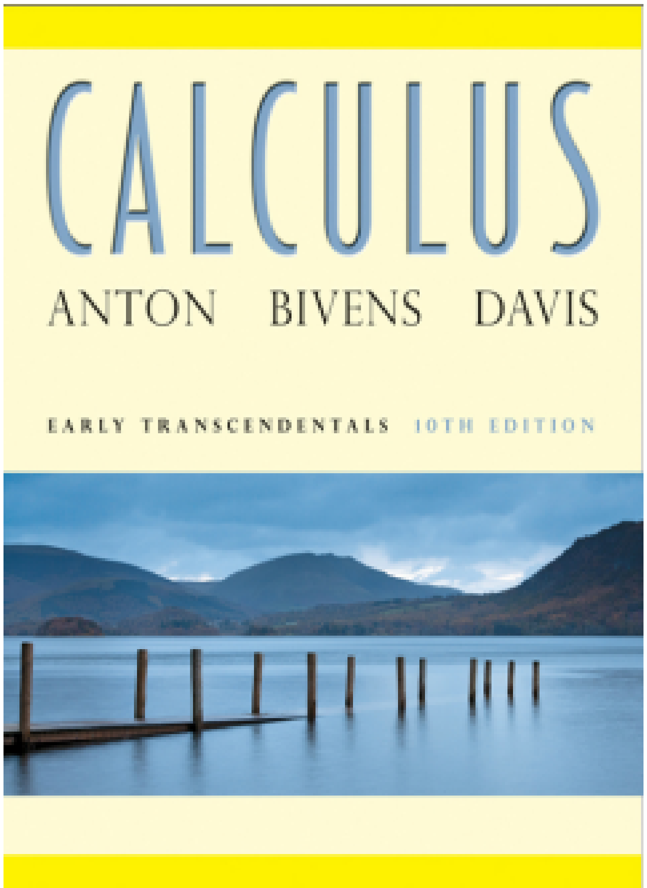

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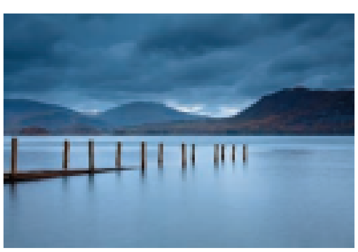

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HOWARD ANTON *Drexel University*

IRL BIVENS *Davidson College*

STEPHEN DAVIS *Davidson College*

CALC

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About HOWARD ANTON

Howard Anton obtained his B.A. from Lehigh University, his M.A. from the University of Illinois, and his Ph.D. from the Polytechnic University of Brooklyn, all in mathematics. In the early 1960s he worked for Burroughs Corporation and Avco Corporation at Cape Canaveral, Florida, where he was involved with the manned space program. In 1968 he joined the Mathematics Department at Drexel University, where he taught full time until 1983. Since that time he has been an Emeritus Professor at Drexel and has devoted the majority of his time to textbook writing and activities for mathematical associations. Dr. Anton was president of the EPADEL section of the Mathematical Association of America (MAA), served on the Board of Governors of that organization, and guided the creation of the student chapters of the MAA. He has published numerous research papers in functional analysis, approximation theory, and topology, as well as pedagogical papers. He is best known for his textbooks in mathematics, which are among the most widely used in the world. There are currently more than one hundred versions of his books, including translations into Spanish, Arabic, Portuguese, Italian, Indonesian, French, Japanese, Chinese, Hebrew, and German. His textbook in linear algebra has won both the Textbook Excellence Award and the McGuffey Award from the Textbook Author's Association. For relaxation, Dr. Anton enjoys traveling and photography.

About IRL BIVENS

Irl C. Bivens, recipient of the George Polya Award and the Merten M. Hasse Prize for Expository Writing in Mathematics, received his A.B. from Pfeiffer College and his Ph.D. from the University of North Carolina at Chapel Hill, both in mathematics. Since 1982, he has taught at Davidson College, where he currently holds the position of professor of mathematics. A typical academic year sees him teaching courses in calculus, topology, and geometry. Dr. Bivens also enjoys mathematical history, and his annual History of Mathematics seminar is a perennial favorite with Davidson mathematics majors. He has published numerous articles on undergraduate mathematics, as well as research papers in his specialty, differential geometry. He has served on the editorial boards of the MAA Problem Book series, the MAA Dolciani Mathematical Expositions series and *The College Mathematics Journal*. When he is not pursuing mathematics, Professor Bivens enjoys reading, juggling, swimming, and walking.

About STEPHEN DAVIS Stephen L. Davis received his B.A. from Lindenwood College and his Ph.D. from Rutgers University in mathematics. Having previously taught at Rutgers University and Ohio State University, Dr. Davis came to Davidson College in 1981, where he is currently a professor of mathematics. He regularly teaches calculus, linear algebra, abstract algebra, and computer science. A sabbatical in 1995–1996 took him to Swarthmore College as a visiting associate professor. Professor Davis has published numerous articles on calculus reform and testing, as well as research papers on finite group theory, his specialty. Professor Davis has held several offices in the Southeastern section of the MAA, including chair and secretary-treasurer and has served on the MAA Board of Governors. He is currently a faculty consultant for the Educational Testing Service for the grading of the Advanced Placement Calculus Exam, webmaster for the North Carolina Association of Advanced Placement Mathematics Teachers, and is actively involved in nurturing mathematically talented high school students through leadership in the Charlotte Mathematics Club. For relaxation, he plays basketball, juggles, and travels. Professor Davis and his wife Elisabeth have three children, Laura, Anne, and James, all former calculus students.

To
my wife Pat and my children: Brian, David, and Lauren

In Memory of
my mother Shirley
my father Benjamin
my thesis advisor and inspiration, George Bachman my benefactor in my time of need, Stephen Girard (1750–1831) —*HA*

To
my son Robert
—*IB*

To
my wife Elisabeth
my children: Laura, Anne, and James
—*SD*

PREFACE

This tenth edition of *Calculus* maintains those aspects of previous editions that have led to the series' success—we continue to strive for student comprehension without sacrificing mathematical accuracy, and the exercise sets are carefully constructed to avoid unhappy surprises that can derail a calculus class.

All of the changes to the tenth edition were carefully reviewed by outstanding teachers comprised of both users and nonusers of the previous edition. The charge of this committee was to ensure that all changes did not alter those aspects of the text that attracted users of the ninth edition and at the same time provide freshness to the new edition that would attract new users.

NEW TO THIS EDITION

- Exercise sets have been modified to correspond more closely to questions in *WileyPLUS*. In addition, more *WileyPLUS* questions now correspond to specific exercises in the text.
- New applied exercises have been added to the book and existing applied exercises have been updated.

- Where appropriate, additional skill/practice exercises were added.

OTHER FEATURES

Flexibility This edition has a built-in flexibility that is designed to serve a broad spectrum of calculus philosophies—from traditional to "reform." Technology can be emphasized or not, and the order of many topics can be permuted freely to accommodate each instructor's specific needs.

Rigor The challenge of writing a good calculus book is to strike the right balance between rigor and clarity. Our goal is to present precise mathematics to the fullest extent possible in an introductory treatment. Where clarity and rigor conflict, we choose clarity; however, we believe it to be important that the student understand the difference between a careful proof and an informal argument, so we have informed the reader when the arguments being presented are informal or motivational. Theory involving -δ arguments appears in separate sections so that they can be covered or not, as preferred by the instructor.

Rule of Four The "rule of four" refers to presenting concepts from the verbal, algebraic, visual, and numerical points of view. In keeping with current pedagogical philosophy, we used this approach whenever appropriate.

Visualization This edition makes extensive use of modern computer graphics to clarify concepts and to develop the student's ability to visualize mathematical objects, particularly those in 3-space. For those students who are working with graphing technology, there are

many exercises that are designed to develop the student's ability to generate and analyze mathematical curves and surfaces.

Quick Check Exercises Each exercise set begins with approximately five exercises (answers included) that are designed to provide students with an immediate assessment of whether they have mastered key ideas from the section. They require a minimum of computation and are answered by filling in the blanks.

Focus on Concepts Exercises Each exercise set contains a clearly identified group of problems that focus on the main ideas of the section.

Technology Exercises Most sections include exercises that are designed to be solved using either a graphing calculator or a computer algebra system such as *Mathematica*, *Maple*, or the open source program *Sage*. These exercises are marked with an icon for easy identification.

Applicability of Calculus One of the primary goals of this text is to link calculus to the real world and the student's own experience. This theme is carried through in the examples and exercises.

Career Preparation This text is written at a mathematical level that will prepare stu dents for a wide variety of careers that require a sound mathematics background, including engineering, the various sciences, and business.

Trigonometry Review Deficiencies in trigonometry plague many students, so we have included a substantial trigonometry review in Appendix B.

Appendix on Polynomial Equations Because many calculus students are weak in solving polynomial equations, we have included an appendix (Appendix C) that reviews the Factor Theorem, the Remainder Theorem, and procedures for finding rational roots.

Principles of Integral Evaluation The traditional Techniques of Integration is entitled "Principles of Integral Evaluation" to reflect its more modern approach to the material. The chapter emphasizes general methods and the role of technology rather than specific tricks for evaluating complicated or obscure integrals.

Historical Notes The biographies and historical notes have been a hallmark of this text from its first edition and have been maintained. All of the biographical materials have been distilled from standard sources with the goal of capturing and bringing to life for the student the personalities of history's greatest mathematicians.

Margin Notes and Warnings These appear in the margins throughout the text to clarify or expand on the text exposition or to alert the reader to some pitfall.

SUPPLEMENTS

The **Student Solutions Manual**, which is printed in two volumes, provides detailed solutions to the odd-numbered exercises in the text. The structure of the step-by-step solutions matches those of the worked examples in the textbook. The Student Solutions Manual is also provided in digital format to students in *WileyPLUS*.

Volume I (Single-Variable Calculus, Early Transcendentals) ISBN: 978-1-118-17381-7 Volume II (Multivariable Calculus, Early Transcendentals) ISBN: 978-1-118-17383-1

The Student Study Guide is available for download from the book companion Web site at www.wiley.com/college/anton or at www.howardanton.com and to users of *WileyPLUS*.

The **Instructor's Solutions Manual**, which is printed in two volumes, contains detailed solutions to all of the exercises in the text. The Instructor's Solutions Manual is also available in PDF format on the password-protected Instructor Companion Site at www.wiley.com/ college/anton or at www.howardanton.com and in *WileyPLUS*.

Volume I (Single-Variable Calculus, Early Transcendentals) ISBN: 978-1-118-17378-7 Volume II (Multivariable Calculus, Early Transcendentals) ISBN: 978-1-118-17379-4

The **Instructor's Manual** suggests time allocations and teaching plans for each section in the text. Most of the teaching plans contain a bulleted list of key points to emphasize. The discussion of each section concludes with a sample homework assignment. The Instructor's Manual is available in PDF format on the password-protected Instructor Companion Site at www.wiley.com/college/anton or at www.howardanton.com and in *WileyPLUS*.

The **Web Projects (Expanding the Calculus Horizon)** referenced in the text can also be downloaded from the companion Web sites and from *WileyPLUS*.

Instructors can also access the following materials from the book companion site or *WileyPLUS*:

- **Interactive Illustrations** can be used in the classroom or computer lab to present and explore key ideas graphically and dynamically. They are especially useful for display of three-dimensional graphs in multivariable calculus.
- The **Computerized Test Bank** features more than 4000 questions—mostly algorithmically generated—that allow for varied questions and numerical inputs.
- The **Printable Test Bank** features questions and answers for every section of the text.
- **PowerPoint lecture slides** cover the major concepts and themes of each section of the book.

Personal-Response System questions ("Clicker Questions") appear at the end of each PowerPoint presentation and provide an easy way to gauge classroom understanding.

- **Additional calculus content** covers analytic geometry in calculus, mathematical mod eling with differential equations and parametric equations, as well as an introduction to linear algebra.

WileyPLUS

WileyPLUS, Wiley's digital-learning environment, is loaded with all of the supplements listed on the previous page, and also features the following:

- **Homework management tools**, which easily allow you to assign and grade algorithmic questions, as well as gauge student comprehension.
- **Algorithmic** questions with randomized numeric values and an answer-entry palette for symbolic notation are provided online though *WileyPLUS*. Students can click on "help" buttons for hints, link to the relevant section of the text, show their work or query their instructor using a white board, or see a step-by-step solution (depending on instructor selecting settings).
- **Interactive Illustrations** can be used in the classroom or computer lab, or for student practice.
- **QuickStart** predesigned reading and homework assignments. Use them as-is or customize them to fit the needs of your classroom.
- The **e-book**, which is an exact version of the print text but also features hyperlinks to questions, definitions, and supplements for quicker and easier support.
- **Guided Online (GO) Tutorial Exercises** that prompt students to build solutions step by step. Rather than simply grading an exercise answer as wrong, GO tutorial problems show students precisely where they are making a mistake.
- **Are You Ready?** quizzes gauge student mastery of chapter concepts and techniques and provide feedback on areas that require further attention.
- **Algebra and Trigonometry Refresher** quizzes provide students with an opportunity to brush up on the material necessary to master calculus, as well as to determine areas that require further review.

WileyPLUS. Learn more at www.wileyplus.com.

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It has been our good fortune to have the advice and guidance of many talented people whose knowledge and skills have enhanced this book in many ways. For their valuable help we thank the following people.

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WEB APPENDICES (online only)

Available for download at www.wiley.com/college/anton or at www.howardanton.com and in *WileyPLUS*.

E REAL NUMBERS, INTERVALS, AND INEQUALITIES

F ABSOLUTE VALUE

G COORDINATE PLANES, LINES, AND LINEAR FUNCTIONS
H DISTANCE, CIRCLES, AND QUADRATIC EQUATIONS
I EARLY PARAMETRIC EQUATIONS OPTION
J MATHEMATICAL MODELS
K THE DISCRIMINANT
L SECOND-ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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BLAMMO THE HUMAN CANNONBALL
COMET COLLISION
HURRICANE MODELING
ITERATION AND DYNAMICAL SYSTEMS
RAILROAD DESIGN
ROBOTICS

THE ROOTS OF CALCULUS

Today's exciting applications of calculus have roots that can be traced to the work of the Greek mathematician Archimedes, but the actual discovery of the fundamental principles of cal culus was made independently by Isaac Newton (English) and Gottfried Leibniz (German) in the late seventeenth century. The work of Newton and Leibniz was motivated by four major classes of scientific and mathematical problems of the time:

- Find the tangent line to a general curve at a given point.
- Find the area of a general region, the length of a general curve, and the volume of a general solid.
- Find the maximum or minimum value of a quantity—for example, the maximum and minimum distances of a planet from the Sun, or the maximum range attainable for a pro jectile by varying its angle of fire.
- Given a formula for the distance traveled by a body in any specified amount of time, find the velocity and acceleration of the body at any instant. Conversely, given a formula that specifies the acceleration of velocity at any instant, find the distance traveled by the body in a specified period of time.

Newton and Leibniz found a fundamental relationship be tween the problem of finding a tangent line to a curve and the problem of determining the area of a region. Their real ization of this connection is considered to be the "discovery of calculus." Though Newton saw how these two problems are related ten years before Leibniz did, Leibniz published his work twenty years before Newton. This situation led to a stormy debate over who was the rightful discoverer of calculus. The debate engulfed Europe for half a century, with the scien tists of the European continent supporting Leibniz and those from England supporting Newton. The conflict was extremely unfortunate because Newton's inferior notation badly ham pered scientific development in England, and the Continent in turn lost the benefit of Newton's discoveries in astronomy and physics for nearly fifty years. In spite of it all, Newton and Leibniz were sincere admirers of each other's work.

