{1}{4}$. This determinant equals zero for all λ values for which the vectors are linearly dependent. Since we already know that $\lambda = -\frac{1}{2}$ is one of those values, we can divide $\lambda + \frac{1}{2}$ into $\lambda^3 - \frac{3}{4}\lambda - \frac{1}{4}$ to obtain
$\lambda^3 - \frac{3}{4}\lambda - \frac{1}{4} = \left(\lambda + \frac{1}{2}\right)\left(\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}\right) = \left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)(\lambda - 1)$.
We conclude that the vectors form a linearly dependent set for $\lambda = -\frac{1}{2}$ and for $\lambda = 1$.

12. By part (b) of Theorem 4.3.2, a set with one vector is linearly independent if that vector is not $\mathbf{0}$.

13. (a) We calculate $T_A(1,2) = (-1,4)$ and $T_A(-1,1) = (-2,2)$. The vector equation

$$k_1(-1,4) + k_2(-2,2) = (0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcl} -1k_1 & - & 2k_2 & = & 0 \\ 4k_1 & + & 2k_2 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} -1 & -1 \\ 4 & 1 \end{vmatrix} = 6 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly independent set.

(b) We calculate $T_A(1,2) = (-1,2)$ and $T_A(-1,1) = (-2,4)$. Since $(-2,4) = 2(-1,2)$, it follows by Definition 1 that $T_A(\mathbf{u}_1)$ and $T_A(\mathbf{u}_2)$ form a linearly dependent set.

14. (a) We calculate $T_A(1,0,0) = (1,1,2)$, $T_A(2,-1,1) = (3,-1,2)$, and $T_A(0,1,1) = (3,-3,2)$. The vector equation

$$k_1(1,1,2) + k_2(3,-1,2) + k_3(3,-3,2) = (0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcl} 1k_1 & + & 3k_2 & + & 3k_3 & = & 0 \\ 1k_1 & - & 1k_2 & - & 3k_3 & = & 0 \\ 2k_1 & + & 2k_2 & + & 2k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 3 & 3 \\ 1 & -1 & -3 \\ 2 & 2 & 2 \end{vmatrix} = -8 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent.

(b) We calculate $T_A(1,0,0) = (1,1,2)$, $T_A(2,-1,1) = (2,-2,2)$, and $T_A(0,1,1) = (2,-2,2)$. Since $T_A(\mathbf{u}_2) = 1T_A(\mathbf{u}_3)$, it follows that the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly dependent.

15. Three vectors in R^3 lie in a plane if and only if they are linearly dependent when they have their initial points at the origin. (See the discussion following Example 6.)

(a) After the three vectors are moved so that their initial points are at the origin, the resulting vectors do not lie on the same plane. Hence these vectors are linearly independent.

(b) After the three vectors are moved so that their initial points are at the origin, the resulting vectors lie on the same plane. Hence these vectors are linearly dependent.

16. **(a)** From the identity $\sin^2 x + \cos^2 x = 1$ we have $(-1)(6) + (2)(3\sin^2 x) + (3)(2\cos^2 x) = 0$ for all real x. Therefore, the set is linearly dependent.

(b) The equality $ax + b\cos x = 0$ is to hold for all real x. Taking $x = 0$ yields $b = 0$, whereas taking $x = \frac{\pi}{2}$ implies $a = 0$. The set is linearly independent.

(c) The equality $(a)(1) + b\sin x + c\sin 2x = 0$ is to hold for all real x. Taking $x = 0$ yields $a = 0$. When $x = \frac{\pi}{2}$, we obtain $b = 0$. Finally, substituting $x = \frac{\pi}{4}$ results in $c = 0$. The set is linearly independent.

(d) From the identity $\cos^2 x - \sin^2 x = \cos 2x$ we have $(1)(\cos 2x) + (1)(\sin^2 x) + (-1)(\cos^2 x) = 0$ for all real x. Therefore, the set is linearly dependent.

(e) Since $(3-x)^2 = 9 - 6x + x^2$ we can write $(3-x)^2 - (x^2 - 6x) - 9 = 0$ or $(1)(3-x)^2 + (-1)(x^2 - 6x) + \left(-\frac{9}{5}\right)(5) = 0$. The set is linearly dependent.

(f) From Theorem 4.3.2(a), this set is linearly dependent.

17. The Wronskian is $W(x) = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x\sin x - \cos x$. Since $W(x)$ is not identically 0 on $(-\infty, \infty)$ (e.g., $W(0) = -1 \neq 0$), the functions x and $\cos x$ are linearly independent.

18. The Wronskian is $W(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$. Since $W(x)$ is not identically 0 on $(-\infty, \infty)$, $\sin x$ and $\cos x$ are linearly independent.

19. **(a)** The Wronskian is $W(x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x$. Since $W(x)$ is not identically 0 on $(-\infty, \infty)$ (e.g., $W(0) = 1 \neq 0$), the functions 1, x and e^x are linearly independent.

(b) The Wronskian is $W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$. Since $W(x)$ is not identically 0 on $(-\infty, \infty)$, the functions 1, x and x^2 are linearly independent.

20.

$$W(x) = \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{vmatrix} \longleftarrow \text{The Wronskian}$$

$$= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x + x^2 \\ 1 & 2+x & 2 + 4x + x^2 \end{vmatrix} \longleftarrow \text{A common factor of } e^x \text{ from each row was taken through the determinant sign.}$$

$$= e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 2 + 4x \end{vmatrix} \longleftarrow -1 \text{ times the first row was added to the second row and to the third row.}$$

$$= (e^{3x})(1)\begin{vmatrix} 1 & 2x \\ 2 & 2+4x \end{vmatrix} \quad \longleftarrow \text{Cofactor expansion along the first column}$$

$$= (e^{3x})(1)(2+4x-4x) = 2e^{3x}$$

Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

21. $$W(x) = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix} \quad \longleftarrow \text{The Wronskian}$$

$$= \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ 0 & 0 & -2\sin x \end{vmatrix} \quad \longleftarrow \text{The first row was added to the third.}$$

$$= -2\sin x \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \quad \longleftarrow \text{Cofactor expansion along the third row}$$

$$= -2\sin x\,(-\sin^2 x - \cos^2 x)$$

$$= (-2\sin x)(-1) = 2\sin x$$

Since $W(x)$ is not identically 0 on $(-\infty,\infty)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly independent.

True-False Exercises

(a) False. By part (b) of Theorem 4.3.2, a set containing a single *nonzero* vector is linearly independent.

(b) True. This follows directly from Definition 1.

(c) False. For instance $\{(1,1),(2,2)\}$ is a linearly dependent set that does not contain $(0,0)$.

(d) True. If $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$ has only one solution $a=b=c=0$ then $a(k\mathbf{v}_1)+b(k\mathbf{v}_2)+c(k\mathbf{v}_3) = k(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3)$ can only equal $\mathbf{0}$ when $a=b=c=0$ as well.

(e) True. Since the vectors must be nonzero, $\{\mathbf{v}_1\}$ must be linearly independent.
Let us begin adding vectors to the set until the set $\{\mathbf{v}_1,\dots,\mathbf{v}_k\}$ becomes linearly dependent, therefore, by construction, $\{\mathbf{v}_1,\dots,\mathbf{v}_{k-1}\}$ is linearly independent. The equation $c_1\mathbf{v}_1+\cdots+c_{k-1}\mathbf{v}_{k-1}+c_k\mathbf{v}_k=\mathbf{0}$ must have a solution with $c_k \neq 0$, therefore $\mathbf{v}_k = -\frac{c_1}{c_k}\mathbf{v}_1 - \cdots - \frac{c_{k-1}}{c_k}\mathbf{v}_{k-1}$. Let us assume there exists another representation $\mathbf{v}_k = d_1\mathbf{v}_1+\cdots+d_{k-1}\mathbf{v}_{k-1}$. Subtracting both sides yields $\mathbf{0} = \left(d_1 + \frac{c_1}{c_k}\right)\mathbf{v}_1 + \cdots + \left(d_{k-1}+\frac{c_{k-1}}{c_k}\right)\mathbf{v}_{k-1}$. By linear independence of $\{\mathbf{v}_1,\dots,\mathbf{v}_{k-1}\}$, we must have $d_1 = -\frac{c_1}{c_k}$, …, $d_{k-1} = -\frac{c_{k-1}}{c_k}$, which shows that $\mathbf{v}_k$ is a *unique* linear combination of $\mathbf{v}_1,\dots,\mathbf{v}_{k-1}$.

(f) False. The set $\left\{\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix},\begin{bmatrix}1&0\\1&0\end{bmatrix},\begin{bmatrix}0&1\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\}$ is linearly dependent since $\begin{bmatrix}1&1\\0&0\end{bmatrix} = (-1)\begin{bmatrix}0&0\\1&1\end{bmatrix} + \begin{bmatrix}1&0\\1&0\end{bmatrix} + \begin{bmatrix}0&1\\0&1\end{bmatrix}$.

(g) True. Requiring that for all x values $a(x-1)(x+2)+bx(x+2)+cx(x-1)=0$ holds true implies that the equality must be true for any specific x value. Setting $x=0$ yields $a=0$. Likewise, $x=1$ implies $b=0$, and $x=-2$ implies $c=0$. Since $a=b=c=0$ is required, we conclude that the three given polynomials are linearly independent.

(h) False. The functions f_1 and f_2 are linearly dependent if there exist scalars k_1 and k_2, not both equal 0, such that $k_1f_1(x)+k_2f_2(x)=0$ for all real numbers x.

4.4 Coordinates and Basis

1. Vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1)+c_2(3,0)=(0,0)$$

has only the trivial solution. For these vectors to span R^2, it must be possible to express every vector $\mathbf{b}=(b_1,b_2)$ in R^2 as

$$c_1(2,1)+c_2(3,0)=(b_1,b_2)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & 0 \\ c_1 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcl} 2c_1 & + & 3c_2 & = & b_1 \\ c_1 & & & = & b_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}=-3\neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1 and b_2. Therefore the vectors $(2,1)$ and $(3,0)$ are linearly independent and span R^2 so that they form a basis for R^2.

2. Vectors $(3,1,-4)$, $(2,5,6)$, and $(1,4,8)$ are linearly independent if the vector equation

$$c_1(3,1,-4)+c_2(2,5,6)+c_3(1,4,8)=(0,0,0)$$

has only the trivial solution. For these vectors to span R^3, it must be possible to express every vector $\mathbf{b}=(b_1,b_2,b_3)$ in R^3 as

$$c_1(3,1,-4)+c_2(2,5,6)+c_3(1,4,8)=(b_1,b_2,b_3)$$

These two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 3c_1 & + & 2c_2 & + & 1c_3 & = & 0 \\ 1c_1 & + & 5c_2 & + & 4c_3 & = & 0 \\ -4c_1 & + & 6c_2 & + & 8c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 3c_1 & + & 2c_2 & + & 1c_3 & = & b_1 \\ 1c_1 & + & 5c_2 & + & 4c_3 & = & b_2 \\ -4c_1 & + & 6c_2 & + & 8c_3 & = & b_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix}=26\neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values b_1, b_2, and b_3. Therefore the vectors $(3,1,-4)$, $(2,5,6)$, and $(1,4,8)$ are linearly independent and span R^3 so that they form a basis for R^3.

3. Polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent if the equation

$$c_1(x^2+1)+c_2(x^2-1)+c_3(2x-1)=0$$

has only the trivial solution. For these polynomials to span P_2, it must be possible to express every polynomial $a_0+a_1x+a_2x^2$ as

$$c_1(x^2+1)+c_2(x^2-1)+c_3(2x-1)=a_0+a_1x+a_2x^2$$

Grouping the terms on the left hand side of both equations as $(c_1-c_2-c_3)+(2c_3)x+(c_1+c_2)x^2$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & 0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} 1c_1 & - & 1c_2 & - & 1c_3 & = & a_0 \\ 0c_1 & + & 0c_2 & + & 2c_3 & = & a_1 \\ 1c_1 & + & 1c_2 & + & 0c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = -4 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the polynomials x^2+1, x^2-1, and $2x-1$ are linearly independent and span P_2 so that they form a basis for P_2.

4. Polynomials $1+x,\ 1-x, 1-x^2$, and $1-x^3$ are linearly independent if the equation

$$c_1(1+x)+c_2(1-x)+c_3(1-x^2)+c_4(1-x^3)=0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0+a_1x+a_2x^2+a_3x^3$ as

$$c_1(1+x)+c_2(1-x)+c_3(1-x^2)+c_4(1-x^3)=a_0+a_1x+a_2x^2+a_3x^3$$

Grouping the terms on the left hand side of both equations as $(c_1+c_2+c_3+c_4)+(c_1-c_2)x-c_3x^2-c_4x^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & - & 1c_2 & + & 0c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & - & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & - & 1c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 1c_3 & + & 1c_4 & = & a_0 \\ 1c_1 & - & 1c_2 & + & 0c_3 & + & 0c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & - & 1c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & - & 1c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -2 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials $1+x,\ 1-x, 1-x^2$ and $1-x^3$ are linearly independent and span P_3 so that they form a basis for P_3.

5. Matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides yields linear systems

$$\begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & 0 \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & 0 \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 3c_1 & + & 0c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 6c_1 & - & 1c_2 & - & 8c_3 & + & 0c_4 & = & a_{12} \\ 3c_1 & - & 1c_2 & - & 12c_3 & - & 1c_4 & = & a_{21} \\ -6c_1 & + & 0c_2 & - & 4c_3 & + & 2c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix} = 48 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

6. Matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Equating corresponding entries on both sides in each equation yields linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 0c_3 & + & 1c_4 & = & 0 \\ 1c_1 & - & 1c_2 & - & 1c_3 & + & 0c_4 & = & 0 \\ 1c_1 & + & 0c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 1c_1 & + & 0c_2 & + & 0c_3 & + & 0c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 1c_2 & + & 0c_3 & + & 1c_4 & = & a_{11} \\ 1c_1 & - & 1c_2 & - & 1c_3 & + & 0c_4 & = & a_{12} \\ 1c_1 & + & 0c_2 & + & 1c_3 & + & 0c_4 & = & a_{21} \\ 1c_1 & + & 0c_2 & + & 0c_3 & + & 0c_4 & = & a_{22} \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_{11}, a_{12}, a_{21}, and a_{22}. Therefore the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are linearly independent and span M_{22} so that they form a basis for M_{22}.

7. **(a)** Vectors $(2,-3,1)$, $(4,1,1)$, and $(0,-7,1)$ are linearly independent if the vector equation

$$c_1(2,-3,1) + c_2(4,1,1) + c_3(0,-7,1) = (0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcr} 2c_1 & + & 4c_2 & + & 0c_3 & = & 0 \\ -3c_1 & + & 1c_2 & - & 7c_3 & = & 0 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(2, -3, 1)$, $(4, 1, 1)$, and $(0, -7, 1)$ are linearly dependent, they do not form a basis for R^3.

(b) Vectors $(1, 6, 4)$, $(2, 4, -1)$, and $(-1, 2, 5)$ are linearly independent if the vector equation

$$c_1(1, 6, 4) + c_2(2, 4, -1) + c_3(-1, 2, 5) = (0,0,0)$$

has only the trivial solution. This equation can be rewritten as a linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 2c_2 & - & 1c_3 & = & 0 \\ 6c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ 4c_1 & - & 1c_2 & + & 5c_3 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $(1, 6, 4)$, $(2, 4, -1)$, and $(-1, 2, 5)$ are linearly dependent, they do not form a basis for R^3.

8. Vectors $\mathbf{p}_1 = 1 - 3x + 2x^2$, $\mathbf{p}_2 = 1 + x + 4x^2$, and $\mathbf{p}_3 = 1 - 7x$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.
By grouping the terms on the left hand side as $c_1(1 - 3x + 2x^2) + c_2(1 + x + 4x^2) + c_3(1 - 7x) = (c_1 + c_2 + c_3) + (-3c_1 + c_2 - 7c_3)x + (2c_1 + 4c_2)x^2$ this equation can be rewritten as the linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & c_3 & = & 0 \\ -3c_1 & + & c_2 & - & 7c_3 & = & 0 \\ 2c_1 & + & 4c_2 & & & = & 0 \end{array}$$

The coefficient matrix of this system has determinant $\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 0$, thus it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly dependent, we conclude that they do not form a basis for P_2.

9. Matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly independent if the equation

$$c_1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} + c_3\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has only the trivial solution. Equating corresponding entries on both sides yields a linear system

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 2c_2 & + & 1c_3 & + & 0c_4 & = & 0 \\ 0c_1 & - & 2c_2 & - & 1c_3 & - & 1c_4 & = & 0 \\ 1c_1 & + & 3c_2 & + & 1c_3 & + & 1c_4 & = & 0 \\ 1c_1 & + & 2c_2 & + & 0c_3 & + & 1c_4 & = & 0 \end{array}$$

Since the determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = 0$, it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the matrices $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ are linearly dependent, we conclude that they do not form a basis for M_{22}.

10. **(a)** The identity $\cos^2 x - \sin^2 x = \cos 2x$ implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, therefore it is not a basis for V.

(b) For the equation $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0$ (required when $x = 0$) and $c_2 = 0$ (required when $x = \frac{\pi}{2}$). Therefore the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ are linearly independent.

Any vector $\mathbf{v}$ in V can be expressed as $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$. However, from the identity $\cos^2 x - \sin^2 x = \cos 2x$ it follows that we can express $\mathbf{v}$ as a linear combination of $\cos^2 x$ and $\sin^2 x$ alone: $\mathbf{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3(\cos^2 x - \sin^2 x) = (k_1 + k_3)\cos^2 x + (k_2 - k_3)\sin^2 x$. This proves that the vectors $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ span V. We conclude that $\mathbf{v}_1 = \cos^2 x$ and $\mathbf{v}_2 = \sin^2 x$ form a basis for V.

(Note that $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are also bases for V.)

11. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,1) = c_1(2,-4) + c_2(3,8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 3c_2 & = & 1 \\ -4c_1 & + & 8c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{5}{28}$, $c_2 = \frac{3}{14}$, therefore the coordinate vector is $(\mathbf{w})_S = (\frac{5}{28}, \frac{3}{14})$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,b) = c_1(1,1) + c_2(0,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 1c_1 & + & 0c_2 & = & a \\ 1c_1 & + & 2c_2 & = & b \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = a$, $c_2 = \frac{b-a}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = (a, \frac{b-a}{2})$.

12. (a) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & 1 \\ -c_1 & + & c_2 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = (\frac{1}{2}, \frac{1}{2})$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(0,1) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ -c_1 & + & c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{1}{2}$, $c_2 = \frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = (-\frac{1}{2}, \frac{1}{2})$.

13. (a) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ we obtain

$$(2,-1,3) = c_1(1,0,0) + c_2(2,2,0) + c_3(3,3,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ & & 2c_2 & + & 3c_3 & = & -1 \\ & & & & 3c_3 & = & 3 \end{array}$$

which can be solved by back-substitution to obtain $c_3 = 1$, $c_2 = -2$, and $c_1 = 3$. The coordinate vector is $(\mathbf{v})_S = (3, -2, 1)$.

(b) Expressing $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ we obtain

$$(5,-12,3) = c_1(1,2,3) + c_2(-4,5,6) + c_3(7,-8,9)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & - & 4c_2 & + & 7c_3 & = & 5 \\ 2c_1 & + & 5c_2 & - & 8c_3 & = & -12 \\ 3c_1 & + & 6c_2 & + & 9c_3 & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = -2, c_2 = 0$, and $c_3 = 1$. The coordinate vector is $(\mathbf{v})_S = (-2, 0, 1)$.

14. **(a)** Since $\mathbf{p} = 4\mathbf{p}_1 + (-3)\mathbf{p}_2 + 1\mathbf{p}_3$ we conclude that the coordinate vector is $(\mathbf{p})_S = (4, -3, 1)$.

(b) Expressing $\mathbf{p}$ as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ we obtain

$$2 - x + x^2 = c_1(1 + x) + c_2(1 + x^2) + c_3(x + x^2)$$

Grouping the terms on the right hand side according to powers of x yields

$$2 - x + x^2 = (c_1 + c_2) + (c_1 + c_3)x + (c_2 + c_3)x^2$$

For this equality to hold for all real x, the coefficients associated with the same power of x on both sides must match. This leads to the linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & & & = & 2 \\ c_1 & & & + & c_3 & = & -1 \\ & & c_2 & + & c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is $c_1 = 0,\ c_2 = 2,\ c_3 = -1$, therefore the coordinate vector is $(\mathbf{p})_S = (0, 2, -1)$.

15. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 & k_1 + k_2 \\ k_1 + k_2 + k_3 & k_1 + k_2 + k_3 + k_4 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcr} k_1 & & & & & & & = & 0 \\ k_1 & + & k_2 & & & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcr} k_1 & & & & & & & = & a \\ k_1 & + & k_2 & & & & & = & b \\ k_1 & + & k_2 & + & k_3 & & & = & c \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & & & & & & & = & 1 \\ k_1 & + & k_2 & & & & & = & 0 \\ k_1 & + & k_2 & + & k_3 & & & = & 1 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & 0 \end{array}$$

which can be solved by forward-substitution to obtain $k_1 = 1,\ k_2 = -1,\ k_3 = 1,\ k_4 = -1$.
This allows us to express $A = 1A_1 - 1A_2 + 1A_3 - 1A_4$.
The coordinate vector is $(A)_S = (1, -1, 1, -1)$.

16. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 + k_2 + k_3 & k_2 \\ k_1 + k_4 & k_3 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 0 \\ & & k_2 & & & & & = & 0 \\ k_1 & & & & & + & k_4 & = & 0 \\ & & & & k_3 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & a \\ & & k_2 & & & & & = & b \\ k_1 & & & & & + & k_4 & = & c \\ & & & & k_3 & & & = & d \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values $a,\ b, c$ and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 6 \\ & & k_2 & & & & & = & 2 \\ k_1 & & & & & + & k_4 & = & 5 \\ & & & & k_3 & & & = & 3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ therefore the solution is $k_1 = 1,\ k_2 = 2,\ k_3 = 3,\ k_4 = 4$.
This allows us to express $A = 1A_1 + 2A_2 + 3A_3 + 4A_4$. The coordinate vector is $(A)_S = (1, 2, 3, 4)$.

17. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1(1 + x + x^2) + c_2(x + x^2) + c_3x^2 = c_1 + (c_1 + c_2)x + (c_1 + c_2 + c_3)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{ccccccc} c_1 & & & & & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccc} c_1 & & & & & = & a_0 \\ c_1 & + & c_2 & & & = & a_1 \\ c_1 & + & c_2 & + & c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.
To express $\mathbf{p} = 7 - x + 2x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccr} c_1 & & & & & = & 7 \\ c_1 & + & c_2 & & & = & -1 \\ c_1 & + & c_2 & + & c_3 & = & 2 \end{array}$$

which can be solved by forward-substitution to obtain $c_1 = 7,\ c_2 = -8,\ c_3 = 3$.
This allows us to express $\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$. The coordinate vector is $(\mathbf{p})_S = (7, -8, 3)$.

18. Vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent if the vector equation

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

has only the trivial solution. For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

Grouping the terms on the left hand sides as $c_1(1 + 2x + x^2) + c_2(2 + 9x) + c_3(3 + 3x + 4x^2) = (c_1 + 2c_2 + 3c_3) + (2c_1 + 9c_2 + 3c_3)x + (c_1 + 4c_3)x^2$ these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & 0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 0 \\ c_1 & & & + & 4c_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & a_0 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & a_1 \\ c_1 & & & + & 4c_3 & = & a_2 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

To express $\mathbf{p} = 2 + 17x - 3x^2$ as a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 2 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & 17 \\ c_1 & & & + & 4c_3 & = & -3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ therefore the solution is $c_1 = 1$, $c_2 = 2$, $c_3 = -1$. This allows us to express $\mathbf{p} = 1\mathbf{p}_1 + 2\mathbf{p}_2 + (-1)\mathbf{p}_3$.

The coordinate vector is $(\mathbf{p})_S = (1, 2, -1)$.

19. **(a)** The third vector is a sum of the first two. This makes the set linearly dependent, hence it cannot be a basis for R^2.

(b) The two vectors generate a plane in R^3, but they do not span all of R^3. Consequently, the set is not a basis for R^3.

(c) For instance, the polynomial $\mathbf{p} = 1$ cannot be expressed as a linear combination of the given two polynomials. This means these two polynomials do not span P_2, hence they do not form a basis for P_2.

(d) For instance, the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ cannot be expressed as a linear combination of the given four matrices. This means these four matrices do not span M_{22}, hence they do not form a basis for M_{22}.

20. If the set contains at least two vectors, then the zero vector can be expressed as a scalar product of any other vector in the set and zero scalar. According to Definition 1 in Section 4.3, this makes the set linearly dependent.

A set with only one vector is linearly dependent if and only if the vector is a zero vector (see the margin note next to Definition 1 in Section 4.3).

21. **(a)** We have $T_A(1,0,0) = (1,0,-1)$, $T_A(0,1,0) = (1,1,2)$, and $T_A(0,0,1) = (1,-3,0)$. The vector equation

$$k_1(1,0,-1) + k_2(1,1,2) + k_3(1,-3,0) = (0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1k_1 & + & 1k_2 & + & 1k_3 & = & 0 \\ 0k_1 & + & 1k_2 & - & 3k_3 & = & 0 \\ -1k_1 & + & 2k_2 & + & 0k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\det(A) = 10 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly independent.

(b) We have $T_A(1,0,0) = (1,0,-1)$, $T_A(0,1,0) = (1,1,2)$, and $T_A(0,0,1) = (2,1,1)$. By inspection,

$$(2,1,1) = (1,0,-1) + (1,1,2)$$

We conclude that the set $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2), T_A(\mathbf{e}_3)\}$ is linearly dependent.

22. (a) Expressing $T_A(\mathbf{u}) = (4,-2,0)$ as a linear combination of the vectors in S we obtain

$$(4,-2,0) = c_1(1,1,0) + c_2(0,1,1) + c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 0c_2 & + & 1c_3 & = & 4 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & -2 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1 = -2$, $c_2 = -6$, and $c_3 = 6$.

The coordinate vector is $(T_A(\mathbf{u}))_S = (-2,-6,6)$.

(b) Expressing $T_A(\mathbf{u}) = (-2,0,-1)$ as a linear combination of the vectors in S we obtain

$$(-2,0,-1) = c_1(1,1,0) + c_2(0,1,1) + c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 1c_1 & + & 0c_2 & + & 1c_3 & = & -2 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & 0 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & -1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$.

The solution of the linear system is $c_1 = 1$, $c_2 = 2$, and $c_3 = -3$.

The coordinate vector is $(T_A(\mathbf{u}))_S = (1,2,-3)$.

23. We have $\mathbf{u}_1 = (\cos 30°, \sin 30°) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\mathbf{u}_2 = (0,1)$.

(a) By inspection, we can express $\mathbf{w} = (\sqrt{3}, 1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(\sqrt{3}, 1) = 2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + 0(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (2,0)$.

(b) Expressing $\mathbf{w} = (\sqrt{3}, 1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0) = c_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} \frac{\sqrt{3}}{2}c_1 & & & = & 1 \\ \frac{1}{2}c_1 & + & c_2 & = & 0 \end{array}$$

The first equation yields $c_1 = \frac{2}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = -\frac{1}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = (\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

(c) By inspection, we can express $\mathbf{w} = (0,1)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$(0,1) = 0\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + 1(0,1)$$

therefore the coordinate vector is $(\mathbf{w})_S = (0, 1)$.

(d) Expressing $\mathbf{w} = (a, b)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a, b) = c_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + c_2(0,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} \frac{\sqrt{3}}{2}c_1 & & & = & a \\ \frac{1}{2}c_1 & + & c_2 & = & b \end{array}$$

The first equation yields $c_1 = \frac{2a}{\sqrt{3}}$, then the second equation can be solved to obtain $c_2 = b - \frac{a}{\sqrt{3}}$. The coordinate vector is $(\mathbf{w})_S = (\frac{2a}{\sqrt{3}}, b - \frac{a}{\sqrt{3}})$.

24. **(a)** $(0, \sqrt{2})$; **(b)** $(1,0)$; **(c)** $(-1, \sqrt{2})$; **(d)** $(a - b, \sqrt{2}b)$

25. **(a)** Polynomials 1, $2t$, $-2 + 4t^2$, and $-12t + 8t^3$ are linearly independent if the equation

$$c_1(1) + c_2(2t) + c_3(-2 + 4t^2) + c_4(-12t + 8t^3) = 0$$

has only the trivial solution. For these polynomials to span P_3, it must be possible to express every polynomial $a_0 + a_1t + a_2t^2 + a_3t^3$ as

$$c_1(1) + c_2(2t) + c_3(-2 + 4t^2) + c_4(-12t + 8t^3) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Grouping the terms on the left hand side of both equations as
$(c_1 - 2c_3) + (2c_2 - 12c_4)t + 4c_3t^2 + 8c_4t^3$ these equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 0 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & a_0 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & a_1 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & a_2 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & a_3 \end{array}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 64 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, a_2, and a_3. Therefore the polynomials 1, $2t, -2+4t^2$, and $-12t+8t^3$ are linearly independent and span P_3 so that they form a basis for P_3.

(b) To express $\mathbf{p} = -1 - 4t + 8t^2 + 8t^3$ as a linear combination of the four vectors in B, we form the nonhomogeneous system as was done in part (a), with the appropriate right hand side values

$$\begin{array}{rcrcrcrcr} 1c_1 & + & 0c_2 & - & 2c_3 & + & 0c_4 & = & -1 \\ 0c_1 & + & 2c_2 & + & 0c_3 & - & 12c_4 & = & -4 \\ 0c_1 & + & 0c_2 & + & 4c_3 & + & 0c_4 & = & 8 \\ 0c_1 & + & 0c_2 & + & 0c_3 & + & 8c_4 & = & 8 \end{array}$$

Back-substitution yields $c_4 = 1, c_3 = 2, c_2 = 4$, and $c_1 = 3$.
The coordinate vector is $(\mathbf{p})_B = (3, 4, 2, 1)$.

26. **(b)** $(\mathbf{p})_B = (2, -8, 0, 1)$

27. **(a)** $\mathbf{w} = 6(3,1,-4) - 1(2,5,6) + 4(1,4,8) = (20,17,2)$

(b) $\mathbf{q} = 3(x^2+1) + 0(x^2-1) + 4(2x-1) = 3x^2 + 8x - 1$

(c) $B = -8\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 7\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -21 & -103 \\ -106 & 30 \end{bmatrix}$

True-False Exercises

(a) False. The set must also be linearly independent.

(b) False. The subset must also span V.

(c) True. This follows from Theorem 4.4.1.

(d) True. For any vector $\mathbf{v} = (a_1, \ldots, a_n)$ in R^n, we have $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ therefore the coordinate vector of $\mathbf{v}$ with respect to the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is $(\mathbf{v})_S = (a_1, \ldots, a_n) = \mathbf{v}$.

(e) False. For instance, $\{1+t^4, t+t^4, t^2+t^4, t^3+t^4, t^4\}$ is a basis for P_4.

4.5 Dimension

1. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = t,\ x_2 = 0,\ x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (t, 0, t) = t(1, 0, 1)$$

therefore the solution space is spanned by a vector $\mathbf{v}_1 = (1, 0, 1)$. This vector is nonzero, therefore it forms a linearly independent set (Theorem 4.3.2(b)). We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

2. The augmented matrix of the linear system $\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$. The general solution is $x_1 = -\frac{1}{4}s,\ x_2 = -\frac{1}{4}s - t,\ x_3 = s,\ x_4 = t$. In vector form

$$(x_1, x_2, x_3, x_4) = \left(-\frac{1}{4}s, -\frac{1}{4}s - t, s, t\right) = s\left(-\frac{1}{4}, -\frac{1}{4}, 1,0\right) + t(0, -1,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = \left(-\frac{1}{4}, -\frac{1}{4}, 1,0\right)$ and $\mathbf{v}_2 = (0, -1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

3. The augmented matrix of the linear system $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The only solution is $x_1 = x_2 = x_3 = 0$.
The solution space has no basis - its dimension is 0.

4. The augmented matrix of the linear system $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 4r - 3s + t,\ x_2 = r,\ x_3 = s,\ x_4 = t$. In vector form

$$(x_1, x_2, x_3, x_4) = (4r - 3s + t, r, s, t) = r(4, 1, 0, 0) + s(-3, 0,1,0) + t(1, 0, 0, 1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = (4, 1, 0, 0)$, $\mathbf{v}_2 = (-3, 0, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 0, 1)$. By inspection, these vectors are linearly independent since $r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$ implies $r = s = t = 0$. We conclude that $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ form a basis for the solution space and that the dimension of the solution space is 3.

5. The augmented matrix of the linear system $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 3s - t,\ x_2 = s,\ x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (3s - t, s, t) = s(3,1,0) + t(-1,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = (3,1,0)$ and $\mathbf{v}_2 = (-1,0,1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

6. The augmented matrix of the linear system $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x = 4t,\ y = -5t,\ z = t$. In vector form

$$(x, y, z) = (4t, -5t, t) = t(4, -5,1)$$

therefore the solution space is spanned by vector $\mathbf{v}_1 = (4, -5,1)$. By Theorem 4.3.2(b), this vector forms a linearly independent set since it is not the zero vector. We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

7. **(a)** If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = \frac{2}{3}s - \frac{5}{3}t$. Expressing the solution in vector form $(x, y, z) = \left(\frac{2}{3}s - \frac{5}{3}t, s, t\right) = s\left(\frac{2}{3}, 1,0\right) + t\left(-\frac{5}{3}, 0,1\right)$. By Theorem 4.3.2(c), $\left\{\left(\frac{2}{3}, 1,0\right), \left(-\frac{5}{3}, 0,1\right)\right\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\left\{\left(\frac{2}{3}, 1,0\right), \left(-\frac{5}{3}, 0,1\right)\right\}$. The dimension of the subspace is 2.

(b) If we let $y = s$ and $z = t$ be arbitrary values, we can solve the plane equation for x: $x = s$. Expressing the solution in vector form $(x, y, z) = (s, s, t) = s(1,1,0) + t(0,0,1)$. By Theorem 4.3.2(c), $\{(1,1,0), (0,0,1)\}$ is linearly independent since neither vector in the set is a scalar multiple of the other. A basis for the subspace is $\{(1,1,0), (0,0,1)\}$. The dimension of the subspace is 2.

(c) In vector form, $(x, y, z) = (2t, -t, 4t) = t(2, -1, 4)$. By Theorem 4.3.2(b), the vector $(2, -1,4)$ forms a linearly independent set since it is not the zero vector. A basis for the subspace is $\{(2, -1, 4)\}$. The dimension of the subspace is 1.

(d) The subspace contains all vectors $(a, a + c, c) = a(1, 1, 0) + c(0, 1, 1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1, 1, 0), (0, 1, 1)\}$. By Theorem 4.3.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

8. **(a)** The given subspace can be expressed as $\text{span}(S)$ where $S = \{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

(b) The subspace contains all vectors $(a,b,a+b,a-b) = a(1,0,1,1) + b(0,1,1,-1)$ thus we can express it as $\text{span}(S)$ where $S = \{(1,0,1,1),(0,1,1,-1)\}$. By Theorem 4.3.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

(c) The subspace contains all vectors $(a,a,a,a) = a(1,1,1,1)$ thus we can express it as as $\text{span}(S)$ where $S = \{(1,1,1,1)\}$. By Theorem 4.3.2(b), S is linearly independent since it contains a single nonzero vector. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 1.

9. **(a)** Let W be the space of all diagonal $n \times n$ matrices. We can write

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1 \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_1} + d_2 \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_2} + \cdots + d_n \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{A_n}$$

The matrices $A_1,\ldots,A_n$ are linearly independent and they span W; hence, $A_1,\ldots,A_n$ form a basis for W. Consequently, the dimension of W is n.

(b) A basis for this space can be constructed by including the n matrices $A_1,\ldots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1 = \frac{n(n-1)}{2}$ matrices B_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) and (j,i) entries, which are both 1.
For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{B_{12}}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B_{13}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{B_{23}}$$

The dimension is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

(c) A basis for this space can be constructed by including the n matrices $A_1,\ldots,A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1 = \frac{n(n-1)}{2}$ matrices C_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) entry, which is 1.
For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{12}}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{13}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{C_{23}}$$

The dimension is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

10. The given subspace can be expressed as $\text{span}(S)$ where $S = \{x, x^2, x^3\}$ is a set of linearly independent vectors in P_3. Therefore S forms a basis for the subspace. The dimension of the subspace is 3.

11. **(a)** W is the set of all polynomials $a_0 + a_1x + a_2x^2$ for which $a_0 + a_1 + a_2 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 + a_1x + a_2x^2$.
Adding two polynomials in W results in another polynomial in W
$(-a_1 - a_2 + a_1x + a_2x^2) + (-b_1 - b_2 + b_1x + b_2x^2)$
$= (-a_1 - a_2 - b_1 - b_2) + (a_1 + b_1)x + (a_2 + b_2)x^2$
since we have $(-a_1 - a_2 - b_1 - b_2) + (a_1 + b_1) + (a_2 + b_2) = 0$.
Likewise, a scalar multiple of a polynomial in W is also in W
$k(-a_1 - a_2 + a_1x + a_2x^2) = -ka_1 - ka_2 + ka_1x + ka_2x^2$
since it meets the condition $(-ka_1 - ka_2) + (ka_1) + (ka_2) = 0$.
According to Theorem 4.2.1, W is a subspace of P_2.

(c) From part (a), an arbitrary polynomial in W can be expressed in the form

$$-a_1 - a_2 + a_1x + a_2x^2 = a_1(-1 + x) + a_2(-1 + x^2)$$

therefore, the polynomials $-1 + x$ and $-1 + x^2$ span W. Also, $a_1(-1 + x) + a_2(-1 + x^2) = 0$ implies $a_1 = a_2 = 0$, so $-1 + x$ and $-1 + x^2$ are linearly independent, hence they form a basis for W. The dimension of W is 2.

12. **(a)** Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$
(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{vmatrix} = 2 \neq 0$
then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.5.4)

(b) Any of the three standard basis vector for R^3 can be used since none of them is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$
(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} 1 & 3 & 1 \\ -1 & 1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 2 \neq 0$
then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.5.4)

13. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 + k_6\mathbf{e}_4 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcrcl} k_1 & - & 3k_2 & + & k_3 & & & & & & & = & 0 \\ -4k_1 & + & 8k_2 & & & + & k_4 & & & & & = & 0 \\ 2k_1 & - & 4k_2 & & & & & + & k_5 & & & = & 0 \\ -3k_1 & + & 6k_2 & & & & & & & + & k_6 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{bmatrix}$.

Based on the leading entries in the first, second, fourth, and fifth columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent. Since $\dim(R^4) = 4$, it follows by Theorem 4.5.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^4. (The answer is not unique.)

14. The equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = 0$ implies $c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 0$, i.e., $(c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$, which by linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ requires that

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Solving this system by back-substitution yields $c_1 = c_2 = c_3 = 0$ therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Since the dimension of V is 3 (as its basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains three vectors), by Theorem 4.5.4 $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ must also be a basis for V.

15. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcl} k_1 & & & + & k_3 & & & & & = & 0 \\ -2k_1 & + & 5k_2 & & & + & k_4 & & & = & 0 \\ 3k_1 & - & 3k_2 & & & & & + & k_5 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} & 0 \end{bmatrix}$.

Based on the leading entries in the first three columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ are linearly independent. Since $\dim(R^3) = 3$, it follows by Theorem 4.5.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ form a basis for R^3. (The answer is not unique.)

16. One of the infinitely many ways to enlarge the given set to a basis for R^4 is by adding the vectors $(0,0,1,0)$ and $(0,0,0,1)$ to the set. Since the resulting set contains $\dim(R^4) = 4$ vectors, by Theorem 4.5.4 we only need to establish the linear independence of the set to be able to conclude that it forms a basis for R^4. The homogeneous equation $k_1(1,0,0,0) + k_2(1,1,0,0) + k_3(0,0,1,0) + k_4(0,0,0,1) = (0,0,0,0)$ can be rewritten as a linear system whose coefficient matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has determinant 1. Using parts (b) and (g) of Theorem 2.3.8, we conclude that there is only the trivial solution, therefore the enlarged set of four vectors is linearly independent (and, consequently , forms a basis for R^4).

17. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{ccccccccc} 1k_1 & + & 1k_2 & + & 2k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 0k_2 & + & 0k_3 & + & 0k_4 & = & 0 \\ 0k_1 & + & 1k_2 & + & 1k_3 & - & 1k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1 = -s - t, k_2 = -s + t, x_3 = s, k_4 = t$.
Letting $s = 1$ and $t = 0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.
Letting $s = 0$ and $t = 1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2$.
By part (b) of Theorem 4.5.3, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. (The answer is not unique.)

18. The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{ccccccccc} 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 2k_2 & + & 0k_3 & + & 3k_4 & = & 0 \\ 1k_1 & + & 0k_2 & + & 3k_3 & + & 4k_4 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

For arbitrary values of s and t, we have $k_1 = -3s - 4t, k_2 = \frac{3}{2}s + \frac{1}{2}t, k_3 = s, k_4 = t$.

Letting $s = 1$ and $t = 0$ allows us to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_3 = 3\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2$.

Letting $s = 0$ and $t = 1$ allows us to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$: $\mathbf{v}_4 = 4\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$.

By part (b) of Theorem 4.5.3, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Based on the leading entries in the first two columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. (The answer is not unique.)

19. The space of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ for which $T_A(\mathbf{x}) = \mathbf{0}$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -t, x_2 = t, x_3 = t$. In vector form, $(x_1, x_2, x_3) = (-t, t, t) = t(-1,1,1)$. Since $\{(-1,1,1)\}$ is a basis for the space, the dimension is 1.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -2s, x_2 = s, x_3 = t$. In vector form, $(x_1, x_2, x_3) = (-2s, s, t) = s(-2,1,0) + t(0,0,1)$. Since $\{(-2,1,0), (0,0,1)\}$ is a basis for the space, the dimension is 2.

(c) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = 0, x_2 = -t, x_3 = t$. In vector form, $(x_1, x_2, x_3) = (0, -t, t) = t(0, -1, 1)$. Since $\{(0, -1, 1)\}$ is a basis for the space, the dimension is 1.

20. The space of all vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ for which $T_A(\mathbf{x}) = \mathbf{0}$ is the solution space of $A\mathbf{x} = \mathbf{0}$.

(a) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$ so $x_1 = -2s + t, x_2 = -\frac{1}{2}s + \frac{1}{4}t, x_3 = s$, $x_4 = t$. In vector form,
$(x_1, x_2, x_3, x_4) = \left(-2s + t, -\frac{1}{2}s + \frac{1}{4}t, s, t\right) = s\left(-2, -\frac{1}{2}, 1,0\right) + t\left(1, \frac{1}{4}, 0,1\right)$.
Since $\{\left(-2, -\frac{1}{2}, 1,0\right), \left(1, \frac{1}{4}, 0,1\right)\}$ is a basis for the space, the dimension is 2.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ so $x_1 = x_2 = x_3 = -t, x_4 = t$. In vector form, $(x_1, x_2, x_3, x_4) = (-t, -t, -t, t) = t(-1, -1, -1, 1)$.
Since $\{(-1, -1, -1, 1)\}$ is a basis for the space, the dimension is 1.

27. In parts (a) and (b), we will use the results of Exercises 18 and 19 by working with coordinate vectors with respect to the standard basis for P_2, $S = \{1, x, x^2\}$.

(a) Denote $\mathbf{v}_1 = -1 + x - 2x^2$, $\mathbf{v}_2 = 3 + 3x + 6x^2$, $\mathbf{v}_3 = 9$.
Then $(\mathbf{v}_1)_S = (-1,1,-2)$, $(\mathbf{v}_2)_S = (3,3,6)$, $(\mathbf{v}_3)_S = (9,0,0)$.
Setting $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S + k_3(\mathbf{v}_3)_S = \mathbf{0}$ we obtain a linear system with augmented matrix $\begin{bmatrix} -1 & 3 & 9 & 0 \\ 1 & 3 & 0 & 0 \\ -2 & 6 & 0 & 0 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Since there is only the trivial solution, it follows that the three coordinate vectors are linearly independent, and, by the result of Exercise 22, so are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Because the number of these vector matches $\dim(P_2) = 3$, from Theorem 4.5.4 the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for P_2.

(b) Denote $\mathbf{v}_1 = 1 + x$, $\mathbf{v}_2 = x^2$, $\mathbf{v}_3 = 2 + 2x + 3x^2$.
Then $(\mathbf{v}_1)_S = (1,1,0)$, $(\mathbf{v}_2)_S = (0,0,1)$, $(\mathbf{v}_3)_S = (2, 2, 3)$.
Setting $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S + k_3(\mathbf{v}_3)_S = \mathbf{0}$ we obtain a linear system with augmented matrix

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$ whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

This yields solutions $k_1 = -2t, k_2 = -3t, k_3 = t$. Taking $t = 1$, we can express $(\mathbf{v}_3)_S$ as a linear combination of $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$: $(\mathbf{v}_3)_S = 2(\mathbf{v}_1)_S + 3(\mathbf{v}_2)_S$ - the same relationship holds true for the vectors themselves: $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$. By part (b) of Theorem 4.5.3, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Based on the leading entries in the first two columns, the vector equation $k_1(\mathbf{v}_1)_S + k_2(\mathbf{v}_2)_S = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). Therefore the coordinate vectors $(\mathbf{v}_1)_S$ and $(\mathbf{v}_2)_S$ are linearly independent and, by the result of Exercise 18, so are the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

We conclude that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(c) Clearly, $1 + x - 3x^2 = \frac{1}{2}(2 + 2x - 6x^2) = \frac{1}{3}(3 + 3x - 9x^2)$ therefore from Theorem 4.5.3(b), the subspace is spanned by $1 + x - 3x^2$. By Theorem 4.3.2(b), a set containing a single nonzero vector is linearly independent.

We conclude that $1 + x - 3x^2$ forms a basis for this subspace of P_2.

True-False Exercises

(a) True.

(b) True. For instance, $\mathbf{e}_1, \dots, \mathbf{e}_{17}$.

(c) False. This follows from Theorem 4.5.2(b).

(d) True. This follows from Theorem 4.5.4.

(e) True. This follows from Theorem 4.5.4.

(f) True. This follows from Theorem 4.5.5(a).

(g) True. This follows from Theorem 4.5.5(b).

(h) True. For instance, invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for M_{22}.

(i) True. The set has $n^2 + 1$ matrices, which exceeds $\dim(M_{nn}) = n^2$.

(j) False. This follows from Theorem 4.5.6(c).

(k) False. For instance, for any constant c, $\text{span}\{x - c, x^2 - c^2\}$ is a two-dimensional subspace of P_2 consisting of all polynomials in P_2 for which $p(c) = 0$. Clearly, there are infinitely many different subspaces of this type.

4.6 Change of Basis

1. **(a)** In this part, B' is the start basis and B is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{13}{10} & -\frac{1}{2} \\ 0 & 1 & -\frac{2}{5} & 0 \end{array}\right]$$

The transition matrix is $P_{B'\to B} = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{2}{5} & 0 \end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -2 & -\frac{13}{2} \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 4c_2 & = & 3 \\ 2c_1 & - & c_2 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{17}{10} \\ 0 & 1 & \frac{8}{5} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{17}{10}$, $c_2 = \frac{8}{5}$, therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix} \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$ and $\mathbf{u}'_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} c_1 &- c_2 &= 3 \\ 3c_1 &- c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$. The solution of the linear system is $c_1 = -4$, $c_2 = -7$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$. This matches the result obtained in part (c).

2. **(a)** In this part, B' is the start basis and B is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & 1 & 4 \end{array}\right] = [I \mid \text{transition from start to end}]$$

No row operations were necessary to obtain the transition matrix $P_{B'\to B} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$.

(b) In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right]$$

The transition matrix is $P_{B\to B'} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$.

(c) Clearly, $[\mathbf{w}]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$ and $\mathbf{u}'_2$ we obtain

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 2c_1 &- 3c_2 &= 3 \\ c_1 &+ 4c_2 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{3}{11} \\ 0 & 1 & -\frac{13}{11} \end{bmatrix}$. The solution of the linear system is $c_1 = -\frac{3}{11}$, $c_2 = -\frac{13}{11}$, therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

This matches the result obtained in part (c).

3. **(a)** In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{rrr|rrr} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & \frac{5}{2} \\ 0 & 1 & 0 & -2 & -3 & -\frac{1}{2} \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

The transition matrix is $P_{B \to B'} = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2c_1 & + & 2c_2 & + & c_3 & = & -5 \\ c_1 & - & c_2 & + & 2c_3 & = & 8 \\ c_1 & + & c_2 & + & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{bmatrix}$. The solution of the linear system is $c_1 = 9$, $c_2 = -9$, $c_3 = -5$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B \to B'}[\mathbf{w}]_B = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1'$, $\mathbf{u}_2'$ and $\mathbf{u}_3'$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 3c_1 & + & c_2 & - & c_3 & = & -5 \\ c_1 & + & c_2 & & & = & 8 \\ -5c_1 & - & 3c_2 & + & 2c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{2} \\ 0 & 1 & 0 & \frac{23}{2} \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1 = -\frac{7}{2}$, $c_2 = \frac{23}{2}$, $c_3 = 6$ therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} -\frac{7}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$, which matches the result we obtained in part (b).

4. **(a)** In this part, B is the start basis and B' is the end basis:

$$[\text{end basis} \mid \text{start basis}] = \left[\begin{array}{rrr|rrr} -6 & -2 & -2 & -3 & -3 & 1 \\ -6 & -6 & -3 & 0 & 2 & 6 \\ 0 & 4 & 7 & -3 & -1 & -1 \end{array}\right]$$

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right]$$

The transition matrix is $P_{B \to B'} = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} -3c_1 & - & 3c_2 & + & c_3 & = & -5 \\ & & 2c_2 & + & 6c_3 & = & 8 \\ -3c_1 & - & c_2 & - & c_3 & = & -5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution of the linear system is $c_1 = 1$, $c_2 = 1$, $c_3 = 1$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Using Formula (12), $[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{1}{12} \\ -\frac{3}{4} & -\frac{17}{12} & -\frac{17}{12} \\ 0 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}'_1$, $\mathbf{u}'_2$ and $\mathbf{u}'_3$ we obtain

$$\begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = c_1\begin{bmatrix} -6 \\ -6 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} -6c_1 - 2c_2 - 2c_3 &= -5 \\ -6c_1 - 6c_2 - 3c_3 &= 8 \\ 4c_2 + 7c_3 &= -5 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{19}{12} \\ 0 & 1 & 0 & -\frac{43}{12} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$.

The solution of the linear system is $c_1 = \frac{19}{12}$, $c_2 = -\frac{43}{12}$, $c_3 = \frac{4}{3}$ therefore the coordinate vector is $[\mathbf{w}]_{B'} = \begin{bmatrix} \frac{19}{12} \\ -\frac{43}{12} \\ \frac{4}{3} \end{bmatrix}$, which matches the result we obtained in part (b).

5. **(a)** The set $\{\mathbf{f}_1, \mathbf{f}_2\}$ is linearly independent since neither vector is a scalar multiple of the other. Thus $\{\mathbf{f}_1, \mathbf{f}_2\}$ is a basis for V and $\dim(V) = 2$.
Likewise, the set $\{\mathbf{g}_1, \mathbf{g}_2\}$ of vectors in V is linearly independent since neither vector is a scalar multiple of the other. By Theorem 4.5.4, $\{\mathbf{g}_1, \mathbf{g}_2\}$ is a basis for V.

(b) Clearly, $[\mathbf{g}_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $[\mathbf{g}_2]_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ hence $P_{B'\to B} = [[\mathbf{g}_1]_B \mid [\mathbf{g}_2]_B] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

(c) We find the two columns of the transitions matrix $P_{B\to B'} = [[\mathbf{f}_1]_{B'} \mid [\mathbf{f}_2]_{B'}]$

$$\mathbf{f}_1 = a_1\mathbf{g}_1 + a_2\mathbf{g}_2 \qquad\qquad \mathbf{f}_2 = b_1\mathbf{g}_1 + b_2\mathbf{g}_2$$

$$\sin x = a_1(2\sin x + \cos x) + a_2(3\cos x) \qquad\qquad \cos x = b_1(2\sin x + \cos x) + b_2(3\cos x)$$

equate the coefficients corresponding to the same function on both sides of each equation

$$\begin{aligned} 2a_1 \phantom{{}+3a_2} &= 1 \\ a_1 + 3a_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} 2b_1 \phantom{{}+3b_2} &= 0 \\ b_1 + 3b_2 &= 1 \end{aligned}$$

reduced row echelon form of the augmented matrix of each system

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

We obtain the transition matrix $P_{B\to B'} = [[\mathbf{f}_1]_{B'} \mid [\mathbf{f}_2]_{B'}] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.

(An alternate way to solve this part is to use Theorem 4.6.1 to yield

$P_{B\to B'} = P_{B'\to B}^{-1} = \left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\right)^{-1} = \frac{1}{(2)(3)-(0)(1)}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$.)

(d) Clearly, the coordinate vector is $[\mathbf{h}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{h}]_{B'} = P_{B\to B'}[\mathbf{h}]_B = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(e) By inspection, $2\sin x - 5\cos x = (2\sin x + \cos x) - 2(3\cos x)$, hence the coordinate vector is $[\mathbf{p}]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, which matches the result obtained in part (d).

6. **(a)** We find the two columns of the transitions matrix $P_{B'\to B} = [[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B]$

$$\mathbf{q}_1 = a_1\mathbf{p}_1 + a_2\mathbf{p}_2 \qquad \mathbf{q}_2 = b_1\mathbf{p}_1 + b_2\mathbf{p}_2$$

$$2 = a_1(6+3x) + a_2(10+2x) \qquad 3+2x = b_1(6+3x) + b_2(10+2x)$$

equate the coefficients corresponding to like powers of x on both sides of each equation

$$\begin{array}{rcrcl} 6a_1 & + & 10a_2 & = & 2 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \qquad \begin{array}{rcrcl} 6b_1 & + & 10b_2 & = & 3 \\ 3b_1 & + & 2b_2 & = & 2 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{9} \\ 0 & 1 & \frac{1}{3} \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{1}{6} \end{array}\right]$$

We obtain the transition matrix $P_{B'\to B} = [[\mathbf{q}_1]_B \mid [\mathbf{q}_2]_B] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$.

(b) We find the two columns of the transitions matrix $P_{B\to B'} = [[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'}]$

$$\mathbf{p}_1 = a_1\mathbf{q}_1 + a_2\mathbf{q}_2 \qquad \mathbf{p}_2 = b_1\mathbf{q}_1 + b_2\mathbf{q}_2$$

$$6+3x = a_1(2) + a_2(3+2x) \qquad 10+2x = b_1(2) + b_2(3+2x)$$

equate the coefficients corresponding to like powers of x on both sides of each equation

$$\begin{array}{rcrcr} 2a_1 & + & 3a_2 & = & 6 \\ & & 2a_2 & = & 3 \end{array} \qquad \begin{array}{rcrcr} 2b_1 & + & 3b_2 & = & 10 \\ & & 2b_2 & = & 2 \end{array}$$

reduced row echelon form of the augmented matrix of each system

$$\begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & \frac{3}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

We obtain the transition matrix $P_{B\to B'} = [[\mathbf{p}_1]_{B'} \mid [\mathbf{p}_2]_{B'}] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$.

(c) Since $-4 + x = (6 + 3x) - (10 + 2x)$, the coordinate vector is $[\mathbf{p}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Using Formula (12), we obtain $[\mathbf{p}]_{B'} = P_{B\to B'}[\mathbf{p}]_B = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$.

(d) We are looking for the coordinate vector $[\mathbf{p}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ with c_1 and c_2 satisfying the equality

$$-4 + x = c_1(2) + c_2(3 + 2x)$$

for all real values x. Equating the coefficients associated with like powers of x on both sides yields the linear system

$$\begin{array}{rcrcr} 2c_1 & + & 3c_2 & = & -4 \\ & & 2c_2 & = & 1 \end{array}$$

which can easily be solved by back-substitution: $c_2 = \frac{1}{2}$, $c_1 = \frac{-4-3\left(\frac{1}{2}\right)}{2} = -\frac{11}{4}$. We conclude that $[\mathbf{p}]_{B'} = \begin{bmatrix} -\frac{11}{4} \\ \frac{1}{2} \end{bmatrix}$, which matches the result obtained in part (c).

7. **(a)** In this part, B_2 is the start basis and B_1 is the end basis:

[end basis | start basis] $= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid$ transition from start to end$] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right]$.

The transition matrix is $P_{B_2\to B_1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$.

(b) In this part, B_1 is the start basis and B_2 is the end basis:

[end basis | start basis] $= \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid$ transition from start to end$] = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right]$.

The transition matrix is $P_{B_1 \to B_2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$.

(c) Since $\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $P_{B_2 \to B_1}$ and $P_{B_1 \to B_2}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} c_1 & + & 2c_2 & = & 0 \\ 2c_1 & + & 3c_2 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 2$, $c_2 = -1$, therefore the coordinate vector is $[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_2} = P_{B_1 \to B_2}[\mathbf{w}]_{B_1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(e) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ we obtain

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcl} 1c_1 & + & 1c_2 & = & 2 \\ 3c_1 & + & 4c_2 & = & 5 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$. The solution of the linear system is $c_1 = 3$, $c_2 = -1$, therefore the coordinate vector is $[\mathbf{w}]_{B_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_{B_1} = P_{B_2 \to B_1}[\mathbf{w}]_{B_2} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

8. **(a)** By Theorem 4.6.2, $P_{B \to S} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis: [end basis | start basis] = $\left[\begin{array}{rr|rr} 2 & -3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$.

The reduced row echelon form of this matrix is

$$[I \mid \text{transition from start to end}] = \left[\begin{array}{rr|rr} 1 & 0 & \frac{4}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{2}{11} \end{array}\right].$$

The transition matrix is $P_{S\to B} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}$.

(c) Since $\begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Since $(5,-3) = (2,1) - (-3,4)$ the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_S = P_{B\to S}[\mathbf{w}]_B = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. From Formula (12), $[\mathbf{w}]_B = P_{S\to B}[\mathbf{w}]_S = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$.

9. **(a)** By Theorem 4.6.2, $P_{B\to S} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

(b) In this part, S is the start basis and B is the end basis: [end basis | start basis] = $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$.

The reduced row echelon form of this matrix is

$[I \mid \text{transition from start to end}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$.

The transition matrix is $P_{S\to B} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

(c) Since $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ it follows that $P_{B\to S}$ and $P_{S\to B}$ are inverses of one another.

(d) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ we obtain

$$\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} c_1 & + & 2c_2 & + & 3c_3 & = & 5 \\ 2c_1 & + & 5c_2 & + & 3c_3 & = & -3 \\ c_1 & & & + & 8c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -239 \\ 0 & 1 & 0 & 77 \\ 0 & 0 & 1 & 30 \end{bmatrix}$. The solution of the linear system is $c_1 = -239$, $c_2 = 77$, $c_3 = 30$ therefore the coordinate vector is $[\mathbf{w}]_B = \begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix}$. From Formula (12), $[\mathbf{w}]_S = P_{B\to S}[\mathbf{w}]_B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -239 \\ 77 \\ 30 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

(e) By inspection, $[\mathbf{w}]_S = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$.

From Formula (12), $[\mathbf{w}]_B = P_{S\to B}[\mathbf{w}]_S = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} -200 \\ 64 \\ 25 \end{bmatrix}$.

10. Reflecting $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ about the line $y = x$ results in $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Likewise for $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we obtain $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

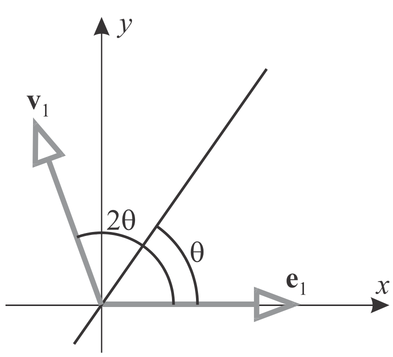
(a) From Theorem 4.6.5, $P_{B\to S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

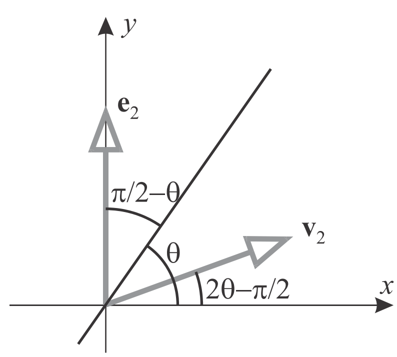
(b) Denoting $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it follows from Theorem 4.6.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

11. (a) Clearly, $\mathbf{v}_1 = (\cos(2\theta), \sin(2\theta))$. Referring to the figure on the right, we see that the angle between the positive x-axis and $\mathbf{v}_2$ is $\frac{\pi}{2} - 2\left(\frac{\pi}{2} - \theta\right) = 2\theta - \frac{\pi}{2}$. Hence,

$$\mathbf{v}_2 = \left(\cos\left(2\theta - \frac{\pi}{2}\right), \sin\left(2\theta - \frac{\pi}{2}\right)\right) = (\sin(2\theta), -\cos(2\theta))$$

From Theorem 4.6.5, $P_{B\to S} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.





(b) Denoting $P = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$, it follows from Theorem 4.6.5 that $P_{S\to B} = P^{-1}$. In our case, $PP = I$ therefore $P = P^{-1}$. Furthermore, since P is symmetric, we also have $P_{S\to B} = P^T$.

12. Since for every vector $\mathbf{v}$ in R^2 we have $[\mathbf{v}]_{B_2} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}[\mathbf{v}]_{B_1}$ and $[\mathbf{v}]_{B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}[\mathbf{v}]_{B_2}$, it follows that $[\mathbf{v}]_{B_3} = \begin{bmatrix} 7 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}[\mathbf{v}]_{B_1} = \begin{bmatrix} 31 & 11 \\ 7 & 2 \end{bmatrix}[\mathbf{v}]_{B_1}$ so that $P_{B_1 \to B_3} = \begin{bmatrix} 31 & 11 \\ 7 & 2 \end{bmatrix}$.

From Theorem 4.6.1, $P_{B_3 \to B_1}$ is the inverse of this matrix: $\begin{bmatrix} -\frac{2}{15} & \frac{11}{15} \\ \frac{7}{15} & -\frac{31}{15} \end{bmatrix}$.

13. Since for every vector $\mathbf{v}$ we have $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$ and $[\mathbf{v}]_C = Q[\mathbf{v}]_B$, it follows that $[\mathbf{v}]_C = QP[\mathbf{v}]_{B'}$ so that $P_{B' \to C} = QP$. From Theorem 4.6.1, $P_{C \to B'} = (QP)^{-1} = P^{-1}Q^{-1}$.

15. **(a)** By Theorem 4.6.2, P is the transition matrix from $B = \{(1,1,0),(1,0,2),(0,2,1)\}$ to S.

(b) By Theorem 4.6.1, $P^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$ is the transition matrix from B to S, hence by Theorem 4.6.2, $B = \left\{\left(\frac{4}{5}, \frac{1}{5}, -\frac{2}{5}\right), \left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}\right), \left(-\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)\right\}$.

16. Let the given basis be denoted as $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, $\mathbf{v}_3 = (1,0,0)$ and denote the unknown basis as $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We have $P_{B \to B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} = [[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid [\mathbf{u}_3]_{B'}]$. Equating the respective columns yields

$$[\mathbf{u}_1]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 = (1,1,1)$$

$$[\mathbf{u}_2]_{B'} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_2 = 0\mathbf{v}_1 + 3\mathbf{v}_2 + 1\mathbf{v}_3 = (4,3,0)$$

$$[\mathbf{u}_3]_{B'} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_3 = 0\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = (3,2,0)$$

Thus the given matrix is the transition matrix from the basis $\{(1,1,1),(4,3,0),(3,2,0)\}$.

17. From $T(1,0) = (2,5)$, $T(0,1) = (3,-1)$, and Theorem 4.6.2 we obtain $P_{B \to S} = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$.

18. From $T(1,0,0) = (1,2,0)$, $T(0,1,0) = (1,-1,1)$, $T(0,0,1) = (0,4,3)$, and Theorem 4.6.2 we obtain $P_{B \to S} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$.

19. By Formula (10), the transition matrix from the standard basis $S = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to B is $P_{S \to B} = [[\mathbf{e}_1]_B | \dots | [\mathbf{e}_n]_B] = [\mathbf{e}_1 | \dots | \mathbf{e}_n] = I_n$ therefore B must be the standard basis.

True-False Exercises

(a) True. The matrix can be constructed according to Formula (10).

(b) True. This follows from Theorem 4.6.1.

(c) True.

(d) True.

(e) False. For instance, $B_1 = \{(0,2),(3,0)\}$ is a basis for R^2 made up of scalar multiples of vectors in the standard basis $B_2 = \{(1,0),(0,1)\}$. However, $P_{B_1 \to B_2} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ (obtained by Theorem 4.6.2) is not a diagonal matrix.

(f) False. A must be invertible.

4.7 Row Space, Column Space, and Null Space

1. **(a)** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

2. **(a)** $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 5\begin{bmatrix} 5 \\ -8 \end{bmatrix}$

3. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, thus $A\mathbf{x} = \mathbf{b}$ is inconsistent. By Theorem 4.7.1, $\mathbf{b}$ is not in the column space of A.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so the system has a unique solution $x_1 = 1, x_2 = -3, x_3 = 1$. By Theorem 4.7.1, $\mathbf{b}$ is in the column space of A. By Formula (2), we can write $\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$.

4. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, thus $A\mathbf{x} = \mathbf{b}$ is inconsistent.
By Theorem 4.7.1, $\mathbf{b}$ is not in the column space of A.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$, so the system has a unique solution $x_1 = -26, x_2 = 13, x_3 = -7, x_4 = 4$.

By Theorem 4.7.1, $\mathbf{b}$ is in the column space of A.

By Formula (2), we can write $-26\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+13\begin{bmatrix}2\\1\\2\\1\end{bmatrix}-7\begin{bmatrix}0\\2\\1\\2\end{bmatrix}+4\begin{bmatrix}1\\1\\3\\2\end{bmatrix}=\begin{bmatrix}4\\3\\5\\7\end{bmatrix}$.

5. **(a)** $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=r\begin{bmatrix}5\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ **(b)** $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3\\0\\-1\\5\end{bmatrix}+r\begin{bmatrix}5\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

6. **(a)** $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$ **(b)** $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-1\\2\\4\\-3\end{bmatrix}+r\begin{bmatrix}-3\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}4\\-1\\0\\1\end{bmatrix}$

7. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&-3&1\\0&0&0\end{bmatrix}$.
The general solution of this system is $x_1=1+3t, x_2=t$; in vector form, $(x_1,x_2)=(1+3t,t)=(1,0)+t(3,1)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1,x_2)=t(3,1)$.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&0&1&-2\\0&1&1&7\\0&0&0&0\end{bmatrix}$. The general solution of this system is $x_1=-2-t, x_2=7-t, x_3=t$; in vector form, $(x_1,x_2,x_3)=(-2-t,7-t,t)=(-2,7,0)+t(-1,-1,1)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1,x_2,x_3)=t(-1,-1,1)$.

8. **(a)** The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&-2&1&2&-1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution of this system is $x_1=-1+2r-s-2t, x_2=r, x_3=s, x_4=t$; in vector form, $(x_1,x_2,x_3,x_4)=(-1+2r-s-2t,r,s,t)=(-1,0,0,0)+r(2,1,0,0)+s(-1,0,1,0)+t(-2,0,0,1)$.
The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is
$(x_1,x_2,x_3,x_4)=r(2,1,0,0)+s(-1,0,1,0)+t(-2,0,0,1)$.

(b) The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix}1&0&-\frac{7}{5}&-\frac{1}{5}&\frac{6}{5}\\0&1&-\frac{4}{5}&\frac{3}{5}&\frac{7}{5}\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution of this system is $x_1=\frac{6}{5}+\frac{7}{5}s+\frac{1}{5}t, x_2=\frac{7}{5}+\frac{4}{5}s-\frac{3}{5}t$, $x_3=s, x_4=t$; in vector form, $(x_1,x_2,x_3,x_4)=\left(\frac{6}{5}+\frac{7}{5}s+\frac{1}{5}t,\frac{7}{5}+\frac{4}{5}s-\frac{3}{5}t,s,t\right)=\left(\frac{6}{5},\frac{7}{5},0,0\right)+s\left(\frac{7}{5},\frac{4}{5},1,0\right)+t\left(\frac{1}{5},-\frac{3}{5},0,1\right)$.

The vector form of the general solution of $A\mathbf{x} = \mathbf{0}$ is
$(x_1, x_2, x_3, x_4) = s\left(\frac{7}{5}, \frac{4}{5}, 1, 0\right) + t\left(\frac{1}{5}, -\frac{3}{5}, 0, 1\right)$.

9. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = 16t,\ x_2 = 19t,\ x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ therefore the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & -16 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -19 \end{bmatrix}$.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = \frac{1}{2}t,\ x_2 = s,\ x_3 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero row of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \end{bmatrix}$.

10. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = -s + \frac{2}{7}t,\ x_2 = -s - \frac{4}{7}t,$ $x_3 = s,\ x_4 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & \frac{4}{7} \end{bmatrix}$.

(b) The reduced row echelon form of A is $\begin{bmatrix}1&0&1&2&1\\0&1&1&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x}=\mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1=-r-2s-t,\ x_2=-r-s-2t,\ x_3=r,\ x_4=s,\ x_5=t$ can be written in the vector form $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=r\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}+s\begin{bmatrix}-2\\-1\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\-2\\0\\0\\1\end{bmatrix}$ therefore the vectors $\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}-1\\-2\\0\\0\\1\end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $[1\quad 0\quad 1\quad 2\quad 1]$ and $[0\quad 1\quad 1\quad 1\quad 2]$.

11. We use Theorem 4.7.5 to obtain the following answers.

(a) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$.

Nonzero rows form a basis for the row space: $[1\quad 0\quad 2], [0\quad 0\quad 1]$.

(b) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}$.

Nonzero rows form a basis for the row space: $[1\quad -3\quad 0\quad 0], [0\quad 1\quad 0\quad 0]$.

12. We use Theorem 4.7.5 to obtain the following answers.

(a) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}4\\-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}5\\0\\-3\\1\\0\end{bmatrix}$.

Nonzero rows form a basis for the row space:
$[1\quad 2\quad 4\quad 5], [0\quad 1\quad -3\quad 0], [0\quad 0\quad 1\quad -3], [0\quad 0\quad 0\quad 1]$.

(b) Columns containing leading 1's form a basis for the column space: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\4\\1\\0\end{bmatrix}, \begin{bmatrix}5\\3\\-7\\1\end{bmatrix}$.

Nonzero rows form a basis for the row space:
$[1\quad 2\quad -1\quad 5], [0\quad 1\quad 4\quad 3], [0\quad 0\quad 1\quad -7], [0\quad 0\quad 0\quad 1]$.

13. **(a)** The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

By Theorems 4.7.4 and 4.7.5, the nonzero rows of B form a basis for the row space of A:
$\mathbf{r}_1 = [1 \quad 0 \quad 11 \quad 0 \quad 3]$, $\mathbf{r}_2 = [0 \quad 1 \quad 3 \quad 0 \quad 0]$, and $\mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 1 \quad 0]$.
By Theorem 4.7.5, columns of B containing leading 1's form a basis for the column space of B:
$\mathbf{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{c}_4' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. By Theorem 4.7.6(b), a basis for the column space of A is formed by the corresponding columns of A: $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 5 \\ 3 \\ 8 \end{bmatrix}$, and $\mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) We begin by transposing the matrix A.
We obtain $A^T = \begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$, whose reduced row echelon form is $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
By Theorem 4.7.5, columns of C containing leading 1's form a basis for the column space of C:
$\mathbf{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}_2' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{c}_3' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. By Theorem 4.7.6(b), a basis for the column space of A^T is formed by the corresponding columns of A^T: $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 5 \\ -7 \\ 0 \\ -6 \end{bmatrix}$, and $\mathbf{c}_3 = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \\ -3 \end{bmatrix}$.
Since columns of A^T are rows of A, a basis for the row space of A is formed by
$\mathbf{r}_1 = [1 \quad -2 \quad 5 \quad 0 \quad 3]$, $\mathbf{r}_2 = [-2 \quad 5 \quad -7 \quad 0 \quad -6]$, and $\mathbf{r}_3 = [-1 \quad 3 \quad -2 \quad 1 \quad -3]$.

14. We construct a matrix whose columns are the given vectors: $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ -4 & 2 & 3 \\ -3 & -2 & 2 \end{bmatrix}$. The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. By Theorem 4.7.5, the three columns of B form a basis for the column space of B. By Theorem 4.7.6(b), the three columns of A form a basis for the column space of A. We conclude that $\{(1, 1, -4, -3), (2, 0, 2, -2), (2, -1, 3, 2)\}$ is a basis for the subspace of R^4 spanned by these vectors.

15. We construct a matrix whose columns are the given vectors: $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By Theorem 4.7.5, the four columns of B form a basis for the column space of B. By Theorem 4.7.6(b), the four columns of A form a basis for the column space of A. We conclude that $\{(1,1,0,0),(0,0,1,1),(-2,0,2,2),(0,-3,0,3)\}$ is a basis for the subspace of R^4 spanned by these vectors.

16. Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$: $A = \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{cccc} \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} -2 \\ 1 \\ 0 \\ 0 \end{matrix}\right] \\ \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 \end{array}$$

contains leading 1's in the first two columns, by Theorems 4.7.5 and 4.7.6(b), the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the column space of A, and for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{w}_4 = -2\mathbf{w}_1 + \mathbf{w}_2$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = -2\mathbf{v}_1 + \mathbf{v}_2$.

17. Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$: $A = \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{ccccc} \left[\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ -1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} -1 \\ 3 \\ 2 \\ 0 \end{matrix}\right] \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{array}$$

contains leading 1's in columns 1, 2, and 4, by Theorems 4.7.5 and 4.7.6(b), the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ form a basis for the column space of A, and for $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

By inspection, the columns of the reduced row echelon form matrix satisfy $\mathbf{w}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$ and $\mathbf{w}_5 = -\mathbf{w}_1 + 3\mathbf{w}_2 + 2\mathbf{w}_4$. Because elementary row operations preserve dependence relations between column vectors, we conclude that $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_5 = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_4$.

18. We are employing the procedure developed in Example 9.

The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.7.5 and 4.7.6(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $[1 \quad 4 \quad 5 \quad 2]$ and $[2 \quad 1 \quad 3 \quad 0]$, form a basis for the row space of A.

19. We are employing the procedure developed in Example 9.

The reduced row echelon form of $A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \frac{13}{14} \\ 0 & 1 & -\frac{2}{7} & \frac{5}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the first two columns of the reduced row echelon form contain leading 1's, by Theorems 4.7.5 and 4.7.6(b) the first two columns of A^T form a basis for the column space of A^T. Consequently, the first two rows of A, $[1 \quad 4 \quad 5 \quad 6 \quad 9]$ and $[3 \quad -2 \quad 1 \quad 4 \quad -1]$, form a basis for the row space of A.

20. Let $B = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -2 \\ 2 & 4 \end{bmatrix}$. We are looking for a matrix A such that $AB = O$. Taking a transpose on both sides results in $B^T A^T = 0^T$. We proceed to solve the homogeneous linear system $B^T\mathbf{u} = \mathbf{0}$. The reduced row echelon form of its augmented matrix $\begin{bmatrix} 1 & -1 & 3 & 2 & 0 \\ 2 & 0 & -2 & 4 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 & 0 \end{bmatrix}$ therefore the general solution in the vector form is $s\begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. We can take $A^T = \begin{bmatrix} 1 & -2 \\ 4 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ thus $A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$.

21. Since $T_A(\mathbf{x}) = A\mathbf{x}$, we are seeking the general solution of the linear system $A\mathbf{x} = \mathbf{b}$.

(a) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 4 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{8}{3} & 0 \\ 0 & 1 & -\frac{4}{3} & 0 \end{bmatrix}$.

The general solution is $x_1 = -\frac{8}{3}t, x_2 = \frac{4}{3}t, x_3 = t$. In vector form, $\mathbf{x} = t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

(b) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 4 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{8}{3} & \frac{7}{3} \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} \end{bmatrix}$.

The general solution is $x_1 = \frac{7}{3} - \frac{8}{3}t, x_2 = -\frac{2}{3} + \frac{4}{3}t, x_3 = t$.

In vector form, $\mathbf{x} = \left(\frac{7}{3}, -\frac{2}{3}, 0\right) + t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

(c) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 4 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{8}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{4}{3} & -\frac{2}{3} \end{bmatrix}$.

The general solution is $x_1 = \frac{1}{3} - \frac{8}{3}t, x_2 = -\frac{2}{3} + \frac{4}{3}t, x_3 = t$.

In vector form, $\mathbf{x} = \left(\frac{1}{3}, -\frac{2}{3}, 0\right) + t\left(-\frac{8}{3}, \frac{4}{3}, 1\right)$ where t is arbitrary.

22. Since $T_A(\mathbf{x}) = A\mathbf{x}$, we are seeking the general solution of the linear system $A\mathbf{x} = \mathbf{b}$.

(a) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The only solution is $x_1 = x_2 = 0$. In vector form, $\mathbf{x} = (0, 0)$.

(b) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The system has no solution; no vector $\mathbf{x}$ exists for which $T_A(\mathbf{x}) = \mathbf{b}$.

(c) The reduced row echelon form of the augmented matrix $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The system has no solution; no vector $\mathbf{x}$ exists for which $T_A(\mathbf{x}) = \mathbf{b}$.

23. **(a)** The reduced row echelon form of A is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution $\mathbf{x} = (x, y, z)$ of $A\mathbf{x} = \mathbf{0}$ is $x = 0, y = 0, z = t$; in vector form, $\mathbf{x} = t(0,0,1)$. This shows that the null space of A consists of all points on the z-axis.

The column space of A, span$\{(1,0,0), (0,1,0)\}$ clearly consists of all points in the xy-plane.

(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an example of such a matrix.

24. **(a)** e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ null space is the origin

(b) e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ null space is the z-axis

(c) e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ null space is the yz-plane

25. **(a)** By inspection, $\begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix}$ has the desired null space. In general, this will hold true for all matrices of the form $\begin{bmatrix} 3a & -5a \\ 3b & -5b \end{bmatrix}$ where a and b are not both zero (if $a = b = 0$ then the null space is the entire plane).

(b) Only the zero vector forms the null space for both A and B (their determinants are nonzero, therefore in each case the corresponding homogeneous system has only the trivial solution).

The line $3x + y = 0$ forms the null space for C.

The entire plane forms the null space for D.

True-False Exercises

(a) True.

(b) False. The column space of A is the space spanned by all column vectors of A.

(c) False. Those column vectors form a basis for the column space of R.

(d) False. This would be true if A were in row echelon form.

(e) False. For instance $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ have the same row space, but different column spaces.

(f) True. This follows from Theorem 4.7.3.

(g) True. This follows from Theorem 4.7.4.

(h) False. Elementary row operations generally can change the column space of a matrix.

(i) True. This follows from Theorem 4.7.1.

(j) False. Let both A and B be $n \times n$ matrices. By Theorem 4.7.4, row operations do not change the row space of a matrix. An invertible matrix can be reduced to I thus its row space is always R^n. On the other hand, a singular matrix cannot be reduced to identity matrix - at least one row in its reduced row echelon form is made up of zeros. Consequently, its row space is spanned by fewer than n vectors, therefore the dimension of this space is less than n.

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

1. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\text{rank}(A) = 1$ (the number of leading 1's)
- $\text{nullity}(A) = 3$ (by Theorem 4.8.2).

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\text{rank}(A) = 2$ (the number of leading 1's)
- $\text{nullity}(A) = 3$ (by Theorem 4.8.2).

2. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\text{rank}(A) = 3$ (the number of leading 1's)
- $\text{nullity}(A) = 2$ (by Theorem 4.8.2).

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. We have

- $\text{rank}(A) = 3$ (the number of leading 1's)
- $\text{nullity}(A) = 1$ (by Theorem 4.8.2).

3. **(a)** $\text{rank}(A) = 3$; $\text{nullity}(A) = 0$

(b) $\text{rank}(A) + \text{nullity}(A) = 3 + 0 = 3 = n \leftarrow$ number of columns of A

(c) 3 leading variables; 0 parameters in the general solution (the solution is unique)

4. **(a)** $\text{rank}(A) = 2$; $\text{nullity}(A) = 1$;

(b) $\text{rank}(A) + \text{nullity}(A) = 2 + 1 = 3 = n \leftarrow$ number of columns of A

(c) 2 leading variables; 1 parameter in the general solution

5. **(a)** $\text{rank}(A) = 1$; $\text{nullity}(A) = 2$

(b) $\text{rank}(A) + \text{nullity}(A) = 1 + 2 = 3 = n \leftarrow$ number of columns of A

(c) 1 leading variable; 2 parameters in the general solution

6. **(a)** $\text{rank}(A) = 3$; $\text{nullity}(A) = 1$;

(b) $\text{rank}(A) + \text{nullity}(A) = 3 + 1 = 4 = n \leftarrow$ number of columns of A

(c) 3 leading variables; 1 parameter in the general solution

7. **(a)** If every column of the reduced row echelon form of a 4×4 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 4
- the nullity of A has the smallest possible value: 0

(b) If every row of the reduced row echelon form of a 3×5 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 3
- the nullity of A has the smallest possible value: 2

(c) If every column of the reduced row echelon form of a 5×3 matrix A contains a leading 1 then

- the rank of A has its largest possible value: 3
- the nullity of A has the smallest possible value: 0

8. The largest possible value for the rank of an $m \times n$ matrix A is the smaller of the two dimensions of A:

- n if $m \geq n$ (when every column of the reduced row echelon form of A contains a leading 1),
- m if $m < n$ (when every row of the reduced row echelon form of A contains a leading 1).

The smallest possible value for the nullity of an $m \times n$ matrix A is

- 0 if $m \geq n$ (when every column of the reduced row echelon form of A contains a leading 1),
- $n - m$ if $m < n$ (when every row of the reduced row echelon form of A contains a leading 1).

9.

			(a)	**(b)**	**(c)**	**(d)**	**(e)**	**(f)**	**(g)**
	Size of A:	$m \times n$	3×3	3×3	3×3	5×9	5×9	4×4	6×2
	rank(A)	$= r$	3	2	1	2	2	0	2
	rank($A \mid \mathbf{b}$)	$= s$	3	3	1	2	3	0	2
(i)	dimension of the row space of A	$= r$	3	2	1	2	2	0	2
	dimension of the column space of A	$= r$	3	2	1	2	2	0	2
	dimension of the null space of A	$= n - r$	0	1	2	7	7	4	0
	dimension of the null space of A^T	$= m - r$	0	1	2	3	3	4	4
(ii)	is the system $A\mathbf{x} = \mathbf{b}$ consistent?	Is $r = s$?	Yes	No	Yes	Yes	No	Yes	Yes
(iii)	number of parameters in the general solution of $A\mathbf{x} = \mathbf{b}$	$= n - r$ if consistent	0	-	2	7	-	4	0

10. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ whereas the reduced row echelon form of A^T is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We conclude that $\text{rank}(A) = \text{rank}(A^T) = 2$.

11. **(a)** Applying Formula (4) to both A and its transpose yields

$$2 + \text{nullity}(A) = 4 \ \text{ and } \ 2 + \text{nullity}(A^T) = 3$$

therefore

$$\text{nullity}(A) - \text{nullity}(A^T) = 1$$

(b) Applying Formula (4) to both A and its transpose yields

$$\text{rank}(A) + \text{nullity}(A) = n \ \text{ and } \ \text{rank}(A^T) + \text{nullity}(A^T) = m$$

By Theorem 4.8.4, $\text{rank}(A^T) = \text{rank}(A)$ therefore

$$\text{nullity}(A) - \text{nullity}(A^T) = n - m$$

12. $T(x_1, x_2) = \begin{bmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Its reduced row echelon form is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) $\operatorname{rank}(A) = 2$ **(b)** $\operatorname{nullity}(A) = 0$

13. $T(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 + x_4 \\ x_4 + x_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$; the standard matrix is

$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

(a) $\operatorname{rank}(A) = 3$ **(b)** $\operatorname{nullity}(A) = 2$

14. (a) The determinant of A is

$$\begin{vmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ t-1 & 0 & 1-t \end{vmatrix}$$ ← -1 times the first row was added to the second row and to the third row.

$$= \begin{vmatrix} 1 & 1+t & t \\ 0 & 0 & 1-t \\ t-1 & 1-t & 1-t \end{vmatrix}$$ ← Last column was added to the second column.

$$= -(1-\mathrm{t})\begin{vmatrix} 1 & 1+t \\ t-1 & 1-t \end{vmatrix}$$ ← Cofactor expansion along the second row.

$$= -(1-t)((1-t) - (1+t)(t-1))$$

$$= -(1-t)^2(2+t)$$

From parts (g) and (n) of Theorem 4.8.4, $\operatorname{rank}(A) = 3$ when $\det(A) \neq 0$, i.e. for all t values other than 1 or -2.

If $t = 1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 1.

If $t = -2$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

(b) The determinant of A is

$$\begin{vmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{vmatrix} = \begin{vmatrix} t & 3 & -1 \\ 3-2t & 0 & 0 \\ -1 & -3 & t \end{vmatrix}$$ ← -2 times the first row was added to the second row.

$$= -(3-2t)\begin{vmatrix} 3 & -1 \\ -3 & t \end{vmatrix}$$ ← Cofactor expansion along the second row.

$$= -(3-2t)(3t-3)$$

$$= 3(2t-3)(t-1)$$

From parts (g) and (n) of Theorem 4.8.4, $\text{rank}(A) = 3$ when $\det(A) \neq 0$, i.e. for all t values other than 1 or $\frac{3}{2}$.

If $t = 1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

If $t = \frac{3}{2}$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

15. By inspection, there must be leading 1's in the first column (because of the first row) and in the third column (because of the fourth row) regardless of the values of r and s, therefore the matrix cannot have rank 1.

It has rank 2 if $r = 2$ and $s = 1$, since there is no leading 1 in the second column in that case.

16. **(a)** e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ - the column space is the xy-plane in R^3

(b) The general solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = (0,0,t)$. The null space is the z-axis.

(c) The row space of A is the xy-plane in R^3

17. No, both row and column spaces of A must be planes through the origin since from $\text{nullity}(A) = 1$, it follows by Formula (4) that $\text{rank}(A) = 3 - 1 = 2$.

18. **(a)** 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A is the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A^T can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A is the zero matrix, then the general solution of $A^T\mathbf{x} = \mathbf{0}$ has three parameters;

19. **(a)** 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A is the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A is the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has three parameters;

20. By part (b) of Theorem 4.8.3, the nullity of A is 0. By Formula(4), $\text{rank}(A) = 6 - 0 = 6$.

21. **(a)** By Formula (4), $\text{nullity}(A) = 7 - 4 = 3$ thus the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is 3.

(b) No, the column space of A is a subspace of R^5 of dimension 4, therefore there exist vectors $\mathbf{b}$ in R^5 that are outside this column space. For any such vector, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

22. The rank of A is 2 if and only if the two row vectors of A are not scalar multiples of one another, i.e. they are nonparallel nonzero vectors. This is equivalent to the cross product of these vectors being nonzero, i.e.

$$\left(\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right) \neq (0,0,0)$$

23. From the result of Exercise 22, the rank of the matrix being less than 2 implies that

$$\begin{vmatrix} x & y \\ 1 & x \end{vmatrix} = x^2 - y = 0, \qquad \begin{vmatrix} x & z \\ 1 & y \end{vmatrix} = xy - z = 0, \qquad \begin{vmatrix} y & z \\ x & y \end{vmatrix} = y^2 - xz = 0$$

therefore $y = x^2$ and $z = xy = x^3$. Letting $x = t$, we obtain $y = t^2$ and $z = t^3$.

24. For instance, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, therefore $\text{rank}(A^2) = 1 \neq 0 = \text{rank}(B^2)$ even though $\text{rank}(A) = 1 = \text{rank}(B)$.

25. The reduced row echelon form of A^T is $\begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution of $A^T\mathbf{x} = 0$ has components $x_1 = -10t, x_2 = 5t, x_3 = t, x_4 = 0$, so in vector form $\mathbf{x} = t(-10,5,1,0)$. Evaluating dot products of columns of A and $\mathbf{v} = (-10,5,1,0)$, which forms a basis for the null space of A^T we obtain

$\mathbf{c}_1 \cdot \mathbf{v} = (1,2,0,2) \cdot (-10,5,1,0) = (1)(-10) + (2)(5) + (0)(1) + (2)(0) = 0$

$\mathbf{c}_2 \cdot \mathbf{v} = (3,6,0,6) \cdot (-10,5,1,0) = (3)(-10) + (6)(5) + (0)(1) + (6)(0) = 0$

$\mathbf{c}_3 \cdot \mathbf{v} = (-2,-5,5,0) \cdot (-10,5,1,0) = (-2)(-10) + (-5)(5) + (5)(1) + (2)(0) = 0$

$\mathbf{c}_4 \cdot \mathbf{v} = (0,-2,10,8) \cdot (-10,5,1,0) = (0)(-10) + (-2)(5) + (10)(1) + (8)(0) = 0$

$\mathbf{c}_5 \cdot \mathbf{v} = (2,4,0,4) \cdot (-10,5,1,0) = (2)(-10) + (4)(5) + (0)(1) + (4)(0) = 0$

$\mathbf{c}_6 \cdot \mathbf{v} = (0,-3,15,18) \cdot (-10,5,1,0) = (0)(-10) + (-3)(5) + (15)(1) + (18)(0) = 0$

Since the column space of A is $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5, \mathbf{c}_6\}$ and the null space of A^T is $\text{span}\{\mathbf{v}\}$, we conclude that the two spaces are orthogonal complements in R^4.

27. **(a)** $m = 3 > 2 = n$ so the system is overdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & b_1 + b_3 \\ 0 & 1 & b_3 \\ 0 & 0 & 3b_1 + b_2 + 2b_3 \end{bmatrix}$ hence the system is inconsistent for all b's that satisfy $3b_1 + b_2 + 2b_3 \neq 0$.

(b) $m = 2 < 3 = n$ so the system is underdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b_1 - \frac{1}{4}b_2 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{6}b_1 - \frac{1}{12}b_2 \end{bmatrix}$ hence the system has infinitely many solutions for all b's (no values of b's can make this system inconsistent).

(c) $m = 2 < 3 = n$ so the system is underdetermined. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & -\frac{3}{2} & -\frac{1}{2}b_1 - \frac{3}{2}b_2 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2}b_1 - \frac{1}{2}b_2 \end{bmatrix}$ hence the system has infinitely many solutions for all b's (no values of b's can make this system inconsistent).

28. The augmented matrix of the system is row equivalent to $\begin{bmatrix} 1 & 0 & -2b_1 + 3b_2 \\ 0 & 1 & -b_1 + b_2 \\ 0 & 0 & 3b_1 - 4b_2 + b_3 \\ 0 & 0 & -2b_1 + b_2 + b_4 \\ 0 & 0 & 7b_1 - 8b_2 + b_5 \end{bmatrix}$. For the system to be consistent, we must have $3b_1 - 4b_2 + b_3 = 0$, $-2b_1 + b_2 + b_4 = 0$, and $7b_1 - 8b_2 + b_5 = 0$.
For arbitrary s and t, the b's must satisfy $b_1 = s, b_2 = t, b_3 = -3s + 4t, b_4 = 2s - t, b_5 = -7s + 8t$.

True-False Exercises

(a) False. For instance, in $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, neither row vectors nor column vectors are linearly independent.

(b) True. In an $m \times n$ matrix, if $m < n$ then by Theorem 4.5.2(a), the n columns in R^m must be linearly dependent. If $m > n$, then by the same theorem, the m rows in R^n must be linearly dependent. We conclude that $m = n$.

(c) False. The nullity in an $m \times n$ matrix is at most n.

(d) False. For instance, if the column contains all zeros, adding it to a matrix does not change the rank.

(e) True. In an $n \times n$ matrix A with linearly dependent rows, $\text{rank}(A) \le n - 1$.
By Formula (4), $\text{nullity}(A) = n - \text{rank}(A) \ge 1$.

(f) False. By Theorem 4.8.7, the nullity must be nonzero.

(g) False. This follows from Theorem 4.8.1.

(h) False. By Theorem 4.8.4, $\text{rank}(A^T) = \text{rank}(A)$ for any matrix A.

(i) True. Since each of the two spaces has dimension 1, these dimensions would add up to 2 instead of 3 as required by Formula (4).

(j) False. For instance, if $n = 3$, $V = \text{span}\{\mathbf{i}, \mathbf{j}\}$ (the xy-plane), and $W = \text{span}\{\mathbf{i}\}$ (the x-axis) then $W^\perp = \text{span}\{\mathbf{j}, \mathbf{k}\}$ (the yz-plane) is not a subspace of $V^\perp = \text{span}\{\mathbf{k}\}$ (the z-axis).
(Note that it is true that $V^\perp$ is a subspace of $W^\perp$.)

4.9 Matrix Transformations from $\mathbf{R^n}$ to $\mathbf{R^m}$

1. **(a)** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2. **(a)** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$

3. **(a)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}2\\-5\\3\end{bmatrix}=\begin{bmatrix}2\\-5\\-3\end{bmatrix}$ **(b)** $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2\\-5\\3\end{bmatrix}=\begin{bmatrix}2\\5\\3\end{bmatrix}$ **(c)** $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2\\-5\\3\end{bmatrix}=\begin{bmatrix}-2\\-5\\3\end{bmatrix}$

4. **(a)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a\\b\\-c\end{bmatrix}$ **(b)** $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a\\-b\\c\end{bmatrix}$ **(c)** $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-a\\b\\c\end{bmatrix}$

5. **(a)** $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}0\\-5\end{bmatrix}$

6. **(a)** $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}a\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\b\end{bmatrix}$

7. **(a)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}-2\\0\\3\end{bmatrix}$ **(c)** $\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\3\end{bmatrix}$

8. **(a)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a\\b\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}a\\0\\c\end{bmatrix}$ **(c)** $\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\b\\c\end{bmatrix}$

9. **(a)** $\begin{bmatrix}\cos 30^\circ&-\sin 30^\circ\\\sin 30^\circ&\cos 30^\circ\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{\sqrt3}{2}&-\frac12\\\frac12&\frac{\sqrt3}{2}\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{3\sqrt3}{2}+2\\\frac32-2\sqrt3\end{bmatrix}\approx\begin{bmatrix}4.60\\-1.96\end{bmatrix}$

(b) $\begin{bmatrix}\cos(-60^\circ)&-\sin(-60^\circ)\\\sin(-60^\circ)&\cos(-60^\circ)\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac12&\frac{\sqrt3}{2}\\-\frac{\sqrt3}{2}&\frac12\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac32-2\sqrt3\\-\frac{3\sqrt3}{2}-2\end{bmatrix}\approx\begin{bmatrix}-1.96\\-4.60\end{bmatrix}$

(c) $\begin{bmatrix}\cos 45^\circ&-\sin 45^\circ\\\sin 45^\circ&\cos 45^\circ\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{\sqrt2}{2}&-\frac{\sqrt2}{2}\\\frac{\sqrt2}{2}&\frac{\sqrt2}{2}\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{7\sqrt2}{2}\\-\frac{\sqrt2}{2}\end{bmatrix}\approx\begin{bmatrix}4.95\\-0.71\end{bmatrix}$

(d) $\begin{bmatrix}\cos 90^\circ&-\sin 90^\circ\\\sin 90^\circ&\cos 90^\circ\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}$

10. **(a)** $\begin{bmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}v_1\cos\alpha-v_2\sin\alpha\\v_1\sin\alpha+v_2\cos\alpha\end{bmatrix}$

(b) $\begin{bmatrix}\cos(-\alpha)&-\sin(-\alpha)\\\sin(-\alpha)&\cos(-\alpha)\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}v_1\cos(-\alpha)-v_2\sin(-\alpha)\\v_1\sin(-\alpha)+v_2\cos(-\alpha)\end{bmatrix}=\begin{bmatrix}v_1\cos\alpha+v_2\sin\alpha\\-v_1\sin\alpha+v_2\cos\alpha\end{bmatrix}$

11. **(a)** $\begin{bmatrix}1&0&0\\0&\cos(-30^\circ)&-\sin(-30^\circ)\\0&\sin(-30^\circ)&\cos(-30^\circ)\end{bmatrix}\begin{bmatrix}2\\-1\\2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\frac{\sqrt3}{2}&\frac12\\0&-\frac12&\frac{\sqrt3}{2}\end{bmatrix}\begin{bmatrix}2\\-1\\2\end{bmatrix}=\begin{bmatrix}2\\1-\frac{\sqrt3}{2}\\\frac12+\sqrt3\end{bmatrix}$

(b) $\begin{bmatrix}\cos 30^\circ&0&\sin 30^\circ\\0&1&0\\-\sin 30^\circ&0&\cos 30^\circ\end{bmatrix}\begin{bmatrix}2\\-1\\2\end{bmatrix}=\begin{bmatrix}\frac{\sqrt3}{2}&0&\frac12\\0&1&0\\-\frac12&0&\frac{\sqrt3}{2}\end{bmatrix}\begin{bmatrix}2\\-1\\2\end{bmatrix}=\begin{bmatrix}\sqrt3+1\\-1\\-1+\sqrt3\end{bmatrix}$

(c) $\begin{bmatrix} \cos(-45°) & 0 & \sin(-45°) \\ 0 & 1 & 0 \\ -\sin(-45°) & 0 & \cos(-45°) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2\sqrt{2} \end{bmatrix}$

(d) $\begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

12. **(a)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 - \frac{\sqrt{3}}{2} \\ -\frac{1}{2} + \sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} \cos(-30°) & 0 & \sin(-30°) \\ 0 & 1 & 0 \\ -\sin(-30°) & 0 & \cos(-30°) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 1 \\ -1 \\ 1 + \sqrt{3} \end{bmatrix}$

(c) $\begin{bmatrix} \cos 45° & 0 & \sin 45° \\ 0 & 1 & 0 \\ -\sin 45° & 0 & \cos 45° \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ -1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} \cos(-90°) & -\sin(-90°) & 0 \\ \sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

13. **(a)** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

14. **(a)** $\begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{a}{\alpha} \\ \frac{b}{\alpha} \end{bmatrix}$ **(b)** $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$

15. **(a)** $\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

16. **(a)** $\begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{a}{\alpha} \\ \frac{b}{\alpha} \\ \frac{c}{\alpha} \end{bmatrix}$ **(b)** $\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}$

17. **(a)** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

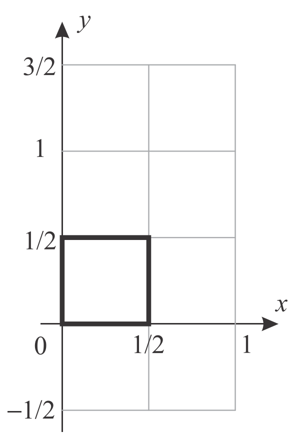
18. **(a)** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

19. **(a)** $\begin{bmatrix} \frac{1}{\alpha} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{a}{\alpha} \\ b \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ \alpha b \end{bmatrix}$

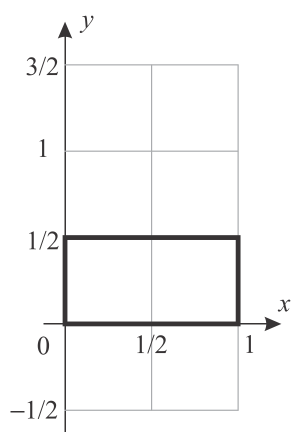
20.

Operator	**Standard Matrix**
Compression in R^3 in the x-direction with factor k $(0 \le k \le 1)$	$\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Compression in R^3 in the y-direction with factor k $(0 \le k \le 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Compression in R^3 in the z-direction with factor k $(0 \le k \le 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$

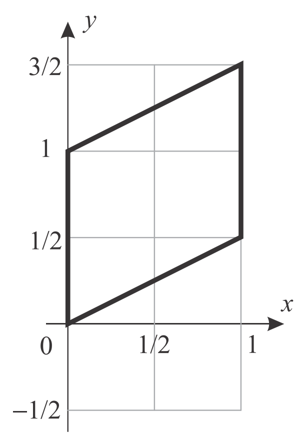
21. **(a)** the matrix A_1 corresponds to the contraction with factor $\frac{1}{2}$



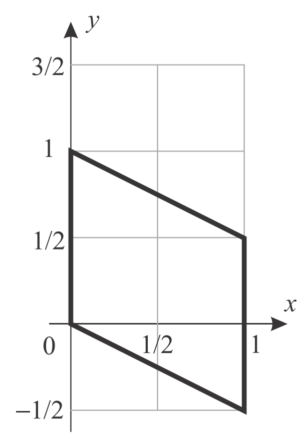
(b) the matrix A_2 corresponds to the compression in the y-direction with factor $\frac{1}{2}$



(c) the matrix A_3 corresponds to the shear in the y-direction by a factor $\frac{1}{2}$



(d) the matrix A_4 corresponds to the shear in the y-direction by a factor $-\frac{1}{2}$

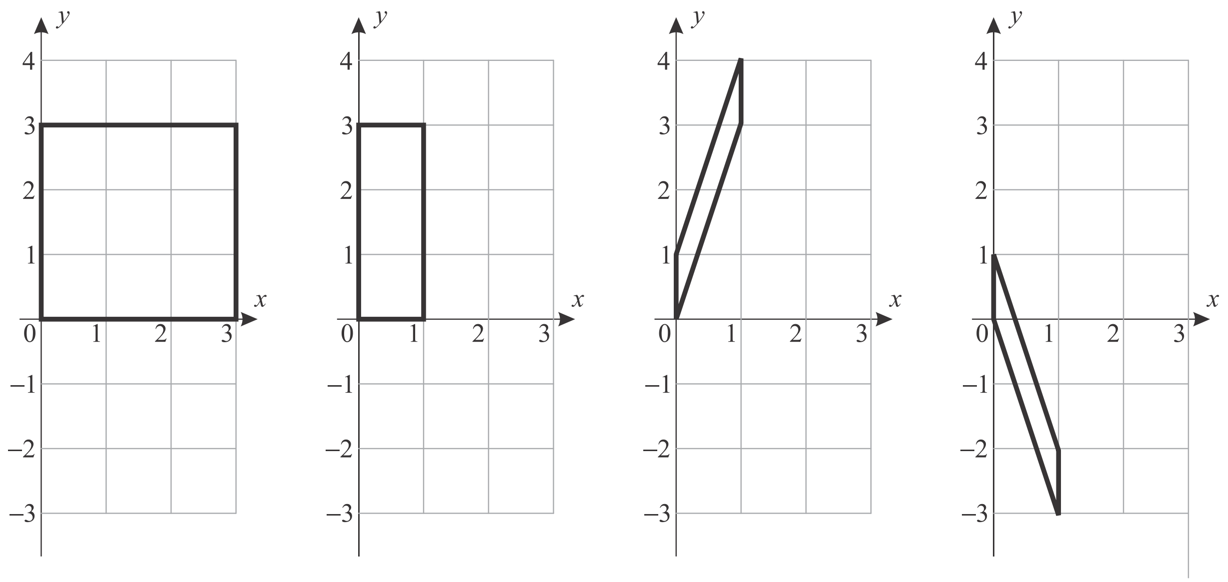


22. **(a)** the matrix A_1 corresponds to the dilation with factor 3

(b) the matrix A_2 corresponds to the expansion in the y-direction with factor 3

(c) the matrix A_3 corresponds to the shear in the y-direction by a factor 3

(d) the matrix A_4 corresponds to the shear in the y-direction by a factor -3



23. **(a)** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (dilation with factor 2)

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (shear in the x-direction by a factor 2)

24. **(a)** $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ (shear in the x-direction by a factor -2)

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (expansion in the x-direction with factor 2)

25. By Formula (4), the standard matrix for the projection is $P_{\pi/3} = \begin{bmatrix} \cos^2 \frac{\pi}{3} & \sin\frac{\pi}{3}\cos\frac{\pi}{3} \\ \sin\frac{\pi}{3}\cos\frac{\pi}{3} & \sin^2\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$.

From $\begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \sqrt{3} \\ \frac{3\sqrt{3}}{4} + 3 \end{bmatrix} \approx \begin{bmatrix} 2.48 \\ 4.30 \end{bmatrix}$ we obtain $P_{\pi/3}(3,4) \approx (2.48, 4.30)$.

26. By Formula (4), the standard matrix for the projection is $P_{\pi/3} = \begin{bmatrix} \cos^2 \frac{\pi}{4} & \sin\frac{\pi}{4}\cos\frac{\pi}{4} \\ \sin\frac{\pi}{4}\cos\frac{\pi}{4} & \sin^2\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

From $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ we obtain $P_{\pi/4}(1,2) = \left(\frac{3}{2}, \frac{3}{2}\right)$.

27. By Formula (6), the standard matrix for the reflection $H_{\pi/3} = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

From $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} + 2\sqrt{3} \\ \frac{3\sqrt{3}}{2} + 2 \end{bmatrix} \approx \begin{bmatrix} 1.96 \\ 4.60 \end{bmatrix}$ we obtain $H_{\pi/3}(3,4) \approx (1.96, 4.60)$.

28. By Formula (6), the standard matrix for the reflection $H_{\pi/4} = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

From $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ we obtain $H_{\pi/4}(1,2) = (2, 1)$.

29. Reflection about the xy-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Reflection about the xz-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Reflection about the yz-plane: $T(1,2,3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

30. Orthogonal projection onto the xy-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Orthogonal projection onto the xz-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Orthogonal projection onto the yz-plane: $T(1,2,3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

31. **(a)** $\begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & -\sin 45° \\ 0 & \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(c) $\begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

32. **(a)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

33. A unit vector in the direction of (2,2,1) is $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$ so in Formula (3), we take $a = b = \frac{2}{3}$ and $c = \frac{1}{3}$. Formula (3) yields the standard matrix

$$\begin{bmatrix} \frac{4}{9}(1-\cos\pi)+\cos\pi & \frac{4}{9}(1-\cos\pi)-\frac{1}{3}\sin\pi & \frac{2}{9}(1-\cos\pi)+\frac{2}{3}\sin\pi \\ \frac{4}{9}(1-\cos\pi)+\frac{1}{3}\sin\pi & \frac{4}{9}(1-\cos\pi)+\cos\pi & \frac{2}{9}(1-\cos\pi)-\frac{2}{3}\sin\pi \\ \frac{2}{9}(1-\cos\pi)-\frac{2}{3}\sin\pi & \frac{2}{9}(1-\cos\pi)+\frac{2}{3}\sin\pi & \frac{1}{9}(1-\cos\pi)+\cos\pi \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} & \frac{8}{9} & \frac{4}{9} \\ \frac{8}{9} & -\frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & -\frac{7}{9} \end{bmatrix}$$

34. A unit vector in the direction of (1,1,1) is $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{3}}(1,1,1)$ so in Formula (3), we take $a = b = c = \frac{1}{\sqrt{3}}$. Formula (3) yields the standard matrix

$$\begin{bmatrix} \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)-\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} \\ \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)-\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} \\ \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)-\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\frac{1}{\sqrt{3}}\sin\frac{\pi}{2} & \frac{1}{3}\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{3}+\frac{1}{\sqrt{3}} \\ \frac{1}{3}+\frac{1}{\sqrt{3}} & \frac{1}{3} & \frac{1}{3}-\frac{1}{\sqrt{3}} \\ \frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{3}+\frac{1}{\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

35. For the rotation about the x-axis, we take $\mathbf{v} = (1,0,0)$, so in Formula (3) we have $a = 1, b = c = 0$:

$$\begin{bmatrix} (1)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)-(0)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+(0)\sin\frac{\pi}{2} \\ (0)\left(1-\cos\frac{\pi}{2}\right)+(0)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)-(1)\sin\frac{\pi}{2} \\ (0)\left(1-\cos\frac{\pi}{2}\right)-(0)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+(1)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For the rotation about the y-axis, we take $\mathbf{v} = (0,1,0)$, so in Formula (3) we have $b = 1, a = c = 0$:

$$\begin{bmatrix} (0)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)-(0)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+(1)\sin\frac{\pi}{2} \\ (0)\left(1-\cos\frac{\pi}{2}\right)+(0)\sin\frac{\pi}{2} & (1)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)-(0)\sin\frac{\pi}{2} \\ (0)\left(1-\cos\frac{\pi}{2}\right)-(1)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+(0)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

For the rotation about the z-axis, we take $\mathbf{v} = (0,0,1)$, so in Formula (3) we have $a = b = 0, c = 1$:

$$\begin{bmatrix} (0)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)-(1)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+(0)\sin\frac{\pi}{2} \\ (0)\left(1-\cos\frac{\pi}{2}\right)+(1)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)-(0)\sin\frac{\pi}{2} \\ (0)\left(1-\cos\frac{\pi}{2}\right)-(0)\sin\frac{\pi}{2} & (0)\left(1-\cos\frac{\pi}{2}\right)+(0)\sin\frac{\pi}{2} & (1)\left(1-\cos\frac{\pi}{2}\right)+\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

37. Since $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ and $2\sin\theta\cos\theta = \sin(2\theta)$, we have $A = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$. The geometric effect of multiplying A by $\mathbf{x}$ is to rotate the vector through the angle 2θ.

38. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$ (since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$). The geometric effect of multiplying A^T by $\mathbf{x}$ is to rotate the vector through the angle $-\theta$ (i.e., to rotate through the angle θ clockwise).

39. The terminal point of the vector is first rotated about the origin through the angle θ, then it is translated by the vector $\mathbf{x}_0$. No, this is not a matrix transformation, for instance it fails the additivity property: $T(\mathbf{u}+\mathbf{v}) = \mathbf{x}_0 + R_\theta(\mathbf{u}+\mathbf{v}) = \mathbf{x}_0 + R_\theta\mathbf{u} + R_\theta\mathbf{v} \neq \mathbf{x}_0 + R_\theta\mathbf{u} + \mathbf{x}_0 + R_\theta\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$.

40. **(a)** $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $[T_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $[T_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4.10 Properties of Matrix Transformations

1. **(a)** From Tables 1 and 3 in Section 4.9, $[T_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 \neq T_2 \circ T_1$.

(b) From Table 1 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 \neq T_2 \circ T_1$.

2. **(a)** From Table 3 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 = T_2 \circ T_1$.

(b) From Tables 5 and 1 in Section 4.9, $[T_1] = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $[T_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 \neq T_2 \circ T_1$.