[*Image: Public domain image from http://commons.wikimedia.org/ wiki/File:Hw-newton.jpg. Image provided courtesy of the University of Texas Libraries, The University of Texas at Austin.*]

ISAAC NEWTON (1642–1727)

Newton was born in the village of Woolsthorpe, England. His father died before he was born and his mother raised him on the family farm. As a youth he showed little evidence of his later brilliance, except for an unusual talent with mechanical devices—he apparently built a working water clock and a toy flour mill powered by a mouse. In 1661 he entered Trinity College in Cambridge with a deficiency in geometry. Fortunately, Newton caught the eye of Isaac Barrow, a gifted mathematician and teacher. Under Barrow's guidance Newton immersed himself in mathematics and science, but he graduated without any special distinction. Because the bubonic plague was spreading rapidly through London, Newton returned to his home in Woolsthorpe and stayed there during the years of 1665 and 1666. In those two momentous years the entire framework of modern science was miraculously created in Newton's mind. He discovered calculus, recognized the underlying principles of planetary motion and gravity, and determined that "white" sunlight was composed of all colors, red to violet. For whatever reasons he kept his discoveries to himself. In 1667 he returned to Cambridge to obtain his Master's degree and upon graduation became a teacher at Trinity. Then in 1669 Newton succeeded his teacher, Isaac Barrow, to the Lucasian chair of mathematics at Trinity, one of the most honored chairs of mathematics in the world. Thereafter, brilliant discoveries flowed from Newton steadily. He formulated the law of gravitation and used it to explain the motion of the moon, the planets, and the tides; he formulated basic theories of light, thermodynamics, and hydrodynamics; and he devised and constructed the first modern reflecting telescope. Throughout his life Newton was hesitant to publish his major discoveries, revealing them only to a select

GEOMETRY FORMULAS

A = area, S = lateral surface area, V = volume, h = height, B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram
Triangle Trapezoid Circle Sector r

h

$A = bh$

h

$A = bh \frac{1}{2}$

h

$A = (a + b)h \frac{1}{2}$

r

$A = pr^2$, $C = 2pr$

s u

r
1
$A = r^2u$, $s = ru$
2
(u in radians)

Right Circular Cylinder Right Circular Cone Any Cylinder or Prism with Parallel Bases Sphere

r

l

r

h

B

B

4

h

h

1

h

$V = pr^2h$, $S = 2prh$ $V = pr^3$, $S = 4pr^2$

$V = Bh$

THE QUADRATIC

$V = pr^2h$, $S = prl$

3

3

ALGEBRA FORMULAS

FORMULA THE BINOMIAL FORMULA

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = -b \pm$
$\sqrt{b^2 - 4ac}$
$2a$

$1 \cdot 2 \cdot 3$ $x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$
$(x - y)^n = x^n - nx^{n-1}y + n(n-1)$

$1 \cdot 2$ $x^{n-2}y^2 - n(n-1)(n-2)$
$1 \cdot 2 \cdot 3$ $x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

BASIC FUNCTIONS

$(x + y)^n = x^n + nx^{n-1}y + n(n-1)$
$1 \cdot 2$ $x^{n-2}y^2 + n(n-1)(n-2)$

TABLE OF INTEGRALS

$n + 1 + C$ $\ln a + C$ $u = \ln|u| + C$

1. $u^n\, du = u^{n+1}$

10. $a^u\, du = a^u$

2. du

11. $\ln u\,du = u \ln u - u + C$

3. 4. 5. 6. $\sin u\,du = -\cos u + C$ $\cos u\,du = \sin u + C$ $\tan u\,du = \ln|\sec u| + C$ $e^u\,du = e^u + C$

12. 13. $\csc u\,du = \ln|\csc u - \cot u| + C = \ln|\tan \frac{1}{2}u| + C$ $\sec u\,du = \ln|\sec u + \tan u| + C = \ln|\tan \frac{1}{4}\pi + \frac{1}{2}u| + C$

14. $\cot u\,du = \ln|\sin u| + C$

7. 8. 9. $\sin^{-1} u\,du = u \sin^{-1} u + \sqrt{1 - u^2} + C$ $\cos^{-1} u\,du = u \cos^{-1} u - \sqrt{1 - u^2} + C$ $\tan^{-1} u\,du = u \tan^{-1} u - \ln\sqrt{1 + u^2} + C$

15. 16. 17. $\cot^{-1} u\,du = u \cot^{-1} u + \ln\sqrt{1 + u^2} + C$ $\sec^{-1} u\,du = u \sec^{-1} u - \ln|u + \sqrt{u^2 - 1}| + C$ $\csc^{-1} u\,du = u \csc^{-1} u + \ln|u + \sqrt{u^2 - 1}| + C$

RECIPROCALS OF BASIC FUNCTIONS

18. 19. 20. 21.

$\frac{1}{1 \pm \sin u}\,du = \tan u \mp \sec u + C$

$\frac{1}{1 \pm \cos u}\,du = -\cot u \pm \csc u + C$

$\frac{1}{1 \pm \tan u}\,du = \frac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

$\frac{1}{\sin u \cos u}\,du = \ln|\tan u| + C$

22. 23. 24. 25.

$\frac{1}{1 \pm \cot u}\,du = \frac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

$\frac{1}{1 \pm \sec u}\,du = u + \cot u \mp \csc u + C$

$\frac{1}{1 \pm \csc u}\,du = u - \tan u \pm \sec u + C$

$\frac{1}{1 \pm e^u}\,du = u - \ln(1 \pm e^u) + C$

POWERS OF TRIGONOMETRIC FUNCTIONS

26. 27. 28. 29.

$\sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

$\cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

$\tan^2 u\,du = \tan u - u + C$

$\sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\sin^{n-2} u\,du$

32. 33. 34. 35.

$\cot^2 u\,du = -\cot u - u + C$

$\sec^2 u\,du = \tan u + C$

$\csc^2 u\,du = -\cot u + C$

$\cot^n u\,du = -\frac{1}{n-1}\cot^{n-1} u - \cot^{n-2} u\,du$

30. $\cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\cos^{n-2} u\,du$

31. $\tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \tan^{n-2} u\,du$

$\sec^n u\,du = \frac{1}{n-1}\sec^{n-2} u \tan u + \frac{n-2}{n-1}\sec^{n-2} u\,du$

36. $\csc^n u\,du = -\frac{1}{n-1}$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS

37.

$n-1$ $\csc^{n-2} u \cot u + n-2$ $n-$ $\csc^{n-2} udu$

1

$2(m+n)$ $+\sin(m-n)u$

$2(m+n)$ $-\cos(m-n)u$

38.

$-\sin(m+n)u$ $\cos mu$ $n)u$

$\sin mu \sin nu\, du =$ $\cos nu\, du = \sin(m+$ $2(m-n) + C$

39.

$-n) + C$

$2(m+n)$ $+\sin(m-n)u$ $2(m$

40.

$-\cos(m+n)u$ $\sin^m u$ $\cos^{n+1} u$

$\cos^n udu = -\sin^{m-1} u$ $2(m-n) + C$

$\sin mu \cos nu\, du =$

41.

$m+n+m-1$ $m+n$

$\sin^{m-2} u \cos^n udu$

$= \sin^{m+1} u \cos^{n-1} u$

$\sin^m u \cos^{n-2} udu$

$m+n+n-1$

$m+n$

PRODUCTS OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

42.

$e^{au} \sin bu\, du = e^{au}$

$a^2 + b^2$ $(a \sin bu - b \cos bu) + C$ 43.

$e^{au} \cos bu\, du = e^{au}$

$a^2 + b^2$ $(a \cos bu + b \sin bu) + C$

$ue^u du = e^u(u-1) + C$

POWERS OF u MULTIPLYING OR DIVIDING BASIC FUNCTIONS

44. 45. 46.

$u^2 \sin udu = 2u \sin u + (2-u^2)$

$\cos u + C$

$u^n a^u du = u^n a^u$

$u \sin udu = \sin u - u \cos u + C$

51. 52. 53.

$u \cos udu = \cos u + u \sin u + C$

$\ln a -$ $\ln a$ n $u^{n-1} a^u du + C$

$u^n e^u du = u^n e^u - n u^{n-1} e^u du$

47.

$u^2 \cos udu = 2u \cos u +$

$(u^2 - 2)\sin u + C$

54.

u^n $= -e^u$

$(n-1)u^{n-1}$ $+1$

$e^u du$

$e^u du$ u^{n-1}

$u^{n-1} \cos udu$

u^n $= -a^u$

$a^u du$

$u^n \sin udu = -u^n \cos u + n$

$a^u du$

$n-1$

48.

$u^n \cos udu = u^n \sin u - n$ 55.

$(n-1)u^{n-1}$ $+\ln a$ $n-1$ u^{n-1}

$u^{n-1} \sin udu$

du

49. 50.

$u^n \ln udu = u^{n+1}$

$u \ln u = \ln |\ln u| + C$

56.

$(n+1)^2$ $[(n+1)\ln u - 1] + C$

POLYNOMIALS MULTIPLYING BASIC FUNCTIONS

57. 58. 59.

$p(u)e^{au} du = \frac{1}{a} p(u)e^{au} - \frac{1}{a^2} p'(u)e^{au} + \frac{1}{a^3} p''(u)e^{au} - \cdots$ [signs alternate: $+-+-\cdots$]

$p(u)\sin au\, du = -\frac{1}{a} p(u) \cos au + \frac{1}{a^2} p'(u)\sin au + \frac{1}{a^3} p''(u) \cos au - \cdots$ [signs alternate in pairs after first term: $++--++--\cdots$]

$p(u) \cos au\, du = \frac{1}{a} p(u)\sin au + \frac{1}{a^2} p'(u) \cos au - \frac{1}{a^3} p''(u)\sin au - \cdots$ [signs alternate in pairs: $++--++--\cdots$]

FOR THE STUDENT

Calculus provides a way of viewing and analyzing the physi cal world. As with all mathematics courses, calculus involves equations and formulas. However, if you successfully learn to use all the formulas and solve all of the problems in the text but do not master the underlying *ideas*, you will have missed the most important part of calculus. If you master these ideas, you will have a widely applicable tool that goes far beyond textbook exercises.

Before starting your studies, you may find it helpful to leaf through this text to get a general feeling for its different parts:

- The opening page of each chapter gives you an overview of what that chapter is about, and the opening page of each section within a chapter gives you an overview of what that section is about. To help you locate specific information, sections are subdivided into topics that are marked with a box like this .

- Each section ends with a set of exercises. The answers to most odd-numbered exercises appear in the back of the book. If you find that your answer to an exercise does not match that in the back of the book, do not assume immedi ately that yours is incorrect—there may be more than one way to express the answer. For example, if your answer is

$\sqrt{2}/2$ and the text answer is $1/\sqrt{2}$, then both are correct since your answer can be obtained by "rationalizing" the text answer. In general, if your answer does not match that in the text, then your best first step is to look for an algebraic manipulation or a trigonometric identity that might help you determine if the two answers are equivalent. If the answer is in the form of a decimal approximation, then your answer might differ from that in the text because of a difference in the number of decimal places used in the computations.

- The section exercises include regular exercises and four special categories: *Quick Check*, *Focus on Concepts*, *True/False*, and *Writing*.
  - The *Quick Check* exercises are intended to give you quick feedback on whether you understand the key ideas in the section; they involve relatively little computation, and have answers provided at the end of the exercise set.
  - The *Focus on Concepts* exercises, as their name suggests, key in on the main ideas in the section.
  - *True/False* exercises focus on key ideas in a different way. You must decide whether the statement is true in *all possible circumstances*, in which case you would declare it to be "true," or whether there are some circumstances in which it is not true, in which case you would declare it to be "false." In each such exercise you are asked to "Explain your answer." You might do this by noting a theorem in the text that shows the statement to be true or by finding a particular example in which the statement is not true.
  - *Writing* exercises are intended to test your ability to ex plain mathematical ideas in words rather than relying solely on numbers and symbols. All exercises requiring writing should be answered in complete, correctly punc tuated logical sentences—not with fragmented phrases and formulas.

- Each chapter ends with two additional sets of exercises: *Chapter Review Exercises*, which, as the name suggests, is a select set of exercises that provide a review of the main concepts and techniques in the chapter, and *Making Con nections*, in which exercises require you to draw on and combine various ideas developed throughout the chapter.

- Your instructor may choose to incorporate technology in your calculus course. Exercises whose solution involves the use of some kind of technology are tagged with icons to alert you and your instructor. Those exercises tagged with the icon require graphing technology—either a graphing calculator or a computer program that can graph equations. Those exercises tagged with the icon C require a com puter algebra system (CAS) such as *Mathematica*, *Maple*, or available on some graphing calculators.

- At the end of the text you will find a set of four appen dices covering various topics such as a detailed review of trigonometry and graphing techniques using technology. Inside the front and back covers of the text you will find endpapers that contain useful formulas.

- The ideas in this text were created by real people with in teresting personalities and backgrounds. Pictures and bio graphical sketches of many of these people appear through out the book.

- Notes in the margin are intended to clarify or comment on important points in the text.

A Word of Encouragement

As you work your way through this text you will find some ideas that you understand immediately, some that you don't understand until you have read them several times, and others that you do not seem to understand, even after several readings. Do not become discouraged—some ideas are intrinsically dif ficult and take time to "percolate." You may well find that a hard idea becomes clear later when you least expect it.

Web Sites for this Text

www.antontextbooks.com

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The development of calculus in the seventeenth and eighteenth centuries was motivated by the need to understand physical phenomena such as the tides, the phases of the moon, the nature of *light, and gravity.*

FUNCTIONS

BEFORE CALCULUS

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a "function," which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus, including polynomials, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

In this section we will define and develop the concept of a "function," which is the basic mathematical object that scientists and mathematicians use to describe relationships between variable quantities. Functions play a central role in calculus and its applications.

DEFINITION OF A FUNCTION

Many scientific laws and engineering principles describe how one quantity depends on another. This idea was formalized in 1673 by Gottfried Wilhelm Leibniz (see p. xx) who coined the term *function* to indicate the dependence of one quantity on another, as described in the following definition.

0.1.1 definition If a variable y depend
value of x determines exactly one valu

Four common methods for representing functions are:

- Numerically by tables • Geometrically by graphs
- Algebraically by formulas • Verbally

Table 0.1.1
indianapolis 500
qualifying speeds
year t speed S
(mi/h)

The method of representation often depends on how the function arises. For example:

• Table 0.1.1 shows the top qualifying speed S for the Indianapolis 500 auto race as a function of the year t. There is exactly one value of S for each value of t. • Figure 0.1.1 is a graphical record of an earthquake recorded on a seismograph. The

1

1994	1998	2002
1995	1999	2003
1996	2000	2004
1997	2001	2005

2006
2007
2008
2009
2010
2011

D
228.011 231.604 233.100 218.263 223.503
225.179 223.471 226.037 231.342 231.725
222.024 227.598 228.985 225.817 226.366
224.864 227.970 227.472

graph describes the deflection D of the seismograph needle as a function of the time T elapsed since the wave left the earthquake's epicenter. There is exactly one value of D for each value of T.

• Some of the most familiar functions arise from formulas; for example, the formula $C = 2\pi r$ expresses the circumference C of a circle as a function of its radius$r$. There is exactly one value of C for each value of r.

• Sometimes functions are described in words. For example, Isaac Newton's Law of Universal Gravitation is often stated as follows: The gravitational force of attraction between two bodies in the Universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This is the verbal description of the formula

$$F = G\frac{m_1 m_2}{r^2}$$

in which F is the force of attraction, m_1 and m_2 are the masses, r is the distance be tween them, and G is a constant. If the masses are constant, then the verbal description defines F as a function of r. There is exactly one value of F for each value of r.

earthquake shock
Arrival of
P-waves
9.4
Arrival of S-waves
Surface waves
Time of
11.8
minutes
minutes
minutes
Time in

0 10 20 30 40 50 60 70 80 T

Figure 0.1.1

f
Computer
Program
Input x Output y

In the mid-eighteenth century the Swiss mathematician Leonhard Euler (pronounced "oiler") conceived the idea of denoting functions by letters of the alphabet, thereby making it possible to refer to functions without stating specific formulas, graphs, or tables. To understand Euler's idea, think of a function as a computer program that takes an *input* x, operates on it in some way, and produces exactly one *output* y. The computer program is an object in its own right, so we can give it a name, say f. Thus, the function f (the computer program) associates a unique output y with each input x (Figure 0.1.2). This suggests the following definition.