3. From Table 8 in Section 4.9, $[T_1] = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1/k & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1/k \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For these transformations, $T_1 \circ T_2 = T_2 \circ T_1$.

4. From Table 6 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1 \\ \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2 & \cos\theta_1 \end{bmatrix}$;

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} \cos\theta_2 & -\cos\theta_1 \sin\theta_2 & \sin\theta_1 \sin\theta_2 \\ \sin\theta_2 & \cos\theta_1 \cos\theta_2 & -\sin\theta_1 \cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}.$$

For these transformations, $T_1 \circ T_2 \neq T_2 \circ T_1$.

5. $[T_B \circ T_A] = [T_B][T_A] = BA = \begin{bmatrix} -10 & -7 \\ 5 & -10 \end{bmatrix}$; $[T_A \circ T_B] = [T_A][T_B] = AB = \begin{bmatrix} -8 & -3 \\ 13 & -12 \end{bmatrix}$

6. $[T_B \circ T_A] = [T_B][T_A] = BA = \begin{bmatrix} 40 & 0 & 20 \\ 12 & -9 & 18 \\ 38 & -18 & 43 \end{bmatrix}$; $[T_A \circ T_B] = [T_A][T_B] = AB = \begin{bmatrix} 19 & 18 & 22 \\ 10 & -3 & 16 \\ 31 & -33 & 58 \end{bmatrix}$

7. **(a)** We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is a rotation of $90°$ and T_2 is a reflection about the line $y = x$. From Tables 5 and 1 in Section 4.9, $[T_1] = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is an orthogonal projection on the y-axis and T_2 is a contraction with factor $k = \frac{1}{2}$. From Tables 3 and 7 in Section 4.9,

$[T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a reflection about the x-axis, T_2 is a dilation with factor $k = 3$, and T_3 is a rotation of $60°$. From Tables 1, 7, and 5 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} \frac{3}{2} & \frac{3\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & -\frac{3}{2} \end{bmatrix}$.

8. **(a)** We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a rotation of $60°$, T_2 is an orthogonal projection on the x-axis, and T_3 is a reflection about the line $y = x$. From Tables 5, 1, and 3 in Section 4.9, $[T_1] = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a dilation with factor $k = 2$, T_2 is a rotation of $45°$, and T_3 is a reflection about the y-axis. From Tables 7, 5, and 1 in Section 4.9, $[T_1] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, and $[T_3] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a rotation of 15°, T_2 is a rotation of 105°, and T_3 is a rotation of 60°. The net effect of the three rotations is a single rotation of $15° + 105° + 60° = 180°$. From Table 5 in Section 4.9, $[T] = \begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

9. **(a)** We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is a reflection about the yz-plane and T_2 is an orthogonal projection on the xz-plane. From Tables 2 and 4 in Section 4.9, $[T_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is a rotation of 45° about the y-axis and T_2 is a dilation with factor $k = \sqrt{2}$. From Tables 6 and 8 in Section 4.9,

$[T_1] = \begin{bmatrix} \cos 45° & 0 & \sin 45° \\ 0 & 1 & 0 \\ -\sin 45° & 0 & \cos 45° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is an orthogonal projection on the xy-plane and T_2 is a reflection about the yz-plane. From Tables 4 and 2 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

10. **(a)** We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a rotation of 30° about the x-axis, T_2 is a rotation of 30° about the z-axis, and T_3 is a contraction with factor $k = \frac{1}{4}$.

From Tables 6 and 8 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$.

Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{16} & -\frac{\sqrt{3}}{16} \\ 0 & \frac{1}{8} & \frac{\sqrt{3}}{8} \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a reflection about the xy-plane, T_2 is a reflection about the xz-plane, and T_3 is an orthogonal projection on the yz-

plane. From Tables 2 and 4 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is a rotation of $270°$ about the x-axis, T_2 is a rotation of 90° about the y-axis, and T_3 is a rotation of 180° about the z-axis. From Table 6 in Section 4.9, $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 270° & -\sin 270° \\ 0 & \sin 270° & \cos 270° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} \cos 180° & -\sin 180° & 0 \\ \sin 180° & \cos 180° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$.

11. **(a)** In vector form, $T_1(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $[T_1] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
Likewise, $T_2(x_1, x_2) = \begin{bmatrix} 3x_1 \\ 2x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $[T_2] = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$.

(b) $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & -2 \end{bmatrix}$
$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & -4 \end{bmatrix}$

(c) $T_1(T_2(x_1, x_2)) = (5x_1 + 4x_2,\ x_1 - 4x_2)$; $T_2(T_1(x_1, x_2)) = (3x_1 + 3x_2,\ 6x_1 - 2x_2)$

12. **(a)** In vector form, $T_1(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so that $[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$.
Likewise, $T_2(x_1, x_2, x_3) = \begin{bmatrix} x_1 + 2x_2 \\ -x_3 \\ 4x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so that $[T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$.

(b) $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$
$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$

(c) $T_1(T_2(x_1, x_2, x_3)) = (4x_1 + 8x_2, -2x_1 - 4x_2 - x_3, -x_1 - 2x_2 + 3x_3)$
$T_2(T_1(x_1, x_2, x_3)) = (2x_2, x_1 + 3x_2, 17x_1 + 3x_2)$

13. **(a)** Not one-to-one (maps distinct vectors with the same x components into the same vector).

(b) One-to-one (distinct vectors that are reflected have distinct images).

(c) One-to-one (distinct vectors that are reflected have distinct images).

(d) One-to-one (distinct vectors that are contracted have distinct images).

14. **(a)** One-to-one (distinct vectors that are rotated have distinct images).

(b) One-to-one (distinct vectors that are reflected have distinct images).

(c) One-to-one (distinct vectors that are contracted have distinct images).

(d) Not one-to-one (maps distinct vectors with the same x and z components into the same vector).

15. **(a)** The reflection about the x-axis in R^2 is its own inverse.

(b) The rotation through an angle of $-\pi/4$ in R^2 (i.e., the clockwise rotation through an angle $\pi/4$) is the desired inverse.

(c) The contraction by a factor of $\frac{1}{3}$ in R^2 is the desired inverse.

16. **(a)** The reflection about the yz-plane in R^3 is its own inverse.

(b) The dilation by a factor of 5 in R^3 is the desired inverse.

(c) The rotation through an angle of 18° about the z-axis is the desired inverse.

17. **(a)** $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 8x_1 + 4x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$; since $\begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix} = 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is not one-to-one

(b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 3x_2 + 2x_3 \\ 2x_1 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}$; since $\begin{vmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{vmatrix} = 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is not one-to-one

18. **(a)** $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 \\ 5x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$; since $\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = 17 \neq 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is one-to-one

(b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 5x_2 + 3x_3 \\ x_1 + 8x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$; since $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix} = -1 \neq 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is one-to-one

19. **(a)** $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$; since $\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 \neq 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is one-to-one;

the standard matrix of T^{-1} is $\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$; $T^{-1}(w_1, w_2) = \left(\frac{1}{3}w_1 - \frac{2}{3}w_2,\ \frac{1}{3}w_1 + \frac{1}{3}w_2\right)$

(b) $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 6x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$; since $\begin{vmatrix} 4 & -6 \\ -2 & 3 \end{vmatrix} = 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is not one-to-one

20. (a) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$;

since $\begin{vmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1 \neq 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is one-to-one;

the reduced row echelon form of the matrix $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ is $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$

therefore the standard matrix of T^{-1} is $\begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}$;

$T^{-1}(w_1, w_2, w_3) = (w_1 - 2w_2 + 4w_3, -w_1 + 2w_2 - 3w_3, -w_1 + 3w_2 - 5w_3)$

(b) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 + 4x_3 \\ -x_1 + x_2 + x_3 \\ -2x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix}$; since $\begin{vmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{vmatrix} = 0$, it follows from parts (g) and (s) of Theorem 4.10.2 that the operator is not one-to-one

21. (a) Suppose $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Subtracting both equations yields $A\begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ therefore the transformation is one-to-one if and only if the nullity of A is 0 since that is equivalent to stating that $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ implies $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so we conclude that the nullity of A is 0, thus T_A is one-to-one.

(b) Proceeding as in part (a), we determine the reduced row echelon form of A to be $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$.

Therefore A has nullity 1 and T_A is not one-to-one (e.g., $A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} -8 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$).

22. (a) Proceeding as in part (a) of Exercise 21, we determine the reduced row echelon form of A to be $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore A has nullity 1 and T_A is not one-to-one (e.g., $A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$).

(b) Proceeding as in part (a) of Exercise 21, we determine the reduced row echelon form of A to be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore A has nullity 0 and T_A is one-to-one.

23. **(a)** The range of $T(x) = Ax$ consists of all vectors (y_1, y_2, y_3) that are images of at least one vector (x_1, x_2, x_3) under this transformation $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$.

Since the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & \frac{14}{11} \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{bmatrix}$, by Theorem 4.7.6 the vectors $\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ are linearly independent, and they span $R(T)$. We conclude that $\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ form a basis for $R(T)$.

(b) The kernel of T consists of all vectors (x_1, x_2, x_3) such that $\begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Based on the reduced row echelon form of A obtained above, the general solution is $x_1 = -\frac{14}{11}t$, $x_2 = \frac{19}{11}t$, $x_3 = t$. Therefore a basis for $\ker(T)$ is formed by the vector $(-14, 19, 11)$.

(c) From part (a), $\text{rank}(T) = \dim(R(T)) = 2$. From part (b), $\text{nullity}(T) = \dim(\ker(T)) = 1$.

(d) Based on the reduced row echelon form of A obtained above, $\text{rank}(A) = 2$ and $\text{nullity}(A) = 1$.

24. **(a)** The range of $T(x) = Ax$ consists of all vectors (y_1, y_2, y_3) that are images of at least one vector (x_1, x_2, x_3) under this transformation $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 4 \\ 20 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$.

Since the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, by Theorem 4.7.6 the vectors $\begin{bmatrix} 2 \\ 4 \\ 20 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ are linearly independent, and they span $R(T)$. We conclude that $\begin{bmatrix} 2 \\ 4 \\ 20 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ form a basis for $R(T)$.

(b) The kernel of T consists of all vectors (x_1, x_2, x_3) such that $\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Based on the reduced row echelon form of A obtained above, the general solution is $x_1 = 0$, $x_2 = t$, $x_3 = 0$. Therefore a basis for $\ker(T)$ is formed by the vector $(0,1,0)$.

(c) From part (a), $\text{rank}(T) = \dim(R(T)) = 2$. From part (b), $\text{nullity}(T) = \dim(\ker(T)) = 1$.

(d) Based on the reduced row echelon form of A obtained above, $\text{rank}(A) = 2$ and $\text{nullity}(A) = 1$.

25. The kernel of T_A consists of all vectors (x_1, x_2, x_3, x_4) such that $\begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{10}{7}t, x_2 = \frac{2}{7}t, x_3 = 0, x_4 = t$. Therefore a basis for $\ker(T_A)$ is formed by the vector $(10, 2, 0, 7)$.

The range of $T_A(\mathbf{x}) = A\mathbf{x}$ consists of all vectors (y_1, y_2, y_3) that are images of at least one vector (x_1, x_2, x_3, x_4) under this transformation

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + x_4\begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$. Based on the reduced row echelon form of A obtained above, by Theorem 4.7.6 the vectors $\begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ are linearly independent, and they span $R(T_A)$. We conclude that $\begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ form a basis for $R(T_A)$.

26. The kernel of T_A consists of all vectors (x_1, x_2, x_3, x_4) such that $\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since the reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = \frac{1}{3}s - \frac{1}{3}t$, $x_2 = -\frac{1}{3}s - \frac{2}{3}t, x_3 = s, x_4 = t$. In vector form, $(x_1, x_2, x_3, x_4) = s\left(\frac{1}{3}, -\frac{1}{3}, 1, 0\right) + t(-\frac{1}{3}, -\frac{2}{3}, 0, 1)$. Therefore a basis for $\ker(T_A)$ is formed by the vectors $(1, -1, 3, 0)$ and $(-1, -2, 0, 3)$.

The range of $T_A(\mathbf{x}) = A\mathbf{x}$ consists of all vectors (y_1, y_2, y_3) that are images of at least one vector (x_1, x_2, x_3, x_4) under this transformation

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & 4 & 2 & 2 \\ -1 & 8 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. Based on the reduced row echelon form of A obtained above, by Theorem 4.7.6 the vectors $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$ are linearly independent, and they span $R(T_A)$. We conclude that $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$ form a basis for $R(T_A)$.

27. (a) By parts (g) and (s) of Theorem 4.10.2, the range of T cannot be R^n - it must be a proper subset of R^n instead.

For instance $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the standard matrix of the orthogonal projection onto the x-axis; the range of this transformation is the x-axis - a proper subset of R^2.

(b) By parts (g) and (r) of Theorem 4.10.2, the kernel of T must contain at least one nonzero vector $\mathbf{v}$. Consequently T maps infinitely many vectors (e.g., scalar multiples $k\mathbf{v}$) into $\mathbf{0}$.

28. (a) By parts (g) and (s) of Theorem 4.10.2, the range of T is R^n. For instance $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is the standard matrix of the reflection about the y-axis; the range of this transformation is R^2.

(b) By parts (g) and (r) of Theorem 4.10.2, the kernel of T must contain only the zero vector. Consequently T maps only $\mathbf{0}$ into $\mathbf{0}$.

29. (a) Yes. If $T_1: R^n \to R^m$ and $T_2: R^m \to R^k$ are both one-to-one then for any vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n, $T_2(T_1(\mathbf{u})) = T_2(T_1(\mathbf{v}))$ must imply $T_1(\mathbf{u}) = T_1(\mathbf{v})$ (since T_2 is one-to-one), which further implies that $\mathbf{u} = \mathbf{v}$ (since T_1 is one-to-one), therefore the composition $T_2 \circ T_1: R^n \to R^k$ is also one-to-one.

(b) Yes. For instance, $T_1(x_1, x_2) = (x_1, x_2, 0)$ is one-to-one but $T_2(x_1, x_2, x_3) = (x_1, x_2)$ is not. However, the composition $T_2(T_1(x_1, x_2)) = T_2(x_1, x_2, 0) = (x_1, x_2)$ is obviously one-to-one. However, if T_1 is not one-to-one, then the composition $T_2 \circ T_1$ is not one-to-one since there must exist two vectors $\mathbf{u} \neq \mathbf{v}$ such that $T_1(\mathbf{u}) = T_1(\mathbf{v})$ leading to $T_2(T_1(\mathbf{u})) = T_2(T_1(\mathbf{v}))$.

30. (a) Since $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ and $2\sin\theta\cos\theta = \sin(2\theta)$, we have $A = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$. The geometric effect of applying this transformation to $\mathbf{x}$ is to rotate the vector through the angle 2θ.

(b) For instance, if $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (the standard matrix of the rotation through an angle θ) then $T_A = T_B \circ T_B$.

31. (a) From Table 1, the standard matrix of the reflection about the line $y = x$ is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
The inverse is $A^{-1} = \frac{1}{(0)(0)-(1)(1)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$.

(b) By Table 9, the standard matrix of the compression in the x-direction with the factor k (such that $0 < k < 1$) is $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$. The inverse is $A^{-1} = \frac{1}{(k)(1)-(0)(0)} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1/k & 0 \\ 0 & 1 \end{bmatrix}$.
Since $\frac{1}{k} > 1$, this is a standard matrix of an expansion in the x-direction with the factor $1/k$. (Compressions in the y-direction can be treated analogously.)

32. (a) From Table 1, the standard matrix of the reflection about the y-axis is $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
The inverse is $A^{-1} = \frac{1}{(-1)(1)-(0)(0)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = A$.
(Reflections about the x-axis can be treated analogously.)

(b) From Table 10, the standard matrix of the shear in the x-direction by a factor k is $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.
The inverse is $A^{-1} = \frac{1}{(1)(1)-(k)(0)}\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ is the standard matrix of a shear in the same direction by the factor $-k$.
(Shears in the y-direction can be treated analogously.)

True-False Exercises

(a) False. For instance, Example 1 shows two matrix operators on R^2 whose composition is not commutative.

(b) True. This is stated as Formula (4).

(c) True. This was established in Example 2.

(d) False. For instance, composition of any reflection operator with itself is the identity operator, which is not a reflection.

(e) True. This is stated in Formula (6).

(f) True. This follows from parts (b) and (d) of Theorem 4.10.1.

(g) True. This follows from parts (b) and (s) of Theorem 4.10.2.

4.11 Geometry of Matrix Operators on R²

1. Coordinates (x, y) are being transformed to coordinates (x', y') according to the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$ so $x = x' - 2y'$ and $y = -2x' + 5y'$. Substituting these into $y = 4x$ yields $-2x' + 5y' = 4(x' - 2y')$ or equivalently $y' = \frac{6}{13}x'$.

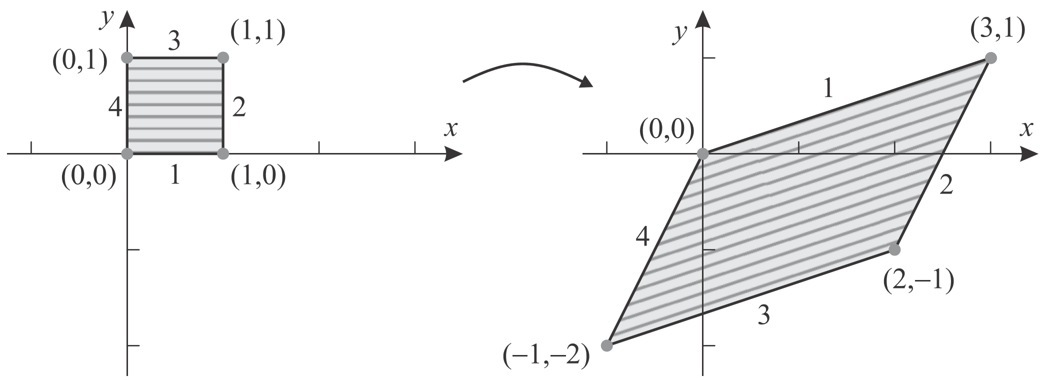
2. Coordinates (x, y) are being transformed to coordinates (x', y') according to the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$ so $x = -2x' + 3y'$ and $y = -3x' + 4y'$. Substituting these into $y = -4x + 3$ yields $-3x' + 4y' = -4(-2x' + 3y') + 3$ or equivalently $y' = \frac{11}{16}x' + \frac{3}{16}$.

3. From Table 1, the standard matrix for the shear in the x-direction by a factor 3 is $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.
Coordinates (x, y) are being transformed to coordinates (x', y') according to the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$ so $x = x' - 3y'$ and $y = y'$. Substituting these into $y = 2x$ yields $y' = 2(x' - 3y')$ or equivalently $y' = \frac{2}{7}x'$.

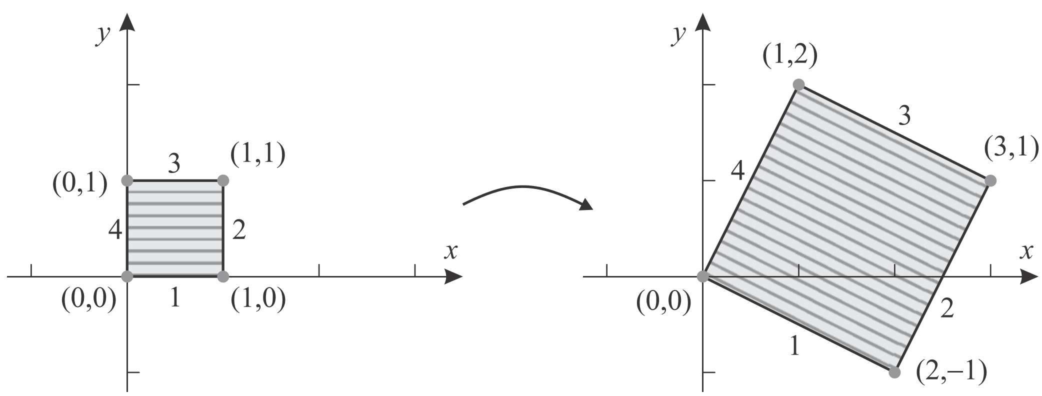
4. From Table 1, the standard matrix for the compression in the y-direction by a factor $\frac{1}{2}$ is $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

Coordinates (x, y) are being transformed to coordinates (x', y') according to the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ so $x = x'$ and $y = 2y'$. Substituting these into $y = 2x$ yields $2y' = 2x'$ or equivalently $y' = x'$.

5. Since $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, the image of the unit square is a parallelogram with vertices $(0,0)$, $(3,1)$, $(-1,-2)$, and $(2,-1)$.



6. Since $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, the image of the unit square is a square with vertices $(0,0)$, $(2,-1)$, $(1,2)$, and $(3,1)$.



7. **(a)** We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is the compression by a factor of $\frac{1}{2}$ in the x-direction and T_2 is the expansion by a factor of 5 in the y-direction. From Table 1, $[T_1] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is the expansion by a factor of 5 in the y-direction and T_2 is the shear by a factor of 2 in the y-direction. From Table 1, $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$.

(c) We are looking for the standard matrix of $T = T_2 \circ T_1$ where T_1 is the reflection about the line $y = x$ and T_2 is rotation through an angle of $180°$ about the origin . From Table 1, $[T_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} \cos 180° & -\sin 180° \\ \sin 180° & \cos 180° \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Therefore, $[T] = [T_2][T_1] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

8. **(a)** We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the reflection about the y-axis, T_2 is the expansion by a factor of 5 in the x-direction, and T_3 is the reflection about the line $y = x$. From Table 1, $[T_1] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$.

(b) We are looking for the standard matrix of $T = T_3 \circ T_2 \circ T_1$ where T_1 is the rotation of $30°$, T_2 is the shear by a factor of -2 in the y-direction, and T_3 is the expansion by a factor of 3 in the y-direction. From Table 1, $[T_1] = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, and $[T_3] = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Therefore, $[T] = [T_3][T_2][T_1] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -3\sqrt{3} + \frac{3}{2} & 3 + \frac{3\sqrt{3}}{2} \end{bmatrix}$.

9. **(a)** From Table 1, T_1, a reflection about the x-axis has the standard matrix $[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and T_2, a compression by a factor of $\frac{1}{3}$ in the x-direction has the standard matrix $[T_2] = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$.

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -1 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -1 \end{bmatrix}$.

Since $T_1 \circ T_2 = T_2 \circ T_1$, these operators commute.

(b) From Table 1, T_1, a reflection about the line $y = x$ has the standard matrix $[T_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and T_2, an expansion by a factor of 2 in the x-direction has the standard matrix $[T_2] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

Since $T_1 \circ T_2 \neq T_2 \circ T_1$, these operators do not commute.

10. **(a)** From Table 1, T_1, a shear in the y-direction with factor of $\frac{1}{4}$ has the standard matrix $[T_1] = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix}$ and T_2, a shear in the y-direction with factor of $\frac{3}{5}$ has the standard matrix $[T_2] =$

$\begin{bmatrix} 1 & 0 \\ \frac{3}{5} & 1 \end{bmatrix}$. $[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 1 & 0 \\ \frac{17}{20} & 1 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 0 \\ \frac{17}{20} & 1 \end{bmatrix}$.

Since $T_1 \circ T_2 = T_2 \circ T_1$, these operators commute.

(b) From Table 1, T_1, a shear in the y-direction with factor of $\frac{1}{4}$ has the standard matrix $[T_1] = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix}$ and T_2, a shear in the x-direction with factor of $\frac{3}{5}$ has the standard matrix $[T_2] = \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & 1 \end{bmatrix}$. $[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} \frac{23}{20} & \frac{3}{5} \\ \frac{1}{4} & 1 \end{bmatrix}$; $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & \frac{3}{5} \\ \frac{1}{4} & \frac{23}{20} \end{bmatrix}$.

Since $T_1 \circ T_2 \neq T_2 \circ T_1$, these operators do not commute.

11. $A = \begin{bmatrix} 4 & 4 \\ 0 & -2 \end{bmatrix} \underset{\substack{\text{Multiply} \\ \text{the first} \\ \text{row by } \frac{1}{4}}}{\rightarrow} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \underset{\substack{\text{Multiply} \\ \text{the second} \\ \text{row by } -\frac{1}{2}}}{\rightarrow} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underset{\substack{\text{Add} -1 \\ \text{times the} \\ \text{second} \\ \text{row to} \\ \text{the first} \\ \text{row}}}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

therefore $E_3E_2E_1A = I$ with $E_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ so that $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Multiplication by A has the geometric effect of shearing by a factor of 1 in the x-direction, then reflection about the x-axis, then expanding by a factor of 2 in the y-direction, then expanding by a factor of 4 in the x-direction.

12. $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \underset{\substack{\text{Add } -2 \\ \text{times the} \\ \text{first row} \\ \text{to the} \\ \text{second} \\ \text{row}}}{\rightarrow} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \underset{\substack{\text{Add} -4 \\ \text{times the} \\ \text{second} \\ \text{row to} \\ \text{the first} \\ \text{row}}}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

therefore $E_2E_1A = I$ with $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ so that $A = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.

Multiplication by A has the geometric effect of shearing by a factor of 4 in the x-direction, then shearing by a factor of 2 in the y-direction.

13. $A = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix} \underset{\substack{\text{Interchange} \\ \text{the first row} \\ \text{and the} \\ \text{second row}}}{\rightarrow} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \underset{\substack{\text{Multiply} \\ \text{the first} \\ \text{row by } \frac{1}{4}}}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \underset{\substack{\text{Multiply} \\ \text{the second} \\ \text{row by } -\frac{1}{2}}}{\rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

therefore $E_3E_2E_1A = I$ with $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ so that $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Multiplication by A has the geometric effect of reflection about the x-axis, then expanding by a factor of 2 in the y-direction, then expanding by a factor of 4 in the x-direction, then reflection about the line $y = x$.

14. $A = \begin{bmatrix} 1 & -3 \\ 4 & 6 \end{bmatrix} \underset{\substack{\text{Add } -4 \text{ times} \\ \text{the first row} \\ \text{to the} \\ \text{second row}}}{\to} \begin{bmatrix} 1 & -3 \\ 0 & 18 \end{bmatrix} \underset{\substack{\text{Multiply} \\ \text{the second} \\ \text{row by } \frac{1}{18}}}{\to} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \underset{\substack{\text{Add 3 times} \\ \text{the second row} \\ \text{to the} \\ \text{first row}}}{\to} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

therefore $E_3E_2E_1A = I$ with $E_1 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{18} \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ so that

$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 18 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

Multiplication by A has the geometric effect of shearing by a factor of -3 in the x-direction, then expanding by a factor of 18 in the y-direction, then shearing by a factor of 4 in the y-direction.

15. **(a)** The unit square is expanded in the x-direction by a factor of 3.

(b) The unit square is reflected about the x-axis and expanded in the y-direction by a factor of 5.

16. **(a)** The unit square is reflected about the y-axis and expanded in the x-direction by a factor of 2.

(b) The unit square is reflected about the x-axis, reflected about the y-axis and expanded in the x-direction by a factor of 3.

17. **(a)** Coordinates (a, b) are being transformed to coordinates (x, y) according to the equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a + b \\ 6a + 2b \end{bmatrix}$. It follows that $x = 3a + b$ and $y = 6a + 2b$ satisfy the equation of the line $y = 2x$ (since $6a + 2b = 2(3a + b)$).

(b) $\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 0$ so $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ is not invertible; Theorem 4.11.1 applies only to invertible matrices.

18. We need a real number k such that the shear of factor k in the x-direction transforms (2,1) into (0,1), i.e., $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Equating the corresponding components results in two equations, of which the second one $1 = 1$ is satisfied for all k. Solving the first equation $0 = 2 + k$ for k yields $k = -2$, therefore the matrix for the shear is $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

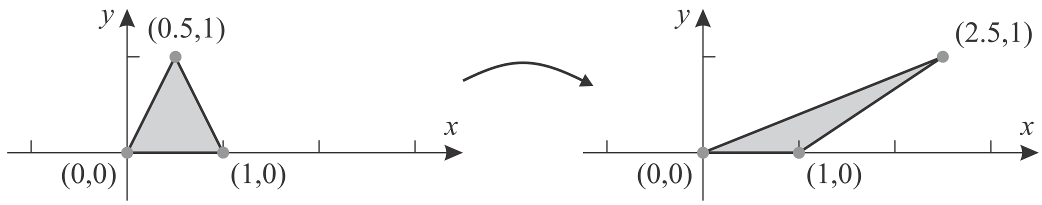
19. Coordinates (x, y) are being transformed to coordinates (x', y') according to the equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$ so $x = x' - 2y'$ and $y = -x' + 3y'$.

Substituting these into $y = 3x + 1$ yields $-x' + 3y' = 3(x' - 2y') + 1$ or equivalently $y' = \frac{4}{9}x' + \frac{1}{9}$.

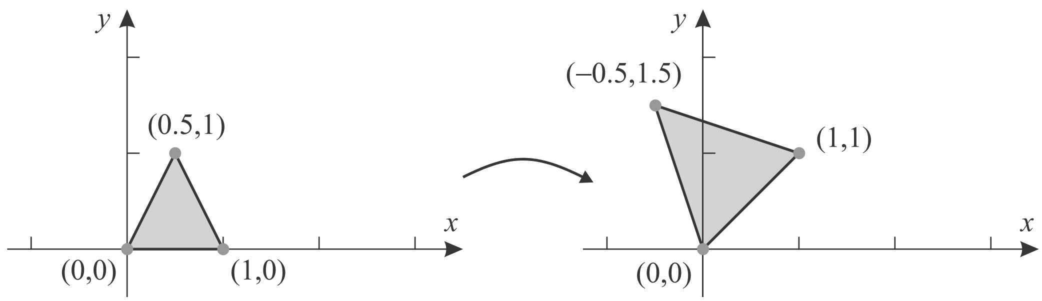
Substituting $x = x' - 2y'$ and $y = -x' + 3y'$ into $y = 3x - 2$ yields $-x' + 3y' = 3(x' - 2y') - 2$ or equivalently $y' = \frac{4}{9}x' - \frac{2}{9}$. Since both lines we obtained in the (x', y') coordinates have the same slope (4/9), we conclude that the given parallel lines are mapped into parallel lines.

20. The standard matrix of a shear by a factor 2 in the x-direction is $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$.

We obtain $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}0.5\\1\end{bmatrix}=\begin{bmatrix}2.5\\1\end{bmatrix}$.



21. **(a)** We obtain $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$, and $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}0.5\\1\end{bmatrix}=\begin{bmatrix}-0.5\\1.5\end{bmatrix}$.



(b) $A=\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} \underset{\substack{\text{Add } -1 \text{ times}\\ \text{the first row}\\ \text{to the}\\ \text{second row}}}{\to} \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix} \underset{\substack{\text{Multiply}\\ \text{the second}\\ \text{row by } \frac{1}{2}}}{\to} \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix} \underset{\substack{\text{Add}\\ \text{the second row}\\ \text{to the}\\ \text{first row}}}{\to} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

therefore $E_3E_2E_1A=I$ with $E_1=\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$, $E_2=\begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix}$, and $E_3=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ so that

$A=E_1^{-1}E_2^{-1}E_3^{-1}=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$.

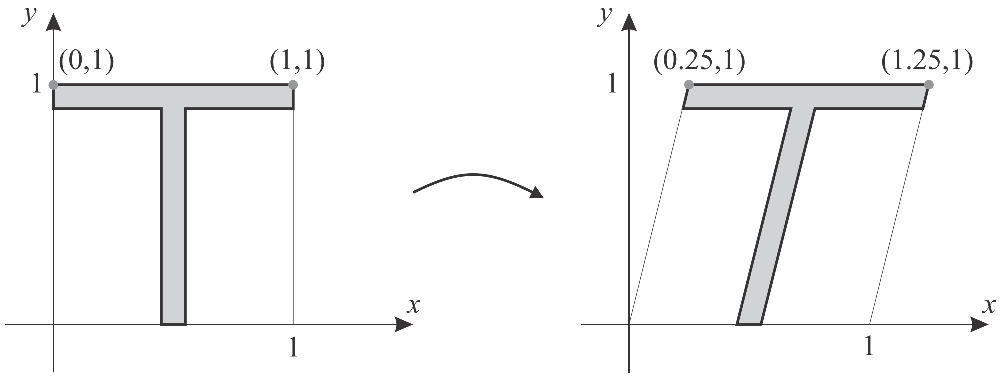
Shearing by a factor of -1 in the x-direction, then expanding by a factor of 2 in the y-direction, then shearing by a factor of 1 in the y-direction will produce the same image as in part (a).

22. **(a)** We calculate $\begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$ therefore the endpoints are (1,1) and (3,2).

(b) The standard matrix of rotation of 30° about the origin is $\begin{bmatrix}\cos 30° & -\sin 30°\\\sin 30° & \cos 30°\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$.

We calculate $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2}-1\\\frac{1}{2}+\sqrt{3}\end{bmatrix}$ and $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}\frac{3\sqrt{3}}{2}-2\\\frac{3}{2}+2\sqrt{3}\end{bmatrix}$ therefore the endpoints are $(\frac{\sqrt{3}}{2}-1,\frac{1}{2}+\sqrt{3})$ and $(\frac{3\sqrt{3}}{2}-2,\frac{3}{2}+2\sqrt{3})$.

23. We calculate the positions of corners of the image of the unit square in which the figure is inscribed:
$\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ 1 \end{bmatrix}$.



(The following calculations determine positions of the remaining endpoints of segments comprising the outline of the figure: $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.45 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.55 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 0.2375 \\ 0.95 \end{bmatrix}$,
$\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.45 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 0.6875 \\ 0.95 \end{bmatrix}$, $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.55 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 0.7875 \\ 0.95 \end{bmatrix}$, and $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 1.2375 \\ 0.95 \end{bmatrix}$.)

24. No. When multiplying by a 2×2 invertible matrix, it follows from part (e) of Theorem 4.11.1, that three points that are not collinear cannot map onto three collinear points, therefore the image of a square cannot be a triangle. (By using part (c) of the same theorem as well, we can conclude that the image of a square has to be a parallelogram instead.)

25. Since $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, the image of the given triangle is the line segment from (0,0) to (2,0). Theorem 4.11.1 does not apply here because A is singular.

26. **(a)** $\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$

(b) A shear in the xz-direction with factor k moves each point (x, y, z) to the new position $(x + ky, y, z + ky)$. The standard matrix for this transformation is $\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$.
A shear in the yz-direction with factor k moves each point (x, y, z) to the new position $(x, y + kx, z + kx)$. The standard matrix for this transformation is $\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$.

True-False Exercises

(a) False. The image is a parallelogram.

(b) True. This is a consequence of Theorem 4.11.2.

(c) True. This is the statement of part (a) of Theorem 4.11.1.

(d) True. Performing the same reflection twice amounts to no change (identity transformation).

(e) False. The matrix represents a composition of a reflection and a dilation.

(f) False. This matrix does not represent a shear in either x or y direction.

(g) True. This matrix represents an expansion by a factor of 3 in the y-direction.

Chapter 4 Supplementary Exercises

1. **(a)** $\mathbf{u}+\mathbf{v}=(3+1,-2+5,4-2)=(4,3,2)$; $\quad k\mathbf{u}=(-1\cdot 3,0,0)=(-3,0,0)$

(b) For any $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,u_3+v_3)$ is an ordered triple of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2,u_3)$ in V and for any scalar k, $k\mathbf{u}=(ku_1,0,0)$ is an ordered triple of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^3.

(d) Axiom 7: $k\big((u_1,u_2,u_3)+(v_1,v_2,v_3)\big)=k(u_1+v_1,u_2+v_2,u_3+v_3)=(k(u_1+v_1),0,0)=$ $k(u_1,u_2,u_3)+k(v_1,v_2,v_3)$ for all real $k,\ u_1,u_2,u_3,\ v_1,v_2,$ and v_3.

Axiom 8: $\quad (k+m)(u_1,u_2,u_3)=\big((k+m)u_1,0,0\big)=(ku_1+mu_1,0,0)=k(u_1,u_2,u_3)+$ $m(u_1,u_2,u_3)$ for all real $k,\ m,u_1,u_2,$ and u_3;

Axiom 9: $\quad k\big(m(u_1,u_2,u_3)\big)=k(mu_1,0,0)=(kmu_1,0,0)=(km)(u_1,u_2,u_3)$ for all real $k,\ m,$ $u_1,u_2,$ and u_3;

(e) Axiom 10 fails to hold: $1(u_1,u_2,u_3)=(u_1,0,0)$ does not generally equal (u_1,u_2,u_3). Consequently, V is not a vector space.

2. **(a)** The solution space is R^3 since all vectors (x,y,z) satisfy the system.

(b) The augmented matrix of the system has the reduced row echelon form $\begin{bmatrix}1 & -\frac{3}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

therefore the general solution is $x=\frac{3}{2}s-\frac{1}{2}t, y=s, z=t$. The solution space is a plane in R^3; its equation is
$2x-3y+z=0$, the first equation in our system (the other two equations were its multiples).

(c) The augmented matrix of the system has the reduced row echelon form $\begin{bmatrix}1 & -2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

therefore the general solution is $x=2t, y=t, z=0$ - these form parametric equations for a line in R^3.

(d) The augmented matrix of the system has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$ therefore the homogeneous system has only the trivial solution (0,0,0) - the origin.

3.

$$\begin{aligned}
A = \begin{bmatrix} 1 & 1 & s \\ 1 & s & 1 \\ s & 1 & 1 \end{bmatrix} &\longleftarrow \text{The coefficient matrix of the system} \\
\begin{bmatrix} 1 & 1 & s \\ 0 & s-1 & 1-s \\ 0 & 1-s & 1-s^2 \end{bmatrix} &\longleftarrow \begin{array}{l} -1 \text{ times the first row was added to the second row and} \\ -s \text{ times the first row was added to the third row.} \end{array} \\
\begin{bmatrix} 1 & 1 & s \\ 0 & s-1 & 1-s \\ 0 & 0 & 2-s-s^2 \end{bmatrix} &\longleftarrow \text{The second row was added to the third row.}
\end{aligned}$$

After factoring $2 - s - s^2 = (2+s)(1-s)$, we conclude that

- the solution space is a plane through the origin if $s = 1$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $\operatorname{nullity}(A) = 2$),
- the solution space is a line through the origin if $s = -2$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so $\operatorname{nullity}(A) = 1$),
- the solution space is the origin if $s \neq -2$ and $s \neq 1$ (the reduced row echelon form becomes $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $\operatorname{nullity}(A) = 0$),
- there are no values of s for which the solution space is R^3.

4. **(a)** $(4a, a-b, a+2b) = a(4,1,1) + b(0,-1,2)$

(b) $(3a+b+3c, -a+4b-c, 2a+b+2c) = a(3,-1,2) + b(1,4,1) + c(3,-1,2)$
$= (a+c)(3,-1,2) + b(1,4,1)$

(c) $(2a-b+4c, 3a-c, 4b+c) = a(2,3,0) + b(-1,0,4) + c(4,-1,1)$

5. **(a)** Using trigonometric identities we can write

$$\begin{aligned}
\mathbf{f}_1 &= \sin(x+\theta) = \sin x \cos\theta + \cos x \sin\theta = (\cos\theta)\mathbf{f} + (\sin\theta)\mathbf{g} \\
\mathbf{g}_1 &= \cos(x+\theta) = \cos x \cos\theta - \sin x \sin\theta = (-\sin\theta)\mathbf{f} + (\cos\theta)\mathbf{g}
\end{aligned}$$

which shows that $\mathbf{f}_1$ and $\mathbf{g}_1$ are both in $W = \operatorname{span}\{\mathbf{f}, \mathbf{g}\}$.

(b) The functions $\mathbf{f}_1 = \sin(x+\theta)$ and $\mathbf{f}_2 = \cos(x+\theta)$ are linearly independent since neither function is a scalar multiple of the other. By Theorem 4.5.4, these functions form a basis for W.

6. **(a)** We are looking for scalars c_1, c_2, and c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$, i.e.,

$$\begin{array}{rcrcrcr}
1c_1 & + & 3c_2 & + & 2c_3 & = & 1 \\
-1c_1 & & & + & c_3 & = & 1
\end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & \frac{2}{3} \end{bmatrix}$ so that the general solution is $c_1 = -1 + t$, $c_2 = \frac{2}{3} - t$, $c_3 = t$.

E.g., letting $t = 0$ yields $-1\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{v}$, whereas with $t = 1$ we obtain $0\mathbf{v}_1 - \frac{1}{3}\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{v}$.

(b) The vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ do not form a basis for R^2 therefore Theorem 4.4.1 does not apply here.

7. Denoting $B = [\ \mathbf{v}_1 \ \mid \cdots \mid \ \mathbf{v}_n\ ]$ we can write $AB = [\ A\mathbf{v}_1 \ \mid \cdots \mid \ A\mathbf{v}_n\ ]$.
By parts (g) and (h) of Theorem 4.8.7, the columns of AB are linearly independent if and only if $\det(AB) \neq 0$. This implies that $\det(A) \neq 0$, i.e., the matrix A must be invertible.

8. No, e.g., $x + 1$ and $x - 1$ form a basis for P_1 even though both are of degree 1.

9. **(a)** The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the rank is 2 and the nullity is 1.

(b) The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank is 2 and the nullity is 2.

(c) For $n = 1$, the rank is 1 and the nullity is 0.
For $n \geq 2$, the reduced row echelon form will always have two nonzero rows; the rank is 2 and the nullity is $n - 2$.

10. **(a)** Adding -1 times the first row to the third row yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; we conclude that the matrix has rank 2 and nullity 1.

(b) Adding -1 times the first row to the fifth row and adding -1 times the second row to the fourth row yields the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ therefore the matrix has rank 3 and nullity 2.

(c) After performing n elementary row operations which follow the same pattern as in parts (a) and (b):

- add -1 times row 1 to row $2n + 1$,
- add -1 times row 2 to row $2n$,
- add -1 times row 3 to row $2n - 1$,
- ...
- add -1 times row n to row $n + 2$,

the reduced row echelon form will be obtained: its top $n + 1$ rows are identical to those in the original X-matrix, whereas the bottom n rows are completely filled with zeros.
We conclude that the matrix has rank $n + 1$ and nullity n.

11. **(a)** Let W be the set of all polynomials p in P_n for which $p(-x) = p(x)$. In order for a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to be in W, we must have
$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = a_0 + a_1(-x) + a_2(-x)^2 + \cdots + a_n(-x)^n = p(-x)$ which implies that for all x, $2a_1x + 2a_3x^3 + \cdots = 0$ so $a_1 = a_3 = \cdots = 0$.
Any polynomial of the form $p(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2\lfloor n/2 \rfloor}x^{2\lfloor n/2 \rfloor}$ satisfies $p(-x) = p(x)$ (the notation $\lfloor t \rfloor$ represents the largest integer less than or equal to t).
This means $W = \operatorname{span}\{1, x^2, x^4, ..., x^{2\lfloor n/2 \rfloor}\}$, so W is a subspace of P_n by Theorem 4.2.3(a).
The polynomials in $\{1, x^2, x^4, ..., x^{2\lfloor n/2 \rfloor}\}$ are linearly independent (since they form a subset of the standard basis for P_n), consequently they form a basis for W.

(b) Let W be the set of all polynomials p in P_n for which $p(0) = p(1)$.
In order for a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to be in W, we must have
$p(0) = a_0 = a_0 + a_1 + a_2 + \cdots + a_n = p(1)$ which implies that $a_1 + a_2 + \cdots + a_n = 0$.
Therefore any polynomial in W can be expressed as
$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + (-a_1 - a_2 - \cdots - a_{n-1})x^n$
$= a_0 + a_1(x - x^n) + a_2(x^2 - x^n) + \cdots + a_{n-1}(x^{n-1} - x^n)$.
This means $W = \operatorname{span}\{1, x - x^n, x^2 - x^n, \dots, x^{n-1} - x^n\}$, so W is a subspace of P_n by Theorem 4.2.3(a). Since $a_0 + a_1(x - x^n) + a_2(x^2 - x^n) + \cdots + a_{n-1}(x^{n-1} - x^n) = 0$ implies $a_0 = a_1 = a_2 = \cdots = a_{n-1} = 0$, it follows that $\{1, x - x^n, x^2 - x^n, \dots, x^{n-1} - x^n\}$ is linearly independent, hence it is a basis for W.

12. For $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ to have a horizontal tangent at $x = 0$, we must have $p'(0) = 0$.
Since $p'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$ it follows that $p'(0) = a_1 = 0$. The set of all polynomials $p(x)$ for which $a_1 = 0$ is $\operatorname{span}\{1, x^2, x^3, \dots, x^n\}$ and therefore a subspace of P_n.
Since the set $\{1, x^2, x^3, \dots, x^n\}$ is clearly linearly independent and spans the subspace, it forms a basis for the subspace.

13. **(a)** A general 3×3 symmetric matrix can be expressed as $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$
$= a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Clearly the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
span the space of all 3×3 symmetric matrices. Also, these matrices are linearly indpendent,
since $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ requires that all six coefficients in the linear combination above
must be zero. We conclude that the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ form a basis for the space of all 3×3 symmetric matrices.

(b) A general 3×3 skew-symmetric matrix can be expressed as

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Clearly the matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ span the space of all 3×3 skew-symmetric matrices. Also, these matrices are linearly indpendent, since

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

requires that all three coefficients in the linear combination above must be zero. We conclude that the matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ form a basis for the space of all 3×3 skew-symmetric matrices.

14. **(a)** A submatrix $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ has nonzero determinant -1 therefore the rank of the original matrix is 2.

(b) All three 2×2 submatrices have zero determinant: $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$. Since determinant of any 1×1 submatrix of the original matrix is nonzero, the original matrix has rank 1.

(c) The original 3×3 matrix has zero determinant. A submatrix $\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$ has nonzero determinant -1 therefore the rank of the original matrix is 2.

(d) The original 3×3 matrix has zero determinant 30 therefore the rank of the original matrix is 3.

15. All submatrices of size 3×3 or larger contain at least two rows that are scalar multiples of each other, so their determinants are 0. Therefore the rank cannot exceed 2. The possible values are:

- $\operatorname{rank}(A) = 2$, e.g., if $a_{51} = a_{16} = 1$ regardless of the other values,
- $\operatorname{rank}(A) = 1$, e.g., if $a_{16} = a_{26} = a_{36} = a_{46} = 0$ and $a_{56} = 1$ regardless of the other values, and
- $\operatorname{rank}(A) = 0$ if all entries are 0.

17. The standard matrices for D_k, R_θ, and S_k are $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (assuming a shear in the x-direction).

(a) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ therefore D_k and R_θ commute.

(b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & k\cos\theta - \sin\theta \\ \sin\theta & k\sin\theta + \cos\theta \end{bmatrix}$ does not generally equal
$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta + k\sin\theta & -\sin\theta + k\cos\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
therefore R_θ and S_k do not commute (same result is obtained if a shear in the y-direction is taken instead)

(c) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ therefore D_k and S_k commute (same result is obtained if a shear in the y-direction is taken instead)

18. **(b)** Every vector (x, y, z) in R^3 can be expressed in exactly one way as a sum of a vector $(x, y, 0)$ in U and $(0,0, z)$ in W. Consequently, $R^3 = U \oplus W$.

(c) Every vector (x, y, z) in R^3 can be expressed as a sum of a vector in U and a vector in V. However, in this case, this representation is not unique, for instance,

$$(1,2,3) = \underbrace{(1,1,0)}_{\text{vector in } U} + \underbrace{(0,1,3)}_{\text{vector in } V} = \underbrace{(1,3,0)}_{\text{vector in } U} + \underbrace{(0,-1,3)}_{\text{vector in } V}$$

We conclude that R^3 is not a direct sum of the xy-plane and the yz-plane.

CHAPTER 5: EIGENVALUES AND EIGENVECTORS

5.1 Eigenvalues and Eigenvectors

1. $A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\mathbf{x}$ therefore $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue -1.

2. $A\mathbf{x} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\mathbf{x}$ therefore $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 4.

3. $A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\mathbf{x}$ therefore $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 5.

4. $A\mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\mathbf{x}$ therefore $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 0.

5. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) - (-4)(-2) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$.

The characteristic equation is $(\lambda - 5)(\lambda + 1) = 0$. The eigenvalues are $\lambda = 5$ and $\lambda = -1$.

The reduced row echelon form of $5I - A = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(5I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = t, x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 5$ is $\{(1,1)\}$.

The reduced row echelon form of $-1I - A = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. The general solution of $(-1I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = -2t, x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = -1$ is $\{(-2,1)\}$.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 7 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda + 2)(\lambda - 2) - (7)(-1) = \lambda^2 + 3$.

The characteristic equation is $\lambda^2 + 3 = 0$. There are no real eigenvalues.

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2$.

The characteristic equation is $(\lambda - 1)^2 = 0$. The eigenvalue is $\lambda = 1$.

The matrix $I - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is already in reduced row echelon form. The general solution of $(I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = s, x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(1,0),(0,1)\}$.

(d) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2$.

The characteristic equation is $(\lambda - 1)^2 = 0$. The eigenvalue is $\lambda = 1$.

The reduced row echelon form of $I - A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = t, x_2 = 0$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(1,0)\}$.

6. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - (-1)^2 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$.
The characteristic equation is $(\lambda - 1)(\lambda - 3) = 0$. The eigenvalues are $\lambda = 1$ and $\lambda = 3$.

The reduced row echelon form of $I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = -t, x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(-1,1)\}$.

The reduced row echelon form of $3I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(3I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = t, x_2 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
A basis for the eigenspace corresponding to $\lambda = 3$ is $\{(1,1)\}$.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2$.
The characteristic equation is $(\lambda - 2)^2 = 0$. The eigenvalue is $\lambda = 2$.

The reduced row echelon form of $2I - A = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(2I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = t, x_2 = 0$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(1,0)\}$.

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2$.
The characteristic equation is $(\lambda - 2)^2 = 0$. The eigenvalue is $\lambda = 2$.

The matrix $2I - A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is already in reduced row echelon form.
The general solution of $(2I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = s, x_2 = t$. In vector form,
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(1,0), (0,1)\}$.

(d) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ 2 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1) - (-2)(2) = \lambda^2 + 3$.
The characteristic equation is $\lambda^2 + 3 = 0$. There are no real eigenvalues.

7. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix}$
$= (\lambda - 1)\begin{vmatrix} \lambda - 4 & -1 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)[(\lambda - 4)(\lambda - 1) - (-1)(2)] = (\lambda - 1)(\lambda^2 - 5\lambda + 6)$
$= (\lambda - 1)(\lambda - 2)(\lambda - 3)$. The characteristic equation is $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$.
The eigenvalues are $\lambda = 1, \lambda = 2$, and $\lambda = 3$.

The reduced row echelon form of $I - A = \begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = 0, x_2 = t, x_3 = 0$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(0,1,0)\}$.

The reduced row echelon form of $2I - A = \begin{bmatrix} -2 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(2I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = -\frac{1}{2}t, x_2 = t, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(-1,2,2)\}$ (scaled by a factor of 2 for convenience).

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(3I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = -t, x_2 = t, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 3$ is $\{(-1,1,1)\}$.

8. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda - 4 \end{vmatrix}$

$= \lambda\begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 4 \end{vmatrix} = \lambda[(\lambda - 1)(\lambda - 4) - (2)(2)] = \lambda(\lambda^2 - 5\lambda) = \lambda^2(\lambda - 5)$.

The characteristic equation is $\lambda^2(\lambda - 5) = 0$. The eigenvalues are $\lambda = 0$ and $\lambda = 5$.

The reduced row echelon form of $0I - A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(0I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = 2t, x_2 = s, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 0$ is $\{(0,1,0), (2,0,1)\}$.

The reduced row echelon form of $5I - A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(5I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = -\frac{1}{2}t, x_2 = 0, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 5$ is $\{(-1,0,2)\}$ (scaled by a factor of 2 for convenience).

9. Cofactor expansion along the second row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda - 6 & -3 & 8 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda + 3 \end{vmatrix}$

$= (\lambda + 2)\begin{vmatrix} \lambda - 6 & 8 \\ -1 & \lambda + 3 \end{vmatrix} = (\lambda + 2)[(\lambda - 6)(\lambda + 3) - (8)(-1)] = (\lambda + 2)(\lambda^2 - 3\lambda - 10)$

$=(\lambda+2)(\lambda+2)(\lambda-5)$. The characteristic equation is $(\lambda+2)^2(\lambda-5)=0$.
The eigenvalues are $\lambda=-2$ and $\lambda=5$.

The reduced row echelon form of $-2I-A=\begin{bmatrix}-8 & -3 & 8\\ 0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of $(-2I-A)\mathbf{x}=\mathbf{0}$ is $x_1=t, x_2=0, x_3=t$. In vector form, $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\0\\t\end{bmatrix}=t\begin{bmatrix}1\\0\\1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=-2$ is $\{(1,0,1)\}$.

The reduced row echelon form of $5I-A=\begin{bmatrix}-1 & -3 & 8\\ 0 & 7 & 0\\ -1 & 0 & 8\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -8\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of $(5I-A)\mathbf{x}=\mathbf{0}$ is $x_1=8t, x_2=0, x_3=t$. In vector form, $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}8t\\0\\t\end{bmatrix}=t\begin{bmatrix}8\\0\\1\end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda=5$ is $\{(8,0,1)\}$.

10. Cofactor expansion along the first row yields $\det(\lambda I-A)=\begin{vmatrix}\lambda & -1 & -1\\ -1 & \lambda & -1\\ -1 & -1 & \lambda\end{vmatrix}$

$$=\lambda\begin{vmatrix}\lambda & -1\\ -1 & \lambda\end{vmatrix}-(-1)\begin{vmatrix}-1 & -1\\ -1 & \lambda\end{vmatrix}+(-1)\begin{vmatrix}-1 & \lambda\\ -1 & -1\end{vmatrix}=\lambda(\lambda^2-1)+(-\lambda-1)-(1+\lambda)$$
$$=\lambda(\lambda-1)(\lambda+1)-(\lambda+1)-(\lambda+1)=(\lambda+1)[\lambda(\lambda-1)-1-1]=(\lambda+1)(\lambda^2-\lambda-2)$$

$=(\lambda+1)(\lambda+1)(\lambda-2)$. We conclude that the eigenvalues are -1 and 2.

The reduced row echelon form of $-1I-A=\begin{bmatrix}-1 & -1 & -1\\ -1 & -1 & -1\\ -1 & -1 & -1\end{bmatrix}$ is $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. The general solution of $(-1I-A)\mathbf{x}=\mathbf{0}$ is $x_1=-s-t, x_2=s, x_3=t$. In vector form, $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-s-t\\s\\t\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$.
A basis for the eigenspace corresponding to $\lambda=-1$ is $\{(-1,1,0),(-1,0,1)\}$.

The reduced row echelon form of $2I-A=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$.

The general solution of $(2I-A)\mathbf{x}=\mathbf{0}$ is $x_1=t, x_2=t, x_3=t$. In vector form, $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\t\\t\end{bmatrix}=t\begin{bmatrix}1\\1\\1\end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda=2$ is $\{(1,1,1)\}$.

11. Cofactor expansion along the second column yields $\det(\lambda I-A)=\begin{vmatrix}\lambda-4 & 0 & 1\\ 0 & \lambda-3 & 0\\ -1 & 0 & \lambda-2\end{vmatrix}$

$=(\lambda-3)\begin{vmatrix}\lambda-4 & 1\\ -1 & \lambda-2\end{vmatrix}=(\lambda-3)[(\lambda-4)(\lambda-2)-(1)(-1)]=(\lambda-3)(\lambda^2-6\lambda+9)=(\lambda-3)^3$.
The characteristic equation is $(\lambda-3)^3=0$. The eigenvalue is $\lambda=3$.

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of $(3I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = t, x_2 = s, x_3 = t$. In vector form,

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 3$ is $\{(0,1,0), (1,0,1)\}$.

12. We use the arrow technique to evaluate the determinant: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 3 & -3 \\ -3 & \lambda + 5 & -3 \\ -6 & 6 & \lambda - 4 \end{vmatrix}$

$= [(\lambda - 1)(\lambda + 5)(\lambda - 4) + 54 + 54] - [18(\lambda + 5) - 18(\lambda - 1) - 9(\lambda - 4)] = \lambda^3 - 12\lambda - 16.$

The characteristic equation is $\lambda^3 - 12\lambda - 16 = 0$.

Following the procedure described in Example 3, we determine that the only integer solutions of the characteristic equation are $\pm 1, \pm 2, \pm 4, \pm 8$, and ± 16. Successively substituting these into the characteristic polynomial, we find $\det(-2I - A) = 0$ so that $\lambda + 2$ must be a factor of the polynomial. Dividing $\lambda + 2$ into $\lambda^3 - 12\lambda - 16$ we obtain

$\det(\lambda I - A) = (\lambda + 2)(\lambda^2 - 2\lambda - 8) = (\lambda + 2)(\lambda + 2)(\lambda - 4)$.

We conclude that the eigenvalues are -2 and 4.

The reduced row echelon form of $-2I - A = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(-2I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = s - t, x_2 = s, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s - t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = -2$ is $\{(1,1,0), (-1,0,1)\}$.

The reduced row echelon form of $4I - A = \begin{bmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(4I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = \frac{1}{2}t, x_2 = \frac{1}{2}t, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 4$ is $\{(1,1,2)\}$ (scaled by a factor of 2 for convenience).

13. The matrix $\lambda I - A$ is lower triangular, therefore by Theorem 2.1.2 its determinant is the product of the entries on the main diagonal. Therefore the characteristic equation is $(\lambda - 3)(\lambda - 7)(\lambda - 1) = 0$.

14. The matrix $\lambda I - A$ is upper triangular, therefore by Theorem 2.1.2 its determinant is the product of the entries on the main diagonal. Therefore the characteristic equation is $(\lambda - 9)(\lambda + 1)(\lambda - 3)(\lambda - 7) = 0$.

15. $T(x, y) = \begin{bmatrix} x + 4y \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$; the standard matrix for the operator T is $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

The following results were obtained in the solution of Exercise 5(a). These statements apply to the matrix A, therefore they also apply to the associated operator T:

- the eigenvalues are $\lambda = 5$ and $\lambda = -1$,

- a basis for the eigenspace corresponding to $\lambda = 5$ is $\{(1,1)\}$,
- a basis for the eigenspace corresponding to $\lambda = -1$ is $\{(-2,1)\}$.

16. $T(x,y,z) = \begin{bmatrix} 2x-y-z \\ x-z \\ -x+y+2z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; the standard matrix for T is $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

We use the arrow technique to evaluate the determinant: $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 1 \\ -1 & \lambda & 1 \\ 1 & -1 & \lambda-2 \end{vmatrix}$

$= [(\lambda-2)\lambda(\lambda-2)+1+1] - [\lambda - (\lambda-2) - (\lambda-2)] = \lambda^3 - 4\lambda^2 + 5\lambda - 2.$

The characteristic equation is $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$.

Following the procedure described in Example 3, we determine that the only possible integer solutions of the characteristic equation are ± 1 and ± 2.

Since $\det(I - A) = 0$, $\lambda - 1$ must be a factor of the characteristic polynomial. Dividing $\lambda - 1$ into $\lambda^3 - 4\lambda^2 + 5\lambda - 2$ leads to $\det(\lambda I - A) = (\lambda-1)(\lambda^2 - 3\lambda + 2) = (\lambda-1)(\lambda-1)(\lambda-2)$.

We conclude that the eigenvalues are 1 and 2.

The reduced row echelon form of $1I - A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(1I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = s + t, x_2 = s, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 1$ is $\{(1,1,0),(1,0,1)\}$.

The reduced row echelon form of $2I - A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The general solution of $(2I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = -t, x_2 = -t, x_3 = t$. In vector form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to $\lambda = 2$ is $\{(-1,-1,1)\}$.

17. **(a)** The transformation D^2 maps any function $\mathbf{f} = f(x)$ in $C^\infty(-\infty,\infty)$ into its second derivative, i.e. $D^2(\mathbf{f}) = f''(x)$. From calculus, we have

$D^2(\mathbf{f} + \mathbf{g}) = \frac{d}{dx}\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f'(x) + g'(x)\big) = f''(x) + g''(x) = D^2(\mathbf{f}) + D^2(\mathbf{g})$ and

$D^2(k\mathbf{f}) = \frac{d}{dx}\frac{d}{dx}\big(kf(x)\big) = \frac{d}{dx}\big(kf'(x)\big) = kf''(x) = kD^2(\mathbf{f})$. We conclude that D^2 is linear.

(b) Denote $\mathbf{f} = \sin\sqrt{\omega}x$ and $\mathbf{g} = \cos\sqrt{\omega}x$. We have

$D^2(\mathbf{f}) = \frac{d}{dx}\frac{d}{dx}\big(\sin\sqrt{\omega}x\big) = \frac{d}{dx}\big(\sqrt{\omega}\cos\sqrt{\omega}x\big) = \big(\sqrt{\omega}\big)\big(-\sqrt{\omega}\ \sin\sqrt{\omega}x\big) = -\omega\ \sin\sqrt{\omega}x = -\omega\mathbf{f}$ and

$D^2(\mathbf{g}) = \frac{d}{dx}\frac{d}{dx}\big(\cos\sqrt{\omega}x\big) = \frac{d}{dx}\big(-\sqrt{\omega}\sin\sqrt{\omega}x\big) = \big(-\sqrt{\omega}\big)\big(\sqrt{\omega}\ \cos\sqrt{\omega}x\big) = -\omega\ \cos\sqrt{\omega}x = -\omega\mathbf{g}$.

It follows that $\mathbf{f} = \sin\sqrt{\omega}x$ and $\mathbf{g} = \cos\sqrt{\omega}x$ are eigenvectors of D^2; $\lambda = -\omega$ is the eigenvalue associated with both of these eigenvectors.

18. Denote $\mathbf{f} = \sinh\sqrt{\omega}x$ and $\mathbf{g} = \cosh\sqrt{\omega}x$. We have

$D^2(\mathbf{f}) = \frac{d}{dx}\frac{d}{dx}\big(\sinh\sqrt{\omega}x\big) = \frac{d}{dx}\big(\sqrt{\omega}\cosh\sqrt{\omega}x\big) = \big(\sqrt{\omega}\big)\big(\sqrt{\omega}\ \sinh\sqrt{\omega}x\big) = \omega\ \sinh\sqrt{\omega}x = \omega\mathbf{f}$ and

$D^2(\mathbf{g}) = \frac{d}{dx}\frac{d}{dx}\big(\cosh\sqrt{\omega}x\big) = \frac{d}{dx}\big(\sqrt{\omega}\sinh\sqrt{\omega}x\big) = \big(\sqrt{\omega}\big)\big(\sqrt{\omega}\ \cosh\sqrt{\omega}x\big) = \omega\ \cosh\sqrt{\omega}x = \omega\mathbf{g}$.

It follows that $\mathbf{f} = \sinh\sqrt{\omega}x$ and $\mathbf{g} = \cosh\sqrt{\omega}x$ are eigenvectors of D^2; $\lambda = \omega$ is the eigenvalue associated with both of these eigenvectors.

19. **(a)** The reflection of any vector on the line $y = x$ is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,1)\}$.
The reflection of any vector perpendicular to the line $y = x$ (i.e., on the line $y = -x$) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(-1,1)\}$.

(b) The projection of any vector on the x-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.
The projection of any vector perpendicular to the x-axis (i.e., on the y-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.

(c) The result of the rotation through 90° of a nonzero vector is never a scalar multiple of the original vector. Consequently, this operator has no real eigenvalues.

(d) The result of the contraction of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^2.

(e) The result of the shear applied to any vector on the x-axis is the same vector whereas the result of the shear applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(1,0)\}$.

20. **(a)** The reflection of any vector on the y-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.
The reflection of any vector perpendicular to the y-axis (i.e., on the x-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.

(b) The result of the rotation through 180° of any vector is the negative of the original vector: the only eigenvalue is $\lambda = -1$ and the corresponding eigenspace is the entire space R^2.

(c) The result of the dilation of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^2.

(d) The result of the expansion applied to any vector on the x-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0)\}$.
The result of the expansion applied to any vector $\mathbf{v}$ on the y-axis is the scalar multiple $k\mathbf{v}$: an eigenvalue $\lambda = k$ corresponds to the eigenspace $\text{span}\{(0,1)\}$.

(e) The result of the shear applied to any vector on the y-axis is the same vector whereas the result of the shear applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(0,1)\}$.

21. **(a)** The reflection of any vector on the xy-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0),(0,1,0)\}$.
The reflection of any vector perpendicular to the xy-plane (i.e., on the z-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(0,0,1)\}$.

(b) The projection of any vector on the xz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0), (0,0,1)\}$.
The projection of any vector perpendicular to the xz-plane (i.e., on the y-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.

(c) The result of the rotation applied to any vector on the x-axis is the same vector whereas the result of the rotation applied to a nonzero vector in any other direction is not a scalar multiple of the original vector. The only eigenvalue is $\lambda = 1$ and the corresponding eigenspace is $\text{span}\{(1,0,0)\}$.

(d) The result of the contraction of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^3.

22. **(a)** The reflection of any vector on the xz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(1,0,0), (0,0,1)\}$.
The reflection of any vector perpendicular to the xz-plane (i.e., on the y-axis) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.

(b) The projection of any vector on the yz-plane is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(0,1,0), (0,0,1)\}$.
The projection of any vector perpendicular to the yz-plane (i.e., on the x-axis) is the zero vector: an eigenvalue $\lambda = 0$ corresponds to the eigenspace $\text{span}\{(1,0,0)\}$.

(c) The result of the rotation applied to any vector on the y-axis is the same vector: an eigenvalue $\lambda = 1$ corresponds to the eigenspace $\text{span}\{(0,1,0)\}$.
The result of the rotation applied to any vector perpendicular to the y-axis (i.e., on the xz-plane) is the negative of the original vector: an eigenvalue $\lambda = -1$ corresponds to the eigenspace $\text{span}\{(1,0,0), (0,0,1)\}$.

(d) The result of the dilation of any vector $\mathbf{v}$ is a scalar multiple $k\mathbf{v}$ therefore the only eigenvalue is $\lambda = k$ and the corresponding eigenspace is the entire space R^3.

23. A line through the origin in the direction of $\mathbf{x} \neq \mathbf{0}$ is invariant under A if and only if $\mathbf{x}$ is an eigenvector of A.

(a) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 1 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda - 1) - (1)(-2) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$.
The characteristic equation is $(\lambda - 2)(\lambda - 3) = 0$. The eigenvalues are $\lambda = 2$ and $\lambda = 3$.

The reduced row echelon form of $2I - A = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$. The general solution of $(2I - A)\mathbf{x} = \mathbf{0}$ is $x = \frac{1}{2}t, y = t$. Therefore $y = 2x$ is an equation of the corresponding invariant line.

The reduced row echelon form of $3I - A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The general solution of $(3I - A)\mathbf{x} = \mathbf{0}$ is $x_1 = t, x_2 = t$. Therefore $y = x$ is an equation of the corresponding invariant line.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - (-1)(1) = \lambda^2 + 1.$

There are no real eigenvalues and no invariant lines.

24. Since $p(\lambda) = \det(\lambda I - A)$, it follows that $p(0) = \det(-A) = (-1)^n \det(A)$.

(a) $n = 3$ and $p(0) = 5$ therefore $\det(A) = -5$

(b) $n = 4$ and $p(0) = 7$ therefore $\det(A) = 7$

25. (a) Since the degree of $p(\lambda)$ is 6, A is a 6×6 matrix (see Exercise 37)

(b) $p(0) \neq 0$, therefore 0 is not an eigenvalue of A. From parts (a) and (t) of Theorem 5.1.6, A is invertible.

(c) A has three eigenspaces since it has three distinct eigenvalues, each corresponding to an eigenspace.

27. Substituting the given eigenvectors $\mathbf{x}$ and the corresponding eigenvalues λ into $A\mathbf{x} = \lambda\mathbf{x}$ yields

$A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, and $A\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

We can combine these three equations into a single equation $A\left[\begin{array}{c|c|c} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{array}\right]$.

Since the matrix $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible, we can multiply both sides on the right by its inverse,

$\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix}$, resulting in $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(Note that this exercise could also be solved by assigning nine unknown values to the elements of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then solving the system of nine equations in nine unknowns resulting from the equation $A\mathbf{x} = \lambda\mathbf{x}$.)

28. Denoting $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have $\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix}$

$= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = \lambda^2 - (\underbrace{a_{11} + a_{22}}_{\mathrm{tr}(A)})\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{22}}_{\det(A)}.$

31. It follows from Exercise 28 that if the characteristic polynomial $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $p(\lambda) = \lambda^2 + c_1\lambda + c_2$ then $c_1 = -\mathrm{tr}(A) = -a_{11} - a_{22}$ and $c_2 = \det(A) = a_{11}a_{22} - a_{12}a_{21}$. Therefore

$p(A) = A^2 + c_1A + c_2I$

$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + (-a_{11} - a_{22})\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + (a_{11}a_{22} - a_{12}a_{21})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix} + \begin{bmatrix} -a_{11}^2 - a_{22}a_{11} & -a_{11}a_{12} - a_{22}a_{12} \\ -a_{11}a_{21} - a_{22}a_{21} & -a_{11}a_{22} - a_{22}^2 \end{bmatrix}$

$$+\begin{bmatrix} a_{11}a_{22}-a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

33. By Theorem 5.1.5, it follows from A being invertible that A cannot have a zero eigenvalue. Multiplying both sides of the equation $A\mathbf{x}=\lambda\mathbf{x}$ by A^{-1} on the left and applying Theorem 1.4.1 yields $A^{-1}A\mathbf{x}=\lambda A^{-1}\mathbf{x}$. Since $A^{-1}A=I$, dividing both sides of the equation by λ we obtain $\frac{1}{\lambda}\mathbf{x}=A^{-1}\mathbf{x}$. This shows that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} associated with an eigenvector $\mathbf{x}$.

34. Subtracting $sI\mathbf{x}=s\mathbf{x}$ from both sides of the equation $A\mathbf{x}=\lambda\mathbf{x}$ and applying Theorem 1.4.1 yields $(A-sI)\mathbf{x}=(\lambda-s)\mathbf{x}$. This shows that $\lambda-s$ is an eigenvalue of $A-sI$ associated with the eigenvector $\mathbf{x}$.

35. Multiplying both sides of the equation $A\mathbf{x}=\lambda\mathbf{x}$ by the scalar s yields $s(A\mathbf{x})=s(\lambda\mathbf{x})$. By Theorem 1.4.1, the equation can be rewritten as $(sA)\mathbf{x}=(s\lambda)\mathbf{x}$. This shows that $s\lambda$ is an eigenvalue of sA associated with the eigenvector $\mathbf{x}$.

36. $\det(\lambda I-A)=\begin{vmatrix} \lambda+2 & -2 & -3 \\ 2 & \lambda-3 & -2 \\ 4 & -2 & \lambda-5 \end{vmatrix}=\lambda^3-6\lambda^2+11\lambda-6;$

The only possibilities for integer solutions of the characteristic equation are $\pm1,\pm2,\pm3,\pm6$.
Since $\det(1I-A)=0$, we divide $\lambda-1$ into $\lambda^3-6\lambda^2+11\lambda-6$ to obtain
$\det(\lambda I-A)=(\lambda-1)(\lambda^2-5\lambda+6)=(\lambda-1)(\lambda-2)(\lambda-3)$.

The reduced row echelon form of $I-A=\begin{bmatrix} 3 & -2 & -3 \\ 2 & -2 & -2 \\ 4 & -2 & -4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
The general solution of $(I-A)\mathbf{x}=\mathbf{0}$ is $x_1=t, x_2=0, x_3=t$.
A basis for the eigenspace of the matrix A corresponding to $\lambda=1$ is $\{(1,0,1)\}$.

The reduced row echelon form of $2I-A=\begin{bmatrix} 4 & -2 & -3 \\ 2 & -1 & -2 \\ 4 & -2 & -3 \end{bmatrix}$ is $\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
The general solution of $(2I-A)\mathbf{x}=\mathbf{0}$ is $x_1=\frac{1}{2}t, x_2=t, x_3=0$.
A basis for the eigenspace of the matrix A corresponding to $\lambda=2$ is $\{(1,2,0)\}$.

The reduced row echelon form of $3I-A=\begin{bmatrix} 5 & -2 & -3 \\ 2 & 0 & -2 \\ 4 & -2 & -2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.
The general solution of $(3I-A)\mathbf{x}=\mathbf{0}$ is $x_1=t, x_2=t, x_3=t$.
A basis for the eigenspace of the matrix A corresponding to $\lambda=3$ is $\{(1,1,1)\}$.

(a) From the result of Exercise 33, the matrix A^{-1} has

- eigenvalue 1 with a basis for the corresponding eigenspace $\{(1,0,1)\}$,
- eigenvalue $\frac{1}{2}$ with a basis for the corresponding eigenspace $\{(1,2,0)\}$,
- eigenvalue $\frac{1}{3}$ with a basis for the corresponding eigenspace $\{(1,1,1)\}$.

(b) From the result of Exercise 34, the matrix $A-3I$ has

- eigenvalue -2 with a basis for the corresponding eigenspace $\{(1,0,1)\}$,
- eigenvalue -1 with a basis for the corresponding eigenspace $\{(1,2,0)\}$,
- eigenvalue 0 with a basis for the corresponding eigenspace $\{(1,1,1)\}$.

(c) From the result of Exercise 34, the matrix $A+2I$ has

- eigenvalue 3 with a basis for the corresponding eigenspace $\{(1,0,1)\}$,
- eigenvalue 4 with a basis for the corresponding eigenspace $\{(1,2,0)\}$,
- eigenvalue 5 with a basis for the corresponding eigenspace $\{(1,1,1)\}$.

True-False Exercises

(a) False. The vector $\mathbf{x}$ must be nonzero (without that requirement, $A\mathbf{x} = \lambda\mathbf{x}$ holds true for <u>all</u> $n \times n$ matrices A and <u>all</u> values λ by taking $\mathbf{x} = \mathbf{0}$).

(b) False. If λ is an eigenvalue of A then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ must have nontrivial solutions.

(c) True. Since $p(0) = 1 \neq 0$, zero is not an eigenvalue of A. By Theorem 5.1.5, we conclude that A is invertible.

(d) False. Every eigenspace must include the zero vector, which is not an eigenvector.

(e) False. E.g., The only eigenvalue of $A = 2I$ is 2. However, the reduced row echelon form of A is I, whose only eigenvalue is 1.

(f) False. By Theorem 5.1.5, the set of columns of A must be linearly dependent.

5.2 Diagonalization

1. $\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1$ does not equal $\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} = -2$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

2. $\begin{vmatrix} 4 & -1 \\ 2 & 4 \end{vmatrix} = 18$ does not equal $\begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} = 14$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

3. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (1)(1)(1) = 1$ does not equal $\begin{vmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - 0 + 1\begin{vmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{vmatrix} = 0$ therefore, by Table 1 in Section 5.2, A and B are not similar matrices.

4. The reduced row echelon forms of the matrices A and B are $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, respectively. Since $\operatorname{rank}(A) = 1$ does not equal $\operatorname{rank}(B) = 2$, by Table 1 in Section 5.2, A and B are not similar matrices.

5. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1)$ therefore A has eigenvalues 1 and -1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{3}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = 0, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors $\mathbf{p}_1$ and $\mathbf{p}_2$: $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

Calculating $P^{-1} = \frac{1}{(1)(1)-(0)(3)} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and performing matrix multiplications we check that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

6. $\det(\lambda I - A) = \begin{vmatrix} \lambda + 14 & -12 \\ 20 & \lambda - 17 \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$ therefore A has eigenvalues 1 and 2.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -\frac{4}{5} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{4}{5}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{3}{4}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors $\mathbf{p}_1$ and $\mathbf{p}_2$: $P = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

Calculating $P^{-1} = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$ and performing matrix multiplications we check that

$P^{-1}AP = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 3)^2$ thus A has eigenvalues 2 and 3 (with algebraic multiplicity 2).

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = 0, x_3 = 0$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix}1&0&2\\0&0&0\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = \lambda_3 = 3$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1 = -2t, x_2 = s, x_3 = t$. We can write

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-2t\\s\\t\end{bmatrix} = s\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\1\end{bmatrix}$ therefore vectors $\mathbf{p}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$ form a basis for this eigenspace. Note that the geometric multiplicity of this eigenvalue matches its algebraic multiplicity.

We form a matrix P using the column vectors $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$: $P = \begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

To invert the matrix P, we can employ the procedure introduced in Section 1.5: since the reduced row echelon form of the matrix $\begin{bmatrix}1&0&-2&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix}$ is $\begin{bmatrix}1&0&0&1&0&2\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix}$, we have $P^{-1} = \begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$.

We check that $P^{-1}AP = \begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&-2\\0&3&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}2&0&0\\0&3&0\\0&0&3\end{bmatrix} = \begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$.

8. Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix}\lambda-1&0&0\\0&\lambda-1&-1\\0&-1&\lambda-1\end{vmatrix} =$ $(\lambda-1)\begin{vmatrix}\lambda-1&-1\\-1&\lambda-1\end{vmatrix} = (\lambda-1)[(\lambda-1)^2-1] = (\lambda-1)(\lambda-2)\lambda$ thus A has eigenvalues 0, 1 and 2.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix}1&0&0\\0&1&1\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 0$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1 = 0, x_2 = -t, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 1$ contains vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1 = t, x_2 = 0, x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 =$

2 contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$: $P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

Calculating $P^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and performing matrix multiplications we check that

$$P^{-1}AP = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

9. **(a)** Cofactor expansion along the second column yields $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 0 & -1 \\ -2 & \lambda - 3 & -2 \\ -1 & 0 & \lambda - 4 \end{vmatrix} =$ $(\lambda - 3)\begin{vmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{vmatrix} = (\lambda - 3)[(\lambda - 4)^2 - 1] = (\lambda - 3)^2(\lambda - 5)$ therefore A has eigenvalues 3 (with algebraic multiplicity 2) and 5.

(b) The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, consequently $\operatorname{rank}(3I - A) = 1$.

The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, consequently $\operatorname{rank}(5I - A) = 2$.

(c) Based on part (b), the geometric multiplicities of the eigenvalues $\lambda = 3$ and $\lambda = 5$ are $3 - 1 = 2$ and $3 - 2 = 1$, respectively. Since these are equal to the corresponding algebraic multiplicities, by Theorem 5.2.4(b) A is diagonalizable.

10. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & -1 & \lambda - 2 \end{vmatrix} = (\lambda - 3)(\lambda - 2)^2$ therefore A has eigenvalues 2 (with algebraic multiplicity 2) and 3.

(b) The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, consequently $\operatorname{rank}(2I - A) = 2$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, consequently $\operatorname{rank}(3I - A) = 2$.

(c) Based on part (b) , the geometric multiplicity of the eigenvalue $\lambda = 3$ is $3 - 2 = 1$, which is equal to the algebraic multiplicity of this eigenvalue. However, the geometric multiplicity of the eigenvalue $\lambda = 2$ is $3 - 2 = 1$, which is less than the corresponding algebraic multiplicity (2), therefore by Theorem 5.2.4(b) A is not diagonalizable.

11. Cofactor expansion along the second row yields

$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix} = -3\begin{vmatrix} -4 & 2 \\ -1 & \lambda-3 \end{vmatrix} + (\lambda-4)\begin{vmatrix} \lambda+1 & 2 \\ 3 & \lambda-3 \end{vmatrix}$

$= (-3)[(-4)(\lambda-3)-(2)(-1)] + (\lambda-4)[(\lambda+1)(\lambda-3)-(2)(3)] = \lambda^3 - 6\lambda^2 + 11\lambda - 6$.

Following the procedure described in Example 3 of Section 5.1, we determine that the only possibilities for integer solutions of the characteristic equation are $\pm 1, \pm 2, \pm 3$, and ± 6.

Since $\det(1I - A) = 0$, $\lambda - 1$ must be a factor of the characteristic polynomial. Dividing $\lambda - 1$ into $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ leads to $\det(\lambda I - A) = (\lambda-1)(\lambda^2 - 5\lambda + 6) = (\lambda-1)(\lambda-2)(\lambda-3)$.

We conclude that the eigenvalues are 1, 2, and 3 - each of them has the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{2}{3}t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{4}t, x_2 = \frac{3}{4}t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has geometric multiplicity 1.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable.

We form a matrix P using the column vectors $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$: $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

12. Since $\det(\lambda I - A) = \begin{vmatrix} \lambda-19 & -9 & 6 \\ -25 & \lambda+11 & 9 \\ -17 & 9 & \lambda+4 \end{vmatrix} = \lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda-1)^2(\lambda-2)$, A has eigenvalues $\lambda = 1$ with the algebraic multiplicity 2 and $\lambda = 2$ with the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that $\lambda = 1$ has the geometric multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$ so that $\lambda = 2$ has the geometric multiplicity 1.

Since for $\lambda = 1$ the geometric multiplicity and the algebraic multiplicity are not equal, we conclude from Theorem 5.2.4(b) that A is not diagonalizable.

13. $\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ -3 & 0 & \lambda - 1 \end{vmatrix} = \lambda^2(\lambda - 1)$ so the eigenvalues are $\lambda = 0$ with the algebraic multiplicity 2 and $\lambda = 1$ with the algebraic multiplicity 1.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{1}{3}t, x_2 = s, x_3 = t$. We can write $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$ therefore vectors $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$ form a basis for this eigenspace. This eigenvalue has the geometric multiplicity 2.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. This eigenvalue has the geometric multiplicity 1.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable.

We form a matrix P using the column vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$: $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. (Note that this answer is not unique.)

$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

14. From Theorem 5.1.2, A has only one eigenvalue $\lambda = 5$ with the algebraic multiplicity 3.

The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that $\lambda = 5$ has the geometric multiplicity 1.

Since the geometric multiplicity and the algebraic multiplicity are not equal, we conclude from Theorem 5.2.4(b) that A is not diagonalizable.

15. **(a)** The degree of the characteristic polynomial of A is 3 therefore A is a 3×3 matrix.
All three eigenspaces (for $\lambda = 1$, $\lambda = -3$, and $\lambda = 5$) must have dimension 1.

(b) The degree of the characteristic polynomial of A is 6 therefore A is a 6×6 matrix.
The possible dimensions of the eigenspace corresponding to $\lambda = 0$ are 1 or 2.
The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.
The possible dimensions of the eigenspace corresponding to $\lambda = 2$ are 1, 2, or 3.

16. **(a)** The degree of the characteristic polynomial of A is 5 therefore A is a 5×5 matrix.
In factored form, $\lambda^3(\lambda^2 - 5\lambda - 6) = \lambda^3(\lambda - 6)(\lambda + 1)$.
The possible dimensions of the eigenspace corresponding to $\lambda = 0$ are 1, 2, or 3.
The dimension of the eigenspace corresponding to $\lambda = 6$ must be 1.
The dimension of the eigenspace corresponding to $\lambda = -1$ must be 1.

(b) The degree of the characteristic polynomial of A is 3 therefore A is a 3×3 matrix.
In factored form, $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$.
The possible dimensions of the eigenspace corresponding to $\lambda = 1$ are 1, 2, or 3.

17. $\det(\lambda I - A) = \begin{vmatrix} \lambda & -3 \\ -2 & \lambda + 1 \end{vmatrix} = \lambda(\lambda + 1) - (-3)(-2) = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$ therefore A has eigenvalues 2 and -3, each with the algebraic multiplicity 1.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{3}{2}t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and calculate $P^{-1} = \frac{1}{(3)(1)-(-1)(2)} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$ so that $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = D$.

Therefore $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1{,}024 & 0 \\ 0 & 59{,}049 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$
$= \begin{bmatrix} 24{,}234 & -34{,}815 \\ -23{,}210 & 35{,}839 \end{bmatrix}$.

18. From Theorem 5.1.2, A has eigenvalues 1 and 2, each with the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and calculate $P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ so that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = D$.

Therefore $A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$.

19. To invert the matrix P, we can employ the procedure introduced in Section 1.5: since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & -5 & 1 \\ 0 & 1 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$, we have

$$P^{-1} = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We verify that $P^{-1}AP = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$ is a diagonal matrix therefore P diagonalizes A.

$$A^{11} = PD^{11}P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} (-2)^{11} & 0 & 0 \\ 0 & (-1)^{11} & 0 \\ 0 & 0 & 1^{11} \end{bmatrix}\begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} -2{,}048 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 10{,}237 & -2{,}047 \\ 0 & 1 & 0 \\ 0 & 10{,}245 & -2{,}048 \end{bmatrix}$$

20. After calculating $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$, we verify that

$P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$ is a diagonal matrix therefore P diagonalizes A.

(a) $A^{1000} = PD^{1000}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} (-1)^{1000} & 0 & 0 \\ 0 & (-1)^{1000} & 0 \\ 0 & 0 & 1^{1000} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $A^{-1000} = PD^{-1000}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} (-1)^{-1000} & 0 & 0 \\ 0 & (-1)^{-1000} & 0 \\ 0 & 0 & 1^{-1000} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $A^{2301} = PD^{2301}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} (-1)^{2301} & 0 & 0 \\ 0 & (-1)^{2301} & 0 \\ 0 & 0 & 1^{2301} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(d) $A^{-2301} = PD^{-2301}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} (-1)^{-2301} & 0 & 0 \\ 0 & (-1)^{-2301} & 0 \\ 0 & 0 & 1^{-2301} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

21. Cofactor expansion along the first row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix}$
$= (\lambda-3)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & \lambda-3 \end{vmatrix} = (\lambda-3)[(\lambda-2)(\lambda-3)-1] - (\lambda-3)$
$= (\lambda-3)[\lambda^2-5\lambda+6-1-1] = (\lambda-3)(\lambda^2-5\lambda+4) = (\lambda-1)(\lambda-3)(\lambda-4)$
therefore A has eigenvalues 1, 3, and 4, each with the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = 2t, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 4$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = -t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix $P = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and find its inverse using the procedure introduced in Section 1.5. Since the reduced row echelon form of the matrix $\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$, we have $P^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

We conclude that $A^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}\begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

22. Since $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda-3)$, A has eigenvalues $\lambda = 3$ with the algebraic multiplicity 1 and $\lambda = 0$ with the algebraic multiplicity 2.

The geometric multiplicity of $\lambda_1 = 3$ must be 1.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the geometric multiplicity of $\lambda_2 = \lambda_3 = 0$ is 2.

Since for each eigenvalue the geometric multiplicity matches the algebraic multiplicity, by Theorem 5.2.4(b) A is diagonalizable, i.e., there exists an invertible matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

Consequently, the matrices A and B are similar.

23. By inspection, both A and B have rank 1 (both matrices are in reduced row echelon form).
Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$.
Setting $AP = PB$ requires that $a = 0$, $b = a$, and $c = 0$.
For any value d, the matrix $P = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ satisfies the equality $AP = PB$. However, $P = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$ has zero determinant therefore it is not invertible so that the similarity condition $B = P^{-1}AP$ cannot be met.

24. By inspection, both A and B have only one eigenvalue, $\lambda = 1$.
Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and $PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Setting $AP = PB$ requires that $a + c = a$ and $b + d = b$, consequently $c = d = 0$.
For any a and b, the matrix $P = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ satisfies the equality $AP = PB$. However, $P = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has zero determinant therefore it is not invertible so that the similarity condition $B = P^{-1}AP$ cannot be met.

25. Since there exist matrices P and Q such that $B = P^{-1}AP$ and $C = Q^{-1}BQ$, we can write $C = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ)$. Consequently, A is similar to C.

26. **(a)** Any square matrix A is similar to itself since $A = I^{-1}AI$.

(b) If $O_{n\times n} = P^{-1}AP$ then $A = PO_{n\times n}P^{-1} = O_{n\times n}$.

(c) Table 1 lists invertibility as one of the similarity invariants. Consequently, a nonsingular matrix cannot be similar to a singular matrix.

27. **(a)** The dimension of the eigenspace must be at least 1, but cannot exceed the algebraic multiplicity of the corresponding eigenvalue. Since the algebraic multiplicities of the eigenvalues 1,3, and 4 are 1, 2, and 3, respectively, we conclude that

- The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.
- The possible dimensions of the eigenspace corresponding to $\lambda = 3$ are 1 or 2.
- The possible dimensions of the eigenspace corresponding to $\lambda = 4$ are 1, 2, or 3.

(b) If A is diagonalizable then by Theorem 5.2.4(b) for each eigenvalue the dimension of the eigenspace must be equal to the algebraic multiplicity. Therefore

- The dimension of the eigenspace corresponding to $\lambda = 1$ must be 1.

- The dimension of the eigenspace corresponding to $\lambda = 3$ must be 2.
- The dimension of the eigenspace corresponding to $\lambda = 4$ must be 3.

(c) If the dimension of the eigenspace were smaller than 3 then by Theorem 4.5.2(a), a set of three vectors from that eigenspace would have to be linearly dependent. Consequently, for the set of the three vectors to be linearly independent, the eigenspace containing the set must be of dimension at least 3. This is only possible for the eigenspace corresponding to the eigenvalue $\lambda = 4$.

29. Using the result obtained in Exercise 30 of Section 5.1, we can take $P = \begin{bmatrix} -b & -b \\ a-\lambda_1 & a-\lambda_2 \end{bmatrix}$ where $\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right]$ and $\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a-d)^2 + 4bc}\right]$.

30. $T(x_1, x_2) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix for the operator T is $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.
$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda-2)(\lambda-1) - (1)(-1) = \lambda^2 - 3\lambda + 3$
The discriminant of this quadratic polynomial, $b^2 - 4ac = (-3)^2 - 4(1)(3) = -3$ is negative, therefore the characteristic polynomial has no real zeros. Consequently, A has no real eigenvalues and it cannot be diagonalized.

31. $T(x_1, x_2) = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix for the operator T is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.
$\det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$ thus A has eigenvalues 1 and -1, both with algebraic multiplicities 1.
The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.
The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.
We form a matrix P using the column vectors $\mathbf{p}_1$ and $\mathbf{p}_2$: $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the two columns can be interchanged.)

32. $T(x_1, x_2, x_3) = \begin{bmatrix} 8x_1 + 3x_2 - 4x_3 \\ -3x_1 + x_2 + 3x_3 \\ 4x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -4 \\ -3 & 1 & 3 \\ 4 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$;
the standard matrix for T is $A = \begin{bmatrix} 8 & 3 & -4 \\ -3 & 1 & 3 \\ 4 & 3 & 0 \end{bmatrix}$. Since $\det(\lambda I - A) = \begin{vmatrix} \lambda - 8 & -3 & 4 \\ 3 & \lambda - 1 & -3 \\ -4 & -3 & \lambda \end{vmatrix} =$
$(\lambda - 4)^2(\lambda - 1)$, thus A has eigenvalues $\lambda = 1$ with algebraic multiplicity 1 and $\lambda = 4$ with algebraic multiplicity 2.
The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that $\lambda = 4$ has geometric multiplicity 1.

Since the geometric multiplicity and the algebraic multiplicity are not equal, we conclude from Theorem 5.2.4(b) that A is not diagonalizable.

33. $T(x_1,x_2,x_3) = \begin{bmatrix} 3x_1 \\ x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix for T is $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

Since $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ -1 & 1 & \lambda \end{vmatrix} = (\lambda - 3)(\lambda - 1)\lambda$, thus A has eigenvalues 0, 1, and 3, each with algebraic multiplicity 1.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = -t, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 3t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

We form a matrix P using the column vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$: $P = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. (Note that this answer is not unique. Any nonzero multiples of these columns would also form a valid matrix P. Furthermore, the columns can be interchanged.)

True-False Exercises

(a) False. E.g., $A = I_2$ has only one eigenvalue $\lambda = 1$, but it is diagonalizable with $P = I_2$.

(b) True. This follows from Theorem 5.2.1.

(c) True. Multiplying $A = P^{-1}BP$ on the left by P yields $PA = BP$.

(d) False. The matrix P is not unique. For instance, interchanging two columns of P results in a different matrix which also diagonalizes A.

(e) True. Since A is invertible, we can take the inverse on both sides of the equality $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ obtaining $P^{-1}A^{-1}P = D^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_n \end{bmatrix}$. Consequently, P diagonalizes both A and A^{-1}.

(f) True. We can transpose both sides of the equality $P^{-1}AP = D$ obtaining $P^T A^T (P^T)^{-1} = D^T = D$, i.e., $((P^T)^{-1})^{-1} A^T (P^T)^{-1} = D$. Consequently, $(P^T)^{-1}$ diagonalizes A^T.

(g) True. A basis for R^n must be a linearly independent set of n vectors, so by Theorem 5.2.1 A is diagonalizable.

(h) True. This follows from Theorem 5.2.2(b).

(i) True. From Theorem 5.1.5 we have $\det(A) = 0$. Since $\det(A^2) = (\det(A))^2 = 0^2 = 0$, it follows from the same Theorem that A^2 is singular.

5.3 Complex Vector Spaces

1. $\overline{\mathbf{u}} = \left(\overline{2-i}, \overline{4i}, \overline{1+i}\right) = (2+i, -4i, 1-i)$; $\mathrm{Re}(\mathbf{u}) = (2,0,1)$; $\mathrm{Im}(\mathbf{u}) = (-1,4,1)$;
$\|\mathbf{u}\| = \sqrt{|2-i|^2 + |4i|^2 + |1+i|^2} = \sqrt{(2^2 + (-1)^2) + (0^2 + 4^2) + (1^2 + 1^2)} = \sqrt{5 + 16 + 2} = \sqrt{23}$

2. $\overline{\mathbf{u}} = (6, 1-4i, 6+2i)$; $\mathrm{Re}(\mathbf{u}) = (6,1,6)$; $\mathrm{Im}(\mathbf{u}) = (0,4,-2)$;
$\|\mathbf{u}\| = \sqrt{|6|^2 + |1+4i|^2 + |6+2i|^2} = \sqrt{36 + 17 + 40} = \sqrt{93}$

3. **(a)** $\overline{\overline{\mathbf{u}}} = \overline{\overline{(3-4i, 2+i, -6i)}} = \overline{(3+4i, 2-i, 6i)} = (3-4i, 2+i, -6i) = \mathbf{u}$

(b) $\overline{k\mathbf{u}} = \overline{i(3-4i, 2+i, -6i)} = \overline{(4+3i, -1+2i, +6)} = (4-3i, -1-2i, 6)$
$\bar{k}\overline{\mathbf{u}} = \bar{i}\,\overline{(3-4i, 2+i, -6i)} = -i(3+4i, 2-i, 6i) = (4-3i, -1-2i, 6)$

(c) $\overline{\mathbf{u}+\mathbf{v}} = \overline{(4-3i, 4, 4-6i)} = (4+3i, 4, 4+6i)$
$\overline{\mathbf{u}} + \overline{\mathbf{v}} = (3+4i, 2-i, 6i) + (1-i, 2+i, 4) = (4+3i, 4, 4+6i)$

(d) $\overline{\mathbf{u}-\mathbf{v}} = \overline{(2-5i, 2i, -4-6i)} = (2+5i, -2i, -4+6i)$
$\overline{\mathbf{u}} - \overline{\mathbf{v}} = (3+4i, 2-i, 6i) - (1-i, 2+i, 4) = (2+5i, -2i, -4+6i)$

4. **(a)** $\overline{\overline{\mathbf{u}}} = \overline{\overline{(6, 1+4i, 6-2i)}} = \overline{(6, 1-4i, 6+2i)} = (6, 1+4i, 6-2i) = \mathbf{u}$

(b) $\overline{k\mathbf{u}} = \overline{(-i)(6, 1+4i, 6-2i)} = \overline{(-6i, 4-i, -2-6i)} = (6i, 4+i, -2+6i)$
$\bar{k}\overline{\mathbf{u}} = \overline{(-i)}\,\overline{(6, 1+4i, 6-2i)} = i(6, 1-4i, 6+2i) = (6i, 4+i, -2+6i)$

(c) $\overline{\mathbf{u}+\mathbf{v}} = \overline{(10, 4+6i, 3-i)} = (10, 4-6i, 3+i)$
$\overline{\mathbf{u}} + \overline{\mathbf{v}} = (6, 1-4i, 6+2i) + (4, 3-2i, -i-3) = (10, 4-6i, 3+i)$

(d) $\overline{\mathbf{u}-\mathbf{v}} = \overline{(2, -2+2i, 9-3i)} = (2, -2-2i, 9+3i)$
$\overline{\mathbf{u}} - \overline{\mathbf{v}} = (6, 1-4i, 6+2i) - (4, 3-2i, -i-3) = (2, -2-2i, 9+3i)$

5. $i\mathbf{x} - 3\mathbf{v} = \overline{\mathbf{u}}$ can be rewritten as $i\mathbf{x} = 3\mathbf{v} + \overline{\mathbf{u}}$; multiplying both sides by $-i$ and using the fact that $(-i)(i) = 1$, we obtain $\mathbf{x} = (-i)(3\mathbf{v} + \overline{\mathbf{u}}) = (-i)[(3+3i, 6-3i, 12) + (3+4i, 2-i, 6i)]$
$= (-i)(6+7i, 8-4i, 12+6i) = (7-6i, -4-8i, 6-12i)$

6. $(1+i)\mathbf{x} + 2\mathbf{u} = \overline{\mathbf{v}}$ can be rewritten as $(1+i)\mathbf{x} = \overline{\mathbf{v}} - 2\mathbf{u}$; multiplying both sides by $1-i$ and using the fact that $(1-i)(1+i) = 2$, we obtain $2\mathbf{x} = (1-i)(\overline{\mathbf{v}} - 2\mathbf{u})$ therefore
$\mathbf{x} = \frac{1}{2}(1-i)(\overline{\mathbf{v}} - 2\mathbf{u}) = \frac{1}{2}(1-i)[(4,3-2i,-i-3) - (12,2+8i,12-4i)]$
$= \frac{1}{2}(1-i)(-8,1-10i,-15+3i) = (-4+4i, -\frac{9}{2} - \frac{11}{2}i, -6+9i)$

7. $\overline{A} = \begin{bmatrix} \overline{-5i} & \overline{4} \\ \overline{2-i} & \overline{1+5i} \end{bmatrix} = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix}$; $\mathrm{Re}(A) = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$; $\mathrm{Im}(A) = \begin{bmatrix} -5 & 0 \\ -1 & 5 \end{bmatrix}$;
$\det(A) = (-5i)(1+5i) - (4)(2-i) = -5i + 25 - 8 + 4i = 17 - i$; $\mathrm{tr}(A) = -5i + (1+5i) = 1$

8. $\overline{A} = \begin{bmatrix} -4i & 2+3i \\ 2-3i & 1 \end{bmatrix}$; $\mathrm{Re}(A) = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$; $\mathrm{Im}(A) = \begin{bmatrix} 4 & -3 \\ 3 & 0 \end{bmatrix}$;
$\det(A) = (4i)(1) - (2-3i)(2+3i) = -13 + 4i$; $\mathrm{tr}(A) = 1 + 4i$

9. **(a)** $\overline{\overline{A}} = \begin{bmatrix} \overline{5i} & \overline{4} \\ \overline{2+i} & \overline{1-5i} \end{bmatrix} = \begin{bmatrix} -5i & 4 \\ 2-i & 1+5i \end{bmatrix} = A$

(b) $\overline{(A^T)} = \begin{bmatrix} \overline{-5i} & \overline{2-i} \\ \overline{4} & \overline{1+5i} \end{bmatrix} = \begin{bmatrix} 5i & 2+i \\ 4 & 1-5i \end{bmatrix}$; $\left(\overline{A}\right)^T = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix}^T = \begin{bmatrix} 5i & 2+i \\ 4 & 1-5i \end{bmatrix}$

(c) From $AB = \begin{bmatrix} -5i & 4 \\ 2-i & 1+5i \end{bmatrix} \begin{bmatrix} 1-i \\ 2i \end{bmatrix} = \begin{bmatrix} (-5i)(1-i) + (4)(2i) \\ (2-i)(1-i) + (1+5i)(2i) \end{bmatrix}$
$= \begin{bmatrix} -5i - 5 + 8i \\ 2 - 2i - i - 1 + 2i - 10 \end{bmatrix} = \begin{bmatrix} -5+3i \\ -9-i \end{bmatrix}$ we obtain $\overline{AB} = \begin{bmatrix} -5-3i \\ -9+i \end{bmatrix}$
$\overline{A}\,\overline{B} = \begin{bmatrix} 5i & 4 \\ 2+i & 1-5i \end{bmatrix} \begin{bmatrix} 1+i \\ 2i \end{bmatrix} = \begin{bmatrix} (5i)(1+i) + (4)(2i) \\ (2+i)(1+i) + (1-5i)(2i) \end{bmatrix}$
$= \begin{bmatrix} 5i - 5 - 8i \\ 2 + 2i + i - 1 - 2i - 10 \end{bmatrix} = \begin{bmatrix} -5-3i \\ -9+i \end{bmatrix}$

10. **(a)** $\overline{\overline{A}} = \begin{bmatrix} \overline{-4i} & \overline{2+3i} \\ \overline{2-3i} & \overline{1} \end{bmatrix} = \begin{bmatrix} 4i & 2-3i \\ 2+3i & 1 \end{bmatrix} = A$

(b) $\overline{(A^T)} = \begin{bmatrix} \overline{-4i} & \overline{2+3i} \\ \overline{2-3i} & \overline{1} \end{bmatrix} = \begin{bmatrix} 4i & 2-3i \\ 2+3i & 1 \end{bmatrix}$; $\left(\overline{A}\right)^T = \begin{bmatrix} -4i & 2+3i \\ 2-3i & 1 \end{bmatrix}^T = \begin{bmatrix} 4i & 2-3i \\ 2+3i & 1 \end{bmatrix}$

(c) From $AB = \begin{bmatrix} 4i & 2-3i \\ 2+3i & 1 \end{bmatrix} \begin{bmatrix} 5i \\ 1-4i \end{bmatrix} = \begin{bmatrix} -30-11i \\ -14+6i \end{bmatrix}$ we obtain $\overline{AB} = \begin{bmatrix} -30+11i \\ -14-6i \end{bmatrix}$
$\overline{A}\,\overline{B} = \begin{bmatrix} -4i & 2+3i \\ 2-3i & 1 \end{bmatrix} \begin{bmatrix} -5i \\ 1+4i \end{bmatrix} = \begin{bmatrix} -30+11i \\ -14-6i \end{bmatrix}$

11. $\mathbf{u} \cdot \mathbf{v} = (i)\left(\overline{4}\right) + (2i)\left(\overline{-2i}\right) + (3)\left(\overline{1+i}\right) = (i)(4) + (2i)(2i) + (3)(1-i) = 4i - 4 + 3 - 3i = -1 + i$
$\mathbf{u} \cdot \mathbf{w} = (i)\left(\overline{2-i}\right) + (2i)\left(\overline{2i}\right) + (3)\left(\overline{5+3i}\right) = (i)(2+i) + (2i)(-2i) + (3)(5-3i)$
$= 2i - 1 + 4 + 15 - 9i = 18 - 7i$
$\mathbf{v} \cdot \mathbf{w} = (4)\left(\overline{2-i}\right) + (-2i)\left(\overline{2i}\right) + (1+i)\left(\overline{5+3i}\right) = (4)(2+i) + (-2i)(-2i) + (1+i)(5-3i)$
$= 8 + 4i - 4 + 5 - 3i + 5i + 3 = 12 + 6i$

Since both $\mathbf{u}^T\overline{\mathbf{v}} = [i \quad 2i \quad 3]\begin{bmatrix} 4 \\ 2i \\ 1-i \end{bmatrix} = [-1+i]$ and $\overline{\mathbf{v}}^T\mathbf{u} = [4 \quad 2i \quad 1-i]\begin{bmatrix} i \\ 2i \\ 3 \end{bmatrix} = [-1+i]$ are equal to $\mathbf{u}\cdot\mathbf{v} = -1+i$, Formula (5) holds.

(a) $\overline{\mathbf{v}\cdot\mathbf{u}} = \overline{(4)(\overline{i}) + (-2i)(\overline{2i}) + (1+i)(\overline{3})} = \overline{(4)(-i) + (-2i)(-2i) + (1+i)(3)}$
$= \overline{-4i - 4 + 3 + 3i} = \overline{-1-i} = -1+i = \mathbf{u}\cdot\mathbf{v}$

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (i)(\overline{4+2-i}) + (2i)(\overline{-2i+2i}) + (3)(\overline{1+i+5+3i})$
$= (i)(6+i) + (2i)(0) + (3)(6-4i) = 6i - 1 + 18 - 12i = 17 - 6i$ equals
$\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = -1 + i + 18 - 7i = 17 - 6i$

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (2i)(-1+i) = -2-2i$ equals
$(k\mathbf{u})\cdot\mathbf{v} = (-2)(\overline{4}) + (-4)(\overline{-2i}) + (6i)(\overline{1+i})$
$=(-2)(4) + (-4)(2i) + (6i)(1-i) = -8 - 8i + 6i + 6 = -2-2i$

12. $\mathbf{u}\cdot\mathbf{v} = (1+i)(\overline{3}) + (4)(\overline{-4i}) + (3i)(\overline{2+3i}) = (1+i)(3) + (4)(4i) + (3i)(2-3i) = 12+25i$
$\mathbf{u}\cdot\mathbf{w} = (1+i)(\overline{1-i}) + (4)(\overline{4i}) + (3i)(\overline{4-5i}) = (1+i)(1+i) + (4)(-4i) + (3i)(4+5i)$
$= -15 - 2i$
$\mathbf{v}\cdot\mathbf{w} = (3)(\overline{1-i}) + (-4i)(\overline{4i}) + (2+3i)(\overline{4-5i}) = (3)(1+i) + (-4i)(-4i) + (2+3i)(4+5i)$
$= -20 + 25i$

Since both $\mathbf{u}^T\overline{\mathbf{v}} = [1+i \quad 4 \quad 3i]\begin{bmatrix} 3 \\ 4i \\ 2-3i \end{bmatrix} = [12+25i]$ and

$\overline{\mathbf{v}}^T\mathbf{u} = [3 \quad 4i \quad 2-3i]\begin{bmatrix} 1+i \\ 4 \\ 3i \end{bmatrix} = [12+25i]$ are equal to $\mathbf{u}\cdot\mathbf{v} = 12+25i$, Formula (5) holds.

(a) $\overline{\mathbf{v}\cdot\mathbf{u}} = \overline{(3)(\overline{1+i}) + (-4i)(\overline{4}) + (2+3i)(\overline{3i})} = \overline{(3)(1-i) + (-4i)(4) + (2+3i)(-3i)}$
$= \overline{-20-25i} = -20+25i = \mathbf{u}\cdot\mathbf{v}$

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (1+i)(\overline{3+1-i}) + (4)(\overline{-4i+4i}) + (3i)(\overline{2+3i+4-5i})$
$= (1+i)(4+i) + (4)(0) + (3i)(6+2i) = -3+23i$ equals
$\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = 12 + 25i - 15 - 2i = -3+23i$

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (1+i)(12+25i) = -13+37i$ equals
$(k\mathbf{u})\cdot\mathbf{v} = (2i)(\overline{3}) + (4+4i)(\overline{-4i}) + (-3+3i)(\overline{2+3i})$
$= (2i)(3) + (4+4i)(4i) + (-3+3i)(2-3i) = -13+37i$

13. $\mathbf{u}\cdot\overline{\mathbf{v}} = (i)(4) + (2i)(-2i) + (3)(1+i) = 4i + 4 + 3 + 3i = 7+7i$
$\overline{\mathbf{w}\cdot\mathbf{u}} = \overline{(2-i)(\overline{i}) + (2i)(\overline{2i}) + (5+3i)(\overline{3})} = \overline{(2-i)(-i) + (2i)(-2i) + (5+3i)(3)}$
$= \overline{-2i - 1 + 4 + 15 + 9i} = \overline{18+7i} = 18-7i$
$\overline{(\mathbf{u}\cdot\overline{\mathbf{v}}) - \overline{\mathbf{w}\cdot\mathbf{u}}} = \overline{7+7i-18+7i} = \overline{-11+14i} = -11-14i$

14. $\overline{i\mathbf{u}}\cdot\mathbf{w} = (-1-i)(1+i) - 4i(-4i) + (-3)(4+5i) = -28-17i$
$\|\mathbf{u}\| = \sqrt{|1+i|^2 + |4|^2 + |3i|^2} = \sqrt{2+16+9} = \sqrt{27} = 3\sqrt{3}$

$(\|\mathbf{u}\|\mathbf{v})\cdot\mathbf{u} = (9\sqrt{3})(1-i) + (-12\sqrt{3}i)(4) + (6\sqrt{3}+9\sqrt{3}i)(-3i) = 36\sqrt{3} - 75\sqrt{3}i$

$\overline{(i\mathbf{u}\cdot\mathbf{w})} + \overline{(\|\mathbf{u}\|\mathbf{v})\cdot\mathbf{u}} = \overline{-28 - 17i + 36\sqrt{3} - 75\sqrt{3}i} = -28 + 36\sqrt{3} + (17 + 75\sqrt{3})i$

15. $\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 5 \\ -1 & \lambda \end{vmatrix} = (\lambda-4)\lambda - (5)(-1) = \lambda^2 - 4\lambda + 5$

Solving the characteristic equation $\lambda^2 - 4\lambda + 5 = 0$ using the quadratic formula yields $\lambda = \frac{4\pm\sqrt{4^2-4(5)}}{2} = \frac{4\pm\sqrt{-4}}{2} = 2 \pm i$ therefore A has eigenvalues $\lambda = 2 + i$ and $\lambda = 2 - i$.

For the eigenvalue $\lambda = 2 + i$, the augmented matrix of the homogeneous system $((2+i)I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -2+i & 5 & 0 \\ -1 & 2+i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $-x_1 + (2+i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (2+i)t, x_2 = t$. The vector $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 2 + i$.

According to Theorem 5.3.4, the vector $\overline{\begin{bmatrix} 2+i \\ 1 \end{bmatrix}} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 2 - i$.

16. The characteristic equation of A is $\lambda^2 - 6\lambda + 13 = 0$. Solving this equation using the quadratic formula yields $\lambda = \frac{6\pm\sqrt{(-6)^2-4(13)}}{2} = \frac{6\pm\sqrt{-16}}{2} = 3 \pm 2i$ therefore A has eigenvalues $\lambda = 3 + 2i$ and $\lambda = 3 - 2i$.

For the eigenvalue $\lambda = 3 + 2i$, the augmented matrix of the homogeneous system $((3+2i)I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 4+2i & 5 & 0 \\ -4 & -4+2i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + \left(1 - \frac{1}{2}i\right)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \left(-1 + \frac{1}{2}i\right)t, x_2 = t$. The vector $\begin{bmatrix} -2+i \\ 2 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 3 + 2i$.