Figure 0.1.2

W (pounds) Weight
225
200
175
150
125
100
75
50
10 15 20 25 30 Age A (years)

Figure 0.1.3

0.1.2
input

In this definition the term *unique* means "exactly one." Thus, a function cannot assign two different outputs to the same input. For example, Figure 0.1.3 shows a plot of weight versus age for a random sample of 100 college students. This plot does *not* describe W as a function of A because there are some values of A with more than one corresponding value of W. This is to be expected, since two people with the same age can have different weights.

INDEPENDENT AND DEPENDENT VARIABLES

For a given input x, the output of a function f is called the ***value*** of f at x or the ***image*** of x under f. Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x; the variable x is called the ***independent variable*** (or ***argument***) of f, and the variable y is called the ***dependent variable*** of f. This terminology is intended to suggest that x is free to vary, but that once x has a specific value a corresponding value of y is determined. For now we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a ***real-valued function of a real variable***. Later, we will consider other kinds of functions.

Table 0.1.2 Example 1 Table 0.1.2 describes a functional relationship $y = f(x)$ for which

x y

0 3

1

4

2

−1

$f(0) = 3$

3

$f(1) = 4$

6

$f(2) = -1$

$f(3) = 6$

Example 2

The equation

f associates y

f associates y $y = 3x^2 - 4x + 2$

f associates y

f associates $y = 6$ with $x = 3$.

has the form $y = f(x)$ in which the function f is given by the formula $f(x) = 3x^2 - 4x + 2$

Leonhard Euler (1707–1783) Euler was probably the most prolific mathematician who ever lived. It has been said that "Euler wrote mathematics as effortlessly as most men breathe." He was born in Basel, Switzerland, and was the son of a Protestant minister who had himself studied mathematics. Euler's genius developed early. He attended the University of Basel, where by age 16 he obtained both a Bachelor of Arts degree and a Master's degree in philosophy. While at Basel, Euler had the good fortune to be tutored one day a week in mathematics by a distinguished mathematician, Johann Bernoulli. At the urging of his father, Euler then began to study theology. The lure of mathematics was too great, however, and by age 18 Euler had begun to do mathematical research. Nevertheless, the influence of his father and his theological studies remained, and throughout his life Euler was a deeply religious, unaffected person. At various times Euler taught at St. Petersburg Academy of Sciences (in Rus sia), the University of Basel, and the Berlin Academy of Sciences. Euler's energy and capacity for work were virtually boundless. His collected works form more than 100 quarto-sized volumes and it is believed that much of his work has been lost. What is astonishing is that Euler was blind for the last 17 years of his life, and this was one of his most productive periods! Euler's flawless memory was phenomenal. Early in his life he memorized the entire *Aeneid* by Virgil, and at age 70 he could not only recite the entire work but could also state the first and last sentence on each page of the book from which he memorized the work. His ability to particularly solve problems in his head was beyond belief. He worked out in his head major problems of lunar motion that baffled Isaac Newton and once did a complicated calculation in his head to settle an argument between two students whose computations differed in the fiftieth decimal place.

Following the development of calculus by Leibniz and Newton, results in mathematics developed rapidly in a disorganized way. Eu ler's genius gave coherence to the mathematical landscape. He was the first mathematician to bring the full power of calculus to bear on problems from physics. He made major contributions to virtu ally every branch of mathematics as well as to the theory of optics, planetary motion, electricity, magnetism, and general mechanics.

[Image: http://commons.wikimedia.org/wiki/File:Leonhard_Euler_by_Handmann_.png]

For each input x, the corresponding output y is obtained by substituting x in this formula. For example,

$$f(0) = 3(0)^2 - 4(0) + 2 = 2$$

$$f(-1.7) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$$

$$f(\sqrt{2}) = 3(\sqrt{2})^2 - 4\sqrt{2} + 2 = 8 - 4\sqrt{2}$$

GRAPHS OF FUNCTIONS

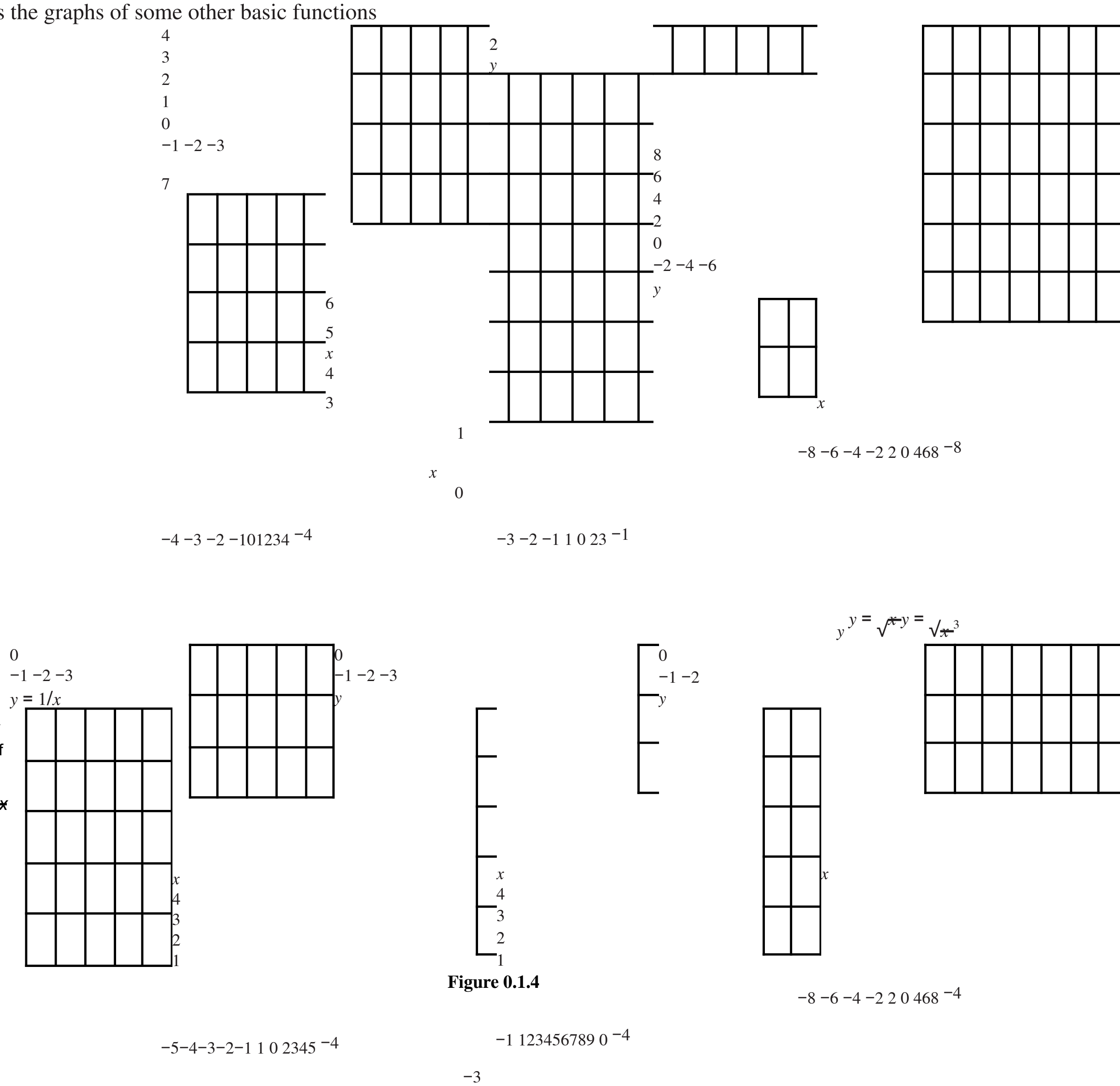
f associates $y = 2$ with $x = 0$.

f associates $y = 17.47$ with $x = -1.7$.

f associates $y = 8 - 4\sqrt{2}$ with $x = \sqrt{2}$.

Figure 0.1.4 shows only portions of the graphs. Where appropriate, and unless indicated otherwise, it is understood that graphs shown in this text extend indefinitely beyond the boundaries of the displayed figure.

If f is a real-valued function of a real variable, then the ***graph*** of f in the *xy*-plane is defined to be the graph of the equation $y = f(x)$. For example, the graph of the function $f(x) = x$ is the graph of the equation $y = x$, shown in Figure 0.1.4. That figure also shows the graphs of some other basic functions that may already be familiar to you. In Appendix A we discuss techniques for graphing functions using graphing technology.



Since $\sqrt{x}$ is imaginary for negative values of x, there are no points on the graph of $y = \sqrt{x}$ in the region where $x < 0$.

4
3
2
1

Figure 0.1.4

y

$(x, f(x))$

$f(x)$ $y = f(x)$

x

x

Graphs can provide valuable visual information about a function. For example, since the graph of a function f in the *xy*-plane is the graph of the equation $y = f(x)$, the points on the graph of f are of the form $(x, f(x))$; that is, *the y-coordinate of a point on the graph of f is the value of f at the corresponding x-coordinate* (Figure 0.1.5). The values of x for which $f(x) = 0$ are the x-coordinates of the points where the graph of f intersects the x-axis (Figure 0.1.6). These values are called the ***zeros*** of f, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of the graph of $y = f(x)$.

Figure 0.1.5 The y-coordinate of a point on the graph of $y = f(x)$ is the value of f at the corresponding x-coordinate.

THE VERTICAL LINE TEST

Not every curve in the xy-plane is the graph of a function. For example, consider the curve in Figure 0.1.7, which is cut at two distinct points, (a, b) and (a, c), by a vertical line. This curve cannot be the graph of $y = f(x)$ for any function f; otherwise, we would have

$$f(a) = b \text{ and } f(a) = c$$

y

$y = f(x)$

x

x_1 0 x_2 x_3

which is impossible, since f cannot assign two different values to a. Thus, there is no function f whose graph is the given curve. This illustrates the following general result, which we will call the ***vertical line test***.

0.1.3 the vertical line test *A curve in the and only if no vertical line intersects the c*

Figure 0.1.6 f has zeros at x_1, 0, x_2, and x_3.

y

(a, c)

(a, b) a

Figure 0.1.7 This curve cannot be the graph of a function.

Example 3 The graph of the equation

$$x^2 + y^2 = 25$$

is a circle of radius 5 centered at the origin and hence there are vertical lines that cut the graph more than once (Figure 0.1.8). Thus this equation does not define y as a function of x.

THE ABSOLUTE VALUE FUNCTION

x

Recall that the ***absolute value*** or ***magnitude*** of a real number x is defined by

$$|x| = x,\ x \geq 0\ -x,\ x < 0$$

Symbols such as $+x$ and $-x$ are de ceptive, since it is tempting to conclude that $+x$ is positive and $-x$ is negative. However, this need not be so, since x itself can be positive or negative. For example, if x is negative, say $x = -3$, then $-x = 3$ is positive and $+x = -3$ is negative.

y

The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative. Thus, $|5| = 5$, $\left|-\frac{4}{7}\right| = \frac{4}{7}$, $|0| = 0$

A more detailed discussion of the properties of absolute value is given in Web Appendix F. However, for convenience we provide the following summary of its algebraic properties.

0.1.4 properties of absolute value *I*

$|-a|=|a|$ A number and its negative have the san

value of a product is the product of the absolute valu

of a ratio is the ratio of the absolute values. (d) $|a$

$x^2 + y^2 = 25$

6

x

−6 6

−6

Figure 0.1.8

The graph of the function $f(x) = |x|$ can be obtained by graphing the two parts of the equation

$$y = x,\ x \geq 0\ -x,\ x < 0$$

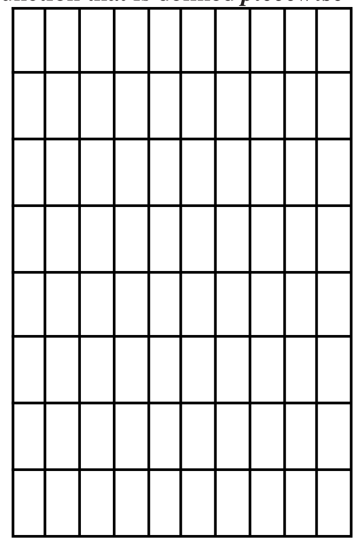
separately. Combining the two parts produces the V-shaped

WARNING

To denote the negative square root you must write $-\sqrt{x}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, whereas the negative square root of 9 is $-\sqrt{9} = -3$. (Do not make the mis take of writing $\sqrt{9} = \pm 3$.)

graph in Figure 0.1.9. Absolute values have important relationships to square roots. To see why this is so, recall from algebra that every positive real number x has two square roots, one positive and one negative. By definition, the symbol $\sqrt{x}$ denotes the *positive* square root of x. Care must be exercised in simplifying expressions of the form $\sqrt{x^2}$, since it is *not* always true that $\sqrt{x^2} = x$. This equation is correct if x is nonnegative, but it is false if x is negative. For example, if $x = -4$, then

The absolute value function $f(x) = |x|$ is an example of a function that is defined ***piecewise***



$$\sqrt{x^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq x$$

4
3
2
1
x
0

TECHNOLOGY MASTERY

Verify (1) by using a graphing utility to show that the equations $y = \sqrt{x^2}$ and $y = |x|$ have the same graph.

in the sense that the formula for f changes, depending on the value of x.

y y = |x|
5

A statement that is correct for all real values of x is

$$\sqrt{x^2} = |x| \quad (1)$$

Example 4 Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \le -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \ge 1 \end{cases}$$

PIECEWISE-DEFINED FUNCTIONS

−1
−2

−5 −4 −3 −2 −1 1 0 2345 −3

Figure 0.1.9 2

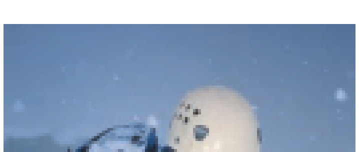
1
y

x

Solution. The formula for f changes at the points $x = -1$ and $x = 1$. (We call these the ***breakpoints*** for the formula.) A good procedure for graphing functions defined piecewise is to graph the function separately over the open intervals determined by the breakpoints, and then graph f at the breakpoints themselves. For the function f in this example the graph is the horizontal ray $y = 0$ on the interval $(-\infty, -1]$, it is the semicircle $y = \sqrt{1-x^2}$ on the interval $(-1, 1)$, and it is the ray $y = x$ on the interval $[1, +\infty)$. The formula for f specifies that the equation $y = 0$ applies at the breakpoint -1 [so $y = f(-1) = 0$], and it specifies that the equation $y = x$ applies at the breakpoint 1 [so $y = f(1) = 1$]. The graph of f is shown in Figure 0.1.10.

−2 12 −1 **Figure 0.1.10**

REMARK In Figure 0.1.10 the solid dot and open circle at the breakpoint $x = 1$ serve to emphasize that the point on the graph lies on the ray and not the semicircle. There is no ambiguity at the breakpoint $x = -1$ because the two parts of the graph join together continuously there.



Example 5 Increasing the speed at which air moves over a person's skin increases the rate of moisture evaporation and makes the person feel cooler. (This is why we fan

ourselves in hot weather.) The ***wind chill index*** is the temperature at a wind speed of 4 mi/h that would produce the same sensation on exposed skin as the current temperature and wind speed combination. An empirical formula (i.e., a formula based on experimental data) for the wind chill index W at 32°F for a wind speed of v mi/h is

$$W = \begin{cases} 55.628 - 22.07v^{0.16}, & 3 < v \\ 32, & 0 \le v \le 3 \end{cases}$$

A computer-generated graph of $W(v)$ is shown in Figure 0.1.11.

The wind chill index measures the sensation of coldness that we feel from the combined effect of temperature and wind speed.