According to Theorem 5.3.4, the vector $\overline{\begin{bmatrix} -2+i \\ 2 \end{bmatrix}} = \begin{bmatrix} -2-i \\ 2 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 3 - 2i$.

17. $\det(\lambda I - A) = \begin{vmatrix} \lambda-5 & 2 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-5)(\lambda-3) - (2)(-1) = \lambda^2 - 8\lambda + 17$

Solving the characteristic equation $\lambda^2 - 8\lambda + 17 = 0$ using the quadratic formula yields $\lambda = \frac{8\pm\sqrt{8^2-4(17)}}{2} = \frac{8\pm\sqrt{-4}}{2} = 4 \pm i$ therefore A has eigenvalues $\lambda = 4 + i$ and $\lambda = 4 - i$.

For the eigenvalue $\lambda = 4 + i$, the augmented matrix of the homogeneous system $((4+i)I - A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -1+i & 2 & 0 \\ -1 & 1+i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $-x_1 + (1+i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (1+i)t, x_2 = t$. The vector $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 4+i$.

According to Theorem 5.3.4, the vector $\begin{bmatrix} \overline{1+i} \\ \overline{1} \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 4-i$.

18. The characteristic equation of A is $\lambda^2 - 10\lambda + 34 = 0$. Solving this equation using the quadratic formula yields $\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4(34)}}{2} = \frac{10 \pm \sqrt{-36}}{2} = 5 \pm 3i$ therefore A has eigenvalues $\lambda = 5+3i$ and $\lambda = 5-3i$.

For the eigenvalue $\lambda = 5+3i$, the augmented matrix of the homogeneous system $\big((5+3i)I - A\big)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -3+3i & -6 & 0 \\ 3 & 3+3i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + (1+i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (-1-i)t, x_2 = t$. The vector $\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 5+3i$.

According to Theorem 5.3.4, the vector $\begin{bmatrix} \overline{1+i} \\ \overline{-1} \end{bmatrix} = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 5-3i$.

19. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ implies $a = b = 1$. We have $|\lambda| = |1+i| = \sqrt{1+1} = \sqrt{2}$.
The angle inside the interval $(-\pi, \pi]$ from the positive x-axis to the ray that joins the origin to the point (1,1) is $\phi = \frac{\pi}{4}$.

20. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$ implies $a = 0$ and $b = -5$. We have $|\lambda| = |0+5i| = \sqrt{0+25} = 5$.
The angle inside the interval $(-\pi, \pi]$ from the positive x-axis to the ray that joins the origin to the point $(0, -5)$ is $\phi = -\frac{\pi}{2}$.

21. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$ implies $a = 1$ and $b = -\sqrt{3}$. We have $|\lambda| = \left|1 - \sqrt{3}i\right| = \sqrt{1+3} = 2$.
The angle inside the interval $(-\pi, \pi]$ from the positive x-axis to the ray that joins the origin to the point $(1, -\sqrt{3})$ is $\phi = -\frac{\pi}{3}$.

22. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$ implies $a = \sqrt{2}$ and $b = -\sqrt{2}$. We have $|\lambda| = \left|\sqrt{2} - \sqrt{2}\,i\right| = \sqrt{2+2} = 2$.
The angle inside the interval $(-\pi, \pi]$ from the positive x-axis to the ray that joins the origin to the point $(\sqrt{2}, -\sqrt{2})$ is $\phi = -\frac{\pi}{4}$.

23. $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 5 \\ -4 & \lambda-7 \end{vmatrix} = (\lambda+1)(\lambda-7) - (5)(-4) = \lambda^2 - 6\lambda + 13$
Solving the characteristic equation $\lambda^2 - 6\lambda + 13 = 0$ using the quadratic formula yields $\lambda =$

$\frac{6\pm\sqrt{6^2-4(13)}}{2} = \frac{6\pm\sqrt{-16}}{2} = 3 \pm 2i$ therefore A has eigenvalues $\lambda = 3+2i$ and $\lambda = 3-2i$.
For the eigenvalue $\lambda = 3-2i$, the augmented matrix of the homogeneous system $\big((3-2i)I - A\big)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 4-2i & 5 & 0 \\ -4 & -4-2i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + \left(1+\frac{1}{2}i\right)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \left(-1-\frac{1}{2}i\right)t$, $x_2 = t$. Since $\begin{bmatrix} -2-i \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 3 - 2i$, it follows from Theorem 5.3.8 that the matrices $P = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ satisfy $A = PCP^{-1}$.

24. The characteristic equation of A is $\lambda^2 - 4\lambda + 5 = 0$. Solving this equation using the quadratic formula yields $\lambda = \frac{4\pm\sqrt{(-4)^2-4(5)}}{2} = \frac{4\pm\sqrt{-4}}{2} = 2 \pm i$ therefore A has eigenvalues $\lambda = 2+i$ and $\lambda = 2-i$.

For the eigenvalue $\lambda = 2-i$, the augmented matrix of the homogeneous system $\big((2-i)I - A\big)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -2-i & 5 & 0 \\ -1 & 2-i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + (-2+i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (2-i)t$, $x_2 = t$. Since $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 2-i$, it follows from Theorem 5.3.8 that the matrices $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ satisfy $A = PCP^{-1}$.

25. $\det(\lambda I - A) = \begin{vmatrix} \lambda-8 & -6 \\ 3 & \lambda-2 \end{vmatrix} = (\lambda-8)(\lambda-2) - (-6)(3) = \lambda^2 - 10\lambda + 34$
Solving the characteristic equation $\lambda^2 - 10\lambda + 34 = 0$ using the quadratic formula yields $\lambda = \frac{10\pm\sqrt{10^2-4(34)}}{2} = \frac{10\pm\sqrt{-36}}{2} = 5 \pm 3i$ therefore A has eigenvalues $\lambda = 5+3i$ and $\lambda = 5-3i$.
For the eigenvalue $\lambda = 5-3i$, the augmented matrix of the homogeneous system $\big((5-3i)I - A\big)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -3-3i & -6 & 0 \\ 3 & 3-3i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + (1-i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (-1+i)t$, $x_2 = t$. Since $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 5-3i$, it follows from Theorem 5.3.8 that the matrices $P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -3 \\ 3 & 5 \end{bmatrix}$ satisfy $A = PCP^{-1}$.

26. The characteristic equation of A is $\lambda^2 - 8\lambda + 17 = 0$. Solving this equation using the quadratic formula yields $\lambda = \frac{8\pm\sqrt{(-8)^2-4(17)}}{2} = \frac{8\pm\sqrt{-4}}{2} = 4 \pm i$ therefore A has eigenvalues $\lambda = 4+i$ and $\lambda = 4-i$.

For the eigenvalue $\lambda = 4-i$, the augmented matrix of the homogeneous system $\big((4-i)I - A\big)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -1-i & 2 & 0 \\ -1 & 1-i & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + (-1+i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = (1-i)t, x_2 = t$. Since $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4 - i$, it follows from Theorem 5.3.8 that the matrices $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ satisfy $A = PCP^{-1}$.

27. **(a)** Letting $k = a + bi$ we have $\mathbf{u} \cdot \mathbf{v} = (2i)(\overline{i}) + (i)(\overline{6i}) + (3i)(\overline{a+bi})$
$= (2i)(-i) + (i)(-6i) + (3i)(a - bi) = 2 + 6 + 3ai + 3b = (8 + 3b) + (3a)i$. Setting this equal to zero yields $a = 0$ and $b = -\frac{8}{3}$ therefore the only complex scalar which satisfies our requirements is $k = -\frac{8}{3}i$.

(b) $\mathbf{u} \cdot \mathbf{v} = (k)(\overline{1}) + (k)(\overline{-1}) + (1+i)(\overline{1-i}) = (k)(1) + (k)(-1) + (1+i)(1+i) = 2i \neq 0$ therefore no complex scalar k satisfies our requirements.

True-False Exercises

(a) False. By Theorem 5.3.4, complex eigenvalues of a real matrix occur in conjugate pairs, so the total number of complex eigenvalues must be even. Consequently, in a 5×5 matrix at least one eigenvalue must be real.

(b) True. $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$ is the characteristic equation of a 2×2 complex matrix A.

(c) False. By Theorem 5.3.5, A has two complex conjugate eigenvalues if $\operatorname{tr}(A)^2 < 4\det(A)$.

(d) True. This follows from Theorem 5.3.4.

(e) False. E.g., $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ is symmetric, but its eigenvalue $\lambda = i$ is not real.

(f) False. (This would be true if we assumed $|\lambda| = 1$.)

5.4 Differential Equations

1. **(a)** We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
The characteristic polynomial of A is
$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) - (-4)(-2) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$
thus the eigenvalues of A are $\lambda = 5$ and $\lambda = -1$.
The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 5$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.
The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -2t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.
The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = 5u_1$ and $u_2' = -1u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^{5x}, u_2 = c_2e^{-x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^{5x} \\ c_2e^{-x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} =$ $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{5x} \\ c_2e^{-x} \end{bmatrix} = \begin{bmatrix} c_1e^{5x} - 2c_2e^{-x} \\ c_1e^{5x} + c_2e^{-x} \end{bmatrix}$ thus $y_1 = c_1e^{5x} - 2c_2e^{-x}$ and $y_2 = c_1e^{5x} + c_2e^{-x}$.

(b) Substituting the initial conditions into the general solution obtained in part (a) yields a system

$$\begin{aligned} c_1e^{5(0)} - 2c_2e^{-0} &= 0 \\ c_1e^{5(0)} + c_2e^{-0} &= 0 \end{aligned}$$

which can be rewritten as

$$\begin{array}{rcrcl} c_1 & - & 2c_2 & = & 0 \\ c_1 & + & c_2 & = & 0 \end{array}$$

The reduced row echelon form of this system's augmented matrix $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ therefore $c_1 = 0$ and $c_2 = 0$.
The solution satisfying the given initial conditions can be expressed as $y_1 = 0$ and $y_2 = 0$.

2. **(a)** We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 5 \end{vmatrix} = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$ thus the eigenvalues of A are $\lambda = 7$ and $\lambda = -1$.
The reduced row echelon form of $7I - A$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 7$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{2}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.
The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{3}{2}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.
Therefore $P = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$.
The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = 7u_1$ and $u_2' = -1u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^{7x}, u_2 = c_2e^{-x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^{7x} \\ c_2e^{-x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} =$ $\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} c_1e^{7x} \\ c_2e^{-x} \end{bmatrix} = \begin{bmatrix} c_1e^{7x} - 3c_2e^{-x} \\ 2c_1e^{7x} + 2c_2e^{-x} \end{bmatrix}$ thus $y_1 = c_1e^{7x} - 3c_2e^{-x}$ and $y_2 = 2c_1e^{7x} + 2c_2e^{-x}$.

(b) Substituting the initial conditions into the general solution obtained in part (a) yields

$$\begin{array}{rcrcr} c_1 & - & 3c_2 & = & 2 \\ 2c_1 & + & 2c_2 & = & 1 \end{array}$$

The reduced row echelon form of this system's augmented matrix $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & \frac{7}{8} \\ 0 & 1 & -\frac{3}{8} \end{bmatrix}$

therefore $c_1 = \frac{7}{8}$ and $c_2 = -\frac{3}{8}$.

The solution satisfying the given initial conditions can be expressed as

$y_1 = \frac{7}{8}e^{7x} + \frac{9}{8}e^{-x}$ and $y_2 = \frac{7}{4}e^{7x} - \frac{3}{4}e^{-x}$.

3. **(a)** We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

Cofactor expansion along the second column yields

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)\begin{vmatrix} \lambda - 4 & -1 \\ 2 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 1)[(\lambda - 4)(\lambda - 1) - (-1)(2)]$$

$$= (\lambda - 1)(\lambda^2 - 5\lambda + 6) = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

The characteristic equation of A is $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$ thus the eigenvalues of A are 1, 2, and 3 (each with the algebraic multiplicity 1).

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{1}{2}t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = u_1, u_2' = 2u_2$, and $u_3' = 3u_3$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^x, u_2 = c_2e^{2x}, u_3 = c_3e^{3x}$ i.e., $\mathbf{u} = \begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix} = \begin{bmatrix} -c_2e^{2x} - c_3e^{3x} \\ c_1e^x + 2c_2e^{2x} + c_3e^{3x} \\ 2c_2e^{2x} + c_3e^{3x} \end{bmatrix}$ thus $y_1 = -c_2e^{2x} - c_3e^{3x},\ y_2 = c_1e^x + 2c_2e^{2x} + c_3e^{3x}$, and $y_3 = 2c_2e^{2x} + c_3e^{3x}$.

(b) Substituting the initial conditions into the general solution obtained in part (a) yields a system

$$\begin{aligned} -c_2e^{2(0)} - c_3e^{3(0)} &= -1 \\ c_1e^0 + 2c_2e^{2(0)} + c_3e^{3(0)} &= 1 \\ 2c_2e^{2(0)} + c_3e^{3(0)} &= 0 \end{aligned}$$

which can be rewritten as

$$\begin{array}{ccccccc} & - & c_2 & - & c_3 & = & -1 \\ c_1 & + & 2c_2 & + & c_3 & = & 1 \\ & & 2c_2 & + & c_3 & = & 0 \end{array}$$

The reduced row echelon form of this system's augmented matrix $\begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ therefore $c_1 = 1, c_2 = -1,$ and $c_3 = 2$.

The solution satisfying the given initial conditions can be expressed as $y_1 = e^{2x} - 2e^{3x},\ y_2 = e^x - 2e^{2x} + 2e^{3x}$, and $y_3 = -2e^{2x} + 2e^{3x}$.

4. We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 36\lambda - 32 = (\lambda - 8)(\lambda - 2)^2$ thus the eigenvalues of A are $\lambda = 8$ and $\lambda = 2$.

The reduced row echelon form of $8I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 8$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$

consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = 8u_1, u_2' = 2u_2$, and $u_3' = 2u_3$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^{8x}, u_2 = c_2e^{2x}, u_3 = c_3e^{2x}$ i.e., $\mathbf{u} = \begin{bmatrix} c_1e^{8x} \\ c_2e^{2x} \\ c_3e^{2x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{8x} \\ c_2e^{2x} \\ c_3e^{2x} \end{bmatrix} = \begin{bmatrix} c_1e^{8x} - c_2e^{2x} - c_3e^{2x} \\ c_1e^{8x} + c_2e^{2x} \\ c_1e^{8x} + c_3e^{2x} \end{bmatrix}$ thus

$y_1 = c_1e^{8x} - c_2e^{2x} - c_3e^{2x},\ y_2 = c_1e^{8x} + c_2e^{2x}$, and $y_3 = c_1e^{8x} + c_3e^{2x}$.

5. Assume $y = f(x)$ is a solution of $y' = ay$ so that $f'(x) = af(x)$.
We have $\frac{d}{dx}(f(x)e^{-ax}) = f'(x)e^{-ax} + f(x)(-a)e^{-ax} = af(x)e^{-ax} - af(x)e^{-ax} = 0$ for all x therefore there exists a constant c for which $f(x)e^{-ax} = c$, i.e., $f(x) = \frac{c}{e^{-ax}} = ce^{ax}$. We conclude that every solution of $y' = ay$ must have the form $f(x) = ce^{ax}$.

7. Substituting $y_1 = y$ and $y_2 = y'$ allows us to rewrite the equation $y'' - y' - 6y = 0$ as $y_2' - y_2 - 6y_1 = 0$. Also, $y_2 = y' = y_1'$ so we obtain the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 6y_1 + y_2 \end{aligned}$$

The coefficient matrix of this system is $A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -6 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1) - (-1)(-6) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$ thus the eigenvalues of A are $\lambda = 3$ and $\lambda = -2$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{3}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-2I - A$ is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{1}{2}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = 3u_1$ and $u_2' = -2u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^{3x}$,

$u_2 = c_2e^{-2x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^{3x} \\ c_2e^{-2x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} c_1e^{3x} \\ c_2e^{-2x} \end{bmatrix} =$ $\begin{bmatrix} c_1e^{3x} - c_2e^{-2x} \\ 3c_1e^{3x} + 2c_2e^{-2x} \end{bmatrix}$ thus $y_1 = c_1e^{3x} - c_2e^{-2x}$ and $y_2 = 3c_1e^{3x} + 2c_2e^{-2x}$.

We conclude that the original equation $y'' - y' - 6y = 0$ has the solution $y = c_1e^{3x} - c_2e^{-2x}$.

8. Substituting $y_1 = y$ and $y_2 = y'$ yields the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 12y_1 - y_2 \end{aligned}$$

The coefficient matrix of this system is $A = \begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -12 & \lambda + 1 \end{vmatrix} = \lambda^2 + \lambda - 12 = (\lambda - 3)(\lambda + 4)$ thus the eigenvalues of A are $\lambda = 3$ and $\lambda = -4$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{3}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-4I - A$ is $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{1}{4}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = 3u_1$ and $u_2' = -4u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^{3x}$, $u_2 = c_2e^{-4x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^{3x} \\ c_2e^{-4x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c_1e^{3x} \\ c_2e^{-4x} \end{bmatrix} =$ $\begin{bmatrix} c_1e^{3x} - c_2e^{-4x} \\ 3c_1e^{3x} + 4c_2e^{-4x} \end{bmatrix}$ thus $y_1 = c_1e^{3x} - c_2e^{-4x}$ and $y_2 = 3c_1e^{3x} + 4c_2e^{-4x}$.

We conclude that the original equation $y'' + y' - 12y = 0$ has the solution $y = c_1e^{3x} - c_2e^{-4x}$.

9. Substituting $y_1 = y, y_2 = y'$, and $y_3 = y''$ allows us to rewrite the equation $y''' - 6y'' + 11y' - 6y = 0$ as $y_3' - 6y_3 + 11y_2 - 6y_1 = 0$. With $y_2 = y' = y_1'$ and $y_3 = y'' = y_2{}'$ we obtain the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 6y_1 - 11y_2 + 6y_3 \end{aligned}$$

The coefficient matrix of this system is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$.

The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -6 & 11 & \lambda - 6 \end{vmatrix}$

$= \lambda \begin{vmatrix} \lambda & -1 \\ 11 & \lambda-6 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ -6 & \lambda-6 \end{vmatrix} = \lambda[\lambda(\lambda-6)+11] - 6 = \lambda^3 - 6\lambda^2 + 11\lambda - 6.$

Following the procedure described in Example 3 of Section 5.1, we determine that the only possibilities for integer solutions of the characteristic equation are $\pm 1, \pm 2, \pm 3$, and ± 6.

Since $\det(1I - A) = 0$, $\lambda - 1$ must be a factor of the characteristic polynomial. Dividing $\lambda - 1$ into $\lambda^3 - 6\lambda^2 + 11\lambda - 6$ leads to $\det(\lambda I - A) = (\lambda-1)(\lambda^2 - 5\lambda + 6) = (\lambda-1)(\lambda-2)(\lambda-3)$.

We conclude that the eigenvalues are 1, 2, and 3 - each of them has the algebraic multiplicity 1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{4}t, x_2 = \frac{1}{2}t, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{9}t, x_2 = \frac{1}{3}t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{u}$ consisting of equations $u_1' = u_1$, $u_2' = 2u_2$, and $u_3' = 3u_3$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^x$, $u_2 = c_2e^{2x}$, and $u_3 = c_3e^{3x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} c_1e^x \\ c_2e^{2x} \\ c_3e^{3x} \end{bmatrix} = \begin{bmatrix} c_1e^x + c_2e^{2x} + c_3e^{3x} \\ c_1e^x + 2c_2e^{2x} + 3c_3e^{3x} \\ c_1e^x + 4c_2e^{2x} + 9c_3e^{3x} \end{bmatrix}$ thus $y_1 = c_1e^x + c_2e^{2x} + c_3e^{3x}$, $y_2 = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$, and $y_3 = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$.

We conclude that the original equation $y''' - 6y'' + 11y' - 6y = 0$ has the solution

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}$$

10. From Formula (2) in Section 5.4, the second equation of the system has the solution $y_2 = c_2e^x$. With this, the first equation becomes $y_1' = y_1 + c_2e^x$. Using the terminology of differential equations, this is an example of a linear equation (note that the term carries a different meaning compared to linear algebra). One of the methods that can be applied here involves rewriting the equation as $y_1' - y_1 = c_2e^x$ and multiplying by the integrating factor e^{-x} so that the equation becomes $(y_1e^{-x})' = c_2$ leading to $y_1e^{-x} = c_2x + c_1$ and subsequently to $y_1 = c_1e^x + c_2xe^x$ (note that this method is typically discussed in differential equations textbooks, and is outside the scope of this text).
The solution of the nondiagonalizable system is $y_1 = c_1e^x + c_2xe^x,\ y_2 = c_2e^x$.

12. **(a)** From (11) in Section 5.4, we have

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1e^{2x} - \frac{1}{4}c_2e^{-3x} \\ c_1e^{2x} + c_2e^{-3x} \end{bmatrix} = \begin{bmatrix} c_1e^{2x} \\ c_1e^{2x} \end{bmatrix} + \begin{bmatrix} -\frac{1}{4}c_2e^{-3x} \\ c_2e^{-3x} \end{bmatrix} = c_1e^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-3x}\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

14. **(a)** Let y and z be functions in $C^\infty(-\infty,\infty)$ and let k be a real number. From calculus, we have $L(y+z) = \frac{d^2}{dx^2}(y+z) + 2\frac{d}{dx}(y+z) - 3(y+z) = y'' + z'' + 2y' + 2z' - 3y - 3z = L(y) + L(z)$ and
$L(ky) = \frac{d^2}{dx^2}(ky) + 2\frac{d}{dx}(ky) - 3(ky) = ky'' + 2ky' - 3ky = kL(y)$ therefore L is a linear operator.

(b) Substituting $y_1 = y$ and $y_2 = y'$ we can rewrite $y'' + 2y' - 3y = 0$ as the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 3y_1 - 2y_2 \end{aligned}$$

This system can be expressed in the form $\mathrm{y}' = A\mathrm{y}$ with $\mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$.

The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -3 & \lambda+2 \end{vmatrix} = \lambda^2 + 2\lambda - 3 = (\lambda-1)(\lambda+3)$ thus the eigenvalues of A are $\lambda = 1$ and $\lambda = -3$.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{1}{3}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = u_1$ and $u_2' = -3u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^x, u_2 = c_2e^{-3x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^x \\ c_2e^{-3x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} =$

$\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix}\begin{bmatrix}c_1e^x\\ c_2e^{-3x}\end{bmatrix}=\begin{bmatrix}c_1e^x-c_2e^{-3x}\\ c_1e^x+3c_2e^{-3x}\end{bmatrix}$ thus $y_1=c_1e^x-c_2e^{-3x}$ and $y_2=c_1e^x+3c_2e^{-3x}$.

We conclude that the differential equation $L(y)=0$ has the solution $y=c_1e^x-c_2e^{-3x}$.

15. (a) Let y and z be functions in $C^\infty(-\infty,\infty)$ and let k be a real number. From calculus, we have

$L(y+z)=\frac{d^3}{dx^3}(y+z)-2\frac{d^2}{dx^2}(y+z)-\frac{d}{dx}(y+z)+2(y+z)$

$=y'''+z'''-2y''-2z''-y'-z'+2y+2z=L(y)+L(z)$ and

$L(ky)=\frac{d^3}{dx^3}(ky)-2\frac{d^2}{dx^2}(ky)-\frac{d}{dx}(ky)+2(ky)=ky'''-2ky''-ky'+2ky=kL(y)$

therefore L is a linear operator.

(b) Substituting $y_1=y,\ y_2=y'$, and $y_3=y''$ we can rewrite $y'''-2y''-y'+2y=0$ as the system

$$\begin{aligned}y_1'&=y_2\\ y_2'&=y_3\\ y_3'&=-2y_1+y_2+2y_3\end{aligned}$$

This system can be expressed in the form $\mathbf{y}'=A\mathbf{y}$ where $\mathbf{y}=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$ and $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 1 & 2\end{bmatrix}$.

Cofactor expansion along the third column yields $\det(\lambda I-A)=\begin{vmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ 2 & -1 & \lambda-2\end{vmatrix}$

$=\begin{vmatrix}\lambda & -1\\ 2 & -1\end{vmatrix}+(\lambda-2)\begin{vmatrix}\lambda & -1\\ 0 & \lambda\end{vmatrix}=(-\lambda+2)+(\lambda-2)\lambda^2=(\lambda-2)(\lambda^2-1)=(\lambda-2)(\lambda-1)(\lambda+1)$.

The characteristic equation is $(\lambda-2)(\lambda-1)(\lambda+1)=0$ therefore the eigenvalues are 2, 1, and -1 – each of them has the algebraic multiplicity 1.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1 & 0 & -\frac{1}{4}\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=2$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1=\frac{1}{4}t, x_2=\frac{1}{2}t, x_3=t$. A vector $\mathbf{p}_1=\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I-A$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2=1$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1=t, x_2=t, x_3=t$. A vector $\mathbf{p}_2=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I-A$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = -1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = -t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = 2u_1, u_2' = u_2$, and $u_3' = -u_3$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^{2x}, u_2 = c_2e^{x}$, and $u_3 = c_3e^{-x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^{2x} \\ c_2e^{x} \\ c_3e^{-x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{2x} \\ c_2e^{x} \\ c_3e^{-x} \end{bmatrix} = \begin{bmatrix} c_1e^{2x} + c_2e^{x} + c_3e^{-x} \\ 2c_1e^{2x} + c_2e^{x} - c_3e^{-x} \\ 4c_1e^{2x} + c_2e^{x} + c_3e^{-x} \end{bmatrix}$ thus $y_1 = c_1e^{2x} + c_2e^{x} + c_3e^{-x}$, $y_2 = 2c_1e^{2x} + c_2e^{x} - c_3e^{-x}$, and $y_3 = 4c_1e^{2x} + c_2e^{x} + c_3e^{-x}$.

We conclude that the differential equation $L(y) = 0$ has the solution $y = c_1e^{2x} + c_2e^{x} + c_3e^{-x}$.

True-False Exercises

(a) True. $\mathbf{y} = \mathbf{0}$ is always a solution (called the trivial solution).

(b) False. If a system has a solution $\mathbf{x} \neq \mathbf{0}$ then any for any real number k, $\mathbf{y} = k\mathbf{x}$ is also a solution.

(c) True. $(c\mathbf{x} + d\mathbf{y})' = c\mathbf{x}' + d\mathbf{y}' = c(A\mathbf{x}) + d(A\mathbf{y}) = A(c\mathbf{x}) + A(d\mathbf{y}) = A(c\mathbf{x} + d\mathbf{y})$

(d) True. The solution can be obtained by following the four-step procedure preceding Example 2.

(e) False. If $P = Q^{-1}AQ$ then $\mathbf{u}' = Q^{-1}AQ\mathbf{u}$ implies $(Q\mathbf{u})' = A(Q\mathbf{u})$. Generally, $\mathbf{u}$ and $\mathbf{y} = Q\mathbf{u}$ are not the same.

5.5 Dynamical Systems and Markov Chains

1. **(a)** A is a stochastic matrix: each column vector has nonnegative entries that add up to 1

(b) A is not a stochastic matrix since entries in its columns do not add up to 1

(c) A is a stochastic matrix: each column vector has nonnegative entries that add up to 1

(d) A is not a stochastic matrix since $(A)_{23} = -\frac{1}{2}$ fails to be nonnegative

2. **(a)** A is a stochastic matrix: each column vector has nonnegative entries that add up to 1

(b) A is not a stochastic matrix since entries in its columns do not add up to 1

(c) A is a stochastic matrix: each column vector has nonnegative entries that add up to 1

(d) A is not a stochastic matrix since $(A)_{11} = -1$ fails to be nonnegative

3. $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$ $\qquad$ $\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix}$

$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix} = \begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix}$ $\qquad$ $\mathbf{x}_4 = P\mathbf{x}_3 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix} = \begin{bmatrix} 0.54545 \\ 0.45455 \end{bmatrix}$

An alternate approach is to determine $P^4 = \begin{bmatrix} 0.5455 & 0.5454 \\ 0.4545 & 0.4546 \end{bmatrix}$ then calculate $\mathbf{x}_4 = P^4\mathbf{x}_0 = \begin{bmatrix} 0.54545 \\ 0.45455 \end{bmatrix}$.

4. $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ $\qquad$ $\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$

$\mathbf{x}_3 = P\mathbf{x}_2 = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix} = \begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix}$ $\qquad$ $\mathbf{x}_4 = P\mathbf{x}_3 = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}\begin{bmatrix} 0.722 \\ 0.278 \end{bmatrix} = \begin{bmatrix} 0.7166 \\ 0.2834 \end{bmatrix}$

An alternate approach is to determine $P^4 = \begin{bmatrix} 0.7166 & 0.7085 \\ 0.2834 & 0.2915 \end{bmatrix}$ then calculate $\mathbf{x}_4 = P^4\mathbf{x}_0 = \begin{bmatrix} 0.7166 \\ 0.2834 \end{bmatrix}$.

5. **(a)** P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since P has all positive entries, it is also a regular matrix.

(b) By Theorem 1.7.1(b), the product of lower triangular matrices is also lower triangular. Consequently, for all positive integers k, the matrix P^k will have 0 in the first row second column entry. Therefore P is not a regular matrix.

(c) P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since $P^2 = \begin{bmatrix} \frac{21}{25} & \frac{1}{5} \\ \frac{4}{25} & \frac{4}{5} \end{bmatrix}$ has all positive entries, we conclude that P is a regular matrix.

6. **(a)** P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since $P^2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ has all positive entries, we conclude that P is a regular matrix.

(b) By Theorem 1.7.1(b), the product of upper triangular matrices is also upper triangular. Consequently, for all positive integers k, the matrix P^k will have 0 in the second row first column entry. Therefore P is not a regular matrix.

(c) P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since P has all positive entries, it is also a regular matrix.

7. P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since P has all positive entries, it is also a regular matrix.

To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{3}{4} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & -\frac{8}{9} \\ 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{8}{9}t, q_2 = t$. For $\mathbf{q}$ to be a probability vector, its components must add up to 1: $q_1 +$

$q_2 = 1$. Solving the resulting equation $\frac{8}{9}t + t = 1$ for t results in $t = \frac{9}{17}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{8}{17} \\ \frac{9}{17} \end{bmatrix}$.

8. P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since P has all positive entries, it is also a regular matrix.

To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} 0.8 & -0.6 \\ -0.8 & 0.6 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & -0.75 \\ 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{3}{4}t, q_2 = t$. For $\mathbf{q}$ to be a probability vector, its components must add up to 1: $q_1 + q_2 = 1$. Solving the resulting equation $\frac{3}{4}t + t = 1$ for t results in $t = \frac{4}{7}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$.

9. P is a stochastic matrix: each column vector has nonnegative entries that add up to 1;

since $P^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{3}{8} & \frac{7}{18} \\ \frac{7}{24} & \frac{1}{8} & \frac{4}{9} \end{bmatrix}$ has all positive entries, we conclude that P is a regular matrix.

To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{4}{3}t, q_2 = \frac{4}{3}t, q_3 = t$. For $\mathbf{q}$ to be a probability vector, we must have $q_1 + q_2 + q_3 = 1$. Solving the resulting equation $\frac{4}{3}t + \frac{4}{3}t + t = 1$ for t results in $t = \frac{3}{11}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix}$.

10. P is a stochastic matrix: each column vector has nonnegative entries that add up to 1; since $P^2 = \begin{bmatrix} \frac{17}{45} & \frac{13}{48} & \frac{47}{150} \\ \frac{4}{15} & \frac{9}{16} & \frac{19}{50} \\ \frac{16}{45} & \frac{1}{6} & \frac{23}{75} \end{bmatrix}$ has all positive entries, we conclude that P is a regular matrix.

To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{3}{5} \\ 0 & \frac{1}{4} & -\frac{2}{5} \\ -\frac{2}{3} & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{8}{5} \\ 0 & 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{6}{5}t, q_2 = \frac{8}{5}t, q_3 = t$. For $\mathbf{q}$ to be a probability vector, we must have $q_1 + q_2 + q_3 = 1$. Solving the resulting equation $\frac{6}{5}t + \frac{8}{5}t + t = 1$ for t results in $t = \frac{5}{19}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{6}{19} \\ \frac{8}{19} \\ \frac{5}{19} \end{bmatrix}$.

11. **(a)** The entry 0.2 represents the probability that the system will stay in state 1 when it is in state 1.

(b) The entry 0.1 represents the probability that the system will move to state 1 when it is in state 2.

(c) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$ therefore if the system is in state 1 initially, there is 0.8 probability that it will be in state 2 at the next observation.

(d) $\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$ therefore if the system has a 50% chance of being in state 1 initially, it will be in state 2 at the next observation with probability 0.85.

12. **(a)** The entry $\frac{6}{7}$ represents the probability that the system will stay in state 2 when it is in state 2.

(b) The entry 0 represents the probability that the system will stay in state 1 when it is in state 1.

(c) $\begin{bmatrix} 0 & \frac{1}{7} \\ 1 & \frac{6}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ therefore if the system is in state 1, there is 0 probability that it will remain in state 1.

(d) $\begin{bmatrix} 0 & \frac{1}{7} \\ 1 & \frac{6}{7} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{14} \\ \frac{13}{14} \end{bmatrix}$ therefore if the system has a 50% chance of being in state 1 initially, it will be in state 2 at the next observation with probability $\frac{13}{14}$.

13. **(a)** The transition matrix is $\begin{array}{c c} & \begin{array}{cc} \textbf{good} & \textbf{bad} \end{array} \\ \begin{array}{c} \textbf{good} \\ \textbf{bad} \end{array} & \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \end{array}$

(b) $\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.93 \\ 0.07 \end{bmatrix}$ therefore if the air quality is good today, it will also be good two days from now with probability 0.93 .

(c) $\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.858 \\ 0.142 \end{bmatrix}$ therefore if the air quality is bad today, it will also be bad three days from now with probability 0.142 .

(d) $\begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.63 \\ 0.37 \end{bmatrix}$ therefore if there is a 20% chance that air quality will be good today, it will be good tomorrow with probability 0.63 .

14. **(a)** The transition matrix is $\begin{array}{c c} & \begin{array}{cc} \textbf{type I} & \textbf{type II} \end{array} \\ \begin{array}{c} \textbf{type I} \\ \textbf{type II} \end{array} & \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \end{array}$

(b) $\begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6875 \\ 0.3125 \end{bmatrix}$ therefore if the mouse chooses type I today, it will choose the same type two days from now with probability 0.6875 .

(c) $\begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.65625 \\ 0.34375 \end{bmatrix}$ therefore if the mouse chooses type II today, it will choose the same type three days from now with probability 0.34375 .

(d) $\begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.525 \\ 0.475 \end{bmatrix}$ therefore if there is a 10% chance that the mouse chooses type I today, it will choose type I tomorrow with probability 0.525 .

15. **(a)** The transition matrix is $\begin{array}{c c} & \begin{array}{cc} \textbf{city} & \textbf{suburbs} \end{array} \\ \begin{array}{c} \textbf{city} \\ \textbf{suburbs} \end{array} & \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \end{array} = P$

The initial state vector $\mathbf{x}_0 = \begin{bmatrix} \frac{100{,}000}{125{,}000} \\ \frac{25{,}000}{125{,}000} \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ represents the fractions of the total population (125,000) living in the city and in the suburbs, respectively.

After one year, the corresponding fractions are contained in the state vector $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$. To determine the populations living in the city and in the suburbs at that time, we can calculate the scalar multiple of the state vector: $125{,}000\ \mathbf{x}_1 = \begin{bmatrix} 95{,}750 \\ 29{,}250 \end{bmatrix}$.

After the second year, the state vector becomes $\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} 0.73472 \\ 0.26528 \end{bmatrix}$, and the corresponding population counts are $125{,}000\,\mathbf{x}_2 = \begin{bmatrix} 91{,}840 \\ 33{,}160 \end{bmatrix}$.

Repeating this process three more times results in

	initial state $k = 0$	after 1 year $k = 1$	after 2 years $k = 2$	after 3 years $k = 3$	after 4 years $k = 4$	after 5 years $k = 5$
state vector $\mathbf{x}_k \approx$	$\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$	$\begin{bmatrix} 0.73472 \\ 0.26528 \end{bmatrix}$	$\begin{bmatrix} 0.705942 \\ 0.294058 \end{bmatrix}$	$\begin{bmatrix} 0.679467 \\ 0.320533 \end{bmatrix}$	$\begin{bmatrix} 0.655110 \\ 0.344890 \end{bmatrix}$
city population suburb population	100,000 25,000	95,750 29,250	91,840 33,160	88,243 36,757	84,933 40,067	81,889 43,111

(b) Since P is a regular stochastic matrix, there exists a unique steady-state probability vector. To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} 0.05 & -0.03 \\ -0.05 & 0.03 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & -\frac{3}{5} \\ 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{3}{5}t, q_2 = t$. The components of the vector $\mathbf{q}$ must add up to 1: $q_1 + q_2 = 1$. Solving the resulting equation $\frac{3}{5}t + t = 1$ for t results in $t = \frac{5}{8}$, consequently over the long term the fractions of the total population living in the city and in the suburbs will approach $\frac{3}{5} \cdot \frac{5}{8} = \frac{3}{8}$ and $\frac{5}{8}$, respectively.

We conclude that the city population will approach $\frac{3}{8} \cdot 125{,}000 = 46{,}875$ and the suburbs population will approach $\frac{5}{8} \cdot 125{,}000 = 78{,}125$.

16. **(a)** The transition matrix is

$$\begin{array}{r c} & \begin{array}{cc} \textbf{station 1} & \textbf{station 2} \end{array} \\ \begin{array}{r} \textbf{station 1} \\ \textbf{station 2} \end{array} & \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix} = P \end{array}$$

Multiplying this matrix by the initial state vector $\mathbf{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ results in $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} 0.475 \\ 0.525 \end{bmatrix}$. After the second year, the state vector becomes $\mathbf{x}_2 = P\mathbf{x}_1 \approx \begin{bmatrix} 0.454 \\ 0.546 \end{bmatrix}$. Repeating this process three more times results in

	initial state	after 1 year	after 2 years	after 3 years	after 4 years	after 5 years
	$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$
market share of station 1 market share of station 2	0.5 0.5	0.475 0.525	0.454 0.546	0.436 0.564	0.420 0.580	0.407 0.593

(b) Since P is a regular stochastic matrix, there exists a unique steady-state probability vector. To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} 0.1 & -0.05 \\ -0.1 & 0.05 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & -0.5 \\ 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{1}{2}t, q_2 = t$. For $\mathbf{q}$ to be a probability vector, its components must add up to 1: $q_1 + q_2 = 1$. Solving the resulting equation $\frac{1}{2}t + t = 1$ for t results in $t = \frac{2}{3}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

17. (a) The transition matrix is $P = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} & \frac{3}{5} \\ \frac{4}{5} & \frac{3}{10} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{2} & \frac{1}{5} \end{bmatrix}$. Multiplying this matrix by the initial state vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ results in $\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{1}{10} \end{bmatrix}$. After the second rental, the state vector becomes $\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} \frac{23}{100} \\ \frac{17}{50} \\ \frac{43}{100} \end{bmatrix}$.

The conclude that the car originally rented from location 1 will after two rentals be at location 1 with probability 0.23 .

(b) Since P is a regular stochastic matrix, there exists a unique steady-state probability vector. To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{9}{10} & -\frac{1}{5} & -\frac{3}{5} \\ -\frac{4}{5} & \frac{7}{10} & -\frac{1}{5} \\ -\frac{1}{10} & -\frac{1}{2} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & -\frac{46}{47} \\ 0 & 1 & -\frac{66}{47} \\ 0 & 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = \frac{46}{47}t$, $q_2 = \frac{66}{47}t$, $q_3 = t$. For $\mathbf{q}$ to be a probability vector, we must have $q_1 + q_2 + q_3 = 1$. Solving the resulting equation $\frac{46}{47}t + \frac{66}{47}t + t = 1$ for t results in $t = \frac{47}{159}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{46}{159} \\ \frac{66}{159} \\ \frac{47}{159} \end{bmatrix} = \begin{bmatrix} \frac{46}{159} \\ \frac{22}{53} \\ \frac{47}{159} \end{bmatrix} \approx \begin{bmatrix} 0.289 \\ 0.415 \\ 0.296 \end{bmatrix}$.

(c) Over the long term, the number of cars will approach $120\begin{bmatrix}\frac{46}{159}\\\frac{22}{53}\\\frac{47}{159}\end{bmatrix} \approx \begin{bmatrix}35\\50\\35\end{bmatrix}$ so the rental agency should allocate no fewer than 35, 50, and 35 parking spaces at locations 1, 2, and 3, respectively.

18. **(a)** Denoting $P = \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & 0\\\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\0 & \frac{1}{4} & \frac{1}{2}\end{bmatrix}$ we can calculate $P^2 = \begin{bmatrix}\frac{3}{8} & \frac{1}{4} & \frac{1}{8}\\\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{1}{8} & \frac{1}{4} & \frac{3}{8}\end{bmatrix}$ which shows that the matrix P is regular.

(b) Since P is a regular stochastic matrix, there exists a unique steady-state probability vector.

To find the steady-state vector, we solve the system $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix}\frac{1}{2} & -\frac{1}{4} & 0\\-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\0 & -\frac{1}{4} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix}1 & 0 & -1\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$ thus the general solution is $q_1 = t, q_2 = 2t, q_3 = t$. For $\mathbf{q}$ to be a probability vector, we must have $q_1 + q_2 + q_3 = 1$. Solving the resulting equation $t + 2t + t = 1$ for t results in $t = \frac{1}{4}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix}\frac{1}{4}\\\frac{1}{2}\\\frac{1}{4}\end{bmatrix}$.

After parents of known genotype (and their offsprings) continue to be crossed with random parents of unknown genotype for a long time, the probability of an offspring to have a genotype AA, Aa, and aa will approach 0.25, 0.5, and 0.25, respectively.

19. For the matrix $P = \begin{bmatrix}\frac{7}{10} & p_{12} & \frac{1}{5}\\p_{21} & \frac{3}{10} & p_{23}\\\frac{1}{10} & \frac{3}{5} & \frac{3}{10}\end{bmatrix}$ to be stochastic, each column vector must be a probability vector: a vector with nonnegative entries that add up to one. Applying the latter condition to each column results in three equations, which can be used to solve for the missing entries:

$$\begin{array}{llll}\text{column 1:} & \frac{7}{10} + p_{21} + \frac{1}{10} = 1 & \text{yields} & p_{21} = 1 - \frac{7}{10} - \frac{1}{10} = \frac{2}{10} = \frac{1}{5}\\ \text{column 2:} & p_{12} + \frac{3}{10} + \frac{3}{5} = 1 & \text{yields} & p_{12} = 1 - \frac{3}{10} - \frac{3}{5} = \frac{1}{10}\\ \text{column 3:} & \frac{1}{5} + p_{23} + \frac{3}{10} = 1 & \text{yields} & p_{23} = 1 - \frac{1}{5} - \frac{3}{10} = \frac{5}{10} = \frac{1}{2}\end{array}$$

The resulting transition matrix is $P = \begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{bmatrix}$. Since P is a regular stochastic matrix, there exists a unique steady-state probability vector. To find the steady-state vector, we solve the system $(I-P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{3}{10} & -\frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{7}{10} & -\frac{1}{2} \\ -\frac{1}{10} & -\frac{3}{5} & \frac{7}{10} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ thus the general solution is $q_1 = t, q_2 = t, q_3 = t$. For $\mathbf{q}$ to be a probability vector, we must have $q_1 + q_2 + q_3 = 1$. Solving the resulting equation $t + t + t = 1$ for t results in $t = \frac{1}{3}$, consequently the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

20. $MP = M$ since each entry in the product MP is a sum of all entries in a column of P, which must be 1.

21. From Theorem 5.5.1(a), we have $P\mathbf{q} = \mathbf{q}$. Therefore for any positive integer k,

$$P^k\mathbf{q} = P^{k-1}(P\mathbf{q}) = P^{k-1}\mathbf{q} = P^{k-2}(P\mathbf{q}) = P^{k-2}\mathbf{q} = \cdots = \mathbf{q}$$

22. **(a)** From Theorem 5.5.1, for each $i = 1,2,\ldots,n$, the sequence $P\mathbf{e}_i, P^2\mathbf{e}_i, \ldots, P^k\mathbf{e}_i, \ldots$approaches $\mathbf{q}$.

(b) As $k \to \infty$, P^k approaches the $n \times n$ matrix $[\,\mathbf{q} \mid \mathbf{q} \mid \ldots \mid \mathbf{q}\,]$.

23. Let A and B be two $n \times n$ stochastic matrices, and let B be partitioned into columns: $B = [\,\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_n\,]$. Using Formula (6) in Section 1.3, we can now see that the product

$$AB = A[\,\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_n\,] = [A\,\mathbf{b}_1 \mid A\,\mathbf{b}_2 \mid \ldots \mid A\,\mathbf{b}_n\,]$$

has columns that are probability vectors (since each of them is a product of a stochastic matrix and a probability vector). We conclude that AB is stochastic.

True-False Exercises

(a) True. All entries are nonnegative and their sum is 1.

(b) True. This is a stochastic matrix since its columns are probability vectors.

Furthermore, $\begin{bmatrix} 0.2 & 1 \\ 0.8 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0.84 & 0.2 \\ 0.16 & 0.8 \end{bmatrix}$ has all positive entries.

(c) True. By definition, a transition matrix is a stochastic matrix.

(d) False. For $\mathbf{q}$ to be a steady-state vector of a regular Markov chain, it must also be a probability vector.

(e) True. (See Exercise 23.)

(f) False. The entries must be nonnegative.

(g) True. This follows from Theorem 5.5.1(a).

Chapter 5 Supplementary Exercises

1. **(a)** The characteristic polynomial is

$\det(\lambda I - A) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = (\lambda - \cos\theta)^2 + (\sin\theta)^2$.

For a real eigenvalue λ to exist, we must have $\lambda = \cos\theta$ and $\sin\theta = 0$. However, the latter equation has no solutions on the given interval $0 < \theta < \pi$, therefore A has no real eigenvalues, and consequently no real eigenvectors.

(b) According to Table 1 in Section 4.11, A is the standard matrix of the rotation in the plane about the origin through a positive angle θ. Unless the angle is an integer multiple of π, no vector resulting from such a rotation is a scalar multiple of the original nonzero vector.

2. Since $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -k^3 & 3k^2 & \lambda - 3k \end{vmatrix} = \lambda^3 - 3k\lambda^2 + 3k^2\lambda - k^3 = (\lambda - k)^3$, A has only one eigenvalue: $\lambda = k$.

3. **(a)** If $D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$ with $d_{ii} \geq 0$ for all i then we can take $S = \begin{bmatrix} \sqrt{d_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{d_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{d_{nn}} \end{bmatrix}$ so that $S^2 = D$ holds true. (Note that the answer is not unique: the main diagonal entries of S could be negative square roots instead.)

(b) From our assumptions it follows that there exists a matrix P such that $A = P^{-1}DA$ where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ with $\lambda_i \geq 0$ for all i. Taking $R = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}$ so that $R^2 = D$ (see part (a)), we can form the matrix $S = PRP^{-1}$ so that $S^2 = PRP^{-1}PRP^{-1} = PR^2P^{-1} = PDP^{-1} = A$.

(c) By Theorem 5.1.2, A has eigenvalues $\lambda_1 = 1, \lambda_2 = 4$, and $\lambda_3 = 9$.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = 0, x_3 = 0$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $9I - A$ is $\begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 9$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = \frac{1}{2}t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix}1\\ 2\\ 2\end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 9\end{bmatrix}$.

Since the reduced row echelon form of $\begin{bmatrix}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 0 & 1\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & 0 & 1 & -1 & \frac{1}{2}\\ 0 & 1 & 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2}\end{bmatrix}$, we have $P^{-1} = \begin{bmatrix}1 & -1 & \frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$. As described in the solution of part (b) we can let $R = \begin{bmatrix}\sqrt{1} & 0 & 0\\ 0 & \sqrt{4} & 0\\ 0 & 0 & \sqrt{9}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$ and form $S = PRP^{-1} = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & -1 & \frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 3\end{bmatrix}$. This matrix satisfies $S^2 = A$.

4. We assume there exists a matrix P such that $B = P^{-1}AP$.

(a) $B^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T = P^T A^T (P^T)^{-1}$ therefore A^T and B^T are similar.

(b) $B^k = (P^{-1}AP)^k = P^{-1}APP^{-1}AP \cdots P^{-1}AP = P^{-1}A^kP$ therefore A^k and B^k are similar.

(c) $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$ therefore A^{-1} and B^{-1} are similar.

7. **(a)** The characteristic polynomial is $\det(\lambda I - A) = \begin{vmatrix}\lambda - 3 & -6\\ -1 & \lambda - 2\end{vmatrix} = -5\lambda + \lambda^2$.

We verify that $-5A + A^2 = -5\begin{bmatrix}3 & 6\\ 1 & 2\end{bmatrix} + \begin{bmatrix}3 & 6\\ 1 & 2\end{bmatrix}^2 = -\begin{bmatrix}15 & 30\\ 5 & 10\end{bmatrix} + \begin{bmatrix}15 & 30\\ 5 & 10\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

(b) The characteristic polynomial is $\det(\lambda I - A) = \begin{vmatrix}\lambda & -1 & 0\\ 0 & \lambda & -1\\ -1 & 3 & \lambda - 3\end{vmatrix} = -1 + 3\lambda - 3\lambda^2 + \lambda^3$.

We verify that

$$-I_3 + 3A - 3A^2 + A^3 = -\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + 3\begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix} - 3\begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix}^2 + \begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix}^3$$
$$= -\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&3&0\\0&0&3\\3&-9&9\end{bmatrix} - \begin{bmatrix}0&0&3\\3&-9&9\\9&-24&18\end{bmatrix} + \begin{bmatrix}1&-3&3\\3&-8&6\\6&-15&10\end{bmatrix}$$
$$= \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

9. Since $\det(\lambda I - A) = \begin{vmatrix}\lambda-3&-6\\-1&\lambda-2\end{vmatrix} = \lambda^2 - 5\lambda$, it follows from the Cayley-Hamilton Theorem that $A^2 - 5A = 0$. This yields $A^2 = 5A = 5\begin{bmatrix}3&6\\1&2\end{bmatrix} = \begin{bmatrix}15&30\\5&10\end{bmatrix}$, $A^3 = 5A^2 = 5\begin{bmatrix}15&30\\5&10\end{bmatrix} = \begin{bmatrix}75&150\\25&50\end{bmatrix}$, $A^4 = 5A^3 = 5\begin{bmatrix}75&150\\25&50\end{bmatrix} = \begin{bmatrix}375&750\\125&250\end{bmatrix}$, and $A^5 = 5A^4 = 5\begin{bmatrix}375&750\\125&250\end{bmatrix} = \begin{bmatrix}1875&3750\\625&1250\end{bmatrix}$.

10. We begin by calculating $A^2 = \begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix} = \begin{bmatrix}0&0&1\\1&-3&3\\3&-8&6\end{bmatrix}$.

Since $\det(\lambda I - A) = \begin{vmatrix}\lambda&-1&0\\0&\lambda&-1\\-1&3&\lambda-3\end{vmatrix} = \lambda^3 - 3\lambda^2 + 3\lambda - 1$, it follows from the Cayley-Hamilton Theorem that $A^3 - 3A^2 + 3A - I = 0$. This yields

$A^3 = 3A^2 - 3A + I = 3\begin{bmatrix}0&0&1\\1&-3&3\\3&-8&6\end{bmatrix} - 3\begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix} + \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&-3&3\\3&-8&6\\6&-15&10\end{bmatrix}$ and

$A^4 = 3A^3 - 3A^2 + A = 3\begin{bmatrix}1&-3&3\\3&-8&6\\6&-15&10\end{bmatrix} - 3\begin{bmatrix}0&0&1\\1&-3&3\\3&-8&6\end{bmatrix} + \begin{bmatrix}0&1&0\\0&0&1\\1&-3&3\end{bmatrix} = \begin{bmatrix}3&-8&6\\6&-15&10\\10&-24&15\end{bmatrix}$

11. Method I

For λ to be an eigenvalue of A associated with a nonzero eigenvector $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$, we must have $A\mathbf{x} = \lambda\mathbf{x}$ i.e.

$$\begin{bmatrix}c_1x_1 + c_2x_2 + \cdots + c_nx_n\\c_1x_1 + c_2x_2 + \cdots + c_nx_n\\\vdots\\c_1x_1 + c_2x_2 + \cdots + c_nx_n\end{bmatrix} = \lambda\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$$

There are two possibilities:

- If $\lambda \neq 0$ then $x_1 = x_2 = \cdots = x_n$. This implies $\lambda = c_1 + \cdots + c_n = \operatorname{tr}(A)$.
- If $\lambda = 0$ then $A\mathbf{x} = \lambda\mathbf{x}$ becomes a homogeneous system $A\mathbf{x} = \mathbf{0}$; its coefficient matrix A can be reduced to $\begin{bmatrix}c_1&c_2&\cdots&c_n\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{bmatrix}$. The solution space has dimension of at least $n-1$ therefore $\lambda = 0$ is an eigenvalue whose geometric multiplicity is at least $n-1$.

We conclude that the only eigenvalues of A are 0 and $\operatorname{tr}(A)$.

Method II

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ -c_1 & \lambda - c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ -c_1 & -c_2 & \lambda - c_3 & \cdots & -c_{n-1} & -c_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_1 & -c_2 & -c_3 & \cdots & \lambda - c_{n-1} & -c_n \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & \lambda - c_n \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ -\lambda & \lambda & 0 & \cdots & 0 & 0 \\ -\lambda & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda & 0 & 0 & \cdots & \lambda & 0 \\ -\lambda & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

-1 times the first row was added to each of the remaining rows.

$$= \begin{vmatrix} \lambda - c_1 - c_2 - c_3 - \cdots - c_{n-1} - c_n & -c_2 & -c_3 & \cdots & -c_{n-1} & -c_n \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

Each of the columns from the second to the last was added to the first column.

$$= (\lambda - c_1 - c_2 - c_3 - \cdots - c_{n-1} - c_n)\lambda^{n-1}$$

We conclude that the only eigenvalues of A are 0 and $\operatorname{tr}(A) = c_1 + \cdots + c_n$.

12. **(b)** Using the companion matrix formula introduced in part (a) we obtain $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$.

13. By Theorem 5.1.2, all eigenvalues of $A^n = 0$ are 0. By Theorem 5.2.3, if A had any eigenvalue $\lambda \neq 0$ then λ^n would be an eigenvalue of A^n. We reached a contradiction, therefore all eigenvalues of A must be 0.

15. The three given eigenvectors can be used as columns of a matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ which diagonalizes A, i.e. $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The latter equation is equivalent to $A = PDP^{-1}$. The matrix $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ therefore

$P^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. We conclude that a matrix A satisfying the given conditions is $A =$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

16. **(a)** Since the characteristic polynomial of A is $\det(\lambda I - A) = (\lambda - 1)(\lambda + 2)(\lambda - 3)(\lambda + 3)$, we have $\det(-A) = \det(0I - A) = (-1)(2)(-3)(3) = 18$ and $\det(A) = (-1)^4 \det(-A) = 18$.

(b) Expanding the characteristic polynomial obtained in part (a) yields $\det(\lambda I - A) = \lambda^4 + \lambda^3 - 11\lambda^2 - 9\lambda + 18$. Using the result of Exercise 5, $\operatorname{tr}(A) = -1$.

17. By Theorem 5.2.3, if A had any eigenvalue λ then λ^3 is an eigenvalue of A^3 corresponding to the same eigenvector. From $A^3 = A$ it follows that $\lambda^3 = \lambda$, so the only possible eigenvalues are -1, 0, and 1.

18. **(a)** We begin by diagonalizing the coefficient matrix of the system $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -3 \\ -2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda - 2$. By the quadratic formula, the eigenvalues of A are $\lambda = \frac{5 \pm \sqrt{(-5)^2 - 4(-2)}}{2} = \frac{5 \pm \sqrt{33}}{2}$.

The reduced row echelon form of $\frac{5+\sqrt{33}}{2} I - A$ is $\begin{bmatrix} 1 & \frac{3-\sqrt{33}}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{5+\sqrt{33}}{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{-3+\sqrt{33}}{4} t$, $x_2 = t$.
A vector $\mathbf{p}_1 = \begin{bmatrix} 3 - \sqrt{33} \\ -4 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $\frac{5-\sqrt{33}}{2} I - A$ is $\begin{bmatrix} 1 & \frac{3+\sqrt{33}}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{5-\sqrt{33}}{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{-3-\sqrt{33}}{4} t$, $x_2 = t$.
A vector $\mathbf{p}_2 = \begin{bmatrix} 3 + \sqrt{33} \\ -4 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 3 - \sqrt{33} & 3 + \sqrt{33} \\ -4 & -4 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = \frac{5+\sqrt{33}}{2} u_1$ and $u_2' = \frac{5-\sqrt{33}}{2} u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1 e^{(5+\sqrt{33})x/2}$, $u_2 = c_2 e^{(5-\sqrt{33})x/2}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1 e^{(5+\sqrt{33})x/2} \\ c_2 e^{(5-\sqrt{33})x/2} \end{bmatrix}$.

From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 3-\sqrt{33} & 3+\sqrt{33} \\ -4 & -4 \end{bmatrix} \begin{bmatrix} c_1 e^{(5+\sqrt{33})x/2} \\ c_2 e^{(5-\sqrt{33})x/2} \end{bmatrix} =$
$\begin{bmatrix} (3-\sqrt{33})c_1 e^{(5+\sqrt{33})x/2} + (3+\sqrt{33})c_2 e^{(5-\sqrt{33})x/2} \\ -4c_1 e^{(5+\sqrt{33})x/2} - 4c_2 e^{(5-\sqrt{33})x/2} \end{bmatrix}$ thus $y_1 = (3-\sqrt{33})c_1 e^{(5+\sqrt{33})x/2} +$
$(3+\sqrt{33})c_2 e^{(5-\sqrt{33})x/2}$ and
$y_2 = -4c_1 e^{(5+\sqrt{33})x/2} - 4c_2 e^{(5-\sqrt{33})x/2}$.

(b) Substituting the initial conditions into the general solution obtained in part (a) yields

$$\begin{array}{rcrcl} (3-\sqrt{33})c_1 & + & (3+\sqrt{33})c_2 & = & 5 \\ -4c_1 & - & 4c_2 & = & 6 \end{array}$$

The reduced row echelon form of this system's augmented matrix $\begin{bmatrix} 3-\sqrt{33} & 3+\sqrt{33} & 5 \\ -4 & -4 & 6 \end{bmatrix}$ is
$\begin{bmatrix} 1 & 0 & -\frac{3}{4} - \frac{19\sqrt{33}}{132} \\ 0 & 1 & -\frac{3}{4} + \frac{19\sqrt{33}}{132} \end{bmatrix}$ therefore $c_1 = -\frac{3}{4} - \frac{19\sqrt{33}}{132}$ and $c_2 = -\frac{3}{4} + \frac{19\sqrt{33}}{132}$.
After simplifying, the solution satisfying the given initial conditions can be expressed as
$y_1 = \left(\frac{5}{2} + \frac{7\sqrt{33}}{22}\right) e^{(5+\sqrt{33})x/2} + \left(\frac{5}{2} - \frac{7\sqrt{33}}{22}\right) e^{(5-\sqrt{33})x/2}$ and
$y_2 = \left(3 + \frac{19\sqrt{33}}{33}\right) e^{(5+\sqrt{33})x/2} + \left(3 - \frac{19\sqrt{33}}{33}\right) e^{(5-\sqrt{33})x/2}$.

19. Let a and b denote the two unknown eigenvalues. We solve the system $a + b + 1 = 6$ and $a \cdot b \cdot 1 = 6$. Rewriting the first equation as $b = 5 - a$ and substituting into the second equation yields $a(5-a) = 6$, therefore $(a-2)(a-3) = 0$. Either $a = 2$ (and $b = 3$) or $a = 3$ (and $b = 2$).
We conclude that the unknown eigenvalues are 2 and 3.

CHAPTER 6: INNER PRODUCT SPACES

6.1 Inner Products

1. **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle = 2(1)(3) + 3(1)(2) = 12$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = 2((3)(3))(0) + 3((3)(2))(-1) = -18$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 2(1+3)(0) + 3(1+2)(-1) = -9$

(d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = [2(3)(3) + 3(2)(2)]^{1/2} = \sqrt{30}$

(e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle (-2,-1), (-2,-1) \rangle^{1/2} = [2(-2)(-2) + 3(-1)(-1)]^{1/2} = \sqrt{11}$

(f) $\|\mathbf{u} - k\mathbf{v}\| = \langle (-8,-5), (-8,-5) \rangle^{1/2} = [2(-8)(-8) + 3(-5)(-5)]^{1/2} = \sqrt{203}$

2. **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}(1)(3) + 5(1)(2) = \frac{23}{2}$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}((3)(3))(0) + 5((3)(2))(-1) = -30$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \frac{1}{2}(1+3)(0) + 5(1+2)(-1) = -15$

(d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\frac{1}{2}(3)(3) + 5(2)(2)\right]^{1/2} = \sqrt{\frac{49}{2}} = \frac{7}{\sqrt{2}}$

(e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle (-2,-1), (-2,-1) \rangle^{1/2} = \left[\frac{1}{2}(-2)(-2) + 5(-1)(-1)\right]^{1/2} = \sqrt{7}$

(f) $\|\mathbf{u} - k\mathbf{v}\| = \langle (-8,-5), (-8,-5) \rangle^{1/2} = \left[\frac{1}{2}(-8)(-8) + 5(-5)(-5)\right]^{1/2} = \sqrt{157}$

3. **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} = 34$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 24 \\ 15 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -39$

(c) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -18$

(d) $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} 8 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix}\right)^{1/2} = \sqrt{89}$

(e) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -3 \end{bmatrix}\right)^{1/2} = \sqrt{34}$

(f) $\|\mathbf{u} - k\mathbf{v}\| = \left[\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ -5 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ -5 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -21 \\ -13 \end{bmatrix} \cdot \begin{bmatrix} -21 \\ -13 \end{bmatrix}\right)^{1/2} = \sqrt{610}$

4. **(a)** $\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 7$

(b) $\langle k\mathbf{v}, \mathbf{w} \rangle = \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 12$

(c) $\langle \mathbf{u}+\mathbf{v},\mathbf{w}\rangle = \left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right) = \begin{bmatrix}4\\5\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix} = 5$

(d) $\|\mathbf{v}\| = \langle \mathbf{v},\mathbf{v}\rangle^{1/2} = \left[\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}\right)^{1/2} = \sqrt{25} = 5$

(e) $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left[\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-2\\-3\end{bmatrix}\cdot\begin{bmatrix}-2\\-3\end{bmatrix}\right)^{1/2} = \sqrt{13}$

(f) $\|\mathbf{u}-k\mathbf{v}\| = \left[\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}-8\\-5\end{bmatrix}\right)\cdot\left(\begin{bmatrix}1 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}-8\\-5\end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix}-8\\-11\end{bmatrix}\cdot\begin{bmatrix}-8\\-11\end{bmatrix}\right)^{1/2} = \sqrt{185}$

5. $\begin{bmatrix}\sqrt{2} & 0\\0 & \sqrt{3}\end{bmatrix}$

6. $\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\0 & \sqrt{5}\end{bmatrix}$

7. $\langle \mathbf{u},\mathbf{v}\rangle = \left(\begin{bmatrix}4 & 1\\2 & -3\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}4 & 1\\2 & -3\end{bmatrix}\begin{bmatrix}6\\2\end{bmatrix}\right) = \begin{bmatrix}-3\\9\end{bmatrix}\cdot\begin{bmatrix}26\\6\end{bmatrix} = -24$

8. $\langle \mathbf{u},\mathbf{v}\rangle = \left(\begin{bmatrix}2 & 1\\-1 & 3\end{bmatrix}\begin{bmatrix}0\\-3\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2 & 1\\-1 & 3\end{bmatrix}\begin{bmatrix}6\\2\end{bmatrix}\right) = \begin{bmatrix}-3\\-9\end{bmatrix}\cdot\begin{bmatrix}14\\0\end{bmatrix} = -42$

9. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\langle \mathbf{u},\mathbf{v}\rangle = \mathrm{tr}(U^TV) = \mathrm{tr}\left(\begin{bmatrix}1 & 13\\10 & 2\end{bmatrix}\right) = 3.$

10. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\langle \mathbf{u},\mathbf{v}\rangle = \mathrm{tr}(U^TV) = \mathrm{tr}\left(\begin{bmatrix}4 & -18\\8 & 52\end{bmatrix}\right) = 56.$

11. $\langle \mathbf{p},\mathbf{q}\rangle = (-2)(4) + (1)(0) + (3)(-7) = -29$

12. $\langle \mathbf{p},\mathbf{q}\rangle = (-5)(3) + (2)(2) + (1)(-4) = -15$

13. $\begin{bmatrix}\sqrt{3} & 0\\0 & \sqrt{5}\end{bmatrix}$

14. $\begin{bmatrix}2 & 0\\0 & \sqrt{6}\end{bmatrix}$

15. $\langle \mathbf{p},\mathbf{q}\rangle = p(-2)q(-2) + p(-1)q(-1) + p(0)q(0) + p(1)q(1)$
$= (-10)(5) + (-2)(2) + (0)(1) + (2)(2) = -50$

16. $\langle \mathbf{p},\mathbf{q}\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) + p(2)q(2)$
$= (-2)(2) + (0)(1) + (2)(2) + (10)(5) = 50$

17. $\|\mathbf{u}\| = \langle \mathbf{u},\mathbf{u}\rangle^{1/2} = [2(-3)(-3) + 3(2)(2)]^{1/2} = \sqrt{30}$
$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \langle(-4,-5),(-4,-5)\rangle^{1/2} = [2(-4)(-4) + 3(-5)(-5)]^{1/2} = \sqrt{107}$

18. $\|\mathbf{u}\| = \langle \mathbf{u},\mathbf{u}\rangle^{1/2} = [2(-1)(-1) + 3(2)(2)]^{1/2} = \sqrt{14}$
$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \langle(-3,-3),(-3,-3)\rangle^{1/2} = [2(-3)(-3) + 3(-3)(-3)]^{1/2} = 3\sqrt{5}$

19. $\|\mathbf{p}\| = \langle \mathbf{p},\mathbf{p}\rangle^{1/2} = \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}$; $d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \sqrt{(-6)^2 + 1^2 + 10^2} = \sqrt{137}$

20. $\|\mathbf{p}\| = \langle \mathbf{p},\mathbf{p}\rangle^{1/2} = \sqrt{(-5)^2 + 2^2 + 1^2} = \sqrt{30}$; $d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \sqrt{(-8)^2 + 0^2 + 5^2} = \sqrt{89}$

21. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\|U\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\operatorname{tr}(U^T U)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 25 & 26 \\ 26 & 68 \end{bmatrix}\right)} = \sqrt{93}$ and

$d(U, V) = \|U - V\| = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{1/2} = \sqrt{\operatorname{tr}((U - V)^T (U - V))} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 25 & 1 \\ 1 & 74 \end{bmatrix}\right)} = \sqrt{99} = 3\sqrt{11}$

22. If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\|U\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\operatorname{tr}(U^T U)} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 10 & -13 \\ -13 & 29 \end{bmatrix}\right)} = \sqrt{39}$ and

$d(U, V) = \|U - V\| = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{1/2} = \sqrt{\operatorname{tr}((U - V)^T (U - V))} = \sqrt{\operatorname{tr}\left(\begin{bmatrix} 18 & 21 \\ 21 & 25 \end{bmatrix}\right)} = \sqrt{43}$

23. $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \sqrt{[p(-2)]^2 + [p(-1)]^2 + [p(0)]^2 + [p(1)]^2} = \sqrt{(-10)^2 + (-2)^2 + 0^2 + 2^2} = 6\sqrt{3}$
$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \sqrt{[p(-2) - q(-2)]^2 + [p(-1) - q(-1)]^2 + [p(0) - q(0)]^2 + [p(1) - q(1)]^2}$
$= \sqrt{(-15)^2 + (-4)^2 + (-1)^2 + 0^2} = 11\sqrt{2}$

24. $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \sqrt{[p(-1)]^2 + [p(0)]^2 + [p(1)]^2 + [p(2)]^2} = \sqrt{(-2)^2 + 0^2 + 2^2 + 10^2} = 6\sqrt{3}$
$d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \sqrt{[p(-1) - q(-1)]^2 + [p(0) - q(0)]^2 + [p(1) - q(1)]^2 + [p(2) - q(2)]^2}$
$= \sqrt{(-4)^2 + (-1)^2 + 0^2 + 5^2} = \sqrt{42}$

25. $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[\left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -4 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 7 \end{bmatrix}\right)^{1/2} = \sqrt{65}$

$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -12 \\ -24 \end{bmatrix} \cdot \begin{bmatrix} -12 \\ -24 \end{bmatrix}\right)^{1/2} = 12\sqrt{5}$

26. $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \end{bmatrix}\right)^{1/2} = \sqrt{58}$

$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \left[\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}\right)\right]^{1/2} = \left(\begin{bmatrix} -9 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} -9 \\ -6 \end{bmatrix}\right)^{1/2} = 3\sqrt{13}$

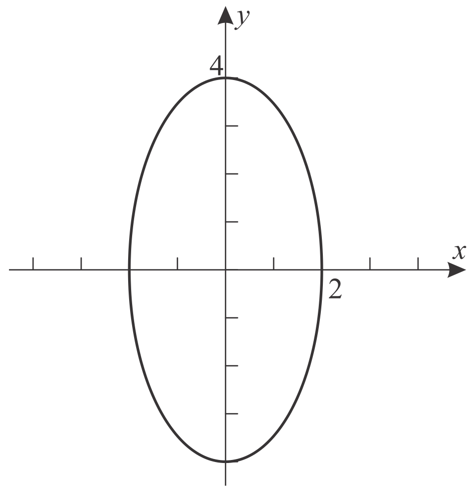
27. **(a)** $\langle 2\mathbf{v} - \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle = \langle 2\mathbf{v}, 3\mathbf{u} + 2\mathbf{w} \rangle - \langle \mathbf{w}, 3\mathbf{u} + 2\mathbf{w} \rangle = \langle 2\mathbf{v}, 3\mathbf{u} \rangle + \langle 2\mathbf{v}, 2\mathbf{w} \rangle - \langle \mathbf{w}, 3\mathbf{u} \rangle - \langle \mathbf{w}, 2\mathbf{w} \rangle$
$= 6\langle \mathbf{v}, \mathbf{u} \rangle + 4\langle \mathbf{v}, \mathbf{w} \rangle - 3\langle \mathbf{w}, \mathbf{u} \rangle - 2\langle \mathbf{w}, \mathbf{w} \rangle = 6\langle \mathbf{u}, \mathbf{v} \rangle + 4\langle \mathbf{v}, \mathbf{w} \rangle - 3\langle \mathbf{u}, \mathbf{w} \rangle - 2\|\mathbf{w}\|^2$
$= 6(2) + 4(-6) - 3(-3) - 2(49) = -101$

(b) $\|\mathbf{u} + \mathbf{v}\| = \sqrt{\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle} = \sqrt{\langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle} = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle}$
$= \sqrt{\|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2} = \sqrt{1 + 2(2) + 4} = 3$

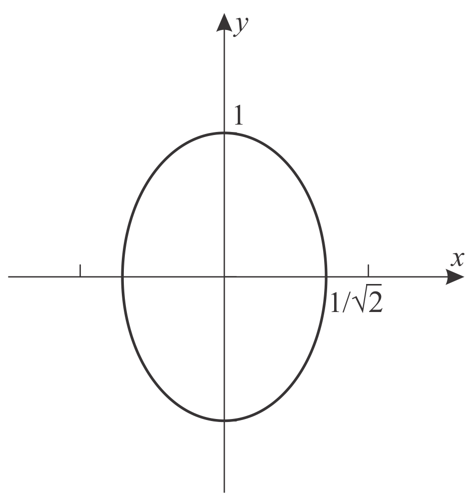
28. **(a)** $\langle \mathbf{u} - \mathbf{v} - 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, 4\mathbf{u} + \mathbf{v} \rangle - \langle \mathbf{v}, 4\mathbf{u} + \mathbf{v} \rangle - \langle 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle$
$= \langle \mathbf{u}, 4\mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, 4\mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle - \langle 2\mathbf{w}, 4\mathbf{u} \rangle - \langle 2\mathbf{w}, \mathbf{v} \rangle$
$= 4\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - 4\langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle - 8\langle \mathbf{w}, \mathbf{u} \rangle - 2\langle \mathbf{w}, \mathbf{v} \rangle$
$= 4\|\mathbf{u}\|^2 - 3\langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{v}\|^2 - 8\langle \mathbf{u}, \mathbf{w} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle = 4 - 3(2) - 4 - 8(-3) - 2(-6) = 30$

(b) $\|2\mathbf{w} - \mathbf{v}\| = \sqrt{\langle 2\mathbf{w} - \mathbf{v}, 2\mathbf{w} - \mathbf{v} \rangle} = \sqrt{\langle 2\mathbf{w}, 2\mathbf{w} - \mathbf{v} \rangle - \langle \mathbf{v}, 2\mathbf{w} - \mathbf{v} \rangle}$
$= \sqrt{\langle 2\mathbf{w}, 2\mathbf{w} \rangle - \langle 2\mathbf{w}, \mathbf{v} \rangle - \langle \mathbf{v}, 2\mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{4\|\mathbf{w}\|^2 - 4\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{v}\|^2} = \sqrt{4(49) - 4(-6) + 4}$
$= \sqrt{224} = 4\sqrt{14}$

29. If $\mathbf{u} = (x, y)$ then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\frac{1}{4}x^2 + \frac{1}{16}y^2}$, so the equation of the unit circle is $\frac{x^2}{4} + \frac{y^2}{16} = 1$.



30. If $\mathbf{u} = (x, y)$ then $\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{2x^2 + y^2}$, so the equation of the unit circle is $2x^2 + y^2 = 1$, which can be rewritten as $\frac{x^2}{1/2} + \frac{y^2}{1} = 1$.



31. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9}u_1v_1 + u_2v_2$ (see Example 3)

32. $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{16}{9}u_1v_1 + u_2v_2$ (see Example 3)

33. Axiom 2 does not hold, e.g., with $\mathbf{u} = \mathbf{v} = \mathbf{w} = (1,0,0)$ we have $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 4$ but $\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle = 1 + 1 = 2$;
Axiom 3 does not hold either, e.g., with $\mathbf{u} = \mathbf{v} = (1,0,0)$ and $k = 2$, $\langle k\mathbf{u}, \mathbf{v} \rangle = 4$ does not equal $k\langle \mathbf{u}, \mathbf{v} \rangle = 2$;
This is not an inner product on R^3.

34. Axiom 4 does not hold, e.g., $\langle(0,1,0),(0,1,0)\rangle = -1 < 0$; this is not an inner product on R^3.

35. By Definition 1, Definition 2, and Theorem 6.1.2, we have

$$\begin{aligned}\langle 2\mathbf{v}-4\mathbf{u}, \mathbf{u}-3\mathbf{v}\rangle &= \langle 2\mathbf{v}-4\mathbf{u}, \mathbf{u}\rangle - \langle 2\mathbf{v}-4\mathbf{u}, 3\mathbf{v}\rangle \\ &= \langle 2\mathbf{v}, \mathbf{u}\rangle - \langle 4\mathbf{u}, \mathbf{u}\rangle - \langle 2\mathbf{v}, 3\mathbf{v}\rangle + \langle 4\mathbf{u}, 3\mathbf{v}\rangle \\ &= 2\langle \mathbf{v}, \mathbf{u}\rangle - 4\langle \mathbf{u}, \mathbf{u}\rangle - 6\langle \mathbf{v}, \mathbf{v}\rangle + 12\langle \mathbf{u}, \mathbf{v}\rangle \\ &= 2\langle \mathbf{u}, \mathbf{v}\rangle - 4\langle \mathbf{u}, \mathbf{u}\rangle - 6\langle \mathbf{v}, \mathbf{v}\rangle + 12\langle \mathbf{u}, \mathbf{v}\rangle \\ &= 14\langle \mathbf{u}, \mathbf{v}\rangle - 4\|\mathbf{u}\|^2 - 6\|\mathbf{v}\|^2\end{aligned}$$

36. By Definition 1, Definition 2, and Theorem 6.1.2, we have

$$\begin{aligned}\langle 5\mathbf{u}+6\mathbf{v}, 4\mathbf{v}-3\mathbf{u}\rangle &= \langle 5\mathbf{u}+6\mathbf{v}, 4\mathbf{v}\rangle - \langle 5\mathbf{u}+6\mathbf{v}, 3\mathbf{u}\rangle \\ &= \langle 5\mathbf{u}, 4\mathbf{v}\rangle + \langle 6\mathbf{v}, 4\mathbf{v}\rangle - \langle 5\mathbf{u}, 3\mathbf{u}\rangle - \langle 6\mathbf{v}, 3\mathbf{u}\rangle \\ &= 20\langle \mathbf{u}, \mathbf{v}\rangle + 24\langle \mathbf{v}, \mathbf{v}\rangle - 15\langle \mathbf{u}, \mathbf{u}\rangle - 18\langle \mathbf{v}, \mathbf{u}\rangle \\ &= 20\langle \mathbf{u}, \mathbf{v}\rangle + 24\langle \mathbf{v}, \mathbf{v}\rangle - 15\langle \mathbf{u}, \mathbf{u}\rangle - 18\langle \mathbf{u}, \mathbf{v}\rangle \\ &= 2\langle \mathbf{u}, \mathbf{v}\rangle - 15\|\mathbf{u}\|^2 + 24\|\mathbf{v}\|^2\end{aligned}$$

37. **(a)** $\langle \mathbf{p}, \mathbf{q}\rangle = \left(\int_{-1}^{1} x^2\, dx\right) = \left(\frac{x^3}{3}\Big]_{-1}^{1}\right) = \frac{2}{3}$

(b) $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \left(\int_{-1}^{1} (1-x^2)^2\, dx\right)^{1/2} = \left(\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big]_{-1}^{1}\right)^{1/2} = \sqrt{\frac{16}{15}} = \frac{4}{\sqrt{15}}$

(c) $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p}\rangle^{1/2} = \left(\int_{-1}^{1} 1\, dx\right)^{1/2} = (x]_{-1}^{1})^{1/2} = \sqrt{2}$

(d) $\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q}\rangle^{1/2} = \left(\int_{-1}^{1} x^4\, dx\right)^{1/2} = \left(\frac{x^5}{5}\Big]_{-1}^{1}\right)^{1/2} = \sqrt{\frac{2}{5}}$

38. **(a)** $\langle \mathbf{p}, \mathbf{q}\rangle = \left(\int_{-1}^{1} 2x^3(1-x^3)\, dx\right) = \left(\left(\frac{x^4}{2} - \frac{2x^7}{7}\right)\Big]_{-1}^{1}\right) = \frac{-4}{7}$

(b) $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p}-\mathbf{q}\| = \left(\int_{-1}^{1} (3x^3-1)^2\, dx\right)^{1/2} = \left(\left(\frac{9x^7}{7} - \frac{3x^4}{2} + x\right)\Big]_{-1}^{1}\right)^{1/2} = \sqrt{\frac{32}{7}} = \frac{4\sqrt{2}}{\sqrt{7}}$

(c) $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p}\rangle^{1/2} = \left(\int_{-1}^{1} (2x^3)^2\, dx\right)^{1/2} = \left(\frac{4x^7}{7}\Big]_{-1}^{1}\right)^{1/2} = \sqrt{\frac{8}{7}} = \frac{2\sqrt{2}}{\sqrt{7}}$

(d) $\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q}\rangle^{1/2} = \left(\int_{-1}^{1} (1-x^3)^2\, dx\right)^{1/2} = \left(\left(x - \frac{x^4}{2} + \frac{x^7}{7}\right)\Big]_{-1}^{1}\right)^{1/2} = \sqrt{\frac{16}{7}} = \frac{4}{\sqrt{7}}$

39. $\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^1 \cos 2\pi x \sin 2\pi x\, dx = \frac{1}{2\pi}\frac{(\sin 2\pi x)^2}{2}\Big]_0^1 = 0$ (substituted $u = \sin 2\pi x$)

40. $\langle \mathbf{f}, \mathbf{g}\rangle = \int_0^1 xe^x dx = (xe^x - e^x)]_0^1 = 1$ (used integration by parts with $u = x$ and $dv = e^x dx$)

41. Part (a) follows directly from Definition 2 and Axiom 4 of Definition 1.

To prove part (b), write

$$\|k\mathbf{v}\| \underset{\text{Def.2}}{=} \sqrt{\langle k\mathbf{v}, k\mathbf{v}\rangle} \underset{\text{Axiom 3}}{=} \sqrt{k\langle \mathbf{v}, k\mathbf{v}\rangle} \underset{\text{Axiom 1}}{=} \sqrt{k\langle k\mathbf{v}, \mathbf{v}\rangle} \underset{\text{Axiom 3}}{=} \sqrt{k^2\langle \mathbf{v}, \mathbf{v}\rangle} \underset{\text{Def.2}}{=} |k|\|\mathbf{v}\|$$

42. Part (c) follows from Definition 2 and part (b) of Theorem 6.1.1, since

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(-1)(\mathbf{v}-\mathbf{u})\| = |-1|\|\mathbf{v}-\mathbf{u}\| = \|\mathbf{v}-\mathbf{u}\| = d(\mathbf{v}, \mathbf{u})$$

Part (d) follows from Definition 2 and Axiom 4 of Definition 1.

43. **(b)** k_1 and k_2 must both be positive in order for $\langle \mathbf{u}, \mathbf{v} \rangle$ to satisfy the positivity axiom. (Refer to the discussion following Theorem 6.1.1.)

44. By using Definition 2 and Axioms 1, 2, and 3 of Definition 1, we have

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \frac{1}{2}(\mathbf{u}+\mathbf{v}) + \frac{1}{2}(\mathbf{u}-\mathbf{v}), \frac{1}{2}(\mathbf{u}+\mathbf{v}) - \frac{1}{2}(\mathbf{u}-\mathbf{v}) \rangle \\ &= \frac{1}{4}\langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle + \frac{1}{4}\langle \mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle - \frac{1}{4}\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle - \frac{1}{4}\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle \\ &= \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2\end{aligned}$$

45. By using Definition 2 and Axioms 1, 2, and 3 of Definition 1, we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2\end{aligned}$$

48. **(b)** $T(1,1,1) = \langle (1,1,1), (1,0,2) \rangle = (1)(1) + (1)(0) + (1)(2) = 3$

(c) $T(x + x^2) = \langle x + x^2, 1 + x \rangle = (0)(1) + (1)(1) + (1)(0) = 1$

(d) $T(x + x^2) = \langle x + x^2, 1 + x \rangle = (1 + 1^2)(1 + 1) + (0 + 0^2)(1 + 0) + (-1 + (-1)^2)(1 - 1) = 4$

True-False Exercises

(a) True. The dot product is the special case of the weighted inner product with all the weights equal to 1.

(b) False. For example, if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} = (1,1)$, and $\mathbf{v} = (-2,1)$ then $\langle \mathbf{u}, \mathbf{v} \rangle = -1$.

(c) True. This follows from Axioms 1 and 2 of Definition 1 since $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{v}+\mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$.

(d) True. This follows from Axiom 3 of Definition 1 as well as part (e) of Theorem 6.1.2.

(e) False. For example, if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} = (1,1)$, and $\mathbf{v} = (-1,1)$ then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ even though both vectors are nonzero.

(f) True. By Definition 2, $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$ so by Axiom 4 of Definition 1, $\|\mathbf{v}\|^2 = 0$ implies $\mathbf{v} = \mathbf{0}$.

(g) False. A must be invertible; otherwise $A\mathbf{v} = \mathbf{0}$ has nontrivial solutions $\mathbf{v} \neq \mathbf{0}$ even though $\langle \mathbf{v}, \mathbf{v} \rangle = A\mathbf{v} \cdot A\mathbf{v} = \mathbf{0}$ which would violate Axiom 4 of Definition 1.

6.2 Angle and Orthogonality in Inner Product Spaces

1. **(a)** $\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(1)(2)+(-3)(4)}{\sqrt{1^2+(-3)^2}\sqrt{2^2+4^2}} = -\frac{10}{\sqrt{10}\sqrt{20}} = -\frac{10}{\sqrt{200}} = -\frac{10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}}$

(b) $\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-1)(2)+(5)(4)+(2)(-9)}{\sqrt{(-1)^2+5^2+2^2}\sqrt{2^2+4^2+(-9)^2}} = 0$

(c) $\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(1)(-3)+(0)(-3)+(1)(-3)+(0)(-3)}{\sqrt{1^2+0^2+1^2+0^2}\sqrt{(-3)^2+(-3)^2+(-3)^2+(-3)^2}} = -\frac{6}{\sqrt{2}\sqrt{36}} = -\frac{1}{\sqrt{2}}$

2. **(a)** $\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(-1)(3)+(0)(8)}{\sqrt{(-1)^2+0^2}\sqrt{3^2+8^2}} = -\frac{3}{\sqrt{73}}$

(b) $\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(4)(1)+(1)(0)+(8)(-3)}{\sqrt{4^2+1^2+8^2}\sqrt{1^2+0^2+(-3)^2}} = -\frac{20}{\sqrt{81}\sqrt{10}} = -\frac{20}{9\sqrt{10}}$

(c) $\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{(2)(4)+(1)(0)+(7)(0)+(-1)(0)}{\sqrt{2^2+1^2+7^2+(-1)^2}\sqrt{4^2+0^2+0^2+0^2}} = \frac{8}{4\sqrt{55}} = \frac{2}{\sqrt{55}}$

3. $\cos\theta = \frac{\langle \mathbf{p},\mathbf{q}\rangle}{\|\mathbf{p}\|\|\mathbf{q}\|} = \frac{(-1)(2)+(5)(4)+(2)(-9)}{\sqrt{(-1)^2+5^2+2^2}\sqrt{2^2+4^2+(-9)^2}} = 0$

4. $\cos\theta = \frac{\langle \mathbf{p},\mathbf{q}\rangle}{\|\mathbf{p}\|\|\mathbf{q}\|} = \frac{(0)(7)+(1)(3)+(-1)(3)}{\sqrt{0^2+1^2+(-1)^2}\sqrt{7^2+3^2+3^2}} = 0$

5. $\cos\theta = \frac{\langle U,V\rangle}{\|U\|\|V\|} = \frac{\operatorname{tr}(U^TV)}{\sqrt{\operatorname{tr}(U^TU)}\sqrt{\operatorname{tr}(V^TV)}} = \frac{(2)(3)+(6)(2)+(1)(1)+(-3)(0)}{\sqrt{2^2+6^2+1^2+(-3)^2}\sqrt{3^2+2^2+1^2+0^2}} = \frac{19}{\sqrt{50}\sqrt{14}} = \frac{19}{10\sqrt{7}}$

6. $\cos\theta = \frac{\langle U,V\rangle}{\|U\|\|V\|} = \frac{\operatorname{tr}(U^TV)}{\sqrt{\operatorname{tr}(U^TU)}\sqrt{\operatorname{tr}(V^TV)}} = \frac{(2)(-3)+(4)(1)+(-1)(4)+(3)(2)}{\sqrt{2^2+4^2+(-1)^2+3^2}\sqrt{(-3)^2+1^2+4^2+2^2}} = 0$

7. **(a)** orthogonal: $\langle \mathbf{u},\mathbf{v}\rangle = -4+6-2 = 0$

(b) not orthogonal: $\langle \mathbf{u},\mathbf{v}\rangle = -2-2-2 = -6 \neq 0$

(c) orthogonal: $\langle \mathbf{u},\mathbf{v}\rangle = (a)(-b)+(b)(a) = 0$

8. **(a)** orthogonal: $\langle \mathbf{u},\mathbf{v}\rangle = 0+0+0 = 0$

(b) not orthogonal: $\langle \mathbf{u},\mathbf{v}\rangle = -8+6+20+9 = 27 \neq 0$

(c) orthogonal: $\langle \mathbf{u},\mathbf{v}\rangle = -ac+0+ac = 0$

9. $\langle \mathbf{p},\mathbf{q}\rangle = (-1)(0)+(-1)(2)+(2)(1) = 0$

10. $\langle \mathbf{p},\mathbf{q}\rangle = (2)(4)+(-3)(2)+(1)(-2) = 0$

11. $\langle U,V\rangle = (2)(-3)+(1)(0)+(-1)(0)+(3)(2) = 0$

12. $\langle U,V\rangle = (5)(1)+(-1)(3)+(2)(-1)+(-2)(0) = 0$

13. The vectors are not orthogonal with respect to the Euclidean inner product since $\langle \mathbf{u},\mathbf{v}\rangle = (1)(2)+(3)(-1) = -1 \neq 0$. Using the weighted inner product instead yields $\langle \mathbf{u},\mathbf{v}\rangle = 2(1)(2)+k(3)(-1) = 4-3k$, so the vectors are orthogonal with respect to this inner product if $k = \frac{4}{3}$.

14. The vectors are not orthogonal with respect to the Euclidean inner product since $\langle \mathbf{u},\mathbf{v}\rangle = (2)(0)+(-4)(3) = -12 \neq 0$. Using the weighted inner product instead yields $\langle \mathbf{u},\mathbf{v}\rangle = 2(2)(0)+k(-4)(3) = -12k$. This would equal 0 if $k = 0$, however, the corresponding formula $\langle \mathbf{u},\mathbf{v}\rangle = 2u_1v_1$ does not represent an inner product since it violates Axiom 4. Consequently, no k values can be found for which the vectors are orthogonal.

15. The orthogonality of the two vectors implies $(w_1)(1)(2)+(w_2)(2)(-4) = 0$. The weights must be positive numbers such that $w_1 = 4w_2$.

16. Begin by forming a matrix A whose rows are the given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$: $A = \begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix}$.

The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & \frac{34}{11} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{6}{11} \end{bmatrix}$ therefore the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is $x_1 = -\frac{34}{11}t, x_2 = 4t, x_3 = -\frac{6}{11}t, x_4 = t$, so that all solution vectors are scalar multiples of $(-34,44,-6,11)$. All solution vectors of the homogeneous system are orthogonal to every row vector of the coefficient matrix (see Example 6 in Section 6.2).
The magnitude of the vector $(-34,44,-6,11)$ is $\sqrt{(-34)^2 + 44^2 + (-6)^2 + 11^2} = 57$.
We conclude that the two unit vectors that are orthogonal to all three of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are $\frac{1}{57}(-34,44,-6,11) = (-\frac{34}{57}, \frac{44}{57}, -\frac{2}{19}, \frac{11}{57})$ and $-\frac{1}{57}(-34,44,-6,11) = (\frac{34}{57}, -\frac{44}{57}, \frac{2}{19}, -\frac{11}{57})$.

17. Orthogonality of $\mathbf{p}_1$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_1, \mathbf{p}_3 \rangle = (2)(1) + (k)(2) + (6)(3) = 2k + 20 = 0$ so $k = -10$.
Likewise, orthogonality of $\mathbf{p}_2$ and $\mathbf{p}_3$ implies $\langle \mathbf{p}_2, \mathbf{p}_3 \rangle = (l)(1) + (5)(2) + (3)(3) = l + 19$ so $l = -19$.
Substituting the values of k and l obtained above yields the polynomials $\mathbf{p}_1 = 2 - 10x + 6x^2$ and $\mathbf{p}_2 = -19 + 5x + 3x^2$ which are not orthogonal since $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = (2)(-19) + (-10)(5) + (6)(3) = -70 \neq 0$. We conclude that no scalars k and l exist that make the three vectors mutually orthogonal.

18. $\langle \mathbf{u}, \mathbf{v} \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) = \begin{bmatrix} 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0$

19. $\langle \mathbf{p}, \mathbf{q} \rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2) = (-2)(4) + (0)(0) + (2)(4) = 0$

20. For A to be a in the subspace of M_{22} spanned by U and V, there must exist scalars a and b such that

$$a \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

Equating corresponding entries in the second column on both sides yields $a = -1$ and $b = 1$.
However, for these values, the remaining entries on both sides do not equal (e.g., $(-1)(1) + (1)(4) \neq -1$). We conclude that A is not in the subspace of M_{22} spanned by U and V.

21. $|\langle \mathbf{u}, \mathbf{v} \rangle| = |2(1)(2) + 3(0)(1) + (3)(-1)| = |1| = 1$;
$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{2(1)(1) + 3(0)(0) + (3)(3)} = \sqrt{11}$; $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{2(2)(2) + 3(1)(1) + (-1)(-1)} = \sqrt{12}$;
since $\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{132} \geq 1 = |\langle \mathbf{u}, \mathbf{v} \rangle|$, we conclude that the Cauchy-Schwarz inequality holds

22. $|\langle U, V \rangle| = |(-1)(1) + (2)(0) + (6)(3) + (1)(3)| = |20| = 20$;
$\|U\| = \sqrt{\langle U, U \rangle} = \sqrt{(-1)(-1) + (2)(2) + (6)(6) + (1)(1)} = \sqrt{42}$;
$\|V\| = \sqrt{\langle V, V \rangle} = \sqrt{(1)(1) + (0)(0) + (3)(3) + (3)(3)} = \sqrt{19}$;
since $\|U\|\|V\| = \sqrt{798} \geq \sqrt{400} = 20 = |\langle U, V \rangle|$, we conclude that the Cauchy-Schwarz inequality holds

23. $|\langle \mathbf{p}, \mathbf{q} \rangle| = |(-1)(2) + (2)(0) + (1)(-4)| = |-6| = 6$;
$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{(-1)(-1) + (2)(2) + (1)(1)} = \sqrt{6}$; $\|\mathbf{q}\| = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle} =$

$\sqrt{(2)(2)+(0)(0)+(-4)(-4)}=\sqrt{20}$;
since $\|\mathbf{p}\|\|\mathbf{q}\|=\sqrt{120}\geq\sqrt{36}=6=|\langle\mathbf{p},\mathbf{q}\rangle|$, we conclude that the Cauchy-Schwarz inequality holds

24. $|\langle\mathbf{u},\mathbf{v}\rangle|=\left|\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right)\right|=\left|\begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}\right|=|3|=3$;
$\|\mathbf{u}\|=\sqrt{\langle\mathbf{u},\mathbf{u}\rangle}=\sqrt{\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)}=\sqrt{\begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}3\\2\end{bmatrix}}=\sqrt{13}$;
$\|\mathbf{v}\|=\sqrt{\langle\mathbf{v},\mathbf{v}\rangle}=\sqrt{\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right)\cdot\left(\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right)}=\sqrt{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\end{bmatrix}}=\sqrt{1}=1$;
since $\|\mathbf{u}\|\|\mathbf{v}\|=\sqrt{13}\geq\sqrt{9}=3=|\langle\mathbf{u},\mathbf{v}\rangle|$, we conclude that the Cauchy-Schwarz inequality holds

25. By inspection, $\langle\mathbf{u},\mathbf{w}_1\rangle=-2\neq 0$. Since $\mathbf{u}$ is not orthogonal to $\mathbf{w}_1$, it is not orthogonal to the subspace.

26. We have $\langle\mathbf{p},\mathbf{w}_1\rangle=(-1)(2)+(-1)(0)+(2)(-1)+(4)(1)=0$ and
$\langle\mathbf{p},\mathbf{w}_2\rangle=(-1)(0)+(-1)(4)+(2)(-2)+(4)(2)=0$ therefore for all scalars a and b
$\langle\mathbf{p},a\mathbf{w}_1+b\mathbf{w}_2\rangle=a\langle\mathbf{p},\mathbf{w}_1\rangle+b\langle\mathbf{p},\mathbf{w}_2\rangle=0$. We conclude that $\mathbf{p}$ is orthogonal to span$\{\mathbf{w}_1,\mathbf{w}_2\}$.

27. Begin by forming a matrix A whose rows are the given vectors:
$A=\begin{bmatrix}1&4&5&2\\2&1&3&0\\-1&3&2&2\end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix}1&0&1&-\frac{2}{7}\\0&1&1&\frac{4}{7}\\0&0&0&0\end{bmatrix}$. The general solution of the homogeneous system $A\mathbf{x}=\mathbf{0}$ is $x_1=-s+\frac{2}{7}t$, $x_2=-s-\frac{4}{7}t$, $x_3=s$, $x_4=t$ therefore $\mathbf{x}=s(-1,-1,1,0)+t(\frac{2}{7},-\frac{4}{7},0,1)$.
A basis for the orthogonal complement is formed by vectors $(-1,-1,1,0)$ and $(\frac{2}{7},-\frac{4}{7},0,1)$.

28. Begin by forming a matrix A whose rows are the given vectors:
$A=\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix}1&0&1&2&1\\0&1&1&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution of the homogeneous system $A\mathbf{x}=\mathbf{0}$ is $x_1=-r-2s-t$, $x_2=-r-s-2t$, $x_3=r$, $x_4=s$, $x_5=t$, therefore $\mathbf{x}=r(-1,-1,1,0,0)+s(-2,-1,0,1,0)+t(-1,-2,0,0,1)$.
A basis for the orthogonal complement is formed by vectors $(-1,-1,1,0,0)$, $(-2,-1,0,1,0)$, and $(-1,-2,0,0,1)$.

29. **(a)** Every vector in W has a form $(x,y)=(x,2x)$, i.e., $W=\text{span}\{(1,2)\}$. By inspection, all vectors in R^2 orthogonal to $(1,2)$ are scalar multiples of the vector $(2,-1)$. Eliminating t from $(x,y)=t(2,-1)=(2t,-t)$ we obtain $x=2(-y)$, i.e. $W^\perp$ can be represented using an equation $y=-\frac{1}{2}x$.

(An alternate method of solving this exercise is to follow the procedure of Example 6: letting $A=[1\quad 2]$, the general solution of $A\begin{bmatrix}x\\y\end{bmatrix}=0$ is $x=-2t$, $y=t$. Eliminating t yields $y=-\frac{1}{2}x$.)

(b) $W^\perp$ will have dimension 1. A normal to the plane is $\mathbf{u}=(1,-2,-3)$, so $W^\perp$ will consist of all scalar multiples of $\mathbf{u}$ or $t\mathbf{u}=(t,-2t,-3t)$ so parametric equations for $W^\perp$ are $x=t$, $y=-2t$, $z=-3t$.

31. **(a)** $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big]_0^1 = \frac{1}{4}$

(b) $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left(\int_0^1 x^2\,dx\right)^{\frac{1}{2}} = \left(\frac{x^3}{3}\Big]_0^1\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}$

$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left(\int_0^1 x^4\,dx\right)^{1/2} = \left(\frac{x^5}{5}\Big]_0^1\right)^{1/2} = \frac{1}{\sqrt{5}}$

32. **(a)** Using the results obtained in Exercise 31 we have $\cos\theta = \frac{\langle \mathbf{p},\mathbf{q} \rangle}{\|\mathbf{p}\|\|\mathbf{q}\|} = \frac{\frac{1}{4}}{\frac{1}{\sqrt{3}}\frac{1}{\sqrt{5}}} = \frac{\sqrt{15}}{4}$.

(b) $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \left(\int_0^1 (x - x^2)^2\,dx\right)^{1/2} = \left(\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right)\Big]_0^1\right)^{1/2} = \sqrt{\frac{1}{30}}$

33. **(a)** $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 (x^2 - x)(x + 1)\,dx = \int_{-1}^1 (x^3 - x)\,dx = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big]_{-1}^1 = 0$

(b) $\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{\frac{1}{2}} = \left(\int_{-1}^1 (x^2 - x)^2\,dx\right)^{\frac{1}{2}} = \left(\left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right)\Big]_{-1}^1\right)^{\frac{1}{2}} = \frac{4}{\sqrt{15}}$

$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left(\int_{-1}^1 (x + 1)^2\,dx\right)^{1/2} = \left(\left(\frac{x^3}{3} + x^2 + x\right)\Big]_{-1}^1\right)^{1/2} = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$

34. **(a)** Using the results obtained in Exercise 33 we have $\cos\theta = \frac{\langle \mathbf{p},\mathbf{q} \rangle}{\|\mathbf{p}\|\|\mathbf{q}\|} = 0$.

(b) $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\| = \left(\int_{-1}^1 (x^2 - 2x - 1)^2\,dx\right)^{1/2} = \left(\left(\frac{x^5}{5} - x^4 + \frac{2x^3}{3} + 2x^2 + x\right)\Big]_{-1}^1\right)^{1/2} = 2\sqrt{\frac{14}{15}}$

35. **(a)** $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 \left(\frac{1}{2} - x\right)dx = \left(\frac{1}{2}x - \frac{x^2}{2}\right)\Big]_0^1 = 0$

(b) $\|\mathbf{p} + \mathbf{q}\|^2 = \int_0^1 \left(\frac{3}{2} - x\right)^2 dx = \left(\frac{9}{4}x - \frac{3x^2}{2} + \frac{x^3}{3}\right)\Big]_0^1 = \frac{13}{12}$; $\|\mathbf{p}\|^2 = \int_0^1 (1)^2\,dx = x]_0^1 = 1$; $\|\mathbf{q}\|^2 = \int_0^1 \left(\frac{1}{2} - x\right)^2 dx = \left(\frac{x}{4} - \frac{x^2}{2} + \frac{x^3}{3}\right)\Big]_0^1 = \frac{1}{12}$; we conclude that $\|\mathbf{p} + \mathbf{q}\|^2 = \frac{13}{12} = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$

36. **(a)** $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 (x^3 - x)\,dx = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big]_{-1}^1 = 0$

(b) $\|\mathbf{p} + \mathbf{q}\|^2 = \int_{-1}^1 (x^2 + x - 1)^2\,dx = \left(\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} - x^2 + x\right)\Big]_{-1}^1 = \frac{26}{15}$;

$\|\mathbf{p}\|^2 = \int_{-1}^1 (x)^2\,dx = \frac{x^3}{3}\Big]_{-1}^1 = \frac{2}{3}$; $\|\mathbf{q}\|^2 = \int_{-1}^1 (x^2 - 1)^2\,dx = \left(\frac{x^5}{5} - \frac{2x^3}{3} + x\right)\Big]_{-1}^1 = \frac{16}{15}$;

we conclude that $\|\mathbf{p} + \mathbf{q}\|^2 = \frac{26}{15} = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$

37. $\|\mathbf{u} - \mathbf{v}\|^2 = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} - \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle$
$= 1 - 0 - 0 + 1 = 2$ therefore $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$.

38. Assuming $\langle \mathbf{w}, \mathbf{u}_1 \rangle = \langle \mathbf{w}, \mathbf{u}_2 \rangle = 0$, it follows that

$$\langle \mathbf{w}, k_1\mathbf{u}_1 + k_2\mathbf{u}_2 \rangle = \langle \mathbf{w}, k_1\mathbf{u}_1 \rangle + \langle \mathbf{w}, k_2\mathbf{u}_2 \rangle = k_1\langle \mathbf{w}, \mathbf{u}_1 \rangle + k_2\langle \mathbf{w}, \mathbf{u}_2 \rangle = (k_1)(0) + (k_2)(0) = 0$$

When V is R^3 with the Euclidean inner product this means that a vector perpendicular to two vectors is also perpendicular to the entire plane spanned by these vectors (assuming they are noncollinear).

39. Using the trigonometric identity $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$ we obtain
$\langle \mathbf{f}_k, \mathbf{f}_l \rangle = \frac{1}{2}\int_0^\pi \cos((k-l)x)\,dx + \frac{1}{2}\int_0^\pi \cos((k+l)x)\,dx$ where both $k-l$ and $k+l$ are nonzero integers.
Substituting $u = (k-l)x$ in the first integral, and $t = (k+l)x$ in the second integral yields
$\langle \mathbf{f}_k, \mathbf{f}_l \rangle = \frac{1}{2}\frac{\sin((k-l)x)}{k-l}\Big]_0^\pi + \frac{1}{2}\frac{\sin((k+l)x)}{k+l}\Big]_0^\pi = 0 + 0 = 0$ since $\sin(m\pi) = 0$ for any integer m.

40. We are looking for positive real numbers w_1 and w_2 for which the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2$ satisfies $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = 1$, and $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$ when applied to the given two vectors. These three equations yield a linear system

$$\begin{array}{rcrcl} -w_1 & + & 3w_2 & = & 0 \\ w_1 & + & 3w_2 & = & 1 \\ w_1 & + & 3w_2 & = & 1 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix}$. Since the system has only one solution $w_1 = \frac{1}{2}, w_2 = \frac{1}{6}$, we conclude that the weighted Euclidean inner product that satisfies the given conditions is $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}u_1v_1 + \frac{1}{6}u_2v_2$.

41. $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ contains all linear combinations $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r$ where $k_1, k_2, \ldots, k_r$ are arbitrary scalars. Let $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.

$\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{w}, k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r \rangle = k_1\langle \mathbf{w}, \mathbf{u}_1 \rangle + k_2\langle \mathbf{w}, \mathbf{u}_2 \rangle + \cdots + k_r\langle \mathbf{w}, \mathbf{u}_r \rangle = 0 + 0 + \cdots + 0 = 0$

Thus if $\mathbf{w}$ is orthogonal to each vector $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$, then $\mathbf{w}$ must be orthogonal to every vector in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$.

42. Let $\mathbf{w}$ be a vector orthogonal to all of the basis vectors. It must be possible to express $\mathbf{w}$ as a linear combination of the basis vectors: $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$. By the Axioms 1-3 of Definition 1 in Section 6.1, we can write

$$\begin{aligned} \langle \mathbf{w}, \mathbf{w} \rangle &= \langle \mathbf{w}, k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \rangle \\ &= k_1\langle \mathbf{w}, \mathbf{v}_1 \rangle + k_2\langle \mathbf{w}, \mathbf{v}_2 \rangle + \cdots + k_r\langle \mathbf{w}, \mathbf{v}_r \rangle \\ &= k_1 \cdot 0 + k_2 \cdot 0 + \cdots + k_r \cdot 0 \\ &= 0 \end{aligned}$$

By Axiom 4 of Definition 1 in Section 6.1, we must have $\mathbf{w} = \mathbf{0}$.

43. Suppose that $\mathbf{v}$ is orthogonal to every basis vector. Then, as in Exercise 41, $\mathbf{v}$ is orthogonal to the span of the set of basis vectors, which is all of W, hence $\mathbf{v}$ is in $W^\perp$. If $\mathbf{v}$ is not orthogonal to every basis vector, then $\mathbf{v}$ clearly cannot be in $W^\perp$. Thus $W^\perp$ consists of all vectors orthogonal to every basis vector.

44. This result can be established by induction.

46. Using the Euclidean inner product, apply the Cauchy-Schwarz inequality to $\mathbf{u} = (a, b)$ and $\mathbf{v} = (\cos\theta, \sin\theta)$.

47. Using the weighted Euclidean inner product of Formula (2) in Section 6.1, the desired inequality follows from the Cauchy-Schwarz inequality.

49. Using the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)dx$, part (a) follows from the Cauchy-Schwarz inequality and part (b) follows from the triangle inequality (part (a) of Theorem 6.2.2).

50. Squaring both sides of the Cauchy-Schwarz inequality and applying Definition 2 of Section 6.1 yields Formula (4).

51. **(a)** We are looking for all vectors $\mathbf{v} = (a, b)$ such that $\langle \mathbf{x}, \mathbf{v} \rangle = a + b$ is equal to $\langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} a+b \\ -a+b \end{bmatrix} = 2a + 2b$. The equation $a + b = 2a + 2b$ yields $a + b = 0$, i.e. $b = -a$. Vectors that satisfy $\langle \mathbf{x}, \mathbf{v} \rangle = \langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle$ must have a form $a(1, -1)$ where a is an arbitrary scalar.

(b) We are looking for all vectors $\mathrm{v} = (a, b)$ such that $\langle \mathbf{x}, \mathbf{v} \rangle = 2a + 3b$ is equal to $\langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle = \langle \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} a+b \\ -a+b \end{bmatrix} \rangle = 4a + 4b$. The equation $2a + 3b = 4a + 4b$ yields $2a + b = 0$, i.e. $b = -2a$. Vectors that satisfy $\langle \mathbf{x}, \mathbf{v} \rangle = \langle T_A(\mathbf{x}), T_A(\mathbf{v}) \rangle$ must have a form $a(1, -2)$ where a is an arbitrary scalar.

52. **(a)** We are looking for all vectors $\mathbf{q} = a + bx + cx^2$ such that $\langle \mathbf{p}, \mathbf{q} \rangle = a + b$ is equal to $\langle T(\mathbf{p}), T(\mathbf{q}) \rangle = \langle 3, 3a - cx^2 \rangle = 9a$. The equation $a + b = 9a$ yields $b = 8a$. Vectors that satisfy $\langle \mathbf{p}, \mathbf{q} \rangle = \langle T(\mathbf{p}), T(\mathbf{q}) \rangle$ must have a form $a + 8ax + cx^2$ where a and c are arbitrary scalars.

(b) We are looking for all vectors $\mathbf{q} = a + bx + cx^2$ such that
$\langle \mathbf{p}, \mathbf{q} \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) = (0)(a - b + c) + (1)(a) + (2)(a + b + c) =$
$3a + 2b + 2c$ is equal to
$\langle T(\mathbf{p}), T(\mathbf{q}) \rangle = \langle 3, 3a - cx^2 \rangle = (3)(3a - c) + (3)(3a) + (3)(3a - c) = 27a - 6c$.
The equation $3a + 2b + 2c = 27a - 6c$ yields $b = 12a - 4c$. Vectors that satisfy $\langle \mathbf{p}, \mathbf{q} \rangle = \langle T(\mathbf{p}), T(\mathbf{q}) \rangle$ must have a form $a + (12a - 4c)x + cx^2$ where a and c are arbitrary scalars.

True-False Exercises

(a) False. If $\mathbf{u}$ is orthogonal to every vector of a subspace W, then $\mathbf{u}$ is in $W^\perp$.

(b) True. $W \cap W^\perp = \{\mathbf{0}\}$.

(c) True. For any vector $\mathbf{w}$ in W, $\langle \mathbf{u} + \mathbf{v}, \ \mathbf{w} \rangle = \langle \mathbf{u}, \ \mathbf{w} \rangle + \langle \mathbf{v}, \ \mathbf{w} \rangle = 0$, so $\mathbf{u} + \mathbf{v}$ is in $W^\perp$.

(d) True. For any vector $\mathbf{w}$ in W, $\langle k\mathbf{u}, \ \mathbf{w} \rangle = k\langle \mathbf{u}, \ \mathbf{w} \rangle = k(0) = 0$, so $k\mathbf{u}$ is in $W^\perp$.