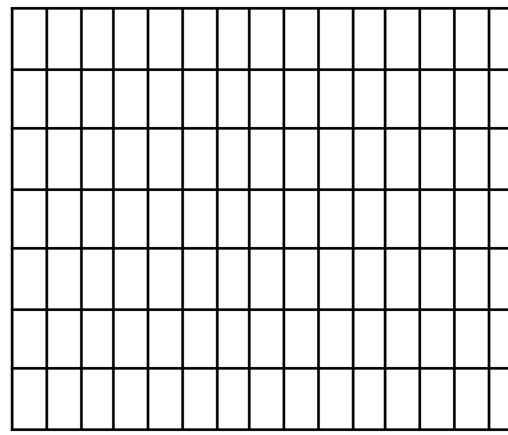


Figure 0.1.11 Wind chill versus wind speed at 32°F

DOMAIN AND RANGE

If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the ***domain*** of f, and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f. For example, if f is the function defined by the table in Example 1, then the domain is the set $\{0, 1, 2, 3\}$ and the range is the set $\{-1, 3, 4, 6\}$.

Sometimes physical or geometric considerations impose restrictions on the allowable inputs of a function. For example, if y denotes the area of a square of side x, then these variables are related by the equation $y = x^2$. Although this equation produces a unique value of y for every real number x, the fact that lengths must be nonnegative imposes the requirement that $x \ge 0$.

One might argue that a physical square cannot have a side of length zero. However, it is often convenient mathe matically to allow zero lengths, and we will do so throughout this text where appropriate.

When a function is defined by a mathematical formula, the formula itself may impose restrictions on the allowable inputs. For example, if $y = 1/x$, then $x = 0$ is not an allowable input since division by zero is undefined, and if $y = \sqrt{x}$, then negative values of x are not allowable inputs because they produce imaginary values for y and we have agreed to consider only real-valued functions of a real variable. In general, we make the following definition.

y

0.1.5 definition If a real and if no domain is sta consists of all real numb the ***natural domain*** of th

The domain and range of a function f can be pictured by projecting the graph of $y = f(x)$ onto the coordinate axes as shown in Figure 0.1.12.

Range

$y = f(x)$

Domain

Figure 0.1.12 The projection of $y = f(x)$ on the x-axis is the set of allowable x-values for f, and the projection on the y-axis is the set of corresponding y-values.

Example 6 Find the natural domain of

(a) $f(x) = x^3$ (b) $f(x) = 1/[(x - 1)(x - 3)]$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

For a review of trigonometry see Appendix B.

Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Solution (d). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x - 3)(x - 2) \geq 0$$

This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty)$$

y $y = x^2$

In some cases we will state the domain explicitly when defining a function. For example, if $f(x) = x^2$ is the area of a square of side x, then we can write

$$f(x) = x^2, \ x \geq 0$$

to indicate that we take the domain of f to be the set of nonnegative real numbers (Figure 0.1.13).

y $y = x^2, x \geq 0$

x

Figure 0.1.13

THE EFFECT OF ALGEBRAIC OPERATIONS ON THE DOMAIN

Algebraic expressions are frequently simplified by canceling common factors in the nu merator and denominator. However, care must be exercised when simplifying formulas for functions in this way, since this process can alter the domain.

6
5
4
3
2
1
x

y

$y = x + 2$

x

Example 7 The natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2} \quad (2)$$

consists of all real x except $x = 2$.

However, if we factor the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2 \quad (3)$$

Since the right side of (3) has a value of $f(2) = 4$ and $f(2)$ was undefined in (2), the algebraic simplification has changed the function. Geometrically, the graph of (3) is the line in Figure 0.1.14*a*, whereas the graph of (2) is the same line but with a hole at $x = 2$, since the function is undefined there (Figure 0.1.14*b*). In short, the geometric effect of the algebraic cancellation is to eliminate the hole in the original graph.

−3−2−1 12345 (*a*)

y

6 5 4 3 2 1

$y = \frac{x^2 - 4}{x - 2}$ x

−3−2−1 12345

(*b*)

Figure 0.1.14

Sometimes alterations to the domain of a function that result from algebraic simplification are irrelevant to the problem at hand and can be ignored. However, if the domain must be preserved, then one must impose the restrictions on the simplified function explicitly. For example, if we wanted to preserve the domain of the function in Example 7, then we would have to express the simplified form of the function as

$$f(x) = x + 2, \quad x \neq 2$$

Example 8 Find the domain and range of

(a) $f(x) = 2 + \sqrt{x - 1}$ (b) $f(x) = (x + 1)/(x - 1)$

Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain, $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x - 1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x - 1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are highlighted in green on the x- and y-axes in Figure 0.1.15.

y

$y = 2 + \sqrt{x - 1}$

5

4

3

2

1

x

1 2 3 4 5 6 7 8 9 10 **Figure 0.1.15**

Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is $\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$

To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x + 1}{x - 1} \quad (4)$$

Although the set of possible y-values is not immediately evident from this equation, the graph of (4), which is shown in Figure 0.1.16, suggests that the range of f consists of all y, except $y = 1$. To see that this is so, we solve (4) for x in terms of y:

5 4 3 2 1

y

$y = \frac{x + 1}{x - 1}$ x

−3 −2 −1 123456 −1

−2

Figure 0.1.16

$$(x - 1)y = x + 1$$

$$xy - y = x + 1$$

$$xy - x = y + 1$$
$$x(y - 1) = y + 1$$
$$x = \frac{y + 1}{y - 1}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y = 1\} = (- , 1) \cup (1, +)$, which agrees with the result obtained graphically.

DOMAIN AND RANGE IN APPLIED PROBLEMS

In applications, physical considerations often impose restrictions on the domain and range of a function.

Example 9 An open box is to be made from a 16-inch by 30-inch piece of card board by cutting out squares of equal size from the four corners and bending up the sides (Figure 0.1.17*a*).

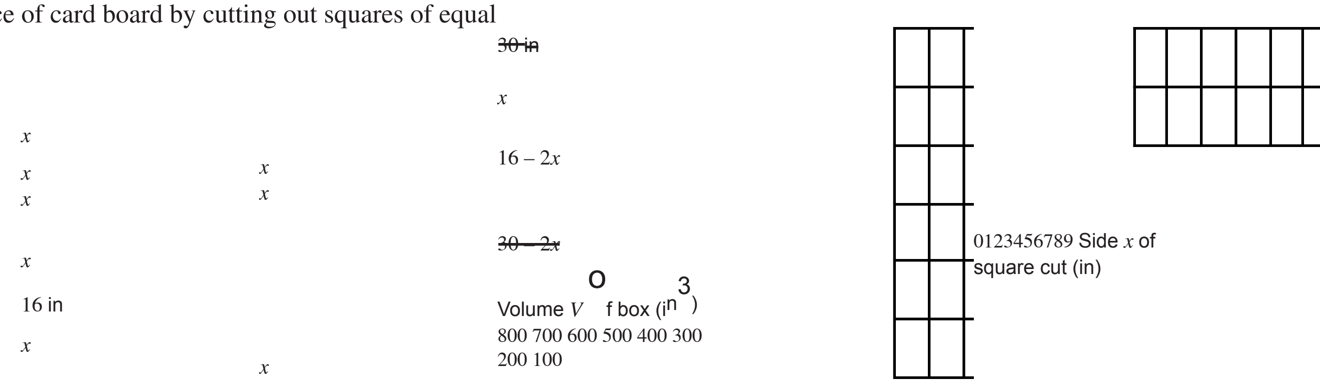
(a) Let V be the volume of the box that results when the squares have sides of length x. Find a formula for V as a function of x.

(b) Find the domain of V.

(c) Use the graph of V given in Figure 0.1.17*c* to estimate the range of V. (d) Describe in words what the graph tells you about the volume.

Solution (a). As shown in Figure 0.1.17*b*, the resulting box has dimensions $16 - 2x$ by $30 - 2x$ by x, so the volume $V(x)$ is given by

$$V(x) = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3$$



(*a*) (*b*) (*c*) **Figure 0.1.17**

Solution (b). The domain is the set of x-values and the range is the set of V-values. Because x is a length, it must be nonnegative, and because we cannot cut out squares whose sides are more than 8 in long (why?), the x-values in the domain must satisfy

$$0 \le x \le 8$$

Solution (c). From the graph of V versus x in Figure 0.1.17*c* we estimate that the V-values in the range satisfy

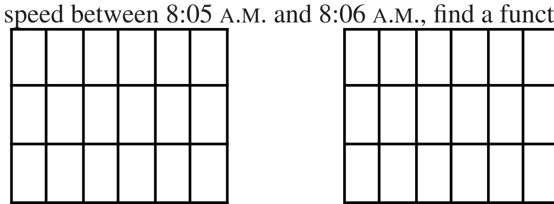
$$0 \le V \le 725$$

Note that this is an approximation. Later we will show how to find the range exactly.

Solution (d). The graph tells us that the box of maximum volume occurs for a value of x that is between 3 and 4 and that the maximum volume is approximately 725 in^3. The graph also shows that the volume decreases toward zero as x gets closer to 0 or 8, which should make sense to you intuitively.

In applications involving time, formulas for functions are often expressed in terms of a variable t whose starting value is taken to be $t = 0$.

Example 10 At 8:05 A.M. a car is clocked at 100 ft/s by a radar detector that is positioned at the edge of a straight highway. Assuming that the car maintains a constant speed between 8:05 A.M. and 8:06 A.M., find a function $D(t)$ that expresses the distance traveled by the car during that time interval as a function of



the time t.

Solution. It would be clumsy to use the actual clock time for the variable t, so let us agree to use the *elapsed* time in seconds, starting with $t = 0$ at 8:05 A.M. and ending with $t = 60$ at 8:06 A.M. At each instant, the distance traveled (in ft) is equal to the speed of the car (in ft/s) multiplied by the elapsed time (in s). Thus,

$$D(t) = 100t, \quad 0 \le t \le 60$$

The graph of D versus t is shown in Figure 0.1.18.

8:05 a.m. Time t (s) 8:06 a.m. **Figure 0.1.18**

y

ISSUES OF SCALE AND UNITS

In geometric problems where you want to preserve the "true" shape of a graph, you must use units of equal length on both axes. For example, if you graph a circle in a coordinate system in which 1 unit in the y-direction is smaller than 1 unit in the x-direction, then the circle will be squashed vertically into an elliptical shape (Figure 0.1.19). x However, sometimes it is inconvenient or impossible to display a graph using units of equal length. For example, consider the equation

$$y = x^2$$

If we want to show the portion of the graph over the interval $-3 \le x \le 3$, then there is no problem using units of equal length, since y only varies from 0 to 9 over that interval. However, if we want to show the portion of the graph over the interval $-10 \le x \le 10$, then there is a problem keeping the units equal in length, since the value of y varies between 0 and 100. In this case the only reasonable way to show all of the graph that occurs over the interval $-10 \le x \le 10$ is to compress the unit of length along the y-axis, as illustrated in Figure 0.1.20.

The circle is squashed because 1 unit on the y-axis has a smaller length than 1 unit on the x-axis.

Figure 0.1.19

In applications where the variables on the two axes have unrelated units (say, centimeters on the y-axis and seconds on the x-axis), then nothing is gained by requiring the units to have equal lengths; choose the lengths to make the graph as clear as possible.

x

y
9
8
7
6
5
4
3
2
1

y
100
80
60
40
20
x

−3 −2 −1 1 2 3

−10 −5 5 10

Figure 0.1.20

✔ QUICK CHECK EXERCISES 0.1 (See page 15 for answers.)

1. Let $f(x) = \sqrt{x + 1} + 4$.
(a) The natural domain of f is .
(b) $f(3) =$
(c) $f(t^2 - 1) =$
(d) $f(x) = 7$ if $x =$
(e) The range of f is .

2. Line segments in an xy-plane form "letters" as depicted.

4. The accompanying table gives a 5-day forecast of high and low temperatures in degrees Fahrenheit (°F).
(a) Suppose that x and y denote the respective high and low temperature predictions for each of the 5 days. Is y a function of x? If so, give the domain and range of this function.
(b) Suppose that x and y denote the respective low and high temperature predictions for each of the 5 days. Is

y a function of x? If so, give the domain and range of this function.

mon tue wed thurs fri

letters

(a) If the y-axis is parallel to the letter I, which of the high

75

73

represent the graph of $y = f(x)$ for some function f ? low50

(b) If the y-axis is perpendicular to the letter I, which of the letters represent 52 the graph of $y = f(x)$ for some function f ?

Table Ex-4

5648

52

(e) The solutions to $f(x) = -\frac{3}{2}$ are $x =$ and $x =$.

y

2

3. The accompanying figure shows the complete graph of $y = f(x)$.

(a) The domain of f is .

(b) The range of f is .

(c) $f(-3) =$

(d) $f\left(\frac{1}{2}\right) =$

5. Let l, w, and A denote the length, width, and area of a rectangle, respectively, and suppose that the width of the rectangle is half the length.

(a) If l is expressed as a function of w, then $l =$. (b) If A is expressed as a function of l, then $A =$. (c) If w is expressed as a function of A, then $w =$.

1

x

−3 −2 −1 3 21

−1

−2

Figure Ex-3

EXERCISE SET 0.1 Graphing Utility

1. Use the accompanying graph to answer the following questions, making reasonable approximations where needed.

(a) For what values of x is $y = 1$?

(b) For what values of x is $y = 3$?

(c) For what values of y is $x = 3$?

(d) For what values of x is $y \leq 0$?

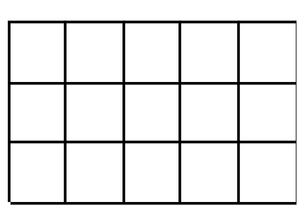
(e) What are the maximum and minimum values of y and for what values of x do they occur?

4. In each part, compare the natural domains of f and g.

(a) $f(x) = \frac{x^2 + x}{x + 1}$; $g(x) = x$

(b) $f(x) = \frac{x\sqrt{x} + \sqrt{x}}{x + 1}$; $g(x) = \sqrt{x}$

y



3

2

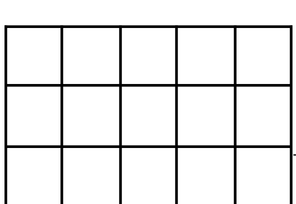
1

0

x

−1 −2 −3

Figure Ex-1



−3 −2 −1 1 0 23

5. The accompanying graph
in U.S. households (adju
1990 and 2005. Use
following ques tion
approximations where r
median income at its max
and what was the me
occurred? (b) When was
minimum value, and wha

when that occurred? (c) T
declining during the 2-ye
2002. Was it declining m
year or the second year o
reasoning.

Median U.S. Hous
Thousands of Cons

Median U.S. Household Inco
48

46

44

42

1990 1995 2000 2005

Source: U.S. Census Bureau, Au

Figure Ex-5

6. Use the median income graph in Exercise
the following questions, making
approximations where needed.

(a) What was the average yearly growth o
come between 1993 and 1999?

(b) The median income was increasing
6-year period between 1993 and 1
increasing more rapidly during the
or the last 3 years of that period? E
reasoning.

(c) Consider the statement: "After year
me dian income this year was finally
that of last year." In what years
statement have been correct?

2. Use the accompanying table to answer the questions posed in Exercise 1.

x	y
−2	5
−1	1
0	−2
2	7
3	−1
4	1
5	0
6	9

Table Ex-2

3. In each part of the accompanying figure, determine whether the graph defines y as a function of x.

y

x x

(c) (b)

(a)

Figure Ex-3

y

y

x x

(*d*)

y

7. Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$, and $f(3t)$.

(a) $f(x) = 3x^2 - 2$ (b) $f(x) = \begin{cases} \frac{1}{x}, & x > 3 \\ 2x, & x \leq 3 \end{cases}$

8. Find $g(3)$, $g(-1)$, $g(\pi)$, $g(-1.1)$, and $g(t^2 - 1)$.

(a) $g(x) = \frac{x+1}{x-1}$ (b) $g(x) = \begin{cases} \sqrt{x+1}, & x \geq 1 \\ 3, & x < 1 \end{cases}$

9–10 Find the natural domain and determine the range of each function. If you have a graphing utility, use it to confirm that your result is consistent with the graph produced by your graphing utility. [*Note:* Set your graphing utility in radian mode when graphing trigonometric functions.] ■

9. (a) $f(x) = \frac{1}{x-3}$ (b) $F(x) = \frac{x}{|x|}$

(c) $g(x) = \sqrt{x^2 - 3}$ (d) $G(x) = \sqrt{x^2 - 2x + 5}$

(e) $h(x) = \frac{1}{1 - \sin x}$ (f) $H(x) = \frac{x^2 - 4}{x - 2}$

10. (a) $f(x) = \sqrt{3 - x}$ (b) $F(x) = \sqrt{4 - x^2}$ (c) $g(x) = 3 + \sqrt{x}$ (d) $G(x) = x^3 + 2$ (e) $h(x) = 3 \sin x$ (f) $H(x) = (\sin \sqrt{x})^{-2}$

11. (a) If you had a device that could record the Earth's population continuously, would you expect the graph of population versus time to be a continuous (unbroken) curve? Explain what might cause breaks in the curve.

(b) Suppose that a hospital patient receives an injection of an antibiotic every 8 hours and that between injections the concentration C of the antibiotic in the bloodstream decreases as the antibiotic is absorbed by the tissues. What might the graph of C versus the elapsed time t look like?

12. (a) If you had a device that could record the temperature of a room continuously over a 24-hour period, would you expect the graph of temperature versus time to be a continuous (unbroken) curve? Explain your reasoning.

(b) If you had a computer that could track the number of boxes of cereal on the shelf of a market continuously over a 1-week period, would you expect the graph of the number of boxes on the shelf versus time to be a continuous (unbroken) curve? Explain your reasoning.

13. A boat is bobbing up and down on some gentle waves. Suddenly it gets hit by a large wave and sinks. Sketch a rough graph of the height of the boat above the ocean floor as a function of time.

14. A cup of hot coffee sits on a table. You pour in some cool milk and let it sit for an hour. Sketch a rough graph of the temperature of the coffee as a function of time.

15–18 As seen in Example 3, the equation $x^2 + y^2 = 25$ does not define y as a function of x. Each graph in these exercises is a portion of the circle $x^2 + y^2 = 25$. In each case, determine whether the graph defines y as a function of x, and if so, give a formula for y in terms of x. ■

15. *y* 5 *x* −5 5 −5

16. *y* 5 *x* −5 5 −5

17. *y* 5 *x* −5 5 −5

18. *y* 5 *x* −5 5 −5

19–22 True–False Determine whether the statement is true or false. Explain your answer. ■

19. A curve that crosses the x-axis at two different points cannot be the graph of a function.

20. The natural domain of a real-valued function defined by a formula consists of all those real numbers for which the formula yields a real value.

21. The range of the absolute value function is all positive real numbers.

22. If $g(x) = 1/\sqrt{f(x)}$, then the domain of g consists of all those real numbers x for which $f(x) \neq 0$.

23. Use the equation $y = x^2 - 6x + 8$ to answer the following questions.
(a) For what values of x is $y = 0$?
(b) For what values of x is $y = -10$?
(c) For what values of x is $y \geq 0$?
(d) Does y have a minimum value? A maximum value? If so, find them.

24. Use the equation $y = 1 + \sqrt{x}$ to answer the following questions.
(a) For what values of x is $y = 4$?
(b) For what values of x is $y = 0$?
(c) For what values of x is $y \geq 6$? (cont.)

(d) Does y have a minimum value? A maximum value? If so, find them.

25. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ.

26. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).

u
L
h

Figure Ex-25

L
u
10 cm

Figure Ex-26

27–28 Express the function in piecewise form without using absolute values. [*Suggestion:* It may help to generate the graph of the function.] ■

27. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x|+|x - 1|$

28. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2|-|x + 1|$

29. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 in by 15 in, by cutting out squares with sides of length x from each corner and bending up the sides.
(a) Express the volume V as a function of x.
(b) Find the domain of V.
(c) Plot the graph of the function V obtained in part (a) and estimate the range of this function.
(d) In words, describe how the volume V varies with x, and discuss how one might construct boxes of maximum volume.

x x
x x
x x
x x
8 in
~~15 in~~

Figure Ex-29

30. Repeat Exercise 29 assuming the box is constructed in the same fashion from a 6-inch-square sheet of metal.

31. A construction company has adjoined a 1000 ft^2 rectangular enclosure to its office building. Three sides of the enclosure are fenced in. The side of the building adjacent to the enclosure is 100 ft long and a portion of this side is used as the fourth side of the enclosure. Let x and y be the dimensions of the enclosure, where x is measured parallel to the building, and let L be the length of fencing required for those dimensions.
(a) Find a formula for L in terms of x and y.
(d) Plot the function in part (b) and estimate the dimensions of the enclosure that minimize the amount of fencing required.

32. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad. The rocket rises vertically when launched, and the camera's elevation angle is continually adjusted to follow the bottom of the rocket.
(a) Express the height x as a function of the elevation angle θ.
(b) Find the domain of the function in part (a).
(c) Plot the graph of the function in part (a) and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height.

Camera
Rocket
u
x
~~3000 ft~~

Figure Ex-32

33. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/cm^2, and the material for the sides costs 0.01 cent/cm^2.
(a) Estimate the radius r and the height h of the can that costs the least to manufacture. [*Suggestion:* Express the cost C in terms of r.]
(b) Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.
(c) Estimate the radius, height, and cost of the can in part

(b) Find a formula that expresses L as a function of x alone. (c) What is the domain of the function in part (b)?

(b), and determine whether your conjecture was correct. **34.** The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure on the next page. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles, with the straightaways extending at least the length of the football field.

(a) Show that it is possible to construct a quarter-mile track around the football field. [*Suggestion:* Find the shortest track that can be constructed around the field.]

(b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the foot ball field and a straightaway. Make a graph of L ver sus x. (*cont.*)

(c) Use the graph to estimate the value of x that produces the shortest straightaways, and then find this value of x exactly.

(d) Use the graph to estimate the length of the longest pos sible straightaways, and then find that length exactly.

WCT =

$$\begin{cases} T, & 0 \le v \le 3 \\ 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275T\,v^{0.16}, & 3 < v \end{cases}$$

160′

Figure Ex-34

~~360′~~

35–36 (i) Explain why the function f has one or more holes in its graph, and state the x-values at which those holes occur. (ii) Find a function g whose graph is identical to that of f, but without the holes. ■

35. $f(x) = \dfrac{(x+2)(x^2-1)}{(x+2)(x-1)}$ **36.** $f(x) = \dfrac{x^2 + |x|}{|x|}$

37. In 2001 the National Weather Service introduced a new wind chill temperature (WCT) index. For a given outside temperature T and wind speed v, the wind chill temperature index is the equivalent temperature that exposed skin would feel with a wind speed of v mi/h. Based on a more accurate model of cooling due to wind, the new formula is

where T is the temperature in °F, v is the wind speed in mi/h, and WCT is the equivalent temperature in °F. Find the WCT to the nearest degree if $T = 25$°F and (a) $v = 3$ mi/h (b) $v = 15$ mi/h (c) $v = 46$ mi/h.

Source: Adapted from UMAP Module 658, *Windchill*, W. Bosch and L. Cobb, COMAP, Arlington, MA.

38–40 Use the formula for the wind chill temperature index described in Exercise 37. ■

38. Find the air temperature to the nearest degree if the WCT is reported as −60°F with a wind speed of 48 mi/h.

39. Find the air temperature to the nearest degree if the WCT is reported as −10°F with a wind speed of 48 mi/h.

40. Find the wind speed to the nearest mile per hour if the WCT is reported as 5°F with an air temperature of 20°F.

✔ QUICK CHECK ANSWERS 0.1

1. (a) $[-1, +\infty)$ (b) 6 (c) $|t| + 4$ (d) 8 (e) $[4, +\infty)$ **2.** (a) M (b) I **3.** (a) $[-3, 3)$ (b) $[-2, 2]$ (c) −1 (d) 1 (e) $-\frac{3}{4}$; $-\frac{3}{2}$ **4.** (a) yes; domain: {65, 70, 71, 73, 75}; range: {48, 50, 52, 56} (b) no **5.** (a) $l = 2w$ (b) $A = l^2/2$ (c) $w = \sqrt{A/2}$

NEW FUNCTIONS FROM OLD

Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f + g$, $f - g$, fg, and f/g. For example, $f + g$ is defined by the formula

$$(f + g)(x) = f(x) + g(x) \quad (1)$$

which states that for each input the value of $f + g$ is obtained by adding the values of f and g. Equation (1) provides a formula for $f + g$ but does not say anything about the domain of $f + g$. However, for the right side of this equation to be defined, x must lie in the domains of both f and g, so we define the domain of $f + g$ to be the intersection of these two domains. More generally, we make the following definition.

0.2.1 definition Given functions f and g

$(f + g)(x)$

$(f - g)(x)$

$(fg)(x)$

$(f/g)(x)$

For the functions $f + g$, $f - g$, and fg we the domains of f and g, and for the fu intersection of the domains of f and g l (to avoid division by zero).

If f is a constant function, that is, $f(x) = c$ for all x, then the product of f and g is cg, so multiplying a func tion by a constant is a special case of multiplying two functions.

Example 1 Let

$$f(x) = 1 + \sqrt{x - 2} \text{ and } g(x) = x - 3$$

Find the domains and formulas for the functions $f + g$, $f - g$, fg, f/g, and $7f$.

Solution. First, we will find the formulas and then the domains. The formulas are

$$(f + g)(x) = f(x) + g(x) = (1 + \sqrt{x - 2}) + (x - 3) = x - 2 + \sqrt{x - 2} \quad (2)$$

$$(f - g)(x) = f(x) - g(x) = (1 + \sqrt{x - 2}) - (x - 3) = 4 - x + \sqrt{x - 2} \quad (3)$$

$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x - 2})(x - 3) \quad (4)$$

$$(f/g)(x) = f(x)/g(x) = \frac{1 + \sqrt{x - 2}}{x - 3} \quad (5)$$

$$(7f)(x) = 7f(x) = 7 + 7\sqrt{x - 2} \quad (6)$$

The domains of f and g are $[2, +)$ and $(- , +)$, respectively (their natural domains). Thus, it follows from Definition 0.2.1 that the domains of $f + g$, $f - g$, and fg are the intersection of these two domains, namely,

$$[2, +) \cap (- , +) = [2, +) \quad (7)$$

Moreover, since $g(x) = 0$ if $x = 3$, the domain of f/g is (7) with $x = 3$ removed, namely,

$$[2, 3) \cup (3, +)$$

Finally, the domain of $7f$ is the same as the domain of f.

We saw in the last example that the domains of the functions $f + g$, $f - g$, fg, and f/g were the natural domains resulting from the formulas obtained for these functions. The following example shows that this will not always be the case.

Example 2 Show that if $f(x) = \sqrt{x}$, $g(x) = \sqrt{x}$, and $h(x) = x$, then the domain of fg is not the same as the natural domain of h.

Solution. The natural domain of $h(x) = x$ is $(- , +)$. Note that

$$(fg)(x) = \sqrt{x}\sqrt{x} = x = h(x)$$

on the domain of fg. The domains of both f and g are $[0, +)$, so the domain of fg is $[0, +) \cap [0, +) = [0, +)$

Although the domain of $f \circ g$ may seem complicated at first glance, it makes sense intuitively: To compute $f(g(x))$ one needs x in the domain of g to compute $g(x)$, and one needs $g(x)$ in the domain of f to compute $f(g(x))$.

by Definition 0.2.1. Since the domains of fg and h are different, it would be misleading to write $(fg)(x) = x$ without including the restriction that this formula holds only for $x \geq 0$.

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that $f(x) = x^2$ and $g(x) = x + 1$

If we substitute $g(x)$ for x in the formula for f, we obtain a new function $f(g(x)) = (g(x))^2 = (x + 1)^2$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x + 1)^2$$

In general, we make the following definition.

0.2.2 definition Given functions f ar
$\circ g$, is the function defined by

$(f \circ g$

The domain of $f \circ g$ is defined to con
is in the domain of f.

Example 3 Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x})^2 + 3 = x + 3$$

Since the domain of g is $[0, +)$ and the domain of f is $(-, +)$, the domain of $f \circ g$ consists of all x in $[0, +)$ such that $g(x) = \sqrt{x}$ lies in $(-, +)$; thus, the domain of $f \circ g$ is $[0, +)$. Therefore,

$(f \circ g)(x) = x + 3, \ x \geq 0$

Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = f(x) = x^2 + 3$$

Since the domain of f is $(-, +)$ and the domain of g is $[0, +)$, the domain of $g \circ f$ consists of all x in $(-, +)$ such that $f(x) = x^2 + 3$ lies in $[0, +)$. Thus, the domain of $g \circ f$ is $(-, +)$. Therefore,

$$(g \circ f)(x) = x^2 + 3$$

There is no need to indicate that the domain is $(-, +)$, since this is the natural domain of $\sqrt{x^2 + 3}$.

Note that the functions $f \circ g$ and $g \circ f$ in Example 3 are not the same. Thus, the order in which functions are composed can (and usually will) make a difference in the end result.

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$, and then find $f(g(h(x)))$.

Example 4 Find $(f \circ g \circ h)(x)$ if

$$f(x) = \sqrt{x},\ g(x) = 1/x,\ h(x) = x^3$$

Solution.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(1/x^3) = 1/x^3 = 1/x^{3/2}$$

EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are solved by "decomposing" functions into compositions of simpler functions. For example, consider the function h given by

$$h(x) = (x + 1)^2$$

To evaluate $h(x)$ for a given value of x, we would first compute $x + 1$ and then square the result. These two operations are performed by the functions

$$g(x) = x + 1 \text{ and } f(x) = x^2$$

We can express h in terms of f and g by writing

$$h(x) = (x + 1)^2 = [g(x)]^2 = f(g(x))$$

so we have succeeded in expressing h as the composition $h = f \circ g$.

The thought process in this example suggests a general procedure for decomposing a function h into a composition $h = f \circ g$:

- Think about how you would evaluate $h(x)$ for a specific value of x, trying to break the evaluation into two steps performed in succession.
- The first operation in the evaluation will determine a function g and the second a function f.
- The formula for h can then be written as $h(x) = f(g(x))$.

For descriptive purposes, we will refer to g as the "inside function" and f as the "outside function" in the expression $f(g(x))$. The inside function performs the first operation and the outside function performs the second.

Example 5 Express $\sin(x^3)$ as a composition of two functions.

Solution. To evaluate $\sin(x^3)$, we would first compute x^3 and then take the sine, so $g(x) = x^3$ is the inside function and $f(x) = \sin x$ the outside function. Therefore,

$$\sin(x^3) = f(g(x))$$

$$\boxed{g(x) = x^3 \text{ and } f(x) = \sin x}$$

Table 0.2.1 gives some more examples of decomposing functions into compositions.