(e) False. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $|\langle \mathbf{u}, \ \mathbf{v} \rangle| = |0| = 0$.

(f) False. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ thus $\|\mathbf{u} + \mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2} \neq \|\mathbf{u}\| + \|\mathbf{v}\|$

6.3 Gram-Schmidt Process; QR-Decomposition

1. **(a)** $\langle (0,1),\ (2,0)\rangle = 0+0=0;$
$\|(0,1)\| = 1;\ \ \|(2,0)\| = 2 \neq 1;$
The set is orthogonal, but is not orthonormal.

(b) $\left\langle \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2}+\frac{1}{2}=0;$
$\left\|\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1;\ \ \left\|\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$
The set is orthogonal and orthonormal.

(c) $\left\langle \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2}-\frac{1}{2}=-1\neq 0;$
The set is not orthogonal (therefore, it is not orthonormal either).

(d) $\langle (0,0),\ (0,1)\rangle = 0+0=0;$
$\|(0,0)\| = 0 \neq 1;\ \ \|(0,1)\| = 1;$
The set is orthogonal, but is not orthonormal.

2. **(a)** $\left\langle \left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\right\rangle = \frac{1}{\sqrt{6}}+0-\frac{1}{\sqrt{6}}=0;\ \ \left\langle \left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right),\ \left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{2}+0+\frac{1}{2}=0;$
$\left\langle \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)\right\rangle = -\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}}\neq 0;$
The set is not orthogonal (therefore, it is not orthonormal either).

(b) $\left\langle \left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right),\ \left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)\right\rangle = \frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0;\ \ \left\langle \left(\frac{2}{3},-\frac{2}{3},\ \frac{1}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\rangle = \frac{2}{9}-\frac{4}{9}+\frac{2}{9}=0;$
$\left\langle \left(\frac{2}{3},\ \frac{1}{3},-\frac{2}{3}\right),\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\rangle = \frac{2}{9}+\frac{2}{9}-\frac{4}{9}=0;$
$\left\|\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)\right\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1;\ \ \left\|\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)\right\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1;\ \ \left\|\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)\right\| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1;$
The set is orthogonal and orthonormal.

(c) $\left\langle (1,0,0),\ \left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\rangle = 0+0+0=0;\ \ \langle (1,0,0),\ (0,0,1)\rangle = 0+0+0=0;$
$\left\langle \left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\ (0,0,1)\right\rangle = 0+0+\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\neq 0;$
The set is not orthogonal (therefore, it is not orthonormal either).

(d) $\left\langle \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right),\ \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right\rangle = \frac{1}{\sqrt{12}}-\frac{1}{\sqrt{12}}+0=0;$
$\left\|\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\right\| = \sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}} = 1\,;\ \ \left\|\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}+0} = 1$
The set is orthogonal and orthonormal.

3. **(a)** $\langle p_1(x),\ p_2(x)\rangle = \frac{2}{3}\left(\frac{2}{3}\right)-\frac{2}{3}\left(\frac{1}{3}\right)+\frac{1}{3}\left(-\frac{2}{3}\right) = \frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0;$
$\langle p_1(x),\ p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right)-\frac{2}{3}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{9}-\frac{4}{9}+\frac{2}{9}=0;$

$\langle p_2(x),\ p_3(x)\rangle = \frac{2}{3}\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) - \frac{2}{3}\left(\frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0;$

The set is orthogonal.

(b) $\langle p_1(x),\ p_2(x)\rangle = 1(0) + 0\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = 0;\quad \langle p_1(x),\ p_3(x)\rangle = 1(0) + 0(0) + 0(1) = 0;$

$\langle p_2(x),\ p_3(x)\rangle = 0(0) + \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}} \neq 0;$

The set is not orthogonal.

4. **(a)** Denoting the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$, $C = \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, and $D = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ we calculate

$\langle A, B\rangle = (1)(0) + (0)\left(\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) + (0)\left(-\frac{2}{3}\right) = 0;$

$\langle A, C\rangle = (1)(0) + (0)\left(\frac{2}{3}\right) + (0)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) = 0;$

$\langle A, D\rangle = (1)(0) + (0)\left(\frac{1}{3}\right) + (0)\left(\frac{2}{3}\right) + (0)\left(\frac{2}{3}\right) = 0;$

$\langle B, \mathrm{C}\rangle = (0)(0) + \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) = 0;$

$\langle B, D\rangle = (0)(0) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) = 0;$

$\langle C, D\rangle = (0)(0) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 0;$

The set is orthogonal.

(b) Denoting the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ we calculate

$\langle A, B\rangle = (1)(0) + (0)(1) + (0)(0) + (0)(0) = 0;$

$\langle A, C\rangle = (1)(0) + (0)(0) + (0)(1) + (0)(1) = 0;$

$\langle A, D\rangle = (1)(0) + (0)(0) + (0)(1) + (0)(-1) = 0;$

$\langle B, \mathrm{C}\rangle = (0)(0) + (1)(0) + (0)(1) + (0)(1) = 0;$

$\langle B, D\rangle = (0)(0) + (1)(0) + (0)(1) + (0)(-1) = 0;$

$\langle C, D\rangle = (0)(0) + (0)(0) + (1)(1) + (1)(-1) = 0;$

The set is orthogonal.

5. Let us denote the column vectors $\mathbf{u}_1 = (1, 0, -1)$, $\mathbf{u}_2 = (2, 0, 2)$, and $\mathbf{u}_3 = (0, 5, 0)$. These vectors are orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2\rangle = 2 + 0 - 2 = 0$, $\langle \mathbf{u}_1, \mathbf{u}_3\rangle = 0 + 0 + 0 = 0$, and $\langle \mathbf{u_2}, \mathbf{u_3}\rangle = 0 + 0 + 0 = 0$. It follows from Theorem 6.3.1 that the column vectors are linearly independent, therefore they form an orthogonal basis for the column space of A. We proceed to normalize each column vector: $\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{1+0+1}}(1, 0, -1) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$; $\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{4+0+4}}(2, 0, 2) = \left(\frac{2}{2\sqrt{2}}, 0,\ \frac{2}{2\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$; $\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{0+25+0}}(0, 5, 0) = (0,\ 1,\ 0)$. A resulting orthonormal basis for the column space is $\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,\ 1,\ 0)\}$.

6. Let us denote the column vectors $\mathbf{u}_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $\mathbf{u}_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$, and $\mathbf{u}_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$. These vectors are orthogonal since $\langle \mathbf{u}_1, \mathbf{u}_2\rangle = -\frac{1}{10} + \frac{1}{10} + 0 = 0$, $\langle \mathbf{u}_1, \mathbf{u}_3\rangle = \frac{1}{15} + \frac{1}{15} - \frac{2}{15} = 0$, and $\langle \mathbf{u_2}, \mathbf{u_3}\rangle = -\frac{1}{6} + \frac{1}{6} + 0 = 0$. It follows from Theorem 6.3.1 that the column vectors are linearly independent, therefore they form an orthogonal basis for the column space of A. We proceed to normalize each column vector:

$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{\frac{1}{25}+\frac{1}{25}+\frac{1}{25}}}\left(\frac{1}{5},\frac{1}{5},\frac{1}{5}\right) = \frac{5}{\sqrt{3}}\left(\frac{1}{5},\frac{1}{5},\frac{1}{5}\right) = \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right);$

$\frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+0}}\left(-\frac{1}{2},\frac{1}{2},0\right) = \sqrt{2}\left(-\frac{1}{2},\frac{1}{2},0\right) = \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right);$

$\frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}}}\left(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right) = \frac{3}{\sqrt{6}}\left(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right) = \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right).$

A resulting orthonormal basis for the column space is $\left\{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right),\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)\right\}$.

7. $\langle \mathbf{v}_1,\mathbf{v}_2\rangle = -\frac{12}{25}+\frac{12}{25}+0 = 0;\ \ \langle \mathbf{v}_1,\mathbf{v}_3\rangle = 0+0+0 = 0;\ \ \langle \mathbf{v}_2,\mathbf{v}_3\rangle = 0+0+0 = 0$;

$\|\mathbf{v}_1\| = \sqrt{\frac{9}{25}+\frac{16}{25}+0} = 1;\ \ \|\mathbf{v}_2\| = \sqrt{\frac{16}{25}+\frac{9}{25}+0} = 1;\ \ \|\mathbf{v}_3\| = \sqrt{0+0+1} = 1;$

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^3) = 3$, this set forms a basis for R^3 by Theorem 4.5.4. This basis is orthonormal, so by Theorem 6.3.2(b),

$$\begin{aligned}\mathbf{u} &= \langle \mathbf{u},\mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u},\mathbf{v}_2\rangle\mathbf{v}_2 + \langle \mathbf{u},\mathbf{v}_3\rangle\mathbf{v}_3\\ &= \left(-\frac{3}{5}-\frac{8}{5}+0\right)\mathbf{v}_1 + \left(\frac{4}{5}-\frac{6}{5}+0\right)\mathbf{v}_2 + (0+0+2)\mathbf{v}_3\\ &= -\frac{11}{5}\mathbf{v}_1 - \frac{2}{5}\mathbf{v}_2 + 2\mathbf{v}_3\end{aligned}$$

8. It was shown in Exercise 7 that the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ form an orthonormal basis for R^3. By Theorem 6.3.2(b),

$$\begin{aligned}\mathbf{u} &= \langle \mathbf{u},\mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u},\mathbf{v}_2\rangle\mathbf{v}_2 + \langle \mathbf{u},\mathbf{v}_3\rangle\mathbf{v}_3\\ &= \left(-\frac{9}{5}-\frac{28}{5}+0\right)\mathbf{v}_1 + \left(\frac{12}{5}-\frac{21}{5}+0\right)\mathbf{v}_2 + (0+0+4)\mathbf{v}_3\\ &= -\frac{37}{5}\mathbf{v}_1 - \frac{9}{5}\mathbf{v}_2 + 4\mathbf{v}_3\end{aligned}$$

9. $\langle \mathbf{v}_1,\mathbf{v}_2\rangle = 4-2-2 = 0;\ \ \langle \mathbf{v}_1,\mathbf{v}_3\rangle = 2-4+2 = 0;\ \ \langle \mathbf{v}_2,\mathbf{v}_3\rangle = 2+2-4 = 0$;

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^3) = 3$, this set forms a basis for R^3 by Theorem 4.5.4. By Theorem 6.3.2(a),

$$\begin{aligned}\mathbf{u} &= \frac{\langle \mathbf{u},\mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u},\mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u},\mathbf{v}_3\rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3\\ &= \frac{-2+0+2}{4+4+1}\mathbf{v}_1 + \frac{-2+0-4}{4+1+4}\mathbf{v}_2 + \frac{-1+0+4}{1+4+4}\mathbf{v}_3\\ &= 0\mathbf{v}_1 - \frac{2}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3\end{aligned}$$

10. $\langle \mathbf{v}_1,\mathbf{v}_2\rangle = -2-2+6-2 = 0 \quad \langle \mathbf{v}_1,\mathbf{v}_3\rangle = 1-2+0+1 = 0 \quad \langle \mathbf{v}_1,\mathbf{v}_4\rangle = 1+0+0-1 = 0$

$\langle \mathbf{v}_2,\mathbf{v}_3\rangle = -2+4+0-2 = 0 \quad \langle \mathbf{v}_2,\mathbf{v}_4\rangle = -2+0+0+2 = 0 \quad \langle \mathbf{v}_3,\mathbf{v}_4\rangle = 1+0+0-1 = 0$

Since this is an orthogonal set of nonzero vectors, it follows from Theorem 6.3.1 that the set is linearly independent. Because the number of vectors in the set matches $\dim(R^4) = 4$, this set forms a basis for R^4 by Theorem 4.5.4. We use Theorem 6.3.2(a):

$$\begin{aligned}\mathbf{u} &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 + \frac{\langle \mathbf{u}, \mathbf{v}_4 \rangle}{\|\mathbf{v}_4\|^2}\mathbf{v}_4 \\ &= \frac{1-1+2-1}{1+1+4+1}\mathbf{v}_1 + \frac{-2+2+3+2}{4+4+9+4}\mathbf{v}_2 + \frac{1+2+0-1}{1+4+0+1}\mathbf{v}_3 + \frac{1+0+0+1}{1+0+0+1}\mathbf{v}_4 \\ &= \frac{1}{7}\mathbf{v}_1 + \frac{5}{21}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3 + 1\mathbf{v}_4\end{aligned}$$

11. $(\mathbf{u})_S = \left(-\frac{11}{5}, -\frac{2}{5}, 2\right)$

12. $(\mathbf{u})_S = \left(-\frac{37}{5}, -\frac{9}{5}, 4\right)$

13. $(\mathbf{u})_S = \left(0, -\frac{2}{3}, \frac{1}{3}\right)$

14. $(\mathbf{u})_S = \left(\frac{1}{7}, \frac{5}{21}, \frac{1}{3}, 1\right)$

15. **(a)** $\|\mathbf{v}\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, so $\mathbf{v}$ forms an orthonormal basis for the line $W = \text{span}\{\mathbf{v}\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v} = \left(-\frac{3}{5} + \frac{24}{5}\right)\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{21}{5}\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{63}{25}, \frac{84}{25}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (-1, 6) - \left(\frac{63}{25}, \frac{84}{25}\right) = \left(-\frac{88}{25}, \frac{66}{25}\right)$;

$\mathbf{w}_2$ is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = -\frac{264}{125} + \frac{264}{125} = 0$

16. **(a)** $\|\mathbf{v}\| = \sqrt{\frac{25}{169} + \frac{144}{169}} = 1$, so $\mathbf{v}$ forms an orthonormal basis for the line $W = \text{span}\{\mathbf{v}\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{v} = \left(\frac{10}{13} + \frac{36}{13}\right)\left(\frac{5}{13}, \frac{12}{13}\right) = \frac{46}{13}\left(\frac{5}{13}, \frac{12}{13}\right) = \left(\frac{230}{169}, \frac{552}{169}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (2, 3) - \left(\frac{230}{169}, \frac{552}{169}\right) = \left(\frac{108}{169}, -\frac{45}{169}\right)$;

$\mathbf{w}_2$ is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = \frac{540}{2197} - \frac{540}{2197} = 0$

17. **(a)** $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{2+3}{1+1}(1, 1) = \frac{5}{2}(1, 1) = \left(\frac{5}{2}, \frac{5}{2}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (2, 3) - \left(\frac{5}{2}, \frac{5}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$;

$\mathbf{w}_2$ is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = -\frac{1}{2} + \frac{1}{2} = 0$

18. **(a)** $\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{9-4}{9+16}(3, 4) = \frac{5}{25}(3, 4) = \left(\frac{3}{5}, \frac{4}{5}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3, -1) - \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{12}{5}, -\frac{9}{5}\right)$;

$\mathbf{w}_2$ is orthogonal to the line since $\langle \mathbf{w}_2, \mathbf{v} \rangle = \frac{36}{5} - \frac{36}{5} = 0$

19. **(a)** $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle \mathbf{v}_2 = 2\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right) + 4\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{10}{3}, \frac{8}{3}, \frac{4}{3}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (4, 2, 1) - \left(\frac{10}{3}, \frac{8}{3}, \frac{4}{3}\right) = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$;
$\mathbf{w}_2$ is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1\rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2\rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$.

20. **(a)** $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle \mathbf{v}_2 = -\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) + \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
$= \left(-\frac{2}{6}, -\frac{2}{6}, \frac{4}{6}\right) + \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right) + \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = (1,1,2)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3, -1, 2) - (1,1,2) = (2, -2, 0)$;
$\mathbf{w}_2$ is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1\rangle = \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} + 0 = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2\rangle = \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} = 0$.

21. **(a)** $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{4}{6}(1, -2, 1) + \frac{2}{5}(2, 1, 0)$
$= \left(\frac{2}{3}, -\frac{4}{3}, \frac{2}{3}\right) + \left(\frac{4}{5}, \frac{2}{5}, 0\right) = \left(\frac{22}{15}, -\frac{14}{15}, \frac{2}{3}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (1, 0, 3) - \left(\frac{22}{15}, -\frac{14}{15}, \frac{2}{3}\right) = \left(-\frac{7}{15}, \frac{14}{15}, \frac{7}{3}\right)$;
$\mathbf{w}_2$ is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1\rangle = -\frac{7}{15} - \frac{28}{15} + \frac{7}{3} = \frac{-7-28+35}{15} = 0$ and
$\langle \mathbf{w}_2, \mathbf{v}_2\rangle = -\frac{14}{15} + \frac{14}{15} + 0 = 0$.

22. **(a)** $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for the plane $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\mathbf{w}_1 = \text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{7}{14}(3, 1, 2) + \frac{1}{3}(-1, 1, 1) = \left(\frac{3}{2}, \frac{1}{2}, 1\right) + \left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
$= \left(\frac{7}{6}, \frac{5}{6}, \frac{4}{3}\right)$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (1, 0, 2) - \left(\frac{7}{6}, \frac{5}{6}, \frac{4}{3}\right) = \left(-\frac{1}{6}, -\frac{5}{6}, \frac{2}{3}\right)$;
$\mathbf{w}_2$ is orthogonal to the plane since $\langle \mathbf{w}_2, \mathbf{v}_1\rangle = -\frac{3}{6} - \frac{5}{6} + \frac{4}{3} = 0$ and $\langle \mathbf{w}_2, \mathbf{v}_2\rangle = \frac{1}{6} - \frac{5}{6} + \frac{2}{3} = 0$.

23. $\text{proj}_W \mathbf{b} = \frac{\langle \mathbf{b}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{b}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{1+2+0-2}{1+1+1+1}(1,1,1,1) + \frac{1+2+0+2}{1+1+1+1}(1,1,-1,-1)$
$= \frac{1}{4}(1,1,1,1) + \frac{5}{4}(1,1,-1,-1) = \left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)$

24. $\text{proj}_W \mathbf{b} = \frac{\langle \mathbf{b}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{b}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{0+2+0+2}{0+1+16+1}(0,1,-4,-1) + \frac{3+10+0-2}{9+25+1+1}(3,5,1,1)$
$= \frac{2}{9}(0,1,-4,-1) + \frac{11}{36}(3,5,1,1) = \left(\frac{11}{12}, \frac{7}{4}, -\frac{7}{12}, \frac{1}{12}\right)$

25. $\text{proj}_W \mathbf{b} = \langle \mathbf{b}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{b}, \mathbf{v}_2\rangle \mathbf{v}_2 + \langle \mathbf{b}, \mathbf{v}_3\rangle \mathbf{v}_3$
$= \left(0 + \frac{2}{\sqrt{18}} + 0 + \frac{1}{\sqrt{18}}\right)\mathbf{v}_1 + \left(\frac{1}{2} + \frac{10}{6} + 0 - \frac{1}{6}\right)\mathbf{v}_2 + \left(\frac{1}{\sqrt{18}} + 0 + 0 + \frac{4}{\sqrt{18}}\right)\mathbf{v}_3 = \frac{3}{\sqrt{18}}\mathbf{v}_1 + 2\mathbf{v}_2 + \frac{5}{\sqrt{18}}\mathbf{v}_3$
$= \left(0, \frac{3}{18}, -\frac{12}{18}, -\frac{3}{18}\right) + \left(1, \frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right) + \left(\frac{5}{18}, 0, \frac{5}{18}, -\frac{20}{18}\right) = \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)$

26. $\text{proj}_W\mathbf{b} = \langle \mathbf{b}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{b}, \mathbf{v}_2\rangle\mathbf{v}_2 + \langle \mathbf{b}, \mathbf{v}_3\rangle\mathbf{v}_3$

$= \left(\frac{1}{2} + 1 + 0 - \frac{1}{2}\right)\mathbf{v}_1 + \left(\frac{1}{2} + 1 + 0 + \frac{1}{2}\right)\mathbf{v}_2 + \left(\frac{1}{2} - 1 + 0 + \frac{1}{2}\right)\mathbf{v}_3 = 1\mathbf{v}_1 + 2\mathbf{v}_2 + 0\mathbf{v}_3$

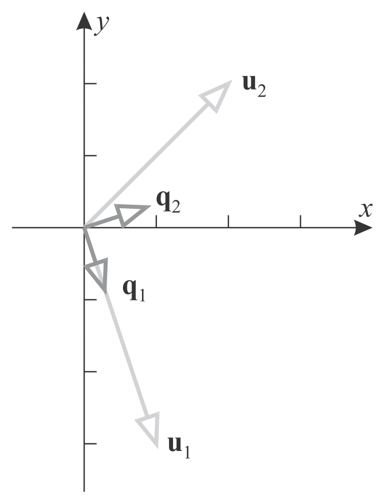
$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + (1,\ 1, -1, -1) = \left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

27. $\mathbf{v}_1 = \mathbf{u}_1 = (1, -3)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (2, 2) - \frac{2-6}{10}(1, -3) = (2, 2) - \left(-\frac{2}{5}, \frac{6}{5}\right) = \left(\frac{12}{5}, \frac{4}{5}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{10}}(1, -3) = \left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$ and

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{\frac{144}{25} + \frac{16}{25}}}\left(\frac{12}{5}, \frac{4}{5}\right) = \frac{1}{\sqrt{\frac{160}{25}}}\left(\frac{12}{5}, \frac{4}{5}\right) = \frac{5}{4\sqrt{10}}\left(\frac{12}{5}, \frac{4}{5}\right) = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right).$

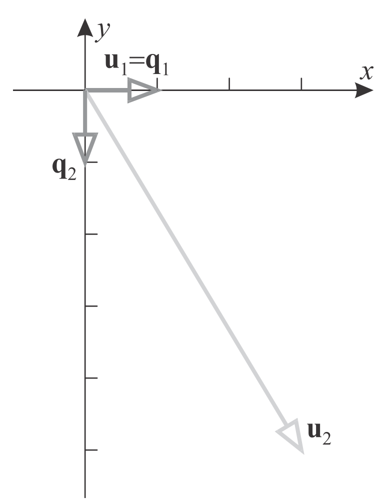


28. $\mathbf{v}_1 = \mathbf{u}_1 = (1,0)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (3, -5) - \frac{3+0}{1}(1, 0) = (3, -5) - (3, 0) = (0, -5)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{1}}(1,0) = (1,0)$ and

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{25}}(0, -5) = (0, -1).$



29. $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (-1,1,0) - \frac{-1+1+0}{1+1+1}(1,1,1) = (-1,1,0) - 0(1,1,1) = (-1,1,0)$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,2,1) - \frac{1+2+1}{1+1+1}(1,1,1) - \frac{-1+2+0}{1+1+0}(-1,1,0)$

$= (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0) = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{3}}(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$,

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2}}(-1,1,0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{1/\sqrt{6}}\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$.

30. $\mathbf{v}_1 = \mathbf{u}_1 = (1,0,0)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (3,7,-2) - \frac{3+0+0}{1+0+0}(1,0,0) = (3,7,-2) - 3(1,0,0) = (0,7,-2)$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (0,4,1) - \frac{0+0+0}{1+0+0}(1,0,0) - \frac{0+28-2}{0+49+4}(0,7,-2)$

$= (0,4,1) - 0(1,0,0) - \frac{26}{53}(0,7,-2) = \left(0, \frac{30}{53}, \frac{105}{53}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{1}(1,0,0) = (1,0,0)$,

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{53}}(0,7,-2) = \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right)$, and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{15/\sqrt{53}}\left(0, \frac{30}{53}, \frac{105}{53}\right) = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$.

31. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$\mathbf{v}_1 = \mathbf{u}_1 = (0,2,1,0)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,\ -1,\ 0,\ 0) - \frac{0-2+0+0}{5}(0,\ 2,\ 1,0) = (1,\ -1,\ 0,\ 0) + \left(0,\ \frac{4}{5},\ \frac{2}{5},\ 0\right)$

$= \left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,\ 2,\ 0,\ -1) - \frac{0+4+0+0}{5}(0,\ 2,\ 1,\ 0) - \frac{1-\frac{2}{5}+0+0}{\frac{6}{5}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right)$

$= (1,\ 2,\ 0,\ -1) - \left(0,\ \frac{8}{5},\ \frac{4}{5},\ 0\right) - \left(\frac{1}{2},\ -\frac{1}{10},\ \frac{1}{5},\ 0\right) = \left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$

$$\mathbf{v}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$$

$= (1,\ 0,\ 0,\ 1) - \frac{0+0+0+0}{5}\mathrm{v}_1 - \frac{1+0+0+0}{\frac{6}{5}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$

$= (1,\ 0,\ 0,\ 1) - \frac{0+0+0+0}{5}\mathrm{v}_1 - \frac{1+0+0+0}{\frac{6}{5}}\left(1,\ -\frac{1}{5},\ \frac{2}{5},\ 0\right) - \frac{\frac{1}{2}+0+0-1}{\frac{5}{2}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right) = \left(\frac{4}{15},\ \frac{4}{15},\ -\frac{8}{15},\ \frac{4}{5}\right)$

An orthonormal basis is formed by the vectors $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(0,2,1,0)}{\sqrt{5}} = \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)$,

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(1,-\frac{1}{5},\frac{2}{5},0\right)}{\frac{\sqrt{30}}{5}} = \left(\frac{5}{\sqrt{30}},\ -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$, $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{1}{2},\frac{1}{2},-1,-1\right)}{\frac{\sqrt{10}}{2}} = \left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$, $\mathbf{q}_4 =$
$\frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)}{\frac{4}{\sqrt{15}}} = \left(\frac{1}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ -\frac{2}{\sqrt{15}},\ \frac{3}{\sqrt{15}}\right)$.

32. We begin by forming a matrix whose columns are the three given vectors: $\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. The reduced row echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. By Theorem 4.7.6(b), vectors $\mathbf{u}_1 = (0,1,2)$ and $\mathbf{u}_2 = (-1,0,1)$ form a basis for the span of the three given vectors (the third vector can be expressed as a linear combination of the first two).
We proceed to use the Gram-Schmidt process to transform the basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ to an orthonormal basis.
$\mathbf{v}_1 = \mathbf{u}_1 = (0,1,2)$
$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (-1,0,1) - \frac{0+0+2}{0+1+4}(0,1,2) = (-1,0,1) - \frac{2}{5}(0,1,2) = (-1, -\frac{2}{5}, \frac{1}{5})$
An orthonormal basis is formed by the vectors
$\mathbf{q_1} = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{5}}(0,1,2) = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6/5}}\left(-1, -\frac{2}{5}, \frac{1}{5}\right) = \left(-\frac{5}{\sqrt{30}}, -\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)$.

33. From Exercise 23, $\mathbf{w}_1 = \text{proj}_W \mathbf{b} = \left(\frac{3}{2}, \frac{3}{2}, -1, -1\right)$, so $\mathbf{w}_2 = \mathbf{b} - \text{proj}_W \mathbf{b} = \left(-\frac{1}{2}, \frac{1}{2}, 1, -1\right)$.

34. From Exercise 25, $\mathbf{w}_1 = \text{proj}_W \mathbf{b} = \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)$, so $\mathbf{w}_2 = \mathbf{b} - \mathbf{w}_1 = \left(-\frac{5}{18}, \frac{1}{6}, \frac{1}{18}, -\frac{1}{18}\right)$.

35. Let W be the plane spanned by the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.
$\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$
$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (2, 0, -1) - \frac{2+0-1}{1+1+1}(1, 1, 1) = (2, 0, -1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$
$\mathbf{w}_1 = \text{proj}_W \mathbf{w} = \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{1+2+3}{3}\mathbf{v}_1 + \frac{\frac{5}{3} - \frac{2}{3} - \frac{12}{3}}{\frac{42}{9}}\mathbf{v}_2 = 2(1,\ 1,\ 1) - \frac{9}{14}\left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$
$= (2,\ 2,\ 2) - \left(\frac{15}{14},\ -\frac{3}{14},\ -\frac{6}{7}\right) = \left(\frac{13}{14},\ \frac{31}{14},\ \frac{20}{7}\right)$
$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = (1,2,3) - \left(\frac{13}{14},\ \frac{31}{14},\ \frac{20}{7}\right) = \left(\frac{1}{14},\ -\frac{3}{14},\ \frac{1}{7}\right)$

36. The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are not orthogonal, therefore we begin by using the Gram-Schmidt process to transform the basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ to an orthonormal basis.
$\mathbf{v}_1 = \mathbf{u}_1 = (-1,0,1,2)$
$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (0,1,0,1) - \frac{0+0+0+2}{1+0+1+4}(-1,0,1,2) = (0,1,0,1) - \frac{1}{3}(-1,0,1,2) = \left(\frac{1}{3}, 1, -\frac{1}{3}, \frac{1}{3}\right)$
An orthonormal basis is formed by the vectors
$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{6}}(-1,0,1,2) = \left(-\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{4/3}}\left(\frac{1}{3}, 1, -\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$.
$\mathbf{w}_1 = \text{proj}_W \mathbf{w} = \langle \mathbf{w}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{w}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \frac{1+0+6+0}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) + \frac{-1+6-6+0}{2\sqrt{3}}\left(\frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, -\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$
$= \left(-\frac{7}{6}, 0, \frac{7}{6}, \frac{7}{3}\right) + \left(-\frac{1}{12}, -\frac{1}{4}, \frac{1}{12}, -\frac{1}{12}\right) = \left(-\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4}\right)$
$\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1 = (-1,2,6,0) - \left(-\frac{5}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{9}{4}\right) = \left(\frac{1}{4}, \frac{9}{4}, \frac{19}{4}, -\frac{9}{4}\right)$.

37. First, transform the given basis into an orthogonal basis $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$.
$\mathbf{v}_1 = \mathbf{u}_1 = (1,\ 1,\ 1)$
$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{1^2 + 2(1)^2 + 3(1)^2} = \sqrt{1+2+3} = \sqrt{6}$
$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,\ 1,\ 0) - \frac{1(1)+2(1)(1)+3(0)(1)}{6}(1,\ 1,\ 1) = (1,\ 1,\ 0) - \frac{1}{2}(1,\ 1,\ 1) = \left(\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$

$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \sqrt{\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right)^2} = \sqrt{6\left(\frac{1}{4}\right)} = \frac{\sqrt{6}}{2}$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,\ 0,\ 0) - \frac{1+0+0}{6}(1,\ 1,\ 1) - \frac{\frac{1}{2}+0+0}{\frac{6}{4}}\left(\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$

$= (1,\ 0,\ 0) - \left(\frac{1}{6},\ \frac{1}{6},\ \frac{1}{6}\right) - \left(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{6}\right) = \left(\frac{2}{3},\ -\frac{1}{3},\ 0\right)$

$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle} = \sqrt{\left(\frac{2}{3}\right)^2 + 2\left(-\frac{1}{3}\right)^2 + 3(0)^2} = \sqrt{\frac{4}{9} + \frac{2}{9}} = \frac{\sqrt{6}}{3}$

The orthonormal basis is $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,1,1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)}{\frac{\sqrt{6}}{2}} = \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$,

and $\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\left(\frac{2}{3},-\frac{1}{3},0\right)}{\frac{\sqrt{6}}{3}} = \left(\frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right)$.

38. Let us denote $\mathbf{w}_1 = (1,0)$ and $\mathbf{w}_2 = (0,1)$. The set is orthogonal because $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 4(1)(0) + (0)(1) = 0$.

$\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{\mathbf{w}_1}{\sqrt{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}} = \frac{1}{\sqrt{4(1)(1)+(0)(0)}}(1,0) = \left(\frac{1}{2}, 0\right)$; $\quad \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{\mathbf{w}_2}{\sqrt{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}} = \frac{1}{\sqrt{4(0)(0)+(1)(1)}}(0,1) = (0,1)$

By normalizing the given orthogonal set, we obtained an orthonormal set $\left\{\left(\frac{1}{2}, 0\right), (0,1)\right\}$.

39. For example, $\mathbf{x} = \left(\frac{1}{\sqrt{3}}, 0\right)$ and $\mathbf{y} = \left(0, \frac{1}{\sqrt{2}}\right)$.

40. The line through the origin making $\frac{\pi}{6}$ angle with the x-axis is a subspace of R^2 spanned by the vector $\mathbf{u} = \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Since this vector has norm 1, the orthogonal projection of $\mathbf{x} = (1,5)$ onto $W = \text{span}\{\mathbf{u}\}$ is $\text{proj}_W \mathbf{x} = \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} = \left(\frac{\sqrt{3}}{2} + \frac{5}{2}\right)\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\frac{3+5\sqrt{3}}{4}, \frac{5+\sqrt{3}}{4}\right)$.

41. **(a)** By inspection, $\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_2' = \mathbf{v}_1 - 2\mathbf{v}_2$ so $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are in W. The dimension of W is 2 since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for W. By Theorem 6.3.1, $\{\mathbf{v}_1', \mathbf{v}_2'\}$ is linearly independent, so by Theorem 4.5.4 it is a basis for W, hence it spans W.

(b) Calculating $\text{proj}_W \mathbf{u}$ using $\{\mathbf{v}_1, \mathbf{v}_2\}$ we obtain

$\frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{-3+0+7}{1+0+1}(1,0,1) + \frac{0+1+0}{0+1+0}(0,1,0) = (2,0,2) + (0,1,0) = (2,1,2)$.

Calculating $\text{proj}_W \mathbf{u}$ using $\{\mathbf{v}_1', \mathbf{v}_2'\}$ instead yields the same vector:

$\frac{\langle \mathbf{u}, \mathbf{v}_1' \rangle}{\|\mathbf{v}_1'\|^2}\mathbf{v}_1' + \frac{\langle \mathbf{u}, \mathbf{v}_2' \rangle}{\|\mathbf{v}_2'\|^2}\mathbf{v}_2' = \frac{-3+1+7}{1+1+1}(1,1,1) + \frac{-3-2+7}{1+4+1}(1,-2,1) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right) + \left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right) = (2,1,2)$.

42. The first three Legendre polynomials are $q_1(x) = 1$, $q_2(x) = x$, and $q_3(x) = \frac{1}{2}(3x^2 - 1) = -\frac{1}{2} + \frac{3}{2}x^2$. By Example 8 and the Remark that follows it, these polynomials form an orthogonal set in the vector space P_2 with the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

We have $\|\mathbf{q}_1\|^2 = \int_{-1}^{1}(1)^2\,dx = x]_{-1}^{1} = 2$, $\|\mathbf{q}_2\|^2 = \int_{-1}^{1}(x)^2\,dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3}$, and

$\|\mathbf{q}_3\|^2 = \int_{-1}^{1}\left(-\frac{1}{2} + \frac{3}{2}x^2\right)^2 dx = \left(\frac{x}{4} - \frac{x^3}{2} + \frac{9x^5}{20}\right)\Big]_{-1}^{1} = \frac{2}{5}$.

(a) For $p(x) = 1 + x + 4x^2$, Theorem 6.3.2(a) yields

$$\mathbf{p} = \frac{\langle \mathbf{p}, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 + \frac{\langle \mathbf{p}, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 + \frac{\langle \mathbf{p}, \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2}\mathbf{q}_3$$

$$= \frac{\int_{-1}^{1}(1 + x + 4x^2)\,dx}{2}\mathbf{q}_1 + \frac{\int_{-1}^{1}(x + x^2 + 4x^3)\,dx}{\frac{2}{3}}\mathbf{q}_2 + \frac{\int_{-1}^{1}\left(-\frac{1}{2} - \frac{x}{2} - \frac{x^2}{2} + \frac{3x^3}{2} + 6x^4\right)dx}{\frac{2}{5}}\mathbf{q}_3$$

$$= \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{4x^3}{3}\right)\Bigg]_{-1}^{1}\mathbf{q}_1 + \frac{3}{2}\left(\frac{x^2}{2} + \frac{x^3}{3} + x^4\right)\Bigg]_{-1}^{1}\mathbf{q}_2 + \frac{5}{2}\left(-\frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{6} + \frac{3x^4}{8} + \frac{6x^5}{5}\right)\Bigg]_{-1}^{1}\mathbf{q}_3$$

$$= \left(\frac{1}{2}\right)\left(\frac{14}{3}\right)\mathbf{q}_1 + \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\mathbf{q}_2 + \left(\frac{5}{2}\right)\left(\frac{16}{15}\right)\mathbf{q}_3$$

$$= \frac{7}{3}\mathbf{q}_1 + 1\mathbf{q}_2 + \frac{8}{3}\mathbf{q}_3$$

(b) For $p(x) = 2 - 7x^2$, Theorem 6.3.2(a) yields

$$\mathbf{p} = \frac{\langle \mathbf{p}, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 + \frac{\langle \mathbf{p}, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 + \frac{\langle \mathbf{p}, \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2}\mathbf{q}_3$$

$$= \frac{\int_{-1}^{1}(2 - 7x^2)\,dx}{2}\mathbf{q}_1 + \frac{\int_{-1}^{1}(2x - 7x^3)\,dx}{\frac{2}{3}}\mathbf{q}_2 + \frac{\int_{-1}^{1}\left(-1 + \frac{13x^2}{2} - \frac{21x^4}{2}\right)dx}{\frac{2}{5}}\mathbf{q}_3$$

$$= \frac{1}{2}\left(2x - \frac{7x^3}{3}\right)\Bigg]_{-1}^{1}\mathbf{q}_1 + \frac{3}{2}\left(x^2 - \frac{7x^4}{4}\right)\Bigg]_{-1}^{1}\mathbf{q}_2 + \frac{5}{2}\left(-x + \frac{13x^3}{6} - \frac{21x^5}{10}\right)\Bigg]_{-1}^{1}\mathbf{q}_3$$

$$= \left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)\mathbf{q}_1 + \left(\frac{3}{2}\right)(0)\mathbf{q}_2 + \left(\frac{5}{2}\right)\left(-\frac{28}{15}\right)\mathbf{q}_3$$

$$= -\frac{1}{3}\mathbf{q}_1 + 0\mathbf{q}_2 - \frac{14}{3}\mathrm{q}_3$$

(c) For $p(x) = 4 + 3x$, Theorem 6.3.2(a) yields

$$\mathbf{p} = \frac{\langle \mathbf{p}, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 + \frac{\langle \mathbf{p}, \mathbf{q}_2 \rangle}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 + \frac{\langle \mathbf{p}, \mathbf{q}_3 \rangle}{\|\mathbf{q}_3\|^2}\mathbf{q}_3$$

$$= \frac{\int_{-1}^{1}(4 + 3x)\,dx}{2}\mathbf{q}_1 + \frac{\int_{-1}^{1}(4x + 3x^2)\,dx}{\frac{2}{3}}\mathbf{q}_2 + \frac{\int_{-1}^{1}\left(-2 - \frac{3x}{2} + 6x^2 + \frac{9x^3}{2}\right)dx}{\frac{2}{5}}\mathbf{q}_3$$

$$= \left(\frac{1}{2}\right)\left(4x + \frac{3x^2}{2}\right)\Bigg]_{-1}^{1}\mathbf{q}_1 + \frac{3}{2}(2x^2 + x^3)]_{-1}^{1}\mathbf{q}_2 + \frac{5}{2}\left(-2x - \frac{3x^2}{4} + 2x^3 + \frac{9x^4}{8}\right)\Bigg]_{-1}^{1}\mathbf{q}_3$$

$$= \left(\frac{1}{2}\right)(8)\mathbf{q}_1 + \left(\frac{3}{2}\right)(2)\mathbf{q}_2 + \left(\frac{5}{2}\right)(0)\mathbf{q}_3$$

$$= 4\mathbf{q}_1 + 3\mathbf{q}_2 + 0\mathrm{q}_3$$

43. First transform the basis $\mathrm{S} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, x, x^2\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$\mathbf{v}_1 = \mathbf{p}_1 = 1$

$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{\int_0^1 1^2 \mathrm{d}x} = \sqrt{x|_0^1} = 1$

$\langle \mathbf{p}_2,\ \mathbf{v}_1 \rangle = \int_0^1 x \cdot 1\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

$\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - \frac{\frac{1}{2}}{1}(1) = x - \frac{\frac{1}{2}}{1}(1) = -\frac{1}{2} + x$

$\|\mathbf{v}_2\|^2 = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \int_0^1 \left(-\frac{1}{2} + x\right)^2 dx = \int_0^1 \left(\frac{1}{4} - x + x^2\right) dx = \left(\frac{1}{4}x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{12}$

$\|\mathbf{v}_2\| = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$

$\langle \mathbf{p}_3, \mathbf{v}_1 \rangle = \int_0^1 x^2 \cdot 1\, dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$

$\langle \mathbf{p}_3, \mathbf{v}_2 \rangle = \int_0^1 x^2\left(-\frac{1}{2} + x\right) dx = \int_0^1 \left(-\frac{1}{2}x^2 + x^3\right) dx = \left(-\frac{1}{6}x^3 + \frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{12}$

$\mathbf{v}_3 = \mathbf{p}_3 - \frac{\langle \mathbf{p}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{p}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{\frac{1}{3}}{1}(1) - \frac{\frac{1}{12}}{\frac{1}{12}}\left(-\frac{1}{2} + x\right) = x^2 - \frac{1}{3} + \frac{1}{2} - x = \frac{1}{6} - x + x^2$

$\|\mathbf{v}_3\|^2 = \langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \int_0^1 \left(\frac{1}{6} - x + x^2\right)^2 dx = \left(\frac{1}{36}x - \frac{1}{6}x^2 + \frac{4}{9}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right)\Big|_0^1 = \frac{1}{180}$

$\|\mathbf{v}_3\| = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$

The orthonormal basis is

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{1} = 1$

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{-\frac{1}{2}+x}{\frac{1}{2\sqrt{3}}} = 2\sqrt{3}\left(-\frac{1}{2} + x\right) = \sqrt{3}(-1 + 2x)$

$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{\frac{1}{6}-x+x^2}{\frac{1}{6\sqrt{5}}} = 6\sqrt{5}\left(\frac{1}{6} - x + x^2\right) = \sqrt{5}(1 - 6x + 6x^2)$

44. The reduced row echelon form of the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ therefore the three column vectors of the original matrix, $\mathbf{u}_1 = (6, 2, -2, 6)$, $\mathbf{u}_2 = (1, 1, -2, 8)$, and $\mathbf{u}_3 = (-5, 1, 5, -7)$ form a basis for the column space. Applying the Gram-Schmidt process yields an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

$\mathbf{v}_1 = \mathbf{u}_1 = (6, 2, -2, 6)$

$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1, 1, -2, 8) - \frac{6+2+4+48}{36+4+4+36}(6, 2, -2, 6) = (1, 1, -2, 8) - \frac{3}{4}(6, 2, -2, 6)$

$= \left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right)$

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (-5, 1, 5, -7) - \frac{-30+2-10-42}{36+4+4+36}(6, 2, -2, 6) - \frac{\frac{35}{2}-\frac{1}{2}-\frac{5}{2}-\frac{49}{2}}{\frac{49}{4}+\frac{1}{4}+\frac{1}{4}+\frac{49}{4}}\left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right)$

$= (-5, 1, 5, -7) + (6, 2, -2, 6) + \frac{2}{5}\left(-\frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right) = \left(-\frac{2}{5}, \frac{14}{5}, \frac{14}{5}, \frac{2}{5}\right)$

45. Let $\mathbf{u}_1 = (1, 2)$, $\mathbf{u}_2 = (-1, 3)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$, and $\mathbf{q}_2 = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$. A QR-decomposition of the matrix A is formed by the given matrix Q and the matrix

$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_1 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} & -\frac{1}{\sqrt{5}} + \frac{6}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}.$

46. Let $\mathbf{u}_1 = (1,0,1)$, $\mathbf{u}_2 = (2,1,4)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, and $\mathbf{q}_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. A QR-decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} + 0 + \frac{4}{\sqrt{2}} \\ 0 & -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

47. Let $\mathbf{u}_1 = (1,0,1)$, $\mathbf{u}_2 = (0,1,2)$, $\mathbf{u}_3 = (2,1,0)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\mathbf{q}_2 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and $\mathbf{q}_3 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. A QR-decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \langle \mathbf{u}_3, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \langle \mathbf{u}_3, \mathbf{q}_2\rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & 0 + 0 + \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} + 0 + 0 \\ 0 & 0 + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 \\ 0 & 0 & \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{4}{\sqrt{6}} \end{bmatrix}.$$

48. Let $\mathbf{u}_1 = (1,1,0)$, $\mathbf{u}_2 = (2,1,3)$, $\mathbf{u}_3 = (1,1,1)$, $\mathbf{q}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\mathbf{q}_2 = \left(\frac{\sqrt{2}}{2\sqrt{19}}, -\frac{\sqrt{2}}{2\sqrt{19}}, \frac{3\sqrt{2}}{\sqrt{19}}\right)$, and $\mathbf{q}_3 = \left(-\frac{3}{\sqrt{19}}, \frac{3}{\sqrt{19}}, \frac{1}{\sqrt{19}}\right)$. A QR-decomposition of the matrix A is formed by the given matrix Q and the matrix

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \langle \mathbf{u}_3, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \langle \mathbf{u}_3, \mathbf{q}_2\rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 \\ 0 & \frac{2\sqrt{2}}{2\sqrt{19}} - \frac{\sqrt{2}}{2\sqrt{19}} + \frac{9\sqrt{2}}{\sqrt{19}} & \frac{\sqrt{2}}{2\sqrt{19}} - \frac{\sqrt{2}}{2\sqrt{19}} + \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & -\frac{3}{\sqrt{19}} + \frac{3}{\sqrt{19}} + \frac{1}{\sqrt{19}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}.$$

49. In partitioned form, $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$. By inspection, $\mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2$, so the column vectors of A are not linearly independent and A does not have a QR-decomposition.

51. The proof of part (a) mirrors the proof of part (b) in the book. By Theorem 6.3.1, an orthogonal set of nonzero vectors in W is linearly independent. It follows from part (b) of Theorem 4.5.5 that this set can be enlarged to form a basis for W. Applying the Gram-Schmidt process (without the normalization step) will yield an enlarged orthogonal set (the original orthogonal set will not be affected).

52. If $\mathbf{v}_3 = \mathbf{0}$ then

$$\begin{aligned} \mathbf{u}_3 &= \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ &= \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{u}_1 + \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{u}_2 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{u}_1 \end{aligned}$$

$$= \left(\frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2} - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\right)\mathbf{u}_1 + \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{u}_2$$

making $\mathbf{u}_3$ a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, which contradicts the assumption of the linear independence of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

53. The diagonal entries of R are $\langle \mathbf{u}_i, \mathbf{q}_i\rangle$ for i = 1, 2, ..., n, where $\mathbf{q}_i = \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ is the normalization of a vector $\mathbf{v}_i$ that is the result of applying the Gram-Schmidt process to $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. Thus, $\mathbf{v}_i$ is $\mathbf{u}_i$ minus a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}$, so $\mathbf{u}_i = \mathbf{v}_i + k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_{i-1}\mathbf{v}_{i-1}$. Thus, $\langle \mathbf{u}_i, \mathbf{v}_i\rangle = \langle \mathbf{v}_i, \mathbf{v}_i\rangle$ and $\langle \mathbf{u}_i, \mathbf{q}_i\rangle = \left\langle \mathbf{u}_i, \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}\right\rangle = \frac{1}{\|\mathbf{v}_i\|}\langle \mathbf{v}_i, \mathbf{v}_i\rangle = \|\mathbf{v}_i\|$. Since each vector $\mathbf{v}_i$ is nonzero, each diagonal entry of R is nonzero.

55. **(b)** The range of T is W; the kernel of T is $W^{\perp}$.

True-False Exercises

(a) False. For example, the vectors (1, 0) and (1, 1) in R^2 are linearly independent but not orthogonal.

(b) False. The vectors must be nonzero for this to be true.

(c) True. A nontrivial subspace of R^3 will have a basis, which can be transformed into an orthonormal basis with respect to the Euclidean inner product.

(d) True. A nonzero finite-dimensional inner product space will have finite basis which can be transformed into an orthonormal basis with respect to the inner product via the Gram-Schmidt process with normalization.

(e) False. $\text{proj}_W\mathbf{x}$ is a vector in W.

(f) True. Every invertible $n \times n$ matrix has a QR-decomposition.

6.4 Best Approximation; Least Squares

1. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$; $A^TA = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$;

The associated normal equation is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$.

2. $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$; $A^TA = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \\ -1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix}$;

$A^T\mathbf{b} = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 5 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$;

The associated normal equation is $\begin{bmatrix} 15 & -1 & 5 \\ -1 & 22 & 30 \\ 5 & 30 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 13 \end{bmatrix}$.

3. $A^TA = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{20}{11} \\ 0 & 1 & -\frac{8}{11} \end{bmatrix}$.

The solution of this system $x_1 = \frac{20}{11}$, $x_2 = -\frac{8}{11}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

4. $A^TA = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 14 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{3}{7} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix}$.

The solution of this system $x_1 = \frac{3}{7}$, $x_2 = -\frac{2}{3}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

5. $A^TA = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$;

$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ -9 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$.

The solution of this system $x_1 = 12$, $x_2 = -3$, $x_3 = 9$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

6. $A^TA = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & 6 \end{bmatrix}$;

$A^T\mathbf{b} = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 9 & -4 & 0 \\ -4 & 6 & -5 \\ 0 & -5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 26 \end{bmatrix}$.

The solution of this system $x_1 = 14$, $x_2 = 30$, $x_3 = 26$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

7. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} \frac{20}{11} \\ -\frac{8}{11} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} \frac{28}{11} \\ \frac{16}{11} \\ \frac{40}{11} \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} \\ -\frac{27}{11} \\ \frac{15}{11} \end{bmatrix}$; $A^T\mathbf{e} =$

$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} -\frac{6}{11} \\ -\frac{27}{11} \\ \frac{15}{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector in the column space of A.

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{\left(-\frac{6}{11}\right)^2 + \left(-\frac{27}{11}\right)^2 + \left(\frac{15}{11}\right)^2} = \frac{3}{11}\sqrt{110} \approx 2.86$.

8. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{7} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{46}{21} \\ -\frac{5}{21} \\ \frac{13}{21} \end{bmatrix} = \begin{bmatrix} -\frac{4}{21} \\ -\frac{16}{21} \\ \frac{8}{21} \end{bmatrix}$; $A^T\mathbf{e} =$

$\begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{21} \\ -\frac{16}{21} \\ \frac{8}{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector in the column space of A.

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{\left(-\frac{4}{21}\right)^2 + \left(-\frac{16}{21}\right)^2 + \left(\frac{8}{21}\right)^2} = \frac{4}{\sqrt{21}} \approx 0.873$.

9. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}$; $A^T\mathbf{e} =$

$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; therefore the least squares error vector is orthogonal to every vector in the column space of A.

Least squares error: $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{3^2 + (-3)^2 + 0^2 + 3^2} = 3\sqrt{3} \approx 5.196$.

10. Least squares error vector: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 14 \\ 30 \\ 26 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix}$;

$A^T\mathbf{e} = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ therefore the least squares error vector is orthogonal to every vector in the column space of A.

Least squares error: $\|\mathbf{b}-A\mathbf{x}\| = \sqrt{(-2)^2+0^2+2^2+2^2} = 2\sqrt{3} \approx 3.464$.

11. $A^TA = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 24 & 12 \\ 12 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = \frac{1}{2} - \frac{1}{2}t$, $x_2 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} -$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} - \frac{1}{2}t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

12. $A^TA = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 42 \\ 42 & 126 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 14 & 42 \\ 42 & 126 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 3 & \frac{2}{7} \\ 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = \frac{2}{7} - 3t$, $x_2 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions: $\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} -$

$\begin{bmatrix} 1 & 3 \\ -2 & -6 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} \frac{2}{7} - 3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ \frac{6}{7} \end{bmatrix} = \begin{bmatrix} \frac{5}{7} \\ \frac{4}{7} \\ \frac{1}{7} \end{bmatrix}$.

13. $A^TA = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 5 & -1 & 4 \\ -1 & 11 & 10 \\ 4 & 10 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 14 \\ 7 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 1 & -\frac{7}{6} \\ 0 & 1 & 1 & \frac{7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = -\frac{7}{6} - t$, $x_2 = \frac{7}{6} - t$, $x_3 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{7}{6} - t \\ \frac{7}{6} - t \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \begin{bmatrix} \frac{14}{3} \\ -\frac{7}{6} \\ \frac{7}{6} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{7}{6} \\ -\frac{49}{6} \end{bmatrix}.$$

14. $A^TA = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -4 & 10 \\ -1 & 3 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & -\frac{5}{7} & \frac{13}{84} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution of the normal system is $x_1 = \frac{2}{7} - \frac{1}{7}t$, $x_2 = \frac{13}{84} + \frac{5}{7}t$, $x_3 = t$. All of these are least squares solutions of $A\mathbf{x} = \mathbf{b}$. The error vector is the same for all solutions:

$$\mathbf{e} = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix} \begin{bmatrix} \frac{2}{7} - \frac{1}{7}t \\ \frac{13}{84} + \frac{5}{7}t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{6} \\ -\frac{1}{3} \\ \frac{11}{6} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{5}{3} \\ -\frac{5}{6} \end{bmatrix}.$$

15. $A^TA = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & \frac{17}{95} \\ 0 & 1 & \frac{143}{285} \end{bmatrix}$.

The solution of this system $x_1 = \frac{17}{95}$, $x_2 = \frac{143}{285}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

By Theorem 6.4.2, $\text{proj}_W\mathbf{b} = A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{17}{95} \\ \frac{143}{285} \end{bmatrix} = \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$.

This matches the result obtained using Theorem 6.4.4:

$$\begin{aligned} \text{proj}_W\mathbf{b} &= A(A^TA)^{-1}A^T\mathbf{b} \\ &= \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \left(\begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \left(\frac{1}{(14)(21) - (-3)(-3)} \begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix} \right) \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{285}\begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 21 & 3 \\ 3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$
$$= \frac{1}{285}\begin{bmatrix} -92 \\ 439 \\ 470 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{bmatrix}$$

16. $A^TA = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}$; $A^T\mathbf{b} = \begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & -\frac{1}{7} \\ 0 & 1 & -\frac{2}{7} \end{bmatrix}$.

The solution of this system $x_1 = -\frac{1}{7}, x_2 = -\frac{2}{7}$ is the unique least squares solution of $A\mathbf{x} = \mathbf{b}$.

By Theorem 6.4.2, $\text{proj}_W\mathbf{b} = A\mathbf{x} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$.

This matches the result obtained using Theorem 6.4.4:

$$\begin{aligned} \text{proj}_W\mathbf{b} &= A(A^TA)^{-1}A^T\mathbf{b} \\ &= \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix}\left(\begin{bmatrix} 42 & 0 \\ 0 & 14 \end{bmatrix}\right)^{-1}\begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 1 \\ 1 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1/42 & 0 \\ 0 & 1/14 \end{bmatrix}\begin{bmatrix} 5 & 1 & 4 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

17. We follow the procedure of Example 2. For $A = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}$, we have $A^TA =$

$\begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix}$ and $A^T\mathbf{u} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$;

The normal system $A^TA\mathbf{x} = A^T\mathbf{u}$ is $\begin{bmatrix} 6 & 6 \\ 6 & 24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$ so that the least squares solution of $A\mathbf{x} = \mathbf{u}$ is $\mathbf{x} = \begin{bmatrix} -\frac{7}{3} \\ \frac{1}{3} \end{bmatrix}$. Denoting $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ we obtain $\text{proj}_W \mathbf{u} = A\mathbf{x} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -\frac{7}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$.

18. Let $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$. $A^TA = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}$;
$A^T\mathbf{u} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$.
The normal system $A^TA\mathbf{x} = A^T\mathbf{u}$ is $\begin{bmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 21 \\ -21 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ so that the least squares solution of $A\mathbf{x} = \mathbf{u}$ is $\mathbf{x} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$. Denoting $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ we obtain $\text{proj}_W \mathbf{u} = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 9 \\ 5 \end{bmatrix}$.

19. Letting $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}([1])^{-1}[1 \quad 0] =$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1][1 \quad 0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \quad 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. This matches the matrix in Table 3 of Section 4.9.

20. Letting $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $P = A(A^TA)^{-1}A^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}([1])^{-1}[0 \quad 1] =$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1][0 \quad 1] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0 \quad 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. This matches the matrix in Table 3 of Section 4.9.

21. Letting $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have $A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matches the matrix in Table 4 of Section 4.9.

22. Letting $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have $A^TA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matches the matrix in Table 4 of Section 4.9.

23. We use Theorem 6.4.6: $\mathbf{x} = R^{-1}Q^T\mathbf{b} = \frac{1}{(5)\left(\frac{7}{5}\right)}\begin{bmatrix} \frac{7}{5} & \frac{1}{5} \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{35} \\ 0 & \frac{5}{7} \end{bmatrix}\begin{bmatrix} \frac{1}{5} \\ \frac{18}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ \frac{18}{7} \end{bmatrix}$.

24. We use Theorem 6.4.6: $\mathbf{x} = R^{-1}Q^T\mathbf{b} = \frac{1}{(5)(1)}\begin{bmatrix} 1 & 10 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

25. **(a)** If $x = s$ and $y = t$, then a point on the plane is $(s, t, -5s + 3t) = s(1,0,-5) + t(0,1,3)$. $\mathbf{w}_1 = (1,0,-5)$ and $\mathbf{w}_2 = (0,1,3)$ form a basis for W (they are linearly independent since neither of them is a scalar multiple of the other).

(b) Letting $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}$, Formula (11) yields

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}\left(\begin{bmatrix} 26 & -15 \\ -15 & 10 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}\left(\frac{1}{(26)(10)-(-15)(-15)}\begin{bmatrix} 10 & 15 \\ 15 & 26 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 10 & 15 \\ 15 & 26 \end{bmatrix}\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 10 & 15 & -5 \\ 15 & 26 & 3 \\ -5 & 3 & 34 \end{bmatrix}.$$

26. **(a)** $W = \text{span}\{(2,-1,4)\}$ so that the vector $(2,-1,4)$ forms a basis for W (its linear independence follows from Theorem 4.3.2(b))

(b) Letting $A = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, Formula (11) yields $P = A(A^TA)^{-1}A^T =$

$$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}[21]^{-1}\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}.$$

27. The reduced row echelon form of the augmented matrix of the given homogeneous system is $\begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ so that the general solution is $x_1 = -\frac{1}{2}s + \frac{1}{2}t$, $x_2 = -\frac{1}{2}s - \frac{1}{2}t$, $x_3 = s$, $x_4 = t$.

The solution space W is spanned by vectors $(-1,-1,2,0)$ and $(1,-1,0,2)$.

We construct the matrix with these vectors as its columns $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$ then follow the procedure of Example 2 in Section 6.4.

$$A^TA = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix};\ A^T\mathbf{u} = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix};$$

The normal system $A^T A\mathbf{x} = A^T\mathbf{u}$ is $\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$. The reduced row echelon form of the augmented matrix of the normal system is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{1}{2}\end{bmatrix}$ so that the least squares solution of $A\mathbf{x} = \mathbf{u}$ is $\mathbf{x} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$. We obtained $\text{proj}_W\mathbf{u} = A\mathbf{x} = \begin{bmatrix}-1 & 1\\-1 & -1\\2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$.

28. Letting $A = \begin{bmatrix}a\\b\\c\end{bmatrix}$, Formula (11) in Section 6.4 yields

$$\begin{aligned}P &= A(A^T A)^{-1}A^T\\&= \begin{bmatrix}a\\b\\c\end{bmatrix}\left(\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\right)^{-1}\begin{bmatrix}a & b & c\end{bmatrix}\\&= \begin{bmatrix}a\\b\\c\end{bmatrix}[a^2+b^2+c^2]^{-1}\begin{bmatrix}a & b & c\end{bmatrix}\\&= \frac{1}{a^2+b^2+c^2}\begin{bmatrix}a^2 & ab & ac\\ab & b^2 & bc\\ac & bc & c^2\end{bmatrix}\end{aligned}$$

29. Let W be the row space of A. Since W is also the column space of A^T, by Formula (11) we have $P = A^T((A^T)^T A^T)^{-1}(A^T)^T = A^T(AA^T)^{-1}A$.

30. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^T yields $A^T A\mathbf{x} = A^T\mathbf{b}$. By Theorem 6.4.4, the least squares solution is $(A^T A)^{-1}A^T\mathbf{b} = (A^T A)^{-1}A^T A\mathbf{x} = \mathbf{x}$.

31. Since $\mathbf{b}$ is orthogonal to the column space of A, it follows that $A^T\mathbf{b} = \mathbf{0}$. By Theorem 6.4.4, the least squares solution is $\mathbf{x} = (A^T A)^{-1}A^T\mathbf{b} = (A^T A)^{-1}\mathbf{0} = \mathbf{0}$.

32. Partitioning A into columns $A = [\mathbf{u}_1|\cdots|\mathbf{u}_n]$, we have $A^T A = [A^T\mathbf{u}_1|\cdots|A^T\mathbf{u}_n]$.

Assume the columns of A are linearly <u>dependent</u>, i.e., there exist scalars $k_1, \dots, k_n$, not all equal 0, such that

$$k_1\mathbf{u}_1 + \cdots + k_n\mathbf{u}_n = \mathbf{0}$$

Multiplying both sides on the left by A^T we obtain

$$k_1 A^T\mathbf{u}_1 + \cdots + k_n A^T\mathbf{u}_n = \mathbf{0}$$

Since scalars $k_1, \dots, k_n$ are not all equal 0, that means the columns of $A^T A$ are linearly dependent. However, this contradicts the assumption of the invertibility of $A^T A$.

Consequently, the columns of A must be linearly <u>independent</u>.

True-False Exercises

(a) True. $A^T A$ is an $n \times n$ matrix.

(b) False. Only square matrices have inverses, but $A^T A$ can be invertible when A is not a square matrix.

(c) True. If A is invertible, so is A^T, so the product $A^T A$ is also invertible.

(d) True. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^T yields $A^T A\mathbf{x} = A^T\mathbf{b}$.

(e) False. By Theorem 6.4.2, the normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent.

(f) True. This follows from Theorem 6.4.2.

(g) False. There may be more than one least squares solution as shown in Example 2.

(h) True. This follows from Theorem 6.4.4.

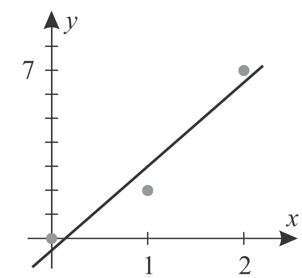
6.5 Mathematical Modeling Using Least Squares

1. We have $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $M^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $M^T M = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$,

$(M^T M)^{-1} = \frac{1}{15-9}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$, and

$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$

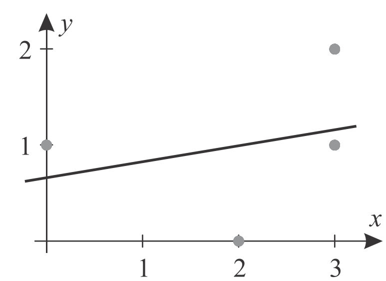
so the least squares straight line fit to the given data points is $y = -\frac{1}{2} + \frac{7}{2}x$.



2. We have $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$, $M^T M = \begin{bmatrix} 4 & 8 \\ 8 & 22 \end{bmatrix}$, $(M^T M)^{-1} = \frac{1}{24}\begin{bmatrix} 22 & -8 \\ -8 & 4 \end{bmatrix}$,

and $\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{24}\begin{bmatrix} 22 & -8 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \end{bmatrix}$ so

the least squares straight line fit to the given data points is $y = \frac{2}{3} + \frac{1}{6}x$.



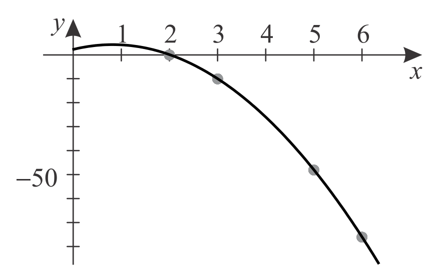
3. We have $M = \begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 5 & 5^2 \\ 1 & 6 & 6^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$, $M^T M =$

$\begin{bmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{bmatrix}$, $(M^T M)^{-1} = \frac{1}{90}\begin{bmatrix} 1989 & -1116 & 135 \\ -1116 & 649 & -80 \\ 135 & -80 & 10 \end{bmatrix}$, and

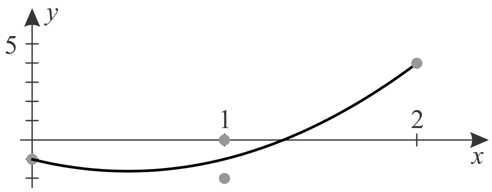
$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} =$

$\frac{1}{90}\begin{bmatrix} 1989 & -1116 & 135 \\ -1116 & 649 & -80 \\ 135 & -80 & 10 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix}\begin{bmatrix} 0 \\ -10 \\ -48 \\ -76 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$

so the least squares quadratic fit to the given data points is $y = 2 + 5x - 3x^2$.



4. We have $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $M^T M = \begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix}$,



$(M^T M)^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{9}{2} & -2 \\ \frac{1}{2} & -2 & 1 \end{bmatrix}$, and

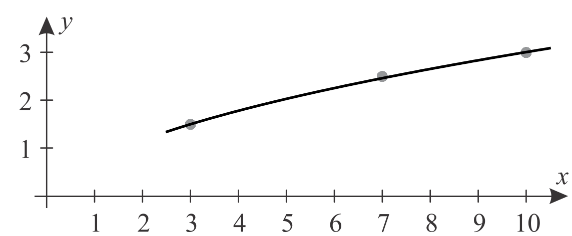
$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{9}{2} & -2 \\ \frac{1}{2} & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{5}{2} \\ \frac{5}{2} \end{bmatrix}$ so the least squares quadratic fit to the given data points is $y = -1 - \frac{5}{2}x + \frac{5}{2}x^2$.

5. With the substitution $X = \frac{1}{x}$, the problem becomes to find a line of the form $y = a + b \cdot X$ that best fits the data points $(1, 7)$, $\left(\frac{1}{3}, 3\right)$, $\left(\frac{1}{6}, 1\right)$.

We have $M = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{3} \\ 1 & \frac{1}{6} \end{bmatrix}$, $M^T M = \begin{bmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & \frac{41}{36} \end{bmatrix}$, $(M^T M)^{-1} = \frac{1}{42}\begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix}$, and

$\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} = \frac{1}{42}\begin{bmatrix} 41 & -54 \\ -54 & 108 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{21} \\ \frac{48}{7} \end{bmatrix}$. The line in terms of X is $y = \frac{5}{21} + \frac{48}{7}X$, so the required curve is $y = \frac{5}{21} + \frac{48}{7x}$.

6. With the substitution $X = \sqrt{x}$, the problem becomes to find a line of the form $y = a + b \cdot X$ that best fits the data points $\left(\sqrt{3}, \frac{3}{2}\right)$, $\left(\sqrt{7}, \frac{5}{2}\right)$, $\left(\sqrt{10}, 3\right)$.



We have $M = \begin{bmatrix} 1 & \sqrt{3} \\ 1 & \sqrt{7} \\ 1 & \sqrt{10} \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 3 \end{bmatrix}$, and

$M^T M = \begin{bmatrix} 3 & \sqrt{3} + \sqrt{7} + \sqrt{10} \\ \sqrt{3} + \sqrt{7} + \sqrt{10} & 20 \end{bmatrix}$. We conduct the remaining computations approximately: $\mathbf{v}^* = (M^T M)^{-1} M^T \mathbf{y} \approx \begin{bmatrix} -0.316 \\ 1.054 \end{bmatrix}$. The line in terms of X is approximately $y = -0.316 + 1.054X$, so the required curve is approximately $y = -0.316 + 1.054\sqrt{x}$.

7. The two column vectors of M are linearly independent if and only if neither is a multiple of the other. Since all the entries in the first column are equal, the columns are linearly independent if and only if the second column has at least two different entries, i.e., if and only if at least two of the numbers x_1, $x_2, \ldots, x_n$ are distinct.

True-False Exercises

(a) False. There is only a unique least squares straight line fit if the data points do not all lie on a vertical line.

(b) True. If the points are not collinear, there is no solution to the system.

(c) True.

(d) False. The line minimizes the sum of the *squares* of the data errors.

6.6 Function Approximation; Fourier Series

1. $a_0 = \frac{1}{\pi}\int_0^{2\pi}(1+x)\,dx = \frac{1}{\pi}\left(x+\frac{x^2}{2}\right)\Big]_0^{2\pi} = 2+2\pi$

Using integration by parts to integrate both $x\cos(kx)$ and $x\sin(kx)$ we obtain

$a_k = \frac{1}{\pi}\int_0^{2\pi}(1+x)\cos(kx)\,dx = \left(\frac{1+x}{k\pi}\sin(kx)+\frac{1}{k^2\pi}\cos(kx)\right)\Big]_0^{2\pi} = 0$ and

$b_k = \frac{1}{\pi}\int_0^{2\pi}(1+x)\sin(kx)\,dx = \left(-\frac{1+x}{k\pi}\cos(kx)+\frac{1}{k^2\pi}\sin(kx)\right)\Big]_0^{2\pi} = -\frac{2}{k}$

(a) $1+x \approx \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + a_3\cos(3x) + b_1\sin x + b_2\sin(2x) + b_3\sin(3x)$ yields $1+x \approx 1+\pi + 0\cos x + 0\cos(2x) - \frac{2}{1}\sin x - \frac{2}{2}\sin(2x) = 1+\pi-2\sin x - \sin 2x$

(b) $1+x \approx \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + \cdots + a_n\cos(nx) + b_1\sin x + b_2\sin(2x) + \cdots + b_n\sin(nx)$ yields $1+x \approx 1+\pi - \frac{2}{1}\sin x - \frac{2}{2}\sin(2x) - \cdots - \frac{2}{n}\sin(nx)$

2. $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{x^3}{3\pi}\Big]_0^{2\pi} = \frac{8\pi^2}{3}$

Using integration by parts twice to integrate both $x^2\cos(kx)$ and $x^2\sin(kx)$ we obtain

$a_k = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(kx)\,dx = \left(\frac{x^2}{k\pi}\sin(kx)+\frac{2x}{k^2\pi}\cos(kx)-\frac{2}{k^3\pi}\sin(kx)\right)\Big]_0^{2\pi} = \frac{4}{k^2}$ and

$b_k = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(kx)\,dx = \left(-\frac{x^2}{k\pi}\cos(kx)+\frac{2x}{k^2\pi}\sin(kx)+\frac{2}{k^3\pi}\cos(kx)\right)\Big]_0^{2\pi} = -\frac{4\pi}{k}$

(a) $x^2 \approx \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + a_3\cos(3x) + b_1\sin x + b_2\sin(2x) + b_3\sin(3x)$ yields

$x^2 \approx \frac{4\pi^2}{3} + 4\cos x + \cos(2x) + \frac{4}{9}\cos(3x) - 4\pi\sin x - 2\pi\sin(2x) - \frac{4\pi}{3}\sin(3x)$

(b) $x^2 \approx \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + \cdots + a_n\cos(nx) + b_1\sin x + b_2\sin(2x) + \cdots + b_n\sin(nx)$ yields

$x^2 \approx \frac{4\pi^2}{3} + \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos(2x) + \cdots + \frac{4}{n^2}\cos(nx) - \frac{4\pi}{1}\sin x - \frac{4\pi}{2}\sin(2x) - \cdots - \frac{4\pi}{n}\sin(nx)$

3. **(a)** Let us denote $W = \text{span}\{1, e^x\}$. Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$ and $\mathbf{u}_2 = e^x$ we obtain an orthogonal basis

$\mathbf{v}_1 = 1, \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = e^x - \frac{\int_0^1 e^x dx}{\int_0^1 1 dx}1 = e^x - \frac{e^x]_0^1}{x]_0^1}1 = e^x - (e-1)1 = e^x - e + 1.$

Since $\int_0^1 (e^x - e + 1)^2 \, dx = \int_0^1 (1 - 2e + e^2 + 2e^x - 2ee^x + e^{2x}) dx$

$= \left(x - 2ex + e^2 x + 2e^x - 2e^{x+1} + \frac{1}{2}e^{2x}\right)\Big|_0^1 = -\frac{3}{2} + 2e - \frac{1}{2}e^2 = \frac{1}{2}(e-1)(3-e)$, an

orthonormal basis is $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_0^1 1 dx}} = \frac{1}{\sqrt{x]_0^1}} = 1$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{e^x - e + 1}{\sqrt{\frac{1}{2}(e-1)(3-e)}}$.

The least squares approximation to $f(x) = x$ from W is

$\text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \int_0^1 x \, dx + \frac{2}{(e-1)(3-e)}\left(\int_0^1 x(e^x - e + 1) \, dx\right)(e^x - e + 1)$

$= \frac{1}{2} + \frac{2}{(e-1)(3-e)}\left(xe^x - e^x - \frac{x^2 e}{2} + \frac{x^2}{2}\right)\Big]_0^1 (e^x - e + 1) = \frac{1}{2} + \frac{2}{(e-1)(3-e)}\left(-\frac{e}{2} + \frac{3}{2}\right)(e^x - e + 1)$

$= \frac{1}{2} + \frac{e^x - e + 1}{e-1} = \frac{1}{2} + \frac{e^x}{e-1} - 1 = \frac{e^x}{e-1} - \frac{1}{2}$

(b) The mean square error is $\int_0^1 \left(x - \left(\frac{e^x}{e-1} - \frac{1}{2}\right)\right)^2 dx = \frac{7e-19}{12e-12} \approx 0.00136$

4. **(a)** Let us denote $W = \text{span}\{1, x\}$.

Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$ and $\mathbf{u}_2 = x$ we obtain an orthogonal basis $\mathbf{v}_1 = 1, \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - \frac{\int_0^1 x dx}{\int_0^1 1 dx}1 = x - \frac{\frac{x^2}{2}\Big]_0^1}{x]_0^1} = x - \frac{1}{2}$ and an orthonormal basis

$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_0^1 1 dx}} = \frac{1}{\sqrt{x]_0^1}} = 1$, $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx}} = \frac{x - \frac{1}{2}}{\sqrt{\frac{\left(x - \frac{1}{2}\right)^3}{3}\Big]_0^1}} = 2\sqrt{3}(x - \frac{1}{2})$

The least squares approximation to $f(x) = e^x$ from W is

$\text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2 \rangle \mathbf{q}_2 = \int_0^1 e^x \, dx + \left(\int_0^1 2\sqrt{3}\left(x - \frac{1}{2}\right)e^x \, dx\right) 2\sqrt{3}(x - \frac{1}{2})$

$= e - 1 + \left(2\sqrt{3}\left(xe^x - \frac{3}{2}e^x\right)\Big]_0^1\right) 2\sqrt{3}\left(x - \frac{1}{2}\right) = 4e - 10 + 6(3-e)x$

(b) The mean square error is $\int_0^1 \left(e^x - (4e - 10 + 6(3-e)x)\right)^2 dx = -\frac{57}{2} + 20e - \frac{7e^2}{2} \approx 0.00394$

5. **(a)** Let us denote $W = \text{span}\{1, x, x^2\}$.

Applying the Gram-Schmidt process to the basis $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, and $\mathbf{u}_3 = x^2$ we obtain an orthogonal basis $\mathbf{v}_1 = 1, \mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = x - \frac{\int_{-1}^1 x dx}{\int_{-1}^1 1 dx}1 = x - \frac{\frac{x^2}{2}\Big]_{-1}^1}{x]_{-1}^1} = x - 0 = x$,

$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 1 dx}1 - \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 x^2 dx}x = x^2 - \frac{\frac{x^3}{3}\Big]_{-1}^1}{x]_{-1}^1} - \frac{\frac{x^4}{4}\Big]_{-1}^1}{\frac{x^3}{3}\Big]_{-1}^1}x = x^2 - \frac{1}{3}$

and an orthonormal basis $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{\int_{-1}^1 1 dx}} = \frac{1}{\sqrt{x]_{-1}^1}} = \frac{1}{\sqrt{2}}$,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \frac{x}{\sqrt{\left.\frac{x^3}{3}\right]_{-1}^{1}}} = \sqrt{\frac{3}{2}}x, \quad \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx}} = \frac{x^2-\frac{1}{3}}{\frac{2}{3}\sqrt{\frac{2}{5}}}.$$

The least squares approximation to $f(x) = \sin \pi x$ from W is

$\text{proj}_W \mathbf{f} = \langle \mathbf{f}, \mathbf{q}_1\rangle \mathbf{q}_1 + \langle \mathbf{f}, \mathbf{q}_2\rangle \mathbf{q}_2 + \langle \mathbf{f}, \mathbf{q}_3\rangle \mathbf{q}_3$

$$= \frac{1}{4}\int_{-1}^{1} \sin \pi x \ dx + \frac{3}{2}\left(\int_{-1}^{1} x \sin \pi x \, dx\right) x + \frac{45}{8}\left(\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)\sin \pi x \, dx\right)\left(x^2 - \frac{1}{3}\right)$$

$$= 0 + \frac{3}{2}\left(-\frac{x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right)\Bigg|_{-1}^{1} x + 0 = \frac{3}{2}\cdot\frac{2}{\pi}x = \frac{3x}{\pi}$$

(b) The mean square error is $\int_{-1}^{1}\left(\sin \pi x - \frac{3x}{\pi}\right)^2 dx = 1 - \frac{6}{\pi^2} \approx 0.392$

8. We use Formulas (8) in Section 6.6

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\, dx = -\left.\frac{(\pi-x)^2}{2\pi}\right]_0^{2\pi} = 0$$

Using integration by parts to integrate both $(\pi - x)\cos(kx)$ and $(\pi - x)\sin(kx)$ we obtain

$$a_k = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\cos(kx)\, dx = \left.\left(\frac{\pi-x}{k\pi}\sin(kx) - \frac{1}{k^2\pi}\cos(kx)\right)\right]_0^{2\pi} = 0 \text{ and}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\sin(kx)\, dx = \left.\left(-\frac{\pi-x}{k\pi}\cos(kx) - \frac{1}{k^2\pi}\sin(kx)\right)\right]_0^{2\pi} = \frac{2}{k}$$

The Fourier series for $\pi - x$ over the interval $[0,2\pi]$ is $\sum_{k=1}^{\infty} \frac{2}{k}\sin(kx)$.

9. Let $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} dx = 1$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\, dx = \frac{1}{\pi}\int_0^{\pi}\cos kx\, dx = 0$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\, dx = \frac{1}{\pi}\int_0^{\pi}\sin kx \ dx = \frac{1}{k\pi}(1 - (-1)^k)$$

So the Fourier series is $\frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{k\pi}\left(1 - (-1)^k\right)\sin kx$.

10. All the coefficients of the series are zero except for $b_3 = 1$, i.e., $\sin(3x)$ is its own Fourier series.

True-False Exercises

(a) False. The area between the graphs is the error, not the mean square error.

(b) True.

(c) True.

(d) False. $\|1\| = \langle 1,\ 1\rangle = \int_0^{2\pi} 1^2 dx = 2\pi \neq 1$.

(e) True.

Chapter 6 Supplementary Exercises

1. **(a)** Let $\mathbf{v} = (v_1,\ v_2,\ v_3,\ v_4)$.
$\langle \mathbf{v},\ \mathbf{u}_1\rangle = v_1, \langle \mathbf{v},\ \mathbf{u}_2\rangle = v_2, \langle \mathbf{v},\ \mathbf{u}_3\rangle = v_3, \langle \mathbf{v},\ \mathbf{u}_4\rangle = v_4$
If $\langle \mathbf{v},\ \mathbf{u}_1\rangle = \langle \mathbf{v},\ \mathbf{u}_4\rangle = 0$, then $v_1 = v_4 = 0$ and $\mathbf{v} = (0,\ v_2,\ v_3,\ 0)$. Since the angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$, $\mathbf{v}$ making equal angles with $\mathbf{u}_2$ and $\mathbf{u}_3$ means that $v_2 = v_3$. In order for the angle between $\mathbf{v}$ and $\mathbf{u}_3$ to be defined $\|\mathbf{v}\| \neq 0$. Thus, $\mathbf{v} = (0, a, a, 0)$ with $a \neq 0$.

(b) As in part (a), since $\langle \mathbf{x},\ \mathbf{u}_1\rangle = \langle \mathbf{x},\ \mathbf{u}_4\rangle = 0$, $x_1 = x_4 = 0$.
Since $\|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$ and we want $\|\mathbf{x}\| = 1$, the cosine of the angle between $\mathbf{x}$ and $\mathbf{u}_2$ is $\cos\theta_2 = \langle \mathbf{x},\ \mathbf{u}_2\rangle = x_2$ and, similarly, $\cos\theta_3 = \langle \mathbf{x},\ \mathbf{u}_3\rangle = x_3$, so we want $x_2 = 2x_3$, and $\mathbf{x} = \langle 0,\ x_2,\ 2x_2,\ 0\rangle$.

$$\|\mathbf{x}\| = \sqrt{x_2^2 + 4x_2^2} = \sqrt{5x_2^2} = |x_2|\sqrt{5}.$$

If $\|\mathbf{x}\| = 1$, then $x_2 = \pm\frac{1}{\sqrt{5}}$, so $\mathbf{x} = \pm\left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}},\ 0\right)$.

3. Recall that if $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$, then $\langle U,\ V\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$.

(a) If U is a diagonal matrix, then $u_2 = u_3 = 0$ and $\langle U,\ V\rangle = u_1v_1 + u_4v_4$.
For V to be in the orthogonal complement of the subspace of all diagonal matrices, then it must be the case that $v_1 = v_4 = 0$ and V must have zeros on the main diagonal.

(b) If U is a symmetric matrix, then $u_2 = u_3$ and $\langle U,\ V\rangle = u_1v_1 + u_2(v_2 + v_3) + u_4v_4$.
Since u_1 and u_4 can take on any values, for V to be in the orthogonal complement of the subspace of all symmetric matrices, it must be the case that $v_1 = v_4 = 0$ and $v_2 = -v_3$, thus V must be skew-symmetric.

5. Let $\mathbf{u} = \left(\sqrt{a_1},\ \ldots,\ \sqrt{a_n}\right)$ and $\mathbf{v} = \left(\frac{1}{\sqrt{a_1}},\ \ldots,\ \frac{1}{\sqrt{a_n}}\right)$. By the Cauchy-Schwarz Inequality,
$\langle \mathbf{u}\cdot\mathbf{v}\rangle^2 = (\underbrace{1 + \cdots + 1}_{n\text{ terms}})^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$ or $n^2 \le (a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right)$.

7. Let $\mathbf{x} = (x_1,\ x_2,\ x_3)$.

$$\begin{aligned}\langle \mathbf{x}, \mathbf{u}_1\rangle &= x_1 + x_2 - x_3\\ \langle \mathbf{x}, \mathbf{u}_2\rangle &= -2x_1 - x_2 + 2x_3\\ \langle \mathbf{x}, \mathbf{u}_3\rangle &= -x_1 + x_3\end{aligned}$$

$\langle \mathbf{x},\ \mathbf{u}_3\rangle = 0 \Rightarrow -x_1 + x_3 = 0$, so $x_1 = x_3$. Then $\langle \mathbf{x},\ \mathbf{u}_1\rangle = x_2$ and $\langle \mathbf{x},\ \mathbf{u}_2\rangle = -x_2$, so $x_2 = 0$ and $\mathbf{x} = (x_1,\ 0,\ x_1)$. Then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_1^2} = \sqrt{2x_1^2} = |x_1|\sqrt{2}$.
If $\|\mathbf{x}\| = 1$ then $x_1 = \pm\frac{1}{\sqrt{2}}$ and the vectors are $\pm\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$.

8. By inspection, the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + \frac{1}{2}u_2v_2 + \frac{1}{3}u_3v_3 + \cdots + \frac{1}{n}u_nv_n$ satisfies
$\langle \mathbf{v}_1, \mathbf{v}_1\rangle = \langle \mathbf{v}_2, \mathbf{v}_2\rangle = \langle \mathbf{v}_3, \mathbf{v}_3\rangle = \cdots = \langle \mathbf{v}_n, \mathbf{v}_n\rangle = 1$ and $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for all $i \neq j$.

9. For $\mathbf{u} = (u_1,\ u_2), \mathbf{v} = (v_1,\ v_2)$ in R^2, let $\langle \mathbf{u},\ \mathbf{v}\rangle = au_1v_1 + bu_2v_2$ be a weighted inner product.
If $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, -1)$ form an orthonormal set, then $\|\mathbf{u}\|^2 = a(1)^2 + b(2)^2 = a + 4b = 1$, $\|\mathbf{v}\|^2 = a(3)^2 + b(-1)^2 = 9a + b = 1$, and $\langle \mathbf{u},\ \mathbf{v}\rangle = a(1)(3) + b(2)(-1) = 3a - 2b = 0$.
This leads to the system $\begin{bmatrix}1 & 4\\ 9 & 1\\ 3 & -2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$.
Since $\begin{bmatrix}1 & 4 & 1\\ 9 & 1 & 1\\ 3 & -2 & 0\end{bmatrix}$ reduces to $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, the system is inconsistent and there is no such weighted inner product.

11. **(a)** Let $\mathbf{u}_1 = (k,\ 0,\ 0,\ \ldots,\ 0), \mathbf{u}_2 = (0,\ k,\ 0,\ \ldots,\ 0), \ldots, \mathbf{u}_n = (0,\ 0,\ 0,\ \ldots,\ k)$ be the edges of the 'cube' in R^n and $\mathbf{u} = (k, k, k, \ldots, k)$ be the diagonal.
Then $\|\mathbf{u}_i\| = k$, $\|\mathbf{u}\| = k\sqrt{n}$, and $\langle \mathbf{u}_i,\ \mathbf{u}\rangle = k^2$, so $\cos\theta = \frac{\langle \mathbf{u}_i, \mathbf{u}\rangle}{\|\mathbf{u}_i\|\|\mathbf{u}\|} = \frac{k^2}{k(k\sqrt{n})} = \frac{1}{\sqrt{n}}$.

(b) As n approaches ∞, $\frac{1}{\sqrt{n}}$ approaches 0, so θ approaches $\frac{\pi}{2}$.

13. Recall that $\mathbf{u}$ can be expressed as the linear combination $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ where $a_i = \langle \mathbf{u},\ \mathbf{v}_i\rangle$ for $i = 1, \ldots, n$. Since $\|\mathbf{v}_i\| = 1$, we have $\cos^2\alpha_i = \left(\frac{\langle \mathbf{u}, \mathbf{v}_i\rangle}{\|\mathbf{u}\|\|\mathbf{v}_i\|}\right)^2 = \left(\frac{a_i}{\|\mathbf{u}\|}\right)^2 = \frac{a_i^2}{a_1^2 + a_2^2 + \cdots + a_n^2}$.
Therefore $\cos^2\alpha_1 + \cdots + \cos^2\alpha_n = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = 1$.

15. To show that $(W^\perp)^\perp = W$, we first show that $W \subseteq (W^\perp)^\perp$. If $\mathbf{w}$ is in W, then $\mathbf{w}$ is orthogonal to every vector in $W^\perp$, so that $\mathbf{w}$ is in $(W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp$.
To show that $(W^\perp)^\perp \subseteq W$, let $\mathbf{v}$ be in $(W^\perp)^\perp$. Since $\mathbf{v}$ is in V, we have, by the Projection Theorem, that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^\perp$. By definition, $\langle \mathbf{v},\ \mathbf{w}_2\rangle = \langle \mathbf{w}_1,\ \mathbf{w}_2\rangle = 0$. But $\langle \mathbf{v},\ \mathbf{w}_2\rangle = \langle \mathbf{w}_1 + \mathbf{w}_2,\ \mathbf{w}_2\rangle = \langle \mathbf{w}_1,\ \mathbf{w}_2\rangle + \langle \mathbf{w}_2,\ \mathbf{w}_2\rangle = \langle \mathbf{w}_2,\ \mathbf{w}_2\rangle$
so that $\langle \mathbf{w}_2,\ \mathbf{w}_2\rangle = 0$. Hence $\mathbf{w}_2 = \mathbf{0}$ and therefore $\mathbf{v} = \mathbf{w}_1$, so that $\mathbf{v}$ is in W. Thus $(W^\perp)^\perp \subseteq W$.

17. $A = \begin{bmatrix}1 & -1\\ 2 & 3\\ 4 & 5\end{bmatrix}$, $A^T = \begin{bmatrix}1 & 2 & 4\\ -1 & 3 & 5\end{bmatrix}$, $A^TA = \begin{bmatrix}21 & 25\\ 25 & 35\end{bmatrix}$, $A^T\mathbf{b} = \begin{bmatrix}1 & 2 & 4\\ -1 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\ 1\\ s\end{bmatrix} = \begin{bmatrix}4s + 3\\ 5s + 2\end{bmatrix}$
The associated normal system is $\begin{bmatrix}21 & 25\\ 25 & 35\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}4s + 3\\ 5s + 2\end{bmatrix}$.
If the least squares solution is $x_1 = 1$ and $x_2 = 2$, then $\begin{bmatrix}21 & 25\\ 25 & 35\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}71\\ 95\end{bmatrix} = \begin{bmatrix}4s + 3\\ 5s + 2\end{bmatrix}$.
The resulting equations have solutions $s = 17$ and $s = 18.6$, respectively, so no such value of s exists.

18. Using the trigonometric identity $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$ we obtain
$\langle \mathbf{f}, \mathbf{g}\rangle = \frac{1}{2}\int_0^{2\pi}\cos((p - q)x)\,dx - \frac{1}{2}\int_0^{2\pi}\cos((p + q)x)\,dx$ where both $p + q$ and $p - q$ are nonzero integers.

Substituting $u = (p - q)x$ in the first integral, and $t = (p + q)x$ in the second integral yields
$\langle \mathbf{f}, \mathbf{g} \rangle = \frac{1}{2}\frac{\sin((p-q)x)}{p-q}\Big]_0^{2\pi} - \frac{1}{2}\frac{\sin((p+q)x)}{p+q}\Big]_0^{2\pi} = 0 - 0 = 0$ since $\sin(m\pi) = 0$ for any integer m.

CHAPTER 7: DIAGONALIZATION AND QUADRATIC FORMS

7.1 Orthogonal Matrices

1. **(a)** $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$ and $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = I$ therefore A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = I$ and $A^TA = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = I$ therefore A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

2. **(a)** $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ therefore A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $AA^T = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{bmatrix} \neq I$ therefore A is not an orthogonal matrix.

3. **(a)** $\|\mathbf{r}_1\| = \sqrt{0^2 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1$ so the matrix is not orthogonal.

(b) $AA^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I$ and $A^TA = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = I$

therefore A is an orthogonal matrix; $A^{-1} = A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

4. **(a)** $AA^T = A^TA = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} = I$ therefore A is an orthogonal matrix;

$$A^{-1} = A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

(b) $\|\mathbf{r}_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{7}{12}} \neq 1.$ The matrix is not orthogonal.

5. $A^TA = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} = I;$

row vectors of A, $\mathbf{r}_1 = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \end{bmatrix}$, $\mathbf{r}_3 = \begin{bmatrix} \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$, form an orthonormal set since $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$ and $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$;

column vectors of A, $\mathbf{c}_1 = \begin{bmatrix} \frac{4}{5} \\ -\frac{9}{25} \\ \frac{12}{25} \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 0 \\ \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} -\frac{3}{5} \\ -\frac{12}{25} \\ \frac{16}{25} \end{bmatrix}$, form an orthonormal set since

$\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$ and $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$.

6. $A^TA = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = I$

row vectors of A, $\mathbf{r}_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $\mathbf{r}_3 = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$, form an orthonormal set since $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$ and $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = \|\mathbf{r}_3\| = 1$;

column vectors of A, $\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$, $\mathbf{c}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$, form an orthonormal set since

$\mathbf{c}_1 \cdot \mathbf{c}_2 = \mathbf{c}_1 \cdot \mathbf{c}_3 = \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$ and $\|\mathbf{c}_1\| = \|\mathbf{c}_2\| = \|\mathbf{c}_3\| = 1$.

7. $T_A(\mathbf{x}) = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{23}{5} \\ \frac{18}{25} \\ \frac{101}{25} \end{bmatrix}$; $\|T_A(\mathbf{x})\| = \sqrt{\frac{529}{25} + \frac{324}{625} + \frac{10201}{625}} = \sqrt{38}$ equals $\|\mathbf{x}\| = \sqrt{4+9+25} = \sqrt{38}$

8. $T_A(\mathbf{x}) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ \frac{2}{3} \\ \frac{7}{3} \end{bmatrix}$; $\|T_A(\mathbf{x})\| = \sqrt{\frac{100}{9} + \frac{4}{9} + \frac{49}{9}} = \sqrt{17}$ equals $\|\mathbf{x}\| = \sqrt{0+1+16} = \sqrt{17}$

9. Yes, by inspection, the column vectors in each of these matrices form orthonormal sets. By Theorem 7.1.1, these matrices are orthogonal.

10. No. Each of these matrices contains a zero column. Consequently, the column vectors do not form orthonormal sets. By Theorem 7.1.1, these matrices are not orthogonal.

11. Let $A = \begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$. Then $A^TA = \begin{bmatrix} 2(a^2+b^2) & 0 \\ 0 & 2(a^2+b^2) \end{bmatrix}$, so a and b must satisfy $a^2 + b^2 = \frac{1}{2}$.

12. All main diagonal entries must be ± 1 in order for the column vectors to form an orthonormal set.

13. **(a)** Formula (3) in Section 7.1 yields the transition matrix $P = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$; since P is orthogonal, $P^{-1} = P^T$ therefore $\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -1+3\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix}$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \sqrt{3} \\ 1 + \frac{5}{2}\sqrt{3} \end{bmatrix}$

14. **(a)** Formula (3) in Section 7.1 yields the transition matrix $P = \begin{bmatrix} \cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$; since P is orthogonal, $P^{-1} = P^T$ therefore $\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \end{bmatrix} = P\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$

15. **(a)** Following the method of Example 6 in Section 7.1 (also see Table 6 in Section 4.9), we use the orthogonal matrix $P = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to obtain

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 5 \end{bmatrix}$$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \\ -3 \end{bmatrix}$

16. **(a)** We follow the method of Example 6 in Section 7.1, with the appropriate orthogonal matrix obtained from Table 6 in Section 4.9: $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4} \\ 0 & \sin\frac{3\pi}{4} & \cos\frac{3\pi}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{\sqrt{2}} \\ -\frac{7}{\sqrt{2}} \end{bmatrix}$$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{\sqrt{2}} \\ \frac{9}{\sqrt{2}} \end{bmatrix}$

17. **(a)** We follow the method of Example 6 in Section 7.1, with the appropriate orthogonal matrix obtained from Table 6 in Section 4.9: $P = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{5\sqrt{3}}{2} \\ 2 \\ -\frac{\sqrt{3}}{2} + \frac{5}{2} \end{bmatrix}$$

(b) Using the matrix P we obtained in part (a), $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{3\sqrt{3}}{2} \\ 6 \\ -\frac{\sqrt{3}}{2} - \frac{3}{2} \end{bmatrix}$

18. If $B = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is the standard basis for R^3 and $B' = \{\mathbf{u}'_1,\ \mathbf{u}'_2,\ \mathbf{u}'_3\}$, then $[\mathbf{u}'_1]_B = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix}$, $[\mathbf{u}'_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $[\mathbf{u}'_3]_B = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$. Thus the transition matrix from B' to B is $P = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$. Then $A = P^{-1} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$.

19. If $B = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is the standard basis for R^3 and $B' = \{\mathbf{u}'_1,\ \mathbf{u}'_2,\ \mathbf{u}'_3\}$, then $[\mathbf{u}'_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[\mathbf{u}'_2]_B = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$, and $[\mathbf{u}'_3]_B = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$, so the transition matrix from B' to B is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$.

20. We obtain the relevant orthogonal matrices using the formulas in Table 6 of Section 4.9:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = P_1^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ with } P_1 = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = P_2^{-1}\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \text{ with } P_2 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Therefore, the matrix A such that $\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is obtained from

$$A = P_2^{-1}P_1^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

21. **(a)** Rotations about the origin, reflections about any line through the origin, and any combination of these are rigid operators.

(b) Rotations about the origin, dilations, contractions, reflections about lines through the origin, and combinations of these are angle preserving.

(c) All rigid operators on R^2 are angle preserving. Dilations and contractions are angle preserving operators that are not rigid.

22. No. If A is orthogonal then by part (c) of Theorem 7.1.3, $\mathbf{u}\cdot\mathbf{v} \neq 0$ implies $A\mathbf{u}\cdot A\mathbf{v} \neq 0$.

23. **(a)** Denoting $\mathbf{p}_1 = p_1(x) = \frac{1}{\sqrt{3}}$, $\mathbf{p}_2 = p_2(x) = \frac{1}{\sqrt{2}}x$, and $\mathbf{p}_3 = p_3(x) = \sqrt{\frac{3}{2}}x^2 - \sqrt{\frac{2}{3}}$ we have

$\langle \mathbf{p}, \mathbf{p}_1 \rangle = p(-1)p_1(-1) + p(0)p_1(0) + p(1)p_1(1) = (1)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) + (3)\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{\sqrt{3}}$

$\langle \mathbf{p}, \mathbf{p}_2 \rangle = p(-1)p_2(-1) + p(0)p_2(0) + p(1)p_2(1) = (1)\left(\frac{-1}{\sqrt{2}}\right) + (1)(0) + (3)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$\langle \mathbf{p}, \mathbf{p}_3 \rangle = p(-1)p_3(-1) + p(0)p_3(0) + p(1)p_3(1) = (1)\left(\frac{1}{\sqrt{6}}\right) + (1)\left(-\frac{2}{\sqrt{6}}\right) + (3)\left(\frac{1}{\sqrt{6}}\right) = \frac{\sqrt{2}}{\sqrt{3}}$

$\langle \mathbf{q}, \mathbf{p}_1 \rangle = q(-1)p_1(-1) + q(0)p_1(0) + q(1)p_1(1) = (-3)\left(\frac{1}{\sqrt{3}}\right) + (0)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}}$

$\langle \mathbf{q}, \mathbf{p}_2 \rangle = q(-1)p_2(-1) + q(0)p_2(0) + q(1)p_2(1) = (-3)\left(\frac{-1}{\sqrt{2}}\right) + (0)(0) + (1)\left(\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$

$\langle \mathbf{q}, \mathbf{p}_3 \rangle = q(-1)p_3(-1) + q(0)p_3(0) + q(1)p_3(1) = (-3)\left(\frac{1}{\sqrt{6}}\right) + (0)\left(-\frac{2}{\sqrt{6}}\right) + (1)\left(\frac{1}{\sqrt{6}}\right) = -\frac{\sqrt{2}}{\sqrt{3}}$

$(\mathbf{p})_S = (\langle \mathbf{p}, \mathbf{p}_1 \rangle, \langle \mathbf{p}, \mathbf{p}_2 \rangle, \langle \mathbf{p}, \mathbf{p}_3 \rangle) = \left(\frac{5}{\sqrt{3}}, \sqrt{2}, \frac{\sqrt{2}}{\sqrt{3}}\right)$

$(\mathbf{q})_S = (\langle \mathbf{q}, \mathbf{p}_1 \rangle, \langle \mathbf{q}, \mathbf{p}_2 \rangle, \langle \mathbf{q}, \mathbf{p}_3 \rangle) = \left(-\frac{2}{\sqrt{3}}, 2\sqrt{2}, -\frac{\sqrt{2}}{\sqrt{3}}\right)$

(b) $\|\mathbf{p}\| = \sqrt{\left(\frac{5}{\sqrt{3}}\right)^2 + \left(\sqrt{2}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{\frac{25}{3} + 2 + \frac{2}{3}} = \sqrt{11}$

$d(\mathbf{p}, \mathbf{q}) = \sqrt{\left(\frac{5}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)^2 + \left(\sqrt{2} - 2\sqrt{2}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\right)^2} = \sqrt{\frac{49}{3} + 2 + \frac{8}{3}} = \sqrt{21}$

$\langle \mathbf{p}, \mathbf{q} \rangle = \left(\frac{5}{\sqrt{3}}\right)\left(-\frac{2}{\sqrt{3}}\right) + (\sqrt{2})(2\sqrt{2}) + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(-\frac{\sqrt{2}}{\sqrt{3}}\right) = -\frac{10}{3} + 4 - \frac{2}{3} = 0$

25. We have $A^T = \left(I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T\right)^T = I_n^T - \frac{2}{\mathbf{x}^T\mathbf{x}}(\mathbf{x}\mathbf{x}^T)^T = I_n^T - \frac{2}{\mathbf{x}^T\mathbf{x}}(\mathbf{x}^T)^T\mathbf{x}^T = I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T = A$ therefore

$A^TA = AA^T = \left(I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T\right)\left(I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T\right) = I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T + \frac{4}{(\mathbf{x}^T\mathbf{x})^2}\mathbf{x}\mathbf{x}^T\mathbf{x}\mathbf{x}^T$

$= I_n - \frac{4}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T + \frac{4(\mathbf{x}^T\mathbf{x})}{(\mathbf{x}^T\mathbf{x})^2}\mathbf{x}\mathbf{x}^T = I_n - \frac{4}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T + \frac{4}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T = I_n$

26. Every unit vector in R^2 can be expressed as $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ for some angle θ. Thus for a 2×2 matrix A to have orthonormal columns, we must have $A = \begin{bmatrix}\cos\theta & \cos\beta\\ \sin\theta & \sin\beta\end{bmatrix}$ for some θ and β such that

$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \cdot \begin{bmatrix}\cos\beta\\ \sin\beta\end{bmatrix} = \cos\theta\cos\beta + \sin\theta\sin\beta = \cos(\theta - \beta) = 0$ so

either $\beta = \theta - \frac{\pi}{2} + 2k\pi$ or $\beta = \theta - \frac{3\pi}{2} + 2k\pi$.

Trigonometric identities imply that either $A = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ or $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

27. (a) Multiplication by $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ is a rotation through θ.

In this case, $\det(A) = \cos^2\theta + \sin^2\theta = 1$.

The determinant of $A = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ is $\det(A) = -\cos^2\theta + \sin^2\theta = -1$.

We can express this matrix as a product $\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

Multiplying by $\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ is a reflection about the x-axis followed by a rotation through θ.

(b) By Formula (6) of Section 4.9, multiplication by $\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ is a reflection about the line through the origin that makes the angle $\frac{\theta}{2}$ with the positive x-axis.

28. (a) Multiplication by $A = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\cos\frac{5\pi}{4} & -\sin\frac{5\pi}{4}\\ \sin\frac{5\pi}{4} & \cos\frac{5\pi}{4}\end{bmatrix}$ is a rotation through $\frac{5\pi}{4}$.

(b) Multiplication by $A = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\cos\frac{2\pi}{3} & \sin\frac{2\pi}{3}\\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3}\end{bmatrix}$ is a reflection about the x-axis followed by a rotation through $\frac{2\pi}{3}$. Also, multiplication by A is a reflection about the line through the origin that makes the angle $\frac{\pi}{3}$ with the positive x-axis.

29. Let A and B be 3×3 standard matrices of two rotations in R^3: T_A and T_B, respectively.
The result stated in this Exercise implies that A and B are both orthogonal and $\det(A) = \det(B) = 1$.
The product AB is a standard matrix of the composition of these rotations $T_A \circ T_B$.
By part (c) of Theorem 7.1.2, AB is an orthogonal matrix.
Furthermore, by Theorem 2.3.4, $\det(AB) = \det(A)\det(B) = 1$.
We conclude that $T_A \circ T_B$ is a rotation in R^3.
(One can show by induction that a composition of more than two rotations in R^3 is also a rotation.)

30. (a) We have $A^TA = AA^T = I$ and $\det(A) = 1$, so by Euler's Axis of Rotation Theorem T_A is a rotation about a line in R^3.
We can rewrite $A\mathbf{u} = \mathbf{u}$ as a homogeneous system $(A - I)\mathbf{u} = \mathbf{0}$. The reduced row echelon form of the coefficient matrix $A - I = \begin{bmatrix}-\frac{5}{7} & \frac{3}{7} & \frac{6}{7}\\ \frac{3}{7} & -\frac{13}{7} & \frac{2}{7}\\ \frac{6}{7} & \frac{2}{7} & -\frac{10}{7}\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -\frac{3}{2}\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$. For $\mathbf{u} = \left(\frac{3}{2}t, \frac{1}{2}t, t\right)$ to be a unit vector, we must have $\|\mathbf{u}\| = \sqrt{\frac{9}{4}t^2 + \frac{1}{4}t^2 + t^2} = \sqrt{\frac{14}{4}t^2} = \frac{\sqrt{14}}{2}|t| = 1$ so $t = \pm\frac{2}{\sqrt{14}}$.
We obtain $\mathbf{u} = \left(\pm\frac{3}{\sqrt{14}}, \pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}\right)$.

31. It follows directly from Definition 1 that the transpose of an orthogonal matrix is orthogonal as well (this is also stated as part (a) of Theorem 7.1.2). Since rows of A are columns of A^T, the equivalence of statements (a) and (c) follows from the equivalence of statements (a) and (b) which is shown in the book.

True-False Exercises

(a) False. Only square matrices can be orthogonal.

(b) False. The row and column vectors are not unit vectors.

(c) False. Only square matrices can be orthogonal. (The statement would be true if $m = n$.)

(d) False. The column vectors must form an orthonormal set.

(e) True. Since $A^TA = I$ for an orthogonal matrix A, A must be invertible (and $A^{-1} = A^T$).

(f) True. A product of orthogonal matrices is orthogonal, so A^2 is orthogonal; furthermore, $\det(A^2) = (\det A)^2 = (\pm 1)^2 = 1$.

(g) True. Since $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for an orthogonal matrix.

(h) True. This follows from Theorem 7.1.3.

7.2 Orthogonal Diagonalization

1. $\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{vmatrix} = \lambda^2 - 5\lambda$

The characteristic equation is $\lambda^2 - 5\lambda = 0$ and the eigenvalues are λ = 0 and λ = 5.
Both eigenspaces are one-dimensional.

2. $\begin{vmatrix} \lambda-1 & 4 & -2 \\ 4 & \lambda-1 & 2 \\ -2 & 2 & \lambda+2 \end{vmatrix} = \lambda^3 - 27\lambda - 54 = (\lambda-6)(\lambda+3)^2$

The characteristic equation is $\lambda^3 - 27\lambda - 54 = 0$ and the eigenvalues are λ = 6 and λ = –3.
The eigenspace for λ = 6 is one-dimensional; the eigenspace for λ = –3 is two-dimensional.

3. $\begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda-3)$

The characteristic equation is $\lambda^3 - 3\lambda^2 = 0$ and the eigenvalues are λ = 3 and λ = 0.
The eigenspace for λ = 3 is one-dimensional; the eigenspace for λ = 0 is two-dimensional.

4. $\begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 36\lambda - 32 = (\lambda-8)(\lambda-2)^2$

The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ and the eigenvalues are λ = 8 and λ = 2.
The eigenspace for λ = 8 is one-dimensional; the eigenspace for λ = 2 is two-dimensional.

5. $\begin{vmatrix} \lambda-4 & -4 & 0 & 0 \\ -4 & \lambda-4 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4 - 8\lambda^3 = \lambda^3(\lambda-8)$

The characteristic equation is $\lambda^4 - 8\lambda^3 = 0$ and the eigenvalues are λ = 0 and λ = 8.
The eigenspace for λ = 0 is three-dimensional; the eigenspace for λ = 8 is one-dimensional.

6. $\begin{vmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda-2 & 1 \\ 0 & 0 & 1 & \lambda-2 \end{vmatrix} = \lambda^4 - 8\lambda^3 + 22\lambda^2 - 24\lambda + 9 = (\lambda-1)^2(\lambda-3)^2$

The characteristic equation is $\lambda^4 - 8\lambda^3 + 22\lambda^2 - 24\lambda + 9 = 0$ and the eigenvalues are λ = 1 and λ = 3.
Both eigenspaces are two-dimensional.

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 6 & -2\sqrt{3} \\ -2\sqrt{3} & \lambda - 7 \end{vmatrix} = \lambda^2 - 13\lambda + 30 = (\lambda - 3)(\lambda - 10)$ therefore A has eigenvalues 3 and 10.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & \frac{2}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{2}{\sqrt{3}}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -2 \\ \sqrt{3} \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 10$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{\sqrt{3}}{2}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{bmatrix}$. We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$.

8. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$ therefore A has eigenvalues 2 and 4.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

9. Cofactor expansion along the second row yields $\det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 0 & 36 \\ 0 & \lambda + 3 & 0 \\ 36 & 0 & \lambda + 23 \end{vmatrix}$

$= (\lambda + 3)\begin{vmatrix} \lambda + 2 & 36 \\ 36 & \lambda + 23 \end{vmatrix} = (\lambda - 25)(\lambda + 3)(\lambda + 50)$ therefore A has eigenvalues $25, -3,$ and -50.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{4}{3}t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = -3$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = 0, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-50I - A$ is $\begin{bmatrix}1 & 0 & -\frac{3}{4}\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = -50$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = \frac{3}{4}t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix}3\\ 0\\ 4\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix}-\frac{4}{5} & 0 & \frac{3}{5}\\ 0 & 1 & 0\\ \frac{3}{5} & 0 & \frac{4}{5}\end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix}25 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -50\end{bmatrix}$.

10. $\det(\lambda I - A) = \begin{vmatrix}\lambda - 6 & 2\\ 2 & \lambda - 3\end{vmatrix} = (\lambda - 2)(\lambda - 7)$ therefore A has eigenvalues 2 and 7.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1 = \frac{1}{2}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix}1\\ 2\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $7I - A$ is $\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 7$ consists of vectors $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ where $x_1 = -2t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix}-2\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 7\end{bmatrix}$.

11. $\det(\lambda I - A) = \begin{vmatrix}\lambda - 2 & 1 & 1\\ 1 & \lambda - 2 & 1\\ 1 & 1 & \lambda - 2\end{vmatrix} = \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda - 3)^2$ therefore A has eigenvalues 3 and 0.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 =$

$\lambda_2 = 3$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A resulting in

$P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

12. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ -1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda - 2)$ therefore A has eigenvalues 0 and 2.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors since the three vectors are already orthogonal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

13. $\det(\lambda I - A) = \begin{vmatrix} \lambda+7 & -24 & 0 & 0 \\ -24 & \lambda-7 & 0 & 0 \\ 0 & 0 & \lambda+7 & -24 \\ 0 & 0 & -24 & \lambda-7 \end{vmatrix} = (\lambda+25)^2(\lambda-25)^2$ therefore A has eigenvalues -25 and 25.

The reduced row echelon form of $-25I - A$ is $\begin{bmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = -25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = -\frac{4}{3}s, x_2 = s, x_3 = -\frac{4}{3}t, x_4 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = \lambda_4 = 25$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = \frac{3}{4}s, x_2 = s, x_3 = \frac{3}{4}t, x_4 = t$. Vectors $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{p}_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$ form a basis for this eigenspace.