Table 0.2.1

composing functions

$f(x)$

$g(x)$

function composition

inside $x^2 + 1$ outside

$(x^2 + 1)^{10}$ $\sin^3 x$

tan x

tan (x^5) $\sqrt{4 - 3x}$ $\sin x$

$\sqrt{x}$

x^{10}

x^5

$8 + x$

$8 + \sqrt{x}$

x^3

$4 - 3x$ $\sqrt{x}$

$(x^2 + 1)^{10} =$

$f(g(x))\ \sin^3 x = \ f(g(x))\ \tan(x^5)\ \ = f(g(x))\ \sqrt{4 -}\ \ \sqrt{\cancel{x}} = f(g(x))$

$3x = f(g(x))\ 8 +$

$\frac{1}{x+1}$ $\qquad \frac{1}{x+1} = f(g(x))\ \frac{1}{}$ $\qquad \frac{x}{x+1}$

REMARK There is always more than one way to express a function as a composition. For example, here are two ways to express $(x^2 + 1)^{10}$ as a composition that differ from that in Table 0.2.1:

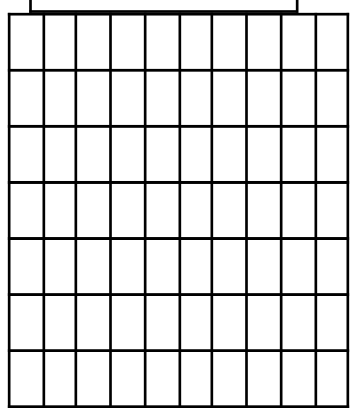
$(x^2 + 1)^{10} = [(x^2 + 1)^2]^5 = f(g(x))$

$(x^2 + 1)^{10} = [(x^2 + 1)^3]^{10/3} = f(g(x))$ NEW

$\boxed{g(x) = (x^2 + 1)^3 \text{ and } f(x) = x^{10/3}}$

FUNCTIONS FROM OLD

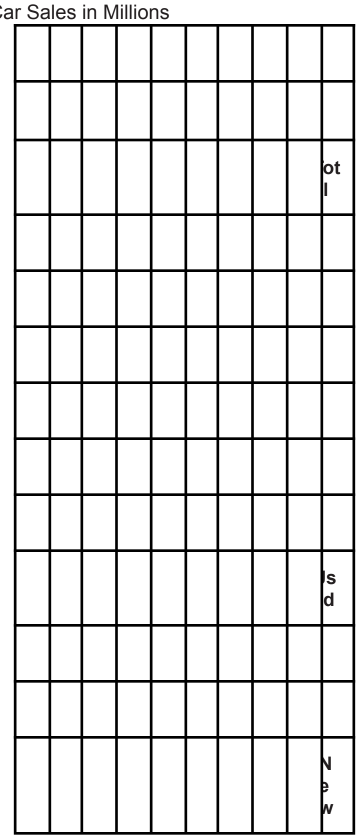
$\boxed{g(x) = (x^2 + 1)^2 \text{ and } f(x) = x^5}$



40 36 32 28 24 20 16 12 8

4

Car Sales in Millions



1995 2005 2000***Source:*** NADA.

Figure 0.2.1

$T(t) = N(t) + U(t)$

by adding the values of $N(t)$ and $U(t)$ for each value of t. In general, the graph of $y = f(x) + g(x)$ can be constructed from the graphs of $y = f(x)$ and $y = g(x)$ by adding corresponding y-values for each x.

Used

The remainder of this section will be devoted to considering the geometric effect of perform ing basic operations on functions. This will enable us to use known graphs of functions to visualize or sketch graphs of related functions. For example, Figure 0.2.1 shows the graphs of yearly new car sales $N(t)$ and used car sales $U(t)$ over a certain time period. Those graphs can be used to construct the graph of the total car sales

New

Example 6 Referring to Figure 0.1.4 for the graphs of $y = \sqrt{\cancel{x}}$ and $y = 1/x$, make a sketch that shows the general shape of the graph of $y = \sqrt{\cancel{x}} +$ $1/x$ for $x \geq 0$. ***Solution.*** To add the corresponding y-values of $y = \sqrt{\cancel{x}}$ and $y = 1/x$ graphically, just imagine them to be "stacked" on top of one another. This yields the sketch in Figure 0.2.2.

y

y

Use the technique in Example 6 to sketch the graph of the function

$\sqrt{\cancel{x}} - \frac{1}{x}$

Figure 0.2.2

Add the y-coordinates of $\sqrt{\cancel{x}}$ and $1/x$ to obtain the y-coordinate of $\sqrt{\cancel{x}} + 1/x$.

y

x

$\sqrt{x}$ − 1/x

x

x

$\sqrt{x}$ + 1/x

TRANSLATIONS

Table 0.2.2 illustrates the geometric effect on the graph of $y = f(x)$ of adding or subtracting a *positive* constant c to f or to its independent variable x. For example, the first result in the table illustrates that adding a positive constant c to a function f adds c to each y-coordinate of its graph, thereby shifting the graph of f up by c units. Similarly, subtracting c from f shifts the graph down by c units. On the other hand, if a positive constant c is added to x, then the value of $y = f(x + c)$ at $x - c$ is $f(x)$; and since the point $x - c$ is c units to the left of x on the x-axis, the graph of $y = f(x + c)$ must be the graph of $y = f(x)$ shifted left by c units. Similarly, subtracting c from x shifts the graph of $y = f(x)$ right by c units.

Table 0.2.2
translation principles

operation on $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
new equation	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
geometric effect	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
example	$y = x^2 + 2$, $y = x^2$	$y = x^2$, $y = x^2 - 2$	$y = x^2$, $y = (x + 2)^2$	$y = x^2$, $y = (x - 2)^2$

y y y y x x x x 2 −2 −2 2

y 3 x 9 $y = \sqrt{x}$ y 3 x

Before proceeding to the next examples, it will be helpful to review the graphs in Figures 0.1.4 and 0.1.9.

Example 7 Sketch the graph of

(a) $y = \sqrt{x - 3}$ (b) $y = \sqrt{x + 3}$

Solution. Using the translation principles given in Table 0.2.2, the graph of the equation $y = \sqrt{x - 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units. The graph of $y = \sqrt{x + 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ left 3 units (Figure 0.2.3).

3 12

$y = \sqrt{x - 3}$

y 3

x
$-3\ 6$

$y = \sqrt{x} + 3$

Figure 0.2.3

Example 8 Sketch the graph of $y = x^2 - 4x + 5$.

Solution. Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$$

(see Web Appendix H for a review of this technique). In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$ (Figure 0.2.4).

x
$-5\ 5\ 2$
y
8
y
8
1
x
y
8

x
$-5\ 5\ 2$
$-5\ 5$
$y = (x - 2)^2$ $y = (x - 2)^2 + 1$

Figure 0.2.4
$y = x^2$

REFLECTIONS

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(-x, y)$. Similarly, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(x, -y)$ [the equation $y = -f(x)$ is equivalent to $-y = f(x)$]. This is summarized in Table 0.2.3.

Table 0.2.3
reflection principles

operation on $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
new equation	$y = f(-x)$	$y = -f(x)$
geometric effect	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
example	y; $y = \sqrt{-x}$ $y = \sqrt{x}$ 3; x; -3 -3	$-6\ 6$; y; $y = \sqrt{x}$; 3; x; $-6\ 6$

2 − x.

y = −√x

Example 9 Sketch the graph of $y = \sqrt[3]{2 - x}$.

Solution. Using the translation and reflection principles in Tables 0.2.2 and 0.2.3, we can obtain the graph by a reflection followed by a translation as follows: First reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x-2)} = \sqrt[3]{2-x}$ (Figure 0.2.5).

y
6
x
x

−10 10 −6
−10 10 −6
−10 10 −6

y
6
y
6
x

$y = \sqrt[3]{2-x}$ $y = \sqrt[3]{-x}$ $y = \sqrt[3]{x}$

Figure 0.2.5

Example 10 Sketch the graph of $y = 4 - |x - 2|$.

Solution. The graph can be obtained by a reflection and two translations: First translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x - 2|$; then reflect this graph about the x-axis to obtain the graph of $y = -|x - 2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x - 2| + 4 = 4 - |x - 2|$ (Figure 0.2.6).

8
8
x
x
x

y
8
x
y
y
y
8

−8 8

−6 10

Figure 0.2.6

−6 10

−8

−8

−8

−8

$y = 4 - |x - 2|$

$y = |x|$

$y = -|x - 2|$

−6 10

$y = |x - 2|$

Describe the geometric effect of multiplying a function f by a *negative* constant in terms of reflection and stretching or compressing. What is the geometric effect of multiplying the independent variable of a function f by a *negative* constant?

STRETCHES AND COMPRESSIONS

Multiplying $f(x)$ by a *positive* constant c has the geometric effect of stretching the graph of $y = f(x)$ in the y-direction by a factor of c if $c > 1$ and compressing it in the y direction by a factor of $1/c$ if $0 < c < 1$. For example, multiplying $f(x)$ by 2 doubles each y-coordinate, thereby stretching the graph vertically by a factor of 2, and multiplying by $\frac{1}{2}$ cuts each y-coordinate in half, thereby compressing the graph vertically by a factor of 2. Similarly, multiplying x by a *positive* constant c has the geometric effect of compressing the graph of $y = f(x)$ by a factor of c in the x-direction if $c > 1$ and stretching it by a factor of $1/c$ if $0 < c < 1$. [If this seems backwards to you, then think of it this way: The value of $2x$ changes twice as fast as x, so a point moving along the x-axis from the origin will only have to move half as far for $y = f(2x)$ to have the same value as $y = f(x)$, thereby creating a horizontal compression of the graph.] All of this is summarized in Table 0.2.4.

stretching and compressing principles

Table 0.2.4

operation on $y = f(x)$	**new equation**	**geometric effect**	**example**
Multiply $f(x)$ by c ($c > 1$)	$y = cf(x)$	Stretches the graph of $y = f(x)$ vertically by a factor of c	
Multiply $f(x)$ by c ($0 < c < 1$)	$y = cf(x)$	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	
Multiply x by c ($c > 1$)	$y = f(cx)$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	
Multiply x by c ($0 < c < 1$)	$y = f(cx)$	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$	

y y y y

2

$y = 2 \cos x$ $y = \cos x$ $y = \cos x$ $y = \cos 2x$

1

$y = \cos x$ $y = \frac{1}{2}\cos x$ 2 $\frac{1}{2}$ -1

1

$y = \cos x$

x

x

1

x x $y = \cos x$

SYMMETRY

Figure 0.2.7 illustrates three types of symmetries: ***symmetry about the x-axis***, ***symmetry about the y-axis***, and ***symmetry about the origin***. As illustrated in the figure, a curve is symmetric about the x-axis if for each point (x, y) on the graph the point $(x, -y)$ is also on the graph, and it is symmetric about the y-axis if for each point (x, y) on the graph the point $(-x, y)$ is also on the graph. A curve is symmetric about the origin if for each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. (Equivalently, a graph is symmetric about the origin if rotating the graph 180° about the origin leaves it unchanged.) This suggests the following symmetry tests.

Explain why the graph of a nonzero function cannot be symmetric about the x-axis.

Symmetric about the x-axis

$(x, -y)$

y

$(-x, y)$ (x, y)

x

Symmetric about the y-axis

y

$(-x, -y)$

Symmetric about the origin

(x, y) x

Figure 0.2.7

y

(x, y)

x

y $x = y^2$

x

Figure 0.2.8

0.2.3 theorem ***(Symmetry T***

(*a*) *A plane curve is symm*
its equation produce

(*b*) *A plane curve is symm*
its equation produce

(*c*) *A plane curve is symm*
$-x$ and y by $-y$ in it

Example 11 Use Theorem 0.2.3 to identify symmetries in the graph of $x = y^2$.

Solution. Replacing y by $-y$ yields $x = (-y)^2$, which simplifies to the original equation $x = y^2$. Thus, the graph is symmetric about the x-axis. The graph is not symmetric about the y-axis because replacing x by $-x$ yields $-x = y^2$, which is not equivalent to the original equation $x = y^2$. Similarly, the graph is not symmetric about the origin because replacing x by $-x$ and y by $-y$ yields $-x = (-y)^2$, which simplifies to $-x = y^2$, and this is again not equivalent to the original equation. These results are consistent with the graph of $x = y^2$ shown in Figure 0.2.8.

EVEN AND ODD FUNCTIONS

A function f is said to be an ***even function*** if

$$f(-x) = f(x) \quad (8)$$

and is said to be an ***odd function*** if

$$f(-x) = -f(x) \quad (9)$$

Geometrically, the graphs of even functions are symmetric about the y-axis because replac ing x by $-x$ in the equation $y = f(x)$ yields $y = f(-x)$, which is equivalent to the original

equation $y = f(x)$ by (8) (see Figure 0.2.9). Similarly, it follows from (9) that graphs of odd functions are symmetric about the origin (see Figure 0.2.10). Some examples of even functions are x^2, x^4, x^6, and $\cos x$; and some examples of odd functions are x^3, x^5, x^7, and $\sin x$.

y

f(−x) f(x) x

−x x

y

f(−x)

−x

x

f(x) x

Figure 0.2.9 This is the graph of an even function since $f(-x) = f(x)$.

✔ QUICK CHECK EXERCISES 0.2 *(See page 27 for answers.)*

Figure 0.2.10 This is the graph of an odd function since $f(-x) = -f(x)$.

1. Let $f(x) = 3\sqrt{x} - 2$ and $g(x) = |x|$. In each part, give the formula for the function and state the corresponding domain. (a) $f + g$: Domain:

(b) $f - g$: Domain:

(c) fg: Domain:

(d) f/g: Domain:

2. Let $f(x) = 2 - x^2$ and $g(x) = \sqrt{x}$. In each part, give the formula for the composition and state the corresponding domain.

(a) $f \circ g$: Domain:

(b) $g \circ f$: Domain:

3. The graph of $y = 1 + (x - 2)^2$ may be obtained by shifting the graph of $y = x^2$ (left/right) by unit(s) and then shifting this new graph (up/down) by unit(s).

4. Let

$f(x) =$ $|x + 1|, -2 \le x \le 0$ $|x - 1|, 0 < x \le 2$

(a) The letter of the alphabet that most resembles the graph of f is .

(b) Is f an even function?

EXERCISE SET 0.2 Graphing Utility

1. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations. (a) $y = f(x) - 1$ (b) $y = f(x - 1)$

(c) $y = \frac{1}{2}f(x)$ (d) $y = f\left(-\frac{1}{2}x\right)$

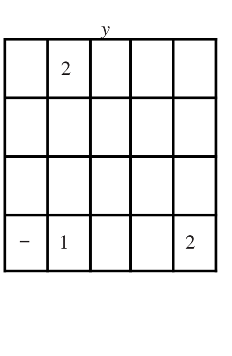


Figure Ex-1

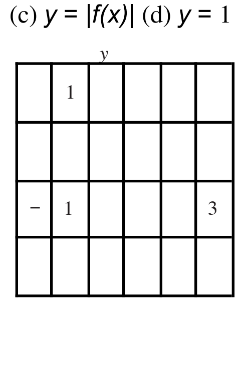
2. Use the graph in Exercise 1 to sketch the graphs of the following equations.

(a) $y = -f(-x)$ (b) $y = f(2 - x)$

(c) $y = 1 - f(2 - x)$ (d) $y = \frac{1}{2}f(2x)$

3. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations. (a) $y = f(x + 1)$ (b) $y = f(2x)$

(c) $y = |f(x)|$ (d) $y = 1 - |f(x)|$



$6x$ **8.** $y = \frac{1}{2}(x^2 - 2x + 3)$ **9.** $y = 3 - \sqrt{x + 1}$ **10.** $y = 1 + \sqrt{x - 4}$

11. $y = \frac{1}{2}\sqrt{x} + 1$ **12.** $y = -\sqrt{3x}$ $x - 3$**14.** $y = 1$

Figure Ex-3

4. Use the graph in Exercise 3 to sketch the graph of the equation $y = f(|x|)$.

5–24 Sketch the graph of the equation by translating, reflect ing, compressing, and stretching the graph of $y = x^2$, $y = \sqrt{x}$, $y = 1/x$, $y = |x|$, or $y = \sqrt[3]{x}$ appropriately. Then use a graph ing utility to confirm that your sketch is correct. ■

37–42 Express f as a composition of two functions; that is, find g and h such that $f = g \circ h$. [*Note:* Each exercise has more than one solution.] ■

37. (a) $f(x) = \sqrt{x + 2}$ (b) $f(x) = |x^2 - 3x + 5|$ **38.** (a) $f(x) = x^2 + 1$ (b) $f(x) = \frac{1}{x - 3}$

5. $y = -2(x + 1)^2 - 3$ **6.** $y = \frac{1}{2}(x - 3)^2 + 2$ **7.** $y = x^2 +$

13. $y = \frac{1}{1 - x}$

$x + 1$**16.** $y = x - 1$

39. (a) $f(x) = \sin^2 x$ (b) $f(x) = 3$ $5 + \cos x$

15. $y = 2 - 1$

40. (a) $f(x) = 3\sin(x^2)$ (b) $f(x) = 3\sin^2 x + 4\sin x$

41. (a) $f(x) = 1 + \sin(x^2)$

(b) $f(x) = 1 - \sqrt[3]{x}$ $\frac{3}{x}$

17. $y = |x + 2| - 2$ **18.** $y = 1 - |x - 3|$ **19.** $y = |2x - 1| + 1$ **20.** $y = \sqrt{x^2 - 4x + 4}$ **21.** $y = 1 - 2\sqrt[3]{x}$ **22.** $y = \sqrt[3]{x - 2} - 3$ **23.** $y = 2 + \sqrt[3]{x + 1}$ **24.** $y + \sqrt[3]{x - 2} = 0$

25. (a) Sketch the graph of $y = x + |x|$ by adding the corre sponding y-coordinates on the graphs of $y = x$ and $y = |x|$.