Applying the Gram-Schmidt process to the two bases $\{\mathbf{p}_1, \mathbf{p}_2\}, \{\mathbf{p}_3, \mathbf{p}_4\}$ amounts to simply normalizing the vectors since the four vectors are already orthogonal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} & 0 \\ 0 & -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} -25 & 0 & 0 & 0 \\ 0 & -25 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$.

14. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 & 0 & 0 \\ -1 & \lambda-3 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda-2)(\lambda-4)$ therefore A has eigenvalues 0, 2, and 4.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = \lambda_2 = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = 0, x_2 = 0, x_3 = s, x_4 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_3 = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = -t, x_2 = t, x_3 = 0, x_4 = 0$. A vector $\mathbf{p}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_4 = 4$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = 0, x_4 = 0$. A vector $\mathbf{p}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the three bases $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3\}$, and $\{\mathbf{p}_4\}$ amounts to simply normalizing the vectors since the four vectors are already orthogonal. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

15. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$ therefore A has eigenvalues 2 and 4.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A:

$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. We have $P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = (2)\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + (4)\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = (2)\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + (4)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

16. In the solution of Exercise 10, we have shown that $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ orthogonally diagonalizes A:

$P^TAP = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$. Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} = (2)\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} + (7)\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = (2)\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} + (7)\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

17. $\det(\lambda I - A) = \begin{vmatrix} \lambda + 3 & -1 & -2 \\ -1 & \lambda + 3 & -2 \\ -2 & -2 & \lambda \end{vmatrix} = (\lambda + 4)^2(\lambda - 2)$ therefore A has eigenvalues -4 and 2.

The reduced row echelon form of $-4I - A$ is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -4$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - 2t, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{2}t, x_2 = \frac{1}{2}t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$ orthogonally diagonalizes A resulting in $P^TAP = D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix} = (-4)\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + (-4)\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} + (2)\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= (-4)\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-4)\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} + (2)\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

18. $\det(\lambda I - A) = \begin{vmatrix} \lambda + 2 & 0 & 36 \\ 0 & \lambda + 3 & 0 \\ 36 & 0 & \lambda + 23 \end{vmatrix} = (\lambda - 25)(\lambda + 3)(\lambda + 50)$ therefore A has eigenvalues 25, -3, and -50.

The reduced row echelon form of $25I - A$ is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 25$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{4}{3}t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-3I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-50I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -50$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{3}{4}t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal.

A matrix $P = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$ orthogonally diagonalizes A resulting in $P^TAP = D = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}$.

Formula (7) of Section 7.2 yields the spectral decomposition of A:

$$\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix} = (25)\begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}\begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + (-50)\begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}\begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

$$= (25)\begin{bmatrix} \frac{16}{25} & 0 & -\frac{12}{25} \\ 0 & 0 & 0 \\ -\frac{12}{25} & 0 & \frac{9}{25} \end{bmatrix} + (-3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-50)\begin{bmatrix} \frac{9}{25} & 0 & \frac{12}{25} \\ 0 & 0 & 0 \\ \frac{12}{25} & 0 & \frac{16}{25} \end{bmatrix}.$$

19. The three vectors are orthogonal, and they can be made into orthonormal vectors by a simple normalization. Forming the columns of a matrix P in this way we obtain an orthogonal matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$. When the diagonal matrix D contains the corresponding eigenvalues on its main diagonal, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, then Formula (2) in Section 7.2 yields $PDP^T = A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$.

20. According to Theorem 7.2.2(b), for every symmetric matrix, eigenvectors corresponding to distinct eigenvalues must be orthogonal. Since $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 1 \neq 0$, it follows that no symmetric matrix can satisfy the given conditions.

21. Yes. The Gram-Schmidt process will ensure that columns of P corresponding to the same eigenvalue are an orthonormal set. Since eigenvectors from distinct eigenvalues are orthogonal, this means that P will be an orthogonal matrix. Then since A is orthogonally diagonalizable, it must be symmetric.

22. $\det(\lambda I - A) = \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - a \end{vmatrix} = (\lambda - a - b)(\lambda - a + b)$ therefore A has eigenvalues $a + b$ and $a - b$.

Assuming $b \neq 0$, the reduced row echelon form of $(a + b)I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = a + b$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Again assuming $b \neq 0$, the reduced row echelon form of $(a-b)I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = a - b$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

23. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = \lambda^2 - 2 = (\lambda - \sqrt{2})(\lambda + \sqrt{2})$ therefore A has eigenvalues $\pm\sqrt{2}$.

A is symmetric, so by Theorem 7.2.2(b), eigenvectors from different eigenspaces are orthogonal.

The reduced row echelon form of $\sqrt{2}I - A$ is $\begin{bmatrix} 1 & 1-\sqrt{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = (\sqrt{2}-1)t, x_2 = t$. A vector $\begin{bmatrix} \sqrt{2}-1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-\sqrt{2}I - A$ is $\begin{bmatrix} 1 & 1+\sqrt{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -\sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = (-\sqrt{2}-1)t, x_2 = t$. A vector $\begin{bmatrix} -\sqrt{2}-1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Unit eigenvectors chosen from two different eigenspaces will meet our desired condition. For instance, let $\mathbf{u}_1 = \begin{bmatrix} \frac{\sqrt{2}-1}{4-2\sqrt{2}} \\ \frac{1}{4-2\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} \frac{-\sqrt{2}-1}{4+2\sqrt{2}} \\ \frac{1}{4+2\sqrt{2}} \end{bmatrix}$.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda-3)$ therefore A has eigenvalues -1 and 3.

A is symmetric, so by Theorem 7.2.2(b), eigenvectors from different eigenspaces are orthogonal.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Unit eigenvectors chosen from two different eigenspaces will meet our desired condition. For instance, let $\mathbf{u}_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

24. **(a)** $\begin{vmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = \lambda^3 - 12\lambda^2 + 36\lambda - 32 = (\lambda-2)^2(\lambda-8)$
so the eigenvalues are $\lambda = 2$ and $\lambda = 8$.

A is symmetric, so by Theorem 7.2.2(b), eigenvectors from different eigenspaces are orthogonal.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t, x_2 = s, x_3 = t$. Vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $8I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 8$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. A vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Unit eigenvectors chosen from two different eigenspaces will meet our desired condition. For instance, let $\mathbf{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

(b) $\begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1)(\lambda-2)$ so the eigenvalues are $\lambda = 0$, $\lambda = 1$, and $\lambda = 2$.

A is symmetric, so by Theorem 7.2.2(b), eigenvectors from different eigenspaces are orthogonal.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 0$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = -t, x_3 = t$. A vector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = 0, x_3 = 0$. A vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace.

Unit eigenvectors chosen from these two different eigenspaces will meet our desired condition. For instance, let $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. (Note that it was not necessary to discuss the third eigenspace.)

25. $A^T A$ is a symmetric $n \times n$ matrix since $(A^T A)^T = A^T (A^T)^T = A^T A$. By Theorem 7.2.1 it has an orthonormal set of n eigenvectors.

28. **(b)** $A = I - \mathbf{v}\mathbf{v}^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ has the characteristic polynomial $\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda - 1 & 0 \\ 1 & 0 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda + 1)$ therefore A has eigenvalues 1 and -1.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -t, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors since the three vectors are already orthogonal.

This yields the columns of a matrix P that orthogonally diagonalizes A: $P = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

29. By Theorem 7.1.3(b), if A is an orthogonal $n \times n$ matrix then $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n. Since the eigenvalues of a symmetric matrix must be real numbers, for every such eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ we have $\|\mathbf{x}\| = \|A\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ hence the only possible eigenvalues for an orthogonal symmetric matrix are 1 and –1.

30. No, a non-symmetric matrix A can have eigenvalues that are real numbers. For instance, the eigenvalues of the matrix $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ are 3 and –1.

True-False Exercises

(a) True. For any square matrix A, both AA^T and $A^T A$ are symmetric, hence orthogonally diagonalizable.

(b) True. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are from distinct eigenspaces of a symmetric matrix, they are orthogonal, so $\|\mathbf{v}_1+\mathbf{v}_2\|^2=\langle \mathbf{v}_1+\mathbf{v}_2,\ \mathbf{v}_1+\mathbf{v}_2\rangle=\langle \mathbf{v}_1,\ \mathbf{v}_1\rangle+2\langle \mathbf{v}_1,\ \mathbf{v}_2\rangle+\langle \mathbf{v}_2,\ \mathbf{v}_2\rangle=\|\mathbf{v}_1\|^2+0+\|\mathbf{v}_2\|^2$.

(c) False. An orthogonal matrix is not necessarily symmetric.

(d) True. By Theorem 1.7.4, if A is an invertible symmetric matrix then A^{-1} is also symmetric.

(e) True. By Theorem 7.1.3(b), if A is an orthogonal $n\times n$ matrix then $\|A\mathbf{x}\|=\|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n. For every eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ we have $\|\mathbf{x}\|=\|A\mathbf{x}\|=\|\lambda\mathbf{x}\|=|\lambda|\|\mathbf{x}\|$ hence $|\lambda|=1$.

(f) True. If A is an $n\times n$ orthogonally diagonalizable matrix, then A has an orthonormal set of n eigenvectors, which form a basis for R^n.

(g) True. This follows from part (a) of Theorem 7.2.2.

7.3 Quadratic Forms

1. **(a)** $3x_1^2+7x_2^2=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

(b) $4x_1^2-9x_2^2-6x_1x_2=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}4 & -3\\ -3 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

(c) $9x_1^2-x_2^2+4x_3^2+6x_1x_2-8x_1x_3+x_2x_3=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}9 & 3 & -4\\ 3 & -1 & \frac{1}{2}\\ -4 & \frac{1}{2} & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

2. **(a)** $5x_1^2+5x_1x_2=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}5 & \frac{5}{2}\\ \frac{5}{2} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

(b) $-7x_1x_2=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}0 & -\frac{7}{2}\\ -\frac{7}{2} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

(c) $x_1^2+x_2^2-3x_3^2-5x_1x_2+9x_1x_3=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}1 & -\frac{5}{2} & \frac{9}{2}\\ -\frac{5}{2} & 1 & 0\\ \frac{9}{2} & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

3. $\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=2x^2+5y^2-6xy$

4. $\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}-2 & \frac{7}{2} & 1\\ \frac{7}{2} & 0 & 6\\ 1 & 6 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=-2x_1^2+3x_3^2+7x_1x_2+2x_1x_3+12x_2x_3$

5. $Q=\mathbf{x}^TA\mathbf{x}=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$; the characteristic polynomial of the matrix A is $\lambda^2-4\lambda+3=(\lambda-3)(\lambda-1)$, so the eigenvalues of A are $\lambda=3,\ 1$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix}-1\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix}1\\1\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors. Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$. In terms of the new variables, we have

$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2]\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = 3y_1^2 + y_2^2.$

6. $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix}5 & 2 & 0\\2 & 2 & 0\\0 & 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$; the characteristic polynomial of the matrix A is

$\det(\lambda I - A) = \begin{vmatrix}\lambda - 5 & -2 & 0\\ -2 & \lambda - 2 & 0\\ 0 & 0 & \lambda - 4\end{vmatrix} = (\lambda - 1)(\lambda - 4)(\lambda - 6)$ so the eigenvalues of A are 1, 4, and 6.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix}1 & \frac{1}{2} & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1 = -\frac{1}{2}t, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_1 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 4$ consists of vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1 = 0, x_2 = 0, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $6I - A$ is $\begin{bmatrix}1 & -2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 6$ consists of vectors $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ where $x_1 = 2t, x_2 = t, x_3 = 0$. A vector $\mathbf{p}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_3\}$ amounts to simply normalizing the vectors; the basis $\{\mathbf{p}_2\}$ is already orthonormal. Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$. In terms of the new variables, we have

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 6\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} = y_1^2 + 4y_2^2 + 6y_3^2.$$

7. $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix}3 & 2 & 0\\ 2 & 4 & -2\\ 0 & -2 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$; the characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \begin{vmatrix}\lambda-3 & -2 & 0\\ -2 & \lambda-4 & 2\\ 0 & 2 & \lambda-5\end{vmatrix} = \lambda^3 - 12\lambda^2 + 39\lambda - 28 = (\lambda-1)(\lambda-4)(\lambda-7)$$

so the eigenvalues of A are 1, 4, and 7.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$ 1 consists of vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = -2t, x_2 = 2t, x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix}-2\\ 2\\ 1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$ 4 consists of vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = t, x_2 = \frac{1}{2}t, x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix}2\\ 1\\ 2\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $7I - A$ is $\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 7$ consists of vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = -\frac{1}{2}t, x_2 = -t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix}-1\\ -2\\ 2\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the three bases amounts to simply normalizing the vectors. Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is

$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$. In terms of the new variables, we have

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2 \quad y_3]\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 7\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} = y_1^2 + 4y_2^2 + 7y_3^2.$$

8. $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the characteristic polynomial of the matrix A is

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -2 & 2 \\ -2 & \lambda - 5 & 4 \\ 2 & 4 & \lambda - 5 \end{vmatrix} = (\lambda - 1)^2(\lambda - 10)$ so the eigenvalues of A are 1 and 10.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$ 1 contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -2s + 2t, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} \frac{2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$ 10 contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{1}{2}t, x_2 = -t, x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. In terms of the new variables, we have

$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2 \quad y_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + y_2^2 + 10y_3^2$.

9. **(a)** $2x^2 + xy + x - 6y + 2 = 0$ can be expressed as $[x \quad y]\underbrace{\begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{[1 \quad -6]}_{K}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{(2)}_{f} = 0$

(b) $y^2 + 7x - 8y - 5 = 0$ can be expressed as $[x \quad y]\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{[7 \quad -8]}_{K}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{(-5)}_{f} = 0$

10. **(a)** $x^2 - xy + 5x + 8y - 3 = 0$ can be expressed as $[x \quad y]\underbrace{\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{[5 \quad 8]}_{K}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{(-3)}_{f} = 0$

(b) $5xy = 8$ should first be rewritten as $5xy - 8 = 0$, then as

$$[x \quad y]\underbrace{\begin{bmatrix} 0 & \frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{[0 \quad 0]}_{K}\begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{(-8)}_{f} = 0$$

11. **(a)** $2x^2 + 5y^2 = 20$ is $\frac{x^2}{10} + \frac{y^2}{4} = 1$ which is an equation of an ellipse.

(b) $x^2 - y^2 - 8 = 0$ is $x^2 - y^2 = 8$ or $\frac{x^2}{8} - \frac{y^2}{8} = 1$ which is an equation of a hyperbola.

(c) $7y^2 - 2x = 0$ is $x = \frac{7}{2}y^2$ which is an equation of a parabola.

(d) $x^2 + y^2 - 25 = 0$ is $x^2 + y^2 = 25$ which is an equation of a circle.

12. **(a)** ellipse (rewrite as $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$); **(b)** hyperbola (rewrite as $\frac{x^2}{5} - \frac{y^2}{4} = 1$);

(c) parabola; **(d)** circle (rewrite as $x^2 + y^2 = 3$)

13. We can rewrite the given equation in the matrix form $\mathbf{x}^T A\mathbf{x} = -8$ with $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$.

The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 2)$ so A has eigenvalues 3 and -2.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -2t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-2I - A$ is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{1}{2}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.

This yields the columns of a matrix P that orthogonally diagonalizes A - of the two possibilities, $\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ we choose the latter, i.e., $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$, since its determinant is 1 so that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. In the rotated coordinates, the equation of the conic becomes $[x' \quad y']\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = -8$, i.e., $3y'^2 - 2x'^2 = 8$; this equation represents a hyperbola.

Solving $P = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we conclude that the angle of rotation is $\theta = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 63.4°$.

14. We can rewrite the given equation in the matrix form $\mathbf{x}^T A\mathbf{x} = 9$ with $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$ so A has eigenvalues 3 and 7.
The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.
The reduced row echelon form of $7I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 7$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.
Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.
This yields the columns of a matrix P that orthogonally diagonalizes A - of the two possibilities, $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ we choose the latter, i.e., $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, since its determinant is 1 so that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. In the rotated coordinates, the equation of the conic becomes $[x' \quad y'] \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 9$, i.e., $7x'^2 + 3y'^2 = 9$; this equation represents an ellipse.
Solving $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we conclude that the angle of rotation is $\theta = \frac{\pi}{4}$.

15. We can rewrite the given equation in the matrix form $\mathbf{x}^T A\mathbf{x} = 15$ with $A = \begin{bmatrix} 11 & 12 \\ 12 & 4 \end{bmatrix}$.
The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 11 & -12 \\ -12 & \lambda - 4 \end{vmatrix} = (\lambda - 20)(\lambda + 5)$ so A has eigenvalues 20 and -5.
The reduced row echelon form of $20I - A$ is $\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 20$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = \frac{4}{3}t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.
The reduced row echelon form of $-5I - A$ is $\begin{bmatrix} 1 & \frac{3}{4} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -5$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{3}{4}t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ forms a basis for this eigenspace.
Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.
This yields the columns of a matrix P that orthogonally diagonalizes A - of the two possibilities,

$\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ and $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$ we choose the former, i.e., $P = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$, since its determinant is 1 so that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. In the rotated coordinates, the equation of the conic becomes $[x' \quad y'] \begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 15$, i.e., $4x'^2 - y'^2 = 3$; this equation represents a hyperbola.

Solving $P = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we conclude that the angle of rotation is

$\theta = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.9°$.

16. We can rewrite the given equation in the matrix form $\mathbf{x}^T A\mathbf{x} = \frac{1}{2}$ with $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{2}\right)$ so A has eigenvalues $\frac{1}{2}$ and $\frac{3}{2}$.

The reduced row echelon form of $\frac{1}{2}I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{1}{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $\frac{3}{2}I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{3}{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.

This yields the columns of a matrix P that orthogonally diagonalizes A - of the two possibilities, $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ we choose the latter, i.e., $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, since its determinant is 1 so that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. In the rotated coordinates, the equation of the conic becomes $[x' \quad y'] \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{2}$, i.e., $\frac{3}{2}x'^2 + \frac{1}{2}y'^2 = \frac{1}{2}$ or equivalently $3x'^2 + y'^2 = 1$. This equation represents an ellipse.

Solving $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ we conclude that the angle of rotation is $\theta = \frac{\pi}{4}$.

17. All matrices in this exercise are diagonal, therefore by Theorem 5.1.2, their eigenvalues are the entries on the main diagonal. We use Theorem 7.3.2 (including the remark below it).

(a) positive definite **(b)** negative definite **(c)** indefinite

(d) positive semidefinite **(e)** negative semidefinite

18. All matrices in this exercise are diagonal, therefore by Theorem 5.1.2, their eigenvalues are the entries on the main diagonal. We use Theorem 7.3.2 (including the remark below it).

(a) indefinite **(b)** negative definite **(c)** positive definite
(d) negative semidefinite **(e)** positive semidefinite

19. For all $(x_1, x_2) \neq (0,0)$, we clearly have $x_1^2 + x_2^2 > 0$ therefore the form is positive definite (an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which are $\lambda_1 = \lambda_2 = 1$ then use Theorem 7.3.2).

20. For all $(x_1, x_2) \neq (0,0)$, we clearly have $-x_1^2 - 3x_2^2 < 0$ therefore the form is negative definite (an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ which are $\lambda = -1$ and $\lambda = -3$ then use Theorem 7.3.2).

21. For all $(x_1, x_2) \neq (0,0)$, we clearly have $(x_1 - x_2)^2 \geq 0$, but cannot claim $(x_1 - x_2)^2 > 0$ when $x_1 = x_2$ therefore the form is positive semidefinite
(an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ which are $\lambda = 2$ and $\lambda = 0$ then use the remark under Theorem 7.3.2).

22. For all $(x_1, x_2) \neq (0,0)$, we clearly have $-(x_1 - x_2)^2 \leq 0$, but cannot claim $-(x_1 - x_2)^2 < 0$ when $x_1 = x_2$ therefore the form is negative semidefinite
(an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which are $\lambda = -2$ and $\lambda = 0$ then use the remark under Theorem 7.3.2).

23. Clearly, the form $x_1^2 - x_2^2$ has both positive and negative values (e.g., $3^2 - 1^2 > 0$ and $2^2 - 4^2 < 0$) therefore this quadratic form is indefinite
(an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which are $\lambda = -1$ and $\lambda = 1$ then use Theorem 7.3.2).

24. Clearly, the form $x_1 x_2$ has both positive and negative values (e.g., $(2)(3) > 0$ and $(-2)(3) < 0$) therefore this quadratic form is indefinite
(an alternate justification would be to calculate eigenvalues of the associated matrix $\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ which are $\lambda = -\frac{1}{2}$ and $\lambda = \frac{1}{2}$ then use Theorem 7.3.2).

25. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$; since both eigenvalues $\lambda = 3$ and $\lambda = 7$ are positive, by Theorem 7.3.2, A is positive definite.

Determinant of the first principal submatrix of A is $\det([5]) = 5 > 0$.
Determinant of the second principal submatrix of A is $\det(A) = 21 > 0$.
By Theorem 7.3.4, A is positive definite.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 & 0 \\ 1 & \lambda-2 & 0 \\ 0 & 0 & \lambda-5 \end{vmatrix} = (\lambda-1)(\lambda-3)(\lambda-5)$; since all three eigenvalues $\lambda = 1, \lambda = 3$, and $\lambda = 5$ are positive, by Theorem 7.3.2, A is positive definite.

Determinant of the first principal submatrix of A is $\det([2]) = 2 > 0$.
Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = 3 > 0$.
Determinant of the third principal submatrix of A is $\det(A) = 15 > 0$.
By Theorem 7.3.4, A is positive definite.

26. **(a)** $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-3)$; since both eigenvalues $\lambda = 1$ and $\lambda = 3$ are positive, by Theorem 7.3.2, A is positive definite.

Determinant of the first principal submatrix of A is $\det([2]) = 2 > 0$.
Determinant of the second principal submatrix of A is $\det(A) = 3 > 0$.
By Theorem 7.3.4, A is positive definite.

(b) $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3)(\lambda-4)$; since all three eigenvalues $\lambda = 1, \lambda = 3$, and $\lambda = 4$ are positive, by Theorem 7.3.2, A is positive definite.

Determinant of the first principal submatrix of A is $\det([3]) = 3 > 0$.
Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\right) = 5 > 0$.
Determinant of the third principal submatrix of A is $\det(A) = 12 > 0$.
By Theorem 7.3.4, A is positive definite.

27. **(a)** Determinant of the first principal submatrix of A is $\det([3]) = 3 > 0$.
Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\right) = -4 < 0$.
Determinant of the third principal submatrix of A is $\det(A) = -19 < 0$.
By Theorem 7.3.4(c), A is indefinite.

(b) Determinant of the first principal submatrix of A is $\det([-3]) = -3 < 0$.
Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}\right) = 5 > 0$.
Determinant of the third principal submatrix of A is $\det(A) = -25 < 0$.
By Theorem 7.3.4(b), A is negative definite.

28. **(a)** Determinant of the first principal submatrix of A is $\det([4]) = 4 > 0$.
Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right) = 7 > 0$.
Determinant of the third principal submatrix of A is $\det(A) = 6 > 0$.
By Theorem 7.3.4(a), A is positive definite.

(b) Determinant of the first principal submatrix of A is $\det([-4]) = -4 < 0$.
Determinant of the second principal submatrix of A is $\det\left(\begin{bmatrix} -4 & -1 \\ -1 & -2 \end{bmatrix}\right) = 7 > 0$.

Determinant of the third principal submatrix of A is $\det(A) = -6 < 0$.
By Theorem 7.3.4(b), A is negative definite.

29. The quadratic form $Q = 5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$ can be expressed in matrix notation as $Q = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & k \end{bmatrix}$. The determinants of the principal submatrices of A are $\det([5]) = 5$, $\det\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 1$, and $\det A = k - 2$. Thus Q is positive definite if and only if $k > 2$.

30. The quadratic form $Q = 3x_1^2 + x_2^2 + 2x_3^2 + 0x_1x_2 - 2x_1x_3 + 2kx_2x_3$ can be expressed in matrix notation as $Q = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & k \\ -1 & k & 2 \end{bmatrix}$. The determinants of the principal submatrices of A are $\det([3]) = 3$, $\det\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = 3$, and $\det A = 5 - 3k^2$. Thus Q is positive definite if and only if $5 - 3k^2 > 0$, i.e., $-\sqrt{\frac{5}{3}} < k < \sqrt{\frac{5}{3}}$.

31. **(a)** We assume A is symmetric so that $\mathbf{x}^T A\mathbf{y} = (\mathbf{x}^T A\mathbf{y})^T = \mathbf{y}^T A^T\mathbf{x} = \mathbf{y}^T A\mathbf{x}$. Therefore
$T(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})^T A(\mathbf{x}+\mathbf{y}) = \mathbf{x}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{x} + \mathbf{x}^T A\mathbf{y} + \mathbf{y}^T A\mathbf{y} = T(\mathbf{x}) + 2\mathbf{x}^T A\mathbf{y} + T(\mathbf{y})$.

(b) $T(c\mathbf{x}) = (c\mathbf{x})^T A(c\mathbf{x}) = c^2(\mathbf{x}^T A\mathbf{x}) = c^2 T(\mathbf{x})$

32. $(c_1x_1 + c_2x_2 + \cdots + c_nx_n)^2 = c_1^2x_1^2 + c_2^2x_2^2 + \cdots + c_n^2x_n^2 + 2c_1c_2x_1x_2 + \cdots + 2c_1c_nx_1x_n + \cdots$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_1^2 & c_1c_2 & \cdots & c_1c_n \\ c_1c_2 & c_2^2 & \cdots & c_2c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1c_n & c_2c_n & \cdots & c_n^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

33. **(a)** For each $i = 1, \ldots, n$ we have

$$\begin{aligned}(x_i - \bar{x})^2 &= x_i^2 - 2x_i\bar{x} + \bar{x}^2 \\ &= x_i^2 - 2x_i\frac{1}{n}\sum_{j=1}^{n} x_j + \frac{1}{n^2}\left(\sum_{j=1}^{n} x_j\right)^2 \\ &= x_i^2 - \frac{2}{n}\sum_{j=1}^{n} x_ix_j + \frac{1}{n^2}\left(\sum_{j=1}^{n} x_j^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n} x_jx_k\right)\end{aligned}$$

Thus in the quadratic form $s_x^2 = \frac{1}{n-1}[(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2]$ the coefficient of x_i^2 is $\frac{1}{n-1}\left[1 - \frac{2}{n} + \frac{1}{n^2}n\right] = \frac{1}{n}$, and the coefficient of x_ix_j for $i \neq j$ is $\frac{1}{n-1}\left[-\frac{2}{n} - \frac{2}{n} + \frac{2}{n^2}n\right] = -\frac{2}{n(n-1)}$.

It follows that $s_x^2 = \mathbf{x}^T A\mathbf{x}$ where $A = \begin{bmatrix} \frac{1}{n} & -\frac{1}{n(n-1)} & \cdots & -\frac{1}{n(n-1)} \\ -\frac{1}{n(n-1)} & \frac{1}{n} & \cdots & -\frac{1}{n(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n(n-1)} & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \end{bmatrix}$.

(b) We have $s_x^2 = \frac{1}{n-1}[(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2] \geq 0$, and $s_x^2 = 0$ if and only if $x_1 = \bar{x}$, $x_2 = \bar{x}, \ldots, x_n = \bar{x}$, i.e., if and only if $x_1 = x_2 = \cdots = x_n$. Thus s_x^2 is a positive semidefinite form.

34. (a) To simplify the equation, multiply both sides by $\frac{3}{2}$ so that $\mathbf{x}^T A\mathbf{x} = \frac{3}{2}$ with $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

We have $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{vmatrix} = (\lambda-1)^2(\lambda-4)$ so the eigenvalues of A are 1 and 4.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ contains vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = -s-t, y = s, z = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 4$ contains vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = t, y = t, z = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to $\{\mathbf{p}_3\}$ amounts to simply normalizing this vector.

Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$. In terms of the new variables, we have

$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [x' \quad y' \quad z'] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = x'^2 + y'^2 + 4z'^2$ so the original equation is expressed as $x'^2 + y'^2 + 4z'^2 = \frac{3}{2}$ or $\frac{2}{3}x'^2 + \frac{2}{3}y'^2 + \frac{8}{3}z'^2 = 1$. The lengths of the three axes in the x', y', and z'-directions are $\sqrt{6}$, $\sqrt{6}$, and $\frac{\sqrt{6}}{2}$, respectively.

(b) A must be positive definite.

35. The eigenvalues of A must be positive and equal to each other. That is, A must have a positive eigenvalue of multiplicity 2.

36. We express the quadratic form in the matrix notation $ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}^T A\mathbf{x}$.
Rotating a coordinate system through an angle θ amounts to the change of variables $\mathbf{x} = P\mathbf{y}$ where $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
The two off-diagonal entries of the matrix $P^TAP = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are both equal to $(c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) = \frac{c-a}{2}\sin 2\theta + b\cos 2\theta$, which equals 0 if $\frac{a-c}{2b} = \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$. Hence the resulting quadratic form $\mathbf{y}^T(P^TAP)\mathbf{y}$ has no cross product terms.

37. If A is an $n \times n$ symmetric matrix such that its eigenvalues $\lambda_1, \ldots, \lambda_n$ are all nonnegative, then by Theorem 7.3.1 there exists a change of variable $\mathbf{y} = P\mathbf{x}$ for which $\mathbf{x}^T A\mathbf{x} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$. The right hand side is always nonnegative, consequently $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x}$ in R^n .

True-False Exercises

(a) True. This follows from part (a) of Theorem 7.3.2 and from the margin note next to Definition 1.

(b) False. The term $4x_1x_2x_3$ cannot be included.

(c) True. One can rewrite $(x_1 - 3x_2)^2 = x_1^2 - 6x_1x_2 + 9x_2^2$.

(d) True. None of the eigenvalues will be 0.

(e) False. A symmetric matrix can also be positive semidefinite or negative semidefinite.

(f) True.

(g) True. $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2$

(h) True. Eigenvalues of A^{-1} are reciprocals of eigenvalues of A. Therefore if all eigenvalues of A are positive, the same is true for all eigenvalues of A^{-1}.

(i) True.

(j) True. This follows from part (a) of Theorem 7.3.4.

(k) True.

(l) False. If $c < 0$, $\mathbf{x}^T A\mathbf{x} = c$ has no graph.

7.4 Optimization Using Quadratic Forms

1. We express the quadratic form in the matrix notation $z = 5x^2 - y^2 = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
$\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 0 \\ 0 & \lambda + 1 \end{vmatrix} = (\lambda - 5)(\lambda + 1)$ therefore the eigenvalues of A are $\lambda = 5$ and $\lambda = -1$.

The reduced row echelon form of $5I - A$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 5$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = t, y = 0$. A vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = 0, y = t$. A vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

We conclude that the constrained extrema are

- constrained maximum: $z = 5$ at $(x, y) = (\pm 1, 0)$;
- constrained minimum: $z = -1$ at $(x, y) = (0, \pm 1)$.

2. We express the quadratic form in the matrix notation $z = xy = \mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda \end{vmatrix} = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)$ therefore the eigenvalues of A are $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$.

The reduced row echelon form of $\frac{1}{2} I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{1}{2}$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = t, y = t$. A vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

The reduced row echelon form of $-\frac{1}{2} I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -\frac{1}{2}$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = -t, y = t$. A vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

We conclude that the constrained extrema are

- constrained maximum: $z = \frac{1}{2}$ at $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(x, y) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$;
- constrained minimum: $z = -\frac{1}{2}$ at $(x, y) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(x, y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

3. We express the quadratic form in the matrix notation $z = 3x^2 + 7y^2 = \mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 7 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$ therefore the eigenvalues of A are $\lambda = 3$ and $\lambda = 7$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = t, y = 0$. A vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

The reduced row echelon form of $7I - A$ is $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 7$ consists of vectors $\begin{bmatrix}x\\ y\end{bmatrix}$ where $x = 0, y = t$. A vector $\begin{bmatrix}0\\ 1\end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

We conclude that the constrained extrema are

- constrained maximum: $z = 7$ at $(x, y) = (0, \pm 1)$;
- constrained minimum: $z = 3$ at $(x, y) = (\pm 1, 0)$.

4. We express the quadratic form in the matrix notation $z = 5x^2 + 5xy = \mathbf{x}^T A\mathbf{x} = [x \quad y]\begin{bmatrix}5 & \frac{5}{2}\\ \frac{5}{2} & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix}\lambda - 5 & -\frac{5}{2}\\ -\frac{5}{2} & \lambda - 5\end{vmatrix} = \lambda^2 - 5\lambda - \frac{25}{4} = \left(\lambda - \frac{5+5\sqrt{2}}{2}\right)\left(\lambda - \frac{5-5\sqrt{2}}{2}\right)$ therefore the eigenvalues of A are $\lambda = \frac{5+5\sqrt{2}}{2}$ and $\lambda = \frac{5-5\sqrt{2}}{2}$.

The reduced row echelon form of $\frac{5+5\sqrt{2}}{2}I - A$ is $\begin{bmatrix}1 & -1-\sqrt{2}\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{5+5\sqrt{2}}{2}$ consists of vectors $\begin{bmatrix}x\\ y\end{bmatrix}$ where $x = (1+\sqrt{2})t, y = t$. A vector $\begin{bmatrix}1+\sqrt{2}\\ 1\end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix}\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}\\ \frac{1}{\sqrt{4+2\sqrt{2}}}\end{bmatrix}$.

The reduced row echelon form of $\frac{5-5\sqrt{2}}{2}I - A$ is $\begin{bmatrix}1 & -1+\sqrt{2}\\ 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{5-5\sqrt{2}}{2}$ consists of vectors $\begin{bmatrix}x\\ y\end{bmatrix}$ where $x = (1-\sqrt{2})t, y = t$. A vector $\begin{bmatrix}1-\sqrt{2}\\ 1\end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix}\frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}\\ \frac{1}{\sqrt{4-2\sqrt{2}}}\end{bmatrix}$.

We conclude that the constrained extrema are

- constrained maximum: $z = \frac{5+5\sqrt{2}}{2} \approx 6.036$ at $(x, y) = \pm\left(\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}, \frac{1}{\sqrt{4+2\sqrt{2}}}\right) \approx \pm(0.924, 0.383)$ and
- constrained minimum: $z = \frac{5-5\sqrt{2}}{2} \approx -1.036$ at $(x, y) = \pm\left(\frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}}, \frac{1}{\sqrt{4-2\sqrt{2}}}\right) \approx \pm(-0.383, 0.924)$.

5. We express the quadratic form in the matrix notation

$w = 9x^2 + 4y^2 + 3z^2 = \mathbf{x}^T A\mathbf{x} = [x \quad y \quad z]\begin{bmatrix}9 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix}\lambda - 9 & 0 & 0\\ 0 & \lambda - 4 & 0\\ 0 & 0 & \lambda - 3\end{vmatrix} = (\lambda - 3)(\lambda - 4)(\lambda - 9)$ therefore the eigenvalues of A are $\lambda = 3, \lambda = 4,$ and $\lambda = 9$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = 0, y = 0, z = t$. A vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace - this vector is already normalized.

The reduced row echelon form of $9I - A$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 9$ consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = t, y = 0, z = 0$. A vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace eigenspace - this vector is already normalized.

We conclude that the constrained extrema are

- constrained maximum: $w = 9$ at $(x, y, z) = (\pm 1,0,0)$;
- constrained minimum: $w = 3$ at $(x, y, z) = (0,0,\pm 1)$.

6. We express the quadratic form in the matrix notation

$w = 2x^2 + y^2 + z^2 + 2xy + 2xz = \mathbf{x}^T A\mathbf{x} = [x \quad y \quad z] \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 & -1 \\ -1 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 3)$ therefore the eigenvalues of A are $\lambda = 0, \lambda = 1,$ and $\lambda = 3$.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 0$ consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = -t, y = t, z = t$. A vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

A normalized eigenvector in this eigenspace is $\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x = 2t, y = t, z = t$. A vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$.

We conclude that the constrained extrema are

- constrained maximum: $w = 3$ at $(x, y, z) = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $(x, y, z) = \left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$;
- constrained minimum: $w = 0$ at $(x, y, z) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $(x, y, z) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

7. The constraint $4x^2 + 8y^2 = 16$ can be rewritten as $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 1$. We define new variables x_1 and y_1 by $x = 2x_1$ and $y = \sqrt{2}y_1$. Our problem can now be reformulated to find maximum and minimum value of $2\sqrt{2}x_1y_1 = [x_1 \quad y_1]\underbrace{\begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ subject to the constraint $x_1^2 + y_1^2 = 1$. We have

$\det(\lambda I - A) = \begin{vmatrix} \lambda & -\sqrt{2} \\ -\sqrt{2} & \lambda \end{vmatrix} = \lambda^2 - 2 = (\lambda - \sqrt{2})(\lambda + \sqrt{2})$ thus A has eigenvalues $\pm\sqrt{2}$.

The reduced row echelon form of $\sqrt{2}I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ where $x_1 = t, y_1 = t$. A vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. In terms of the original variables, this corresponds to $x = 2x_1 = \sqrt{2}$ and $y = \sqrt{2}y_1 = 1$.

The reduced row echelon form of $-\sqrt{2}I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -\sqrt{2}$ consists of vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ where $x_1 = -t, y_1 = t$. A vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. In terms of the original variables, this corresponds to $x = 2x_1 = -\sqrt{2}$ and $y = \sqrt{2}y_1 = 1$.

We conclude that the constrained extrema are

- constrained maximum value: $\sqrt{2}$ at $(x, y) = (\sqrt{2}, 1)$ and $(x, y) = (-\sqrt{2}, -1)$;
- constrained minimum value: $-\sqrt{2}$ at $(x, y) = (-\sqrt{2}, 1)$ and $(x, y) = (\sqrt{2}, -1)$.

8. The constraint $x^2 + 3y^2 = 16$ can be rewritten as $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{\frac{4}{\sqrt{3}}}\right)^2 = 1$. We define new variables x_1 and y_1 by $x = 4x_1$ and $y = \frac{4}{\sqrt{3}}y_1$. Our problem can now be reformulated to find maximum and minimum value of $16x_1^2 + \frac{16}{\sqrt{3}}x_1y_1 + \frac{32}{3}y_1^2 = [x_1 \quad y_1]\underbrace{\begin{bmatrix} 16 & \frac{8}{\sqrt{3}} \\ \frac{8}{\sqrt{3}} & \frac{32}{3} \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ subject to the constraint $x_1^2 + y_1^2 = 1$. We have $\det(\lambda I - A) = \begin{vmatrix} \lambda - 16 & -\frac{8}{\sqrt{3}} \\ -\frac{8}{\sqrt{3}} & \lambda - \frac{32}{3} \end{vmatrix} = (\lambda - 8)\left(\lambda - \frac{56}{3}\right)$ thus A has eigenvalues 8 and $\frac{56}{3}$.

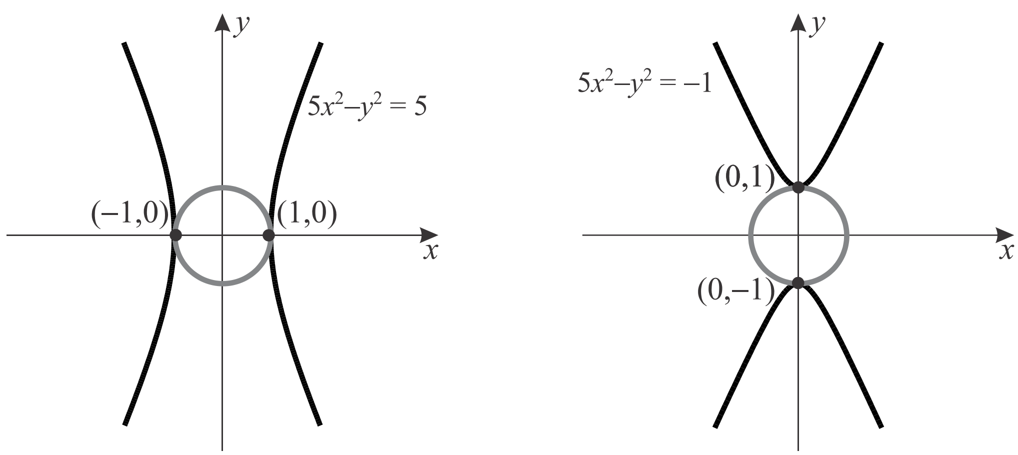
The reduced row echelon form of $\frac{56}{3}I - A$ is $\begin{bmatrix} 1 & -\sqrt{3} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{56}{3}$ consists of vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ where $x_1 = \sqrt{3}t, y_1 = t$. A vector $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$. In terms of the original variables, this corresponds to $x = 4x_1 = 2\sqrt{3}$ and $y = \frac{4}{\sqrt{3}}y_1 = \frac{2}{\sqrt{3}}$.

The reduced row echelon form of $8I - A$ is $\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 8$ consists of vectors $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ where $x_1 = -\frac{1}{\sqrt{3}}t, y_1 = t$. A vector $\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}$ forms a basis for this eigenspace. A normalized eigenvector in this eigenspace is $\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$. In terms of the original variables, this corresponds to $x = 4x_1 = -2$ and $y = \frac{4}{\sqrt{3}}y_1 = 2$.

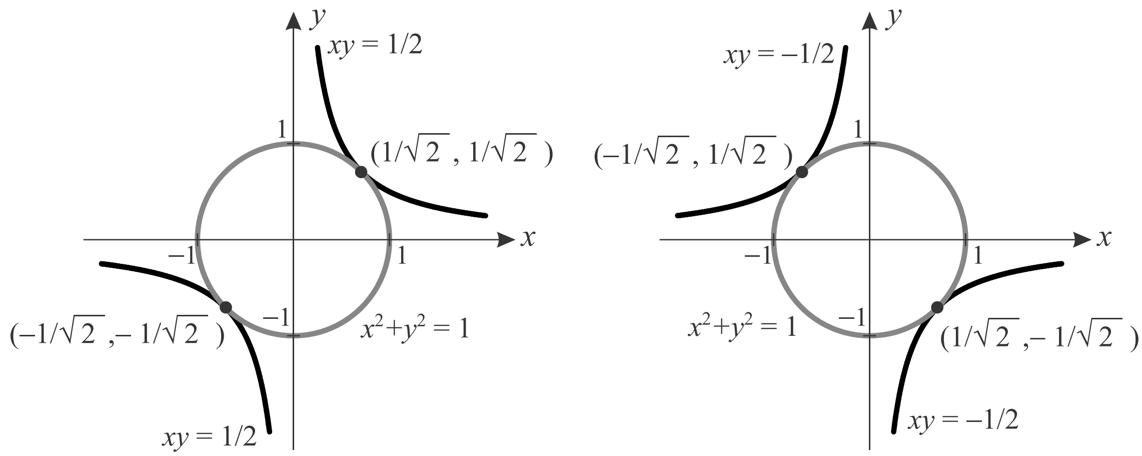
We conclude that the constrained extrema are

- constrained maximum value: $\frac{56}{3}$ at $(x, y) = \left(2\sqrt{3}, \frac{2}{\sqrt{3}}\right)$ and $(x, y) = \left(-2\sqrt{3}, -\frac{2}{\sqrt{3}}\right)$.
- constrained minimum value: 8 at $(x, y) = (-2,2)$ and $(x, y) = (2, -2)$.

9. The following illustration indicates positions of constrained extrema consistent with the solution that was obtained for Exercise 1.



10. The following illustration indicates positions of constrained extrema consistent with the solution that was obtained for Exercise 2.



11. **(a)** The first partial derivatives of $f(x,y)$ are $f_x(x,y) = 4y - 4x^3$ and $f_y(x,y) = 4x - 4y^3$. Since $f_x(0,0) = f_y(0,0) = 0$, $f_x(1,1) = f_y(1,1) = 0$, and $f_x(-1,-1) = f_y(-1,-1) = 0$, f has critical points at (0,0), (1,1), and $(-1,-1)$.

(b) The second partial derivatives of $f(x,y)$ are $f_{xx}(x,y) = -12x^2$, $f_{xy}(x,y) = 4$, and $f_{yy}(x,y) = -12y^2$ therefore the Hessian matrix of f is $H(x,y) = \begin{bmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda & -4 \\ -4 & \lambda \end{vmatrix} = (\lambda - 4)(\lambda + 4)$ so $H(0,0)$ has eigenvalues -4 and 4; since $H(0,0)$ is indefinite, f has a saddle point at (0,0);

$\det(\lambda I - H(1,1)) = \begin{vmatrix} \lambda + 12 & -4 \\ -4 & \lambda + 12 \end{vmatrix} = (\lambda + 8)(\lambda + 16)$ so $H(1,1)$ has eigenvalues -8 and -16; since $H(1,1)$ is negative definite, f has a relative maximum at (1,1);

$\det(\lambda I - H(-1,-1)) = \begin{vmatrix} \lambda + 12 & -4 \\ -4 & \lambda + 12 \end{vmatrix} = (\lambda + 8)(\lambda + 16)$ so $H(-1,-1)$ has eigenvalues -8 and -16; since $H(-1,-1)$ is negative definite, f has a relative maximum at $(-1,-1)$

12. **(a)** The first partial derivatives of $f(x,y)$ are $f_x(x,y) = 3x^2 - 6y$ and $f_y(x,y) = -6x - 3y^2$. Since $f_x(0,0) = f_y(0,0) = 0$ and $f_x(-2,2) = f_y(-2,2) = 0$, f has critical points at (0,0) and $(-2,2)$.

(b) The second partial derivatives of $f(x,y)$ are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -6$, and $f_{yy}(x,y) = -6y$ therefore the Hessian matrix of f is $H(x,y) = \begin{bmatrix} 6x & -6 \\ -6 & -6y \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda & 6 \\ 6 & \lambda \end{vmatrix} = (\lambda - 6)(\lambda + 6)$ so $H(0,0)$ has eigenvalues -6 and 6; since $H(0,0)$ is indefinite, f has a saddle point at (0,0);

$\det(\lambda I - H(-2,2)) = \begin{vmatrix} \lambda + 12 & 6 \\ 6 & \lambda + 12 \end{vmatrix} = (\lambda + 6)(\lambda + 18)$ so $H(-2,2)$ has eigenvalues -6 and -18; since $H(-2,2)$ is negative definite, f has a relative maximum at $(-2,2)$

13. The first partial derivatives of f are $f_x(x,\ y) = 3x^2 - 3y$ and $f_y(x,\ y) = -3x - 3y^2$. To find the critical points we set f_x and f_y equal to zero. This yields the equations $y = x^2$ and $x = -y^2$. From this we conclude that $y = y^4$ and so $y = 0$ or $y = 1$. The corresponding values of x are $x = 0$ and $x = -1$ respectively. Thus there are two critical points: (0, 0) and
(−1, 1).

The Hessian matrix is $H(x,\ y) = \begin{bmatrix} f_{xx}(x,\ y) & f_{xy}(x,\ y) \\ f_{yx}(x,\ y) & f_{yy}(x,\ y) \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & -6y \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda & 3 \\ 3 & \lambda \end{vmatrix} = (\lambda - 3)(\lambda + 3)$ so $H(0,0)$ has eigenvalues -3 and 3; since $H(0,0)$ is indefinite, f has a saddle point at (0,0);

$\det(\lambda I - H(-1,1)) = \begin{vmatrix} \lambda + 6 & 3 \\ 3 & \lambda + 6 \end{vmatrix} = (\lambda + 3)(\lambda + 9)$ so $H(-1,1)$ has eigenvalues -3 and -9; since $H(-1,1)$ is negative definite, f has a relative maximum at $(-1,1)$.

14. The first and second partial derivatives of f are:
$f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = -3x + 3y^2$, $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$.
Setting $f_x = 0$ and $f_y = 0$ results in $y = x^2$ and $x = y^2$; substituting the former equation into the latter yields $x = x^4$. Rewriting this equation as $x^4 - x = 0$ then factoring yields $x(x^3 - 1) = 0$ and $x(x-1)(x^2 + x + 1) = 0$. Thus either $x = 0$ or $x = 1$; from the equation $y = x^2$, the critical points are (0,0) and (1,1).

The Hessian matrix of f is $H(x,y) = \begin{bmatrix} f_{xx}(x,\ y) & f_{xy}(x,\ y) \\ f_{yx}(x,\ y) & f_{yy}(x,\ y) \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda & 3 \\ 3 & \lambda \end{vmatrix} = (\lambda - 3)(\lambda + 3)$ so $H(0,0)$ has eigenvalues -3 and 3; since $H(0,0)$ is indefinite, f has a saddle point at (0,0);

$\det(\lambda I - H(1,1)) = \begin{vmatrix} \lambda - 6 & 3 \\ 3 & \lambda - 6 \end{vmatrix} = (\lambda - 3)(\lambda - 9)$ so $H(1,1)$ has eigenvalues 3 and 9; since $H(1,1)$ is positive definite, f has a relative minimum at (1,1).

15. The first partial derivatives of f are $f_x(x,\ y) = 2x - 2xy$ and $f_y(x,\ y) = 4y - x^2$. To find the critical points we set f_x and f_y equal to zero. This yields the equations $2x(1 - y) = 0$ and $y = \frac{1}{4}x^2$. From the first, we conclude that $x = 0$ or $y = 1$. Thus there are three critical points: (0, 0), (2, 1), and (−2, 1).

The Hessian matrix is $H(x,\ y) = \begin{bmatrix} f_{xx}(x,\ y) & f_{xy}(x,\ y) \\ f_{yx}(x,\ y) & f_{yy}(x,\ y) \end{bmatrix} = \begin{bmatrix} 2 - 2y & -2x \\ -2x & 4 \end{bmatrix}$.

$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$ so $H(0,0)$ has eigenvalues 2 and 4; since $H(0,0)$ is positive definite, f has a relative minimum at (0,0).

$\det(\lambda I - H(2,1)) = \begin{vmatrix} \lambda & 4 \\ 4 & \lambda - 4 \end{vmatrix} = \lambda^2 - 4\lambda - 16$ so the eigenvalues of $H(2,1)$ are $2 \pm 2\sqrt{5}$. One of these is positive and one is negative; thus this matrix is indefinite and f has a saddle point at (2, 1).

$\det(\lambda I - H(-2,1)) = \begin{vmatrix} \lambda & -4 \\ -4 & \lambda - 4 \end{vmatrix} = \lambda^2 - 4\lambda - 16$ so the eigenvalues of $H(-2,1)$ are $2 \pm 2\sqrt{5}$. One of these is positive and one is negative; thus this matrix is indefinite and f has a saddle point at (−2, 1).

16. The first and second partial derivatives of f are:
$f_x(x,y) = 3x^2 - 3$, $f_y(x,y) = 3y^2 - 3$, $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 6y$.

Setting $f_x = 0$ and $f_y = 0$ results in four critical points: $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$.
The Hessian matrix of f is $H(x,y) = \begin{bmatrix} f_{xx}(x,\ y) & f_{xy}(x,\ y) \\ f_{yx}(x,\ y) & f_{yy}(x,\ y) \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$. By Theorem 5.1.2, the eigenvalues of the diagonal matrix $H(x,y)$ are its main diagonal entries, therefore

- at the critical point $(-1,-1)$, $H(-1,-1)$ has eigenvalues $-6,-6$ so f has a relative maximum,
- at the critical point $(-1,1)$, $H(-1,1)$ has eigenvalues $-6,6$ so f has a saddle point,
- at the critical point $(1,-1)$, $H(1,-1)$ has eigenvalues $6,-6$ so f has a saddle point,
- at the critical point $(1,1)$, $H(1,1)$ has eigenvalues $6,6$ so f has a relative minimum.

17. The problem is to maximize $z = 4xy$ subject to $x^2 + 25y^2 = 25$, or $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$.
Let $x = 5x_1$ and $y = y_1$, so that the problem is to maximize $z = 20x_1y_1$ subject to $\|(x_1,\ y_1)\| = 1$.
Write $z = \mathbf{x}^T A\mathbf{x} = [x_1 \quad y_1]\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

$$\begin{vmatrix} \lambda & -10 \\ -10 & \lambda \end{vmatrix} = \lambda^2 - 100 = (\lambda+10)(\lambda-10).$$

The largest eigenvalue of A is $\lambda = 10$ which has positive unit eigenvector $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Thus the maximum value of $z = 20\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 10$ which occurs when $x = 5x_1 = \frac{5}{\sqrt{2}}$ and $y = y_1 = \frac{1}{\sqrt{2}}$, which are the coordinates of one of the corner points of the rectangle.

18. Since $\|\mathbf{x}\| = 1$ and $A\mathbf{x} = 2\mathbf{x}$, it follows that $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(2\mathbf{x}) = 2(\mathbf{x}^T\mathbf{x}) = 2\|\mathbf{x}\|^2 = 2$.

19. **(a)** The first partial derivatives of $f(x,y)$ are $f_x(x,y) = 4x^3$ and $f_y(x,y) = 4y^3$.
Since $f_x(0,0) = f_y(0,0) = 0$, f has a critical point at $(0,0)$.
The second partial derivatives of $f(x,y)$ are $f_{xx}(x,y) = 12x^2$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2$. We have $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 0$ therefore the second derivative test is inconclusive.

The first partial derivatives of $g(x,y)$ are $g_x(x,y) = 4x^3$ and $g_y(x,y) = -4y^3$.
Since $g_x(0,0) = g_y(0,0) = 0$, g has a critical point at $(0,0)$.
The second partial derivatives of $g(x,y)$ are $g_{xx}(x,y) = 12x^2$, $g_{xy}(x,y) = 0$, and $g_{yy}(x,y) = -12y^2$. We have $g_{xx}(0,0)g_{yy}(0,0) - g_{xy}^2(0,0) = 0$ therefore the second derivative test is inconclusive.

(b) Clearly, for all $(x,y) \neq (0,0)$, $f(x,y) > f(0,0) = 0$ therefore f has a relative minimum at $(0,0)$.

For all $x \neq 0$, $g(x,0) > g(0,0) = 0$; however, for all $y \neq 0$, $g(0,y) < g(0,0) = 0$ - consequently, g has a saddle point at $(0,0)$.

20. The general quadratic form on R^2, $f(x,y) = a_1x^2 + a_2y^2 + a_3xy$ has first and second partial derivatives
$f_x(x,y) = 2a_1x + a_3y$, $f_y(x,y) = 2a_2y + a_3x$, $f_{xx}(x,y) = 2a_1$, $f_{xy}(x,y) = a_3$, and $f_{yy}(x,y) = 2a_2$.
The assumption $H(x,y) = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ implies that $a_1 = 1$, $a_2 = 1$, and $a_3 = 4$.
The equations $f_x = 0$ and $f_y = 0$ become $2x + 4y = 0$ and $4x + 2y = 0$ so the only critical point is

(0,0).
$\det(\lambda I - H(0,0)) = \begin{vmatrix} \lambda-2 & -4 \\ -4 & \lambda-2 \end{vmatrix} = (\lambda-6)(\lambda+2)$ so $H(0,0)$ has eigenvalues -2 and 6. We conclude that $f(x,y)$ has a saddle point at (0,0).

21. $\mathbf{x}$ is a unit eigenvector corresponding to λ, then $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(\lambda\mathbf{x}) = \lambda(\mathbf{x}^T\mathbf{x}) = \lambda(1) = \lambda$.

22. Let us assume A is a symmetric matrix.
If $m = M$ then c must be equal to $m = M$; taking $\mathbf{x}_c = \mathbf{u}_m$ we obtain $\mathbf{x}_c^T A\mathbf{x}_c = \mathbf{u}_m^T A\mathbf{u}_m = m = c$.
Now, consider the case $m < M$. With the vectors given in the hint, Theorem 7.4.1 yields
$A\mathbf{u}_M = M\mathbf{u}_M$ and $A\mathbf{u}_m = m\mathbf{u}_m$. Eigenvectors from different eigenspaces must be orthogonal, so $\mathbf{u}_m^T A\mathbf{u}_M = \mathbf{u}_m^T(M\mathbf{u}_M) = M(\mathbf{u}_m^T\mathbf{u}_M) = 0$ and $\mathbf{u}_M^T A\mathbf{u}_m = \mathbf{u}_M^T(m\mathbf{u}_m) = m(\mathbf{u}_M^T\mathbf{u}_m) = 0$. We have

$$\begin{aligned}
\mathbf{x}_c^T A\mathbf{x}_c &= \left(\sqrt{\frac{M-c}{M-m}}\,\mathbf{u}_m^T + \sqrt{\frac{c-m}{M-m}}\mathbf{u}_M^T\right) A \left(\sqrt{\frac{M-c}{M-m}}\,\mathbf{u}_m + \sqrt{\frac{c-m}{M-m}}\mathbf{u}_M\right) \\
&= \frac{M-c}{M-m}\mathbf{u}_m^T A\mathbf{u}_m + \frac{c-m}{M-m}\mathbf{u}_M^T A\mathbf{u}_M \\
&= \frac{M-c}{M-m}m + \frac{c-m}{M-m}M \\
&= \frac{Mm - cm + cM - mM}{M-m} \\
&= \frac{c(M-m)}{M-m} \\
&= c
\end{aligned}$$

True-False Exercises

(a) False. If the only critical point of the quadratic form is a saddle point, then it will have neither a maximum nor a minimum value.

(b) True. This follows from part (b) of Theorem 7.4.1.

(c) True.

(d) False. The second derivative test is inconclusive in this case.

(e) True. If $\det(A) < 0$, then A will have a negative eigenvalue.

7.5 Hermitian, Unitary, and Normal Matrices

1. $\overline{A} = \begin{bmatrix} -2i & 1+i \\ 4 & 3-i \\ 5-i & 0 \end{bmatrix}$ therefore $A^* = \overline{A}^T = \begin{bmatrix} -2i & 4 & 5-i \\ 1+i & 3-i & 0 \end{bmatrix}$

2. $\overline{A} = \begin{bmatrix} -2i & 1+i & -1-i \\ 4 & 5+7i & i \end{bmatrix}$ therefore $A^* = \overline{A}^T = \begin{bmatrix} -2i & 4 \\ 1+i & 5+7i \\ -1-i & i \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & i & 2-3i \\ -i & -3 & 1 \\ 2+3i & 1 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 0 & 3+5i \\ 0 & -4 & -i \\ 3-5i & i & 6 \end{bmatrix}$

5. **(a)** $(A)_{13} = 2-3i$ does not equal $(A^*)_{13} = 2+3i$

(b) $(A)_{22} = i$ does not equal $(A^*)_{22} = -i$

6. **(a)** $(A)_{12} = 1+i$ does not equal $(A^*)_{12} = 1-i$

(b) $(A)_{33} = 2+i$ does not equal $(A^*)_{33} = 2-i$

7. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -2+3i \\ -2-3i & \lambda+1 \end{vmatrix} = \lambda^2 - 2\lambda - 16 = \left(\lambda - \left(1+\sqrt{17}\right)\right)\left(\lambda - \left(1-\sqrt{17}\right)\right)$ so A has real eigenvalues $1+\sqrt{17}$ and $1-\sqrt{17}$.

For the eigenvalue $\lambda = 1+\sqrt{17}$, the augmented matrix of the homogeneous system $\left((1+\sqrt{17})I - A\right)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -2+\sqrt{17} & -2+3i & 0 \\ -2-3i & 2+\sqrt{17} & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + \frac{2+\sqrt{17}}{13}(-2+3i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \frac{2+\sqrt{17}}{13}(2-3i)t$, $x_2 = t$. The vector $\mathbf{v}_1 = \begin{bmatrix} \frac{2+\sqrt{17}}{13}(2-3i) \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 1+\sqrt{17}$.

For the eigenvalue $\lambda = 1-\sqrt{17}$, the augmented matrix of the homogeneous system $\left((1-\sqrt{17})I - A\right)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -2-\sqrt{17} & -2+3i & 0 \\ -2-3i & 2-\sqrt{17} & 0 \end{bmatrix}$.

As before, this yields $x_1 + \frac{2-\sqrt{17}}{13}(-2+3i)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \frac{2-\sqrt{17}}{13}(2-3i)t$, $x_2 = t$. The vector $\mathbf{v}_2 = \begin{bmatrix} \frac{2-\sqrt{17}}{13}(2-3i) \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 1-\sqrt{17}$.

We have

$\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(\frac{2+\sqrt{17}}{13}(2-3i)\right)\left(\overline{\frac{2-\sqrt{17}}{13}(2-3i)}\right) + (1)(\overline{1}) = \left(\frac{2+\sqrt{17}}{13}(2-3i)\right)\left(\frac{2-\sqrt{17}}{13}(2+3i)\right) + (1)(1)$
$= \frac{(2+\sqrt{17})(2-\sqrt{17})}{13^2}(2-3i)(2+3i) + 1 = \frac{4-17}{13^2}(4+9) + 1 = -1+1 = 0$ therefore the eigenvectors from different eigenspaces are orthogonal.

8. $\det(\lambda I - A) = \begin{vmatrix} \lambda & -2i \\ 2i & \lambda-2 \end{vmatrix} = \lambda^2 - 2\lambda - 4 = \left(\lambda - \left(1+\sqrt{5}\right)\right)\left(\lambda - \left(1-\sqrt{5}\right)\right)$ so A has real eigenvalues $1+\sqrt{5}$ and $1-\sqrt{5}$.

For the eigenvalue $\lambda = 1+\sqrt{5}$, the augmented matrix of the homogeneous system $\left((1+\sqrt{5})I - A\right)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 1+\sqrt{5} & -2i & 0 \\ 2i & -1+\sqrt{5} & 0 \end{bmatrix}$. The rows of this matrix must be scalar multiples of each other (see Example 3 in Section 5.3) therefore it suffices to solve the equation corresponding to the second row, which yields $x_1 + \left(\frac{1-\sqrt{5}}{2}i\right)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \left(\frac{-1+\sqrt{5}}{2}i\right)t, x_2 = t$. The vector $\mathbf{v}_1 = \begin{bmatrix} \frac{-1+\sqrt{5}}{2}i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 1+\sqrt{5}$.

For the eigenvalue $\lambda = 1-\sqrt{5}$, the augmented matrix of the homogeneous system $\left((1-\sqrt{5})I - A\right)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 1-\sqrt{5} & -2i & 0 \\ 2i & -1-\sqrt{5} & 0 \end{bmatrix}$. As before, this yields $x_1 + \left(\frac{1+\sqrt{5}}{2}i\right)x_2 = 0$. The general solution of this equation (and, consequently, of the entire system) is $x_1 = \left(\frac{-1-\sqrt{5}}{2}i\right)t, x_2 = t$.

The vector $\mathbf{v}_2 = \begin{bmatrix} \frac{-1-\sqrt{5}}{2}i \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = 1-\sqrt{5}$.

We have $\mathbf{v}_1 \cdot \mathbf{v}_2 = \left(\frac{-1+\sqrt{5}}{2}i\right)\left(\overline{\frac{-1-\sqrt{5}}{2}i}\right) + (1)(\overline{1}) = \left(\frac{-1+\sqrt{5}}{2}i\right)\left(\frac{1+\sqrt{5}}{2}i\right) + (1)(1) = -\left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) + 1 = -\frac{1}{4}(-1+5) + 1 = -1+1 = 0$ therefore the eigenvectors from different eigenspaces are orthogonal.

9. The following computations show that the row vectors of A are orthonormal:

$\|\mathbf{r}_1\| = \sqrt{\left|\frac{3}{5}\right|^2 + \left|\frac{4}{5}i\right|^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$; $\|\mathbf{r}_2\| = \sqrt{\left|-\frac{4}{5}\right|^2 + \left|\frac{3}{5}i\right|^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$;

$\mathbf{r}_1 \cdot \mathbf{r}_2 = \left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) + \left(\frac{4}{5}i\right)\left(-\frac{3}{5}i\right) = -\frac{12}{5} + \frac{12}{5} = 0$

By Theorem 7.5.3, A is unitary, and $A^{-1} = A^* = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5}i & -\frac{3}{5}i \end{bmatrix}$.

10. We will show that the row vectors of A, $\mathbf{r}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$ are orthonormal.

$\|\mathbf{r}_1\| = \sqrt{\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2} = 1$; $\|\mathbf{r}_2\| = \sqrt{\left|-\frac{1}{2}(1+i)\right|^2 + \left|\frac{1}{2}(1+i)\right|^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$;

$\mathbf{r}_1 \cdot \mathbf{r}_2 = \left(\frac{1}{\sqrt{2}}\right)\left(\overline{-\frac{1}{2}(1+i)}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\overline{\frac{1}{2}(1+i)}\right) = \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2} + \frac{1}{2}i\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2} - \frac{1}{2}i\right) = 0$

By Theorem 7.5.3, A is unitary, therefore $A^{-1} = A^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} + \frac{1}{2}i \\ \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{1}{2}i \end{bmatrix}$.

11. The following computations show that the column vectors of A are orthonormal:

$\|\mathbf{c}_1\| = \sqrt{\left|\frac{1}{2\sqrt{2}}(\sqrt{3}+i)\right|^2 + \left|\frac{1}{2\sqrt{2}}(1+i\sqrt{3})\right|^2} = \sqrt{\frac{4}{8} + \frac{4}{8}} = 1$;

$\|\mathbf{c}_2\| = \sqrt{\left|\frac{1}{2\sqrt{2}}(1-i\sqrt{3})\right|^2 + \left|\frac{1}{2\sqrt{2}}(i-\sqrt{3})\right|^2} = \sqrt{\frac{4}{8} + \frac{4}{8}} = 1$;

$\mathbf{c}_1 \cdot \mathbf{c}_2 = \frac{1}{2\sqrt{2}}(\sqrt{3}+i)\frac{1}{2\sqrt{2}}(1+i\sqrt{3}) + \frac{1}{2\sqrt{2}}(1+i\sqrt{3})\frac{1}{2\sqrt{2}}(-i-\sqrt{3}) = 0$

By Theorem 7.5.3, A is unitary, therefore $A^{-1} = A^* = \begin{bmatrix} \frac{1}{2\sqrt{2}}(\sqrt{3}-i) & \frac{1}{2\sqrt{2}}(1-i\sqrt{3}) \\ \frac{1}{2\sqrt{2}}(1+i\sqrt{3}) & \frac{1}{2\sqrt{2}}(-i-\sqrt{3}) \end{bmatrix}$.

12. We will show that the row vectors of A, $\mathbf{r}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}}(-1+i) & \frac{1}{\sqrt{6}}(1-i) \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$ are orthonormal.

$\|\mathbf{r}_1\| = \sqrt{\left|\frac{1}{\sqrt{3}}(-1+i)\right|^2 + \left|\frac{1}{\sqrt{6}}(1-i)\right|^2} = \sqrt{\frac{2}{3}+\frac{2}{6}} = 1$; $\|\mathbf{r}_2\| = \sqrt{\left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{2}{\sqrt{6}}\right|^2} = \sqrt{\frac{1}{3}+\frac{4}{6}} = 1$;

$\mathbf{r}_1 \cdot \mathbf{r}_2 = \left(\frac{1}{\sqrt{3}}(-1+i)\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{6}}(1-i)\right)\left(\frac{2}{\sqrt{6}}\right) = \left(\frac{1}{3}\right)(-1+i) + \left(\frac{2}{6}\right)(1-i) = 0$

By Theorem 7.5.3, A is unitary, therefore $A^{-1} = A^* = \begin{bmatrix} \frac{1}{\sqrt{3}}(-1-i) & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}}(1+i) & \frac{2}{\sqrt{6}} \end{bmatrix}$.

13. $\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & -1+i \\ -1-i & \lambda-5 \end{vmatrix} = (\lambda-3)(\lambda-6)$ thus A has eigenvalues $\lambda = 3$ and $\lambda = 6$.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (-1+i)t$, $y = t$. A vector $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $6I - A$ is $\begin{bmatrix} 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 6$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = \left(\frac{1}{2}-\frac{1}{2}i\right)t$, $y = t$. A vector $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors.

Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$. It follows that $P^{-1}AP = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix} \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$.

14. $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & i \\ -i & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-4)$ thus A has eigenvalues $\lambda = 2$ and $\lambda = 4$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (i)t$, $y = t$. A vector $\begin{bmatrix} i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I - A$ is $\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 4$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (-i)t$, $y = t$. A vector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors.

Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \end{bmatrix}$. It follows that $P^{-1}AP = \begin{bmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

15. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 6 & -2 - 2i \\ -2 + 2i & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 8)$ thus A has eigenvalues $\lambda = 2$ and $\lambda = 8$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & \frac{1}{2} + \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (-\frac{1}{2} - \frac{1}{2}i)t, y = t$. A vector $\begin{bmatrix} -1 - i \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $8I - A$ is $\begin{bmatrix} 1 & -1 - i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 8$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (1 + i)t, y = t$. A vector $\begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors.

Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{-1-i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} \frac{-1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$. It follows that $P^{-1}AP = \begin{bmatrix} \frac{-1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 6 & 2 + 2i \\ 2 - 1i & 4 \end{bmatrix}\begin{bmatrix} \frac{-1-i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$.

16. $\det(\lambda I - A) = \begin{vmatrix} \lambda & -3 - i \\ -3 + i & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda + 5)$ thus A has eigenvalues $\lambda = 2$ and $\lambda = -5$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & -\frac{3}{2} - \frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = \left(\frac{3}{2} + \frac{1}{2}i\right)t, y = t$. A vector $\begin{bmatrix} \frac{3}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-5I - A$ is $\begin{bmatrix} 1 & \frac{3}{5} + \frac{1}{5}i \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -5$ consists of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = (-\frac{3}{5} - \frac{1}{5}i)t, y = t$. A vector $\begin{bmatrix} -\frac{3}{5} - \frac{1}{5}i \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors.

Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{3+i}{\sqrt{14}} & \frac{-3-i}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} \frac{3-i}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{-3+i}{\sqrt{35}} & \frac{5}{\sqrt{35}} \end{bmatrix}$.

It follows that $P^{-1}AP = \begin{bmatrix} \frac{3-i}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{-3+i}{\sqrt{35}} & \frac{5}{\sqrt{35}} \end{bmatrix} \begin{bmatrix} 0 & 3+i \\ 3-i & -3 \end{bmatrix} \begin{bmatrix} \frac{3+i}{\sqrt{14}} & \frac{-3-i}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$.

17. The characteristic polynomial of A is $(\lambda-5)(\lambda^2+\lambda-2) = (\lambda+2)(\lambda-1)(\lambda-5)$; thus the eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 5$. The augmented matrix of the system $(-2I-A)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & -1 & 1-i & 0 \\ 0 & 1+i & -2 & 0 \end{bmatrix}$, which can be reduced to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1+i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda_1 = -2$, and $\mathbf{p}_1 = \begin{bmatrix} 0 \\ \frac{1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ is a unit eigenvector. Similar computations show that $\mathbf{p}_2 = \begin{bmatrix} 0 \\ \frac{-1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$ is a unit eigenvector corresponding to $\lambda_2 = 1$, and $\mathbf{p}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a unit eigenvector corresponding to $\lambda_3 = 5$. The vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ form an orthogonal set, and the unitary matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3]$ diagonalizes the matrix A:

$$P^*AP = \begin{bmatrix} 0 & \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1-i}{\sqrt{3}} & \frac{-1+i}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

18. $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & \lambda-2 & 0 \\ -\frac{1}{\sqrt{2}}i & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3)$ thus A has eigenvalues 1, 2, and 3.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 0 & -\sqrt{2}i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = (\sqrt{2}i)t, x_2 = -t, x_3 = t$. A vector $\begin{bmatrix} \sqrt{2}i \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = 0, x_2 = t, x_3 = t$. A vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 0 & \sqrt{2}i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$

3 consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = (-\sqrt{2}i)t, x_2 = -t, x_3 = t$. A vector $\begin{bmatrix} -\sqrt{2}i \\ -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the three bases amounts to simply normalizing the respective vectors. Therefore A is unitarily diagonalized by $P = \begin{bmatrix} \frac{1}{\sqrt{2}}i & 0 & -\frac{1}{\sqrt{2}}i \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$. Since P is unitary, $P^{-1} = P^* = \begin{bmatrix} -\frac{1}{\sqrt{2}}i & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. It follows that $P^{-1}AP = \begin{bmatrix} -\frac{1}{\sqrt{2}}i & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ -\frac{1}{\sqrt{2}}i & 2 & 0 \\ \frac{1}{\sqrt{2}}i & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}i & 0 & -\frac{1}{\sqrt{2}}i \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

19. $A = \begin{bmatrix} 0 & i & 2-3i \\ i & 0 & 1 \\ -2-3i & -1 & 4i \end{bmatrix}$

20. $A = \begin{bmatrix} 0 & 0 & 3-5i \\ 0 & 0 & -i \\ -3-5i & -i & 0 \end{bmatrix}$

21. **(a)** $(-A)_{12} = -i$ does not equal $(A^*)_{12} = i$;
also, $(-A)_{13} = -2+3i$ does not equal $(A^*)_{13} = 2-3i$

(b) $(-A)_{11} = -1$ does not equal $(A^*)_{11} = 1$;
also, $(-A)_{13} = -3+5i$ does not equal $(A^*)_{13} = -3-5i$ and
$(-A)_{23} = i$ does not equal $(A^*)_{23} = -i$.

22. **(a)** $(-A)_{13} = -2+3i$ does not equal $(A^*)_{13} = 2-3i$;
also, $(-A)_{23} = -1-i$ does not equal $(A^*)_{23} = -1+i$

(b) $(-A)_{13} = -4-7i$ does not equal $(A^*)_{13} = -4+7i$;
also, $(-A)_{33} = -1$ does not equal $(A^*)_{33} = 1$

23. $\det(\lambda I - A) - \begin{bmatrix} \lambda & 1-i \\ -1+i & \lambda - i \end{bmatrix} = \lambda^2 - i\lambda + 2 = (\lambda - 2i)(\lambda + i)$; thus the eigenvalues of A, $\lambda = 2i$ and $\lambda = -i$, are pure imaginary numbers.

24. $\det(\lambda I - A) = \begin{vmatrix} \lambda & -3i \\ -3i & \lambda \end{vmatrix} = \lambda^2 + 9 = (\lambda - 3i)(\lambda + 3i)$; the eigenvalues of A, $\lambda = 3i$ and $\lambda = -3i$, are pure imaginary numbers.

25. $A^* = \begin{bmatrix} 1-2i & 2-i & -2+i \\ 2-i & 1-i & i \\ -2+i & i & 1-i \end{bmatrix}$; we have $AA^* = A^*A = \begin{bmatrix} 15 & 8 & -8 \\ 8 & 8 & -7 \\ -8 & -7 & 8 \end{bmatrix}$

26. $A^* = \begin{bmatrix} 2-2i & -i & 1+i \\ -i & 2i & 1+3i \\ 1+i & 1+3i & -3-8i \end{bmatrix}$; we have $AA^* = A^*A = \begin{bmatrix} 11 & 4 & -14 \\ 4 & 15 & -22 \\ -14 & -22 & 85 \end{bmatrix}$

27. **(a)** If $B = \frac{1}{2}(A + A^*)$, then $B^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + A^{**}) = \frac{1}{2}(A^* + A) = B$. Similarly, $C^* = C$.

(b) We have $B + iC = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) = A$ and $B - iC = \frac{1}{2}(A + A^*) - \frac{1}{2}(A - A^*) = A^*$.

(c) $AA^* = (B + iC)(B - iC) = B^2 - iBC + iCB + C^2$ and $A^*A = B^2 + iBC - iCB + C^2$.
Thus $AA^* = A^*A$ if and only if $-iBC + iCB = iBC - iCB$, or $2iCB = 2iBC$.
Thus A is normal if and only if B and C commute i.e., $CB = BC$.

28. By Theorem 7.5.1 and Formula (5), $A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^*A\mathbf{u} = \mathbf{v}^*(A^*)^*\mathbf{u} = (A^*\mathbf{v})^*\mathbf{u} = \mathbf{u} \cdot A^*\mathbf{v}$.
Also, $\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^*\mathbf{u} = \mathbf{v}^*A^*\mathbf{u} = A^*\mathbf{u} \cdot \mathbf{v}$.

30. $AA^* = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} = \begin{bmatrix} \alpha^2 + \gamma^2 + \beta^2 + \delta^2 & \alpha\beta + \gamma\delta - \alpha\beta - \gamma\delta \\ \alpha\beta + \gamma\delta - \alpha\beta - \gamma\delta & \beta^2 + \delta^2 + \alpha^2 + \gamma^2 \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; thus $A^* = A^{-1}$ and A is unitary.

31. $A\mathbf{x} = \begin{bmatrix} \frac{7}{5} + \frac{11}{5}i \\ -\frac{1}{5} + \frac{2}{5}i \end{bmatrix}$; $\|A\mathbf{x}\| = \sqrt{\left|\frac{7}{5} + \frac{11}{5}i\right|^2 + \left|-\frac{1}{5} + \frac{2}{5}i\right|^2} = \sqrt{\frac{49}{25} + \frac{121}{25} + \frac{1}{25} + \frac{4}{25}} = \sqrt{7}$ equals
$\|\mathbf{x}\| = \sqrt{|1+i|^2 + |2-i|^2} = \sqrt{1+1+4+1} = \sqrt{7}$ which verifies part (b);
$A\mathbf{y} = \begin{bmatrix} \frac{7}{5} + \frac{4}{5}i \\ -\frac{1}{5} + \frac{3}{5}i \end{bmatrix}$; $A\mathbf{x} \cdot A\mathbf{y} = \left(\frac{7}{5} + \frac{11}{5}i\right)\overline{\left(\frac{7}{5} + \frac{4}{5}i\right)} + \left(-\frac{1}{5} + \frac{2}{5}i\right)\overline{\left(-\frac{1}{5} + \frac{3}{5}i\right)}$
$= \left(\frac{7}{5} + \frac{11}{5}i\right)\left(\frac{7}{5} - \frac{4}{5}i\right) + \left(-\frac{1}{5} + \frac{2}{5}i\right)\left(-\frac{1}{5} - \frac{3}{5}i\right) = \left(\frac{93}{25} + \frac{49}{25}i\right) + \left(\frac{7}{25} + \frac{1}{25}i\right) = 4 + 2i$ equals
$\mathbf{x} \cdot \mathbf{y} = (1+i)\overline{(1)} + (2-i)\overline{(1-i)} = (1+i)(1) + (2-i)(1+i) = (1+i) + (3+i) = 4 + 2i$ which verifies part (c).

32. Eigenvectors $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ which were found in the solution of Exercise 14 have the desired properties.

33. $A^* = \begin{bmatrix} \overline{a} & 0 & 0 \\ 0 & 0 & \overline{b} \\ 0 & \overline{c} & 0 \end{bmatrix}$; $AA^* = \begin{bmatrix} a\overline{a} & 0 & 0 \\ 0 & c\overline{c} & 0 \\ 0 & 0 & b\overline{b} \end{bmatrix} = \begin{bmatrix} |a|^2 & 0 & 0 \\ 0 & |c|^2 & 0 \\ 0 & 0 & |b|^2 \end{bmatrix}$; $A^*A = \begin{bmatrix} a\overline{a} & 0 & 0 \\ 0 & b\overline{b} & 0 \\ 0 & 0 & c\overline{c} \end{bmatrix} =$
$\begin{bmatrix} |a|^2 & 0 & 0 \\ 0 & |b|^2 & 0 \\ 0 & 0 & |c|^2 \end{bmatrix}$
A is normal if and only if $|b| = |c|$.

34. From Formulas (3) and (4), such a matrix must be equal to its own inverse.

35. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is both Hermitian and unitary.

36. Applying Theorem 5.3.1 to each column, we have

$$\overline{A+B} = \overline{A} + \overline{B} \tag{*}$$

$$\overline{kA} = \overline{k}\,\overline{A} \tag{**}$$

Part (b): $(A+B)^* \underset{\text{Def.1}}{=} \left(\overline{A+B}\right)^T \underset{(*)}{=} \left(\overline{A}+\overline{B}\right)^T = \overline{A}^T + \overline{B}^T \underset{\text{Def.1}}{=} A^* + B^*$

Part (d): $(kA)^* \underset{\text{Def.1}}{=} \left(\overline{kA}\right)^T \underset{(**)}{=} \left(\overline{k}\ \overline{A}\right)^T = \overline{k}\,\overline{A}^T \underset{\text{Def.1}}{=} \overline{k}A^*$

37. Part (a): $(A^*)^* \underset{\text{Def.1}}{=} \left(\overline{\overline{A}^T}\right)^T \underset{\text{Th. 5.3.2(b)}}{=} \left(\overline{\overline{A^T}}\right)^T \underset{\text{Th. 5.3.2(a)}}{=} (A^T)^T = A$

Part (e): $(AB)^* \underset{\text{Def.1}}{=} \left(\overline{AB}\right)^T \underset{\text{Th. 5.3.2(c)}}{=} \left(\overline{A}\ \overline{B}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T \underset{\text{Def.1}}{=} B^*A^*$

38. A is a real skew-Hermitian matrix whenever $A^* = -A$, which is equivalent to $\left(\overline{A}\right)^T = -A$:

$$\begin{bmatrix} a_{11}-b_{11}i & a_{21}-b_{21}i & \cdots & a_{n1}-b_{n1}i \\ a_{12}-b_{12}i & a_{22}-b_{22}i & \cdots & a_{n2}-b_{n2}i \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}-b_{1n}i & a_{2n}-b_{2n}i & \cdots & a_{nn}-b_{nn}i \end{bmatrix} = \begin{bmatrix} -a_{11}-b_{11}i & -a_{12}-b_{12}i & \cdots & -a_{1n}-b_{1n}i \\ -a_{21}-b_{21}i & -a_{22}-b_{22}i & \cdots & -a_{2n}-b_{2n}i \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}-b_{n1}i & -a_{n2}-b_{n2}i & \cdots & -a_{nn}-b_{nn}i \end{bmatrix}$$

Comparing the main diagonal entries on both sides, we must have $a_{11} = a_{22} = \cdots = a_{nn} = 0$.

39. If A is unitary, then $A^{-1} = A^*$ and so $(A^*)^{-1} = (A^{-1})^* = (A^*)^*$; thus A^* is also unitary.

40. If A is skew-Hermitian then $B = iA$ is Hermitian since

$$B^* = (iA)^* \underset{\text{Th. 7.5.1(d)}}{=} \overline{i}A^* = -iA^* = (-i)(-A) = iA = B$$

For every eigenvalue λ of A there must exist a nonzero vector $\mathbf{x}$ for which

$$A\mathbf{x} = \lambda\mathbf{x}$$

Multiplying both sides by i yields $(iA)\mathbf{x} = (i\lambda)\mathbf{x}$, i.e. $B\mathbf{x} = (\lambda i)\mathbf{x}$. By Theorem 7.5.2(a), λi must be real, consequently, λ is either 0 or purely imaginary.

41. A unitary matrix A has the property that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all x in C^n. Thus if A is unitary and $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$, we must have $|\lambda|\|\mathbf{x}\| = \|A\mathbf{x}\| = \|\mathbf{x}\|$ and so $|\lambda| = 1$.

42. $P^* = (\mathbf{uu}^*)^* \underset{\text{Th. 7.5.1(e)}}{=} (\mathbf{u}^*)^*\mathbf{u}^* \underset{\text{Th. 7.5.1(a)}}{=} \mathbf{uu}^* = P$ therefore P is Hermitian.

43. If $H = I - 2\mathbf{uu}^*$, then $H^* = (I - 2\mathbf{uu}^*)^* = I^* - 2\mathbf{u}^{**}\mathbf{u}^* = I - 2\mathbf{uu}^* = H$; thus H is Hermitian.
$HH^* = (I - 2\mathbf{uu}^*)(I - 2\mathbf{uu}^*) = I - 2\mathbf{uu}^* - 2\mathbf{uu}^* + 4\mathbf{uu}^*\mathbf{uu}^* = I - 4\mathbf{uu}^* + 4\mathbf{u}\|\mathbf{u}\|^2\mathbf{u}^* = I$
so H is unitary.

44. $A^*(A^{-1})^* \underset{\text{Th. 7.5.1(e)}}{=} (A^{-1}A)^* = I^* = I$ therefore A^* is invertible and its inverse is $(A^{-1})^*$.

45. **(a)** This result can be obtained by mathematical induction.

(b) $\det(A^*) = \det\left(\left(\overline{A}\right)^T\right) = \det\left(\overline{A}\right) = \overline{\det(A)}$.

True-False Exercises

(a) False. Denoting $A = \begin{bmatrix} 0 & i \\ i & 2 \end{bmatrix}$, we observe that $(A)_{12} = i$ does not equal $(A^*)_{12} = -i$.

(b) False. For $\mathbf{r}_1 = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 0 & -\frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$,

$\mathbf{r}_1 \cdot \mathbf{r}_2 = -\frac{i}{\sqrt{2}}(\overline{0}) + \frac{i}{\sqrt{6}}\left(\overline{-\frac{i}{\sqrt{6}}}\right) + \frac{i}{\sqrt{3}}\left(\overline{\frac{i}{\sqrt{3}}}\right) = 0 + \left(\frac{i}{\sqrt{6}}\right)^2 - \left(\frac{i}{\sqrt{3}}\right)^2 = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6} \neq 0$

thus the row vectors do not form an orthonormal set and the matrix is not unitary by Theorem 7.5.3.

(c) True. If A is unitary, so $A^{-1} = A^*$, then $(A^*)^{-1} = A = (A^*)^*$.

(d) False. Normal matrices that are not Hermitian are also unitarily diagonalizable.

(e) False. If A is skew-Hermitian, then $(A^2)^* = (A^*)(A^*) = (-A)(-A) = A^2 \neq -A^2$.

Chapter 7 Supplementary Exercises

1. **(a)** For $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $A^{-1} = A^T = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$.

(b) For $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so $A^{-1} = A^T = \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$.

3. Since A is symmetric, there exists an orthogonal matrix P such that $P^T AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$.

Since A is positive definite, all $\lambda's$ must be positive. Let us form a diagonal matrix $C = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}$. Then $A = PDP^T = PCC^TP^T = (PC)(PC)^T$. The matrix $(PC)^T$ is nonsingular (it is a transpose of a product of two nonsingular matrices), therefore it generates an inner product on R^n:

$$\langle \mathbf{u}, \mathbf{v} \rangle = (PC)^T\mathbf{u} \cdot (PC)^T\mathbf{v} = \mathbf{u}^T(PCC^TP^T)\mathbf{v} = \mathbf{u}^T A\mathbf{v}$$

4. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -2 & -2 \\ -2 & \lambda - 3 & -2 \\ -2 & -2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)^2(\lambda - 7)$.

The eigenvalues of A are $\lambda = 1$ and $\lambda = 7$.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -s - t, x_2 = s, x_3 = t$. This eigenspace has dimension 2 (vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form its basis.)

The reduced row echelon form of $7I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$ 7 contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. This eigenspace has dimension 1 ($\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms its basis.)

5. The characteristic equation of A is $\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 2)(\lambda - 1)$, so the eigenvalues are λ = 0, 2, 1.

Orthogonal bases for the eigenspaces are λ = 0: $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; λ = 2: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$; λ = 1: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Thus $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$ orthogonally diagonalizes A, and $P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6. **(a)** $-4x_1^2 + 16x_2^2 - 15x_1x_2 = [x_1 \quad x_2] \begin{bmatrix} -4 & -\frac{15}{2} \\ -\frac{15}{2} & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}$

(b) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3 = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & \frac{1}{2} \\ -4 & \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}$

7. In matrix form, the quadratic form is $\mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2] \begin{bmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The characteristic equation of A is $\lambda^2 - 5\lambda + \frac{7}{4} = 0$ which has solutions $\lambda = \frac{5 \pm 3\sqrt{2}}{2}$ or
λ ≈ 4.62, 0.38. Since both eigenvalues of A are positive, the quadratic form is positive definite.

8. **(a)** $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2] \begin{bmatrix} -3 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the characteristic polynomial of the matrix A is $\det(\lambda I - A) = \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda - 5 \end{vmatrix} = \left(\lambda - \left(1 + \sqrt{17}\right)\right)\left(\lambda - \left(1 - \sqrt{17}\right)\right)$ so A has eigenvalues $1 \pm \sqrt{17}$.

The reduced row echelon form of $\left(1 + \sqrt{17}\right)I - A$ is $\begin{bmatrix} 1 & 4 - \sqrt{17} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1 + \sqrt{17}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = (-4 + \sqrt{17})t, x_2 = t$.

A vector $\mathbf{p}_1 = \begin{bmatrix} -4 + \sqrt{17} \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $(1-\sqrt{17})I - A$ is $\begin{bmatrix} 1 & 4+\sqrt{17} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1-\sqrt{17}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = (-4-\sqrt{17})t$, $x_2 = t$.

A vector $\mathbf{p}_2 = \begin{bmatrix} -4-\sqrt{17} \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases $\{\mathbf{p}_1\}$ and $\{\mathbf{p}_2\}$ amounts to simply normalizing the vectors.

Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-4+\sqrt{17}}{\sqrt{34-8\sqrt{17}}} & \frac{-4-\sqrt{17}}{\sqrt{34+8\sqrt{17}}} \\ \frac{1}{\sqrt{34-8\sqrt{17}}} & \frac{1}{\sqrt{34+8\sqrt{17}}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. In terms of the new variables, we have

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^TAP)\mathbf{y} = [y_1 \quad y_2] \begin{bmatrix} 1+\sqrt{17} & 0 \\ 0 & 1-\sqrt{17} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (1+\sqrt{17})y_1^2 + (1-\sqrt{17})y_2^2.$$

(b) $Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} -5 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the characteristic polynomial of the matrix A is $\det(\lambda I - A) = \begin{vmatrix} \lambda+5 & -2 & -3 \\ -2 & \lambda-1 & 0 \\ -3 & 0 & \lambda+1 \end{vmatrix} = \lambda(\lambda-2)(\lambda+7)$ so the eigenvalues of A are 0, 2, and -7.

The reduced row echelon form of $0I - A$ is $\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 0$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{1}{3}t$, $x_2 = -\frac{2}{3}t$, $x_3 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t$, $x_2 = 2t$, $x_3 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-7I - A$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -7$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -2t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $\mathbf{p}_3 = \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to the three bases amounts to simply normalizing the vectors. Therefore an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms in Q is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{21}} \\ -\frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. In terms of the new variables, we have

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 2y_2^2 - 7y_3^2.$$

9. **(a)** $y - x^2 = 0$ or $y = x^2$ represents a parabola.

(b) $3x - 11y^2 = 0$ or $x = \frac{11}{3}y^2$ represents a parabola.

10. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -1 \\ -1 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda^2 - \lambda + 2)$ thus A has eigenvalues 2 and $\frac{1 \pm \sqrt{3}i}{2}$.

The reduced row echelon form of $2I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda =$ 2 consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = t, x_2 = t, x_3 = t$. A vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $\frac{1+\sqrt{3}i}{2}I - A$ is $\begin{bmatrix} 1 & 0 & \frac{1-\sqrt{3}i}{2} \\ 0 & 1 & \frac{1+\sqrt{3}i}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = \frac{1+\sqrt{3}i}{2}$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = \frac{-1+\sqrt{3}i}{2}t, x_2 = \frac{-1-\sqrt{3}i}{2}t, x_3 = t$. A vector $\begin{bmatrix} \frac{-1+\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{2} \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

By Theorem 5.3.4, a vector $\begin{bmatrix} \frac{-1-\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} \\ 1 \end{bmatrix}$ forms a basis for the eigenspace corresponding to $\lambda = \frac{1-\sqrt{3}i}{2}$.

Applying the Gram-Schmidt process to the three bases amounts to simply normalizing the respective vectors. Therefore A is unitarily diagonalized by $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

Since U is unitary, $U^{-1} = U^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1+\sqrt{3}i}{2\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

It follows that $U^{-1}AU = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1+\sqrt{3}i}{2\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1-\sqrt{3}i}{2\sqrt{3}} & \frac{-1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1+\sqrt{3}i}{2} & 0 \\ 0 & 0 & \frac{1-\sqrt{3}i}{2} \end{bmatrix}$.

11. Partitioning U into columns we can write $U = [\mathbf{u}_1|\mathbf{u}_2|\dots|\mathbf{u}_n]$. The given product can be rewritten in partitioned form as well:

$$A = U\begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} = [\mathbf{u}_1|\mathbf{u}_2|\dots|\mathbf{u}_n]\begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} = [z_1\mathbf{u}_1|z_2\mathbf{u}_2|\dots|z_n\mathbf{u}_n]$$

By Theorem 7.5.3, the columns of U form an orthonormal set. Therefore, columns of A must also be orthonormal: $(z_i\mathbf{u}_i)\cdot(z_j\mathbf{u}_j) = (z_i\overline{z_j})(\mathbf{u}_i\cdot\mathbf{u}_j) = 0$ for all $i \neq j$ and $\|z_i\mathbf{u}_i\| = |z_i|\|\mathbf{u}_i\| = 1$ for all i. By Theorem 7.5.3, A is a unitary matrix.

12. Refer to the solution of Exercise 40 in Section 7.5.

13. Partitioning the given matrix into columns $A = [\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3]$, we must find $\mathbf{u}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$\mathbf{u}_1\cdot\mathbf{u}_2 = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{3}} = 0$, $\mathbf{u}_1\cdot\mathbf{u}_3 = -\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{6}} + \frac{c}{\sqrt{3}} = 0$, and $\|\mathbf{u}_1\|^2 = a^2 + b^2 + c^2 = 1$.

Subtracting the second equation from the first one yields $a = 0$. Therefore $c = -\frac{\sqrt{3}}{\sqrt{6}}b = -\frac{b}{\sqrt{2}}$.

Substituting into $\|\mathbf{u}_1\|^2 = 1$ we obtain $b^2 + \frac{b^2}{2} = 1$ so that $b^2 = \frac{2}{3}$.

There are two possible solutions:

- $a = 0,\ b = \sqrt{\frac{2}{3}},\ c = -\frac{1}{\sqrt{3}}$ and
- $a = 0,\ b = -\sqrt{\frac{2}{3}},\ c = \frac{1}{\sqrt{3}}$.

14. **(a)** Negative definite

(b) Positive definite

(c) Indefinite

(d) Indefinite

(e) Indefinite

(f) Theorem 7.3.4 is inconclusive

CHAPTER 8: LINEAR TRANSFORMATIONS

8.1 General Linear Transformations

1. **(a)** $T\left(2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = T\left(\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}^2 = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$ does not equal
$2T\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^2 = 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$ so T does not satisfy the homogeneity property.
Consequently, T is not a linear transformation.

(b) Let A and B be any 2×2 matrices and let k be any real number. We have
$T(kA) = \text{tr}(kA) = ka_{11} + ka_{22} = k(a_{11} + a_{22}) = k\,\text{tr}(A) = kT(A)$ and
$T(A + B) = \text{tr}(A + B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22})$
$= \text{tr}(A) + \text{tr}(B) = T(A) + T(B)$
therefore T is a linear transformation.

The kernel of T consists of all matrices $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that $T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a + d = 0$, i.e., $d = -a$.
We conclude that the kernel of T consists of all matrices of the form $\begin{bmatrix}a & b\\c & -a\end{bmatrix}$.

(c) Let A and B be any 2×2 matrices and let k be any real number. We have
$T(kA) = kA + (kA)^T = kA + kA^T = k(A + A^T) = kT(A)$ and
$T(A + B) = A + B + (A + B)^T = A + B + A^T + B^T = A + A^T + B + B^T = T(A) + T(B)$
therefore T is a linear transformation.
The kernel of T consists of all matrices $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that
$T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a & b\\c & d\end{bmatrix} + \begin{bmatrix}a & b\\c & d\end{bmatrix}^T = \begin{bmatrix}a & b\\c & d\end{bmatrix} + \begin{bmatrix}a & c\\b & d\end{bmatrix} = \begin{bmatrix}2a & b+c\\b+c & 2d\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ therefore $a = d = 0$ and $c = -b$.
We conclude that the kernel of T consists of all matrices of the form $\begin{bmatrix}0 & b\\-b & 0\end{bmatrix}$.

2. **(a)** Let A and B be any 2×2 matrices and let k be any real number. We have
$T(kA) = (kA)_{11} = ka_{11} = k(A)_{11} = kT(A)$ and
$T(A + B) = (A + B)_{11} = a_{11} + b_{11} = (A)_{11} + (B)_{11} = T(A) + T(B)$
therefore T is a linear transformation.

The kernel of T consists of all matrices $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that $T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a = 0$.
We conclude that the kernel of T consists of all matrices of the form $\begin{bmatrix}0 & b\\c & d\end{bmatrix}$.

(b) Let A and B be any 2×2 matrices and let k be any real number. We have
$T(kA) = O_{2\times 2} = kO_{2\times 2} = kT(A)$ and $T(A + B) = O_{2\times 2} = O_{2\times 2} + O_{2\times 2} = T(A) + T(B)$
therefore T is a linear transformation.

The kernel of T is M_{22}.

(c) Let A and B be any 2×2 matrices and let k be any real number. We have
$T(kA) = c(kA) = k(cA) = kT(A)$ and $T(A+B) = c(A+B) = cA + cB = T(A) + T(B)$
therefore T is a linear transformation.
The kernel of T consists of all matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that
$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
If $c = 0$ then $\ker(T) = M_{22}$, otherwise $\ker(T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$.

3. For $\mathbf{u} \neq \mathbf{0}$, $T(-1\mathbf{u}) = \|-\mathbf{u}\| = \|\mathbf{u}\| = T(\mathbf{u}) \neq -1T(\mathbf{u})$, so the mapping is not a linear transformation.

4. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in R^3 and let k be any real number. By properties of cross product listed in Theorem 3.5.2, we have
$T(k\mathbf{u}) = (k\mathbf{u}) \times \mathbf{v}_0 = k(\mathbf{u} \times \mathbf{v}_0) = kT(\mathbf{u})$ and
$T(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \times \mathbf{v}_0 = (\mathbf{u} \times \mathbf{v}_0) + (\mathbf{v} \times \mathbf{v}_0) = T(\mathbf{u}) + T(\mathbf{v})$
therefore T is a linear transformation.
If $\mathbf{v}_0 = \mathbf{0}$ then $\ker(T) = R^3$. Otherwise $\ker(T) = \operatorname{span}\{\mathbf{v}_0\}$.

5. Let A_1 and A_2 be any 2×2 matrices and let k be any real number. We have
$T(kA_1) = (kA_1)B = k(A_1 B) = kT(A_1)$ and
$T(A_1 + A_2) = (A_1 + A_2)B = A_1 B + A_2 B = T(A_1) + T(A_2)$ therefore T is a linear transformation.
The kernel of T consists of all 2×2 matrices whose rows are orthogonal to all columns of B.

6. **(a)** $T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = 3ka - 4kb + kc - kd = k(3a - 4b + c - d) = kT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$
$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = T\left(\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}\right) = 3(a+a') - 4(b+b') + (c+c') - (d+d')$
$= (3a - 4b + c - d) + (3a' - 4b' + c' - d') = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$
therefore T is a linear transformation.
The kernel of T consists of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $3a - 4b + c - d = 0$.

(b) $T\left(2\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\right) = 16$ does not equal $2T\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) = 2(4) = 8$ therefore T is not a linear transformation.

7. Let $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$.

(a) $T(kp(x)) = ka_0 + ka_1(x+1) + ka_2(x+1)^2 = kT(p(x))$
$T(p(x) + q(x)) = a_0 + b_0 + (a_1 + b_1)(x+1) + (a_2 + b_2)(x+1)^2$
$= a_0 + a_1(x+1) + a_2(x+1)^2 + b_0 + b_1(x+1) + b_2(x+1)^2 = T(p(x)) + T(q(x))$
Thus T is a linear transformation.
The kernel of T consists of all polynomials $a_0 + a_1 x + a_2 x^2$ such that
$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x+1) + a_2(x+1)^2 = 0$. This equality requires that $a_0 = a_1 = a_2 = 0$ therefore $\ker(T) = \{0\}$.

(b) $T(kp(x)) = T(ka_0 + ka_1 x + ka_2 x^2) = (ka_0 + 1) + (ka_1 + 1)x + (ka_2 + 1)x^2 \neq kT(p(x))$ so T is not a linear transformation.

8. **(a)** Since T maps the zero function $f(x) = 0$ to $g(x) = 1$, by Theorem 8.1.1(a), T is not a linear transformation.

(b) Let f and g be any functions in $F(-\infty, \infty)$ and let k be any real number. We have $T(kf(x)) = kf(x+1) = kT(f(x))$ and $T\big(f(x) + g(x)\big) = f(x+1) + g(x+1) = T\big(f(x)\big) + T(g(x))$ therefore T is a linear transformation.
The kernel of T contains only the zero function.

9. $T\big(k(a_0, a_1, a_2, \dots, a_n, \dots)\big) = T(ka_0, ka_1, ka_2, \dots, ka_n, \dots) = (0, ka_0, ka_1, \dots, ka_n, \dots)$
$= k(0, a_0, a_1, \dots, a_n, \dots) = kT(a_0, a_1, a_2, \dots, a_n, \dots)$
$T\big((a_0, a_1, a_2, \dots, a_n, \dots) + (b_0, b_1, b_2, \dots, b_n \dots)\big) = T(a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots, a_n + b_n, \dots)$
$= (0, a_0 + b_0, a_1 + b_1, \dots, a_n + b_n, \dots) = (0, a_0, a_1, \dots, a_n, 0) + (0, b_0, b_1, \dots, b_n, 0)$
$= T(a_0, a_1, a_2, \dots, a_n, \dots) + T(b_0, b_1, b_2, \dots, b_n, \dots)$ therefore T is a linear transformation.
The kernel of T contains only $(0,0,0, \dots)$.

10. **(a)** $T(x^2) = (x)(x^2) = x^3 \neq 0$ therefore x^2 is not in $\ker(T)$.

(b) $T(0) = (x)(0) = 0$ therefore 0 is in $\ker(T)$.

(c) $T(1+x) = (x)(1+x) = x + x^2 \neq 0$ therefore $1 + x$ is not in $\ker(T)$.

(d) $T(-x) = (x)(-x) = -x^2 \neq 0$ therefore $-x$ is not in $\ker(T)$.

11. **(a)** Since $x + x^2 = x(1+x)$, $x + x^2$ is in $R(T)$.

(b) $1 + x$ cannot be expressed in the form $xp(x)$ for any polynomial $p(x)$ therefore $1 + x$ is not in $R(T)$.

(c) $3 - x^2$ cannot be expressed in the form $xp(x)$for any polynomial $p(x)$ therefore $3 - x^2$ is not in $R(T)$.

(d) $-x = x(-1)$ therefore $-x$ is in $R(T)$.

12. **(a)** The equation $3\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$ therefore $\ker(T) = \{\mathbf{0}\}$.

(b) Every vector $\mathbf{w}$ in V is an image of $\frac{1}{3}\mathbf{w}$ under T. Consequently, $R(T) = V$.

13. **(a)** nullity(T) = 5 – rank(T) = 2

(b) $\dim(P_4) = 5$, so nullity(T) = 5 – rank(T) = 4

(c) Since $R(T) = R^3$, T has rank 3; $\dim(M_{mn}) = mn$ so $\text{nullity}(T) = mn - \text{rank}(T) = mn - 3$

(d) nullity(T) = 4 – rank(T) = 1

14. **(a)** $\text{rank}(T) = 7 - \text{nullity}(T) = 5$

(b) $\text{rank}(T) = \dim(P_3) - \text{nullity}(T) = 4 - 1 = 3$

(c) Since $\ker(T) = P_5$, T has nullity 6; $\text{rank}(T) = \dim(P_5) - \text{nullity}(T) = 6 - 6 = 0$

(d) $\text{rank}(T) = \dim(P_n) - \text{nullity}(T) = n + 1 - 3 = n - 2$

15. **(a)** $T\left(\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}\right) = 3\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -12 & 9 \end{bmatrix}$

(b) The only 2×2 matrix A such that $3A = 0$ is the zero matrix. Consequently, $\ker(T) = \{0\}$ so the nullity of T is 0.
By Theorem 8.1.4, $\operatorname{rank}(T) = \dim(M_{22}) - \operatorname{nullity}(T) = 4 - 0 = 4$.

16. **(a)** $T(1 + 4x + 8x^2) = \frac{1}{4} + x + 2x^2$

(b) The only polynomial p in P_2 A such that $\frac{1}{4}p = 0$ is the zero polynomial. Consequently, $\ker(T) = \{0\}$ so the nullity of T is 0.
By Theorem 8.1.4, $\operatorname{rank}(T) = \dim(P_2) - \operatorname{nullity}(T) = 3 - 0 = 3$.

17. **(a)** $T(x^2) = ((-1)^2, 0^2, 1^2) = (1,0,1)$

(b) The kernel of T consists of all polynomials $p(x) = a_0 + a_1x + a_2x^2$ such that $\big(p(-1), p(0), p(1)\big) = (a_0 - a_1 + a_2,\ a_0,\ a_0 + a_1 + a_2) = (0,0,0)$.
Equating the corresponding components we obtain a linear system

$$\begin{array}{ccccccc} a_0 & - & a_1 & + & a_2 & = & 0 \\ a_0 & & & & & = & 0 \\ a_0 & + & a_1 & + & a_2 & = & 0 \end{array}$$

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ hence the system has a unique solution $a_0 = a_1 = a_2 = 0$. We conclude that $\ker(T) = \{0\}$.

(c) It follows from the solution of part (b) that $\operatorname{nullity}(T) = 0$.
By Theorem 8.1.4, $\operatorname{rank}(T) = \dim(P_2) - \operatorname{nullity}(T) = 3 - 0 = 3$. Consequently, $R(T) = R^3$.

18. **(a)** $T(1 + \sin x + \cos x) = (1 + \sin 0 + \cos 0\,, 1 + \sin \pi + \cos \pi\,, 1 + \sin(2\pi) + \cos(2\pi)) = (2,0,2)$

(b) The kernel of T consists of all functions $f(x) = c_1 + c_2 \sin x + c_3 \cos x$ such that
$\big(f(0), f(\pi), f(2\pi)\big)$
$=(c_1 + c_2 \sin 0 + c_3 \cos 0\,, c_1 + c_2 \sin \pi + c_3 \cos \pi\,, c_1 + c_2 \sin(2\pi) + c_3 \cos(2\pi))$
$= (c_1 + c_3, c_1 - c_3, c_1 + c_3) = (0,0,0)$
Equating the corresponding components yields $c_1 = c_3 = 0$. Since c_2 is an arbitrary real number, we conclude that $\ker(T) = \operatorname{span}\{\sin x\}$.

(c) $T(c_1 + c_2 \sin x + c_3 \cos x) = (c_1 + c_3, c_1 - c_3, c_1 + c_3) = c_1(1,1,1) + c_3(1,-1,1)$.
Consequently, $R(T) = \operatorname{span}\{(1,1,1), (1,-1,1)\}$.

19. For $\mathbf{x} = (x_1,\ x_2) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we have $(x_1,\ x_2) = c_1(1,\ 1) + c_2(1,\ 0) = (c_1 + c_2,\ c_1)$ or

$$\begin{array}{ccccc} c_1 & + & c_2 & = & x_1 \\ c_1 & & & = & x_2 \end{array}$$

which has the solution $c_1 = x_2, c_2 = x_1 - x_2$.
$(x_1,\ x_2) = x_2(1,\ 1) + (x_1 - x_2)(1,\ 0) = x_2\mathbf{v}_1 + (x_1 - x_2)\mathbf{v}_2$ and
$T(x_1,\ x_2) = x_2T(\mathbf{v}_1) + (x_1 - x_2)T(\mathbf{v}_2) = x_2(1,\ -2) + (x_1 - x_2)(-4,\ 1) = (-4x_1 + 5x_2,\ x_1 - 3x_2)$
$T(5, -3) = (-20 - 15, 5 + 9) = (-35, 14)$.

20. We begin by expressing (x_1, x_2) as a linear combination of the basis vectors $(-2,1)$ and $(1,3)$:

$$(x_1, x_2) = c_1(-2,1) + c_2(1,3)$$

Equating the corresponding components we obtain a linear system

$$\begin{array}{rcrcl} -2c_1 & + & c_2 & = & x_1 \\ c_1 & + & 3c_2 & = & x_2 \end{array}$$

which yields $c_1 = -\frac{3}{7}x_1 + \frac{1}{7}x_2$, $c_2 = \frac{1}{7}x_1 + \frac{2}{7}x_2$, allowing us to write

$(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-2,1) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(1,3)$ and
$T(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)T(-2,1) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)T(1,3)$
$= \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-1,2,0) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(0,-3,5) = \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)$ so that
$T(2,-3) = \left(\frac{9}{7}, -\frac{6}{7}, -\frac{20}{7}\right)$.

21. For $\mathbf{x} = (x_1,\ x_2,\ x_3) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, we have
$(x_1,\ x_2,\ x_3) = c_1(1,\ 1,\ 1) + c_2(1,\ 1,\ 0) + c_3(1,\ 0,\ 0) = (c_1 + c_2 + c_3,\ c_1 + c_2,\ c_1)$ or

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & x_1 \\ c_1 & + & c_2 & & & = & x_2 \\ c_1 & & & & & = & x_3 \end{array}$$

which has the solution $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - (x_2 - x_3) - x_3 = x_1 - x_2$.
$(x_1,\ x_2,\ x_3) = x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3$
$T(x_1,\ x_2,\ x_3) = x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3)$
$= x_3(2,\ -1,\ 4) + (x_2 - x_3)(3,\ 0,\ 1) + (x_1 - x_2)(-1,\ 5,\ 1)$
$= (-x_1 + 4x_2 - x_3,\ 5x_1 - 5x_2 - x_3,\ x_1 + 3x_3)$
$T(2,\ 4,\ -1) = (-2 + 16 + 1,\ 10 - 20 + 1,\ 2 - 3) = (15,\ -9,\ -1)$

22. We begin by expressing (x_1, x_2, x_3) as a linear combination of the basis vectors $(1,2,1)$, $(2,9,0)$, and $(3,3,4)$:

$$(x_1, x_2, x_3) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

Equating the corresponding components we obtain a linear system

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & x_1 \\ 2c_1 & + & 9c_2 & + & 3c_3 & = & x_2 \\ c_1 & & & + & 4c_3 & = & x_3 \end{array}$$

which yields $c_1 = -36x_1 + 8x_2 + 21x_3$, $c_2 = 5x_1 - x_2 - 3x_3$, $c_3 = 9x_1 - 2x_2 - 5x_3$ allowing us to write

$(x_1, x_2, x_3) = (-36x_1 + 8x_2 + 21x_3)(1,2,1) + (5x_1 - x_2 - 3x_3)(2,9,0) + (9x_1 - 2x_2 - 5x_3)(3,3,4)$ and

$$T(x_1, x_2, x_3)$$

$= (-36x_1 + 8x_2 + 21x_3)T(1,2,1) + (5x_1 - x_2 - 3x_3)T(2,9,0) + (9x_1 - 2x_2 - 5x_3)T(3,3,4)$
$= (-36x_1 + 8x_2 + 21x_3)(1,0) + (5x_1 - x_2 - 3x_3)(-1,1) + (9x_1 - 2x_2 - 5x_3)(0,1)$
$= (-41x_1 + 9x_2 + 24x_3, 14x_1 - 3x_2 - 8x_3)$
so that $T(7,13,7) = (-2,3)$.

23. **(a)** $T\big(k(a_0 + a_1x + a_2x^2 + a_3x^3)\big) = T(ka_0 + ka_1x + ka_2x^2 + ka_3x^3) = 5ka_0 + ka_3x^2$
$= k(5a_0 + a_3x^2) = kT(a_0 + a_1x + a_2x^2 + a_3x^3)$

$T(a_0 + a_1x + a_2x^2 + a_3x^3 + b_0 + b_1x + b_2x^2 + b_3x^3)$
$= T\big((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3\big) = 5(a_0 + b_0) + (a_3 + b_3)x^2$
$= (5a_0 + a_3x^2) + (5b_0 + b_3x^2) = T(a_0 + a_1x + a_2x^2 + a_3x^3) + T(b_0 + b_1x + b_2x^2 + b_3x^3)$
therefore T is linear.

(b) The kernel of T consists of all polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $T(a_0 + a_1x + a_2x^2 + a_3x^3) = 5a_0 + a_3x^2 = 0$, which requires that $a_0 = a_3 = 0$. Therefore every vector in $\ker(T)$ can be written in the form $a_1x + a_2x^2$, i.e., $\ker(T) = \text{span}\{x, x^2\}$. The set $\{x, x^2\}$ is linearly independent since neither polynomial is a scalar multiple of the other one. We conclude that $\{x, x^2\}$ is a basis for $\ker(T)$.

(c) $T(a_0 + a_1x + a_2x^2 + a_3x^3) = 5a_0 + a_3x^2$ so $R(T) = \text{span}\{5, x^2\}$. The set $\{5, x^2\}$ is linearly independent since neither polynomial is a scalar multiple of the other one.
We conclude that $\{5, x^2\}$ is a basis for $R(T)$.

24. **(a)** $T\big(k(a_0 + a_1x + a_2x^2)\big) = T(ka_0 + ka_1x + ka_2x^2) = 3ka_0 + ka_1x + (ka_0 + ka_1)x^2$
$= k(3a_0 + a_1x + (a_0 + a_1)x^2) = kT(a_0 + a_1x + a_2x^2)$
$T(a_0 + a_1x + a_2x^2 + b_0 + b_1x + b_2x^2)$
$= T\big((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\big) = 3(a_0 + b_0) + (a_1 + b_1)x + (a_0 + b_0 + a_1 + b_1)x^2$
$= (3a_0 + a_1x + (a_0 + a_1)x^2) + (3b_0 + b_1x + (b_0 + b_1)x^2)$
$= T(a_0 + a_1x + a_2x^2) + T(b_0 + b_1x + b_2x^2)$
therefore T is linear.

(b) The kernel of T consists of all polynomials $p(x) = a_0 + a_1x + a_2x^2$ such that $T(a_0 + a_1x + a_2x^2) = 3a_0 + a_1x + (a_0 + a_1)x^2 = 0$, which requires that $a_0 = a_1 = 0$. Therefore every vector in $\ker(T)$ can be written in the form a_2x^2, i.e., $\ker(T) = \text{span}\{x^2\}$. The set $\{x^2\}$ is linearly independent since x^2 is not the zero polynomial.
We conclude that $\{x^2\}$ is a basis for $\ker(T)$.

(c) $T(a_0 + a_1x + a_2x^2) = 3a_0 + a_1x + (a_0 + a_1)x^2 = a_0(3 + x^2) + a_1(x + x^2)$
so $R(T) = \text{span}\{3 + x^2, x + x^2\}$.
The set $\{3 + x^2, x + x^2\}$ is linearly independent since neither polynomial is a scalar multiple of the other one. We conclude that $\{3 + x^2, x + x^2\}$ is a basis for $R(T)$.

25. **(a)** If $p'(x) = 0$, then $p(x)$ is a constant, so $\ker(D)$ consists of all constant polynomials.

(b) The kernel of J contains all polynomials $a_0 + a_1x$ such that $\int_{-1}^{1}(a_0 + a_1x)\,dx = 0$. By integration, this condition yields $\left(a_0x + \frac{a_1x^2}{2}\right)\Big]_{-1}^{1} = 0$, i.e., $a_0 + \frac{a_1}{2} + a_0 - \frac{a_1}{2} = 0$, or equivalently, $a_0 = 0$.
The kernel consists of all polynomials of the form a_1x.

26. For any functions f and g in $C[a, b]$ and for any real number k we have
$T(k\mathbf{f}) = 5kf(x) + 3\int_a^x kf(t)\,dt = k\big(5f(x) + \int_a^x f(t)\,dt\big) = kT(\mathbf{f})$ and
$T(\mathbf{f} + \mathbf{g}) = 5\big(f(x) + g(x)\big) + 3\int_a^x\big(f(t) + g(t)\big)dt = \big(5f(x) + 3\int_a^x f(t)\,dt\big) + \big(5g(x) + 3\int_a^x g(t)\,dt\big)$
$= T(\mathbf{f}) + T(\mathbf{g})$
This shows that T is a linear operator.

27. **(a)** If $f^4(x) = 0$, then $f'''(x) = a$ for some constant a. Applying Fundamental Theorem of Calculus, we obtain $f''(x) = ax + b$, then $f'(x) = \frac{a}{2}x^2 + bx + c$, and $f(x) = \frac{a}{6}x^3 + \frac{b}{2}x^2 + cx + d$ for constant b, c, and d. We conclude that P_3 is the kernel of $T(f(x)) = f^{(4)}(x)$.

(b) By similar reasoning, $T(f(x)) = f^{(n+1)}(x)$ has $\ker(T) = P_n$.

28. Since $R(T) = R$, it follows from Theorem 8.1.4 that $\ker(T)$ has dimension $\dim(M_{nn}) - \dim(R) = n^2 - 1$.

29. **(a)** $R(T)$ must be a subspace of R^3, thus the possibilities are a line through the origin, a plane through the origin, the origin only, or all of R^3.

(b) The origin, a line through the origin, a plane through the origin, or the entire space R^3 .

30. **(a)** For all polynomials $p(x)$ and $q(x)$ in P_n and for every scalar k we have
$T\big(kp(x)\big) = kp(x+1) = kT\big(p(x)\big)$ and
$T\big(p(x) + q(x)\big) = p(x+1) + q(x+1) = T\big(p(x)\big) + T\big(q(x)\big)$ therefore T is linear.

(b) Since T maps the zero polynomial $p(x) = 0$ to $q(x) = 1$, by Theorem 8.1.1(a), T is not a linear transformation.

31. $T(2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3) = 2T(\mathbf{v}_1) - 3T(\mathbf{v}_2) + 4T(\mathbf{v}_3) = (2,\ -2,\ 4) - (0,\ 9,\ 6) + (-12,\ 4,\ 8)$
$= (-10,\ -7,\ 6)$

32. Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ be any vector in V. Then
$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) = c_1\mathbf{0} + c_2\mathbf{0} + \cdots + c_n\mathbf{0} = \mathbf{0}$
Since $\mathbf{v}$ was an arbitrary vector in V, T must be the zero transformation.

33. Let $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ be any vector in V. Then
$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{v}$
Since $\mathbf{v}$ was an arbitrary vector in V, T must be the identity operator.

34. For every vector $\mathbf{v}$ in V, there are unique scalars $c_1, c_2, ..., c_n$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.
A linear transformation with the desired properties can be defined by
$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$

True-False Exercises

(a) True. $c_1 = k$, $c_2 = 0$ gives the homogeneity property and $c_1 = c_2 = 1$ gives the additivity property.

(b) False. Every linear transformation will have $T(-\mathbf{v}) = -T(\mathbf{v})$.

(c) True. Only the zero transformation has this property.

(d) False. $T(\mathbf{0}) = \mathbf{v}_0 + \mathbf{0} = \mathbf{v}_0 \neq \mathbf{0}$, so T is not a linear transformation.

(e) True. This follows from part (a) of Theorem 8.1.3.

(f) True. This follows from part (b) of Theorem 8.1.3.

(g) False. T does not necessarily have rank 4.

(h) False. $\det(A + B) \neq \det(A) + \det(B)$ in general.

(i) False. nullity(T) = rank(T) = 2

8.2 Compositions and Inverse Transformations

1. **(a)** By inspection, ker(T) = {$\mathbf{0}$}, so T is one-to-one.

(b) By inspection, ker(T) = {$\mathbf{0}$}, so T is one-to-one.

(c) (x, y, z) is in ker(T) if both $x + y + z = 0$ and $x - y - z = 0$, which is $x = 0$ and $y + z = 0$. Thus, $\ker(T) = \text{span}\{(0, 1, -1)\}$ and T is not one-to-one.

2. **(a)** (x, y) is in ker(T) if $x - y = 0$ or $x = y$, so $\ker(T) = \text{span}\{(1, 1)\}$ and T is not one-to-one.

(b) $T(x, y) = \mathbf{0}$ if $2x + 3y = 0$ or $x = -\frac{3}{2}y$ so $\ker(T) = \text{span}\left\{\left(-\frac{3}{2}, 1\right)\right\}$ and T is not one-to-one.

(c) (x, y) is in ker(T) only if $x + y = 0$ and $x - y = 0$, so $\ker(T) = \{\mathbf{0}\}$ and T is one-to-one.

3. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, so nullity($A$)= 1.

Multiplication by A is not one-to-one.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 30 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 7 \end{bmatrix}$, so nullity($A$)= 1.

Multiplication by A is not one-to-one.

4. **(a)** The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so nullity($A$)= 0.

Multiplication by A is one-to-one.

(b) The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 12 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so nullity($A$)= 1.

Multiplication by A is not one-to-one.

5. **(a)** Since nullity(T)= 0, T is one-to-one.

(b) $\text{nullity}(T) = \dim(V) - \text{rank}(T) = 0$ therefore T is one-to-one.

(c) Since $\text{rank}(T) \le \dim(W) < \dim(V)$, we have $\text{nullity}(T) = \dim(V) - \text{rank}(T) > 0$. We conclude that T is not one-to-one.

6. **(a)** Since nullity(T)= 0, T is one-to-one; $\text{rank}(T) = \dim(V) - \text{nullity}(T) = \dim(V)$ so T is onto.

(b) $\text{nullity}(T) = \dim(V) - \text{rank}(T) > 0$ so T is not one-to-one; $\text{rank}(T) < \dim(V)$ so T is not onto.

(c) $R(T) = V$ so T is onto; $\text{nullity}(T) = \dim(V) - \text{rank}(T) = 0$ so T is one-to-one.

7. For example, $T(1 - x^2) = (1 - (-1)^2, 1 - 1^2) = (0,0)$.
The transformation is onto since for any real numbers a and b, a polynomial $p(x)$ in P_2 can be found such that $p(-1) = a$ and $p(1) = b$.

8. Setting $a_0 + a_1(x+1) + a_2(x+1)^2 = 0$ implies $a_0 = a_1 = a_2 = 0$. Since $\ker(T) = \{\mathbf{0}\}$, by Theorem 8.2.1, T is one-to-one.
$\text{rank}(T) = \dim(P_2) - \text{nullity}(T) = \dim(P_2)$ therefore T is onto.

9. No; T is not one-to-one because $\ker(T) \neq \{\mathbf{0}\}$ as $T(\mathbf{a}) = \mathbf{a} \times \mathbf{a} = \mathbf{0}$.

10. Since elementary matrices are invertible, $EA = 0$ yields $A = E^{-1}0 = 0$ therefore $\ker(T) = \{0\}$. Consequently, by Theorem 8.2.1 T is a one-to-one linear operator.

11. $(T_2 \circ T_1)(x,\ y) = T_2(2x,\ 3y) = (2x - 3y,\ 2x + 3y)$

12. $(T_2 \circ T_1)(x,\ y) = T_2(2x,\ -3y,\ x+y) = (2x+3y,\ -3y+x+y) = (2x+3y,\ x-2y)$

13. $(T_3 \circ T_2 \circ T_1)(x,y) = T_3\left(T_2\big(T_1(x,y)\big)\right) = T_3\big(T_2(-2y,3x,x-2y)\big) = T_3(3x,x-2y,-2y)$
$= \big(3x-2y,x-2y-(-2y)\big) = (3x-2y,x)$

14. $(T_3 \circ T_2 \circ T_1)(x,y) = T_3\left(T_2\big(T_1(x,y)\big)\right) = T_3\big(T_2(x+y,y,-x)\big) = T_3(0,x+y+y-x,3y)$
$= T_3(0,2y,3y) = (4y,12y-6y) = (4y,6y)$

15. **(a)** $(T_1 \circ T_2)(A) = T_1(A^T) = \text{tr}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = a + d$

(b) $(T_2 \circ T_1)(A)$ does not exist because $T_1(A)$ is not a 2×2 matrix.

16. **(a)** $(T_1 \circ T_2)(A) = T_1(A^T) = k\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = \begin{bmatrix} ka & kc \\ kb & kd \end{bmatrix}$

(b) $(T_2 \circ T_1)(A) = T_2\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = \begin{bmatrix} ka & kc \\ kb & kd \end{bmatrix}$

17. $(T_2 \circ T_1)(a_0 + a_1x + a_2x^2) = T_2\big(T_1(a_0 + a_1x + a_2x^2)\big) = T_2(a_0 + a_1(x+1) + a_2(x+1)^2)$
$= x(a_0 + a_1(x+1) + a_2(x+1)^2) = a_0x + a_1x(x+1) + a_2x(x+1)^2$

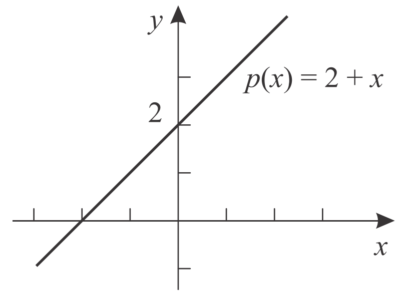
18. $(T_1 \circ T_2)\big(p(x)\big) = T_1\left(T_2\big(p(x)\big)\right) = T_1\big(p(x+1)\big) = p\big((x+1)-1\big) = p(x)$
$(T_2 \circ T_1)\big(p(x)\big) = T_2\left(T_1\big(p(x)\big)\right) = T_2\big(p(x-1)\big) = p\big((x-1)+1\big) = p(x)$

19. **(a)** $T(1-\ 2x)\ =\ (1-2(0), 1-2(1))\ =\ (1,-1)$

(b) $T\big(kp(x)\big) = \big(kp(0), kp(1)\big) = k\big(p(0), p(1)\big) = kT\big(p(x)\big)$;
$T\big(p(x)+q(x)\big) = \big(p(0)+q(0), p(1)+q(1)\big) = \big(p(0), p(1)\big) + \big(q(0), q(1)\big)$
$= T\big(p(x)\big) + T\big(q(x)\big)$

(c) Let $p(x) = a_0 + a_1x$, then $T(p(x)) = (a_0,\ a_0 + a_1)$ so if
$T(p(x))\ =\ (0,0)$, then $a_0 = a_1 = 0$ and p is the zero polynomial, so $\ker(T) = \{\mathbf{0}\}$.

(d) Since $T(p(x)) = (a_0,\ a_0 + a_1)$, then $T^{-1}(2,\ 3)$ has $a_0 = 2$ and $a_0 + a_1 = 3$ or $a_1 = 1$. Thus, $T^{-1}(2,\ 3) = 2 + x$.



20. **(a)** T is not one-to-one, e.g., $T(0, \dots, 0,1) = T(0, \dots, 0,2) = (0, \dots, 0,0)$.

(b) $\ker(T) = \{(0, \dots, 0)\}$ thus T is one-to-one;
$T^{-1}(x_1, x_2, \dots, x_n) = (x_n, x_{n-1}, \dots, x_2, x_1)$ (i.e., $T^{-1} = T$)

(c) $\ker(T) = \{(0, \dots, 0)\}$ thus T is one-to-one; $T^{-1}(x_1, x_2, \dots, x_n) = (x_n, x_1, \dots, x_{n-1})$

21. **(a)** For T to have an inverse, all the a_i's must be nonzero since otherwise T would have a nonzero kernel.

(b) $T^{-1}(x_1,\ x_2,\ \dots,\ x_n) = \left(\frac{1}{a_1}x_1,\ \frac{1}{a_2}x_2,\ \dots,\ \frac{1}{a_n}x_n\right)$

22. **(a)** Setting $T_1(x, y) = (x + y, x - y) = (0,0)$ and $T_2(x, y) = (2x + y, x - 2y) = (0,0)$ yields the systems

$$\begin{array}{rcrcl} x & + & y & = & 0 \\ x & - & y & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcl} 2x & + & y & = & 0 \\ x & - & 2y & = & 0 \end{array}$$

each of which has only the trivial solution $(x, y) = (0,0)$ therefore $\ker(T_1) = \ker(T_2) = \{(0,0)\}$. By Theorem 8.2.1, T_1 and T_2 are one-to-one.

(b) From Formula (9) of Section 4.9, the standard matrices of T_1 and T_2 are $[T_1] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $[T_2] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, respectively. Both matrices are invertible, and from Formula (7) of Section 4.10, we have $[T_1^{-1}] = [T_1]^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ and $[T_2^{-1}] = [T_2]^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

so that

$T_1^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_1^{-1}]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{1}{2}y \end{bmatrix}$, i.e. $T_1^{-1}(x, y) = \left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y\right)$ and

$T_2^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_2^{-1}]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{5}x + \frac{1}{5}y \\ \frac{1}{5}x - \frac{2}{5}y \end{bmatrix}$, i.e. $T_2^{-1}(x, y) = \left(\frac{2}{5}x + \frac{1}{5}y, \frac{1}{5}x - \frac{2}{5}y\right)$.

Since $(T_2 \circ T_1)(x, y) = T_2(T_1(x, y)) = T_2(x + y, x - y) = (3x + y, -x + 3y)$ we have $[T_2 \circ T_1] = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$ which is an invertible matrix with $[T_2 \circ T_1]^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix}$.

Therefore $(T_2 \circ T_1)^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [T_2 \circ T_1]^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{10}x - \frac{1}{10}y \\ \frac{1}{10}x + \frac{3}{10}y \end{bmatrix}$, i.e.,

$(T_2 \circ T_1)^{-1}(x,y) = \left(\frac{3}{10}x - \frac{1}{10}y, \frac{1}{10}x + \frac{3}{10}y\right)$.

(c) $(T_1^{-1} \circ T_2^{-1})(x,y) = T_1^{-1}\left(T_2^{-1}(x,y)\right) = T_1^{-1}\left(\frac{2}{5}x + \frac{1}{5}y, \frac{1}{5}x - \frac{2}{5}y\right)$

$= \left(\frac{1}{2}\left(\frac{2}{5}x + \frac{1}{5}y\right) + \frac{1}{2}\left(\frac{1}{5}x - \frac{2}{5}y\right), \frac{1}{2}\left(\frac{2}{5}x + \frac{1}{5}y\right) - \frac{1}{2}\left(\frac{1}{5}x - \frac{2}{5}y\right)\right) = \left(\frac{3}{10}x - \frac{1}{10}y, \frac{1}{10}x + \frac{3}{10}y\right)$

$= (T_2 \circ T_1)^{-1}(x,y)$

23. **(a)** Since $T_1(p(x)) = xp(x)$, $T_1^{-1}(p(x)) = \frac{1}{x}p(x)$.
Since $T_2(p(x)) = p(x+1)$, $T_2^{-1}(p(x)) = p(x-1)$.
$(T_1^{-1} \circ T_2^{-1})(p(x)) = T_1^{-1}(p(x-1)) = \frac{1}{x}p(x-1)$

(b) Since $(T_2 \circ T_1)(p(x)) = T_2\left(T_1(p(x))\right) = T_2(xp(x)) = (x+1)p(x+1)$, we have
$(T_2 \circ T_1)\left((T_1^{-1} \circ T_2^{-1})(p(x))\right) = (T_2 \circ T_1)\left(\frac{1}{x}p(x-1)\right) = (x+1)\left(\frac{1}{x+1}\right)p(x-1+1) = p(x)$

24. From Table 2 of Section 4.9, we have $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$CBA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is an invertible matrix with the inverse $(CBA)^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ so that by Formula (8) of Section 4.10,

$(T_{CBA})^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T_{(CBA)^{-1}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$.

Since $A^{-1}B^{-1}C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, it follows that

$T_{A^{-1}B^{-1}C^{-1}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix} = (T_{CBA})^{-1}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

25. $T_2(\mathbf{v}) = \frac{1}{4}\mathbf{v}$, then $(T_1 \circ T_2)(\mathbf{v}) = T_1\left(\frac{1}{4}\mathbf{v}\right) = 4\left(\frac{1}{4}\mathbf{v}\right) = \mathbf{v}$ and $(T_2 \circ T_1)(\mathbf{v}) = T_2(4\mathbf{v}) = \frac{1}{4}(4\mathbf{v}) = \mathbf{v}$.

26. **(a)** $(T_2 \circ T_1)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T_2\left((a+b) + (c+d)x\right) = (a+b,\ c+d,\ a+b)$

(b) For example, $(T_2 \circ T_1)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = (T_2 \circ T_1)\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = (1,1,1)$.

(c) For example, $(1,2,3)$ cannot be expressed in the form $(a+b,\ c+d,\ a+b)$.

27. By inspection, $T(x,y,z) = (x,y,0)$. Then $T(T(x,y,z)) = T(x,y,0) = (x,y,0) = T(x,y,z)$ or $T \circ T = T$.

28. $T(k\mathbf{f}) = kf(0) + 2kf'(0) + 3kf'(1) = kT(\mathbf{f})$
$T(\mathbf{f} + \mathbf{g}) = f(0) + g(0) + 2(f'(0) + g'(0)) + 3(f'(1) + g'(1)) = T(\mathbf{f}) + T(\mathbf{g})$

Thus T is a linear transformation.
Let $\mathbf{f} = f(x) = x^2(x-1)^2$, then $f'(x) = 2x(x-1)(2x-1)$ so $f(0) = 0$, $f'(0) = 0$, and $f'(1) = 0$.
$T(\mathbf{f}) = f(0) + 2f'(0) + 3f'(1) = 0$, so $\ker(T) \neq \{\mathbf{0}\}$ and T is not one-to-one.

29. **(a)** $D\big(kp(x)\big) = \frac{d}{dx}\big(kp(x)\big) = kp'(x) = kD\big(p(x)\big)$
$D\big(p(x)+q(x)\big) = \frac{d}{dx}\big(p(x)+q(x)\big) = p'(x) + q'(x) = D\big(p(x)\big) + D\big(q(x)\big)$
$J\big(kp(x)\big) = \int_0^x kp(t)\,dt = k\int_0^x p(t)\,dt = kJ\big(p(x)\big)$
$J\big(p(x)+q(x)\big) = \int_0^x \big(p(t)+q(t)\big)\,dt = \int_0^x p(t)\,dt + \int_0^x q(t)\,dt = J\big(p(x)\big) + J\big(q(x)\big)$

(b) D is not one-to-one (e.g., $D(x^2+3) = D(x^2) = 2x$) so it does not have an inverse.

(c) Yes, this can be accomplished by taking $D : V \to P_{n-1}$ and $J : P_{n-1} \to V$ where V is the set of all polynomials $p(x)$ in P_n such that $p(0) = 0$.

30. **(a)** $(J \circ D)(\mathbf{f}) = J\big(D(x^2+3x+2)\big) = J(2x+3) = \int_0^x (2t+3)\,dt = (t^2+3t)\big]_0^x = x^2 + 3x$

(b) $(J \circ D)(\mathbf{f}) = J\big(D(\sin x)\big) = J(\cos x) = \int_0^x (\cos t)\,dt = \sin t\big]_0^x = \sin x$

31. The kernel of J contains all polynomials $a_0 + a_1x$ such that $\int_{-1}^1 (a_0 + a_1x)dx = 0$. By integration, this condition yields $\left(a_0x + \frac{a_1x^2}{2}\right)\Big]_{-1}^1 = 0$, i.e., $a_0 + \frac{a_1}{2} + a_0 - \frac{a_1}{2} = 0$, or equivalently, $a_0 = 0$.
The kernel consists of all polynomials of the form a_1x.
Since $\ker(J) \neq \{\mathbf{0}\}$, by Theorem 8.2.1, J is not one-to-one.

32. For every polynomial $q(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$ in P_{n-1} we can construct a polynomial $p(x) = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots + \frac{a_{n-2}}{n-1}x^{n-1} + \frac{a_{n-1}}{n}x^n$ in P_n for which $D\big(p(x)\big) = p'(x) = q(x)$. We conclude that D is onto.

35. $T\big(kp(x)\big) = kp\big(q_0(x)\big) = kT\big(p(x)\big)$
$T\big(p_1(x) + p_2(x)\big) = p_1\big(q_0(x)\big) + p_2\big(q_0(x)\big) = T\big(p_1(x)\big) + T\big(p_2(x)\big)$

36. For any linear transformation $T{:}\,V \to W$, $\dim(V) = \text{nullity}(T) + \text{rank}(T) \geq \text{rank}(T)$.
If T is onto then $\text{rank}(T) = \dim(W)$. Consequently, $\dim(V) \geq \dim(W)$.

37. If there were such a transformation T, then it would have nullity 0 (because it would be one-to-one), and have rank no greater than the dimension of W (because its range is a subspace of W). Then by the dimension Theorem, $\dim(V) = \text{rank}(T) + \text{nullity}(T) \leq \dim(W)$ which contradicts the assumption
$\dim(W) < \dim(V)$. Thus there is no such transformation.

True-False Exercises

(a) True. This is the statement of Theorem 8.2.3.

(b) False. For example, with $T_1(x,y) = (y,x)$ and $T_2(x,y) = (x,0)$ we have
$(T_1 \circ T_2)(x,y) = T_1(x,0) = (0,x)$ which does not equal $(T_2 \circ T_1)(x,y) = T_2(y,x) = (y,0)$.

(c) True.

(d) True. For T to have an inverse, it must be one-to-one.

(e) False. T^{-1} does not exist.

(f) True. If T_1 is not one-to-one then there is some nonzero vector $\mathbf{v}_1$ with $T_1(\mathbf{v}_1) = \mathbf{0}$. Thus $(T_2 \circ T_1)(\mathbf{v}_1) = T_2(\mathbf{0}) = \mathbf{0}$ and $\ker(T_2 \circ T_1) \neq \{\mathbf{0}\}$.

8.3 Isomorphism

1. The transformation is an isomorphism.

2. The transformation is not an isomorphism(not onto).

3. The transformation is an isomorphism.

4. The transformation is not an isomorphism(not a linear transformation).

5. The transformation is not an isomorphism (not a linear transformation).

6. The transformation is an isomorphism.

7. The transformation is an isomorphism.

8. The transformation is not an isomorphism(not onto).

9. **(a)** $T\left(\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$

(b) $T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}$.

10. **(a)** If $p(x)$ is a polynomial and $p(0) = 0$, then $p(x)$ has constant term 0.

$$T(ax^3 + bx^2 + cx) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(b) $T(a + b\sin(x) + c\cos(x)) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

11. $\det(A) = -3 \neq 0$, so by Theorem 4.10.2 T_A is one-to-one and onto. Consequently, T_A is an isomorphism.

12. $\det(A) = 0$, so by Theorem 4.10.2 T_A is not one-to-one. Consequently, T_A is not an isomorphism.

13. The reduced row echelon form of A is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution space W contains vectors (x_1, x_2, x_3, x_4) such that $x_1 = -r - s - t,\ x_2 = r, x_3 = s, x_4 = t$ so

$(x_1, x_2, x_3, x_4) = (-r - s - t, r, s, t) = r(-1,1,0,0) + s(-1,0,1,0) + t(-1,0,0,1)$ hence $\dim(W) = 3$.
$(-r - s - t, r, s, t) \to (r, s, t)$ is an isomorphism between W and R^3.

14. The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solution space W contains vectors
(x_1, x_2, x_3, x_4) such that $x_1 = -s,\ x_2 = -t, x_3 = s, x_4 = t$ so
$(x_1, x_2, x_3, x_4) = (-s, -t, s, t) = s(-1,0,1,0) + t(0,-1,0,1)$ hence $\dim(W) = 2$.
$(-s, -t, s, t) \to (s, t)$ is an isomorphism between W and R^2.

15. Let us denote the given transformation by T. T is a linear transformation, since for any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ in M_{22} and for any scalar k, we have

$$T(kA) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = \begin{bmatrix} ka \\ ka + kb \\ ka + kb + kc \\ ka + kb + kc + kd \end{bmatrix} = k\begin{bmatrix} a \\ a + b \\ a + b + c \\ a + b + c + d \end{bmatrix} = kT(A) \text{ and}$$

$$T(A + B) = T\left(\begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix}\right) = \begin{bmatrix} a + a' \\ a + a' + b + b' \\ a + a' + b + b' + c + c' \\ a + a' + b + b' + c + c' + d + d' \end{bmatrix}$$

$$= \begin{bmatrix} a \\ a + b \\ a + b + c \\ a + b + c + d \end{bmatrix} + \begin{bmatrix} a' \\ a' + b' \\ a' + b' + c' \\ a' + b' + c' + d' \end{bmatrix} = T(A) + T(B)$$

By inspection, $\begin{bmatrix} a \\ a + b \\ a + b + c \\ a + b + c + d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ implies $a = b = c = d = 0$, therefore $\ker(T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ so T is one-to-one.
$\operatorname{rank}(T) = \dim(M_{22}) - \operatorname{nullity}(T) = 4 - 0 = 4$ hence T is onto.
We conclude that T is an isomorphism.

16. Let us denote the given transformation by T.
By inspection, $T\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ is in $\ker(T)$ so T is not one-to-one.
We conclude that T is not an isomorphism.

17. Yes, $(a, b) \to (0, a, b)$ is an isomorphism between R^2 and the yz-plane in R^3.

18. **(a)** M_{mn} is isomorphic to R^k if $k = mn$ (since that makes dimensions of both spaces equal).

(b) M_{mn} is isomorphic to P_k if $k = mn - 1$ (since that makes dimensions of both spaces equal).

19. No. By inspection, $T(x^2 - x) = T(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so T is not one-to-one.

20. Let us denote the given transformation by T. T is a linear transformation, since for any $A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_0 & b_1 \\ b_2 & b_3 \end{bmatrix}$ in M_{22} and for any scalar k, we have

$T(kA) = T\left(\begin{bmatrix} ka_0 & ka_1 \\ ka_2 & ka_3 \end{bmatrix}\right) = ka_0 + ka_1x + ka_2x^2 + ka_3x^3 = k(a_0 + a_1x + a_2x^2 + a_3x^3) = kT(A)$

$T(A + B) = T\left(\begin{bmatrix} a_0 + b_0 & a_1 + b_1 \\ a_2 + b_2 & a_3 + b_3 \end{bmatrix}\right) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$

$= (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3) = T(A) + T(B)$.

By inspection, $a_0 + a_1x + a_2x^2 + a_3x^3 = 0$ implies $a_0 = a_1 = a_2 = a_3 = 0$ so $\ker(T) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$.

Therefore T is one-to-one.

$\text{rank}(T) = \dim(M_{22}) - \text{nullity}(T) = 4 - 0 = 4$ hence T is onto.

We conclude that T is an isomorphism.

Furthermore, $\langle T(A), T(B)\rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 = \langle A, B\rangle$, so T is an inner product space isomorphism.

True-False Exercises

(a) False. $\dim(R^2) = 2$ while $\dim(P_2) = 3$.

(b) True. If $\ker(T) = \{\mathbf{0}\}$ then $\text{rank}(T) = 4$ so T is one-to-one and onto.

(c) False. $\dim(M_{33}) = 9$ while $\dim(P_9) = 10$.

(d) True. For instance, if V consists of all matrices of the form $\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$, then $T\left(\begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is an isomorphism $T: V \to R^4$.

(e) True.

(f) True. For instance if V consists of all vectors in R^{n+1} of the form $(x_1, \dots, x_n, 0)$, then $T(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$ is an isomorphism $T: R^n \to V$.

8.4 Matrices for General Linear Transformations

1. **(a)** $T(\mathbf{u}_1) = T(1) = x = \mathbf{v}_2$, $T(\mathbf{u}_2) = T(x) = x^2 = \mathbf{v}_3$, and $T(\mathbf{u}_3) = T(x^2) = x^3 = \mathbf{v}_4$ therefore

$[T(\mathbf{u}_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(\mathbf{u}_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $[T(\mathbf{u}_3)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, thus $[T]_{B',B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) By inspection, $[\mathbf{x}]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ so $[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}$.

On the other hand, $[T(\mathbf{x})]_{B'} = [c_0x + c_1x^2 + c_2x^3]_{B'} = \begin{bmatrix} 0 \\ c_0 \\ c_1 \\ c_2 \end{bmatrix}$. Formula (5) is satisfied.

2. **(a)** From the definition of T we have $T(1) = 1, T(x) = 1 - 2x$, and $T(x^2) = -3x$. By inspection, $[T(1)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[T(x)]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, and $[T(x^2)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, therefore $[T]_{B',B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix}$.

(b) By inspection, $[\mathbf{x}]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ so $[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 \\ -2c_1 - 3c_2 \end{bmatrix}$.

On the other hand, $T(\mathbf{x}) = (c_0 + c_1) - (2c_1 + 3c_2)x$ and, by inspection, $[T(\mathbf{x})]_{B'} = \begin{bmatrix} c_0 + c_1 \\ -(2c_1 + 3c_2) \end{bmatrix}$.

Formula (5) is satisfied.

3. **(a)** $T(1) = 1$; $T(x) = x - 1 = -1 + x$; $T(x^2) = (x-1)^2 = 1 - 2x + x^2$

Thus the matrix for T relative to B is $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $[T]_B[\mathbf{x}]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_1 - 2a_2 \\ a_2 \end{bmatrix}$. For $\mathbf{x} = a_0 + a_1x + a_2x^2$,

$T(\mathbf{x}) = a_0 + a_1(x-1) + a_2(x-1)^2 = a_0 - a_1 + a_2 + (a_1 - 2a_2)x + a_2x^2$,

so $[T(\mathbf{x})]_B = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_1 - 2a_2 \\ a_2 \end{bmatrix}$.

4. **(a)** From the definition of T we have $T(\mathbf{u}_1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $T(\mathbf{u}_2) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. To find the coordinate vectors, we solve the systems

$$\begin{array}{rcrcr} a_1 & - & a_2 & = & 0 \\ a_1 & & & = & 2 \end{array} \quad \text{and} \quad \begin{array}{rcrcr} b_1 & - & b_2 & = & -1 \\ b_1 & & & = & -1 \end{array}$$

so clearly we have $[T(\mathbf{u}_1)]_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $[T(\mathbf{u}_2)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Therefore $[T]_B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$.

(b) For an arbitrary $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in R^2, solving the system

$$\begin{array}{rcrcr} c_1 & - & c_2 & = & x \\ c_1 & & & = & y \end{array}$$

yields $[\mathbf{x}]_B = \begin{bmatrix} y \\ y - x \end{bmatrix}$ therefore $[T]_B[\mathbf{x}]_B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} y \\ y - x \end{bmatrix} = \begin{bmatrix} x + y \\ 2y \end{bmatrix}$.

On the other hand, $T(\mathbf{x}) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$. Solving the system

$$\begin{array}{rcrcl} c_1 & - & c_2 & = & x-y \\ c_1 & & & = & x+y \end{array}$$

yields $[T(\mathbf{x})]_B = \begin{bmatrix} x+y \\ 2y \end{bmatrix}$ showing that Formula (8) holds for every $\mathbf{x}$ in R^2.

5. **(a)** $T(\mathbf{u}_1) = T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} = 0\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 + \frac{8}{3}\mathbf{v}_3$; $T(\mathbf{u}_2) = T\left(\begin{bmatrix} -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = 0\mathbf{v}_1 + \mathbf{v}_2 + \frac{4}{3}\mathbf{v}_3$

$$[T]_{B',B} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \\ \frac{8}{3} & \frac{4}{3} \end{bmatrix}$$

(b) For an arbitrary $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ in R^2, solving the system

$$\begin{array}{rcrcl} c_1 & - & 2c_2 & = & x \\ 3c_1 & + & 4c_2 & = & y \end{array}$$

yields $[\mathbf{x}]_B = \begin{bmatrix} \frac{2x+y}{5} \\ \frac{-3x+y}{10} \end{bmatrix}$ therefore $[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \\ \frac{8}{3} & \frac{4}{3} \end{bmatrix}\begin{bmatrix} \frac{2x+y}{5} \\ \frac{-3x+y}{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{x}{2} \\ \frac{2}{3}x + \frac{2}{3}y \end{bmatrix}$.

On the other hand, $T(\mathbf{x}) = \begin{bmatrix} x+2y \\ -x \\ 0 \end{bmatrix}$. Solving the system

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & x+2y \\ c_1 & + & 2c_2 & & & = & -x \\ c_1 & & & & & = & 0 \end{array}$$

yields $[T(\mathbf{x})]_{B'} = \begin{bmatrix} 0 \\ -\frac{x}{2} \\ \frac{2}{3}x + \frac{2}{3}y \end{bmatrix}$ showing that Formula (5) holds for every $\mathbf{x}$ in R^2.

6. **(a)** We have $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $[T(\mathbf{v}_2)]_B = \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, and $[T(\mathbf{v}_3)]_B = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

Therefore $[T]_B = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

(b) For an arbitrary $\mathbf{x} = (x_1, x_2, x_3)$ in R^3, solving the system

$$\begin{array}{rcrcrcl} a_1 & & & + & a_3 & = & x_1 \\ & & a_2 & + & a_3 & = & x_2 \\ a_1 & + & a_2 & & & = & x_3 \end{array}$$

yields $[\mathbf{x}]_B = \begin{bmatrix} \frac{1}{2}(x_1 - x_2 + x_3) \\ \frac{1}{2}(-x_1 + x_2 + x_3) \\ \frac{1}{2}(x_1 + x_2 - x_3) \end{bmatrix}$ therefore

$$[T]_B[\mathbf{x}]_B = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(x_1 - x_2 + x_3) \\ \frac{1}{2}(-x_1 + x_2 + x_3) \\ \frac{1}{2}(x_1 + x_2 - x_3) \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_1 - x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_3 \end{bmatrix}.$$

On the other hand, solving the system

$$\begin{array}{ccccccc} b_1 & & & + & b_3 & = & x_1 - x_2 \\ & & b_2 & + & b_3 & = & x_2 - x_1 \\ b_1 & + & b_2 & & & = & x_1 - x_3 \end{array}$$

yields $[T(\mathbf{x})]_B = \begin{bmatrix} \frac{3}{2}x_1 - x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_3 \end{bmatrix}$ showing that Formula (8) holds for every $\mathbf{x}$ in R^3.

(c) The augmented matrix of the homogeneous system

$$\begin{array}{ccccccc} x_1 & - & x_2 & & & = & 0 \\ -x_1 & + & x_2 & & & = & 0 \\ x_1 & & & - & x_3 & = & 0 \end{array}$$

has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the system has nontrivial solutions (e.g., (1,1,1)) $\ker(T) \neq \{(0,0,0)\}$ so that T is not one-to-one.

7. (a) We have $T(1) = 1$, $T(x) = 2x + 1 = 1 + 2x$, and $T(x^2) = (2x+1)^2 = 1 + 4x + 4x^2$.

Therefore, $[T]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$.

(b) Step 1. The coordinate vector of $\mathbf{x} = 2 - 3x + 4x^2$ with respect to the basis B is $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

Step 2. $[T]_B[\mathbf{x}]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix} = [T(\mathbf{x})]_B$.

Step 3. Reconstructing $T(\mathbf{x})$ from the coordinate vector obtained in Step 2, we obtain $T(\mathbf{x}) = 3(1) + 10(x) + 16(x^2) = 3 + 10x + 16x^2$.

(c) $T(2 - 3x + 4x^2) = 2 - 3(2x+1) + 4(2x+1)^2 = 2 - 6x - 3 + 16x^2 + 16x + 4$
$= 3 + 10x + 16x^2$

8. (a) From the definition of T we have $T(1) = x$, $T(x) = x^2 - 3x$, and
$T(x^2) = x(x-3)^2 = x^3 - 6x^2 + 9x$.

The corresponding coordinate vectors are $[T(1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(x)]_{B'} = \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$, and

$[T(x^2)]_{B'} = \begin{bmatrix} 0 \\ 9 \\ -6 \\ 1 \end{bmatrix}$, therefore $[T]_{B',B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Step 1. The coordinate vector of $\mathbf{x} = 1 + x - x^2$ with respect to the basis B is $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Step 2. $[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -11 \\ 7 \\ -1 \end{bmatrix} = [T(\mathbf{x})]_{B'}$.

Step 3. Reconstructing $T(\mathbf{x})$ from the coordinate vector obtained in Step 2, we obtain $T(\mathbf{x}) = 0(1) - 11(x) + 7(x^2) - 1(x^3) = -11x + 7x^2 - x^3$.

(c) $T(1 + x - x^2) = x(1 + (x-3) - (x-3)^2) = -11x + 7x^2 - x^3$.

9. **(a)** Since A is the matrix for T relative to B, $A = [[T(\mathbf{v}_1)]_B \quad [T(\mathbf{v}_2)]_B]$.
That is, $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $[T(\mathbf{v}_2)]_B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(b) Since $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $T(\mathbf{v}_1) = 1\mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.
Similarly, $T(\mathbf{v}_2) = 3\mathbf{v}_1 + 5\mathbf{v}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -5 \\ 20 \end{bmatrix} = \begin{bmatrix} -2 \\ 29 \end{bmatrix}$.

(c) If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, then

$$\begin{aligned} x_1 &= c_1 - c_2 \\ x_2 &= 3c_1 + 4c_2 \end{aligned}$$

Solving for c_1 and c_2 gives $c_1 = \frac{4}{7}x_1 + \frac{1}{7}x_2$, $c_2 = -\frac{3}{7}x_1 + \frac{1}{7}x_2$, so

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) = \left(\tfrac{4}{7}x_1 + \tfrac{1}{7}x_2\right)\begin{bmatrix} 3 \\ -5 \end{bmatrix} + \left(-\tfrac{3}{7}x_1 + \tfrac{1}{7}x_2\right)\begin{bmatrix} -2 \\ 29 \end{bmatrix} = \begin{bmatrix} \frac{18}{7}x_1 + \frac{1}{7}x_2 \\ -\frac{107}{7}x_1 + \frac{24}{7}x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{18}{7} & \frac{1}{7} \\ -\frac{107}{7} & \frac{24}{7} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(d) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{18}{7} & \frac{1}{7} \\ -\frac{107}{7} & \frac{24}{7} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{19}{7} \\ -\frac{83}{7} \end{bmatrix}$

10. **(a)** By Formula (4), $[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$, $[T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$, $[T(\mathbf{v}_3)]_{B'} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$, and $[T(\mathbf{v}_4)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) Reconstructing the vectors using their coordinate vectors obtained in part (a) yields

$$T(\mathbf{v}_1) = 3\begin{bmatrix}0\\8\\8\end{bmatrix} + 1\begin{bmatrix}-7\\8\\1\end{bmatrix} - 3\begin{bmatrix}-6\\9\\1\end{bmatrix} = \begin{bmatrix}11\\5\\22\end{bmatrix} \qquad T(\mathbf{v}_2) = -2\begin{bmatrix}0\\8\\8\end{bmatrix} + 6\begin{bmatrix}-7\\8\\1\end{bmatrix} + 0\begin{bmatrix}-6\\9\\1\end{bmatrix} = \begin{bmatrix}-42\\32\\-10\end{bmatrix}$$

$$T(\mathbf{v}_3) = 1\begin{bmatrix}0\\8\\8\end{bmatrix} + 2\begin{bmatrix}-7\\8\\1\end{bmatrix} + 7\begin{bmatrix}-6\\9\\1\end{bmatrix} = \begin{bmatrix}-56\\87\\17\end{bmatrix} \qquad T(\mathbf{v}_4) = 0\begin{bmatrix}0\\8\\8\end{bmatrix} + 1\begin{bmatrix}-7\\8\\1\end{bmatrix} + 1\begin{bmatrix}-6\\9\\1\end{bmatrix} = \begin{bmatrix}-13\\17\\2\end{bmatrix}$$

(c) We follow the procedure of Example 10 in Section 8.1. First of all, we express $\mathbf{x} = (x_1, x_2, x_3, x_4)$ as a linear combination of the vectors in B. The corresponding linear system

$$\begin{array}{rcrcrcrcl} & & 2c_2 & + & c_3 & + & 6c_4 & = & x_1 \\ c_1 & + & c_2 & + & 4c_3 & + & 9c_4 & = & x_2 \\ c_1 & - & c_2 & - & c_3 & + & 4c_4 & = & x_3 \\ c_1 & - & c_2 & + & 2c_3 & + & 2c_4 & = & x_4 \end{array}$$

has the augmented matrix whose reduced row echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{9}{2}x_1 - 4x_2 - \frac{1}{2}x_3 + \frac{11}{2}x_4 \\ 0 & 1 & 0 & 0 & \frac{5}{2}x_1 - 2x_2 - \frac{1}{2}x_3 + \frac{5}{2}x_4 \\ 0 & 0 & 1 & 0 & -\frac{2}{5}x_1 + \frac{2}{5}x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ 0 & 0 & 0 & 1 & -\frac{3}{5}x_1 + \frac{3}{5}x_2 + \frac{1}{5}x_3 - \frac{4}{5}x_4 \end{array}\right].$$

This yields $c_1 = \frac{9}{2}x_1 - 4x_2 - \frac{1}{2}x_3 + \frac{11}{2}x_4$, $c_2 = \frac{5}{2}x_1 - 2x_2 - \frac{1}{2}x_3 + \frac{5}{2}x_4$, $c_3 = -\frac{2}{5}x_1 + \frac{2}{5}x_2 - \frac{1}{5}x_3 - \frac{1}{5}x_4$, $c_4 = -\frac{3}{5}x_1 + \frac{3}{5}x_2 + \frac{1}{5}x_3 - \frac{4}{5}x_4$. Therefore

$$T(x_1, x_2, x_3, x_4) = \left(\tfrac{9}{2}x_1 - 4x_2 - \tfrac{1}{2}x_3 + \tfrac{11}{2}x_4\right)\begin{bmatrix}11\\5\\22\end{bmatrix} + \left(\tfrac{5}{2}x_1 - 2x_2 - \tfrac{1}{2}x_3 + \tfrac{5}{2}x_4\right)\begin{bmatrix}-42\\32\\-10\end{bmatrix}$$

$$+ \left(-\tfrac{2}{5}x_1 + \tfrac{2}{5}x_2 - \tfrac{1}{5}x_3 - \tfrac{1}{5}x_4\right)\begin{bmatrix}-56\\87\\17\end{bmatrix} + \left(-\tfrac{3}{5}x_1 + \tfrac{3}{5}x_2 + \tfrac{1}{5}x_3 - \tfrac{4}{5}x_4\right)\begin{bmatrix}-13\\17\\2\end{bmatrix}$$

$$= \begin{bmatrix} -\frac{253}{10}x_1 + \frac{49}{5}x_2 + \frac{241}{10}x_3 - \frac{229}{10}x_4 \\ \frac{115}{2}x_1 - 39x_2 - \frac{65}{2}x_3 + \frac{153}{2}x_4 \\ 66x_1 - 60x_2 - 9x_3 + 91x_4 \end{bmatrix}$$

(d) $T\left(\begin{bmatrix}2\\2\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-31\\37\\12\end{bmatrix}$

11. **(a)** Since A is the matrix for T relative to B, the columns of A are $[T(\mathbf{v}_1)]_B$, $[T(\mathbf{v}_2)]_B$, and $[T(\mathbf{v}_3)]_B$, respectively. That is, $[T(\mathbf{v}_1)]_B = \begin{bmatrix}1\\2\\6\end{bmatrix}$, $[T(\mathbf{v}_2)]_B = \begin{bmatrix}3\\0\\-2\end{bmatrix}$, and $[T(\mathbf{v}_3)]_B = \begin{bmatrix}-1\\5\\4\end{bmatrix}$.

(b) Since $[T(\mathbf{v}_1)]_B = \begin{bmatrix}1\\2\\6\end{bmatrix}$,

$T(\mathbf{v}_1) = \mathbf{v}_1 + 2\mathbf{v}_2 + 6\mathbf{v}_3 = 3x + 3x^2 - 2 + 6x + 4x^2 + 18 + 42x + 12x^2 = 16 + 51x + 19x^2$.
Similarly, $T(\mathbf{v}_2) = 3\mathbf{v}_1 - 2\mathbf{v}_3 = 9x + 9x^2 - 6 - 14x - 4x^2 = -6 - 5x + 5x^2$, and

$T(\mathbf{v}_3) = -\mathbf{v}_1 + 5\mathbf{v}_2 + 4\mathbf{v}_3 = -3x - 3x^2 - 5 + 15x + 10x^2 + 12 + 28x + 8x^2$
$=7+40x + 15x^2$.

(c) If $a_0 + a_1x + a_2x^2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$
$= c_1(3x + 3x^2) + c_2(-1 + 3x + 2x^2) + c_3(3 + 7x + 2x^2)$ then
$a_0 = -c_2 + 3c_3,\ a_1 = 3c_1 + 3c_2 + 7c_3,\ a_2 = 3c_1 + 2c_2 + 2c_3$.
Solving for c_1, c_2, and c_3 gives
$c_1 = \frac{1}{3}(a_0 - a_1 + 2a_2),\ \ c_2 = \frac{1}{8}(-5a_0 + 3a_1 - 3a_2),\ \ c_3 = \frac{1}{8}(a_0 + a_1 - a_2)$, so
$T(a_0 + a_1x + a_2x^2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3)$
$= \frac{1}{3}(a_0 - a_1 + 2a_2)(16 + 51x + 19x^2) + \frac{1}{8}(-5a_0 + 3a_1 - 3a_2)(-6 - 5x + 5x^2)$
$+\frac{1}{8}(a_0 + a_1 - a_2)(7 + 40x + 15x^2)$
$= \frac{239a_0 - 161a_1 + 289a_2}{24} + \frac{201a_0 - 111a_1 + 247a_2}{8}x + \frac{61a_0 - 31a_1 + 107a_2}{12}x^2$

(d) In $1 + x^2, a_0 = 1, a_1 = 0, a_2 = 1$.
$T(1 + x^2) = \frac{239+289}{24} + \frac{201+247}{8}x + \frac{61+107}{12}x^2 = 22 + 56x + 14x^2$

12. **(a)** $(T_2 \circ T_1)(1) = T_2\big(T_1(1)\big) = T_2(x) = 2x + 1$;
$(T_2 \circ T_1)(x) = T_2\big(T_1(x)\big) = T_2(x^2) = (2x + 1)^2 = 4x^2 + 4x + 1$;
$[(T_2 \circ T_1)(1)]_{B'} = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $[(T_2 \circ T_1)(x)]_{B'} = \begin{bmatrix}1\\4\\4\end{bmatrix}$ therefore $[T_2 \circ T_1]_{B',B} = \begin{bmatrix}1 & 1\\2 & 4\\0 & 4\end{bmatrix}$;

$T_2(1) = 1; T_2(x) = 2x + 1; T_2(x^2) = (2x + 1)^2 = 4x^2 + 4x + 1$;
$[T_2(1)]_{B'} = \begin{bmatrix}1\\0\\0\end{bmatrix}, [T_2(x)]_{B'} = \begin{bmatrix}1\\2\\0\end{bmatrix}$, and $[T_2(x^2)]_{B'} = \begin{bmatrix}1\\4\\4\end{bmatrix}$ therefore $[T_2]_{B'} = \begin{bmatrix}1 & 1 & 1\\0 & 2 & 4\\0 & 0 & 4\end{bmatrix}$;

$T_1(1) = x; T_1(x) = x^2; [T_1(1)]_{B'} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $[T_1(x)]_{B'} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ therefore $[T_1]_{B',B} = \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}$.

(b) By Theorem 8.4.1, $[T_2 \circ T_1]_{B',B} = [T_2]_{B'}[T_1]_{B',B}$

(c) $[T_2]_{B'}[T_1]_{B',B} = \begin{bmatrix}1 & 1 & 1\\0 & 2 & 4\\0 & 0 & 4\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & 4\\0 & 4\end{bmatrix} = [T_2 \circ T_1]_{B',B}$

13. **(a)** $(T_2 \circ T_1)(1) = T_2(2) = 6x$ and $(T_2 \circ T_1)(x) = T_2(-3x) = -9x^2$ so $[T_2 \circ T_1]_{B',B} = \begin{bmatrix}0 & 0\\6 & 0\\0 & -9\\0 & 0\end{bmatrix}$;

$T_1(1) = 2$ and $T_1(x) = -3x$ so $[T_1]_{B'',B} = \begin{bmatrix}2 & 0\\0 & -3\\0 & 0\end{bmatrix}$;

$T_2(1) = 3x,\ T_2(x) = 3x^2$, and $T_2(x^2) = 3x^3$ so $[T_2]_{B',B''} = \begin{bmatrix}0 & 0 & 0\\3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}$.

(b) $[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B}$

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 0 & -9 \\ 0 & 0 \end{bmatrix}$

14. $[T(\mathbf{v}_1)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(\mathbf{v}_2)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $[T(\mathbf{v}_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and $[T(\mathbf{v}_4)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ therefore $[T]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

15. **(a)** Since $T(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $T(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $T(x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $[T(1)]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$[T(x)]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, and $[T(x^2)]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. Consequently, $[T]_{B,B'} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Since $T(1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $T(1+x) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and $T(1+x^2) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, we have $[T(1)]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$[T(1+x)]_B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, and $[T(1+x^2)]_B = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$. Consequently, $[T]_{B,B''} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) Applying the three-step procedure to the bases B' and B we obtain:

Step 1. The coordinate vector of $\mathbf{x} = 2 + 2x + x^2$ with respect to the basis B' is $[\mathbf{x}]_{B'} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Step 2. $[T]_{B,B'}[\mathbf{x}]_{B'} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 2 \end{bmatrix} = [T(\mathbf{x})]_B$.

Step 3. Reconstructing $T(\mathbf{x})$ from the coordinate vector obtained in Step 2, we obtain
$T(\mathbf{x}) = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$.

Applying the three-step procedure to the bases B'' and B we obtain:

Step 1. The coordinate vector of $\mathbf{x} = 2 + 2x + x^2$ with respect to the basis B'' is $[\mathbf{x}]_{B''} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

To find the coordinate vector of $\mathbf{x} = 2 + 2x + x^2$ with respect to the basis B' we solve the system

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 2 \\ & & c_2 & & & = & 2 \\ & & & + & c_3 & = & 1 \end{array}$$

Back-substitution yields $[\mathbf{x}]_{B''} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Step 2. $[T]_{B,B''}[\mathbf{x}]_{B''} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 2 \end{bmatrix} = [T(\mathbf{x})]_B$.

Step 3. Reconstructing $T(\mathbf{x})$ from the coordinate vector obtained in Step 2, we obtain $T(\mathbf{x}) = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$.

(c) $T(2 + 2x + x^2) = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$

16. **(a)** Since $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{B'} = \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{B'} = \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{B'} = \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{B'} = \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{B'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ therefore $[T]_{B',B} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $[T]_{B'',B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$.

(b) Applying the three-step procedure to the bases B and B' we obtain:

Step 1. The coordinate vector of $\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis B is $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Step 2. $[T]_{B',B}[\mathbf{x}]_B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} = [T(\mathbf{x})]_{B'}$.

Step 3. Reconstructing $T(\mathbf{x})$ from the coordinate vector obtained in Step 2, we obtain $T(\mathbf{x}) = 6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Applying the three-step procedure to the bases B and B'' we obtain:

Step 1. The coordinate vector of $\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis B is $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Step 2. $[T]_{B'',B}[\mathbf{x}]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = [T(\mathbf{x})]_{B''}$.

Step 3. Reconstructing $T(\mathbf{x})$ from the coordinate vector obtained in Step 2, we obtain $T(\mathbf{x}) = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

(c) $T\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right) = \begin{bmatrix}6\\4\end{bmatrix}$

17. **(a)** $D(\mathbf{p}_1) = 0$; $D(\mathbf{p}_2) = 1$; $D(\mathbf{p}_3) = 2x$; $[D(\mathbf{p}_1)]_B = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $[D(\mathbf{p}_2)]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}$, and $[D(\mathbf{p}_3)]_B = \begin{bmatrix}0\\2\\0\end{bmatrix}$

therefore $[D]_B = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$.

(b) Denoting $\mathbf{p} = 6 - 6x + 24x^2$, we have $[\mathbf{p}]_B = \begin{bmatrix}6\\-6\\24\end{bmatrix}$;

$[D]_B[\mathbf{p}]_B = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}6\\-6\\24\end{bmatrix} = \begin{bmatrix}-6\\48\\0\end{bmatrix} = [D(\mathbf{p})]_B$ thus $D(\mathbf{p}) = -6\mathbf{p}_1 + 48\mathbf{p}_2 + 0\mathbf{p}_3 = -6 + 48x$.

18. **(a)** $D(\mathbf{p}_1) = 0$; $D(\mathbf{p}_2) = -3$; $D(\mathbf{p}_3) = -3 + 16x$; by inspection we obtain $[D(\mathbf{p}_1)]_B = \begin{bmatrix}0\\0\\0\end{bmatrix}$;

to determine the coordinate vectors $[D(\mathbf{p}_2)]_B$ and $[D(\mathbf{p}_3)]_B$ we solve the systems

$$\begin{array}{rcrcrcr} 2a_1 & + & 2a_2 & + & 2a_3 & = & -3\\ & - & 3a_2 & - & 3a_3 & = & 0\\ & & & & 8a_3 & = & 0 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcr} 2b_1 & + & 2b_2 & + & 2b_3 & = & -3\\ & - & 3b_2 & - & 3b_3 & = & 16\\ & & & & 8b_3 & = & 0 \end{array}$$

obtaining $[D(\mathbf{p}_2)]_B = \begin{bmatrix}-\frac{3}{2}\\0\\0\end{bmatrix}$ and $[D(\mathbf{p}_3)]_B = \begin{bmatrix}\frac{23}{6}\\-\frac{16}{3}\\0\end{bmatrix}$. Therefore $[D]_B = \begin{bmatrix}0 & -\frac{3}{2} & \frac{23}{6}\\0 & 0 & -\frac{16}{3}\\0 & 0 & 0\end{bmatrix}$.

(b) Solving the system

$$\begin{array}{rcrcrcr} 2a_1 & + & 2a_2 & + & 2a_3 & = & 6\\ & - & 3a_2 & - & 3a_3 & = & -6\\ & & & & 8a_3 & = & 24 \end{array}$$

we obtain $a_1 = 1, a_2 = -1, a_3 = 3$ so that $[\mathbf{p}]_B = \begin{bmatrix}1\\-1\\3\end{bmatrix}$;

$[D]_B[\mathbf{p}]_B = \begin{bmatrix}0 & -\frac{3}{2} & \frac{23}{6}\\0 & 0 & -\frac{16}{3}\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\\3\end{bmatrix} = \begin{bmatrix}13\\-16\\0\end{bmatrix} = [D(\mathbf{p})]_B$

therefore $D(\mathbf{p}) = 13\mathbf{p}_1 - 16\mathbf{p}_2 + 0\mathbf{p}_3 = 13(2) - 16(2 - 3x) + 0(2 - 3x + 8x^2) = -6 + 48x$

19. **(a)** $D(\mathbf{f}_1) = D(1) = 0$; $D(\mathbf{f}_2) = D(\sin x) = \cos x$; $D(\mathbf{f}_3) = D(\cos x) = -\sin x$

The matrix for D relative to this basis is $\begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}$.

(b) Since $[2 + 3\sin x - 4\cos x]_B = \begin{bmatrix}2\\3\\-4\end{bmatrix}$, $[D(2 + 3\sin x - 4\cos x)]_B = \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}2\\3\\-4\end{bmatrix} = \begin{bmatrix}0\\4\\3\end{bmatrix}$.

Consequently, $D(2 + 3\sin x - 4\cos x) = (0)(1) + 4\sin x + 3\cos x = 4\sin x + 3\cos x$.

20. $\mathbf{x}$ is in V; $[\mathbf{x}]_B$ is in R^4; $[T(\mathbf{x})]_{B'}$ is in R^7; $T(\mathbf{x})$ is in W

21. **(a)** $[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B}$

(b) $[T_3 \circ T_2 \circ T_1]_{B',B} = [T_3]_{B',B'''}[T_2]_{B''',B''}[T_1]_{B'',B}$

22. Applying T to each basis vector for V yields $\mathbf{0}$ (the zero vector in W). Furthermore, the coordinate vector of $\mathbf{0}$ with respect to any basis for W must be the zero vector in R^n where $n = \dim(W)$. It follows from formula (4) that the matrix for T relative to the two bases is a zero matrix.

23. The matrix for T relative to B is the matrix whose columns are the transforms of the basis vectors in B in terms of the standard basis. Since B is the standard basis for R^n, this matrix is the standard matrix for T. Also, since B' is the standard basis for R^m, the resulting transformation will give vector components relative to the standard basis.

True-False Exercises

(a) False. The conclusion would only be true if $T: V \to V$ were a linear operator, i.e., if $V = W$.

(b) False.

(c) True. Since the matrix for T is invertible, by Theorem 8.4.2 $\ker(T) = \{\mathbf{0}\}$.

(d) False. It follows from Theorem 8.4.1 that the matrix of $S \circ T$ relative to B is $[S]_B[T]_B$.

(e) True. This follows from Theorem 8.4.2.

8.5 Similarity

1. **(a)** $\det(A) = -2$ does not equal $\det(B) = -1$

(b) $\mathrm{tr}(A) = 3$ does not equal $\mathrm{tr}(B) = -2$

2. **(a)** $\det(A) = -1$ does not equal $\det(B) = 0$

(b) $\mathrm{tr}(A) = 2$ does not equal $\mathrm{tr}(B) = 0$

3. Since $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(3)(1)-(2)(1)}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$,

$[T]_{B'} = P_{B\to B'}[T]_B P_{B'\to B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$

4. Since $\begin{bmatrix} 4 & 5 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{(4)(-1)-(5)(1)}\begin{bmatrix} -1 & -5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{5}{9} \\ \frac{1}{9} & -\frac{4}{9} \end{bmatrix}$,

$[T]_{B'} = P_{B\to B'}[T]_B P_{B'\to B} = \begin{bmatrix} \frac{1}{9} & \frac{5}{9} \\ \frac{1}{9} & -\frac{4}{9} \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} & -\frac{17}{9} \\ \frac{26}{9} & \frac{37}{9} \end{bmatrix}$

5. Since $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(3)(1)-(2)(1)}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$,

$[T]_B = P_{B'\to B}[T]_{B'}P_{B\to B'} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix}$

6. Since $\begin{bmatrix} 4 & 5 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{(4)(-1)-(5)(1)}\begin{bmatrix} -1 & -5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{5}{9} \\ \frac{1}{9} & -\frac{4}{9} \end{bmatrix}$,

$[T]_B = P_{B'\to B}[T]_{B'}P_{B\to B'} = \begin{bmatrix} 4 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{9} & \frac{5}{9} \\ \frac{1}{9} & -\frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{20}{9} & -\frac{17}{9} \\ \frac{5}{9} & \frac{16}{9} \end{bmatrix}$

7. From the definition of T we have $T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [T(\mathbf{u}_1)]_B$ and $T(\mathbf{u}_2) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = [T(\mathbf{u}_2)]_B$ therefore $[T]_B = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$. Since $[\mathbf{v}_1]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $[\mathbf{v}_2]_B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$, we have $P_{B'\to B} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$.
The inverse of $P_{B'\to B}$ is $P_{B'\to B}^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$. Using Theorem 8.5.2, we obtain
$[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 20 \\ -6 & -11 \end{bmatrix}$.

8. From the definition of T we have $T(\mathbf{u}_1) = \begin{bmatrix} 16 \\ -2 \end{bmatrix}$ and $T(\mathbf{u}_2) = \begin{bmatrix} -3 \\ 16 \end{bmatrix}$. Solving the linear systems

$$\begin{array}{rcrcr} 2a_1 & + & 4a_2 & = & 16 \\ 2a_1 & - & a_2 & = & -2 \end{array} \quad \text{and} \quad \begin{array}{rcrcr} 2b_1 & + & 4b_2 & = & -3 \\ 2b_1 & - & b_2 & = & 16 \end{array}$$

yields the coordinate vectors $[T(\mathbf{u}_1)]_B = \begin{bmatrix} \frac{4}{5} \\ \frac{18}{5} \end{bmatrix}$ and $[T(\mathbf{u}_2)]_B = \begin{bmatrix} \frac{61}{10} \\ -\frac{19}{5} \end{bmatrix}$ therefore $[T]_B = \begin{bmatrix} \frac{4}{5} & \frac{61}{10} \\ \frac{18}{5} & -\frac{19}{5} \end{bmatrix}$.

From Theorem 8.5.1, $P_{B'\to B} = [I]_{B,B'}$. Solving the linear systems

$$\begin{array}{rcrcr} 2a_1 & + & 4a_2 & = & 18 \\ 2a_1 & - & a_2 & = & 8 \end{array} \quad \text{and} \quad \begin{array}{rcrcr} 2b_1 & + & 4b_2 & = & 10 \\ 2b_1 & - & b_2 & = & 15 \end{array}$$

yields the coordinate vectors $[\mathbf{v}_1]_B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $[\mathbf{v}_2]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ therefore $P_{B'\to B} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$.
The inverse of $P_{B'\to B}$ is $P_{B'\to B}^{-1} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$. Using Theorem 8.5.2, we obtain
$[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} \frac{4}{5} & \frac{61}{10} \\ \frac{18}{5} & -\frac{19}{5} \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 15 & \frac{25}{2} \\ -\frac{98}{2} & -18 \end{bmatrix}$.

9. Denoting $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$, we have $T(\mathbf{e}_1) = (-2,1,0) = \left(T(\mathbf{e}_1)\right)_B$, $T(\mathbf{e}_2) = (-1,0,1) = \left(T(\mathbf{e}_2)\right)_B$, and $T(\mathbf{e}_3) = (0,1,0) = \left(T(\mathbf{e}_3)\right)_B$ therefore $[T]_B = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
Since $(\mathbf{v}_1)_B = (-2,1,0)$, $(\mathbf{v}_2)_B = (-1,0,1)$, and $(\mathbf{v}_3)_B = (0,1,0)$, we have $P_{B'\to B} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Using Theorem 8.5.2, we obtain

$$[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

10. Denoting $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$, we have $T(\mathbf{e}_1) = (1,0,1) = \big(T(\mathbf{e}_1)\big)_B$, $T(\mathbf{e}_2) = (2,-1,0) = \big(T(\mathbf{e}_2)\big)_B$, and $T(\mathbf{e}_3) = (-1,0,7) = \big(T(\mathbf{e}_3)\big)_B$ therefore $[T]_B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}$. Since $(\mathbf{v}_1)_B = (1,0,0)$, $(\mathbf{v}_2)_B = (1,1,0)$, and $(\mathbf{v}_3)_B = (1,1,1)$, we have $P_{B'\to B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

The inverse of $P_{B'\to B}$ is $P_{B'\to B}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Using Theorem 8.5.2, we obtain

$$[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{bmatrix}.$$

11. Denoting $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ we have $T(\mathbf{e}_1) = (\cos 45°, \sin 45°) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \big(T(\mathbf{e}_1)\big)_B$ and $T(\mathbf{e}_2) = (-\sin 45°, \cos 45°) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \big(T(\mathbf{e}_2)\big)_B$ therefore $[T]_B = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Since $(\mathbf{v}_1)_B = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(\mathbf{v}_2)_B = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, we have $P_{B'\to B} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Using Theorem 8.5.2, we obtain $[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

12. Denoting $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ we have $T(\mathbf{e}_1) = (1,0) = \big(T(\mathbf{e}_1)\big)_B$ and $T(\mathbf{e}_2) = (k,1) = \big(T(\mathbf{e}_2)\big)_B$ therefore $[T]_B = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

Since $(\mathbf{v}_1)_B = (k,1)$ and $(\mathbf{v}_2)_B = (1,0)$, we have $P_{B'\to B} = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$.

The inverse of $P_{B'\to B}$ is $P_{B'\to B}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$. Using Theorem 8.5.2, we obtain $[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.

13. Denoting $\mathbf{p}_1 = 1$ and $\mathbf{p}_2 = x$ we have $T(\mathbf{p}_1) = -1 + x$ and $T(\mathbf{p}_2) = x$. Thus $\big(T(\mathbf{p}_1)\big)_B = (-1,1)$ and $\big(T(\mathbf{p}_2)\big)_B = (0,1)$ so $[T]_B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$. Since $(\mathbf{q}_1)_B = (1,1)$ and $(\mathbf{q}_2)_B = (-1,1)$, we have

$P_{B'\to B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The inverse of $P_{B'\to B}$ is $P_{B'\to B}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Using Theorem 8.5.2, we obtain $[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

14. We have $T(\mathbf{p}_1) = 9 + 3x$ and $T(\mathbf{p}_2) = 12 + 2x$. Thus $\left(T(\mathbf{p}_1)\right)_B = \left(\frac{2}{3}, \frac{1}{2}\right)$ and $\left(T(\mathbf{p}_2)\right)_B = \left(-\frac{2}{9}, \frac{4}{3}\right)$ so $[T]_B = \begin{bmatrix} \frac{2}{3} & -\frac{2}{9} \\ \frac{1}{2} & \frac{4}{3} \end{bmatrix}$. Since $(\mathbf{q}_1)_B = \left(-\frac{2}{9}, \frac{1}{3}\right)$ and $(\mathbf{q}_2)_B = \left(\frac{7}{9}, -\frac{1}{6}\right)$, we have $P_{B'\to B} = \begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$. The inverse of $P_{B'\to B}$ is $P_{B'\to B}^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$. Using Theorem 8.5.2, we obtain

$[T]_{B'} = P_{B'\to B}^{-1}[T]_B P_{B'\to B} = \begin{bmatrix} \frac{3}{4} & \frac{7}{2} \\ \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{2}{9} \\ \frac{1}{2} & \frac{4}{3} \end{bmatrix}\begin{bmatrix} -\frac{2}{9} & \frac{7}{9} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

15. **(a)** We have $T(1) = 5 + x^2$, $T(x) = 6 - x$, and $T(x^2) = 2 - 8x - 2x^2$ so the matrix for T relative to the standard basis B is $[T]_B = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$. The characteristic polynomial for $[T]_B$ is $\lambda^3 - 2\lambda^2 - 15\lambda + 36 = (\lambda + 4)(\lambda - 3)^2$ so the eigenvalues of T are $\lambda = -4$ and $\lambda = 3$.

(b) A basis for the eigenspace of $[T]_B$ corresponding to $\lambda = -4$ is $\begin{bmatrix} -2 \\ \frac{8}{3} \\ 1 \end{bmatrix}$, so the polynomial $-2 + \frac{8}{3}x + x^2$ is a basis in P^2 for the corresponding eigenspace of T.

A basis for the eigenspace of $[T]_B$ corresponding to $\lambda = 3$ is $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, so the polynomial $5 - 2x + x^2$ is a basis in P^2 for the corresponding eigenspace of T.

16. **(a)** Let B be the standard basis for M_{22}, $B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. We have $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so by inspection

$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_B = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

therefore $[T]_B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A$.

The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & -2 & 0 \\ -1 & \lambda & -1 & 0 \\ 0 & -1 & \lambda+2 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2(\lambda+1)(\lambda+2).$$

We conclude that the eigenvalues of T are $1, -1$, and -2.

(b) The reduced row echelon form of $I - A$ is $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ contains vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ where $a = 2s, b = 3s, c = s, d = t$. Vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for this eigenspace of $A = [T]_B$.

The reduced row echelon form of $-I - A$ is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ contains vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ where $a = -2t, b = t, c = t, d = 0$. A vector $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace of $A = [T]_B$.

The reduced row echelon form of $-2I - A$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -2$ contains vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ where $a = -t, b = 0, c = t, d = 0$. A vector $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for this eigenspace of $A = [T]_B$.

Reconstructing vectors in M_{22} from their respective coordinate vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, and $\mathbf{u}_4$ yields

- a basis $\left\{\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ for the eigenspace of T corresponding to $\lambda = 1$,
- a basis $\left\{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ for the eigenspace of T corresponding to $\lambda = -1$,
- a basis $\left\{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ for the eigenspace of T corresponding to $\lambda = -2$.

18. E.g., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are not similar since their determinants do not equal.

19. The matrix of T with respect to the standard basis for R^2, $B = \{(1,0),(0,1)\}$, is $[T]_B = \begin{bmatrix} 3 & -4 \\ -1 & 7 \end{bmatrix}$.
By Formula (12), $\det(T) = \det([T]_B) = 17$.
The characteristic polynomial of $[T]_B$ is $\lambda^2 - 10\lambda + 17$. The eigenvalues of T are $5 \pm 2\sqrt{2}$.

20. The matrix of T with respect to the standard basis for R^3, $B = \{(1,0,0),(0,1,0),(0,0,1)\}$, is
$[T]_B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$. By Formula (12), $\det(T) = \det([T]_B) = 0$.
The characteristic polynomial of $[T]_B$ is $\lambda^3 - 3\lambda^2 + 3\lambda$. The eigenvalues of T are 0 and $\frac{3\pm i\sqrt{3}}{2}$.

21. $T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x-1) + a_2(x-1)^2 = (a_0 - a_1 + a_2) + (a_1 - 2a_2)x + a_2x^2$
hence the matrix of T with respect to the standard basis for P_2, $B = \{1, x, x^2\}$, is $[T]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.
By Formula (12), $\det(T) = \det([T]_B) = 1$. The characteristic polynomial of $[T]_B$ is $(\lambda - 1)^3$.
The only eigenvalue of T is 1.

22. **(a)** $T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)$
$= a_0 + a_1(2x+1) + a_2(2x+1)^2 + a_3(2x+1)^3 + a_4(2x+1)^4$
$= (a_0 + a_1 + a_2 + a_3 + a_4) + (2a_1 + 4a_2 + 6a_3 + 8a_4)x + (4a_2 + 12a_3 + 24a_4)x^2$
$+(8a_3 + 32a_4)x^3 + 16a_4x^4$
hence the matrix of T with respect to the standard basis for P_4, $B = \{1, x, x^2, x^3, x^4\}$, is
$[T]_B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 0 & 4 & 12 & 24 \\ 0 & 0 & 0 & 8 & 32 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}$. By Theorem 2.1.2, this matrix has nonzero determinant equal to the product of the entries on the main diagonal. It follows from Theorem 4.8.7 that the rank of T is 5, and the nullity of T is 0.

(b) Theorem 4.8.7 also implies that $[T]_B$ is invertible, therefore by Theorem 8.4.2, T is one-to-one.

23. Step (1) follows from the hypothesis (since $B = P^{-1}AP$).
Step (2) follows from Formula (1) in Section 1.4 (since $I = P^{-1}P$).
Step (3) follows from parts (f), (g), and (m) of Theorem 1.4.1.
Step (4) follows from Theorem 2.3.4.
Step (5) follows from commutativity of real number multiplication.
Step (6) follows from Theorem 2.3.4, Formula (1) in Section 1.4, and from Theorem 2.1.2
(since $\det(P^{-1})\det(P) = \det(P^{-1}P) = \det(I) = 1$).

31. Since $C[\mathbf{x}]_B = D[\mathbf{x}]_B$ for all $\mathbf{x}$ in V, then we can let $\mathbf{x} = \mathbf{v}_i$ for each of the basis vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V. Since $[\mathbf{v}_i]_B = \mathbf{e}_i$ for each i where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for R^n, this yields $C\mathbf{e}_i = D\mathbf{e}_i$ for i = 1, 2, ..., n. But $C\mathbf{e}_i$ and $D\mathbf{e}_i$ are the ith columns of C and D, respectively. Since corresponding columns of C and D are all equal, $C = D$.

True-False Exercises

(a) False. Every matrix is similar to itself since $A = I^{-1}AI$.

(b) True. If $A = P^{-1}BP$ and $B = Q^{-1}CQ$, then $A = P^{-1}(Q^{-1}CQ)P = (QP)^{-1}C(QP)$.

(c) True. Invertibility is a similarity invariant.

(d) True. If $A = P^{-1}BP$, then $A^{-1} = (P^{-1}BP)^{-1} = P^{-1}B^{-1}P$.

(e) True.

(f) False. For example, if T_1 is the zero operator then $[T_1]_B$ with respect to any basis B is a zero matrix.

(g) True. By Theorem 8.5.2, for any basis B' for R^n there exists P such that $[T]_{B'} = P^{-1}[T]_B P = P^{-1}IP = P^{-1}P = I$.

(h) False. If B and B' are different, let $[T]_B$ be given by the matrix $P_{B\to B'}$.
Then $[T]_{B',B} = P_{B\to B'}[T]_B = P_{B\to B'}P_{B'\to B} = I_n$.

Chapter 8 Supplementary Exercises

1. No. $T(\mathbf{x}_1 + \mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2) + B \neq (A\mathbf{x}_1 + B) + (A\mathbf{x}_2 + B) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$

2. **(a)** $A^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$
$= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$

$A^3 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$
$= \begin{bmatrix} \cos\theta\cos(2\theta) - \sin\theta\sin(2\theta) & -\cos\theta\sin(2\theta) - \sin\theta\cos(2\theta) \\ \sin\theta\cos(2\theta) + \cos\theta\sin(2\theta) & -\sin\theta\sin(2\theta) + \cos\theta\cos(2\theta) \end{bmatrix}$
$= \begin{bmatrix} \cos(3\theta) & -\sin(3\theta) \\ \sin(3\theta) & \cos(3\theta) \end{bmatrix}$

(b) $A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$

(c) According to Table 9 of Section 4.9, multiplication by A corresponds to a rotation through an angle θ.
Consequently, multiplication by A^n corresponds to a rotation through an angle $n\theta$, which agrees with the result in part (b).

3. For instance let A and B have all zero entries except for the nonzero entries $(A)_{11} \neq (B)_{11}$ so that their traces are different. E.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not similar.

5. **(a)** The matrix for T relative to the standard basis is $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. A basis for the range of T is a basis for the column space of A. A reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.
Since row operations don't change the dependency relations among columns, the reduced form

of A indicates that $T(\mathbf{e}_3)$ and any two of $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_4)$ form a basis for the range.
The reduced form of A shows that the general solution of $A\mathbf{x} = \mathbf{0}$ is $x = -s$, $x_2 = s, x_3 = 0, x_4 = s$ so a basis for the null space of A, which is the kernel of T, is $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) Since $R(T)$ is three-dimensional and $\ker(T)$ is one-dimensional, $\operatorname{rank}(T) = 3$ and $\operatorname{nullity}(T) = 1$.

6. **(a)** Setting $[x_1 \quad x_2 \quad x_3]\begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix} = [0 \quad 0 \quad 0]$ yields the linear system

$$\begin{array}{rcrcrcl} -x_1 & + & 3x_2 & + & 2x_3 & = & 0 \\ 2x_1 & & & + & 2x_3 & = & 0 \\ 4x_1 & + & x_2 & + & 5x_3 & = & 0 \end{array}$$

The reduced row echelon form of the augmented matrix of this system is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

therefore the kernel of T consists of all vectors $[x_1 \quad x_2 \quad x_3]$ such that $x_1 = -t, x_2 = -t, x_3 = t$.
A basis for $\ker(T)$ is formed by $[-1 \quad -1 \quad 1]$.

(b) The range of T contains all vectors $x_1[-1 \quad 2 \quad 4] + x_2[3 \quad 0 \quad 1] + x_3[2 \quad 2 \quad 5]$.
Based on the reduced row echelon form obtained in part (a), $[-1 \quad 2 \quad 4]$ and $[3 \quad 0 \quad 1]$ form a basis for $R(T)$.

7. **(a)** The matrix for T relative to B is $[T]_B = \begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & -1 & -4 & 6 \\ 1 & 2 & 5 & -6 \\ 3 & 2 & 3 & -2 \end{bmatrix}$.

$[T]_B$ reduces to $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has rank 2 and nullity 2. Thus, $\operatorname{rank}(T) = 2$ and $\operatorname{nullity}(T) = 2$.

(b) Since $\operatorname{nullity}(T) \neq 0$, T is not one-to-one.

9. **(a)** Since $A = P^{-1}BP$, we have
$A^T = (P^{-1}BP)^T = P^T B^T (P^{-1})^T = ((P^T)^{-1})^{-1} B^T (P^{-1})^T = ((P^{-1})^T)^{-1} B^T (P^{-1})^T$.
Thus, A^T and B^T are similar.

(b) $A^{-1} = (P^{-1}BP)^{-1} = P^{-1}B^{-1}P$ thus A^{-1} and B^{-1} are similar.

11. For $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(X) = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b & b \\ d & d \end{bmatrix} = \begin{bmatrix} a+b+c & 2b+d \\ d & d \end{bmatrix}$.
$T(X) = \mathbf{0}$ gives the equations

$$\begin{aligned} a + b + c &= 0 \\ 2b + d &= 0 \\ d &= 0 \end{aligned}$$

Thus $b = d = 0$ and $c = -a$ hence X is in $\ker(T)$ if it has the form $\begin{bmatrix} k & 0 \\ -k & 0 \end{bmatrix}$, so $\ker(T) = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\right\}$ which is one-dimensional. We conclude that $\operatorname{nullity}(T) = 1$ and $\operatorname{rank}(T) = \dim(M_{22}) - \operatorname{nullity}(T) = 3$.

13. The standard basis for M_{22} is $\mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $L(\mathbf{u}_1) = \mathbf{u}_1$, $L(\mathbf{u}_2) = \mathbf{u}_3$, $L(\mathbf{u}_3) = \mathbf{u}_2$, and $L(\mathbf{u}_4) = \mathbf{u}_4$, the matrix of L relative to the standard basis is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

14. **(a)** $P = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} = P_{B' \to B} = [[\mathbf{v}_1]_B \mid [\mathbf{v}_2]_B \mid [\mathbf{v}_3]_B]$ therefore
$\mathbf{v}_1 = 2\mathbf{u}_1 + 1\mathbf{u}_2 + 0\mathbf{u}_3$, $\mathbf{v}_2 = -1\mathbf{u}_1 + 1\mathbf{u}_2 + 1\mathbf{u}_3$, and $\mathbf{v}_3 = 3\mathbf{u}_1 + 4\mathbf{u}_2 + 2\mathbf{u}_3$.

(b) Using the inverse matrix $P^{-1} = \begin{bmatrix} -2 & 5 & -7 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix} = P_{B \to B'} = [[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid [\mathbf{u}_3]_{B'}]$ we obtain
$\mathbf{u}_1 = -2\mathbf{v}_1 - 2\mathbf{v}_2 + 1\mathbf{v}_3$, $\mathbf{u}_2 = 5\mathbf{v}_1 + 4\mathbf{v}_2 - 2\mathbf{v}_3$, and $\mathbf{u}_3 = -7\mathbf{v}_1 - 5\mathbf{v}_2 + 3\mathbf{v}_3$.

15. The transition matrix from B' to B is $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, thus $[T]_{B'} = P^{-1}[T]_B P = \begin{bmatrix} -4 & 0 & 9 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$.

16. Denoting $A = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ we have $\det(\lambda I - A) = \det(\lambda I - B) = \lambda^2 - 5\lambda + 5$.
Both matrices have eigenvalues $\frac{5 \pm \sqrt{5}}{2}$.
$\left(\frac{5+\sqrt{5}}{2} I - A\right)\mathbf{x} = \mathbf{0}$ has solutions $x_1 = \frac{3-\sqrt{5}}{2} t$, $x_2 = t$; $\left(\frac{5-\sqrt{5}}{2} I - A\right)\mathbf{x} = \mathbf{0}$ has solutions $x_1 = \frac{3+\sqrt{5}}{2} t$, $x_2 = t$;
$\left(\frac{5+\sqrt{5}}{2} I - B\right)\mathbf{x} = \mathbf{0}$ has solutions $x_1 = \frac{\sqrt{5}-1}{2} t$, $x_2 = t$; $\left(\frac{5-\sqrt{5}}{2} I - B\right)\mathbf{x} = \mathbf{0}$ has solutions $x_1 = \frac{-\sqrt{5}-1}{2} t$, $x_2 = t$

Letting $P = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{\sqrt{5}-1}{2} & \frac{-\sqrt{5}-1}{2} \\ 1 & 1 \end{bmatrix}$ we have $P^{-1}AP = \begin{bmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{bmatrix} = Q^{-1}BQ$ therefore
$B = Q(P^{-1}AP)Q^{-1} = (QP^{-1})A(PQ^{-1}) = (PQ^{-1})^{-1}A(PQ^{-1})$, which shows that A and B are similar.

$\det\left(\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}\right) = 0$ does not equal $\det\left(\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}\right) = -2$, thus that the two matrices are not similar.

17. $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, thus $[T\}_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.
Note that this can also be read directly from $[T(\mathbf{x})]_B$.

19. **(a)** $D(\mathbf{f} + \mathbf{g}) = (f(x) + g(x))'' = f''(x) + g''(x)$ and $D(k\mathbf{f}) = (kf(x))'' = kf''(x)$.

(b) If $\mathbf{f}$ is in $\ker(D)$, then $\mathbf{f}$ has the form $\mathbf{f} = f(x) = a_0 + a_1 x$, so a basis for $\ker(D)$ is $f(x) = 1, g(x) = x$.

(c) The equation $D(\mathbf{f}) = \mathbf{f}$ can be rewritten as $y'' = y$. Substituting $y_1 = y$ and $y_2 = y'$ yields the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 \end{aligned}$$

The coefficient matrix of this system is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial of A is $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$ thus the eigenvalues of A are $\lambda = 1$ and $\lambda = -1$.

The reduced row echelon form of $1I - A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $-1I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = -1$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for this eigenspace.

Therefore $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ diagonalizes A and $P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The substitution $\mathbf{y} = P\mathbf{u}$ yields the "diagonal system" $\mathbf{u}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{u}$ consisting of equations $u_1' = u_1$ and $u_2' = -u_2$. From Formula (2) in Section 5.4, these equations have the solutions $u_1 = c_1e^x, u_2 = c_2e^{-x}$, i.e., $\mathbf{u} = \begin{bmatrix} c_1e^x \\ c_2e^{-x} \end{bmatrix}$. From $\mathbf{y} = P\mathbf{u}$ we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^x \\ c_2e^{-x} \end{bmatrix} = \begin{bmatrix} c_1e^x - c_2e^{-x} \\ c_1e^x + c_2e^{-x} \end{bmatrix}$ thus $y_1 = c_1e^x - c_2e^{-x}$ and $y_2 = c_1e^x + c_2e^{-x}$.

We conclude that the original equation $y'' = y$ has the solution $y = c_1e^x - c_2e^{-x}$.

Thus, $f(x) = e^x$ and $g(x) = e^{-x}$ form a basis for the subspace of $C^2(-\infty, \infty)$ containing the functions satisfying the equation $D(\mathbf{f}) = \mathbf{f}$. (Other bases are possible, e.g. $\{e^x, -e^{-x}\}$.)

20. (a) $T(x^2 + 5x + 6) = \begin{bmatrix} (-1)^2 + 5(-1) + 6 \\ 0^2 + 5(0) + 6 \\ 1^2 + 5(1) + 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$

(b) For any $p(x)$ and $q(x)$ in P_2 and for any real number k we have

$$T\big(kp(x)\big) = \begin{bmatrix} kp(-1) \\ kp(0) \\ kp(1) \end{bmatrix} = k\begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = kT\big(p(x)\big) \text{ and}$$

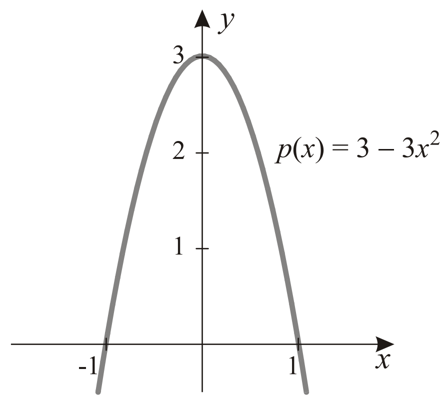
$$T\big(p(x) + q(x)\big) = \begin{bmatrix} p(-1) + q(-1) \\ p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} = T\big(p(x)\big) + T\big(q(x)\big)$$

therefore T is a linear transformation

(c) Solving $T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_0 \\ a_0 + a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields $a_0 = a_1 = a_2 = 0$ thus $\ker(T) = \{0\}$. Consequently, T is one-to-one.

(d) Setting $T(a_0 + a_1x + a_2x^2) = \begin{bmatrix} a_0 - a_1 + a_2 \\ a_0 \\ a_0 + a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ yields a linear system with augmented matrix $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, whose reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$.
Thus $T^{-1}(0,3,0) = 3 - 3x^2$.

(e)



21. **(c)** Note that $a_1P_1(x) + a_2P_2(x) + a_3P_3(x)$ evaluated at x_1, x_2, and x_3 gives the values a_1, a_2, and a_3, respectively, since $P_i(x_i) = 1$ and $P_i(x_j) = 0$ for $i \neq j$.

(d) From the computations in part (c), the points lie on the graph.

22. **(a)** For any $y(x)$ and $z(x)$ in V and for any real number k we have
$L(ky(x)) = \frac{d^2}{dx^2}(ky(x)) + p(x)\frac{d}{dx}(ky(x)) + q(x)(ky(x)) = k(y''(x) + p(x)y'(x) + q(x)y(x))$
$= kL(y(x))$ and
$L(y(x) + z(x)) = \frac{d^2}{dx^2}(y(x) + z(x)) + p(x)\frac{d}{dx}(y(x) + z(x)) + q(x)(y(x) + z(x))$
$= (y''(x) + p(x)y'(x) + q(x)y(x)) + (z''(x) + p(x)z'(x) + q(x)z(x)) = L(y(x)) + L(z(x))$
therefore L is a linear transformation.

(b) Letting $p(x) = 0$ and $q(x) = 1$, we have $L(y(x)) = y''(x) + y(x)$.
$L(\phi(x)) = \frac{d^2}{dx^2}(c_1 \sin x + c_2 \cos x) + c_1 \sin x + c_2 \cos x$
$= \frac{d}{dx}(c_1 \cos x - c_2 \sin x) + c_1 \sin x + c_2 \cos x = -c_1 \sin x - c_2 \cos x + c_1 \sin x + c_2 \cos x = 0$
therefore $\phi(x)$ is in $\ker(L)$ for all real values of c_1 and c_2.

23. $D(1) = 0$
$D(x) = 1$
$D(x^2) = 2x$
$\vdots$
$D(x^n) = nx^{n-1}$

This gives the matrix shown.

24. For every integer $k > 1$, we have $D\left(\frac{(x-c)^k}{k!}\right) = \frac{k(x-c)^{k-1}}{k!} = \frac{(x-c)^{k-1}}{(k-1)!}$; also, $D(x-c) = 1$ and $D(1) = 0$.

Therefore, the matrix of D with respect to the given basis is $\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$.

25. The matrix of J with respect to the given bases is $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n+1} \end{bmatrix}$.

CHAPTER 9: NUMERICAL METHODS

9.1 LU-Decompositions

1. Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1 and y_2 by $\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}$

Step 3. Solving $\underbrace{\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution yields $y_1 = 0, y_2 = 1$.

Step 4. Solving $\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = 2, x_2 = 1$.

2. Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}}$

Step 3. Solving $\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution yields $y_1 = -1, y_2 = -5,$ $y_3 = -3$.

Step 4. Solving $\underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ -5 \\ -3 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = -2, x_2 = 1, x_3 = -3$.

3. $A = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$ $\qquad \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$ (we follow the procedure of Example 3)

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \leftarrow \text{ multiplier } = \tfrac{1}{2} \qquad \begin{bmatrix} 2 & 0 \\ \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \leftarrow \text{ multiplier } = 1 \qquad \begin{bmatrix} 2 & 0 \\ -1 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \leftarrow \text{ multiplier } = \tfrac{1}{3} \qquad L = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1 and y_2 by $\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}$

Step 3. Solving $\underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution yields $y_1 = -1, y_2 = -1$.

Step 4. Solving $\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = 3, x_2 = -1$.

4. $A = \begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}$ $\qquad \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$ (we follow the procedure of Example 3)

$\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \leftarrow$ multiplier $= -\frac{1}{5}$ $\qquad \begin{bmatrix} -5 & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \leftarrow$ multiplier $= -6$ $\qquad \begin{bmatrix} -5 & 0 \\ 6 & \bullet \end{bmatrix}$

$U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \leftarrow$ multiplier $= -\frac{1}{7}$ $\qquad L = \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -10 \\ 19 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1 and y_2 by $\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}$

Step 3. Solving $\underbrace{\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -10 \\ 19 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution yields $y_1 = 2, y_2 = -1$.

Step 4. Solving $\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = 4, x_2 = -1$.

5. $A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$ $\qquad \begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \leftarrow$ multiplier $= \frac{1}{2}$ $\qquad \begin{bmatrix} 2 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ multiplier} = 0 \\ \leftarrow \text{ multiplier} = 1 \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & \bullet & 0 \\ -1 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ multiplier} = -\frac{1}{2} \\ \\ \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{ multiplier} = -4 \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{ multiplier} = \frac{1}{5} \end{matrix} \qquad L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}}$

Step 3. Solving $\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution yields $y_1 = -2, y_2 = 1, y_3 = 0$.

Step 4. Solving $\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = -1, x_2 = 1, x_3 = 0$.

6. $$A = \begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{ multiplier} = -\frac{1}{3} \\ \\ \\ \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ multiplier} = -1 \\ \leftarrow \text{ multiplier} = 0 \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & \bullet & 0 \\ 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ multiplier} = \frac{1}{2} \\ \\ \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{ multiplier} = -1 \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{ multiplier} = 1 \end{matrix} \qquad L = \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}}$

Step 3. Solving $\underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution yields $y_1 = 11, y_2 = -2, y_3 = 1$.

Step 4. Solving $\underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = 1, x_2 = -2, x_3 = 1$.

7. **(a)** $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; $U^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} & -\frac{7}{48} \\ 0 & \frac{1}{4} & \frac{5}{24} \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$ **(b)** $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{5}{48} & -\frac{1}{48} & -\frac{7}{48} \\ -\frac{7}{24} & \frac{11}{24} & \frac{5}{24} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

8. **(a)** $L^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{5}{14} & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$; $U^{-1} = \begin{bmatrix} 1 & -3 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ **(b)** $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -\frac{8}{7} & \frac{3}{7} & \frac{8}{7} \\ \frac{3}{7} & -\frac{2}{7} & -\frac{3}{7} \\ \frac{5}{14} & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$

9. **(a)** Reduce A to upper triangular form.

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$, 2, and -2, which leads to $L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ where the 1's on the diagonal reflect that no multiplication was required on the 2nd and 3rd diagonal entries.

(b) To change the 2 on the diagonal of L to a 1, the first column of L is divided by 2 and the diagonal matrix has a 2 as the 1, 1 entry.

$A = L_1DU_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is the LDU-decomposition of A.

(c) Let $U_2 = DU$, and $L_2 = L_1$, then $A = L_2U_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

10. **(a)** Setting $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ yields $\begin{bmatrix} a & ad \\ b & bd + c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Equating entries in the first row leads to a contradiction, since we cannot simultaneously have $a = 0$ and $ad = 1$. We conclude that the matrix has no LU-decomposition. (This matrix cannot be reduced to a row echelon form without interchanging rows.)

(b) By inspection, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U}$.

11. $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, so the system $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ is

$$\begin{aligned} y_1 \qquad\qquad &= 1 \\ y_2 \quad &= 2 \\ 3y_1 - 5y_2 + y_3 &= 5 \end{aligned}$$

which has the solution $y_1 = 1, y_2 = 2, y_3 = 12$.

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 12 \end{bmatrix}$ is

$$\begin{aligned} &x_1 + 2x_2 + 2x_3 = 1 \\ &x_2 + 4x_3 = 2 \\ &17x_3 = 12 \end{aligned}$$

which gives the solution of the original system: $x_1 = \frac{21}{17}, x_2 = -\frac{14}{17}, x_3 = \frac{12}{17}$.

12. The inverse of $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ thus $P^{-1}\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$.

We rewrite $A\mathbf{x} = \mathbf{b}$ as $P^{-1}(PLU)\mathbf{x} = P^{-1}\mathbf{b}$, i.e., $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}_{P^{-1}\mathbf{b}}$.

Solving $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}_{P^{-1}\mathbf{b}}$ by forward substitution yields $y_1 = 3, y_2 = 0, y_3 = 0$.

Solving $\underbrace{\begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = \frac{3}{4}, x_2 = 0, x_3 = 0$.

13.

$A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \leftarrow$ multiplier $= \frac{1}{2}$ $\qquad \begin{bmatrix} 2 & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \leftarrow$ multiplier $= -4$ $\qquad \begin{bmatrix} 2 & 0 \\ 4 & \bullet \end{bmatrix}$

$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftarrow$ multiplier $= -\frac{1}{3}$ $\qquad \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$

A general 2×2 lower triangular matrix with nonzero main diagonal entries can be factored as $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ therefore $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. We conclude that an LDU-decomposition of A is $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = LDU$.

14. Reduce A to upper triangular form:

$$\begin{bmatrix} 3 & -12 & 6 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{3}$, -6, $\frac{1}{2}$, and 4, which leads to $L_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 6 & -4 & 1 \end{bmatrix}$.

Since $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we conclude that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the LDU-decomposition of A.

15. If rows 2 and 3 of A are interchanged, then the resulting matrix has an LU-decomposition.

For $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $PA = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}$. Reduce PA to upper triangular form:

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{3}$, -3, and $\frac{1}{2}$, so $L = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. Since $P = P^{-1}$, we have $A = PLU =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $P\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, the system to solve is $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$ is

$$\begin{aligned} 3y_1 \qquad\qquad &= -2 \\ 2y_2 \qquad &= 4 \\ 3y_1 \qquad + y_3 &= 1 \end{aligned}$$

which has the solution $y_1 = -\frac{2}{3}$, $y_2 = 2$, $y_3 = 3$.

$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 2 \\ 3 \end{bmatrix}$ is

$$\begin{aligned} x_1 - \frac{1}{3}x_2 &= -\frac{2}{3} \\ x_2 + \frac{1}{2}x_3 &= 2 \\ x_3 &= 3 \end{aligned}$$

which gives the solution to the original system: $x_1 = -\frac{1}{2}$, $x_2 = \frac{1}{2}$, $x_3 = 3$.

16. As discussed in the last subsection of Section 9.1, we introduce a permutation matrix $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Multiplying QA results in interchanging the first two rows of A, so that an LU-decomposition can be found.

$$QA = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & -2 \\ 2 & 2 & 5 \end{bmatrix} \begin{matrix} \leftarrow \text{ multiplier} = 1 \\ \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ multiplier} = 0 \\ \leftarrow \text{ multiplier} = -2 \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bullet & 0 \\ 2 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & -3 \end{bmatrix} \leftarrow \text{ multiplier } = \tfrac{1}{3} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & -3 \end{bmatrix} \leftarrow \text{ multiplier } = 0 \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{ multiplier } = -\tfrac{1}{3} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

Since $P = Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we obtain a PLU-decomposition of A:

$$A = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 1 & 4 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = PLU.$$

Using $P^{-1}\mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}$, we rewrite $A\mathbf{x} = \mathbf{b}$ as $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$, i.e., $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}}_{P^{-1}\mathbf{b}}$.

Solving $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}}_{P^{-1}\mathbf{b}}$ by forward substitution yields $y_1 = 5$, $y_2 = \frac{7}{3}$, $y_3 = 4$.

Solving $\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ \frac{7}{3} \\ 4 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = -16$, $x_2 = 5$, $x_3 = 4$.

17. Approximately $\frac{2}{3}n^3$ additions and multiplications are required – see Section 9.3.

18. **(a)** If A has such an LU-decomposition, it can be written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ wx & wy + z \end{bmatrix}$ which leads to the equations

$$\begin{aligned} x &= a \\ y &= b \\ wx &= c \\ wy + z &= d \end{aligned}$$

Since $a \neq 0$, the system has the unique solution $x = a$, $y = b$, $w = \frac{c}{a}$, and $z = d - \frac{bc}{a} = \frac{ad-bc}{a}$. Because the solution is unique, the LU-decomposition is also unique.

(b) From part (a) the LU-decomposition is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$.

True-False Exercises

(a) False. If the matrix cannot be reduced to row echelon form without interchanging rows, then it does not have an LU-decomposition.

(b) False. If the row equivalence of A and U requires interchanging rows of A, then A does not have an LU-decomposition.

(c) True. This follows from part (b) of Theorem 1.7.1.

(d) True. (Refer to the subsection "LDU-Decompositions" for the relevant result.)

(e) True. The procedure for obtaining a PLU-decomposition of a matrix A has been described in the subsection "PLU-Decompositions".

9.2 The Power Method

1. **(a)** $\lambda_3 = -8$ is the dominant eigenvalue since $|\lambda_3| = 8$ is greater than the absolute values of all remaining eigenvalues

(b) $|\lambda_1| = |\lambda_4| = 5$; no dominant eigenvalue

2. **(a)** $\lambda_3 = -3$ is the dominant eigenvalue since $|\lambda_3| = 3$ is greater than the absolute values of all remaining eigenvalues

(b) $|\lambda_1| = |\lambda_4| = 3$; no dominant eigenvalue

3. $A\mathbf{x}_0 = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ $\qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ -1 \end{bmatrix} \approx \begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix}$

$A\mathbf{x}_1 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix} \approx \begin{bmatrix} 5.09902 \\ -0.78446 \end{bmatrix}$ $\qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix}$

$A\mathbf{x}_2 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix} \approx \begin{bmatrix} 5.09391 \\ -0.83631 \end{bmatrix}$ $\qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix}$

$A\mathbf{x}_3 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix} \approx \begin{bmatrix} 5.09596 \\ -0.82478 \end{bmatrix}$ $\qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.98715 \\ -0.15977 \end{bmatrix}$

$$\lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 = (A\mathbf{x}_1)^T \mathbf{x}_1 \approx 5.15385$$

$$\lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 = (A\mathbf{x}_2)^T \mathbf{x}_2 \approx 5.16185$$

$$\lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 = (A\mathbf{x}_3)^T \mathbf{x}_3 \approx 5.16226$$

$$\lambda^{(4)} = A\mathbf{x}_4 \cdot \mathbf{x}_4 = (A\mathbf{x}_4)^T \mathbf{x}_4 \approx 5.16228$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 5 & 1 \\ 1 & \lambda + 1 \end{vmatrix} = \lambda^2 - 4\lambda - 6 = \left(\lambda - 2 - \sqrt{10}\right)\left(\lambda - 2 + \sqrt{10}\right)$; the dominant eigenvalue is $2 + \sqrt{10} \approx 5.16228$.

The reduced row echelon form of $(2+\sqrt{10})I - A$ is $\begin{bmatrix} 1 & 3+\sqrt{10} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2+\sqrt{10}$ contains vectors (x_1, x_2) where $x_1 = -(3+\sqrt{10})t$, $x_2 = t$,. A vector $(-3-\sqrt{10}, 1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_4$ approximates a unit eigenvector $\frac{1}{\sqrt{20+6\sqrt{10}}}(3\sqrt{10}, -1) \approx (0.98709, -0.16018)$ and $\lambda^{(4)}$ approximates the dominant eigenvalue $2+\sqrt{10} \approx 5.16228$.

4. $A\mathbf{x}_0 = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$ $\quad\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{53}}\begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.96152 \\ -0.27472 \\ 0.00000 \end{bmatrix}$

$A\mathbf{x}_1 \approx \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0.96152 \\ -0.27472 \\ 0.00000 \end{bmatrix} \approx \begin{bmatrix} 7.28011 \\ -3.57137 \\ 0.54944 \end{bmatrix}$ $\quad\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \frac{1}{8.12752}\begin{bmatrix} 7.28011 \\ -3.57137 \\ 0.54944 \end{bmatrix} \approx \begin{bmatrix} 0.89574 \\ -0.43942 \\ 0.06760 \end{bmatrix}$

$A\mathbf{x}_2 \approx \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0.89574 \\ -0.43942 \\ 0.06760 \end{bmatrix} \approx \begin{bmatrix} 7.14898 \\ -4.56318 \\ 1.21685 \end{bmatrix}$ $\quad\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \frac{1}{8.56804}\begin{bmatrix} 7.14898 \\ -4.56318 \\ 1.21685 \end{bmatrix} \approx \begin{bmatrix} 0.83438 \\ -0.53258 \\ 0.14202 \end{bmatrix}$

$A\mathbf{x}_3 \approx \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0.83438 \\ -0.53258 \\ 0.14202 \end{bmatrix} \approx \begin{bmatrix} 6.90581 \\ -5.14829 \\ 1.77527 \end{bmatrix}$ $\quad\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \frac{1}{8.7947}\begin{bmatrix} 6.90581 \\ -5.14829 \\ 1.77527 \end{bmatrix} \approx \begin{bmatrix} 0.78522 \\ -0.58539 \\ 0.20186 \end{bmatrix}$

$$\lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 = (A\mathbf{x}_1)^T\mathbf{x}_1 \approx \begin{bmatrix} 7.28011 & -3.57137 & 0.54944 \end{bmatrix}\begin{bmatrix} 0.96152 \\ -0.27472 \\ 0.00000 \end{bmatrix} \approx 7.98113$$

$$\lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 = (A\mathbf{x}_2)^T\mathbf{x}_2 \approx \begin{bmatrix} 7.14898 & -4.56318 & 1.21685 \end{bmatrix}\begin{bmatrix} 0.89574 \\ -0.43942 \\ 0.06760 \end{bmatrix} \approx 8.49100$$

$$\lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 = (A\mathbf{x}_3)^T\mathbf{x}_3 \approx \begin{bmatrix} 6.90581 & -5.14829 & 1.77527 \end{bmatrix}\begin{bmatrix} 0.83438 \\ -0.53258 \\ 0.14202 \end{bmatrix} \approx 8.75607$$

$$\lambda^{(4)} = A\mathbf{x}_4 \cdot \mathbf{x}_4 = (A\mathbf{x}_4)^T\mathbf{x}_4 \approx \begin{bmatrix} 6.66734 & -5.48648 & 2.18006 \end{bmatrix}\begin{bmatrix} 0.78522 \\ -0.58539 \\ 0.20186 \end{bmatrix} \approx 8.88712$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-7 & 2 & 0 \\ 2 & \lambda-6 & 2 \\ 0 & 2 & \lambda-5 \end{vmatrix} = (\lambda-3)(\lambda-6)(\lambda-9)$; the dominant eigenvalue is 9.

The reduced row echelon form of $9I - A$ is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 9$ contains vectors (x_1, x_2, x_3) where $x_1 = 2t$, $x_2 = -2t$, $x_3 = t$. A vector $(2, -2, 1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_4$ approximates the unit eigenvector $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \approx (0.66667, -0.66667, 0.33333)$ and $\lambda^{(4)}$ approximates the dominant eigenvalue 9.

5. $A\mathbf{x}_0 = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $\quad\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A\mathbf{x}_1 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$

$$A\mathbf{x}_2 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -3.5 \\ 6.5 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} -0.53846 \\ 1 \end{bmatrix}$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -0.53846 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -3.53846 \\ 6.61538 \end{bmatrix} \qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} -0.53488 \\ 1 \end{bmatrix}$$

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = 6$$
$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = 6.6$$
$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 6.60550$$
$$\lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} \approx 6.60555$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 3 \\ 3 & \lambda - 5 \end{vmatrix} = \lambda^2 - 6\lambda - 4$, so the eigenvalues of A are $\lambda = 3 \pm \sqrt{13}$. The dominant eigenvalue is $3 + \sqrt{13} \approx 6.60555$ with corresponding scaled eigenvector $\begin{bmatrix} \frac{2-\sqrt{13}}{3} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.53518 \\ 1 \end{bmatrix}$.

6. $$A\mathbf{x}_0 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix} \qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \frac{1}{7}\begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.57143 \\ 0.85714 \end{bmatrix}$$

$$A\mathbf{x}_1 \approx \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1.00000 \\ 0.57143 \\ 0.85714 \end{bmatrix} \approx \begin{bmatrix} 5.85714 \\ 3.14286 \\ 5.42857 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} \approx \frac{1}{5.85714}\begin{bmatrix} 5.85714 \\ 3.14286 \\ 5.42857 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.53659 \\ 0.92683 \end{bmatrix}$$

$$A\mathbf{x}_2 \approx \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1.00000 \\ 0.53659 \\ 0.92683 \end{bmatrix} \approx \begin{bmatrix} 5.92683 \\ 3.07317 \\ 5.70732 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \frac{1}{5.92683}\begin{bmatrix} 5.92683 \\ 3.07317 \\ 5.70732 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.51852 \\ 0.96296 \end{bmatrix}$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1.00000 \\ 0.51852 \\ 0.96296 \end{bmatrix} \approx \begin{bmatrix} 5.96296 \\ 3.03704 \\ 5.85185 \end{bmatrix} \qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \frac{1}{5.96296}\begin{bmatrix} 5.96296 \\ 3.03704 \\ 5.85185 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.50932 \\ 0.98137 \end{bmatrix}$$

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = \frac{(A\mathbf{x}_1)^T\mathbf{x}_1}{\mathbf{x}_1^T\mathbf{x}_1} \approx \frac{12.30612}{2.06122} \approx 5.97030$$
$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = \frac{(A\mathbf{x}_2)^T\mathbf{x}_2}{\mathbf{x}_2^T\mathbf{x}_2} \approx \frac{12.86556}{2.14694} \approx 5.99252$$
$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} = \frac{(A\mathbf{x}_3)^T\mathbf{x}_3}{\mathbf{x}_3^T\mathbf{x}_3} \approx \frac{13.17284}{2.19616} \approx 5.99813$$
$$\lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} = \frac{(A\mathbf{x}_4)^T\mathbf{x}_4}{\mathbf{x}_4^T\mathbf{x}_4} \approx \frac{13.33386}{2.22248} \approx 5.99953$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -2 & -2 \\ -2 & \lambda-2 & 0 \\ -2 & 0 & \lambda-4 \end{vmatrix} = \lambda(\lambda-3)(\lambda-6)$; the dominant eigenvalue is 6.