(b) Express the equation $y = x + |x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

26. Sketch the graph of $y = x + (1/x)$ by adding correspond ing y-coordinates on the graphs of $y = x$ and $y = 1/x$. Use a graphing utility to confirm that your sketch is correct.

27–28 Find formulas for $f + g$, $f - g$, fg, and f/g, and state the domains of the functions. ■

27. $f(x) = 2\sqrt{x - 1}$, $g(x) = \sqrt{x - 1}$

28. $f(x) = \frac{x}{1 + x^2}$, $g(x) = \frac{1}{x}$

29. Let $f(x) = \sqrt{x}$ and $g(x) = x^3 + 1$. Find
(a) $f(g(2))$ (b) $g(f(4))$ (c) $f(f(16))$ (d) $g(g(0))$ (e) $f(2 + h)$ (f) $g(3 + h)$.

30. Let $g(x) = \sqrt{x}$. Find
(a) $g(5s + 2)$ (b) $g(\sqrt{x} + 2)$ (c) $3g(5x)$ (d) $\frac{1}{g(x)}$ (e) $g(g(x))$ (f) $(g(x))^2 - g(x^2)$ (g) $g(1/\sqrt{x})$ (h) $g((x - 1)^2)$ (i) $g(x + h)$.

31–34 Find formulas for $f \circ g$ and $g \circ f$, and state the domains of the compositions. ■

31. $f(x) = x^2$, $g(x) = \sqrt{1 - x}$

32. $f(x) = \sqrt{x - 3}$, $g(x) = \sqrt{x^2 + 3}$

33. $f(x) = \frac{1 + x}{1 - x}$, $g(x) = \frac{x}{1 - x}$

34. $f(x) = \frac{x}{1 + x^2}$, $g(x) = \frac{1}{x}$

35–36 Find a formula for $f \circ g \circ h$. ■

35. $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$, $h(x) = x^3$

36. $f(x) = \frac{1}{1 + x}$, $g(x) = \sqrt[3]{x}$, $h(x) = \frac{1}{x^3}$

42. (a) $f(x) = \frac{1}{1 - x^2}$ (b) $f(x) = |5 + 2x|$

43–46 True–False Determine whether the statement is true or false. Explain your answer. ■

43. The domain of $f + g$ is the intersection of the domains of f and g.

44. The domain of $f \circ g$ consists of all values of x in the domain of g for which $g(x) = 0$.

45. The graph of an even function is symmetric about the y-axis.

46. The graph of $y = f(x + 2) + 3$ is obtained by translating the graph of $y = f(x)$ right 2 units and up 3 units.

47. Use the data in the accompanying table to make a plot of $y = f(g(x))$.

x	−3	−2	−1	0	1	2	3
$f(x)$	−4	−3	−2	−1	0	1	2
$g(x)$	−1	0	1	2	3	−2	−3

Table Ex-47

48. Find the domain of $g \circ f$ for the functions f and g in Exercise 47.

49. Sketch the graph of $y = f(g(x))$ for the functions graphed in the accompanying figure.

y

3

f

x

−3 3

g

−3

Figure Ex-49

50. Sketch the graph of $y = g(f(x))$ for the functions graphed in Exercise 49.

51. Use the graphs of f and g in Exercise 49 to esti mate the solutions of the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

52. Use the table given in Exercise 47 to solve the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

53–56 Find

$f(x + h) - f(x)$

h and $\dfrac{f(w) - f(x)}{w - x}$

Simplify as much as possible. ■

53. $f(x) = 3x^2 - 5$ **54.** $f(x) = x^2 + 6x$ **55.** $f(x) = 1/x$ **56.** $f(x) = 1/x^2$ **57.** Classify the functions whose values are given in the accom panying table as even, odd, or neither.

x −2 −1 0 1 2

−3 3

f(x) g(x)

5	3	2	3	1	− 3
4	1	−	0	2	−

2

		2			1

h(x) 2 −5 8 −2 8 −5 2

Table Ex-57

58. Complete the accompanying table so that the graph of $y = f(x)$ is symmetric about

(a) the y-axis (b) the origin.

x 3 −2 −1 −1 0 0 1 2 −5

f(x)

−3 1

Table Ex-58

59. The accompanying figure shows a portion of a graph. Com plete the graph so that the entire graph is symmetric about (a) the x-axis (b) the y-axis (c) the origin.

60. The accompanying figure shows a portion of the graph of a function f. Complete the graph assuming that

(a) f is an even function (b) f is an odd function. y

y

63. In each part, classify the function as even, odd, or neither.

(a) $f(x) = x^2$ (b) $f(x) = x^3$

(c) $f(x) = |x|$ (d) $f(x) = x + 1$ (e) $f(x) = \dfrac{x^5 - x}{1 + x^2}$ (f) $f(x) = 2$

64. Suppose that the function f has domain all real numbers. Determine whether each function can be classified as even or odd. Explain.

(a) $g(x) = \dfrac{f(x) + f(-x)}{2}$ (b) $h(x) = \dfrac{f(x) - f(-x)}{2}$

5

65. Suppose that the function f has domain all real numbers. −4

Show that f can be written as the sum of an even function and an odd function. [*Hint:* See Exercise 64.]

66–67 Use Theorem 0.2.3 to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin. ■

66. (a) $x = 5y^2 + 9$ (b) $x^2 - 2y^2 = 3$ (c) $xy = 5$

67. (a) $x^4 = 2y^3 + y$ (b) $y = \dfrac{x}{3 + x^2}$

(c) $y^2 = |x| - 5$

68–69 (i) Use a graphing utility to graph the equation in the first quadrant. [*Note:* To do this you will have to solve the equation for y in terms of x.] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants. ■

x

x

Figure Ex-59

Figure Ex-60

61–62 Classify the functions graphed in the accompanying figures as even, odd, or neither. ■

61.

x
y
y

x
x

(*a*) (*b*) **Figure Ex-61**

62.

y

y

x

x

(*a*)

Figure Ex-62

(*b*)

68. $9x^2 + 4y^2 = 36$ **69.** $4x^2 + 16y^2 = 16$ **70.** The graph of the equation $x^{2/3} + y^{2/3} = 1$, which is shown in the accompanying figure, is called a ***four-cusped hypocycloid***.

(a) Use Theorem 0.2.3 to confirm that this graph is symmetric about the x-axis, the y-axis, and the origin. (b) Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.

(c) Repeat part (b) for the remaining three quadrants.

y

Four-cusped hypocycloid **Figure Ex-70**

71. The equation $y = |f(x)|$ can be written as

$$y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

which shows that the graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by retaining the portion that lies on or above the x-axis and reflecting about the x-axis the portion that lies below the x-axis. Use this method to obtain the graph of $y = |2x - 3|$ from the graph of $y = 2x - 3$.

72–73 Use the method described in Exercise 71.

■ **72.** Sketch the graph of $y = |1 - x^2|$.

73. Sketch the graph of

(a) $f(x) = |\cos x|$ (b) $f(x) = \cos x + |\cos x|$.

74. The ***greatest integer function***, $\lfloor x \rfloor$, is defined to be the greatest integer that is less than or equal to x. For example, $\lfloor 2.7 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$, and $\lfloor 4 \rfloor = 4$. In each part, sketch the graph of $y = f(x)$.

(a) $f(x) = \lfloor x \rfloor$

(b) $f(x) = \lfloor x^2 \rfloor$

(c) $f(x) = \lfloor x \rfloor^2$

(d) $f(x) = \lfloor \sin x \rfloor$

75. Is it ever true that $f \circ g = g \circ f$ if f and g are nonconstant functions? If not, prove it; if so, give some examples for which it is true.

✔ QUICK CHECK ANSWERS 0.2

1. (a) $(f + g)(x) = 3\sqrt{x - 2} + x$; $x \geq 0$ (b) $(f - g)(x) = 3\sqrt{x - 2} - x$; $x \geq 0$ (c) $(fg)(x) = 3x^{3/2} - 2x$; $x \geq 0$ (d) $(f/g)(x) = \dfrac{3\sqrt{x} - 2}{x}$; $x > 0$ **2.** (a) $(f \circ g)(x) = 2 - x$; $x \geq 0$ (b) $(g \circ f)(x) = \sqrt{2 - x^2}$; $-\sqrt{2} \leq x \leq \sqrt{2}$ **3.** right; 2; up; 1 **4.** (a) W (b) yes

FAMILIES OF FUNCTIONS

Functions are often grouped into families according to the form of their defining formulas or other common characteristics. In this section we will discuss some of the most basic families of functions.

FAMILIES OF CURVES

The graph of a constant function $f(x) = c$ is the graph of the equation $y = c$, which is the horizontal line shown in Figure 0.3.1a. If we vary c, then we obtain a set or ***family*** of horizontal lines such as those in Figure 0.3.1b. Constants that are varied to produce families of curves are called ***parameters***. For example, recall that an equation of the form $y = mx + b$ represents a line of slope m and y-intercept b. If we keep b fixed and treat m as a parameter, then we obtain a family of lines whose members all have y-intercept b (Figure 0.3.2a), and if we keep m fixed and treat b as a parameter, we obtain a family of parallel lines whose members all have slope m (Figure 0.3.2b).

(0, c)
y
$y = c$
x
(a)
y
$c = 4$
$c = 3$
$c = 2$
$c = 1$
$c = 0$
$c = -1$
$c = -2$ $c = -3$
$c = -4.5$
x
(b)

Figure 0.3.1

y
x
x
The family $y = mx + b$
(m fixed and b varying)
The family $y = mx + b$
(b fixed and m varying)
(b)
(a)
y

Figure 0.3.2

POWER FUNCTIONS; THE FAMILY $y = x^n$

A function of the form $f(x) = x^p$, where p is constant, is called a ***power function***. For the moment, let us consider the case where p is a positive integer, say $p = n$. The graphs of the curves $y = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 0.3.3. The first graph is the line with slope 1 that passes through the origin, and the second is a parabola that opens up and has its vertex at the origin (see Web Appendix H).

y $y = x^2$ x y $y = x^3$ x
1 1 1 1
1
x
y $y = x$
x
y $y = x^4$
x
y $y = x^5$
−1 1 −1
−1 1 −1
−1 1 −1

Figure 0.3.3

−1 1 −1
−1 1 −1

For $n \geq 2$ the shape of the curve $y = x^n$ depends on whether n is even or odd (Fig ure 0.3.4):

- For even values of n, the functions $f(x) = x^n$ are even, so their graphs are symmetric about the y-axis. The graphs all have the general shape of the graph of $y = x^2$, and each graph passes through the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$. As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x >$

1 and $x < -1$.

- For odd values of n, the functions $f(x) = x^n$ are odd, so their graphs are symmetric about the origin. The graphs all have the general shape of the curve $y = x^3$, and each graph passes through the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$. As n increases, the graphs become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.

y
$y = x^8$
$y = x^4$ $y = x^2$
2 1 $y = x^6$
$y = x^3$ $y = x^5$ $y = x$
1
y
$y = x^7$
x
(n even)
-1 1
x
-1 1
-1
The family $y = x^n$
(n odd)

Figure 0.3.4

The family $y = x^n$

REMARK The flattening and steepening effects can be understood by considering what happens when a number x is raised to higher and higher powers: If $-1 < x < 1$, then the absolute value of x^n *decreases* as n increases, thereby causing the graphs to become flatter on this interval as n increases (try raising $\frac{1}{2}$ or $-\frac{1}{2}$ to higher and higher powers). On the other hand, if $x > 1$ or $x < -1$, then the absolute value of x^n *increases* as n increases, thereby causing the graphs to become steeper on these intervals as n increases (try raising 2 or -2 to higher and higher powers).

THE FAMILY $y = x^{-n}$

If p is a negative integer, say $p = -n$, then the power functions $f(x) = x^p$ have the form $f(x) = x^{-n} = 1/x^n$. Figure 0.3.5 shows the graphs of $y = 1/x$ and $y = 1/x^2$. The graph of $y = 1/x$ is called an ***equilateral hyperbola*** (for reasons to be discussed later).

As illustrated in Figure 0.3.5, the shape of the curve $y = 1/x^n$ depends on whether n is even or odd:

- For even values of n, the functions $f(x) = 1/x^n$ are even, so their graphs are sym metric about the y-axis. The graphs all have the general shape of the curve $y = 1/x^2$, and each graph passes through the points $(-1, 1)$ and $(1, 1)$. As n increases, the graphs become steeper over the intervals $-1 < x < 0$ and $0 < x < 1$ and become flatter over the intervals $x > 1$ and $x < -1$.
- For odd values of n, the functions $f(x) = 1/x^n$ are odd, so their graphs are symmetric about the origin. The graphs all have the general shape of the curve $y = 1/x$, and each graph passes through the points $(1, 1)$ and $(-1, -1)$. As n increases, the graphs become steeper over the intervals $-1 < x < 0$ and $0 < x < 1$ and become flatter over the intervals $x > 1$ and $x < -1$.
- For both even and odd values of n the graph $y = 1/x^n$ has a break at the origin (called a ***discontinuity***), which occurs because division by zero is undefined.

By considering the value of $1/x^n$ for a fixed x as n increases, explain why the graphs become flatter or steeper as de scribed here for increasing values of n.

y
$y = 1/x$ (1, 1)
y
y
$y = 1/x$
$y = 1/x^4$ $y = 1/x^6$
$y = 1/x^3$
(1, 1)
x
$y = 1/x^5$
$y = 1/x^2$
y
$y = 1/x^2$
x
(−1, −1)
(−1, 1) (−1, 1) (1, 1) (−1, −1) (1, 1)
x

x

The family $y = 1/x^n$
(n odd)

Figure 0.3.5

The family $y = 1/x^n$
(n even)

INVERSE PROPORTIONS

Recall that a variable y is said to be ***inversely proportional to a variable x*** if there is a positive constant k, called the ***constant of proportionality***, such that

$$y = \frac{k}{x} \quad (1)$$

Since k is assumed to be positive, the graph of (1) has the same shape as $y = 1/x$ but is compressed or stretched in the y-direction. Also, it should be evident from (1) that doubling x multiplies y by $\frac{1}{2}$, tripling x multiplies y by $\frac{1}{3}$, and so forth.

Equation (1) can be expressed as $xy = k$, which tells us that the product of inversely proportional variables is a positive constant. This is a useful form for identifying inverse proportionality in experimental data.

Table 0.3.1

x	y
0.8	6.25
1	5
2.5	2
4	1.25
6.25	0.8
10	0.5

Example 1 Table 0.3.1 shows some experimental data.

(a) Explain why the data suggest that y is inversely proportional to x.

(b) Express y as a function of x.

(c) Graph your function and the data together for $x > 0$.

Solution. For every data point we have $xy = 5$, so y is inversely proportional to x and $y = 5/x$. The graph of this equation with the data points is shown in Figure 0.3.6.