The reduced row echelon form of $6I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 6$ contains vectors (x_1, x_2, x_3) where $x_1 = t, x_2 = \frac{1}{2}t, x_3 = t$. A vector $(1, \frac{1}{2}, 1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_4$ approximates the eigenvector $\left(1, \frac{1}{2}, 1\right)$ and $\lambda^{(4)}$ approximates the dominant eigenvalue 6.

7. **(a)** $A\mathbf{x}_0 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

$A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix}$ $\quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}$

$A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.6 \end{bmatrix}$ $\quad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}$

(b) $\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = 2.8$; $\quad \lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 2.976$; $\quad \lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 2.997$

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix} = (\lambda-3)(\lambda-1)$; the dominant eigenvalue is 3.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 10$ contains vectors (x_1, x_2) where $x_1 = -t, x_2 = t$. A vector $(-1, 1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_3$ approximates the eigenvector $(1, -1)$ and $\lambda^{(3)}$ approximates the dominant eigenvalue 3.

(d) The percentage error is $\left|\frac{\lambda - \lambda^{(3)}}{\lambda}\right| \approx \left|\frac{3 - 2.997}{3}\right| = 0.001 = 0.1\%$.

8. **(a)** $A\mathbf{x}_0 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix}$ $\quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \frac{1}{10}\begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.3 \\ 1.0 \end{bmatrix}$

$A\mathbf{x}_1 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.3 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.9 \\ 10.0 \end{bmatrix}$ $\quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \frac{1}{10}\begin{bmatrix} 0.9 \\ 0.9 \\ 10.0 \end{bmatrix} = \begin{bmatrix} 0.09 \\ 0.09 \\ 1.00 \end{bmatrix}$

$A\mathbf{x}_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 0.09 \\ 0.09 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 0.27 \\ 0.27 \\ 10.00 \end{bmatrix}$ $\quad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} = \frac{1}{10}\begin{bmatrix} 0.27 \\ 0.27 \\ 10.00 \end{bmatrix} = \begin{bmatrix} 0.027 \\ 0.027 \\ 1.000 \end{bmatrix}$

(b)

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = \frac{(A\mathbf{x}_1)^T \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \approx \frac{10.54}{1.18} \approx 8.932$$

$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = \frac{(A\mathbf{x}_2)^T \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2} \approx \frac{10.049}{1.016} \approx 9.889$$

$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} = \frac{(A\mathbf{x}_3)^T \mathbf{x}_3}{\mathbf{x}_3^T \mathbf{x}_3} \approx \frac{10.004}{1.001} \approx 9.990$$

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ -1 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 10 \end{vmatrix} = (\lambda - 1)(\lambda - 3)(\lambda - 10)$; the dominant eigenvalue is 10.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 10$ contains vectors (x_1, x_2, x_3) where $x_1 = 0, x_2 = 0, x_3 = t$. A vector $(0, 0,1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_3$ approximates the eigenvector (0,0,1) and $\lambda^{(3)}$ approximates the dominant eigenvalue 10.

(d) The percentage error in the approximation $\lambda^{(3)} \approx 9.99$ of the dominant eigenvalue $\lambda = 10$ is

$$\left|\frac{\lambda - \lambda^{(3)}}{\lambda}\right| = \left|\frac{10 - 9.99}{10}\right| = 0.001 = 0.1\%$$

9. By Formula (10), $\mathbf{x}_5 = \frac{A^5\mathbf{x}_0}{\max(A^5\mathbf{x}_0)} \approx \begin{bmatrix} 0.99180 \\ 1 \end{bmatrix}$. Thus $\lambda^{(5)} = \frac{A\mathbf{x}_5 \cdot \mathbf{x}_5}{\mathbf{x}_5 \cdot \mathbf{x}_5} \approx 2.99993$.

10. By Formula (10), $\mathbf{x}_5 = \frac{A^5\mathbf{x}_0}{\max(A^5\mathbf{x}_0)} \approx \begin{bmatrix} 1.00000 \\ 0.99180 \end{bmatrix}$. Thus $\lambda^{(5)} = \frac{A\mathbf{x}_5 \cdot \mathbf{x}_5}{\mathbf{x}_5 \cdot \mathbf{x}_5} \approx 2.99993$.

11. By inspection, A is symmetric and has a dominant eigenvalue -1.
Assuming $a \neq 0$, the power sequence is

$$A\mathbf{x}_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix} \qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{|a|}\begin{bmatrix} -a \\ 0 \end{bmatrix} = \begin{bmatrix} -a/|a| \\ 0 \end{bmatrix}$$

$$A\mathbf{x}_1 \approx \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} a/|a| \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} = \frac{1}{1}\begin{bmatrix} a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} a/|a| \\ 0 \end{bmatrix}$$

$$A\mathbf{x}_2 \approx \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} = \frac{1}{1}\begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} -a/|a| \\ 0 \end{bmatrix}$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} \approx \begin{bmatrix} a/|a| \\ 0 \end{bmatrix} \qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} = \frac{1}{1}\begin{bmatrix} a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} a/|a| \\ 0 \end{bmatrix}$$

$\vdots$ $\qquad\qquad$ $\vdots$

The quantity $a/|a|$ is equal to 1 if $a > 0$ and -1 if $a < 0$. Since the power sequence continues to oscillate between $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it does not converge.

12. **(a)** E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} \approx \begin{bmatrix} 0.28604 \\ -0.09535 \\ 0.95346 \end{bmatrix} \qquad \lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 \approx 10.86364$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.26286 \\ -0.04381 \\ 0.96384 \end{bmatrix} \qquad \lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 \approx 10.90211 \qquad \left|\frac{\lambda^{(2)}-\lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.00353 = 0.353\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.27723 \\ -0.03214 \\ 0.96027 \end{bmatrix} \qquad \lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 \approx 10.90765 \qquad \left|\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00051 = 0.051\%$$

(b) E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} \approx \begin{bmatrix} 0.12217 \\ 0.12217 \\ 0.12217 \\ 0.97736 \end{bmatrix} \qquad \lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 \approx 8.46269$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.14413 \\ 0.12971 \\ 0.17295 \\ 0.96565 \end{bmatrix} \qquad \lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 \approx 8.50187 \qquad \left|\frac{\lambda^{(2)}-\lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.05467 = 5.467\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.15083 \\ 0.12371 \\ 0.19658 \\ 0.96089 \end{bmatrix} \qquad \lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 \approx 8.51040 \qquad \left|\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00461 = 0.461\%$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.15372 \\ 0.11887 \\ 0.20847 \\ 0.95853 \end{bmatrix} \qquad \lambda^{(4)} = A\mathbf{x}_4 \cdot \mathbf{x}_4 \approx 8.51272 \qquad \left|\frac{\lambda^{(4)}-\lambda^{(3)}}{\lambda^{(4)}}\right| \approx 0.00027 = 0.027\%$$

13. **(a)** Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.229 \\ 0.668 \\ 0.668 \end{bmatrix}, \qquad \lambda^{(1)} \approx 7.632$$

$$\mathbf{x}_2 \approx \begin{bmatrix} 0.507 \\ 0.320 \\ 0.800 \end{bmatrix}, \qquad \lambda^{(2)} \approx 9.968$$

$$\mathbf{x}_3 \approx \begin{bmatrix} 0.380 \\ 0.197 \\ 0.904 \end{bmatrix}, \qquad \lambda^{(3)} \approx 10.622$$

$$\mathbf{x}_4 \approx \begin{bmatrix} 0.344 \\ 0.096 \\ 0.934 \end{bmatrix}, \qquad \lambda^{(4)} \approx 10.827$$

$$\mathbf{x}_5 \approx \begin{bmatrix} 0.317 \\ 0.044 \\ 0.948 \end{bmatrix}, \qquad \lambda^{(5)} \approx 10.886$$

$$\mathbf{x}_6 \approx \begin{bmatrix} 0.302 \\ 0.016 \\ 0.953 \end{bmatrix}, \qquad \lambda^{(6)} \approx 10.903$$

$$\mathbf{x}_7 \approx \begin{bmatrix} 0.294 \\ 0.002 \\ 0.956 \end{bmatrix}, \qquad \lambda^{(7)} \approx 10.908$$

$$\mathbf{x}_8 \approx \begin{bmatrix} 0.290 \\ -0.006 \\ 0.957 \end{bmatrix}, \qquad \lambda^{(8)} \approx 10.909$$

(b) Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.577 \\ 0 \\ 0.577 \\ 0.577 \end{bmatrix}, \qquad \lambda^{(1)} \approx 6.333$$

$$\mathbf{x}_2 \approx \begin{bmatrix} 0.249 \\ 0 \\ 0.498 \\ 0.830 \end{bmatrix}, \qquad \lambda^{(2)} \approx 8.062$$

$$\mathbf{x}_3 = \begin{bmatrix} 0.193 \\ 0.041 \\ 0.376 \\ 0.905 \end{bmatrix}, \qquad \lambda^{(3)} \approx 8.382$$

$$\mathbf{x}_4 \approx \begin{bmatrix} 0.175 \\ 0.073 \\ 0.305 \\ 0.933 \end{bmatrix}, \qquad \lambda^{(4)} \approx 8.476$$

$$\mathbf{x}_5 \approx \begin{bmatrix} 0.167 \\ 0.091 \\ 0.266 \\ 0.945 \end{bmatrix}, \qquad \lambda^{(5)} \approx 8.503$$

$$\mathbf{x}_6 \approx \begin{bmatrix} 0.162 \\ 0.101 \\ 0.245 \\ 0.951 \end{bmatrix}, \qquad \lambda^{(6)} \approx 8.511$$

$$\mathbf{x}_7 \approx \begin{bmatrix} 0.159 \\ 0.107 \\ 0.234 \\ 0.953 \end{bmatrix}, \qquad \lambda^{(7)} \approx 8.513$$

$$\mathbf{x}_8 \approx \begin{bmatrix} 0.158 \\ 0.110 \\ 0.228 \\ 0.954 \end{bmatrix}, \qquad \lambda^{(8)} \approx 8.513$$

14. **(a)** E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} \approx \begin{bmatrix} 0.3 \\ -0.1 \\ 1.0 \end{bmatrix} \qquad \lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} \approx 10.86364$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} \approx \begin{bmatrix} 0.27273 \\ -0.04545 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 10.90211 \qquad \left|\frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.00353 = 0.353\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 0.28870 \\ -0.03347 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 10.90765 \qquad \left|\frac{\lambda^{(3)} - \lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00051 = 0.051\%$$

(b) E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 0.125 \\ 0.125 \\ 0.125 \\ 1.000 \end{bmatrix} \qquad \lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} \approx 8.46269$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} \approx \begin{bmatrix} 0.14925 \\ 0.13433 \\ 0.17910 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 8.50187 \qquad \left|\frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.05467 = 5.467\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 0.15697 \\ 0.12875 \\ 0.20459 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 8.51040 \qquad \left|\frac{\lambda^{(3)} - \lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00461 = 0.461\%$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} 0.16037 \\ 0.12401 \\ 0.21749 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} \approx 8.51272 \qquad \left|\frac{\lambda^{(4)} - \lambda^{(3)}}{\lambda^{(4)}}\right| \approx 0.00027 = 0.027\%$$

9.3 Comparison of Procedures for Solving Linear Systems

1. **(a)** For $n = 1000 = 10^3$, the flops for both phases is $\frac{2}{3}(10^3)^3 + \frac{3}{2}(10^3)^2 - \frac{7}{6}(10^3) = 668{,}165{,}500$, which is 0.6681655 gigaflops, so it will take $0.6681655 \times 10^{-1} \approx 0.067$second.

(b) $n = 10{,}000 = 10^4$: $\frac{2}{3}(10^4)^3 + \frac{3}{2}(10^4)^2 - \frac{7}{6}(10^4) = 666{,}816{,}655{,}000$ flops or 666.816655 gigaflops. The time is about 66.68 seconds.

(c) $n = 100{,}000 = 10^5$; $\frac{2}{3}(10^5)^3 + \frac{3}{2}(10^5)^2 - \frac{7}{6}(10^5) \approx 666{,}682 \times 10^9$flops or 666,682 gigaflops. The time is about 66,668 seconds which is about 18.5 hours.

2. **(a)** The number of gigaflops required is $\left(\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\right)10^{-9} \approx 666.817$. At 100 gigaflops per second, the time required to solve the system is approximately 6.66817s.

(b) The number of gigaflops required is $\left(\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\right)10^{-9} \approx 666{,}681.667$. At 100 gigaflops per second, the time required to solve the system is approximately 6666.817 s (i.e., 1 hour, 51 minutes, and 6.817 seconds).

(c) The number of gigaflops required is $\left(\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\right)10^{-9} \approx 6.666682 \cdot 10^8$. At 100 gigaflops per second, the time required to solve the system is approximately 6,666,682 s (i.e., 77 days, 3 hours, 51 minutes, and 22 seconds).

3. $n = 10{,}000 = 10^4$

(a) $\frac{2}{3}n^3 \approx \frac{2}{3}(10^{12}) \approx 666.67 \times 10^9$;
666.67 gigaflops are required, which will take $\frac{666.67}{70} \approx 9.52$ seconds.

(b) $n^2 \approx 10^8 = 0.1 \times 10^9$; 0.1 gigaflop is required, which will take about 0.0014 second.

(c) This is the same as part (a); about 9.52 seconds.

(d) $2n^3 \approx 2 \times 10^{12} = 2000 \times 10^9$;
2000 gigaflops are required, which will take about 28.57 seconds.

4. **(a)** The number of petaflops required is approximately $\frac{2}{3}n^3 10^{-15} \approx 0.66667$, therefore the time required for the forward phase of Gauss-Jordan elimination is approximately 0.041667 s.

(b) The number of petaflops required is approximately $n^2 10^{-15} \approx 0.00001$, therefore the time required for the backward phase of Gauss-Jordan elimination is approximately 0.000000625 s.

(c) The number of petaflops required is approximately $\frac{2}{3}n^3 10^{-15} \approx 0.66667$, therefore the time required for the LU-decomposition is approximately 0.041667 s.

(d) The number of petaflops required is approximately $2n^3 10^{-15} \approx 2$, therefore the time required for the computation of A^{-1} by reducing $[A|I]$ to $[I|A^{-1}]$ is approximately 0.125 s.

5. **(a)** $n = 100{,}000 = 10^5$; $\frac{2}{3}n^3 \approx \frac{2}{3} \times 10^{15} \approx 0.667 \times 10^{15} = 6.67 \times 10^5 \times 10^9$;

Thus, the forward phase would require about 6.67×10^5seconds.
$n^2 = 10^{10} = 10 \times 10^9$; The backward phase would require about 10 seconds.

(b) $n = 10{,}000 = 10^4$; $\frac{2}{3}n^3 \approx \frac{2}{3} \times 10^{12} \approx 0.667 \times 10^{12} \approx 6.67 \times 10^2 \times 10^9$;
About 667 gigaflops are required, so the computer would have to execute 2(667) = 1334 gigaflops per second.

6. The number of teraflops required is approximately $2n^3 10^{-12} \approx 2000$. A computer must be able to execute more than 4000 teraflops per second to be able to find A^{-1} in less than 0.5 s.

7. Multiplying each of the n^2 entries of A by c requires n^2 flops.

8. n^2 flops are required to compute $A + B$

9. Let $C = [c_{ij}] = AB$. Computing c_{ij} requires first multiplying each of the n entries a_{ik} by the corresponding entry b_{kj}, which requires n flops. Then the n terms $a_{ik}b_{kj}$ must be summed, which requires $n - 1$ flops. Thus, each of the n^2 entries in AB requires $2n - 1$ flops, for a total of $n^2(2n-1) = 2n^3 - n^2$ flops.
Note that adding two numbers requires 1 flop, adding three numbers requires 2 flops, and in general, $n - 1$ flops are required to add n numbers.

10. Each diagonal entry can be obtained using $k-1$ multiplications, so the computation of A^k would involve $n(k-1)$ flops overall. (Note that the number of flops can be reduced to $n\lceil \log_2 k \rceil$.)

9.4 Singular Value Decomposition

1. The characteristic polynomial of $A^TA = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is $\lambda^2(\lambda - 5)$; thus the eigenvalues of A^TA are $\lambda_1 = 5$ and $\lambda_2 = 0$, and $\sigma_1 = \sqrt{5}$ and $\sigma_2 = 0$ are singular values of A.

2. $A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 9 & 0 \\ 0 & \lambda - 16 \end{vmatrix} = (\lambda - 9)(\lambda - 16)$;
the eigenvalues of A^TA are $\lambda_1 = 16$ and $\lambda_2 = 9$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = 3$.

3. The eigenvalues of $A^TA = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 5$ (i.e., $\lambda = 5$ is an eigenvalue of multiplicity 2); thus the singular value of A is $\sigma_1 = \sqrt{5}$.

4. $A^TA = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 3 & -\sqrt{2} \\ -\sqrt{2} & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 4)$;
the eigenvalues of A^TA are $\lambda_1 = 4$ and $\lambda_2 = 1$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = 1$.

5. The only eigenvalue of $A^TA = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is $\lambda = 2$ (multiplicity 2), and the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form an orthonormal basis for the eigenspace (which is all of R^2).

The singular values of A are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = \sqrt{2}$. We have $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, and

$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$.

This results in the following singular value decomposition of A:

$$A = U\Sigma V^T = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

6. $A^TA = \begin{bmatrix}-3 & 0\\0 & -4\end{bmatrix}\begin{bmatrix}-3 & 0\\0 & -4\end{bmatrix} = \begin{bmatrix}9 & 0\\0 & 16\end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix}\lambda - 9 & 0\\0 & \lambda - 16\end{vmatrix} = (\lambda - 9)(\lambda - 16)$;
the eigenvalues of A^TA are $\lambda_1 = 16$ and $\lambda_2 = 9$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = 3$.

The reduced row echelon form of $16I - A^TA$ is $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 16$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1 = 0, x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix}0\\1\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $9I - A^TA$ is $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 9$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1 = t, x_2 = 0$. A vector $\mathbf{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ orthogonally diagonalizes A^TA: $V^T(A^TA)V = \begin{bmatrix}16 & 0\\0 & 9\end{bmatrix}$.

From part (d) of Theorem 9.5.4,

$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{4}\begin{bmatrix}-3 & 0\\0 & -4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{3}\begin{bmatrix}-3 & 0\\0 & -4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix}-3 & 0\\0 & -4\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}4 & 0\\0 & 3\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}_{V^T}$$

7. The eigenvalues of $A^TA = \begin{bmatrix}4 & 0\\6 & 4\end{bmatrix}\begin{bmatrix}4 & 6\\0 & 4\end{bmatrix} = \begin{bmatrix}16 & 24\\24 & 52\end{bmatrix}$ are $\lambda_1 = 64$ and $\lambda_2 = 4$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = 8$ and $\sigma_2 = 2$.

We have $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{8}\begin{bmatrix}4 & 6\\0 & 4\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix}$, and $\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix}4 & 6\\0 & 4\end{bmatrix}\begin{bmatrix}-\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}$.

This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}8 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$$

8. $A^TA = \begin{bmatrix}3 & 3\\3 & 3\end{bmatrix}\begin{bmatrix}3 & 3\\3 & 3\end{bmatrix} = \begin{bmatrix}18 & 18\\18 & 18\end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix}\lambda-18 & -18\\-18 & \lambda-18\end{vmatrix} = (\lambda-36)\lambda$;
the eigenvalues of A^TA are $\lambda_1 = 36$ and $\lambda_2 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 6$ and $\sigma_2 = \sqrt{\lambda_2} = 0$.

The reduced row echelon form of $36I - A^TA$ is $\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 36$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $0I - A^TA$ is $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 0$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{v}_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$ orthogonally diagonalizes A^TA: $V^T(A^TA)V = \begin{bmatrix}36 & 0\\0 & 0\end{bmatrix}$.

From part (d) of Theorem 9.5.4,

$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{6}\begin{bmatrix}3 & 3\\3 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

To obtain $\mathbf{u}_2$, we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{2}\mathbf{u}_1 = \begin{bmatrix}1\\1\end{bmatrix}$. A vector $\mathbf{u}_2$ orthogonal to this vector must be a solution of $[1 \quad 1]\begin{bmatrix}x_1\\x_2\end{bmatrix} = [0]$.

An orthonormal basis for the solution space is formed by $\mathbf{u}_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix}3 & 3\\3 & 3\end{bmatrix}}_{A} = \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}6 & 0\\0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}}_{V^T}$$

9. The eigenvalues of $A^TA = \begin{bmatrix}-2 & -1 & 2\\2 & 1 & -2\end{bmatrix}\begin{bmatrix}-2 & 2\\-1 & 1\\2 & -2\end{bmatrix} = \begin{bmatrix}9 & -9\\-9 & 9\end{bmatrix}$ are $\lambda_1 = 18$ and $\lambda_2 = 0$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ respectively. The only nonzero singular value of A is $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, and we have $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix}-2 & 2\\ -1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\\ -\frac{2}{3}\end{bmatrix}$. We must choose the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthonormal basis R^3.

A possible choice is $\mathbf{u}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix}\frac{\sqrt{2}}{6}\\ -\frac{2\sqrt{2}}{3}\\ -\frac{\sqrt{2}}{6}\end{bmatrix}$. This results in the following singular value decomposition: $A = U\Sigma V^T = \begin{bmatrix}\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6}\\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3}\\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6}\end{bmatrix}\begin{bmatrix}3\sqrt{2} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$.

Note: The singular value decomposition is not unique. It depends on the choice of the (extended) orthonormal basis for R^3. This is just one possibility.

10. $A^TA = \begin{bmatrix}-2 & 2\\ -1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}-2 & -1 & 2\\ 2 & 1 & -2\end{bmatrix} = \begin{bmatrix}8 & 4 & -8\\ 4 & 2 & -4\\ -8 & -4 & 8\end{bmatrix}$;
$\det(\lambda I - A^TA) = \begin{vmatrix}\lambda-8 & -4 & 8\\ -4 & \lambda-2 & 4\\ 8 & 4 & \lambda-8\end{vmatrix} = (\lambda-18)\lambda^2$; the eigenvalues of A^TA are $\lambda_1 = 18$ and $\lambda_2 = \lambda_3 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 3\sqrt{2}$, $\sigma_2 = \sqrt{\lambda_2} = 0$, and $\sigma_3 = \sqrt{\lambda_3} = 0$.

The reduced row echelon form of $18I - A^TA$ is $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & \frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 18$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = -t, x_2 = -\frac{1}{2}t, x_3 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix}-\frac{2}{3}\\ -\frac{1}{3}\\ \frac{2}{3}\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $0I - A^TA$ is $\begin{bmatrix}1 & \frac{1}{2} & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 0$ contains vectors $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ where $x_1 = -\frac{1}{2}s + t, x_2 = s, x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix}-1\\ 2\\ 0\end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$ form a basis for this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace: $\mathbf{q}_1 = \mathbf{p}_1 = \begin{bmatrix}-1\\ 2\\ 0\end{bmatrix}$ and $\mathbf{q}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{q}_1\rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 = \begin{bmatrix}-1\\ 2\\ 0\end{bmatrix} - \frac{-1}{5}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}\frac{4}{5}\\ \frac{2}{5}\\ 1\end{bmatrix}$, then proceed to normalize the two vectors to yield an orthonormal basis: $\mathbf{v}_2 = \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \begin{bmatrix} \frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3] = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ -\frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$ orthogonally diagonalizes A^TA:

$V^T(A^TA)V = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

From part (d) of Theorem 9.5.4,

$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

To obtain $\mathbf{u}_2$, we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. A vector $\mathbf{u}_2$ orthogonal to this vector must be a solution of $\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0]$.

An orthonormal basis for the solution space is formed by $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}}_{V^T}$$

11. The eigenvalues of $A^TA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = 2$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{2}$. We have $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. We choose $\mathbf{u}_3 =$

$\begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3. This results in the following singular value decomposition: $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12. $A^TA = \begin{bmatrix} 6 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 52 & 24 \\ 24 & 16 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 52 & -24 \\ -24 & \lambda - 16 \end{vmatrix} = (\lambda - 64)(\lambda - 4)$

the eigenvalues of A^TA are $\lambda_1 = 64$ and $\lambda_2 = 4$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 8$ and $\sigma_2 = \sqrt{\lambda_2} = 2$.

The reduced row echelon form of $64I - A^TA$ is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 64$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = 2t, x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $4I - A^TA$ is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -\frac{1}{2}t, x_2 = t$. A vector $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ orthogonally diagonalizes A^TA: $V^T(A^TA)V = \begin{bmatrix} 64 & 0 \\ 0 & 4 \end{bmatrix}$.

From part (d) of Theorem 9.5.4,

$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{8}\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$.

To obtain $\mathbf{u}_3$, we extend the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{5}\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\sqrt{5}\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. A vector $\mathbf{u}_3$ orthogonal to both of these vectors must be a solution of the homogeneous linear system $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore an orthonormal basis for the solution space is formed by $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{V^T}$$

19. **(b)** In the solution of Exercise 5, we obtained a singular value decomposition

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A polar decomposition of A is

$$\begin{aligned} A &= (U\Sigma U^T)(UV^T) \\ &= \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \right) \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

True-False Exercises

(a) False. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, and $A^T A$ is an $n \times n$ matrix.

(b) True. $(A^T A)^T = A^T (A^T)^T = A^T A$.

(c) False. $A^T A$ may have eigenvalues that are 0.

(d) False. A would have to be symmetric to be orthogonally diagonalizable.

(e) True. This follows since $A^T A$ is a symmetric $n \times n$ matrix.

(f) False. The eigenvalues of $A^T A$ are the squares of the singular values of A.

(g) True. This follows from Theorem 9.4.3.

9.5 Data Compression Using Singular Value Decomposition

1. From Exercise 9 in Section 9.4, A has the singular value decomposition $A =$
$$\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus the reduced singular value decomposition of A is $A = U_1\Sigma_1V_1^T = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} [3\sqrt{2}] \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

2. $\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} [3\sqrt{2}] \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

3. From Exercise 11 in Section 9.4, A has the singular value decomposition $A = U\Sigma V^T =$
$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the reduced singular value decomposition of A is $A = U_1\Sigma_1V_1^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

5. The reduced singular value expansion of A is $3\sqrt{2}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

6. $\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T = 3\sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

7. The reduced singular value decomposition of A is $\sqrt{3}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} [1 \quad 0] + \sqrt{2}\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} [0 \quad 1]$.

8. $\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T = 8\begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} + 2\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

9. A rank 100 approximation of A requires storage space for 100(200 + 500 + 1) = 70,100 numbers, while A has 200(500) = 100,000 entries.

True-False Exercises

(a) True. This follows from the definition of a reduced singular value decomposition.

(b) True. This follows from the definition of a reduced singular value decomposition.

(c) False. V_1 has size $n \times k$ so that V_1^T has size $k \times n$.

Chapter 9 Supplementary Exercises

1. Reduce A to upper triangular form:

$$\begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$ and 2, so $L = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$.

2. $A = \begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & -\frac{1}{3} \\ 6 & 0 \end{bmatrix} \leftarrow$ multiplier $= -\frac{1}{6}$ $\qquad$ $\begin{bmatrix} -6 & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 2 \end{bmatrix} \leftarrow$ multiplier $= -6$ $\qquad$ $\begin{bmatrix} -6 & 0 \\ 6 & \bullet \end{bmatrix}$

$U = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} \leftarrow$ multiplier $= \frac{1}{2}$ $\qquad$ $\begin{bmatrix} -6 & 0 \\ 6 & 2 \end{bmatrix}$

A general 2×2 lower triangular matrix with nonzero main diagonal entries can be factored as $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ therefore $\begin{bmatrix} -6 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$. We conclude that an LDU-decomposition of A is $A = \begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} = LDU$.

3. Reduce A to upper triangular form.

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 4 & 7 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$, -1, -1, $\frac{1}{2}$, -1, and $\frac{1}{2}$ so $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

4. It was shown in the solution of Exercise 3 that $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

A general 3×3 lower triangular matrix with nonzero main diagonal entries can be factored as
$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_{21}/a_{11} & 1 & 0 \\ a_{31}/a_{11} & a_{32}/a_{22} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
therefore $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. We conclude that an LDU-decomposition of A is
$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 4 & 7 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = LDU.$

5. **(a)** The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$ so the dominant eigenvalue of A is $\lambda_1 = 3$, with corresponding positive unit eigenvector

$$\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

(b) $A\mathbf{x}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$
$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.7809 \\ 0.6247 \end{bmatrix}$$
$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.7328 \\ 0.6805 \end{bmatrix}$$
$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.7158 \\ 0.6983 \end{bmatrix}$$
$$\mathbf{x}_5 = \frac{A\mathbf{x}_4}{\|A\mathbf{x}_4\|} \approx \begin{bmatrix} 0.7100 \\ 0.7042 \end{bmatrix}$$

$\mathbf{x}_5 \approx \begin{bmatrix} 0.7100 \\ 0.7042 \end{bmatrix}$ as compared to $\mathbf{v} \approx \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$.

(c) $A\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$
$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$
$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 1 \\ 0.9286 \end{bmatrix}$$

$$\mathbf{x}_4 = \frac{A\mathrm{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} 1 \\ 0.9756 \end{bmatrix}$$
$$\mathbf{x}_5 = \frac{A\mathbf{x}_4}{\max(A\mathbf{x}_4)} \approx \begin{bmatrix} 1 \\ 0.9918 \end{bmatrix}$$

$\mathbf{x}_5 \approx \begin{bmatrix} 1 \\ 0.9918 \end{bmatrix}$ as compared to the exact eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

7. The Rayleigh quotients will converge to the dominant eigenvalue $\lambda_4 = -8.1$. However, since the ratio $\frac{|\lambda_4|}{|\lambda_1|} = \frac{8.1}{8} = 1.0125$ is very close to 1, the rate of convergence is likely to be quite slow.

8. $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $\det(\lambda I - A^T A) = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{vmatrix} = (\lambda - 4)\lambda$;
the eigenvalues of $A^T A$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = 0$.

The reduced row echelon form of $4I - A^T A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $0I - A^T A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 0$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t, x_2 = t$. A vector $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ orthogonally diagonalizes $A^T A$: $V^T(A^T A)V = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$.

From part (d) of Theorem 9.5.4, $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

To obtain $\mathbf{u}_2$, we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A vector $\mathbf{u}_2$ orthogonal to this vector must be a solution of $[1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0]$.

An orthonormal basis for the solution space is formed by $\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$

9. The eigenvalues of $A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, respectively. The only nonzero singular value A is $\sigma_1 = \sqrt{4} = 2$, and we have

$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

We must choose the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthonormal basis for R^3. A possible choice is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. This results in the following singular value decomposition: $A =$

$U\Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

10. $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{vmatrix} = (\lambda - 4)\lambda$;

the eigenvalues of A^TA are $\lambda_1 = 4$ and $\lambda_2 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = 0$.

The reduced row echelon form of $4I - A^TA$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t, x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

From part (d) of Theorem 9.5.4, $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A reduced singular value decomposition of A is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}[2]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

A reduced singular value expansion is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T = 2\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

11. A has rank 2, thus $U_1 = [\mathbf{u}_1 \quad \mathbf{u}_2]$ and $V_1^T = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix}$ and the reduced singular value decomposition of A is

$$A = U_1 \Sigma_1 V_1^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 & 0 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

14. Since A has rank 1 it can be written as $A = \mathbf{u}\mathbf{v}^T$. Thus, $A^2 = (\mathbf{u}\mathbf{v}^T)(\mathbf{u}\mathbf{v}^T) = \mathbf{u}(\mathbf{v}^T\mathbf{u})\mathbf{v}^T$.
But $\mathbf{v}^T\mathbf{u}$ is a scalar, say k. Thus, $A^2 = \mathbf{u}(\mathbf{v}^T\mathbf{u})\mathbf{v}^T = k\mathbf{u}\mathbf{v}^T = kA$.

CHAPTER 10: APPLICATIONS OF LINEAR ALGEBRA

10.1 Constructing Curves and Surfaces Through Specified Points

1. **(a)** Substituting the coordinates of the points into Equation (4) yields $\begin{vmatrix} x & y & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$ which, upon cofactor expansion along the first row, yields $-3x + y + 4 = 0$; that is, $y = 3x - 4$.

(b) As in (a), $\begin{vmatrix} x & y & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$ yields $2x + y - 1 = 0$ or $y = -2x + 1$.

2. **(a)** Equation (9) yields $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 40 & 2 & 6 & 1 \\ 4 & 2 & 0 & 1 \\ 34 & 5 & 3 & 1 \end{vmatrix} = 0$ which, upon first-row cofactor expansion, yields $18(x^2 + y^2) - 72x - 108y + 72 = 0$ or, dividing by 18, $x^2 + y^2 - 4x - 6y + 4 = 0$. Completing the squares in x and y yields the standard form $(x-2)^2 + (y-3)^2 = 9$.

(b) As in (a), $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 8 & 2 & -2 & 1 \\ 34 & 3 & 5 & 1 \\ 52 & -4 & 6 & 1 \end{vmatrix} = 0$ yields $50(x^2 + y^2) + 100x - 200y - 1000 = 0$; that is, $x^2 + y^2 + 2x - 4y - 20 = 0$. In standard form this is $(x+1)^2 + (y-2)^2 = 25$.

3. Using Equation (10) we obtain $\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & -10 & 25 & 2 & -5 & 1 \\ 16 & -4 & 1 & 4 & -1 & 1 \end{vmatrix} = 0$ which is the same as $\begin{vmatrix} x^2 & xy & y^2 & x & y \\ 0 & 0 & 1 & 0 & -1 \\ 4 & 0 & 0 & 2 & 0 \\ 4 & -10 & 25 & 2 & -5 \\ 16 & -4 & 1 & 4 & -1 \end{vmatrix} = 0$ by expansion along the second row (taking advantage of the zeros there). Add column five to column three and take advantage of another row of all but one zero to get $\begin{vmatrix} x^2 & xy & y^2+y & x \\ 4 & 0 & 0 & 2 \\ 4 & -10 & 20 & 2 \\ 16 & -4 & 0 & 4 \end{vmatrix} = 0$.

Now expand along the first row and get $160x^2 + 320xy + 160(y^2 + y) - 320x = 0$; that is, $x^2 + 2xy + y^2 - 2x + y = 0$, which is the equation of a parabola.

4. **(a)** From Equation (11), the equation of the plane is $\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & -3 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{vmatrix} = 0.$

Expansion along the first row yields $-2x - 4y - 2z = 0$; that is, $x + 2y + z = 0$.

(b) As in (a), $\begin{vmatrix} x & y & z & 1 \\ 2 & 3 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix} = 0$ yields $-2x + 2y - 4z + 2 = 0$; that is

$-x + y - 2z + 1 = 0$ for the equation of the plane.

5. **(a)** Equation (11) involves the determinant of the coefficient matrix of the system

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Rows 2 through 4 show that the plane passes through the three points $(x_i,\ y_i,\ z_i)$, $i = 1, 2, 3$, while row 1 gives the equation $c_1x + c_2y + c_3z + c_4 = 0$. For the plane passing through the origin parallel to the plane passing through the three points, the constant term in the final equation will be 0, which is accomplished by using

$$\begin{vmatrix} x & y & z & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

(b) The parallel planes passing through the origin are $x + 2y + z = 0$ and $-x + y - 2z = 0$, respectively.

6. **(a)** Using Equation (12), the equation of the sphere is $\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 14 & 1 & 2 & 3 & 1 \\ 6 & -1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 6 & 1 & 2 & -1 & 1 \end{vmatrix} = 0.$

Expanding by cofactors along the first row yields $16(x^2 + y^2 + z^2) - 32x - 64y - 32z + 32 = 0$; that is, $(x^2 + y^2 + z^2) - 2x - 4y - 2x + 2 = 0$. Completing the squares in each variable yields the standard form $(x - 1)^2 + (y - 2)^2 + (z - 1)^2 = 4$.

Note: When evaluating the cofactors, it is useful to take advantage of the column of ones and elementary row operations; for example, the cofactor of $x^2 + y^2 + z^2$ above can be evaluated as follows: $\begin{vmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 \\ -2 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -4 & 0 \end{vmatrix} = 16$ by cofactor expansion of the latter determinant along the last column.

(b) As in (a), $\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 5 & 0 & 1 & -2 & 1 \\ 11 & 1 & 3 & 1 & 1 \\ 5 & 2 & -1 & 0 & 1 \\ 11 & 3 & 1 & -1 & 1 \end{vmatrix} = 0$ yields

$-24(x^2+y^2+z^2)+48x+48y+72=0$; that is, $x^2+y^2+z^2-2x-2y-3=0$ or in standard form, $(x-1)^2+(y-1)^2+z^2=5$.

7. Substituting each of the points $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3)$, $(x_4,\ y_4)$, and $(x_5,\ y_5)$ into the equation $c_1x^2+c_2xy+c_3y^2+c_4x+c_5y+c_6=0$ yields

$$\begin{aligned} c_1x_1^2+c_2x_1y_1+c_3y_1^2+c_4x_1+c_5y_1+c_6&=0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad & \\ c_1x_5^2+c_2x_5y_5+c_3y_5^2+c_4x_5+c_5y_5+c_6&=0. \end{aligned}$$

These together with the original equation form a homogeneous linear system with a non-trivial solution for $c_1,\ c_2,\ \ldots,\ c_6$. Thus the determinant of the coefficient matrix is zero, which is exactly Equation (10).

8. As in the previous problem, substitute the coordinates $(x_i,\ y_i,\ z_i)$ of each of the three points into the equation $c_1x+c_2y+c_3z+c_4=0$ to obtain a homogeneous system with nontrivial solution for $c_1,\ \ldots,\ c_4$. Thus the determinant of the coefficient matrix is zero, which is exactly Equation (11).

9. Substituting the coordinates $(x_i,\ y_i,\ z_i)$ of the four points into the equation $c_1(x^2+y^2+z^2)+c_2x+c_3y+c_4z+c_5=0$ of the sphere yields four equations, which together with the above sphere equation form a homogeneous linear system for $c_1,\ \ldots,\ c_5$ with a nontrivial solution. Thus the determinant of this system is zero, which is Equation (12).

10. Upon substitution of the coordinates of the three points $(x_1,\ y_1)$, $(x_2,\ y_2)$ and $(x_3,\ y_3)$, we obtain the equations:

$$\begin{aligned} c_1y+c_2x^2+c_3x+c_4&=0 \\ c_1y_1+c_2x_1^2+c_3x_1+c_4&=0 \\ c_1y_2+c_2x_2^2+c_3x_2+c_4&=0 \\ c_1y_3+c_2x_3^2+c_3x_3+c_4&=0. \end{aligned}$$

This is a homogeneous system with a nontrivial solution for $c_1,\ c_2,\ c_3,\ c_4$, so the determinant of the coefficient matrix is zero; that is, $\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0.$

11. Expanding the determinant in Equation (9) by cofactors of the first row makes it apparent that the coefficient of x^2+y^2 in the final equation is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. If the points are collinear, then the columns are linearly dependent ($y_i = mx_i + b$), so the coefficient of $x^2 + y^2 = 0$ and the resulting equation is that of the line through the three points.

12. If the three distinct points are collinear then two of the coordinates can be expressed in terms of the third. Without loss of generality, we can say that y and z can be expressed in terms of x, i.e., x is the parameter. If the line is $(x, ax + b, cx + d)$, then the determinant in Equation (11) is $\begin{vmatrix} x & ax+b & cx+d & 1 \\ x_1 & ax_1+b & cx_1+d & 1 \\ x_2 & ax_2+b & cx_2+d & 1 \\ x_3 & ax_3+b & cx_3+d & 1 \end{vmatrix}$. Performing the following column operations:

- add $-a$ times the first column to the second column,
- add $-b$ times the fourth column to the second column,

yields the determinant $\begin{vmatrix} x & 0 & cx+d & 1 \\ x_1 & 0 & cx_1+d & 1 \\ x_2 & 0 & cx_2+d & 1 \\ x_3 & 0 & cx_3+d & 1 \end{vmatrix}$ equal to the original one. Expanding along the second column, it is clear that the determinant is 0 and Equation (11) becomes $0 = 0$.

13. As in Exercise 11, the coefficient of $x^2 + y^2 + z^2$ will be 0, so Equation (12) gives the equation of the plane in which the four points lie.

10.2 The Earliest Applications of Linear Algebra

1. The number of oxen is 50 per herd, and there are 7 herds, so there are 350 oxen. Hence the total number of oxen and sheep is 350 + 350 = 700.

2. **(a)** The equations are $B = 2A$, $C = 3(A + B)$, $D = 4(A + B + C)$, $300 = A + B + C + D$. Solving this linear system gives $A = 5$ (and $B = 10$, $C = 45$, $D = 240$).

(b) The equations are $B = 2A$, $C = 3B$, $D = 4C$, $132 = A + B + C + D$. Solving this linear system gives $A = 4$ (and $B = 8$, $C = 24$, $D = 96$).

3. Note that this is, effectively, Gaussian elimination applied to the augmented matrix $\begin{bmatrix} 1 & 1 & 10 \\ 1 & \frac{1}{4} & 7 \end{bmatrix}$.

4. **(a)** Let x represent oxen and y represent sheep, then the equations are $5x + 2y = 10$ and $2x + 5y = 8$. The corresponding array is

2	5
5	2
8	10

and the elimination step subtracts twice column 2 from five times column 1, giving

	5
21	2
20	10

and so $y = \frac{20}{21}$ unit for a sheep, and $x = \frac{34}{21}$ units for an ox.

(b) Let x, y, and z represent the number of bundles of each class. Then the equations are

$$\begin{aligned} 2x + y \quad\quad &= 1 \\ 3y + z &= 1 \\ x \quad\quad + 4z &= 1 \end{aligned}$$

and the corresponding array is

	2	1
3	1	
1		4
1	1	1

Subtract two times the numbers in the third column from the second column to get

		1
3	1	
1	−8	4
1	−1	1

Now subtract three times the numbers in the second column from the first column to get

		1
	1	
25	−8	4
4	−1	1

This is equivalent to the linear system

$$\begin{aligned} &x + 4z = 1 \\ &y - 8z = -1. \\ &25z = 4 \end{aligned}$$

From this, the solution is that a bundle of the first class contains $\frac{9}{25}$ measure, second class contains $\frac{7}{25}$ measure, and third class contains $\frac{4}{25}$ measure.

5. **(a)** From equations 2 through n, $x_j = a_j - x_1$ (j = 2, ..., n). Using these equations in equation 1 gives

$$x_1 + (a_2 - x_1) + (a_3 - x_1) + \cdots + (a_n - x_1) = a_1$$
$$x_1 = \frac{a_2 + a_3 + \cdots + a_n - a_1}{n - 2}$$

First find x_1 in terms of the known quantities n and the a_i. Then we can use $x_j = a_j - x_1$ (j = 2, ..., n) to find the other x_i.

(b) Exercise 7(b) may be solved using this technique. x_1 represents gold, x_2 represents brass, x_3 represents tin, and x_4 represents much-wrought iron, so n = 4 and

$$\begin{aligned}
a_1 &= 60,\\
a_2 &= \frac{2}{3}(60) = 40,\\
a_3 &= \frac{3}{4}(60) = 45,\\
a_4 &= \frac{3}{5}(60) = 36.\\
x_1 &= \frac{(a_2 + a_3 + a_4) - a_1}{n - 2} = \frac{40 + 45 + 36 - 60}{4 - 2} = \frac{61}{2}\\
x_2 &= a_2 - x_1 = 40 - \frac{61}{2} = \frac{19}{2}\\
x_3 &= a_3 - x_1 = 45 - \frac{61}{2} = \frac{29}{2}\\
x_4 &= a_4 - x_1 = 36 - \frac{61}{2} = \frac{11}{2}
\end{aligned}$$

The crown was made with $30\frac{1}{2}$ minae of gold, $9\frac{1}{2}$ minae of brass, $14\frac{1}{2}$ minae of tin, and $5\frac{1}{2}$ minae of iron.

6. **(a)** We can write this as

$$\begin{aligned}
5x + y + z - K &= 0\\
x + 7y + z - K &= 0\\
x + y + 8z - K &= 0
\end{aligned}$$

(a 3×4 system). Since the coefficient matrix of equations (5) is invertible (its determinant is 262), there is a unique solution x, y, z for every K; hence, K is an arbitrary parameter.

(b) Gaussian elimination gives $x = \frac{21K}{131}, y = \frac{14K}{131}, z = \frac{12K}{131}$, for any choice of K. Since 131 is prime, we must choose K to be an integer multiple of 131 to get integer solutions. The obvious choice is K = 131, giving x = 21, y = 14, z = 12, and K = 131.

(c) This solution corresponds to $K = 262 = 2 \cdot 131$.

7. **(a)** The system is $x + y = 1000$, $\left(\frac{1}{5}\right)x - \left(\frac{1}{4}\right)y = 10$, with solution $x = 577$ and $\frac{7}{9}$, $y = 422$ and $\frac{2}{9}$. The legitimate son receives $577\frac{7}{9}$ staters, the illegitimate son receives $422\frac{2}{9}$.

(b) The system is $G + B = \left(\frac{2}{3}\right)60$, $G + T = \left(\frac{3}{4}\right)60$, $G + I = \left(\frac{3}{5}\right)60$, $G + B + T + I = 60$, with solution $G = 30.5$, $B = 9.5$, $T = 14.5$, and $I = 5.5$. The crown was made with $30\frac{1}{2}$ minae of gold, $9\frac{1}{2}$ minae of brass, $14\frac{1}{2}$ minae of tin, and $5\frac{1}{2}$ minae of iron.

(c) The system is $A = B + \left(\frac{1}{3}C\right)$, $B = C + \left(\frac{1}{3}\right)A$, $C = \left(\frac{1}{3}\right)B + 10$, with solution $A = 45$, $B = 37.5$, and $C = 22.5$.
The first person has 45 minae, the second has $37\frac{1}{2}$, and the third has $22\frac{1}{2}$.

10.3 Cubic Spline Interpolation

2. **(a)** Set $h = .2$ and

$$\begin{aligned} x_1 &= 0, & y_1 &= .00000 \\ x_2 &= .2, & y_2 &= .19867 \\ x_3 &= .4, & y_3 &= .38942 \\ x_4 &= .6, & y_4 &= .56464 \\ x_5 &= .8, & y_5 &= .71736 \\ x_6 &= 1.0, & y_6 &= .84147 \end{aligned}$$

Then

$$\begin{aligned} \frac{6(y_1 - 2y_2 + y_3)}{h^2} &= -1.1880 \\ \frac{6(y_2 - 2y_3 + y_4)}{h^2} &= -2.3295 \\ \frac{6(y_3 - 2y_4 + y_5)}{h^2} &= -3.3750 \\ \frac{6(y_4 - 2y_5 + y_6)}{h^2} &= -4.2915 \end{aligned}$$

and the linear system (21) for the parabolic runout spline becomes

$$\begin{bmatrix} 5 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ M_5 \end{bmatrix} = \begin{bmatrix} -1.1880 \\ -2.3295 \\ -3.3750 \\ -4.2915 \end{bmatrix}.$$

Solving this system yields $M_2 = -.15676$, $M_3 = -.40421$, $M_4 = -.55592$, $M_5 = -.74712$. From (19) and (20) we have $M_1 = M_2 = -.15676$, $M_6 = M_5 = -.74712$. The specific interval $.4 \le x \le .6$ is the third interval. Using (14) to solve for a_3, b_3, c_3, and d_3 gives

$$a_3 = \frac{(M_4 - M_3)}{6h} = -.12643$$
$$b_3 = \frac{M_3}{2} = -.20211$$
$$c_3 = \frac{(y_4 - y_3)}{h} - \frac{(M_4 + 2M_3)h}{6} = .92158$$
$$d_3 = y_3 = .38942.$$

The interpolating parabolic runout spline for $.4 \le x \le .6$ is thus

$$S(x) = -.12643(x - .4)^3 - .20211(x - .4)^2 + .92158(x - .4) + .38942.$$

(b) $S(.5) = -.12643(.1)^3 - .20211(.1)^2 + .92158(.1) + .38942 = .47943$. Since $\sin(.5) = S(.5) = .47943$ to five decimal places, the percentage error is zero.

3. **(a)** Given that the points lie on a single cubic curve, the cubic runout spline will agree exactly with the single cubic curve.

(b) Set $h = 1$ and

$$\begin{array}{ll} x_1 = 0, & y_1 = 1 \\ x_2 = 1, & y_2 = 7 \\ x_3 = 2, & y_3 = 27 \\ x_4 = 3, & y_4 = 79 \\ x_5 = 4, & y_5 = 181. \end{array}$$

Then

$$\frac{6(y_1 - 2y_2 + y_3)}{h^2} = 84$$
$$\frac{6(y_2 - 2y_3 + y_4)}{h^2} = 192$$
$$\frac{6(y_3 - 2y_4 + y_5)}{h^2} = 300$$

and the linear system (24) for the cubic runout spline becomes $\begin{bmatrix} 6 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 84 \\ 192 \\ 300 \end{bmatrix}$.

Solving this system yields $M_2 = 14, M_3 = 32, M_4 = 50$.

From (22) and (23) we have $M_1 = 2M_2 - M_3 = -4,\ M_5 = 2M_4 - M_3 = 68$.

Using (14) to solve for the a_i's, b_i's, c_i's, and d_i's we have

$a_1 = \frac{(M_2 - M_1)}{6h} = 3,\ a_2 = \frac{(M_3 - M_2)}{6h} = 3,\ a_3 = \frac{(M_4 - M_3)}{6h} = 3,\ a_4 = \frac{(M_5 - M_4)}{6h} = 3,$

$b_1 = \frac{M_1}{2} = -2,\ b_2 = \frac{M_2}{2} = 7,\ b_3 = \frac{M_3}{2} = 16,\ b_4 = \frac{M_4}{2} = 25,$

$c_1 = \frac{(y_2 - y_1)}{h} - \frac{(M_2 + 2M_1)h}{6} = 5,\ c_2 = \frac{(y_3 - y_2)}{h} - \frac{(M_3 + 2M_2)h}{6} = 10,$

$c_3 = \frac{(y_4 - y_3)}{h} - \frac{(M_4 + 2M_3)h}{6} = 33,\ c_4 = \frac{(y_5 - y_4)}{h} - \frac{(M_5 + 2M_4)h}{6} = 74,$

$d_1 = y_1 = 1,\ d_2 = y_2 = 7,\ d_3 = y_3 = 27,\ d_4 = y_4 = 79.$

For $0 \le x \le 1$ we have

$$S(x) = S_1(x) = 3x^3 - 2x^2 + 5x + 1.$$

For $1 \le x \le 2$ we have

$$S(x) = S_2(x) = 3(x-1)^3 + 7(x-1)^2 + 10(x-1) + 7 = 3x^3 - 2x^2 + 5x + 1.$$

For $2 \le x \le 3$ we have

$$S(x) = S_3(x) = 3(x-2)^3 + 16(x-2)^2 + 33(x-2) + 27 = 3x^3 - 2x^2 + 5x + 1.$$

For $3 \le x \le 4$ we have

$$S(x) = S_4(x) = 3(x-3)^3 + 25(x-3)^2 + 74(x-3) + 79 = 3x^3 - 2x^2 + 5x + 1.$$

Thus $S_1(x) = S_2(x) = S_3(x) = S_4(x)$, or $S(x) = 3x^3 - 2x^2 + 5x + 1$ for $0 \le x \le 4$.

4. The linear system (16) for the natural spline becomes $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -.0001116 \\ -.0000816 \\ -.0000636 \end{bmatrix}$.

Solving this system yields $M_2 = -.0000252$, $M_3 = -.0000108$, $M_4 = -.0000132$.

From (17) and (18) we have $M_1 = 0$, $M_5 = 0$.

Solving for the a_i's, b_i's, c_i's, and d_i's from Equations (14) we have $a_1 = \frac{(M_2 - M_1)}{6h} = -.00000042$, $a_2 = \frac{(M_3 - M_2)}{6h} = .00000024$, $a_3 = \frac{(M_4 - M_3)}{6h} = -.00000004$, $a_4 = \frac{(M_5 - M_4)}{6h} = -.00000022$,

$b_1 = \frac{M_1}{2} = 0$, $b_2 = \frac{M_2}{2} = -.0000126$, $b_3 = \frac{M_3}{2} = -.0000054$, $b_4 = \frac{M_4}{2} = -.0000066$, $b_5 = \frac{M_5}{2} = 0$.

$c_1 = \frac{(y_2 - y_1)}{h} - \frac{(M_2 + 2M_1)h}{6} = .000214$, $c_2 = \frac{(y_3 - y_2)}{h} - \frac{(M_3 + 2M_2)h}{6} = .000088$,
$c_3 = \frac{(y_4 - y_3)}{h} - \frac{(M_4 + 2M_3)h}{6} = -.000092$, $c_4 = \frac{(y_5 - y_4)}{h} - \frac{(M_5 + 2M_4)h}{6} = -.000212$,
$d_1 = y_1 = .99815$, $d_2 = y_2 = .99987$, $d_3 = y_3 = .99973$, $d_4 = y_4 = .99823$.

The resulting natural spline is $S(x) =$

$$\begin{cases} -.00000042(x+10)^3 + .000214(x+10) + .99815, & -10 \le x \le 0 \\ .00000024(x)^3 - .0000126(x)^2 + .000088(x) + .99987, & 0 \le x \le 10 \\ -.00000004(x-10)^3 - .0000054(x-10)^2 - .000092(x-10) + .99973, & 10 \le x \le 20 \\ -.00000022(x-20)^3 - .0000066(x-20)^2 - .000212(x-20) + .99823, & 20 \le x \le 30. \end{cases}$$

Assuming the maximum is attained in the interval [0, 10] we set $S'(x)$ equal to zero in this interval: $S'(x) = .00000072x^2 - .0000252x + .000088 = 0$.

To three significant digits the root of this quadratic equation in the interval [0, 10] is $x = 3.93$, and $S(3.93) = 1.00004$.

5. The linear system (24) for the cubic runout spline becomes $\begin{bmatrix} 6 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -.0001116 \\ -.0000816 \\ -.0000636 \end{bmatrix}$.

Solving this system yields $M_2 = -.0000186$, $M_3 = -.0000131$, $M_4 = -.0000106$.

From (22) and (23) we have $M_1 = 2M_2 - M_3 = -.0000241$, $M_5 = 2M_4 - M_3 = -.0000081$.

Solving for the a_i's, b_i's, c_i's, and d_i's from Equations (14) we have $a_1 = \frac{(M_2 - M_1)}{6h} = .00000009$, $a_2 = \frac{(M_3 - M_2)}{6h} = .00000009$, $a_3 = \frac{(M_4 - M_3)}{6h} = .00000004$, $a_4 = \frac{(M_5 - M_4)}{6h} = .00000004$.

$b_1 = \frac{M_1}{2} = -.0000121$, $b_2 = \frac{M_2}{2} = -.0000093$, $b_3 = \frac{M_3}{2} = -.0000066$, $b_4 = \frac{M_4}{2} = -.0000053$,

$c_1 = \frac{(y_2-y_1)}{h} - \frac{(M_2+2M_1)h}{6} = .000282,\ c_2 = \frac{(y_3-y_2)}{h} - \frac{(M_3+2M_2)h}{6} = .000070,$
$c_3 = \frac{(y_4-y_3)}{h} - \frac{(M_4+2M_3)h}{6} = -.000087,\ c_4 = \frac{(y_5-y_4)}{h} - \frac{(M_5+2M_4)h}{6} = -.000207,$

$d_1 = y_1 = .99815,\ d_2 = y_2 = .99987,\ d_3 = y_3 = .99973,\ d_4 = y_4 = .99823.$
The resulting cubic runout spline is $S(x) =$
$$\begin{cases} .00000009(x+10)^3 - .0000121(x+10)^2 + .000282(x+10) + .99815, & -10 \le x \le 0 \\ .00000009(x)^3 - .0000093(x)^2 + .000070(x) + .99987, & 0 \le x \le 10 \\ .00000004(x-10)^3 - .0000066(x-10)^2 - .000087(x-10) + .99973, & 10 \le x \le 20 \\ .00000004(x-20)^3 - .0000053(x-20)^2 - .000207(x-20) + .99823, & 20 \le x \le 30. \end{cases}$$
Assuming the maximum is attained in the interval [0, 10], we set $S'(x)$ equal to zero in this interval:
$$S'(x) = .00000027x^2 - .0000186x + .000070 = 0.$$
To three significant digits the root of this quadratic equation in the interval [0, 10] is 4.00 and $S(4.00)$ = 1.00001.

6. **(a)** Set h = .5 and $x_1 = 0, y_1 = 0,\ x_2 = .5,\ y_2 = 1, x_3 = 1,\ y_3 = 0.$
For a natural spline with n = 3, $M_1 = M_3 = 0$ and $\frac{6(y_1-2y_2+y_3)}{h^2} = -48 = 4M_2$. Thus $M_2 = -12$.
$a_1 = \frac{M_2-M_1}{6h} = -4, a_2 = \frac{M_3-M_2}{6h} = 4, b_1 = \frac{M_1}{2} = 0, b_2 = \frac{M_2}{2} = -6, c_1 = \frac{y_2-y_1}{h} - \frac{(M_2+2M_1)h}{6} = 3,$
$c_2 = \frac{y_3-y_2}{h} - \frac{(M_3+2M_2)h}{6} = 0,\ d_1 = 0,\ d_2 = 1$
For $0 \le x \le .5$, we have $S(x) = S_1(x) = -4x^3 + 3x$.
For $.5 \le x \le 1$, we have
$$S(x) = S_2(x) = 4(x-.5)^3 - 6(x-.5)^2 + 1 = 4x^3 - 12x^2 + 9x - 1.$$
The resulting natural spline is $S(x) = \begin{cases} -4x^3 + 3x & 0 \le x \le 0.5 \\ 4x^3 - 12x^2 + 9x - 1 & 0.5 \le x \le 1 \end{cases}.$

(b) Again h = .5 and
$$\begin{aligned} x_1 &= .5, & y_1 &= 1 \\ x_2 &= 1, & y_2 &= 0 \\ x_3 &= 1.5, & y_3 &= -1 \end{aligned}$$
$$4M_2 = \frac{6(y_1 - 2y_2 + y_3)}{h^2} = 0$$
Thus $M_1 = M_2 = M_3 = 0$, hence all a_i and b_i are also 0.
$c_1 = \frac{y_2-y_1}{h} = -2, c_2 = \frac{y_3-y_2}{h} = -2,\ d_1 = y_1 = 1,\ d_2 = y_2 = 0$
For $.5 \le x \le 1$, we have $S(x) = S_1(x) = -2(x-.5) + 1 = -2x + 2.$
For $1 \le x \le 1.5$, we have $S(x) = S_2(x) = -2(x-1) + 0 = -2x + 2.$
The resulting natural spline is $S(x) = \begin{cases} -2x + 2 & 0.5 \le x \le 1 \\ -2x + 2 & 1 \le x \le 1.5 \end{cases}.$

(c) The three data points are collinear, so the spline is just the line the points lie on.

7. **(b)** Equations (15) together with the three equations in part (a) of the exercise statement give

$$4M_1 + M_2 + M_{n-1} = \frac{6(y_{n-1} - 2y_1 + y_2)}{h^2}$$
$$M_1 + 4M_2 + M_3 = \frac{6(y_1 - 2y_2 + y_3)}{h^2}$$
$$M_2 + 4M_3 + M_4 = \frac{6(y_2 - 2y_3 + y_4)}{h^2}$$
$$\vdots$$
$$M_{n-3} + 4M_{n-2} + M_{n-1} = \frac{6(y_{n-3} - 2y_{n-2} + y_{n-1})}{h^2}$$
$$M_1 + M_{n-2} + 4M_{n-1} = \frac{6(y_{n-2} - 2y_{n-1} + y_1)}{h^2}.$$

The linear system for $M_1,\ M_2,\ \ldots,\ M_{n-1}$ in matrix form is

$$\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_{n-1} - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_1 \end{bmatrix}.$$

8. **(b)** Equations (15) together with the two equations in part (a) give

$$2M_1 + M_2 = \frac{6(y_2 - y_1 - hy'_1)}{h^2}$$
$$M_1 + 4M_2 + M_3 = \frac{6(y_1 - 2y_2 + y_3)}{h^2}$$
$$M_2 + 4M_3 + M_4 = \frac{6(y_2 - 2y_3 + y_4)}{h^2}$$
$$\vdots$$
$$M_{n-2} + 4M_{n-1} + M_n = \frac{6(y_{n-2} - 2y_{n-1} + y_n)}{h^2}$$
$$M_{n-1} + 2M_n = \frac{6(y_{n-1} - y_n + hy'_n)}{h^2}.$$

This linear system for $M_1,\ M_2,\ \ldots,\ M_n$ in matrix form is

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} -hy'_1 - y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ \vdots \\ y_{n-2} - 2y_{n-1} + y_n \\ y_{n-1} - y_n + hy'_n \end{bmatrix}.$$

10.4 Markov Chains

1. **(a)** $\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$, $\mathbf{x}^{(2)} = P\mathbf{x}^{(1)} = \begin{bmatrix} .46 \\ .54 \end{bmatrix}$.

Continuing in this manner yields $\mathbf{x}^{(3)} = \begin{bmatrix} .454 \\ .546 \end{bmatrix}$, $\mathbf{x}^{(4)} = \begin{bmatrix} .4546 \\ .5454 \end{bmatrix}$ and $\mathbf{x}^{(5)} = \begin{bmatrix} .45454 \\ .54546 \end{bmatrix}$.

(b) *P* is regular because all of the entries of *P* are positive. Its steady-state vector **q** solves $(I - P)\mathbf{q} = \mathbf{0}$; that is, $\begin{bmatrix} .6 & -.5 \\ -.6 & .5 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This yields one independent equation, $.6q_1 - .5q_2 = 0$, or $q_1 = \frac{5}{6}q_2$. Solutions are thus of the form $\mathbf{q} = s\begin{bmatrix} \frac{5}{6} \\ 1 \end{bmatrix}$. Set $s = \frac{1}{\frac{5}{6}+1} = \frac{6}{11}$ to obtain $\mathbf{q} = \begin{bmatrix} \frac{5}{11} \\ \frac{6}{11} \end{bmatrix}$.

2. **(a)** $\mathbf{x}^{(1)} = P\mathbf{x}^{(0)} = \begin{bmatrix} .7 \\ .2 \\ .1 \end{bmatrix}$; likewise $\mathbf{x}^{(2)} = \begin{bmatrix} .23 \\ .52 \\ .25 \end{bmatrix}$ and $\mathbf{x}^{(3)} = \begin{bmatrix} .273 \\ .396 \\ .331 \end{bmatrix}$.

(b) *P* is regular because all of its entries are positive. To solve $(I - P)\mathbf{q} = \mathbf{0}$, i.e.
$\begin{bmatrix} .8 & -.1 & -.7 \\ -.6 & .6 & -.2 \\ -.2 & -.5 & .9 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, reduce the coefficient matrix to row-echelon form:

$$\begin{bmatrix} 8 & -1 & -7 \\ -6 & 6 & -2 \\ -2 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 5 & -9 \\ 0 & -21 & 29 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{22}{21} \\ 0 & 1 & -\frac{29}{21} \\ 0 & 0 & 0 \end{bmatrix}.$$

This yields solutions (setting $q_3 = s$) of the form $\begin{bmatrix} \frac{22}{21} \\ \frac{29}{21} \\ 1 \end{bmatrix}s$.

To obtain a probability vector, take $s = \frac{1}{\frac{22}{21}+\frac{29}{21}+1} = \frac{21}{72}$, yielding $\mathbf{q} = \begin{bmatrix} \frac{22}{72} \\ \frac{29}{72} \\ \frac{21}{72} \end{bmatrix}$.

3. **(a)** Solve $(I - P)\mathbf{q} = \mathbf{0}$, i.e., $\begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ -\frac{2}{3} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The only independent equation is $\frac{2}{3}q_1 = \frac{3}{4}q_2$, yielding $\mathbf{q} = \begin{bmatrix} \frac{9}{8} \\ 1 \end{bmatrix}s$.

Setting $s = \frac{8}{17}$ yields $\mathbf{q} = \begin{bmatrix} \frac{9}{17} \\ \frac{8}{17} \end{bmatrix}$.

(b) As in (a), solve $\begin{bmatrix} .19 & -.26 \\ -.19 & .26 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e., $.19q_1 = .26q_2$. Solutions have the form $\mathbf{q} = \begin{bmatrix} \frac{26}{19} \\ 1 \end{bmatrix} s$. Set $s = \frac{19}{45}$ to get $\mathbf{q} = \begin{bmatrix} \frac{26}{45} \\ \frac{19}{45} \end{bmatrix}$.

(c) Again, solve $\begin{bmatrix} \frac{2}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{4} \\ -\frac{1}{3} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by reducing the coefficient matrix to row-echelon form: $\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$ yielding solutions of the form $\mathbf{q} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{3} \\ 1 \end{bmatrix} s$.

Set $s = \frac{12}{19}$ to get $\mathbf{q} = \begin{bmatrix} \frac{3}{19} \\ \frac{4}{19} \\ \frac{12}{19} \end{bmatrix}$.

4. (a) Prove by induction that $p_{12}^{(n)} = 0$: Already true for $n = 1$. If true for $n - 1$, we have $P^n = P^{n-1}P$, so $p_{12}^{(n)} = p_{11}^{(n-1)}p_{12} + p_{12}^{(n-1)}p_{22}$. But $p_{12} = p_{12}^{(n-1)} = 0$ so $p_{12}^{(n)} = 0 + 0 = 0$. Thus, no power of P can have all positive entries, so P is not regular.

(b) If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $P\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_1 + x_2 \end{bmatrix}$, $P^2\mathbf{x} = \begin{bmatrix} \frac{1}{4}x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_1 + x_2 \end{bmatrix}$ etc. We use induction to show

$P^n\mathbf{x} = \begin{bmatrix} \left(\frac{1}{2}\right)^n x_1 \\ \left(1 - \left(\frac{1}{2}\right)^n\right) x_1 + x_2 \end{bmatrix}$.

Already true for $n = 1, 2$. If true for $n - 1$, then

$$P^n = P(P^{n-1}\mathbf{x}) = P\begin{bmatrix} \left(\frac{1}{2}\right)^{n-1} x_1 \\ \left(1 - \left(\frac{1}{2}\right)^{n-1}\right) x_1 + x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} x_1 \\ \left(1 - \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^n\right) x_1 + x_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)^n x_1 \\ \left(1 - \left(\frac{1}{2}\right)^n\right) x_1 + x_2 \end{bmatrix}.$$

Then $\lim_{n\to\infty} P^n\mathbf{x} = \begin{bmatrix} 0 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if $\mathbf{x}$ is a state vector.

(c) The Theorem says that the entries of the steady state vector should be positive; they are not for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

5. Let $\mathbf{q} = \begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \\ \vdots \\ \frac{1}{k} \end{bmatrix}$. Then $(P\mathbf{q})_i = \sum_{j=1}^{k} p_{ij}q_j = \sum_{j=1}^{k} \frac{1}{k} p_{ij} = \frac{1}{k}\sum_{j=1}^{k} p_{ij} = \frac{1}{k}$, since the row sums of P are 1. Thus $(P\mathbf{q})_i = q_i$ for all i. By Theorem 10.4.4, $\mathbf{q} = \begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \\ \vdots \\ \frac{1}{k} \end{bmatrix}$ is the steady-state vector.

6. Since P has zeros entries, consider $P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$, so P is regular. Note that all the rows of P sum to 1. Since P is 3×3, Exercise 5 implies $\mathbf{q} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

7. Let $\mathbf{x} = [x_1 \quad x_2]^{\mathrm{T}}$ be the state vector, with $x_1 =$ probability that John is happy and $x_2 =$ probability that John is sad. The transition matrix P will be $P = \begin{bmatrix} \frac{4}{5} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{3} \end{bmatrix}$ since the columns must sum to one. We find the steady state vector for P by solving $\begin{bmatrix} \frac{1}{5} & -\frac{2}{3} \\ -\frac{1}{5} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $\frac{1}{5}q_1 = \frac{2}{3}q_2$, so $\mathbf{q} = \begin{bmatrix} \frac{10}{3} \\ 1 \end{bmatrix} s$. Let $s = \frac{3}{13}$ and get $\mathbf{q} = \begin{bmatrix} \frac{10}{13} \\ \frac{3}{13} \end{bmatrix}$, so $\frac{10}{13}$ is the probability that John will be happy on a given day.

8. The state vector $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$ will represent the proportion of the population living in regions 1, 2, and 3, respectively. In the transition matrix, p_{ij} will represent the proportion of the people in region j who move to region i, yielding $P = \begin{bmatrix} .90 & .15 & .10 \\ .05 & .75 & .05 \\ .05 & .10 & .85 \end{bmatrix}$.

$$\begin{bmatrix} .10 & -.15 & -.10 \\ -.05 & .25 & -.05 \\ -.05 & -.10 & .15 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

First reduce to row echelon form $\begin{bmatrix} 1 & 0 & -\frac{13}{7} \\ 0 & 1 & -\frac{4}{7} \\ 0 & 0 & 0 \end{bmatrix}$, yielding $\mathbf{q} = \begin{bmatrix} \frac{13}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix} s$. Set $s = \frac{7}{24}$ and get $\mathbf{q} =$

$\begin{bmatrix} \frac{13}{24} \\ \frac{4}{24} \\ \frac{7}{24} \end{bmatrix}$, i.e., in the long run $\frac{13}{24}$ $\left(\text{or } 54\frac{1}{6}\%\right)$ of the people reside in region 1, $\frac{4}{24}$ $\left(\text{or } 16\frac{2}{3}\%\right)$ in region 2, and $\frac{7}{24}$ $\left(\text{or } 29\frac{1}{6}\%\right)$ in region 3.

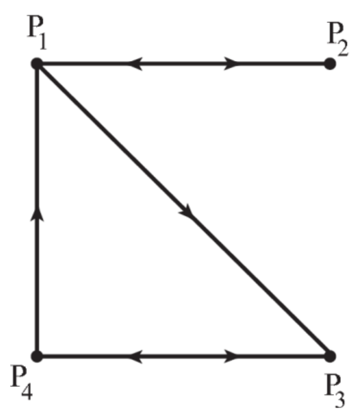
10.5 Graph Theory

1. Note that the matrix has the same number of rows and columns as the graph has vertices, and that ones in the matrix correspond to arrows in the graph.

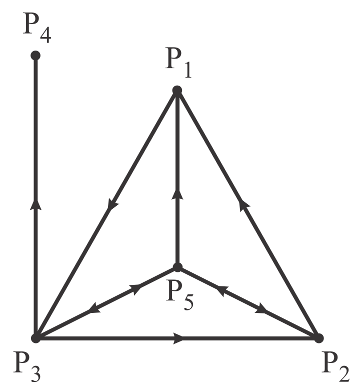
(a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

2. See the remark in problem 1; we obtain

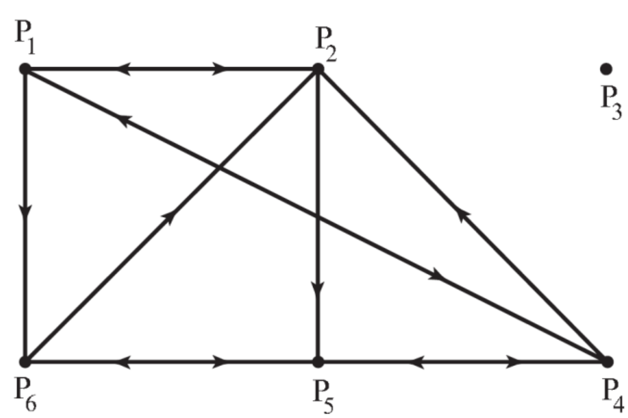
(a)



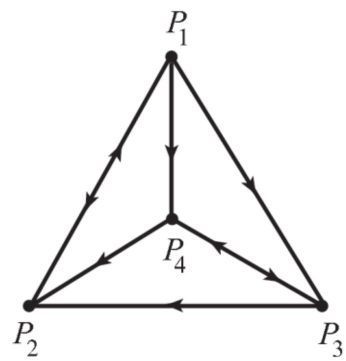
(b)



(c)



3. **(a)**



(b) $m_{12} = 1$, so there is one 1-step connection from P_1 to P_2.

$$M^2 = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } M^3 = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}.$$

So $m_{12}^{(2)} = 2$ and $m_{12}^{(3)} = 3$ meaning there are two 2-step and three 3-step connections from P_1 to P_2 by Theorem 10.6.1. These are:

1-step: $P_1 \to P_2$

2-step: $P_1 \to P_4 \to P_2$ and $P_1 \to P_3 \to P_2$

3-step: $P_1 \to P_2 \to P_1 \to P_2$, $P_1 \to P_3 \to P_4 \to P_2$, and $P_1 \to P_4 \to P_3 \to P_2$.

(c) Since $m_{14} = 1$, $m_{14}^{(2)} = 1$ and $m_{14}^{(3)} = 2$, there are one 1-step, one 2-step and two 3-step connections from P_1 to P_4. These are:

1-step: $P_1 \to P_4$

2-step: $P_1 \to P_3 \to P_4$

3-step: $P_1 \to P_2 \to P_1 \to P_4$ and $P_1 \to P_4 \to P_3 \to P_4$.

4. **(a)** $$M^T M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) The kth diagonal entry of $M^T M$ is $\sum_{i=1}^{5} {m_{ik}}^2$, i.e., the sum of the squares of the entries in column k of M. These entries are 1 if family member i influences member k and 0 otherwise.

(c) The ij entry of $M^T M$ is the number of family members who influence both member i and member j.

5. **(a)** Note that to be contained in a clique, a vertex must have "two-way" connections with at least two other vertices. Thus, P_4 could not be in a clique, so $\{P_1, P_2, P_3\}$ is the only possible clique. Inspection shows that this is indeed a clique.

(b) Not only must a clique vertex have two-way connections to at least two other vertices, but the vertices to which it is connected must share a two-way connection. This consideration eliminates P_1 and P_2, leaving $\{P_3, P_4, P_5\}$ as the only possible clique. Inspection shows that it is indeed a clique.

(c) The above considerations eliminate P_1, P_3 and P_7 from being in a clique. Inspection shows that each of the sets $\{P_2, P_4, P_6\}$, $\{P_4, P_6, P_8\}$, $\{P_2, P_6, P_8\}$, $\{P_2, P_4, P_8\}$ and $\{P_4, P_5, P_6\}$ satisfy conditions (i) and (ii) in the definition of a clique. But note that P_8 can be added to the first set and we still satisfy the conditions. P_5 may not be added, so $\{P_2, P_4, P_6, P_8\}$ is a clique, containing all the other possibilities except $\{P_4, P_5, P_6\}$, which is also a clique.

6. **(a)** With the given M we get $S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$, $S^3 = \begin{bmatrix} 0 & 3 & 1 & 3 & 1 \\ 3 & 0 & 3 & 1 & 1 \\ 1 & 3 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 3 \\ 1 & 1 & 3 & 3 & 0 \end{bmatrix}$.

Since $s_{ii}^{(3)} = 0$ for all i, there are no cliques in the graph represented by M.

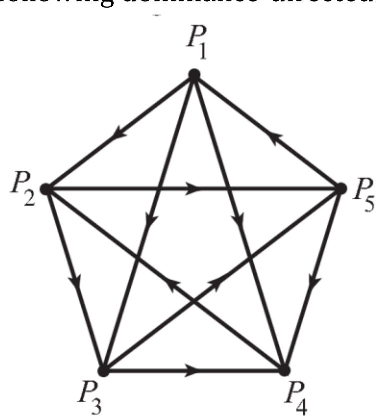
(b) Here $S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$, $S^3 = \begin{bmatrix} 0 & 6 & 1 & 7 & 0 & 2 \\ 6 & 0 & 7 & 1 & 6 & 3 \\ 1 & 7 & 2 & 8 & 1 & 4 \\ 7 & 1 & 8 & 2 & 7 & 5 \\ 0 & 6 & 1 & 7 & 0 & 2 \\ 2 & 3 & 4 & 5 & 2 & 2 \end{bmatrix}$.

The elements along the main diagonal tell us that only P_3, P_4, and P_6 are members of a clique. Since a clique contains at least three vertices, we must have $\{P_3, P_4, P_6\}$ as the only clique.

7. $M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Then $M^2 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $M + M^2 = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

By summing the rows of $M + M^2$, we get that the power of P_1 is 2 + 1 + 2 = 5, the power of P_2 is 3, of P_3 is 4, and of P_4 is 2.

8. Associating vertex P_1 with team A, P_2 with B, …, P_5 with E, the game results yield the following dominance-directed graph:



which has vertex matrix $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$.

Then $M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \end{bmatrix}$, $M + M^2 = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 & 0 \end{bmatrix}$. Summing the rows, we get that the power of A is 8, of B is 6, of C is 5, of D is 3, and of E is 6. Thus ranking in decreasing order we get A in first place, B and E tie for second place, C in fourth place, and D last.

10.6 Games of Strategy

1. (a) From Equation (2), the expected payoff of the game is

$$\mathbf{p}A\mathbf{q} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 & 6 & -4 & 1 \\ 5 & -7 & 3 & 8 \\ -8 & 0 & 6 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = -\frac{5}{8}.$$

(b) If player R uses strategy $[p_1 \quad p_2 \quad p_3]$ against player C's strategy $\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$ his payoff will be $\mathbf{p}A\mathbf{q} = \left(-\frac{1}{4}\right)p_1 + \left(\frac{9}{4}\right)p_2 - p_3$. Since p_1, p_2, and p_3 are nonnegative and add up to 1, this is a weighted average of the numbers $-\frac{1}{4}, \frac{9}{4}$, and -1. Clearly this is the largest if $p_1 = p_3 = 0$ and $p_2 = 1$; that is, $\mathbf{p} = [0 \quad 1 \quad 0]$.

(c) As in (b), if player C uses $[q_1 \quad q_2 \quad q_3 \quad q_4]^{\mathrm{T}}$ against $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$, we get $\mathbf{p}A\mathbf{q} = -6q_1 + 3q_2 + q_3 - \frac{1}{2}q_4$. Clearly this is minimized over all strategies by setting $q_1 = 1$ and $q_2 = q_3 = q_4 = 0$. That is $\mathbf{q} = [1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$.

2. As per the hint, we will construct a 3×3 matrix with two saddle points, say $a_{11} = a_{33} = 1$. Such a matrix is $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 0 \\ 1 & 2 & 1 \end{bmatrix}$. Note that $a_{13} = a_{31} = 1$ are also saddle points.

3. **(a)** Calling the matrix A, we see a_{22} is a saddle point, so the optimal strategies are pure, namely: $\mathbf{p}^* = [0 \quad 1]$, $\mathbf{q}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the value of the game is $v = a_{22} = 3$.

(b) As in (a), a_{21} is a saddle point, so optimal strategies are $\mathbf{p}^* = [0 \quad 1 \quad 0]$, $\mathbf{q}^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the value of the game is $v = a_{21} = 2$.

(c) Here, a_{32} is a saddle point, so optimal strategies are $\mathbf{p}^* = [0 \quad 0 \quad 1]$, $\mathbf{q}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $v = a_{32} = 2$.

(d) Here, a_{21} is a saddle point, so $\mathbf{p}^* = [0 \quad 1 \quad 0 \quad 0]$, $\mathbf{q}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v = a_{21} = -2$.

4. **(a)** Calling the matrix A, the formulas of Theorem 10.6.2 yield $\mathbf{p}^* = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix}$, $\mathbf{q}^* = \begin{bmatrix} \frac{1}{8} \\ \frac{7}{8} \end{bmatrix}$, $v = \frac{27}{8}$ (A has no saddle points).

(b) As in (a), $\mathbf{p}^* = \begin{bmatrix} \frac{40}{60} & \frac{20}{60} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $\mathbf{q}^* = \begin{bmatrix} \frac{10}{60} \\ \frac{50}{60} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}$, $v = \frac{1400}{60} = \frac{70}{3}$ (Again, A has no saddle points).

(c) For this matrix, a_{11} is a saddle point, so $\mathbf{p}^* = [1 \quad 0]$, $\mathbf{q}^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $v = a_{11} = 3$.

(d) This matrix has no saddle points, so, as in (a), $\mathbf{p}^* = \begin{bmatrix} \frac{-3}{-5} & \frac{-2}{-5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$, $\mathbf{q}^* = \begin{bmatrix} \frac{-3}{-5} \\ \frac{-2}{-5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$, and $v = \frac{-19}{-5} = \frac{19}{5}$.

(e) Again, A has no saddle points, so as in (a), $\mathbf{p}^* = \begin{bmatrix} \frac{3}{13} & \frac{10}{13} \end{bmatrix}$, $\mathbf{q}^* = \begin{bmatrix} \frac{1}{13} \\ \frac{12}{13} \end{bmatrix}$, and $v = \frac{-29}{13}$.

5. Let
$a_{11} =$ payoff to R if the black ace and black two are played = 3.
$a_{12} =$ payoff to R if the black ace and red three are played = –4.
$a_{21} =$ payoff to R if the red four and black two are played = –6.
$a_{22} =$ payoff to R if the red four and red three are played = 7.
So, the payoff matrix for the game is $A = \begin{bmatrix} 3 & -4 \\ -6 & 7 \end{bmatrix}$.

A has no saddle points, so from Theorem 10.7.2, $\mathbf{p}^* = \begin{bmatrix} \frac{13}{20} & \frac{7}{20} \end{bmatrix}$, $\mathbf{q}^* = \begin{bmatrix} \frac{11}{20} \\ \frac{9}{20} \end{bmatrix}$; that is, player R should play the black ace 65 percent of the time, and player C should play the black two 55 percent of the time. The value of the game is $-\frac{3}{20}$, that is, player C can expect to collect on the average 15 cents per game.

10.7 Leontief Economic Models

1. **(a)** Calling the given matrix E, we need to solve $(I-E)\mathbf{p} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This yields $\frac{1}{2}p_1 = \frac{1}{3}p_2$, that is, $\mathbf{p} = s\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$. Set $s = 2$ and get $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(b) As in (a), solve $(I-E)\mathbf{p} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{3} & 1 & -\frac{1}{2} \\ -\frac{1}{6} & -1 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

In row-echelon form, this reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solutions of this system have the form $\mathbf{p} = s\begin{bmatrix} 1 \\ \frac{5}{6} \\ 1 \end{bmatrix}$. Set $s = 6$ and get $\mathbf{p} = \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix}$.

(c) As in (a), solve $(I-E)\mathbf{p} = \begin{bmatrix} .65 & -.50 & -.30 \\ -.25 & .80 & -.30 \\ -.40 & -.30 & .60 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, which reduces to

$\begin{bmatrix} 1 & 0 & -\frac{78}{79} \\ 0 & 1 & -\frac{54}{79} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solutions are of the form $\mathbf{p} = s\begin{bmatrix} \frac{78}{79} \\ \frac{54}{79} \\ 1 \end{bmatrix}$.

Let $s = 79$ to obtain $\mathbf{p} = \begin{bmatrix} 78 \\ 54 \\ 79 \end{bmatrix}$.

2. **(a)** By Corollary 10.7.4, this matrix is productive, since each of its row sums is .9.

(b) By Corollary 10.7.5, this matrix is productive, since each of its column sums is less than one.

(c) Try $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Then $C\mathbf{x} = \begin{bmatrix} 1.9 \\ .9 \\ .9 \end{bmatrix}$, i.e., $\mathbf{x} > C\mathbf{x}$, so this matrix is productive by Theorem 10.7.3.

3. Theorem 10.8.2 says there will be one linearly independent price vector for the matrix E if some positive power of E is positive. Since E is not positive, try E^2.

$$E^2 = \begin{bmatrix} .2 & .34 & .1 \\ .2 & .54 & .6 \\ .6 & .12 & .3 \end{bmatrix} > 0$$

4. The exchange matrix for this arrangement (using A, B, and C in that order) is $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$.

For equilibrium, we must solve $(I - E)\mathbf{p} = 0$. That is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Row reduction yields solutions of the form $\mathbf{p} = \begin{bmatrix} \frac{18}{16} \\ \frac{15}{16} \\ 1 \end{bmatrix}s$. Set $s = \frac{1600}{15}$ and obtain $\mathbf{p} = \begin{bmatrix} 120 \\ 100 \\ 106.67 \end{bmatrix}$; i.e., the price of tomatoes was \$120, corn was \$100, and lettuce was \$106.67.

5. Taking the CE, EE, and ME in that order, we form the consumption matrix C, where c_{ij} = the amount (per consulting dollar) of the i-th engineer's services purchased by the j-th engineer. Thus, $C = \begin{bmatrix} 0 & .2 & .3 \\ .1 & 0 & .4 \\ .3 & .4 & 0 \end{bmatrix}$.

We want to solve $(I - C)\mathbf{x} = \mathbf{d}$, where $\mathbf{d}$ is the demand vector, i.e.

$$\begin{bmatrix} 1 & -.2 & -.3 \\ -.1 & 1 & -.4 \\ -.3 & -.4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 500 \\ 700 \\ 600 \end{bmatrix}.$$

In row-echelon form this reduces to $\begin{bmatrix} 1 & -.2 & -.3 \\ 0 & 1 & -.43877 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 500.00 \\ 765.31 \\ 1556.19 \end{bmatrix}$.

Back-substitution yields the solution $\mathbf{x} = \begin{bmatrix} 1256.48 \\ 1448.12 \\ 1556.19 \end{bmatrix}$.

The CE received \$1256, the EE received \$1448, and the ME received \$1556.

6. **(a)** The solution of the system $(I - C)\mathbf{x} = \mathbf{d}$ is $\mathbf{x} = (I - C)^{-1}\mathbf{d}$. The effect of increasing the demand d_i for the ith industry by one unit is the same as adding $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ to $\mathbf{d}$ where the 1 is in the ith row. The new solution is $(I - C)^{-1}\left(\mathbf{d} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \mathbf{x} + (I - C)^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ which has the effect of adding the ith column of $(I - C)^{-1}$ to the original solution.

(b) The increase in value is the second column of $(I-C)^{-1}$, $\frac{1}{503}\begin{bmatrix}542\\690\\170\end{bmatrix}$. Thus the value of the coal-mining operation must increase by $\frac{542}{503}$.

7. The i-th column sum of E is $\sum_{j=1}^{n} e_{ji}$, and the elements of the i-th column of $I-E$ are the negatives of the elements of E, except for the ii-th, which is $1-e_{ii}$. So, the i-th column sum of $I-E$ is $1-\sum_{j=1}^{n} e_{ji} = 1-1 = 0$. Now, $(I-E)^{\mathrm{T}}$ has zero row sums, so the vector $\mathbf{x} = [1 \quad 1 \quad \cdots \quad 1]^{\mathrm{T}}$ solves $(I-E)^{\mathrm{T}}\mathbf{x} = \mathbf{0}$. This implies $\det(I-E)^{\mathrm{T}} = 0$. But $\det(I-E)^{\mathrm{T}} = \det(I-E)$, so $(I-E)\mathbf{p} = \mathbf{0}$ must have nontrivial (i.e., nonzero) solutions.

8. Let C be a consumption matrix whose column sums are less than one; then the row sums of C^T are less than one. By Corollary 10.7.4, C^T is productive so $(\mathrm{I}-C^T)^{-1} \geq 0$. But

$$(I-C)^{-1} = (((I-C)^{\mathrm{T}}))^{-1})^T = ((I-C^T)^{-1})^T \geq 0.$$

Thus, C is productive.

10.8 Forest Management

1. Using Equation (18), we calculate

$$Yld_2 = \frac{30s}{2} = 15s$$

$$Yld_3 = \frac{50s}{2+\frac{3}{2}} = \frac{100s}{7}.$$

So all the trees in the second class should be harvested for an optimal yield (since $s = 1000$) of \$15,000.

2. From the solution to Example 1, we see that for the fifth class to be harvested in the optimal case we must have $\frac{p_5 s}{(.28^{-1}+.31^{-1}+.25^{-1}+.23^{-1})} > 14.7s$, yielding $p_5 > \$222.63$.

3. Assume $p_2 = 1$, then $Yld_2 = \frac{s}{(.28)^{-1}} = .28s$. Thus, for all the yields to be the same we must have

$$\frac{p_3 s}{(.28^{-1}+.31^{-1})} = .28s$$

$$\frac{p_4 s}{(.28^{-1}+.31^{-1}+.25^{-1})} = .28s$$

$$\frac{p_5 s}{(.28^{-1}+.31^{-1}+.25^{-1}+.23^{-1})} = .28s$$

$$\frac{p_6 s}{(.28^{-1}+.31^{-1}+.25^{-1}+.23^{-1}+.37^{-1})} = .28s$$

Solving these successively yields $p_3 = 1.90$, $p_4 = 3.02$, $p_5 = 4.24$ and $p_6 = 5.00$. Thus the ratio $p_2 : p_3 : p_4 : p_5 : p_6 = 1 : 1.90 : 3.02 : 4.24 : 5.00$.

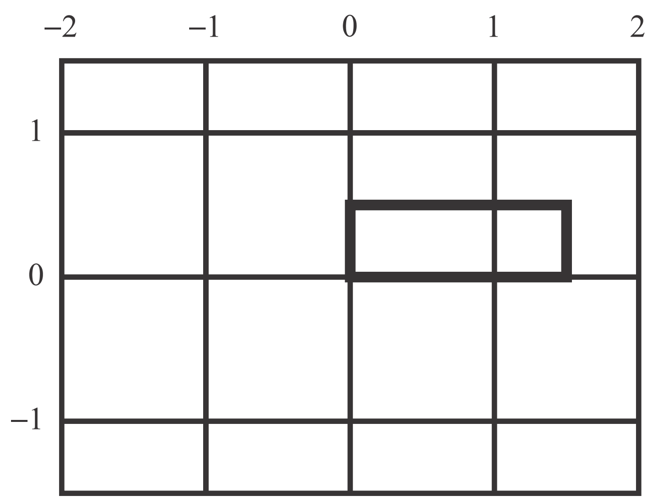
5. Since $\mathbf{y}$ is the harvest vector, $N = \sum_{i=1}^{n} y_i$ is the number of trees removed from the forest. Then Equation (7) and the first of Equations (8) yield $N = g_1 x_1$, and from Equation (17) we obtain $N = \frac{g_1 s}{1+\frac{g_1}{g_2}+\cdots+\frac{g_1}{g_{k-1}}} = \frac{s}{\frac{1}{g_1}+\cdots+\frac{1}{g_{k-1}}}$.

6. Set $g_1 = \cdots = g_{n-1} = g$, and $p_2 = 1$. Then from Equation (18), $Yld_2 = \frac{p_2 s}{\frac{1}{g_1}} = gs$. Since we want all of the Yld_k's to be the same, we need to solve $Yld_k = \frac{p_k s}{(k-1)\frac{1}{g}} = gs$ for p_k for $3 \le k \le n$. Thus $p_k = k - 1$. So the ratio $p_2 : p_3 : p_4 : \cdots : p_n = 1 : 2 : 3 : \cdots : (n-1)$.

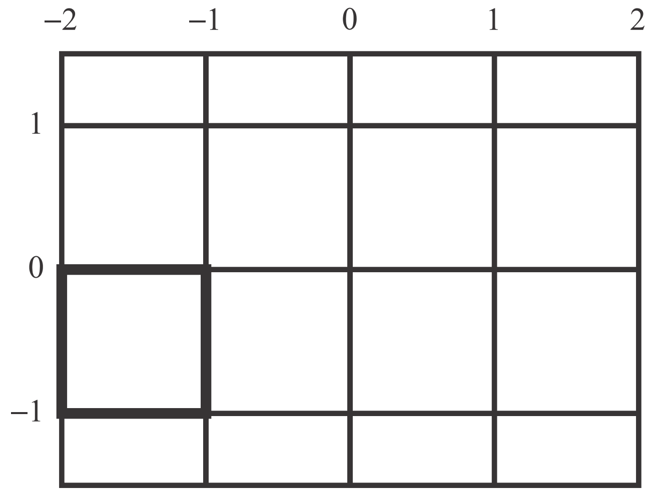
10.9 Computer Graphics

1. **(a)** Using the coordinates of the points as the columns of a matrix we obtain $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) The scaling is accomplished by multiplication of the coordinate matrix on the left by $\begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, resulting in the matrix $\begin{bmatrix} 0 & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which represents the vertices (0, 0, 0), $\left(\frac{3}{2}, 0, 0\right)$, $\left(\frac{3}{2}, \frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}, 0\right)$ as shown below.



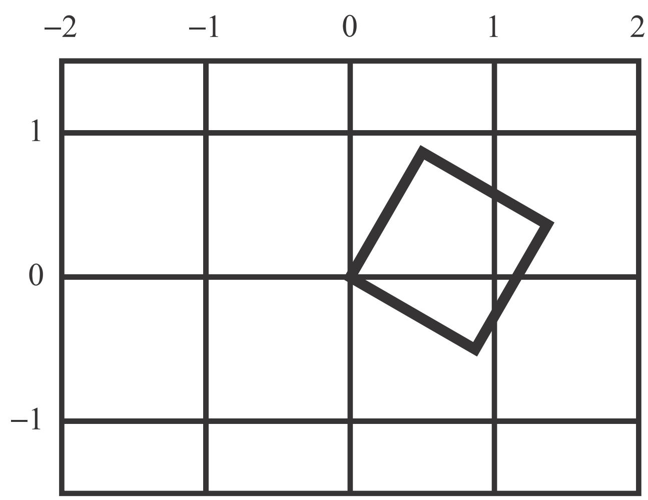
(c) Adding the matrix $\begin{bmatrix} -2 & -2 & -2 & -2 \\ -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & 3 \end{bmatrix}$ to the original matrix yields $\begin{bmatrix} -2 & -1 & -1 & -2 \\ -1 & -1 & 0 & 0 \\ 3 & 3 & 3 & 3 \end{bmatrix}$, which represents the vertices (–2, –1, 3), (–1, –1, 3), (–1, 0, 3), and (–2, 0, 3) as shown below.



(d) Multiplying by the matrix $\begin{bmatrix} \cos(-30°) & -\sin(-30°) & 0 \\ \sin(-30°) & \cos(-30°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we obtain

$$\begin{bmatrix} 0 & \cos(-30°) & \cos(-30°) - \sin(-30°) & -\sin(-30°) \\ 0 & \sin(-30°) & \cos(-30°) + \sin(-30°) & \cos(-30°) \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & .866 & 1.366 & .500 \\ 0 & -.500 & .366 & .866 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

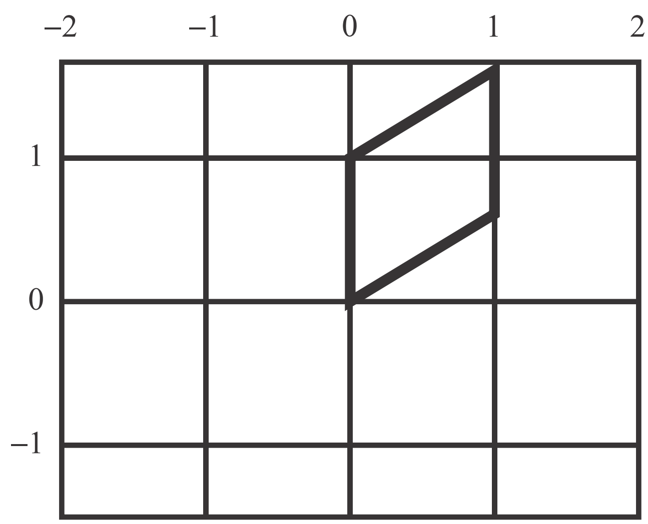
The vertices are then (0, 0, 0), (.866, –.500, 0), (1.366, .366, 0), and (.500, .866, 0) as shown:



2. **(a)** Simply perform the matrix multiplication $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} x_i + \frac{1}{2}y_i \\ y_i \\ z_i \end{bmatrix}$.

(b) We multiply $\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ yielding the vertices (0, 0, 0), (1, 0, 0), $\left(\frac{3}{2},\ 1,\ 0\right)$, and $\left(\frac{1}{2},\ 1,\ 0\right)$.

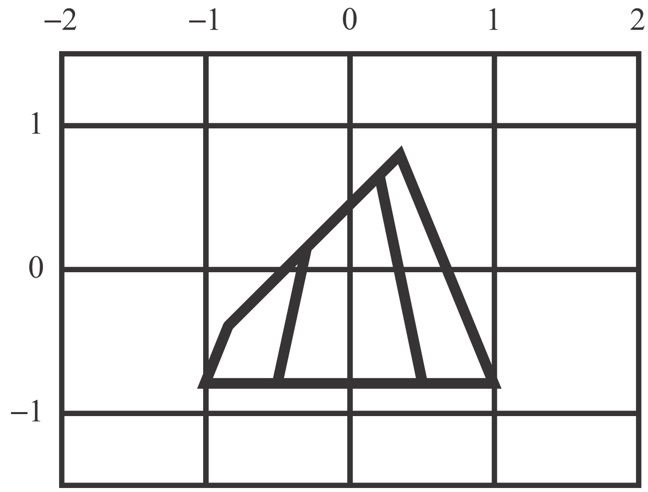
(c) Obtain the vertices via $\begin{bmatrix} 1 & 0 & 0 \\ .6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & .6 & 1.6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, yielding (0, 0, 0), (1, .6, 0), (1, 1.6, 0), and (0, 1, 0), as shown:



3. **(a)** This transformation looks like scaling by the factors 1, –1, 1, respectively and indeed its matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) For this reflection we want to transform $(x_i,\ y_i,\ z_i)$ to $(-x_i,\ y_i,\ z_i)$ with the matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Negating the *x*-coordinates of the 12 points in view 1 yields the 12 points

$(-1.000, -.800, .000)$, $(-.500, -.800, -.866)$, etc., as shown:



(c) Here we want to negate the z-coordinates, with the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

This does not change View 1.

4. **(a)** The formulas for scaling, translation, and rotation yield the matrices $M_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$,

$M_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \cos 20° & -\sin 20° \\ 0 & \sin 20° & \cos 20° \end{bmatrix}$, $M_4 =$ $\begin{bmatrix} \cos(-45°) & 0 & \sin(-45)° \\ 0 & 1 & 0 \\ -\sin(-45°) & 0 & \cos(-45°) \end{bmatrix}$, and $M_5 = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Clearly $P' = M_5 M_4 M_3 (M_1 P + M_2)$.

5. **(a)** As in 4(a), $M_1 = \begin{bmatrix} .3 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & -\sin 45° \\ 0 & \sin 45° & \cos 45° \end{bmatrix}$, $M_3 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$,

$M_4 = \begin{bmatrix} \cos 35° & 0 & \sin 35° \\ 0 & 1 & 0 \\ -\sin 35° & 0 & \cos 35° \end{bmatrix}$, $M_5 = \begin{bmatrix} \cos(-45°) & -\sin(-45°) & 0 \\ \sin(-45°) & \cos(-45°) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M_6 =$ $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$, and $M_7 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) As in 4(b), $P' = M_7(M_6 + M_5 M_4 (M_3 + M_2 M_1 P))$.

6. Using the hint given, we have $R_1 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$, $R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_3 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, $R_4 = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $R_5 = \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix}$.

7. **(a)** We rewrite the formula for v'_i as $v'_i = \begin{bmatrix} 1 \cdot x_i + x_0 \cdot 1 \\ 1 \cdot y_i + y_0 \cdot 1 \\ 1 \cdot z_i + z_0 \cdot 1 \\ 1 \cdot 1 \end{bmatrix}$. So $v'_i = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$.

(b) We want to translate x_i by –5, y_i by +9, z_i by –3, so $x_0 = -5$, $y_0 = 9$, $z_0 = -3$.

The matrix is $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

8. This can be done most easily performing the multiplication RR^T and showing that this is I. For example, for the rotation matrix about the x-axis we obtain

$$RR^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

10.10 Equilibrium Temperature Distributions

1. **(a)** The discrete mean value property yields the four equations

$$\begin{aligned} t_1 &= \frac{1}{4}(t_2 + t_3) \\ t_2 &= \frac{1}{4}(t_1 + t_4 + 1 + 1) \\ t_3 &= \frac{1}{4}(t_1 + t_4) \\ t_4 &= \frac{1}{4}(t_2 + t_3 + 1 + 1). \end{aligned}$$

Translated into matrix notation, this becomes $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$.

(b) To solve the system in part (a), we solve $(I - M)\mathbf{t} = \mathbf{b}$ for $\mathbf{t}$:

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1 & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

In row-echelon form, this is $\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -15 & 4 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ \frac{3}{4} \end{bmatrix}.$

Back substitution yields the result $\mathbf{t} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}.$

(c) $\mathbf{t}^{(1)} = M\mathbf{t}^{(0)} + \mathbf{b} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \mathbf{t}^{(2)} = M\mathbf{t}^{(1)} + \mathbf{b} = \begin{bmatrix} \frac{1}{8} \\ \frac{5}{8} \\ \frac{1}{8} \\ \frac{5}{8} \end{bmatrix},$

$\mathbf{t}^{(3)} = M\mathbf{t}^{(2)} + \mathbf{b} = \begin{bmatrix} \frac{3}{16} \\ \frac{11}{16} \\ \frac{3}{16} \\ \frac{11}{16} \end{bmatrix}, \mathbf{t}^{(4)} = M\mathbf{t}^{(3)} + \mathbf{b} = \begin{bmatrix} \frac{7}{32} \\ \frac{23}{32} \\ \frac{7}{32} \\ \frac{23}{32} \end{bmatrix}, \mathbf{t}^{(5)} = M\mathbf{t}^{(4)} + \mathbf{b} = \begin{bmatrix} \frac{15}{64} \\ \frac{47}{64} \\ \frac{15}{64} \\ \frac{47}{64} \end{bmatrix}$

$$\mathbf{t}^{(5)} - \mathbf{t} = \begin{bmatrix} \frac{15}{64} \\ \frac{47}{64} \\ \frac{15}{64} \\ \frac{47}{64} \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{64} \\ -\frac{1}{64} \\ -\frac{1}{64} \\ -\frac{1}{64} \end{bmatrix}$$

(d) Using percentage error $= \frac{\text{computed value} - \text{actual value}}{\text{actual value}} \times 100\%$ we have that the percentage error for t_1 and t_3 was $\frac{-.0371}{.2871} \times 100\% = -12.9\%$, and for t_2 and t_4 was $\frac{.0371}{.7129} \times 100\% = 5.2\%$.

2. The average value of the temperature on the circle is $\frac{1}{2\pi r}\int_{-\pi}^{\pi} f(\theta)r\,d\theta$, where r is the radius of the circle and $f(\theta)$ is the temperature at the point of the circumference where the radius to that point makes the angle θ with the horizontal. Clearly $f(\theta) = 1$ for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ and is zero otherwise. Consequently, the value of the integral above (which equals the temperature at the center of the circle) is $\frac{1}{2}$.

3. As in 1(c), but using M and $\mathbf{b}$ as in the problem statement, we obtain

$$\mathbf{t}^{(1)} = M\mathbf{t}^{(0)} + \mathbf{b} = \begin{bmatrix} \frac{3}{4} & \frac{5}{4} & \frac{1}{2} & \frac{5}{4} & 1 & \frac{1}{2} & \frac{5}{4} & 1 & \frac{3}{4} \end{bmatrix}^{\mathrm{T}}$$
$$\mathbf{t}^{(2)} = M\mathbf{t}^{(1)} + \mathbf{b} = \begin{bmatrix} \frac{13}{16} & \frac{9}{8} & \frac{9}{16} & \frac{11}{8} & \frac{13}{16} & \frac{7}{16} & \frac{21}{16} & 1 & \frac{5}{8} \end{bmatrix}^{\mathrm{T}}.$$

10.11 Computed Tomography

1. **(c)** The linear system

$$x_{31}^* = \frac{1}{20}[28 + x_{31}^* - x_{32}^*]$$
$$x_{32}^* = \frac{1}{20}[24 + 3x_{31}^* - 3x_{32}^*]$$

can be rewritten as

$$\begin{aligned} 19x_{31}^* + x_{32}^* &= 28 \\ -3x_{31}^* + 23x_{32}^* &= 24, \end{aligned}$$

which has the solution $x_{31}^* = \frac{31}{22}$, $x_{32}^* = \frac{27}{22}$.

2. **(a)** Setting $\mathbf{x}_0^{(1)} = \left(x_{01}^{(1)},\ x_{02}^{(1)}\right) = \left(x_{31}^{(0)},\ x_{32}^{(0)}\right) = (0,\ 0)$, and using part (b) of Exercise 1, we have

$$x_{31}^{(1)} = \frac{1}{20}[28] = 1.40000$$
$$x_{32}^{(1)} = \frac{1}{20}[24] = 1.20000$$
$$x_{31}^{(2)} = \frac{1}{20}[28 + 1.4 - 1.2] = 1.41000$$
$$x_{32}^{(2)} = \frac{1}{20}[24 + 3(1.4) - 3(1.2)] = 1.23000$$
$$x_{31}^{(3)} = \frac{1}{20}[28 + 1.41 - 1.23] = 1.40900$$
$$x_{32}^{(3)} = \frac{1}{20}[24 + 3(1.41) - 3(1.23)] = 1.22700$$
$$x_{31}^{(4)} = \frac{1}{20}[28 + 1.409 - 1.227] = 1.40910$$
$$x_{32}^{(4)} = \frac{1}{20}[24 + 3(1.409) - 3(1.227) = 1.22730$$

$$x_{31}^{(5)} = \frac{1}{20}[28 + 1.4091 - 1.2273] = 1.40909$$
$$x_{32}^{(5)} = \frac{1}{20}[24 + 3(1.4091) - 3(1.2273)] = 1.22727$$
$$x_{31}^{(6)} = \frac{1}{20}[28 + 1.40909 - 1.22727] = 1.40909$$
$$x_{32}^{(6)} = \frac{1}{20}[24 + 3(1.40909) - 3(1.22727)] = 1.22727.$$

(b) $\mathbf{x}_0^{(1)} = (1,\ 1) = \left(x_{31}^{(0)},\ x_{32}^{(0)}\right)$

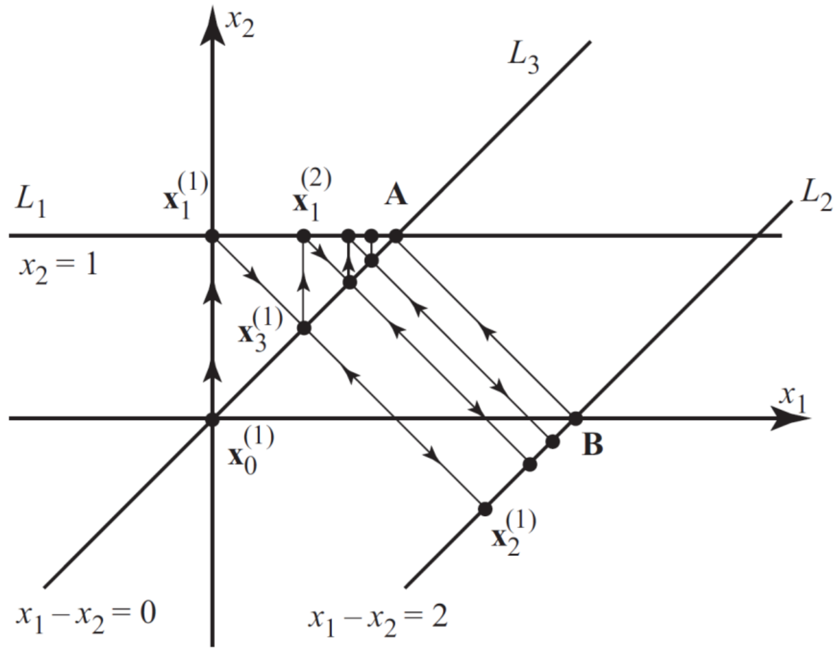
$$x_{31}^{(1)} = \frac{1}{20}[28 + 1 - 1] = 1.4$$
$$x_{32}^{(2)} = \frac{1}{20}[24 + 3(1) - 3(1)] = 1.2$$

Since $\mathbf{x}_3^{(1)}$ in this part is the same as $\mathbf{x}_3^{(1)}$ in part (a), we will get $\mathbf{x}_3^{(2)}$ as in part (a) and therefore $\mathbf{x}_3^{(3)},\ \ldots,\ \mathbf{x}_3^{(6)}$ will also be the same as in part (a).

(c) $\mathbf{x}_0^{(1)} = (148,\ -15) = \left(x_{31}^{(0)},\ x_{32}^{(0)}\right)$

$$x_{31}^{(1)} = \frac{1}{20}[28 + 148 - (-15)] = 9.55000$$
$$x_{32}^{(1)} = \frac{1}{20}[24 + 3(148) - 3(-15)] = 25.65000$$
$$x_{31}^{(2)} = \frac{1}{20}[28 + 9.55 - 25.65] = 0.59500$$
$$x_{32}^{(2)} = \frac{1}{20}[24 + 3(9.55) - 3(25.65)] = -1.21500$$
$$x_{31}^{(3)} = \frac{1}{20}[28 + 0.595 + 1.215] = 1.49050$$
$$x_{32}^{(3)} = \frac{1}{20}[24 + 3(0.595) + 3(1.215)] = 1.47150$$
$$x_{31}^{(4)} = \frac{1}{20}[28 + 1.4905 - 1.4715] = 1.40095$$
$$x_{32}^{(4)} = \frac{1}{20}[24 + 3(1.4905) - 3(1.4715)] = 1.20285$$
$$x_{31}^{(5)} = \frac{1}{20}[28 + 1.40095 - 1.20285] = 1.40991$$
$$x_{32}^{(5)} = \frac{1}{20}[24 + 3(1.40095) - 3(1.20285)] = 1.22972$$
$$x_{31}^{(6)} = \frac{1}{20}[28 + 1.40991 - 1.22972] = 1.40901$$
$$x_{32}^{(6)} = \frac{1}{20}[24 + 3(1.40991) - 3(1.22972)] = 1.22703$$

4. Referring to the figure below and starting with $\mathbf{x}_0^{(1)} = (0,\ 0)$:
$\mathbf{x}_0^{(1)}$ is projected to $\mathbf{x}_1^{(1)}$ on L_1, $\mathbf{x}_1^{(1)}$ is projected to $\mathbf{x}_2^{(1)}$ on L_2, $\mathbf{x}_2^{(1)}$ is projected to $\mathbf{x}_3^{(1)}$ on L_3, and so on.



As seen from the graph the points of the limit cycle are $\mathbf{x}_1^* = \mathbf{A}$, $\mathbf{x}_2^* = \mathbf{B}$, $\mathbf{x}_3^* = \mathbf{A}$.
Since $\mathbf{x}_1^*$ is the point of intersection of L_1 and L_3 it follows on solving the system

$$x_{12}^* = 1$$
$$x_{11}^* - x_{12}^* = 0$$

that $\mathbf{x}_1^* = (1,\ 1)$. Since $\mathbf{x}_2^* = (x_{21}^*,\ x_{22}^*)$ is on L_2, it follows that $x_{21}^* - x_{22}^* = 2$. Now $\overline{\mathbf{x}_1^*\mathbf{x}_2^*}$ is perpendicular to L_2, therefore $\left(\frac{x_{22}^*-1}{x_{21}^*-1}\right)(1) = -1$ so we have $x_{22}^* - 1 = 1 - x_{21}^*$ or $x_{21}^* + x_{22}^* = 2$.
Solving the system

$$x_{21}^* - x_{22}^* = 2$$
$$x_{21}^* + x_{22}^* = 2$$

gives $x_{21}^* = 2$ and $x_{22}^* = 0$. Thus the points on the limit cycle are $\mathbf{x}_1^* = (1,\ 1)$, $\mathbf{x}_2^* = (2,\ 0)$, $\mathbf{x}_3^* = (1,\ 1)$.

7. Let us choose units so that each pixel is one unit wide. Then a_{ij} =length of the center line of the i-th beam that lies in the j-th pixel.
If the i-th beam crosses the j-th pixel squarely, it follows that $a_{ij} = 1$. From Fig. 10.11.11 in the text, it is then clear that

$$a_{17} = a_{18} = a_{19} = 1$$
$$a_{24} = a_{25} = a_{26} = 1$$
$$a_{31} = a_{32} = a_{33} = 1$$
$$a_{73} = a_{76} = a_{79} = 1$$
$$a_{82} = a_{85} = a_{88} = 1$$
$$a_{91} = a_{94} = a_{97} = 1$$

since beams 1, 2, 3, 7, 8, and 9 cross the pixels squarely. Next, the centerlines of beams 5 and 11 lie along the diagonals of pixels 3, 5, 7 and 1, 5, 9, respectively. Since these diagonals have length $\sqrt{2}$,we have

$$a_{53} = a_{55} = a_{57} = \sqrt{2} = 1.41421$$
$$a_{11,1} = a_{11,5} = a_{11,9} = \sqrt{2} = 1.41421.$$

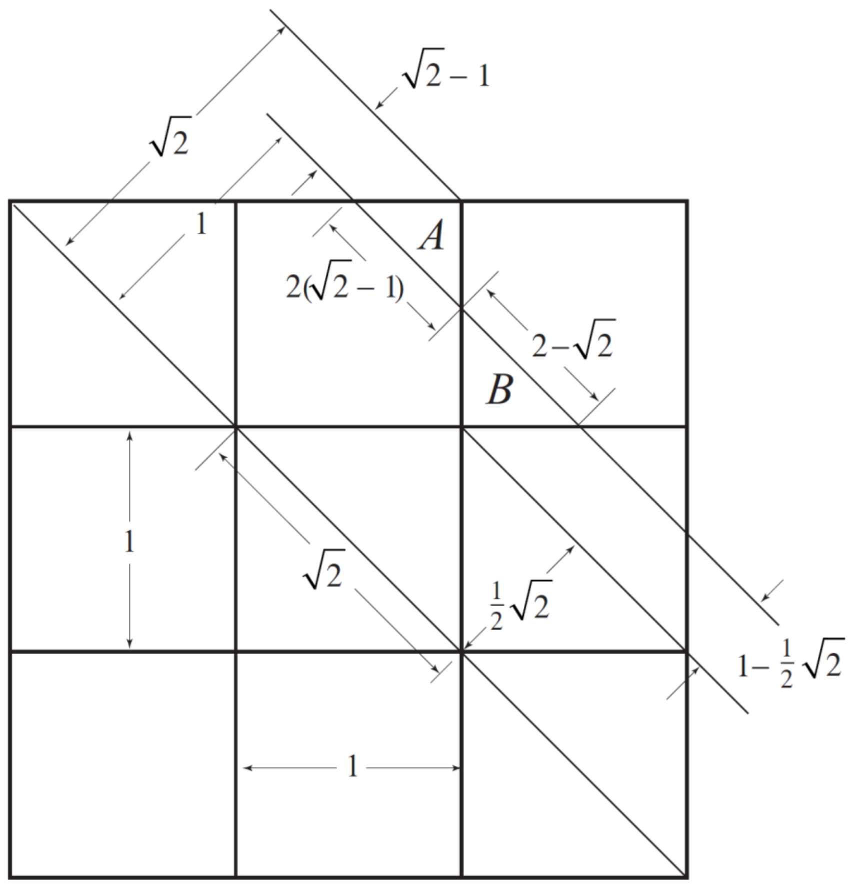
In the following diagram, the hypotenuse of triangle A is the portion of the centerline of the 10th beam that lies in the 2nd pixel. The length of this hypotenuse is twice the height of triangle A, which in turn is $\sqrt{2} - 1$. Thus,

$$a_{10,2} = 2(\sqrt{2} - 1) = .82843.$$

By symmetry we also have

$$a_{10,2} = a_{10,6} = a_{12,4} = a_{12,8} = a_{62} = a_{64} = a_{46} = a_{48} = .82843.$$

Also from the diagram, we see that the hypotenuse of triangle B is the portion of the centerline of the 10th beam that lies in the 3rd pixel. Thus, $a_{10,3} = 2 - \sqrt{2} = .58579$.



By symmetry we have $a_{10,3} = a_{12,7} = a_{61} = a_{49} = .58579$.

The remaining a_{ij}'s are all zero, and so the 12 beam equations (4) are

$$x_7 + x_8 + x_9 = 13.00$$
$$x_4 + x_5 + x_6 = 15.00$$
$$x_1 + x_2 + x_3 = 8.00$$
$$.82843(x_6 + x_8) + .58579x_9 = 14.79$$
$$1.41421(x_3 + x_5 + x_7) = 14.31$$

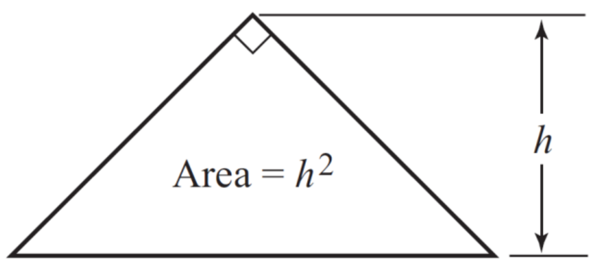
$$.82843(x_2 + x_4) + .58579x_1 = 3.81$$
$$x_3 + x_6 + x_9 = 18.00$$
$$x_2 + x_5 + x_8 = 12.00$$
$$x_1 + x_4 + x_7 = 6.00$$
$$.82843(x_2 + x_6) + .58579x_3 = 10.51$$
$$1.41421(x_1 + x_5 + x_9) = 16.13$$
$$.82843(x_4 + x_8) + .58579x_7 = 7.04$$

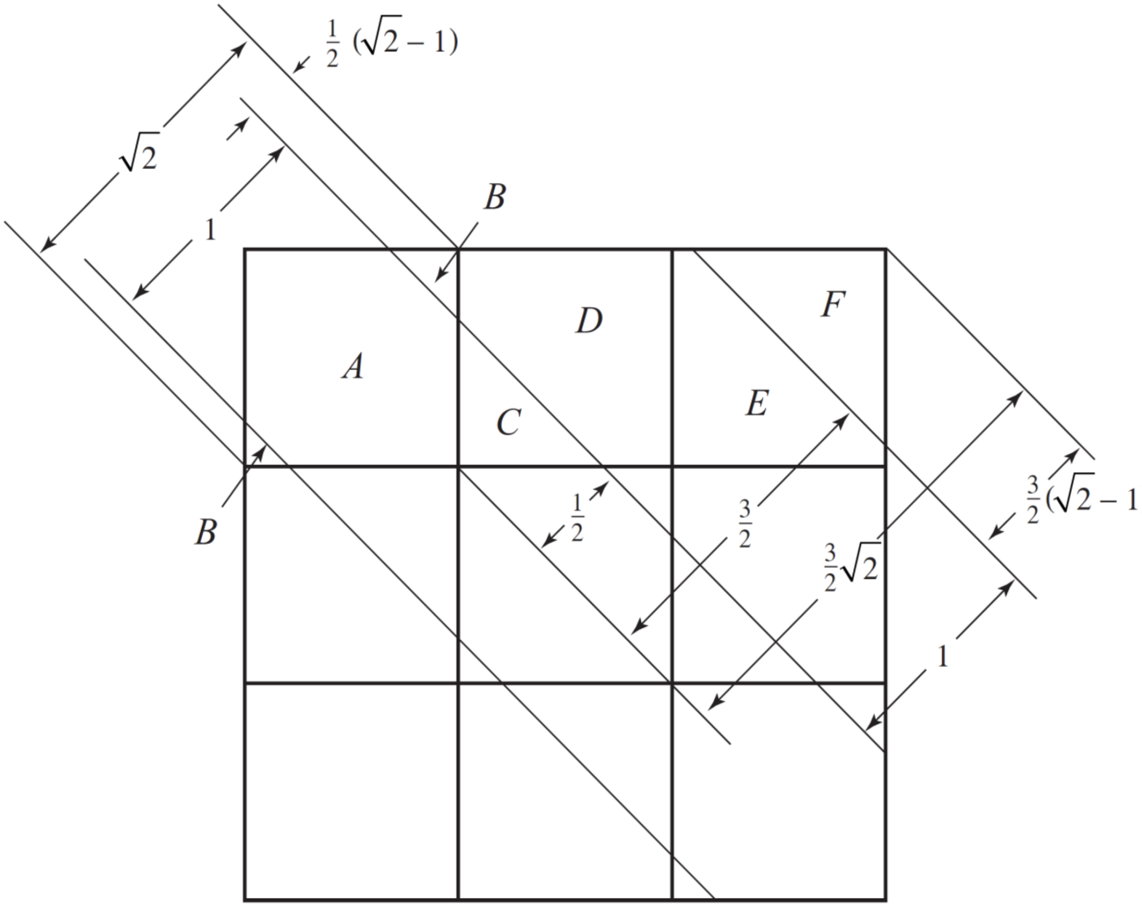
8. Let us choose units so that each pixel is one unit wide. Then a_{ij} =area of the *i*-th beam that lies in the *j*-th pixel. Since the width of each beam is also one unit it follows that $a_{ij} = 1$ if the *i*-th beam crosses the *j*-th pixel squarely. From Fig. 10.12.11 in the text, it is then clear that

$$a_{17} = a_{18} = a_{19} = 1$$
$$a_{24} = a_{25} = a_{26} = 1$$
$$a_{31} = a_{32} = a_{33} = 1$$
$$a_{73} = a_{76} = a_{79} = 1$$
$$a_{82} = a_{85} = a_{88} = 1$$
$$a_{91} = a_{94} = a_{97} = 1$$

since beams 1, 2, 3, 7, 8, and 9 cross the pixels squarely.

For the remaining a_{ij}'s, first observe from the figure that an isosceles right triangle of height h, as indicated, has area h^2. From the diagram of the nine pixels, we then have





$$\text{Area of triangle } B = \left[\frac{1}{2}\left(\sqrt{2}-1\right)\right]^2 = \frac{1}{4}\left(3-2\sqrt{2}\right) = 0.4289$$

$$\text{Area of triangle } C = \left[\frac{1}{2}\right]^2 = \frac{1}{4} = .25000$$

$$\text{Area of triangle } F = \left[\frac{3}{2}\left(\sqrt{2}-1\right)\right]^2 = \frac{9}{4}\left(3-2\sqrt{2}\right) = .38604$$

We also have

$$\text{Area of polygon } A = 1 - 2 \times (\text{Area of triangle } B) = \sqrt{2} - \frac{1}{2} = .91421$$

$$\text{Area of polygon } D = 1 - (\text{Area of triangle } C) = 1 - \frac{1}{4} = \frac{3}{4} = .7500$$

$$\text{Area of polygon } E = 1 - (\text{Area of triangle } F) = \frac{1}{4}\left(18\sqrt{2} - 23\right) = .61396$$

Referring back to Fig. 10.11.11, we see that

$$a_{11,1} = \text{Area of polygon } A = .91421$$
$$a_{10,1} = \text{Area of triangle } B = .04289$$
$$a_{11,2} = \text{Area of triangle } C = .25000$$
$$a_{10,2} = \text{Area of polygon } D = .75000$$
$$a_{10,3} = \text{Area of polygon } E = .61396$$

By symmetry we then have

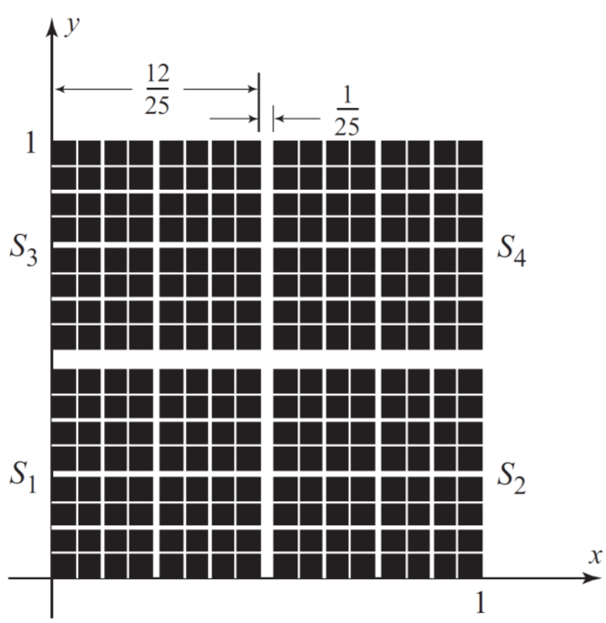
$a_{11,1} = a_{11,5} = a_{11,9} = a_{53} = a_{55} = a_{57} = .91421$
$a_{10,1} = a_{10,5} = a_{10,9} = a_{12,1} = a_{12,5} = a_{12,9} = a_{63} = a_{65} = a_{67} = a_{43} = a_{45} = a_{47} = .04289$
$a_{11,2} = a_{11,4} = a_{11,6} = a_{11,8} = a_{52} = a_{54} = a_{56} = a_{58} = .25000$
$a_{10,2} = a_{10,6} = a_{12,4} = a_{12,8} = a_{62} = a_{64} = a_{46} = a_{48} = .75000$
$a_{10,3} = a_{12,7} = a_{61} = a_{49} = .61396.$

The remaining a_{ij}'s are all zero, and so the 12 beam equations (4) are

$$\begin{aligned}
x_7 + x_8 + x_9 &= 13.00\\
x_4 + x_5 + x_6 &= 15.00\\
x_1 + x_2 + x_3 &= 8.00\\
0.04289(x_3 + x_5 + x_7) + 0.75(x_6 + x_8) + 0.61396x_9 &= 14.79\\
0.91421(x_3 + x_5 + x_7) + 0.25(x_2 + x_4 + x_6 + x_8) &= 14.31\\
0.04289(x_3 + x_5 + x_7) + 0.75(x_2 + x_4) + 0.61396x_1 &= 3.81\\
x_3 + x_6 + x_9 &= 18.00\\
x_2 + x_5 + x_8 &= 12.00\\
x_1 + x_4 + x_7 &= 6.00\\
0.04289(x_1 + x_5 + x_9) + 0.75(x_2 + x_6) + 0.61396x_3 &= 10.51\\
0.91421(x_1 + x_5 + x_9) + 0.25(x_2 + x_4 + x_6 + x_8) &= 16.13\\
0.04289(x_1 + x_5 + x_9) + 0.75(x_4 + x_8) + 0.61396x_7 &= 7.04
\end{aligned}$$

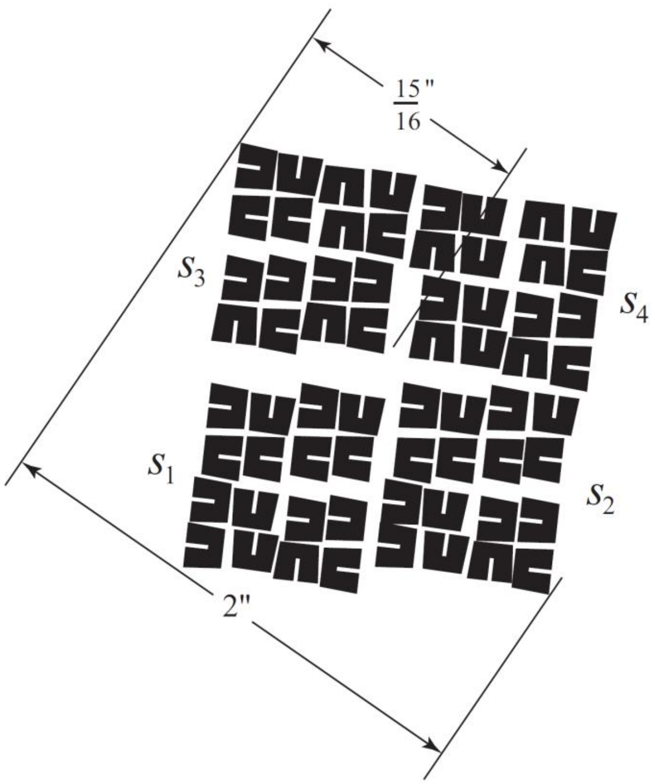
10.12 Fractals

1. Each of the subsets $S_1,\ S_2,\ S_3,\ S_4$ in the figure is congruent to the entire set scaled by a factor of $\frac{12}{25}$. Also, the rotation angles for the four subsets are all $0°$. The displacement distances can be determined from the figure to find the four similitudes that map the entire set onto the four subsets $S_1,\ S_2,\ S_3,\ S_4$. These are, respectively,



$T_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{12}{25}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}$, i = 1, 2, 3, 4, where the four values of $\begin{bmatrix} e_i \\ f_i \end{bmatrix}$ are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \frac{13}{25} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{13}{25} \end{bmatrix}$, and $\begin{bmatrix} \frac{13}{25} \\ \frac{13}{25} \end{bmatrix}$. Because $s = \frac{12}{25}$ and k = 4 in the definition of a self-similar set, the Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(4)}{\ln\left(\frac{25}{12}\right)} = 1.888\ldots$.The set is a fractal because its Hausdorff dimension is not an integer.

2. The rough measurements indicated in the figure give an approximate scale factor of $s \approx \frac{\left(\frac{15}{16}\right)}{2} =$ .47 to two decimal places. Since k = 4, the Hausdorff dimension of the set is approximately $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(4)}{\ln\left(\frac{1}{.47}\right)} = 1.8$ to two significant digits. Examination of the figure reveals rotation angles of 180°, 180°, 0°, and –90° for the sets S_1, S_2, S_3, and S_4, respectively.



3. By inspection, reading left to right and top to bottom, the triplets are:

(0, 0, 0) none are rotated
(1, 0, 0) the upper right iteration is rotated 90°
(2, 0, 0) the upper right iteration is rotated 180°
(3, 0, 0) the upper right iteration is rotated 270°
(0, 0, 1) the lower right iteration is rotated 90°
(0, 0, 2) the lower right iteration is rotated 180°
(1, 2, 0) the upper right iteration is rotated 90° and the lower left is rotated 180°

(2, 1, 3) the upper right iteration is rotated 180°, the lower left is rotated 90°, and the lower right is rotated 270°

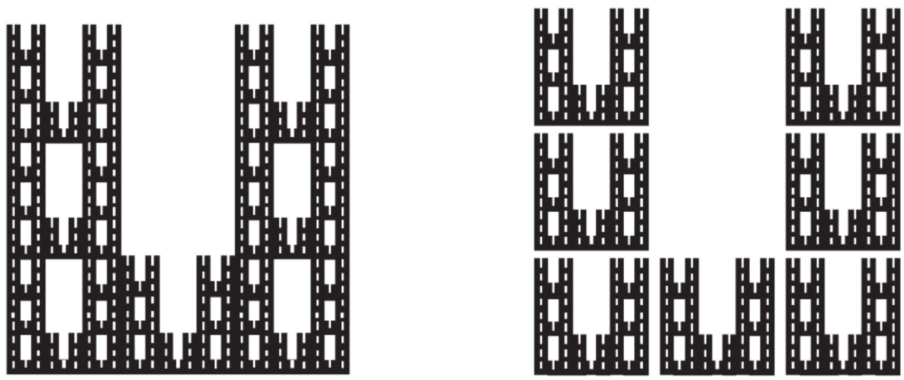
(2, 0, 1) the upper right iteration is rotated 180° and the lower right is rotated 90°

(2, 0, 2) the upper right and lower right iterations are both rotated 180°

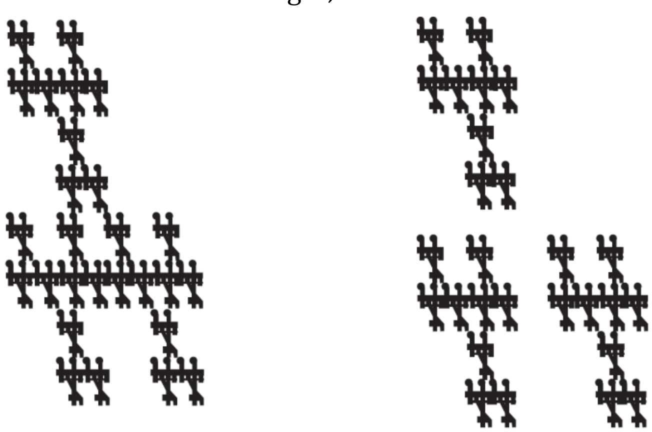
(2, 2, 0) the upper right and lower left iterations are both rotated 180°

(0, 3, 3) the lower left and lower right iterations are both rotated 270°

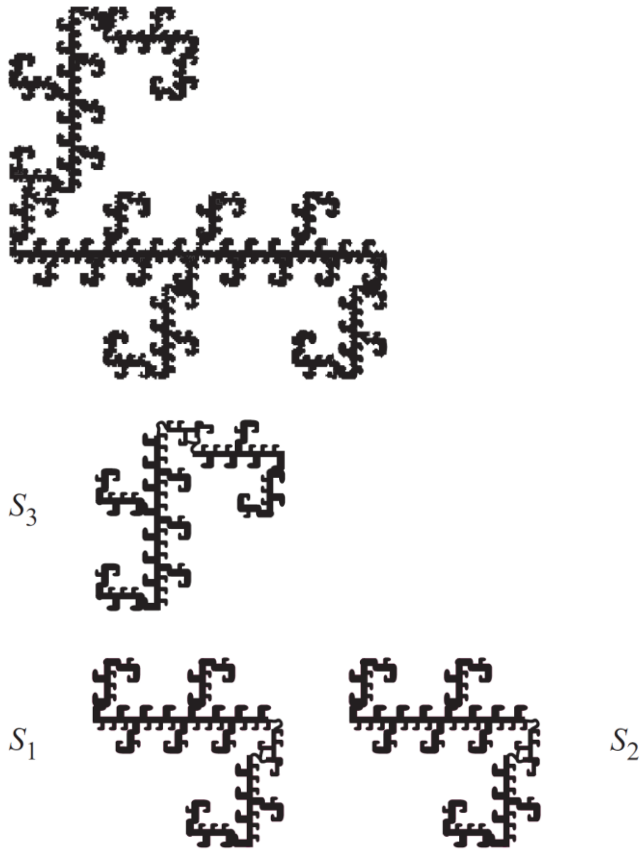
4. **(a)** The figure shows the original self-similar set and a decomposition of the set into seven nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{3}$. By inspection, the rotations angles are 0° for all seven subsets. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(7)}{\ln(3)} = 1.771\ldots$. Because its Hausdorff dimension is not an integer, the set is a fractal.



(b) The figure shows the original self-similar set and a decomposition of the set into three nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{2}$. By inspection, the rotation angles are 180° for all three subsets. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(3)}{\ln(2)} = 1.584\ldots$. Because its Hausdorff dimension is not an integer, the set is a fractal.

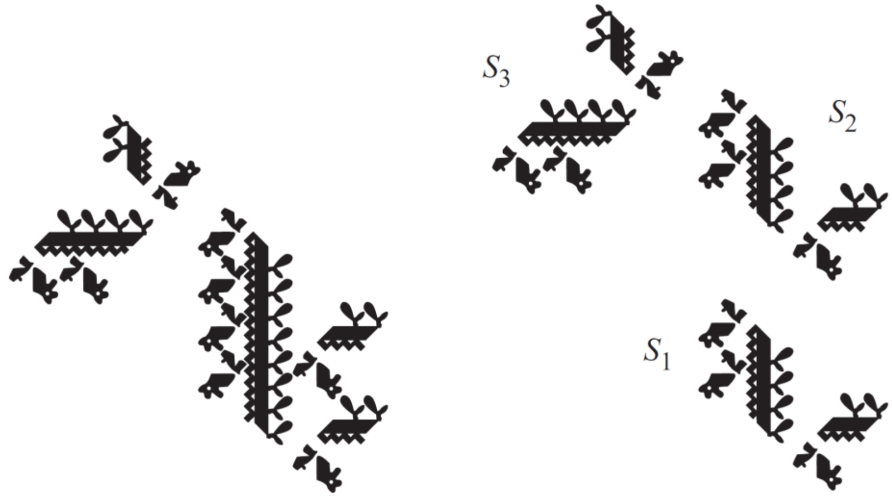


(c) The figure shows the original self-similar set and a decomposition of the set into three nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{2}$. By inspection, the rotation angles are 180°, 180°, and –90° for S_1, S_2, and S_3, respectively. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(3)}{\ln(2)} =$ 1.584. . .. Because its Hausdorff dimension is not an integer, the set is a fractal.



(d) The figure shows the original self-similar set and a decomposition of the set into three nonoverlapping congruent subsets, each of which is congruent to the original set scaled by a factor $s = \frac{1}{2}$. By inspection, the rotation angles are 180°, 180°, and –90° for S_1, S_2, and S_3, respectively. The Hausdorff dimension of the set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(3)}{\ln(2)} =$

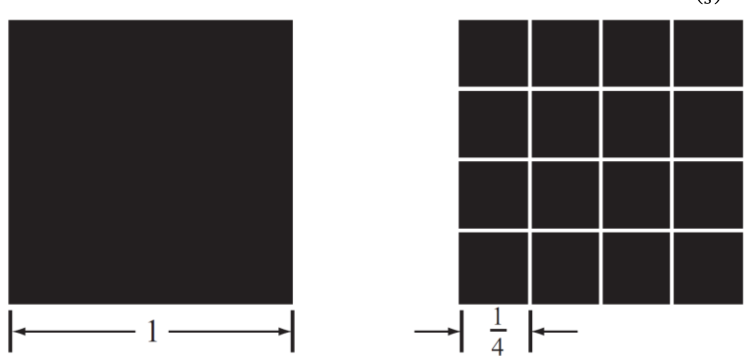
1.584. . .. Because its Hausdorff dimension is not an integer, the set is a fractal.



5. The matrix of the affine transformation in question is $\begin{bmatrix} .85 & .04 \\ -.04 & .85 \end{bmatrix}$. The matrix portion of a similitude is of the form $s\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Consequently, we must have $s\cos\theta = .85$ and $s\sin\theta = -.04$. Solving this pair of equations gives $s = \sqrt{(.85)^2 + (-.04)^2} = .8509...$ and $\theta = \tan^{-1}\left(\frac{-.04}{.85}\right) = -2.69\ldots°$.

6. Letting $\begin{bmatrix} x \\ y \end{bmatrix}$ be the vector to the tip of the fern and using the hint, we have $\begin{bmatrix} x \\ y \end{bmatrix} = T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .85 & .04 \\ -.04 & .85 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} .075 \\ .180 \end{bmatrix}$. Solving this matrix equation gives $\begin{bmatrix} x \\ y \end{bmatrix} =$ $\begin{bmatrix} .15 & -.04 \\ .04 & .15 \end{bmatrix}^{-1}\begin{bmatrix} .075 \\ .180 \end{bmatrix} = \begin{bmatrix} .766 \\ .996 \end{bmatrix}$ rounded to three decimal places.

7. As the figure indicates, the unit square can be expressed as the union of 16 nonoverlapping congruent squares, each of side length $\frac{1}{4}$. Consequently, the Hausdorff dimension of the unit square as given by Equation (2) of the text is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(16)}{\ln(4)} = 2$.

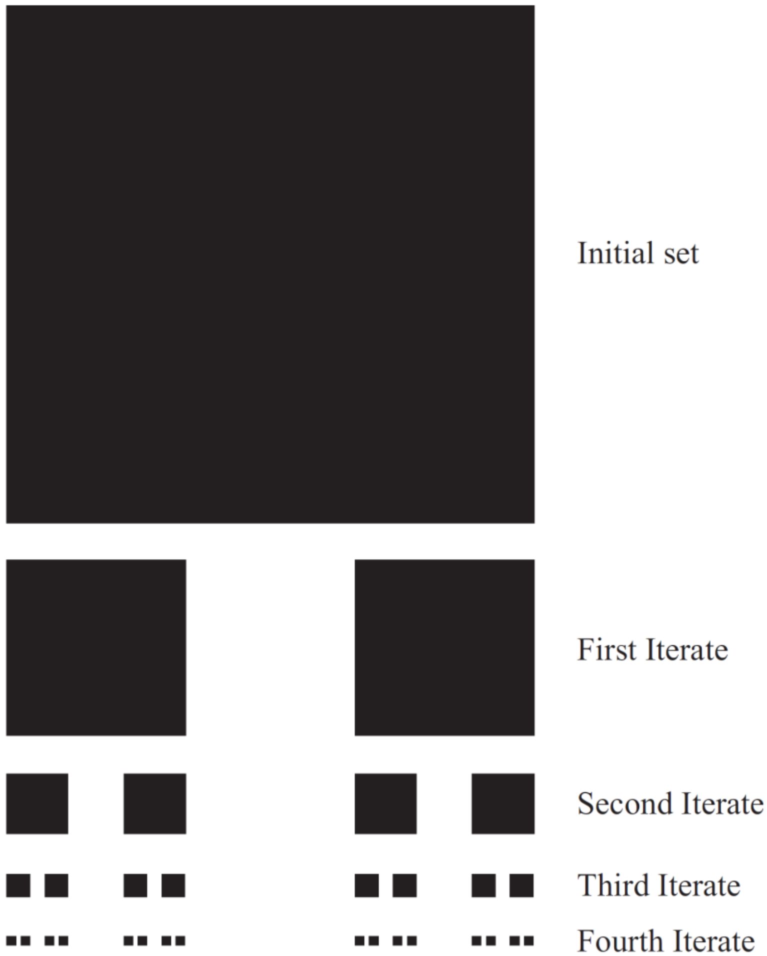


8. The similitude T_1 maps the unit square (whose vertices are (0, 0), (1, 0), (1, 1), and (0, 1)) onto the square whose vertices are (0, 0), $\left(\frac{3}{4}, 0\right)$, $\left(\frac{3}{4}, \frac{3}{4}\right)$, and $\left(0, \frac{3}{4}\right)$. The similitude T_2 maps the unit square onto the square whose vertices are $\left(\frac{1}{4},\ 0\right)$, (1, 0), $\left(1,\ \frac{3}{4}\right)$, and $\left(\frac{1}{4},\ \frac{3}{4}\right)$. The similitude T_3 maps the unit square onto the square whose vertices are $\left(0,\ \frac{1}{4}\right)$, $\left(\frac{3}{4},\ \frac{1}{4}\right)$, $\left(\frac{3}{4},\ 1\right)$, and (0, 1). Finally the similitude T_4 maps the unit square onto the square whose vertices are $\left(\frac{1}{4},\ \frac{1}{4}\right)$, $\left(1,\ \frac{1}{4}\right)$, (1, 1), and $\left(\frac{1}{4},\ 1\right)$. Each of these four smaller squares has side length of $\frac{3}{4}$, so that the common scale factor of the similitudes is $s = \frac{3}{4}$. The right-hand side of Equation (2) of the text gives $\frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(4)}{\ln\left(\frac{4}{3}\right)} = 4.818\ldots$. This is not the correct Hausdorff dimension of the square (which is 2) because the four smaller squares overlap.

9. Because $s = \frac{1}{2}$ and k = 8, Equation (2) of the text gives $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(8)}{\ln(2)} = 3$ for the Hausdorff dimension of a unit cube. Because the Hausdorff dimension of the cube is the same as its topological dimension, the cube is not a fractal.

10. A careful examination of Figure Ex-10 in the text shows that the Menger sponge can be expressed as the union of 20 smaller nonoverlapping congruent Menger sponges each of side length $\frac{1}{3}$. Consequently, k = 20 and $s = \frac{1}{3}$, and so the Hausdorff dimension of the Menger sponge is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(20)}{\ln(3)} = 2.726\ldots$. Because its Hausdorff dimension is not an integer, the Menger sponge is a fractal.

11. The figure shows the first four iterates as determined by Algorithm 1 and starting with the unit square as the initial set. Because k = 2 and $s = \frac{1}{3}$, the Hausdorff dimension of the Cantor set is $d_H(S) = \frac{\ln(k)}{\ln\left(\frac{1}{s}\right)} = \frac{\ln(2)}{\ln(3)} = 0.6309\ldots$. Notice that the Cantor set is a subset of the unit interval along the x-axis and that its topological dimension must be 0 (since the topological dimension of any set is a nonnegative integer less than or equal to its Hausdorff dimension).



12. The area of the unit square S_0 is, of course, 1. Each of the eight similitudes $T_1,\ T_2,\ \ldots,\ T_8$ given in Equation (8) of the text has scale factor $s = \frac{1}{3}$, and so each maps the unit square onto a smaller square of area $\frac{1}{9}$. Because these eight smaller squares are nonoverlapping, their total area is $\frac{8}{9}$, which is then the area of the set S_1. By a similar argument, the area of the set S_2 is $\frac{8}{9}$-th the area of the set S_1. Continuing the argument further, we find that the areas of $S_0,\ S_1,\ S_2,$ $S_3,\ S_4,\ \ldots$, form the geometric sequence $1,\ \frac{8}{9},\ \left(\frac{8}{9}\right)^2, \left(\frac{8}{9}\right)^3, \left(\frac{8}{9}\right)^4,\ \ldots$. (Notice that this implies that the area of the Sierpinski carpet is 0, since the limit of $\left(\frac{8}{9}\right)^n$ as n tends to infinity is 0.)

10.13 Chaos

1. Because $250 = 2 \cdot 5^3$ it follows from (i) that $\Pi(250) = 3 \cdot 250 = 750$.
Because $25 = 5^2$ it follows from (ii) that $\Pi(25) = 2 \cdot 25 = 50$.
Because $125 = 5^3$ it follows from (ii) that $\Pi(125) = 2 \cdot 125 = 250$.

Because $30 = 6 \cdot 5$ it follows from (ii) that $\Pi(30) = 2 \cdot 30 = 60$.
Because $10 = 2 \cdot 5$ it follows from (i) that $\Pi(10) = 3 \cdot 10 = 30$.
Because $50 = 2 \cdot 5^2$ it follows from (i) that $\Pi(50) = 3 \cdot 50 = 150$.
Because $3750 = 6 \cdot 5^4$ it follows from (ii) that $\Pi(3750) = 2 \cdot 3750 = 7500$.
Because $6 = 6 \cdot 5^0$ it follows from (ii) that $\Pi(6) = 2 \cdot 6 = 12$
Because $5 = 5^1$ it follows from (ii) that $\Pi(5) = 2 \cdot 5 = 10$.

2. The point (0, 0) is obviously a 1-cycle. We now choose another of the 36 points of the form $\left(\frac{m}{6}, \frac{n}{6}\right)$, say $\left(0, \frac{1}{6}\right)$. Its iterates produce the 12-cycle

$$\begin{bmatrix} 0 \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \end{bmatrix}.$$

So far we have accounted for 13 of the 36 points. Taking one of the remaining points, say $\left(0, \frac{2}{6}\right)$, we arrive at the 4-cycle $\begin{bmatrix} 0 \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{2}{6} \end{bmatrix}$. We continue in this way, each time starting with some point of the form $\left(\frac{m}{6}, \frac{n}{6}\right)$ that has not yet appeared in a cycle, until we exhaust all such points. This yields a 3-cycle: $\begin{bmatrix} \frac{3}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{3}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} 0 \\ \frac{3}{6} \end{bmatrix}$;

another 4-cycle: $\begin{bmatrix} \frac{4}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{2}{6} \end{bmatrix}$; and another 12-cycle:

$$\begin{bmatrix} \frac{1}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{2}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{1}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} \frac{4}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{1}{6} \\ \frac{4}{6} \end{bmatrix} \to \begin{bmatrix} \frac{5}{6} \\ \frac{3}{6} \end{bmatrix} \to \begin{bmatrix} \frac{2}{6} \\ \frac{5}{6} \end{bmatrix}.$$

The possible periods of points for the form $\left(\frac{m}{6}, \frac{n}{6}\right)$ are thus 1, 3, 4 and 12. The least common multiple of these four numbers is 12, and so $\Pi(6) = 12$.

3. **(a)** We are given that $x_0 = 3$ and $x_1 = 7$. With p = 15 we have

$$
\begin{aligned}
x_2 &= x_1 + x_0 \mod 15 = 7 + 3 \mod 15 = 10 \mod 15 = 10,\\
x_3 &= x_2 + x_1 \mod 15 = 10 + 7 \mod 15 = 17 \mod 15 = 2,\\
x_4 &= x_3 + x_2 \mod 15 = 2 + 10 \mod 15 = 12 \mod 15 = 12\\
x_5 &= x_4 + x_3 \mod 15 = 12 + 2 \mod 15 = 14 \mod 15 = 14,\\
x_6 &= x_5 + x_4 \mod 15 = 14 + 12 \mod 15 = 26 \mod 15 = 11,\\
x_7 &= x_6 + x_5 \mod 15 = 11 + 14 \mod 15 = 25 \mod 15 = 10\\
x_8 &= x_7 + x_6 \mod 15 = 10 + 11 \mod 15 = 21 \mod 15 = 6,\\
x_9 &= x_8 + x_7 \mod 15 = 6 + 10 \mod 15 = 16 \mod 15 = 1,\\
x_{10} &= x_9 + x_8 \mod 15 = 1 + 6 \mod 15 = 7 \mod 15 = 7,\\
x_{11} &= x_{10} + x_9 \mod 15 = 7 + 1 \mod 15 = 8 \mod 15 = 8\\
x_{12} &= x_{11} + x_{10} \mod 15 = 8 + 7 \mod 15 = 15 \mod 15 = 0,\\
x_{13} &= x_{12} + x_{11} \mod 15 = 0 + 8 \mod 15 = 8 \mod 15 = 8,\\
x_{14} &= x_{13} + x_{12} \mod 15 = 8 + 0 \mod 15 = 8 \mod 15 = 8,\\
x_{15} &= x_{14} + x_{13} \mod 15 = 8 + 8 \mod 15 = 16 \mod 15 = 1,\\
x_{16} &= x_{15} + x_{14} \mod 15 = 1 + 8 \mod 15 = 9 \mod 15 = 9,\\
x_{17} &= x_{16} + x_{15} \mod 15 = 9 + 1 \mod 15 = 10 \mod 15 = 10,\\
x_{18} &= x_{17} + x_{16} \mod 15 = 10 + 9 \mod 15 = 19 \mod 15 = 4,\\
x_{19} &= x_{18} + x_{17} \mod 15 = 4 + 10 \mod 15 = 14 \mod 15 = 14,\\
x_{20} &= x_{19} + x_{18} \mod 15 = 14 + 4 \mod 15 = 18 \mod 15 = 3,\\
x_{21} &= x_{20} + x_{19} \mod 15 = 3 + 14 \mod 15 = 17 \mod 15 = 2,\\
x_{22} &= x_{21} + x_{20} \mod 15 = 2 + 3 \mod 15 = 5 \mod 15 = 5,\\
x_{23} &= x_{22} + x_{21} \mod 15 = 5 + 2 \mod 15 = 7 \mod 15 = 7,\\
x_{24} &= x_{23} + x_{22} \mod 15 = 7 + 5 \mod 15 = 12 \mod 15 = 12,\\
x_{25} &= x_{24} + x_{23} \mod 15 = 12 + 7 \mod 15 = 19 \mod 15 = 4,\\
x_{26} &= x_{25} + x_{24} \mod 15 = 4 + 12 \mod 15 = 16 \mod 15 = 1,\\
x_{27} &= x_{26} + x_{25} \mod 15 = 1 + 4 \mod 15 = 5 \mod 15 = 5,\\
x_{28} &= x_{27} + x_{26} \mod 15 = 5 + 1 \mod 15 = 6 \mod 15 = 6,\\
x_{29} &= x_{28} + x_{27} \mod 15 = 6 + 5 \mod 15 = 11 \mod 15 = 11,\\
x_{30} &= x_{29} + x_{28} \mod 15 = 11 + 6 \mod 15 = 17 \mod 15 = 2,\\
x_{31} &= x_{30} + x_{29} \mod 15 = 2 + 11 \mod 15 = 13 \mod 15 = 13,\\
x_{32} &= x_{31} + x_{30} \mod 15 = 13 + 2 \mod 15 = 15 \mod 15 = 0,\\
x_{33} &= x_{32} + x_{31} \mod 15 = 0 + 13 \mod 15 = 13 \mod 15 = 13,\\
x_{34} &= x_{33} + x_{32} \mod 15 = 13 + 0 \mod 15 = 13 \mod 15 = 13,\\
x_{35} &= x_{34} + x_{33} \mod 15 = 13 + 13 \mod 15 = 26 \mod 15 = 11,\\
x_{36} &= x_{35} + x_{34} \mod 15 = 11 + 13 \mod 15 = 24 \mod 15 = 9,\\
x_{37} &= x_{36} + x_{35} \mod 15 = 9 + 11 \mod 15 = 20 \mod 15 = 5,\\
x_{38} &= x_{37} + x_{36} \mod 15 = 5 + 9 \mod 15 = 14 \mod 15 = 14,\\
x_{39} &= x_{38} + x_{37} \mod 15 = 14 + 5 \mod 15 = 19 \mod 15 = 4,\\
x_{40} &= x_{39} + x_{38} \mod 15 = 4 + 14 \mod 15 = 18 \mod 15 = 3,\\
x_{41} &= x_{40} + x_{39} \mod 15 = 3 + 4 \mod 15 = 7 \mod 15 = 7,
\end{aligned}
$$

and finally: $x_{40} = x_0$ and $x_{41} = x_1$. Thus this sequence is periodic with period 40.

(b) Step (ii) of the algorithm is $x_{n+1} = x_n + x_{n-1} \bmod p$.
Replacing n in this formula by n + 1 gives

$$x_{n+2} = x_{n+1} + x_n \bmod p = (x_n + x_{n-1}) + x_n \bmod p = 2x_n + x_{n-1} \bmod p.$$

These equations can be written as

$$x_{n+1} = x_{n-1} + x_n \bmod p$$
$$x_{n+2} = x_{n-1} + 2x_n \bmod p$$

which in matrix form are $\begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \bmod p$.

(c) Beginning with $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, we obtain

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 15 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$
$$\begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \end{bmatrix} \bmod 21 = \begin{bmatrix} 25 \\ 40 \end{bmatrix} \bmod 21 = \begin{bmatrix} 4 \\ 19 \end{bmatrix}$$
$$\begin{bmatrix} x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 19 \end{bmatrix} \bmod 21 = \begin{bmatrix} 23 \\ 42 \end{bmatrix} \bmod 21 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \bmod 21 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bmod 21 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x_{10} \\ x_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \bmod 21 = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \bmod 21 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$
$$\begin{bmatrix} x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 16 \end{bmatrix} \bmod 21 = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$
$$\begin{bmatrix} x_{14} \\ x_{15} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 16 \end{bmatrix} \bmod 21 = \begin{bmatrix} 26 \\ 42 \end{bmatrix} \bmod 21 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x_{16} \\ x_{17} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \bmod 21 = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \bmod 21 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

and we see that $\begin{bmatrix} x_{16} \\ x_{17} \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$.

4. (c) We have that

$$C\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{1}{101} \\ \frac{1}{101} \end{bmatrix},$$
$$C^2\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{101} \\ \frac{1}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{2}{101} \\ \frac{3}{101} \end{bmatrix},$$
$$C^3\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{101} \\ \frac{3}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{5}{101} \\ \frac{8}{101} \end{bmatrix},$$
$$C^4\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{5}{101} \\ \frac{8}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{13}{101} \\ \frac{21}{101} \end{bmatrix},$$
$$C^5\left(\begin{bmatrix} \frac{1}{101} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{13}{101} \\ \frac{21}{101} \end{bmatrix} \bmod 1 = \begin{bmatrix} \frac{34}{101} \\ \frac{55}{101} \end{bmatrix}.$$

Because all five iterates are different, the period of the periodic point $\left(\frac{1}{101},\ 0\right)$ must be greater than 5.

5. If $0 \le x < 1$ and $0 \le y < 1$, then $T(x,\ y) = \left(x + \frac{5}{12},\ y\right)\text{mod}1$, and so
$T^2(x,\ y) = \left(x + \frac{10}{12},\ y\right)\text{mod}1$, $T^3(x,\ y) = \left(x + \frac{15}{12},\ y\right)\text{mod}1$, ...,
$T^{12}(x,\ y) = \left(x + \frac{60}{12},\ y\right)\text{mod}1 = (x+5,\ y)\text{mod}1 = (x,\ y)$.
Thus every point in S returns to its original position after 12 iterations and so every point in S is a periodic point with period at most 12. Because every point is a periodic point, no point can have a dense set of iterates, and so the mapping cannot be chaotic.

6. **(a)** The matrix of Arnold's cat map, $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, is one in which (i) the entries are all integers, (ii) the determinant is 1, and (iii) the eigenvalues, $\frac{3+\sqrt{5}}{2} = 2.6180...$ and $\frac{3-\sqrt{5}}{2} = 0.3819...$, do not have magnitude 1. The three conditions of an Anosov automorphism are thus satisfied.

(b) The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are ± 1, both of which have magnitude 1. By part (iii) of the definition of an Anosov automorphism, this matrix is not the matrix of an Anosov automorphism.
The entries of the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ are integers, its determinant is 1; and neither of its eigenvalues, $2 \pm \sqrt{3}$, has magnitude 1. Consequently, this is the matrix of an Anosov automorphism.
The eigenvalues of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are both equal to 1, and so both have magnitude 1. By part (iii) of the definition, this is not the matrix of an Anosov automorphism.
The entries of the matrix $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ are integers; its determinant is 1; and neither of its eigenvalues, $4 \pm \sqrt{15}$, has magnitude 1. Consequently, this is the matrix of an Anosov automorphism.
The determinant of the matrix $\begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}$ is 2, and so by part (ii) of the definition, this is not the matrix of an Anosov automorphism.

(c) The eigenvalues of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\pm i$; both of which have magnitude 1. By part (iii) of the definition, this cannot be the matrix of an Anosov automorphism.
Starting with an arbitrary point $\begin{bmatrix} x \\ y \end{bmatrix}$ in the interior of S, (that is, with $0 < x < 1$ and $0 < y < 1$) we obtain

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\text{mod}1 = \begin{bmatrix} y \\ -x \end{bmatrix}\text{mod}1 = \begin{bmatrix} y \\ 1-x \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ 1-x \end{bmatrix}\text{mod}1 = \begin{bmatrix} 1-x \\ -y \end{bmatrix}\text{mod}1 = \begin{bmatrix} 1-x \\ 1-y \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1-x \\ 1-y \end{bmatrix}\text{mod}1 = \begin{bmatrix} 1-y \\ -1+x \end{bmatrix}\text{mod}1 = \begin{bmatrix} 1-y \\ x \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1-y \\ x \end{bmatrix}\text{mod}1 = \begin{bmatrix} x \\ -1+y \end{bmatrix}\text{mod}1 = \begin{bmatrix} x \\ y \end{bmatrix}.$$

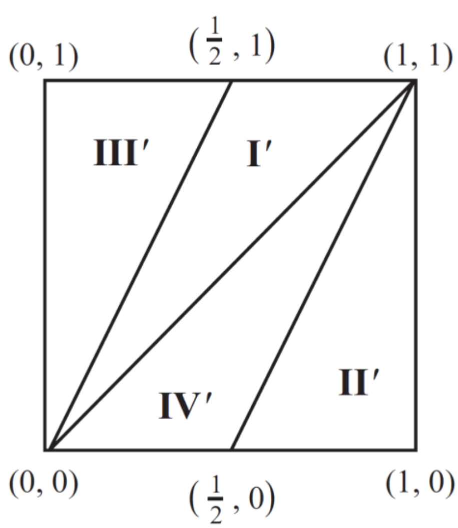
Thus every point in the interior of S is a periodic point with period at most 4. The geometric effect of this transformation, as seen by the iterates, is to rotate each point in the interior of S clockwise by 90° about the center point $\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$ of S. Consequently, each point in the interior of S has period 4 with the exception of the center point $\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$ which is a fixed point.

For points not in the interior of S, we first observe that the origin $\begin{bmatrix}0\\0\end{bmatrix}$ is a fixed point, which can easily be verified. Starting with a point of the form $\begin{bmatrix}x\\0\end{bmatrix}$ with $0 < x < 1$, we obtain a 4-cycle $\begin{bmatrix}x\\0\end{bmatrix} \to \begin{bmatrix}0\\1-x\end{bmatrix} \to \begin{bmatrix}1-x\\0\end{bmatrix} \to \begin{bmatrix}0\\x\end{bmatrix} \to \begin{bmatrix}x\\0\end{bmatrix}$ if $x \neq \frac{1}{2}$, otherwise we obtain the 2-cycle $\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} \to \begin{bmatrix}0\\\frac{1}{2}\end{bmatrix} \to \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$.

Similarly, starting with a point of the form $\begin{bmatrix}0\\y\end{bmatrix}$ with $0 < y < 1$, we obtain a 4-cycle $\begin{bmatrix}0\\y\end{bmatrix} \to \begin{bmatrix}y\\0\end{bmatrix} \to \begin{bmatrix}0\\1-y\end{bmatrix} \to \begin{bmatrix}1-y\\0\end{bmatrix} \to \begin{bmatrix}0\\y\end{bmatrix}$ if $y \neq \frac{1}{2}$, otherwise we obtain the 2-cycle $\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix} \to \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} \to \begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}$.

Thus every point not in the interior of S is a periodic point with 1, 2, or 4. Finally because no point in S can have a dense set of iterates, it follows that the mapping cannot be chaotic.

9. As per the hint, we wish to find the regions in S that map onto the four indicated regions in the figure below.



We first consider region **I′** with vertices (0, 0), $\left(\frac{1}{2},\ 1\right)$, and (1, 1). We seek points $(x_1,\ y_1)$, $(x_2,\ y_2)$, and $(x_3,\ y_3)$, with entries that lie in [0, 1], that map onto these three points under the mapping $\begin{bmatrix}x\\y\end{bmatrix} \to \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}a\\b\end{bmatrix}$ for certain integer values of a and b to be determined.

This leads to the three equations

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_1\\y_2\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix},$$

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_3\\y_3\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

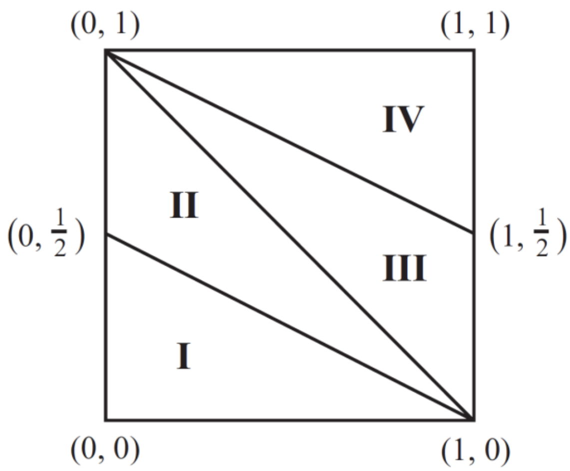
The inverse of the matrix $\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ is $\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$. We multiply the above three matrix equations by this inverse and set $\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$. Notice that c and d must be integers. This leads to

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix}=-\begin{bmatrix}c\\d\end{bmatrix},$$

$$\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix},$$

$$\begin{bmatrix}x_3\\y_3\end{bmatrix}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix}.$$

The only integer values of c and d that will give values of x_i and y_i in the interval [0, 1] are $c = d = 0$. This then gives $a = b = 0$ and the mapping $\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ maps the three points (0, 0), $\left(0,\ \frac{1}{2}\right)$, and (1, 0) to the three points (0, 0), $\left(\frac{1}{2},\ 1\right)$, and (1, 1), respectively. The three points (0, 0), $\left(0,\ \frac{1}{2}\right)$, and (1, 0) define the triangular region labeled **I** in the diagram below, which then maps onto the region **I′**.



For region **II′**, the calculations are as follows:

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix},\quad \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_3\\y_3\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix};$$

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}1\\-\frac{1}{2}\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix},$$

$$\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}c\\d\end{bmatrix},$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}.$$

Only $c = 1$ and $d = -1$ will work. This leads to
$a = 0$, $b = -1$ and the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ maps region **II** with vertices $\left(0, \frac{1}{2}\right)$, (0, 1), and (1, 0) onto region **II′**. For region **III′**, the calculations are as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = -\begin{bmatrix} c \\ d \end{bmatrix},$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix},$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}.$$

Only $c = -1$ and $d = 0$ will work. This leads to
$a = -1$, $b = -1$ and the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ maps region **III** with vertices (1, 0), $\left(1, \frac{1}{2}\right)$, and (0, 1) onto region **III′**.

For region **IV′**, the calculations are as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = -\begin{bmatrix} c \\ d \end{bmatrix},$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix},$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix}.$$

Only $c = 0$ and $d = -1$ will work. This leads to
$a = -1$, $b = -2$ and the mapping $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ maps region **IV** with vertices (0, 1), (1, 1), and $\left(1, \frac{1}{2}\right)$ onto region **IV′**.

12. As per the hint, we want to find all solutions of the matrix equation $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} r \\ s \end{bmatrix}$ where $0 \le x_0 < 1$, $0 \le y_0 < 1$, and r and s are nonnegative integers. This equation can be rewritten as $\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$, which has the solution $x_0 = \frac{-4r+3s}{5}$ and $y_0 = \frac{3r-s}{5}$. First trying r = 0 and s = 0, 1, 2, ..., then r = 1 and
s = 0, 1, 2, ..., etc., we find that the only values of r and s that yield values of x_0 and y_0 lying in [0, 1) are:
r = 1 and s = 2, which give $x_0 = \frac{2}{5}$ and $y_0 = \frac{1}{5}$;
r = 2 and s = 3, which give $x_0 = \frac{1}{5}$ and $y_0 = \frac{3}{5}$;

$r = 2$ and $s = 4$, which give $x_0 = \frac{4}{5}$ and $y_0 = \frac{2}{5}$;

$r = 3$ and $s = 5$, which give $x_0 = \frac{3}{5}$ and $y_0 = \frac{4}{5}$.

We can then check that $\left(\frac{2}{5}, \frac{1}{5}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$ form one 2-cycle and $\left(\frac{1}{5}, \frac{3}{5}\right)$ and $\left(\frac{4}{5}, \frac{2}{5}\right)$ form another 2-cycle.

14. Begin with a 101×101 array of white pixels and add the letter 'A' in black pixels to it. Apply the mapping T to this image, which will scatter the black pixels throughout the image. Then superimpose the letter 'B' in black pixels onto this image. Apply the mapping T again and then superimpose the letter 'C' in black pixels onto the resulting image. Repeat this procedure with the letter 'D' and 'E'. The next application of the mapping will return you to the letter 'A' with the pixels for the letters 'B' through 'E' scattered in the background. Four subsequent applications of T to this image will produce the remaining images.

10.14 Cryptography

1. First group the plaintext into pairs, add the dummy letter T, and get the numerical equivalents from Table 1.

DA	*RK*	*NI*	*GH*	*TT*
4 1	18 11	14 9	7 8	20 20

(a) For the enciphering matrix $A = \begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}$, reducing everything mod 26, we have

$$\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}7\\9\end{bmatrix} \qquad \begin{matrix}G\\I\end{matrix}$$

$$\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}18\\11\end{bmatrix} = \begin{bmatrix}51\\47\end{bmatrix} = \begin{bmatrix}25\\21\end{bmatrix} \qquad \begin{matrix}Y\\U\end{matrix}$$

$$\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}14\\9\end{bmatrix} = \begin{bmatrix}41\\37\end{bmatrix} = \begin{bmatrix}15\\11\end{bmatrix} \qquad \begin{matrix}O\\K\end{matrix}$$

$$\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix} = \begin{bmatrix}31\\22\end{bmatrix} = \begin{bmatrix}5\\22\end{bmatrix} \qquad \begin{matrix}E\\V\end{matrix}$$

$$\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}20\\20\end{bmatrix} = \begin{bmatrix}80\\60\end{bmatrix} = \begin{bmatrix}2\\8\end{bmatrix} \qquad \begin{matrix}B\\H\end{matrix}$$

The Hill cipher is *GIYUOKEVBH*.

(b) For the enciphering matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, reducing everything mod 26, we have

$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 6 \end{bmatrix}$ $\quad \begin{matrix} S \\ F \end{matrix}$

$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 18 \\ 11 \end{bmatrix} = \begin{bmatrix} 105 \\ 40 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \end{bmatrix}$ $\quad \begin{matrix} A \\ N \end{matrix}$

$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 14 \\ 9 \end{bmatrix} = \begin{bmatrix} 83 \\ 32 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ $\quad \begin{matrix} E \\ F \end{matrix}$

$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 52 \\ 23 \end{bmatrix} = \begin{bmatrix} 0 \\ 23 \end{bmatrix}$ $\quad \begin{matrix} Z \\ W \end{matrix}$

$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 140 \\ 60 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$ $\quad \begin{matrix} J \\ H \end{matrix}$

The Hill cipher is *SFANEFZWJH*.

2. **(a)** For $A = \begin{bmatrix} 9 & 1 \\ 7 & 2 \end{bmatrix}$, det($A$) = 18 – 7 = 11, which is not divisible by 2 or 13. Therefore by Corollary 10.14.3, A is invertible. From Equation (2):

$$A^{-1} = (11)^{-1}\begin{bmatrix} 2 & -1 \\ -7 & 9 \end{bmatrix} = 19\begin{bmatrix} 2 & -1 \\ -7 & 9 \end{bmatrix} = \begin{bmatrix} 38 & -19 \\ -133 & 171 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 23 & 15 \end{bmatrix} (\text{mod} 26).$$

Checking:

$$AA^{-1} = \begin{bmatrix} 9 & 1 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 12 & 7 \\ 23 & 15 \end{bmatrix} = \begin{bmatrix} 131 & 78 \\ 130 & 79 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod} 26)$$
$$A^{-1}A = \begin{bmatrix} 12 & 7 \\ 23 & 15 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 157 & 26 \\ 312 & 53 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod} 26).$$

(b) For $A = \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$, det($A$) = 9 – 5 = 4, which is divisible by 2. Therefore by Corollary 10.14.3, A is not invertible.

(c) For $A = \begin{bmatrix} 8 & 11 \\ 1 & 9 \end{bmatrix}$, det($A$) = 72 – 11 = 61 = 9 (mod 26), which is not divisible by 2 or 13. Therefore by Corollary 10.14.3, A is invertible. From (2):

$$A^{-1} = (9)^{-1}\begin{bmatrix} 9 & -11 \\ -1 & 8 \end{bmatrix} = 3\begin{bmatrix} 9 & -11 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 27 & -33 \\ -3 & 24 \end{bmatrix} = \begin{bmatrix} 1 & 19 \\ 23 & 24 \end{bmatrix} (\text{mod} 26).$$

Checking:

$$AA^{-1} = \begin{bmatrix} 8 & 11 \\ 1 & 9 \end{bmatrix}\begin{bmatrix} 1 & 19 \\ 23 & 24 \end{bmatrix} = \begin{bmatrix} 261 & 416 \\ 208 & 235 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod} 26)$$
$$A^{-1}A = \begin{bmatrix} 1 & 19 \\ 23 & 24 \end{bmatrix}\begin{bmatrix} 8 & 11 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 27 & 182 \\ 208 & 469 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod} 26).$$

(d) For $A = \begin{bmatrix} 2 & 1 \\ 1 & 7 \end{bmatrix}$, det($A$) = 14 – 1 = 13, which is divisible by 13. Therefore by Corollary 10.14.4, A is not invertible.

(e) For $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$, det($A$) = 6 – 6 = 0, so that A is not invertible by Corollary 10.14.4.

(f) For $A = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$, det($A$) = 3 – 8 = –5 = 21 (mod 26), which is not divisible by 2 or 13. Therefore by Corollary 10.14.4, A is invertible. From (2):

$$A^{-1} = (21)^{-1}\begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = 5\begin{bmatrix} 3 & -8 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -40 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} (\text{mod} 26).$$

Checking:

$$AA^{-1} = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} = \begin{bmatrix} 183 & 52 \\ 78 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod}26)$$
$$A^{-1}A = \begin{bmatrix} 15 & 12 \\ 21 & 5 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 156 \\ 26 & 183 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\text{mod}26).$$

3. From Table 1 the numerical equivalent of this ciphertext is

19 1 11 14 15 24 1 15 10 24

Now we have to find the inverse of $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

Since det(A) = 8 – 3 = 5, we have by (2):

$$A^{-1} = (5)^{-1} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = 21 \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 42 & -21 \\ -63 & 84 \end{bmatrix} = \begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} (\text{mod}26).$$

To obtain the plaintext, multiply each ciphertext vector by A^{-1} and reduce the results modulo 26.

$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 19 \\ 1 \end{bmatrix} = \begin{bmatrix} 309 \\ 291 \end{bmatrix} = \begin{bmatrix} 23 \\ 5 \end{bmatrix} \quad \begin{matrix} W \\ E \end{matrix}$$
$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 11 \\ 14 \end{bmatrix} = \begin{bmatrix} 246 \\ 249 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \end{bmatrix} \quad \begin{matrix} L \\ O \end{matrix}$$
$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 15 \\ 24 \end{bmatrix} = \begin{bmatrix} 360 \\ 369 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \end{bmatrix} \quad \begin{matrix} V \\ E \end{matrix}$$
$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 15 \end{bmatrix} = \begin{bmatrix} 91 \\ 105 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix} \quad \begin{matrix} M \\ A \end{matrix}$$
$$\begin{bmatrix} 16 & 5 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ 24 \end{bmatrix} = \begin{bmatrix} 280 \\ 294 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \end{bmatrix} \quad \begin{matrix} T \\ H \end{matrix}$$

The plaintext is thus WE LOVE MATH.

4. From Table 1 the numerical equivalent of the known plaintext is

AR *MY*

1 18 13 25

and the numerical equivalent of the corresponding ciphertext is

SL *HK*

19 12 8 11

so the corresponding plaintext and ciphertext vectors are

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 18 \end{bmatrix} \leftrightarrow \mathbf{c}_1 = \begin{bmatrix} 19 \\ 12 \end{bmatrix}$$
$$\mathbf{p}_2 = \begin{bmatrix} 13 \\ 25 \end{bmatrix} \leftrightarrow \mathbf{c}_2 = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

We want to reduce $C = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \end{bmatrix} = \begin{bmatrix} 19 & 12 \\ 8 & 11 \end{bmatrix}$ to I by elementary row operations and simultaneously apply these operations to $P = \begin{bmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 18 \\ 13 & 25 \end{bmatrix}$.

The calculations are as follows:

$\begin{bmatrix} 19 & 12 & 1 & 18 \\ 8 & 11 & 13 & 25 \end{bmatrix}$ Form the matrix $[C \quad P]$.

$\begin{bmatrix} 1 & 132 & 11 & 198 \\ 8 & 11 & 13 & 25 \end{bmatrix}$ Multiply the first row by $19^{-1} = 11(\text{mod}26)$.

$\begin{bmatrix} 1 & 2 & 11 & 16 \\ 8 & 11 & 13 & 25 \end{bmatrix}$ Replace 132 and 198 by their residues modulo 26.

$\begin{bmatrix} 1 & 2 & 11 & 16 \\ 0 & -5 & -75 & -103 \end{bmatrix}$ –8 times the first row to the second.

$\begin{bmatrix} 1 & 2 & 11 & 16 \\ 0 & 21 & 3 & 1 \end{bmatrix}$ Replace the entries in the second row by their residues modulo 26.

$\begin{bmatrix} 1 & 2 & 11 & 16 \\ 0 & 1 & 15 & 5 \end{bmatrix}$ Multiply the second row by $21^{-1} = 5(\text{mod}26)$.

$\begin{bmatrix} 1 & 0 & -19 & 6 \\ 0 & 1 & 15 & 5 \end{bmatrix}$ Add –2 times the second row to the first.

$\begin{bmatrix} 1 & 0 & 7 & 6 \\ 0 & 1 & 15 & 5 \end{bmatrix}$ Replace –19 by its residue modulo 26.

Thus $(A^{-1})^T = \begin{bmatrix} 7 & 6 \\ 15 & 5 \end{bmatrix}$ so the deciphering matrix is $A^{-1} = \begin{bmatrix} 7 & 15 \\ 6 & 5 \end{bmatrix}$ (mod 26).

Since $\det(A^{-1}) = 35 - 90 = -55 = 23(\text{mod}26)$,

$$A = (A^{-1})^{-1} = 23^{-1}\begin{bmatrix} 5 & -15 \\ -6 & 7 \end{bmatrix} = 17\begin{bmatrix} 5 & -15 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} 85 & -255 \\ -102 & 119 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 2 & 15 \end{bmatrix} (\text{mod}26)$$

is the enciphering matrix.

5. From Table 1 the numerical equivalent of the known plaintext is

AT OM

1 20 15 13

and the numerical equivalent of the corresponding ciphertext is

JY QO

10 25 17 15

The corresponding plaintext and ciphertext vectors are:

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \leftrightarrow \mathbf{c}_1 = \begin{bmatrix} 10 \\ 25 \end{bmatrix}$$
$$\mathbf{p}_2 = \begin{bmatrix} 15 \\ 13 \end{bmatrix} \leftrightarrow \mathbf{c}_2 = \begin{bmatrix} 17 \\ 15 \end{bmatrix}$$

We want to reduce $C = \begin{bmatrix} 10 & 25 \\ 17 & 15 \end{bmatrix}$ to I by elementary row operations and simultaneously apply these operations to $P = \begin{bmatrix} 1 & 20 \\ 15 & 13 \end{bmatrix}$.

The calculations are as follows:

$\begin{bmatrix} 10 & 25 & 1 & 20 \\ 17 & 15 & 15 & 13 \end{bmatrix}$ Form the matrix $[C \quad P]$.

$\begin{bmatrix} 27 & 40 & 16 & 33 \\ 17 & 15 & 15 & 13 \end{bmatrix}$ Add the second row to the first (since 10^{-1} does not exist mod 26).

$\begin{bmatrix} 1 & 14 & 16 & 7 \\ 17 & 15 & 15 & 13 \end{bmatrix}$ Replace the entries in the first row by their residues modulo 26.

$\begin{bmatrix} 1 & 14 & 16 & 7 \\ 0 & -223 & -257 & -106 \end{bmatrix}$ Add –17 times the first row to the second.

$\begin{bmatrix} 1 & 14 & 16 & 7 \\ 0 & 11 & 3 & 24 \end{bmatrix}$ Replace the entries in the 2nd row by their residues modulo 26.

$\begin{bmatrix} 1 & 14 & 16 & 7 \\ 0 & 1 & 57 & 456 \end{bmatrix}$ Multiply the second row by $11^{-1} = 19$ (mod 26).

$\begin{bmatrix} 1 & 14 & 16 & 7 \\ 0 & 1 & 5 & 14 \end{bmatrix}$ Replace the entries in the 2nd row by their residues modulo 26.

$\begin{bmatrix} 1 & 0 & -54 & -189 \\ 0 & 1 & 5 & 14 \end{bmatrix}$ Add –14 times the second row to the first.

$\begin{bmatrix} 1 & 0 & 24 & 19 \\ 0 & 1 & 5 & 14 \end{bmatrix}$ Replace –54 and –189 by their residues modulo 26.

Thus $(A^{-1})^T = \begin{bmatrix} 24 & 19 \\ 5 & 14 \end{bmatrix}$, and so the deciphering matrix is $A^{-1} = \begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}$.

From Table 1 the numerical equivalent of the given ciphertext is

LN	*GI*	*HG*	*YB*	*VR*	*EN*	*JY*	*QO*
12 14	7 9	8 7	25 2	22 18	5 14	10 25	17 15

To obtain the plaintext pairs, we multiply each ciphertext vector by A^{-1}:

$$\begin{aligned}
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 12 \\ 14 \end{bmatrix} &= \begin{bmatrix} 358 \\ 424 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \end{bmatrix} \quad \begin{matrix} T \\ H \end{matrix} \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} &= \begin{bmatrix} 213 \\ 259 \end{bmatrix} = \begin{bmatrix} 5 \\ 25 \end{bmatrix} \quad \begin{matrix} E \\ Y \end{matrix} \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} &= \begin{bmatrix} 227 \\ 250 \end{bmatrix} = \begin{bmatrix} 19 \\ 16 \end{bmatrix} \quad \begin{matrix} S \\ P \end{matrix} \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 25 \\ 2 \end{bmatrix} &= \begin{bmatrix} 610 \\ 503 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \end{bmatrix} \quad \begin{matrix} L \\ I \end{matrix} (\text{mod} 26) \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 22 \\ 18 \end{bmatrix} &= \begin{bmatrix} 618 \\ 670 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} \quad \begin{matrix} T \\ T \end{matrix} \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 5 \\ 14 \end{bmatrix} &= \begin{bmatrix} 190 \\ 291 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad \begin{matrix} H \\ E \end{matrix} \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 10 \\ 25 \end{bmatrix} &= \begin{bmatrix} 365 \\ 540 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \quad \begin{matrix} A \\ T \end{matrix} \\
\begin{bmatrix} 24 & 5 \\ 19 & 14 \end{bmatrix}\begin{bmatrix} 17 \\ 15 \end{bmatrix} &= \begin{bmatrix} 483 \\ 533 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \end{bmatrix} \quad \begin{matrix} O \\ M \end{matrix}
\end{aligned}$$

which yields the message *THEY SPLIT THE ATOM*.

6. Since we want a Hill 3-cipher, we will group the letters in triples. From Table 1 the numerical equivalents of the known plaintext are

I	H	A	V	E	C	O	M	E
9	8	1	22	5	3	15	13	5

and the numerical equivalent of the corresponding ciphertext are

H	P	A	F	Q	G	G	D	U
8	16	1	6	17	7	7	4	21

The corresponding plaintext and ciphertext vectors are

$$\mathbf{p}_1 = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix} \leftrightarrow \mathbf{c}_1 = \begin{bmatrix} 8 \\ 16 \\ 1 \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} 22 \\ 5 \\ 3 \end{bmatrix} \leftrightarrow \mathbf{c}_2 = \begin{bmatrix} 6 \\ 17 \\ 7 \end{bmatrix}$$

$$\mathbf{p}_3 = \begin{bmatrix} 15 \\ 13 \\ 5 \end{bmatrix} \leftrightarrow \mathbf{c}_3 = \begin{bmatrix} 7 \\ 4 \\ 21 \end{bmatrix}$$

We want to reduce $C = \begin{bmatrix} 8 & 16 & 1 \\ 6 & 17 & 7 \\ 7 & 4 & 21 \end{bmatrix}$ to I by elementary row operations and simultaneously apply these operations to $P = \begin{bmatrix} 9 & 8 & 1 \\ 22 & 5 & 3 \\ 15 & 13 & 5 \end{bmatrix}$. The calculations are as follows:

$\begin{bmatrix} 8 & 16 & 1 & 9 & 8 & 1 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{bmatrix}$	Form the matrix $[C \quad P]$.
$\begin{bmatrix} 15 & 20 & 22 & 24 & 21 & 6 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{bmatrix}$	Add the third row to the first since 8^{-1} does not exist modulo 26.
$\begin{bmatrix} 1 & 140 & 154 & 168 & 147 & 42 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{bmatrix}$	Multiply the first row by $15^{-1} = 7 \pmod{26}$.
$\begin{bmatrix} 1 & 10 & 24 & 12 & 17 & 16 \\ 6 & 17 & 7 & 22 & 5 & 3 \\ 7 & 4 & 21 & 15 & 13 & 5 \end{bmatrix}$	Replace the entries in the first row by their residues modulo 26.
$\begin{bmatrix} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & -43 & -137 & -50 & -97 & -93 \\ 0 & -66 & -147 & -69 & -106 & -107 \end{bmatrix}$	Add –6 times the first row to the second and –7 times the first row to the third.
$\begin{bmatrix} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & 9 & 19 & 2 & 7 & 11 \\ 0 & 12 & 9 & 9 & 24 & 23 \end{bmatrix}$	Replace the entries in the second and third rows by their residues modulo 26.
$\begin{bmatrix} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & 1 & 57 & 6 & 21 & 33 \\ 0 & 12 & 9 & 9 & 24 & 23 \end{bmatrix}$	Multiply the second row by $9^{-1} = 3 (\bmod\ 26)$.
$\begin{bmatrix} 1 & 10 & 24 & 12 & 17 & 16 \\ 0 & 1 & 5 & 6 & 21 & 7 \\ 0 & 12 & 9 & 9 & 24 & 23 \end{bmatrix}$	Replace the entries in the second row by their residues modulo 26.
$\begin{bmatrix} 1 & 0 & -26 & -48 & -193 & -54 \\ 0 & 1 & 5 & 6 & 21 & 7 \\ 0 & 0 & -51 & -63 & -228 & -61 \end{bmatrix}$	Add –10 times the second row to the first and –12 times the second row to the third.
$\begin{bmatrix} 1 & 0 & 0 & 4 & 15 & 24 \\ 0 & 1 & 5 & 6 & 21 & 7 \\ 0 & 0 & 1 & 15 & 6 & 17 \end{bmatrix}$	Replace the entries in the first and second row by their residues modulo 26.
$\begin{bmatrix} 1 & 0 & 0 & 4 & 15 & 24 \\ 0 & 1 & 0 & -69 & -9 & -78 \\ 0 & 0 & 1 & 15 & 6 & 17 \end{bmatrix}$	Add –5 times the third row to the second.

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 15 & 24 \\ 0 & 1 & 0 & 9 & 17 & 0 \\ 0 & 0 & 1 & 15 & 6 & 17 \end{bmatrix}$$

Replace the entries in the second row by their residues modulo 26.

Thus, $(A^{-1})^T = \begin{bmatrix} 4 & 15 & 24 \\ 9 & 17 & 0 \\ 15 & 6 & 17 \end{bmatrix}$ and so the deciphering matrix is $A^{-1} = \begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}$.

From Table 1 the numerical equivalent of the given ciphertext is

H	P	A	F	Q	G	G	D	U	G	G	D	H	P	G	O	D	Y	N	O	R
8	16	1	6	17	7	7	4	21	7	4	4	8	16	7	15	4	25	14	15	18

To obtain the plaintext triples, we multiply each ciphertext vector by A^{-1}:

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 8 \\ 16 \\ 1 \end{bmatrix} = \begin{bmatrix} 191 \\ 398 \\ 209 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix} \quad \begin{matrix} I \\ H \\ A \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 6 \\ 17 \\ 7 \end{bmatrix} = \begin{bmatrix} 282 \\ 421 \\ 263 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \\ 3 \end{bmatrix} \quad \begin{matrix} V \\ E \\ C \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 21 \end{bmatrix} = \begin{bmatrix} 379 \\ 299 \\ 525 \end{bmatrix} = \begin{bmatrix} 15 \\ 13 \\ 5 \end{bmatrix} \quad \begin{matrix} O \\ M \\ E \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 124 \\ 197 \\ 236 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 2 \end{bmatrix} \quad \begin{matrix} T \\ O \\ B \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 8 \\ 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 281 \\ 434 \\ 311 \end{bmatrix} = \begin{bmatrix} 21 \\ 18 \\ 25 \end{bmatrix} \quad \begin{matrix} U \\ R \\ Y \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 15 \\ 4 \\ 25 \end{bmatrix} = \begin{bmatrix} 471 \\ 443 \\ 785 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \quad \begin{matrix} C \\ A \\ E \end{matrix}$$

$$\begin{bmatrix} 4 & 9 & 15 \\ 15 & 17 & 6 \\ 24 & 0 & 17 \end{bmatrix}\begin{bmatrix} 14 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 461 \\ 573 \\ 642 \end{bmatrix} = \begin{bmatrix} 19 \\ 1 \\ 18 \end{bmatrix} \quad \begin{matrix} S \\ A \\ R \end{matrix}$$

Finally, the message is *I HAVE COME TO BURY CAESAR*.

7. **(a)** Multiply each of the triples of the message by $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and reduce the results modulo 2.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The encoded message is 010110001.

(b) Reduce $[A \quad I]$ to $[I \quad A^{-1}]$ modulo 2.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$ Form the matrix $[A \quad I]$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 1 \end{bmatrix}$$ Add the first row to the third row.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$ Replace 2 by its residue modulo 2.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$ Add the third row to the second row.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$ Replace 2 by its residue modulo 2.

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$ Add the second row to the first row.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$ Replace 2 by its residue modulo 2.

Thus $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The decoded message is 110101111, which is the original message.

8. Since 29 is a prime number, by Corollary 10.15.2 a matrix A with entries in Z_{29} is invertible if and only if $\det(A) \neq 0 \pmod{29}$.

10.15 Genetics

1. Use induction on n, the case $n = 1$ being already given. If the result is true for $n - 1$, then

$$M^n = M^{n-1}M = (PD^{n-1}P^{-1})(PDP^{-1}) = PD^{n-1}(P^{-1}P)DP^{-1} = PD^{n-1}DP^{-1} = PD^nP^{-1},$$

proving the result.

2. Using Table 1 and notations of Example 1, we derive the following equations:

$$\begin{aligned} a_n &= \frac{1}{2}a_{n-1} + \frac{1}{4}b_{n-1} \\ b_n &= \frac{1}{2}a_{n-1} + \frac{1}{2}b_{n-1} + \frac{1}{2}c_{n-1} \\ c_n &= \frac{1}{4}b_{n-1} + \frac{1}{2}c_{n-1}. \end{aligned}$$

The transition matrix is thus $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$. The characteristic polynomial of M is

$$\det(\lambda I - M) = \lambda^3 - \left(\frac{3}{2}\right)\lambda^2 + \left(\frac{1}{2}\right)\lambda = \lambda(\lambda - 1)\left(\lambda - \frac{1}{2}\right),$$

so the eigenvalues of M are $\lambda = 1$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = 0$. Corresponding eigenvectors (found by solving $(\lambda I - M)\mathbf{x} = 0$) are $\mathbf{e}_1 = [1 \quad 2 \quad 1]^T$, $\mathbf{e}_2 = [1 \quad 0 \quad -1]^T$, and $\mathbf{e}_3 = [1 \quad -2 \quad 1]^T$. Thus

$$M^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

This yields

$$\mathbf{x}^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1} & \frac{1}{4} & \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1} & \frac{1}{4} & \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}.$$

Remembering that $a_0 + b_0 + c_0 = 1$, we obtain

$$\begin{aligned} a_n &= \frac{1}{4}a_0 + \frac{1}{4}b_0 + \frac{1}{4}c_0 + \left(\frac{1}{2}\right)^{n+1} a_0 - \left(\frac{1}{2}\right)^{n+1} c_0 = \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1} (a_0 - c_0) \\ b_n &= \frac{1}{2}a_0 + \frac{1}{2}b_0 + \frac{1}{2}c_0 = \frac{1}{2} \end{aligned}$$

$$c_n = \frac{1}{4}a_0 + \frac{1}{4}b_0 + \frac{1}{2}c_0 - \left(\frac{1}{2}\right)^{n+1} a_0 + \left(\frac{1}{2}\right)^{n+1} c_0 = \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1} (a_0 - c_0).$$

Since $\left(\frac{1}{2}\right)^{n+1}$ approaches zero as $n \to \infty$, we obtain $a_n \to \frac{1}{4}$, $b_n \to \frac{1}{2}$, and $c_n \to \frac{1}{4}$ as $n \to \infty$.

3. Call M_1 the matrix of Example 1, and M_2 the matrix of Exercise 2. Then $\mathbf{x}^{(2n)} = (M_2M_1)^n\mathbf{x}^{(0)}$ and $\mathbf{x}^{(2n+1)} = M_1(M_2M_1)^n\mathbf{x}^{(0)}$. We have

$$M_2M_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{8} & \frac{1}{4} \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^3 - \frac{5}{4}\lambda^2 + \frac{1}{4}\lambda$, so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{4}$, $\lambda_3 = 0$. Corresponding eigenvectors are $\mathbf{e}_1 = [5 \quad 6 \quad 1]^T$, $\mathbf{e}_2 = [-1 \quad 0 \quad 1]^T$, and $\mathbf{e}_3 = [1 \quad -2 \quad 1]^T$. Thus,

$$(M_2M_1)^n = PD^nP^{-1} = \begin{bmatrix} 5 & -1 & 1 \\ 6 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{4}\right)^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ -\frac{1}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Using the notation of Example 1 (recall that $a_0 + b_0 + c_0 = 1$), we obtain

$$\begin{aligned} a_{2n} &= \frac{5}{12} + \frac{1}{6 \cdot 4^n}(2a_0 - b_0 - 4c_0) \\ b_{2n} &= \frac{1}{2} \\ c_{2n} &= \frac{1}{12} - \frac{1}{6 \cdot 4^n}(2a_0 - b_0 - 4c_0) \end{aligned}$$

and

$$\begin{aligned} a_{2n+1} &= \frac{2}{3} + \frac{1}{6 \cdot 4^n}(2a_0 - b_0 - 4c_0) \\ b_{2n+1} &= \frac{1}{3} - \frac{1}{6 \cdot 4^n}(2a_0 - b_0 - 4c_0) \\ c_{2n+1} &= 0. \end{aligned}$$

4. The characteristic polynomial of M is $(\lambda - 1)\left(\lambda - \frac{1}{2}\right)$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$. Corresponding eigenvectors are easily found to be $\mathbf{e}_1 = [1 \quad 0]^T$ and $\mathbf{e}_2 = [1 \quad -1]^T$. From this point, the verification of Equation (7) is in the text.

5. From Equation (9), if $b_0 = .25 = \frac{1}{4}$, we get $b_1 = \frac{\frac{1}{4}}{\frac{9}{8}} = \frac{2}{9}$, then $b_2 = \frac{\frac{2}{9}}{\frac{10}{9}} = \frac{1}{5}$, $b_3 = \frac{\frac{1}{5}}{\frac{11}{10}} = \frac{2}{11}$, and, in general, $b_n = \frac{2}{8+n}$. We will reach $\frac{2}{20} = .10$ in 12 generations. According to Equation (8), under the controlled program the percentage would be $\frac{1}{2^{14}}$ in 12 generations, or $\frac{1}{16{,}384} \approx .00006 = .006\%$.

6. $P^{-1}\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{12}(1+\sqrt{5}) \\ \frac{1}{12}(1-\sqrt{5}) \end{bmatrix}$, $D^nP^{-1}\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{3}\left[\frac{1}{4}(1+\sqrt{5})\right]^{n+1} \\ \frac{1}{3}\left[\frac{1}{4}(1-\sqrt{5})\right]^{n+1} \end{bmatrix}$

$$PD^nP^{-1}\mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4^{n+2}}\left[(-3-\sqrt{5})(1+\sqrt{5})^{n+1} + (-3+\sqrt{5})(1-\sqrt{5})^{n+1}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n} + (1-\sqrt{5})^{n}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n} + (1-\sqrt{5})^{n}\right] \\ \frac{1}{3}\cdot\frac{1}{4^{n+1}}\left[(1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}\right] \\ \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4^{n+2}}\left[(-3-\sqrt{5})(1+\sqrt{5})^{n+1} + (-3+\sqrt{5})(1-\sqrt{5})^{n+1}\right] \end{bmatrix}$$

As n tends to infinity, $\frac{1}{4^{n+2}}$and $\frac{1}{4^{n+1}}$ approach 0, so $\mathbf{x}^{(n)}$ approaches $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$.

7. From (13) we have that the probability that the limiting sibling-pairs will be type (A, AA) is $a_0 + \frac{2}{3}b_0 + \frac{1}{3}c_0 + \frac{2}{3}d_0 + \frac{1}{3}e_0$.
The proportion of A genes in the population at the outset is as follows: all the type (A, AA) genes, $\frac{2}{3}$ of the type (A, Aa) genes, $\frac{1}{3}$ the type (A, aa) genes, etc. ...yielding

$$a_0 + \frac{2}{3}b_0 + \frac{1}{3}c_0 + \frac{2}{3}d_0 + \frac{1}{3}e_0.$$

8. From an (A, AA) pair we get only (A, AA) pairs and similarly for (a, aa). From either (A, aa) or (a, AA) pairs we must get an Aa female, who will not mature. Thus no offspring will come from such pairs. The transition matrix is then $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

9. For the first column of M we realize that parents of type (A, AA) can produce offspring only of that type, and similarly for the last column. The fifth column is like the second column, and follows the analysis in the text. For the middle two columns, say the third, note that male offspring from (A, aa) must be of type a, and females are of type Aa, because of the way the genes are inherited.

10.16 Age-Specific Population Growth

1. **(a)** The characteristic polynomial of L is $\lambda^2 - \lambda - \frac{3}{4}$, so the eigenvalues of L are $\lambda = \frac{3}{2}$ and $\lambda = -\frac{1}{2}$, thus $\lambda_1 = \frac{3}{2}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{\frac{1}{2}}{\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$ is the corresponding eigenvector.

(b) $\mathbf{x}^{(1)} = L\mathbf{x}^{(0)} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$, $\mathbf{x}^{(2)} = L\mathbf{x}^{(1)} = \begin{bmatrix} 175 \\ 50 \end{bmatrix}$, $\mathbf{x}^{(3)} \approx L\mathbf{x}^{(2)} \approx \begin{bmatrix} 250 \\ 88 \end{bmatrix}$, $\mathbf{x}^{(4)} \approx L\mathbf{x}^{(3)} \approx \begin{bmatrix} 382 \\ 125 \end{bmatrix}$, $\mathbf{x}^5 \approx L\mathbf{x}^{(4)} \approx \begin{bmatrix} 570 \\ 191 \end{bmatrix}$

(c) $\mathbf{x}^{(6)} \approx L\mathbf{x}^{(5)} \approx \begin{bmatrix} 857 \\ 285 \end{bmatrix}$; $\mathbf{x}^{(6)} \approx \lambda_1 \mathbf{x}^{(5)} \approx \begin{bmatrix} 855 \\ 287 \end{bmatrix}$

5. a_1 is the average number of offspring produced in the first age period. $a_2 b_1$ is the number of offspring produced in the second period times the probability that the female will live into the second period, i.e., it is the expected number of offspring per female during the second period, and so on for all the periods. Thus, the sum of these, which is the net reproduction rate, is the expected number of offspring produced by a given female during her expected lifetime.

7. $R = 0 + 4\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{19}{8} = 2.375$

8. $R = 0 + (.00024)(.99651) + \cdots + (.00240)(.99651) \cdots (.987) = 1.49611.$

10.17 Harvesting of Animal Populations

1. **(a)** The characteristic polynomial of L is $\lambda^3 - 2\lambda - \frac{3}{8} = \left(\lambda - \frac{3}{2}\right)\left[\lambda^2 + \left(\frac{3}{2}\right)\lambda + \frac{1}{4}\right]$, so $\lambda_1 = \frac{3}{2}$. Thus h, the fraction harvested of each age group, is $1 - \frac{2}{3} = \frac{1}{3}$ so the yield is $33\frac{1}{3}\%$ of the population. The eigenvector corresponding to $\lambda = \frac{3}{2}$ is $\begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{18} \end{bmatrix}$; this is the age distribution vector after each harvest.

(b) From Equation (10), the age distribution vector $\mathbf{x}_1$ is $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \end{bmatrix}^T$. Equation (9) tells us that $h_1 = 1 - \frac{1}{\frac{19}{8}} = \frac{11}{19}$, so we harvest $\frac{11}{19}$ or 57.9% of the youngest age class. Since $L\mathbf{x}_1 =$

$\begin{bmatrix} \frac{19}{8} & \frac{1}{2} & \frac{1}{8} \end{bmatrix}^T$, the youngest class contains 79.2% of the population. Thus the yield is 57.9% of 79.2%, or 45.8% of the population.

2. The Leslie matrix of Example 1 has $b_1 = .845$, $b_2 = .975$, $b_3 = .965$, etc. This, together with the harvesting data from Equations (13) and the formula of Equation (5) yields

$$\mathbf{x}_1 = [1 \quad .845 \quad .824 \quad .795 \quad .755 \quad .699 \quad .626 \quad .5323 \quad 0 \quad 0 \quad 0 \quad 0]^T$$
$$L\mathbf{x}_1 = [2.090 \quad .845 \quad .824 \quad .795 \quad .755 \quad .699 \quad .626 \quad .532 \quad .418 \quad 0 \quad 0 \quad 0]^T.$$

The total of the entries of $L\mathbf{x}_1$ is 7.584. The proportion of sheep harvested is $h_1(L\mathbf{x}_1)_2 + h_9(L\mathbf{x}_1)_9 = 1.51$, or 19.9% of the population.

4. In this situation we have $h_I \neq 0$, and $h_1 = h_2 = \cdots = h_{I-1} = h_{I+1} = \cdots = h_n = 0$. Equation (4) then takes the form $a_1 + a_2b_1 + a_3b_1b_2 + \cdots + a_Ib_1b_2\cdots b_{I-1}(1-h_I) + a_{I+1}b_1b_2\cdots b_I(1-h_I) + \cdots + a_nb_1b_2\cdots b_{n-1}(1-h_I) = 1$.

$$\begin{aligned}&(1-h_I)[a_Ib_1b_2\cdots b_{I-1} + a_{I+1}b_1b_2\cdots b_I + \cdots + a_nb_1b_2\cdots b_{n-1}]\\&= 1 - a_1 - a_2b_1 - \cdots - a_{I-1}b_1b_2\cdots b_{I-2}\end{aligned}$$

So,

$$\begin{aligned}h_I &= \frac{a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} - 1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}} + 1\\&= \frac{a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} - 1 + a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}}\\&= \frac{R-1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_nb_1b_2\cdots b_{n-1}}\end{aligned}$$

5. Here $h_J = 1$, $h_I \neq 0$, and all the other h_k's are zero. Then Equation (4) becomes

$$a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} + (1-h_I)[a_Ib_1b_2\cdots b_{I-1} + \cdots + a_{J-1}b_1b_2\cdots b_{J-2}] = 1.$$

We solve for h_I to obtain

$$h_I = \frac{a_1 + a_2b_1 + \cdots + a_{I-1}b_1b_2\cdots b_{I-2} - 1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_{J-1}b_1b_2\cdots b_{J-2}} + 1 = \frac{a_1 + a_2b_1 + \cdots + a_{J-1}b_1b_2\cdots b_{J-2} - 1}{a_Ib_1b_2\cdots b_{I-1} + \cdots + a_{J-1}b_1b_2\cdots b_{J-2}}.$$

10.18 A Least Squares Model for Human Hearing

1. From Theorem 10.18.1, we compute $a_0 = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2 dt = \frac{2}{3}\pi^2$,
$a_k = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2 \cos kt\, dt = \frac{4}{k^2}$, and $b_k = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2 \sin kt\, dt = 0$.
So the least-squares trigonometric polynomial of order 3 is $\frac{\pi^2}{3} + 4\cos t + \cos 2t + \frac{4}{9}\cos 3t$.

2. From Theorem 10.18.2, we compute $a_0 = \frac{2}{T}\int_0^T t^2 dt = \frac{2}{3}T^2$, $a_k = \frac{2}{T}\int_0^T t^2 \cos\frac{2k\pi t}{T} dt = \frac{T^2}{k^2\pi^2}$ and $b_k = \frac{2}{T}\int_0^T t^2 \sin\frac{2k\pi t}{T} dt = -\frac{T^2}{k\pi}$.
So the least-squares trigonometric polynomial of order 4 is

$$\frac{T^2}{3}+\frac{T^2}{\pi^2}\left(\cos\frac{2\pi t}{T}+\frac{1}{4}\cos\frac{4\pi t}{T}+\frac{1}{9}\cos\frac{6\pi t}{T}+\frac{1}{16}\cos\frac{8\pi t}{T}\right)$$
$$-\frac{T^2}{\pi}\left(\sin\frac{2\pi t}{T}+\frac{1}{2}\sin\frac{4\pi t}{T}+\frac{1}{3}\sin\frac{6\pi t}{T}+\frac{1}{4}\sin\frac{8\pi t}{T}\right).$$

3. From Theorem 10.18.2, $a_0=\frac{1}{\pi}\int_0^{2\pi}f(t)dt=\frac{1}{\pi}\int_0^{\pi}\sin t\,dt=\frac{2}{\pi}$. (Note the upper limit on the second integral),

$$a_k=\frac{1}{\pi}\int_0^{\pi}\sin t\cos kt\,dt=\frac{1}{\pi}\left(\frac{1}{k^2-1}[k\sin kt\sin t+\cos kt\cos t]\right)\Big|_0^{\pi}$$
$$=\frac{1}{\pi}\left(\frac{1}{k^2-1}[0+(-1)^{k-1}-1]\right)=\begin{cases}0 & \text{if } k \text{ is odd}\\ -\dfrac{2}{\pi(k^2-1)} & \text{if } k \text{ is even.}\end{cases}$$
$$b_k=\frac{1}{\pi}\int_0^{\pi}\sin kt\sin t\,dt=\begin{cases}\dfrac{1}{2} & \text{if } k=1\\ 0 & \text{if } k>1\end{cases}.$$

So the least-squares trigonometric polynomial of order 4 is $\frac{1}{\pi}+\frac{1}{2}\sin t-\frac{2}{3\pi}\cos 2t-\frac{2}{15\pi}\cos 4t$.

4. From Theorem 10.18.2, $a_0=\frac{1}{\pi}\int_0^{2\pi}\sin\frac{1}{2}t\,dt=\frac{4}{\pi}$.

$$a_k=\frac{1}{\pi}\int_0^{2\pi}\sin\frac{1}{2}t\cos kt\,dt=-\frac{4}{\pi(4k^2-1)}=-\frac{4}{\pi(2k-1)(2k+1)},$$
$$b_k=\frac{1}{\pi}\int_0^{2\pi}\sin\frac{1}{2}t\sin kt\,dt=0.$$

So the least-square trigonometric polynomial of order n is

$$\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos t}{1\cdot 3}+\frac{\cos 2t}{3\cdot 5}+\frac{\cos 3t}{5\cdot 7}+\cdots+\frac{\cos nt}{(2n-1)(2n+1)}\right).$$

5. From Theorem 10.18.2, $a_0=\frac{2}{T}\int_0^{T}f(t)dt=\frac{2}{T}\int_0^{\frac{1}{2}T}t\,dt+\frac{2}{T}\int_{\frac{1}{2}T}^{T}(T-t)dt=\frac{T}{2}$.

$$a_k=\frac{2}{T}\int_0^{\frac{1}{2}T}t\cos\frac{2k\pi t}{T}dt+\frac{2}{T}\int_{\frac{1}{2}T}^{T}(T-t)\cos\frac{2k\pi t}{T}dt=\frac{4T}{4k^2\pi^2}((-1)^k-1)=\begin{cases}0 & \text{if } k \text{ is even}\\ \frac{8T}{(2k)^2\pi^2} & \text{if } k \text{ is odd}\end{cases}$$
$$b_k=\frac{2}{T}\int_0^{\frac{1}{2}T}t\sin\frac{2k\pi t}{T}dt+\frac{2}{T}\int_{\frac{1}{2}T}^{T}(T-t)\sin\frac{2k\pi t}{T}dt=\frac{T(-1)^{k+1}}{2k\pi}+\frac{T(-1)^k}{2k\pi}=\frac{T(-1)^k}{2k\pi}(-1+1)=0$$

So the least-squares trigonometric polynomial of order n is:

$$\frac{T}{4}-\frac{8T}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi t}{T}+\frac{1}{6^2}\cos\frac{6\pi t}{T}+\frac{1}{10^2}\cos\frac{10\pi t}{T}+\cdots+\frac{1}{(2n)^2}\cos\frac{2\pi nt}{T}\right)$$

if n is even; the last term involves $n-1$ if n is odd.

6. **(a)** $\|1\|=\sqrt{\int_0^{2\pi}dt}=\sqrt{2\pi}$

(b) $\|\cos kt\| = \sqrt{\int_0^{2\pi} \cos^2 kt\, dt} = \sqrt{\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt} = \sqrt{\pi}$

(c) $\|\sin kt\| = \sqrt{\int_0^{2\pi} \sin^2 kt\, dt} = \sqrt{\int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right) dt} = \sqrt{\pi}$

10.19 Warps and Morphs

1. **(a)** Equation (2) is $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and Equation (3) is $c_1 + c_2 + c_3 = 1$. These equations can be written in combined matrix form as $\begin{bmatrix} 1 & 3 & 4 \\ 1 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$.

This system has the unique solution $c_1 = \frac{1}{5}$, $c_2 = \frac{2}{5}$, and $c_3 = \frac{2}{5}$. Because these coefficients are all nonnegative, it follows that $\mathbf{v}$ is a convex combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(b) As in part (a) the system for c_1, c_2, and c_3 is $\begin{bmatrix} 1 & 3 & 4 \\ 1 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ which has the unique solution $c_1 = \frac{2}{5}$, $c_2 = \frac{4}{5}$, and $c_3 = -\frac{1}{5}$. Because one of these coefficients is negative, it follows that $\mathbf{v}$ is not a convex combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(c) As in part (a) the system for c_1, c_2, and c_3 is $\begin{bmatrix} 3 & -2 & 3 \\ 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ which has the unique solution $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, and $c_3 = 0$. Because these coefficients are all nonnegative, it follows that $\mathbf{v}$ is a convex combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

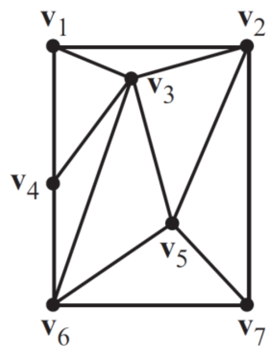
(d) As in part (a) the system for c_1, c_2, and c_3 is $\begin{bmatrix} 3 & -2 & 3 \\ 3 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ which has the unique solution $c_1 = \frac{4}{15}$, $c_2 = \frac{6}{15}$, and $c_3 = \frac{5}{15}$. Because these coefficients are all nonnegative, it follows that $\mathbf{v}$ is a convex combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

2. For both triangulations the number of triangles, m, is equal to 7; the number of vertex points, n, is equal to 7; and the number of boundary vertex points, k, is equal to 5. Equation (7), $m = 2n - 2 - k$, becomes $7 = 2(7) - 2 - 5$, or $7 = 7$.

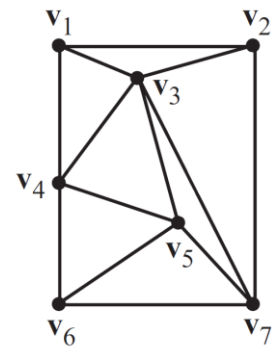
3. Combining everything that is given in the statement of the problem, we obtain:

$$\begin{aligned} \mathbf{w} &= M\mathbf{v} + \mathbf{b} = M(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) + (c_1 + c_2 + c_3)\mathbf{b} \\ &= c_1(M\mathbf{v}_1 + \mathbf{b}) + c_2(M\mathbf{v}_2 + \mathbf{b}) + c_3(M\mathbf{v}_3 + \mathbf{b}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3. \end{aligned}$$

4. **(a)**



(b)



5. **(a)** Let $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then the three matrix equations $M\mathbf{v}_i + \mathbf{b} = \mathbf{w}_i$, $i = 1, 2, 3$, can be written as the six scalar equations

$$m_{11} + m_{12} + b_1 = 4$$
$$m_{21} + m_{22} + b_2 = 3$$

$$2m_{11} + 3m_{12} + b_1 = 9$$
$$2m_{21} + 3m_{22} + b_2 = 5$$

$$2m_{11} + m_{12} + b_1 = 5$$
$$2m_{21} + m_{22} + b_2 = 3$$

The first, third, and fifth equations can be written in matrix form as

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$ and the second, fourth, and sixth equations as

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$.

The first system has the solution $m_{11} = 1, m_{12} = 2, b_1 = 1$ and the second system has the solution $m_{21} = 0, m_{22} = 1, b_2 = 2$. Thus we obtain $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) As in part (a), we are led to the following two linear systems:

$\begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -2 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

Solving these two linear systems leads to $M = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) As in part (a), we are led to the following two linear systems:

$\begin{bmatrix} -2 & 1 & 1 \\ 3 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 & 1 \\ 3 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$.

Solving these two linear systems leads to $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

(d) As in part (a), we are led to the following two linear systems:

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 \\ -4 & -2 & 1 \end{bmatrix}\begin{bmatrix} m_{11} \\ m_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{7}{2} \\ -\frac{7}{2} \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 \\ -4 & -2 & 1 \end{bmatrix}\begin{bmatrix} m_{21} \\ m_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -9 \end{bmatrix}.$$

Solving these two linear systems leads to $M = \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$.

7. **(a)** The vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of a triangle can be written as the convex combinations $\mathbf{v}_1 = (1)\mathbf{v}_1 + (0)\mathbf{v}_2 + (0)\mathbf{v}_3$, $\mathbf{v}_2 = (0)\mathbf{v}_1 + (1)\mathbf{v}_2 + (0)\mathbf{v}_3$, and $\mathbf{v}_3 = (0)\mathbf{v}_1 + (0)\mathbf{v}_2 + (1)\mathbf{v}_3$. In each of these cases, precisely two of the coefficients are zero and one coefficient is one.

(b) If, for example, $\mathbf{v}$ lies on the side of the triangle determined by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then from Exercise 6(a) we must have that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + (0)\mathbf{v}_3$ where $c_1 + c_2 = 1$. Thus at least one of the coefficients, in this example c_3, must equal zero.

(c) From part (b), if at least one of the coefficients in the convex combination is zero, then the vector must lie on one of the sides of the triangle. Consequently, none of the coefficients can be zero if the vector lies in the interior of the triangle.

8. **(a)** Consider the vertex $\mathbf{v}_1$ of the triangle and its opposite side determined by the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$. The midpoint $\mathbf{v}_m$ of this opposite side is $\frac{(\mathbf{v}_2+\mathbf{v}_3)}{2}$ and the point on the line segment from $\mathbf{v}_1$ to $\mathbf{v}_m$ that is two-thirds of the distance to $\mathbf{v}_m$ is given by

$$\frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\mathbf{v}_m = \frac{1}{3}\mathbf{v}_1 + \frac{2}{3}\left(\frac{\mathbf{v}_2 + \mathbf{v}_3}{2}\right) = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3.$$

(b) $\frac{1}{3}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$

10.20 Internet Search Engines

1. **(a)** The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$. Since $I - B = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the eigenspace associated with $\lambda = 1$ is span{(1,2,2)}. The normalized eigenvector in that eigenspace is $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$. We obtain:

- page rank $\frac{2}{5}$ for pages 2 and 3 (a tie),

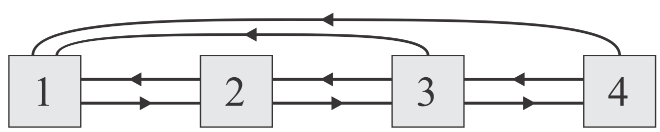
- page rank $\frac{1}{5}$ for page 1.

(b) The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$. Since $I - B = \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the eigenspace associated with $\lambda = 1$ is span$\{(1,0,1)\}$. The normalized eigenvector in that eigenspace is $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$. We obtain:

- page rank $\frac{1}{2}$ for pages 1 and 3 (a tie),
- page rank 0 for page 2.

(Note that the matrix B in this part is not regular and the state vector sequence $\mathbf{x}^{(k)}$ does not converge. However, it could be shown that the fractional page count for this webgraph will approach $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$.)

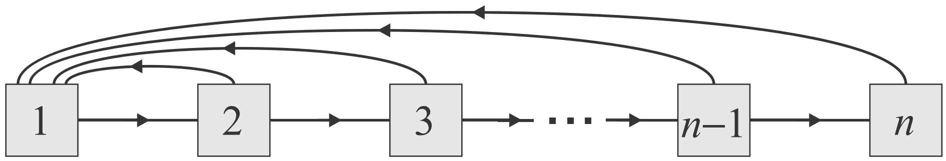
9. The following directed graph represents the given slide show:



The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 1 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}$. Since $I - B = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{3} & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the eigenspace associated with $\lambda = 1$ is span$\{(4,5,3,1)\}$. The normalized eigenvector in that eigenspace is $\left(\frac{4}{13}, \frac{5}{13}, \frac{3}{13}, \frac{1}{13}\right)$. We obtain:

- page rank $\frac{5}{13}$ for slide 2,
- page rank $\frac{4}{13}$ for slide 1,
- page rank $\frac{3}{13}$ for slide 3,
- page rank $\frac{1}{13}$ for slide 4.

10. The following directed graph represents the given slide show:



The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}$.

We have

$$I - B = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & -\frac{1}{2} & -1 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Adding each of the rows 2 through n to the first row results in the zeros in the entire first row. Multiplying the second row by -1, and each of the rows 3 through n by -2 now yields

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

We then perform the following sequence of elementary row operations

- add 2 times row n to the row $n-1$;
- add 2 times row $n-1$ to the row $n-2$;
- ...
- add 2 times the fourth row to the third row;
- add the third row to the second row

then interchange rows 1 and 2, 2 and 3, etc., to obtain the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -2^{n-2} \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -2^{n-2} \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -2^{n-3} \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & -2^{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -2^{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2^{1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$ The eigenspace associated with $\lambda = 1$ is

$\operatorname{span}\{(2^{n-2}, 2^{n-2}, 2^{n-3}, \dots, 2^2, 2^1, 2^0)\}$. We can express

$$\begin{aligned} 2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2^0 &= (2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2^0)\frac{2-1}{2-1} \\ &= 2^{n-1} - 2^{n-2} + 2^{n-2} - 2^{n-3} + \cdots + 2^2 - 2^2 + 2^1 - 2^0 \\ &= 2^{n-1} - 1 \end{aligned}$$

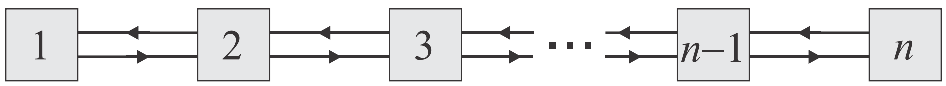
therefore the normalized eigenvector in that eigenspace is

$\frac{1}{2^{n-1}+2^{n-2}-1}(2^{n-2}, 2^{n-2}, 2^{n-3}, \dots, 2^2, 2^1, 2^0)$.

The slides can be arranged in the order of the decreasing rank as follows:

- page rank $\frac{2^{n-2}}{2^{n-1}+2^{n-2}-1}$ for slides 1 and 2 (a tie),
- page rank $\frac{2^{n-3}}{2^{n-1}+2^{n-2}-1}$ for slide 3,
- ...
- page rank $\frac{2^{1}}{2^{n-1}+2^{n-2}-1}$ for slide $n-1$,
- page rank $\frac{2^{0}}{2^{n-1}+2^{n-2}-1}$ for slide n.

11. The following directed graph represents the given slide show:



The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}$.

We have

$$I - B = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Adding the first row to the second row, then the resulting second row to the third, etc. until row $n-1$ is added to row n, we obtain

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

We then multiply each row between 2 and $n-1$ by 2, and perform the following sequence of elementary row operations

- add row $n-1$ to the row $n-2$;
- add row $n-2$ to the row $n-3$;
- ...
- add the third row to the second row;
- add $\frac{1}{2}$ times the second row to the first row

to obtain the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

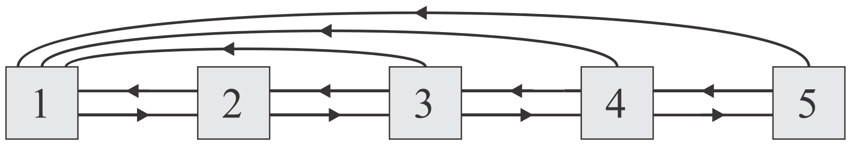
. The eigenspace associated with $\lambda = 1$ is span$\{(1,2,2,\ldots,2,2,1)\}$ therefore the normalized eigenvector in that eigenspace is $\frac{1}{2n-2}(1,2,2,\ldots,2,2,1)$.

The slides can be arranged in the order of the decreasing rank as follows:

- page rank $\frac{2}{2n-2} = \frac{1}{n-1}$ for slides 2 through $n-1$ (an $n-2$-way tie),
- page rank $\frac{1}{2n-2}$ for slides 1 and n.

(Note that the matrix B in this problem is not regular and the state vector sequence $\mathbf{x}^{(k)}$ does not converge. However, it could be shown that the fractional page count for this webgraph will approach $\frac{1}{2n-2}(1,2,2,\ldots,2,2,1)$.)

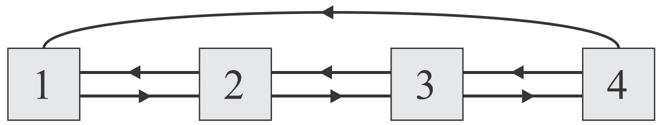
12. The following directed graph represents the given slide show:



The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ 1 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}$. Since $I - B =$ $\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{3} & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & -10.5 \\ 0 & 1 & 0 & 0 & -13 \\ 0 & 0 & 1 & 0 & -7.5 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, the eigenspace associated with $\lambda = 1$ is span$\{(10.5,13,7.5,3,1)\}$. The normalized eigenvector in that eigenspace is $\left(\frac{3}{10},\frac{13}{35},\frac{3}{14},\frac{3}{35},\frac{1}{35}\right)$. We obtain:

- page rank $\frac{13}{35} \approx 0.371$ for slide 2,
- page rank $\frac{3}{10} = 0.3$ for slide 1,
- page rank $\frac{3}{14} \approx 0.214$ for slide 3,
- page rank $\frac{3}{35} \approx 0.086$ for slide 4,
- page rank $\frac{1}{35} \approx 0.029$ for slide 5.

13. The following directed graph represents the given slide show:



The probability transition matrix is $B = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$. Since $I - B = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the eigenspace associated with $\lambda = 1$ is $\text{span}\{(2,3,2,1)\}$. The normalized eigenvector in that eigenspace is $\left(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}\right)$. We obtain:

- page rank $\frac{3}{8} = 0.375$ for slide 2,
- page rank $\frac{1}{4} = 0.25$ for slides 1 and 3 (a tie),
- page rank $\frac{1}{8} = 0.125$ for slide 4.

(Note that the matrix B in this problem is not regular and the state vector sequence $\mathbf{x}^{(k)}$ does not converge. However, it could be shown that the fractional page count for this webgraph will approach $\left(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}\right)$.)