Inverse proportions arise in various laws of physics. For example, ***Boyle's law*** in physics states that *if a fixed amount of an ideal gas is held at a constant temperature, then the product of the pressure P exerted by the gas and the volume V that it occupies is constant*; that is,

$$PV = k$$

This implies that the variables P and V are inversely proportional to one another. Fig ure 0.3.7 shows a typical graph of volume versus pressure under the conditions of Boyle's law. Note how doubling the pressure corresponds to halving the volume, as expected.

x

y

$y = \sqrt{x}$

(1, 1)

(a) y

$y = \sqrt[3]{x}$

10 9

8
7
6
y

3
2
1

$y = \frac{5}{x}$

1 2 3 4 5 6 7 8 9 10

(1, 1)

(b)
x
5
4

x

$y = \sqrt{x}$

y

Figure 0.3.6

P (Pressure)

$2P_0$ P_0

V (Volume)

V_0 V $\frac{1}{2}$ 0

Figure 0.3.7 Doubling pressure corresponds to halving volume

If $p = 1/n$, where n is a positive integer, then the power functions $f(x) = x^p$ have the form $f(x) = x^{1/n} = \sqrt[n]{x}$

In particular, if $n = 2$, then $f(x) = \sqrt{x}$, and if $n = 3$, then $f(x) = \sqrt[3]{x}$. The graphs of these functions are shown in parts (*a*) and (*b*) of Figure 0.3.8.

Since every real number has a real cube root, the domain of the function $f(x) = \sqrt[3]{x}$ is $(-\infty, +\infty)$, and hence the graph of $y = \sqrt[3]{x}$ extends over the entire x-axis. In contrast, the graph of $y = \sqrt{x}$ extends only over the interval $[0, +\infty)$ because $\sqrt{x}$ is imaginary for negative x. As illustrated in Figure 0.3.8*c*, the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ form the upper and lower halves of the parabola $x = y^2$.

In general, the graph of $y = \sqrt[n]{x}$ extends over the entire x-axis if n is odd, but extends only over the interval $[0, +\infty)$ if n is even.

Figure 0.3.8
(*c*)

y
4
3
2
1
x

$y = -\sqrt{x}$

$y = x^{2/3}$

x

POWER FUNCTIONS WITH NONINTEGER EXPONENTS

Power functions can have other fractional exponents. Some examples are $f(x) = x^{2/3}$, $f(x) = \sqrt[5]{x^3}$, $f(x) = x^{-7/8}$ (2)

−4 −3 −2 −1 1234 **Figure 0.3.9**

The graph of $f(x) = x^{2/3}$ is shown in Figure 0.3.9. We will discuss expressions involving irrational exponents later.

TECHNOLOGY MASTERY

Graphing utililties sometimes omit portions of the graph of a function involving fractional exponents (or radicals). If $f(x) = x^{p/q}$, where p/q is a positive fraction in *lowest terms*, then you can circumvent this problem as follows:

- If p is even and q is odd, then graph $g(x) = |x|^{p/q}$ instead of $f(x)$.
- If p is odd and q is odd, then graph $g(x) = (|x|/x)|x|^{p/q}$ instead of $f(x)$.

Use a graphing utility to generate graphs of $f(x) = \sqrt[5]{x^2}$ and $f(x) = x^{-7/8}$ that show all of their signif icant features.

The constant 0 is a polynomial called

POLYNOMIALS

A ***polynomial in x*** is a function that is expressible as a sum of finitely many terms of the form cx^n, where c is a constant and n is a nonnegative integer. Some examples of polynomials are

$2x + 1$, $3x^2 + 5x - \sqrt{2}$, x^3, $4\ (= 4x^0)$, $5x^7 - x^4 + 3$ The function $(x^2 - 4)^3$ is also a polynomial because it can be expanded by the binomial formula (see the inside front cover) and expressed as a sum of terms of the form cx^n:

$$(x^2-4)^3 = (x^2)^3 - 3(x^2)^2(4) + 3(x^2)(4^2) - (4^3) = x^6 - 12x^4 + 48x^2 - 64 \quad (3)$$

A general polynomial can be written in either of the following forms, depending on whether one wants the powers of x in ascending or descending order: $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$

A more detailed review of polynomials appears in Appendix C.

The constants $c_0, c_1, \ldots, c_n$ are called the ***coefficients*** of the polynomial. When a polyno mial is expressed in one of these forms, the highest power of x that occurs with a nonzero coefficient is called the ***degree*** of the polynomial. Nonzero constant polynomials are con sidered to have degree 0, since we can write $c = cx^0$. Polynomials of degree 1, 2, 3, 4, and 5 are described as ***linear***, ***quadratic***, ***cubic***, ***quartic***, and ***quintic***, respectively. For example, $3 + 5x$ $x^2 - 3x + 1$ $2x^3 - 7$

$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$

the ***zero polynomial***. In this text we

Has c (li

Has degree (quadratic

Has degree 3 (cubic)

will take the degree of the zero poly nomial to be undefined. Other texts may use different conventions for the

$8x^4 - 9x^3 + 5x - 3$ $\sqrt{3} + x^3 + x^5$ $(x^2 - 4)^3$

degree of the zero polynomial.

(3)]

Has de (qua

Has degree 5 (quintic)

Has degree 6 [see

The natural domain of a polynomial in x is $(-\infty, +\infty)$, since the only operations involved are multiplication and addition; the range depends on the particular polynomial. We already know that the graphs of polynomials of degree 0 and 1 are lines and that the graphs of polynomials of degree 2 are parabolas. Figure 0.3.10 shows the graphs of some typical polynomials of higher degree. Later, we will discuss polynomial graphs in detail, but for now it suffices to observe that graphs of polynomials are very well behaved in the sense that they have no discontinuities or sharp corners. As illustrated in Figure 0.3.10, the graphs of polynomials wander up and down for awhile in a roller-coaster fashion, but eventually that behavior stops and the graphs steadily rise or fall indefinitely as one travels along the curve in either the positive or negative direction. We will see later that the number of peaks and valleys is less than the degree of the polynomial.

y y y x x x y x

Degree 2 Degree 4 Degree 3 Degree 5 **Figure 0.3.10**

RATIONAL FUNCTIONS

A function that can be expressed as a ratio of two polynomials is called a ***rational function***. If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

consists of all values of x such that $Q(x) \neq 0$. For example, the domain of the rational function

$f(x) = x^2 + 2x\ x^2 - 1$

consists of all values of x, except $x = 1$ and $x = -1$. Its graph is shown in Figure 0.3.11 along with the graphs of two other typical rational functions.

The graphs of rational functions with nonconstant denominators differ from the graphs of polynomials in some essential ways:

• Unlike polynomials whose graphs are continuous (unbroken) curves, the graphs of rational functions have discontinuities at the points where the denominator is zero. • Unlike polynomials, rational functions may have numbers at which they are not defined. Near such points, many rational functions have graphs that closely approxi mate a vertical line, called a ***vertical asymptote***. These are represented by the dashed vertical lines in Figure 0.3.11.

• Unlike the graphs of nonconstant polynomials, which eventually rise or fall indefi nitely, the graphs of many rational functions eventually get closer and closer to some horizontal line, called a ***horizontal asymptote***, as one traverses the curve in either the positive or negative direction. The horizontal asymptotes are represented by the dashed horizontal lines in the first two parts of Figure 0.3.11. In the third part of the figure the x-axis is a horizontal asymptote.

y

4

y

4

−5 5 −4

x

x

y

3

−5 7 −3

x

−4 4

$x^2 - 1\ x^2 - 1$

$y =$

$x^2 + 2x$

$y = 3$

$y =$

$x^2 - 2x - 3$

$x^2 + 1$

Figure 0.3.11

ALGEBRAIC FUNCTIONS

Functions that can be constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, multiplication, division, and root extraction) are called ***algebraic functions***. Some examples are

$$f(x) = x^2 - 4,\ f(x) = 3\sqrt[3]{x}(2 + x),\ f(x) = x^{2/3}(x + 2)^2$$

As illustrated in Figure 0.3.12, the graphs of algebraic functions vary widely, so it is difficult to make general statements about them. Later in this text we will develop general calculus methods for analyzing such functions.

THE FAMILIES $y = A \sin Bx$ AND $y = A \cos Bx$

Many important applications lead to trigonometric functions of the form

In this text we will assume that the in dependent variable of a trigonometric function is in radians unless otherwise stated. A review of trigonometric func tions can be found in Appendix B.

$f(x) = A \sin(Bx - C)$ and $g(x) = A \cos(Bx - C)$ (4)

where A, B, and C are nonzero constants. The graphs of such

functions can be obtained by stretching, compressing, x
translating, and reflecting the graphs of $y = \sin x$ and $y = \cos$

y

4

3

2

1

x

y

5
4
3
2

x
1

x

−5 −4 −3 −2 −1 12345

−3 −2 −1 12 −5

y

15
10
5

−4 −3 −2 −1 1

$y = \sqrt{x^2 - 4}$ $y = x^{2/3}(x + 2)^2$ $y = \sqrt[3]{x}(2 + x)^3$ **Figure 0.3.12**

appropriately. To see why this is so, let us start with the case where $C = 0$ and consider how the graphs of the equations

$$y = A \sin Bx \text{ and } y = A \cos Bx$$

relate to the graphs of $y = \sin x$ and $y = \cos x$. If A and B are positive, then the effect of the constant A is to stretch or compress the graphs of $y = \sin x$ and $y = \cos x$ vertically and the effect of the constant B is to compress or stretch the graphs of $\sin x$ and $\cos x$ horizontally. For example, the graph of $y = 2 \sin 4x$ can be obtained by stretching the graph of $y = \sin x$ vertically by a factor of 2 and compressing it horizontally by a factor of 4. (Recall from Section 0.2 that the multiplier of x *stretches* when it is less than 1 and *compresses* when it is greater than 1.) Thus, as shown in Figure 0.3.13, the graph of $y = 2 \sin 4x$ varies between −2 and 2, and repeats every $2\pi/4 = \pi/2$ units.

y

2

x

1

−2c −c c 2c −1

−2

$y = 2 \sin 4x$ $y = \sin x$

Figure 0.3.13

In general, if A and B are positive numbers, then the graphs of

$$y = A \sin Bx \text{ and } y = A \cos Bx$$

oscillate between $-A$ and A and repeat every $2\pi/B$ units, so we say that these functions have ***amplitude*** A and ***period*** $2\pi/B$. In addition, we define the ***frequency*** of these func tions to be the reciprocal of the period, that is, the frequency is $B/2\pi$. If A or B is negative, then these constants cause reflections of the graphs about the axes as well as compressing or stretching them; and in this case the amplitude, period, and frequency are given by

$|B|$, frequency = $|B|$

amplitude = $|A|$, period =
2π 2π

Example 2 Make sketches of the following graphs that show the period and amplitude. (a) $y = 3 \sin 2\pi x$ (b) $y = -3 \cos 0.5x$ (c) $y = 1 + \sin x$

Solution (a). The equation is of the form $y = A \sin Bx$ with $A = 3$ and $B = 2\pi$, so the graph has the shape of a sine function, but it has an amplitude of $A = 3$ and a period of $2\pi/B = 2\pi/2\pi = 1$ (Figure 0.3.14*a*).

Solution (b). The equation is of the form $y = A \cos Bx$ with $A = -3$ and $B = 0.5$, so the graph has the shape of a cosine curve that has been reflected about the x-axis (be cause $A = -3$ is negative), but with amplitude $|A| = 3$ and period $2\pi/B = 2\pi/0.5 = 4\pi$ (Figure 0.3.14*b*).

Solution (c). The graph has the shape of a sine curve that has been translated up 1 unit (Figure 0.3.14*c*).

y y

3 3 2 y

Amplitude Amplitude

x x

1 4c

x

−3 Period −3 Amplitude Period C c o

Figure 0.3.14
(*a*) (*b*) (*c*)

THE FAMILIES $y = A \sin(Bx - C)$ AND $y = A \cos(Bx - C)$ To investigate the graphs of the more general families

$$y = A \sin(Bx - C) \text{ and } y = A \cos(Bx - C)$$

it will be helpful to rewrite these equations as

$$y = A \sin\left[B\left(x - \frac{C}{B}\right)\right] \text{ and } y = A \cos\left[B\left(x - \frac{C}{B}\right)\right]$$

In this form we see that the graphs of these equations can be obtained by translating the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ to the left or right, depending on the sign of C/B. For example, if $C/B > 0$, then the graph of

$$y = A \sin[B(x - C/B)] = A \sin(Bx - C)$$

can be obtained by translating the graph of $y = A \sin Bx$ to the right by C/B units (Fig ure 0.3.15). If $C/B < 0$, the graph of $y = A \sin(Bx - C)$ is obtained by translating the graph of $y = A \sin Bx$ to the left by $|C/B|$ units.

y C/B

Amplitude = A
x
y = A sin (Bx − C)
y = A sin Bx

Figure 0.3.15

Example 3 Find the amplitude and period of

$$y = 3\cos\left(2x + \frac{\pi}{2}\right)$$

and determine how the graph of $y = 3\cos 2x$ should be translated to produce the graph of this equation. Confirm your results by graphing the equation on a calculator or computer.

Solution. The equation can be rewritten as

y

3

$2x -$

$-\frac{\pi}{2}$ $= 3\cos 2\ x -$

$-\frac{\pi}{4}$

$y = 3\cos$

form

x

which is of the $y = A\cos$ B $x - \frac{C}{B}$

O C #3 c o

−3

Figure 0.3.16

with $A = 3$, $B = 2$, and $C/B = -\pi/4$. It follows that the amplitude is $A = 3$, the period is $2\pi/B = \pi$, and the graph is obtained by translating the graph of $y = 3\cos 2x$ left by $|C/B| = \pi/4$ units (Figure 0.3.16).

✔ QUICK CHECK EXERCISES 0.3 (See page 38 for answers.)

1. Consider the family of functions $y = x^n$, where n is an in teger. The graphs of $y = x^n$ are symmetric with respect to the y-axis if n is . These graphs are symmetric with respect to the origin if n is . The y-axis is a vertical asymptote for these graphs if n is .

2. What is the natural domain of a polynomial?

3. Consider the family of functions $y = x^{1/n}$, where n is a nonzero integer. Find the natural domain of these functions if n is
(a) positive and even (b) positive and odd (c) negative and even (d) negative and odd.

4. Classify each equation as a polynomial, rational, algebraic, or not an algebraic function.
(a) $y = \sqrt{x} + 2$ (b) $y = \sqrt{3x^4 - x + 1}$ (c) $y = 5x^3 + \cos 4x$ (d) $y = \frac{x^2 + 5}{2x - 7}$ (e) $y = 3x^2 + 4x^{-2}$

5. The graph of $y = A\sin Bx$ has amplitude and is periodic with period .

EXERCISE SET 0.3 Graphing Utility

1. (a) Find an equation for the family of lines whose members have slope $m = 3$.
(b) Find an equation for the member of the family that passes through $(-1, 3)$.
(c) Sketch some members of the family, and label them with their equations. Include the line in part (b).

2. Find an equation for the family of lines whose members are perpendicular to those in Exercise 1.

3. (a) Find an equation for the family of lines with y-intercept $b = 2$.
(b) Find an equation for the member of the family whose angle of inclination is 135°.
(c) Sketch some members of the family, and label them with their equations. Include the line in part (b).

4. Find an equation for
(a) the family of lines that pass through the origin
(b) the family of lines with x-intercept $a = 1$
(c) the family of lines that pass through the point $(1, -2)$
(d) the family of lines parallel to $2x + 4y = 1$.

5. Find an equation for the family of lines tangent to the circle with center at the origin and radius 3.

6. Find an equation for the family of lines that pass through the intersection of $5x - 3y + 11 = 0$ and $2x - 9y + 7 = 0$.

7. The U.S. Internal Revenue Service uses a 10-year linear de preciation schedule to determine the value of various busi ness items. This means that an item is assumed to have a value of zero at the end of the tenth year and that at inter mediate times the value is a linear function of the elapsed time. Sketch some typical depreciation lines, and explain the practical significance of the y-intercepts.

8. Find all lines through $(6, -1)$ for which the product of the x- and y-intercepts is 3.

(c) $y = -1/x^8$ (d) $y = \sqrt{x^2} - 1$

(e) $y = \sqrt[4]{x} - 2$ (f) $y = -\sqrt[5]{x^2}$

9–10 State a geometric property common to all lines in the family, and sketch five of the lines. ■

9. (a) The family $y = -x + b$
(b) The family $y = mx - 1$
(c) The family $y = m(x + 4) + 2$
(d) The family $x - ky = 1$

10. (a) The family $y = b$
(b) The family $Ax + 2y + 1 = 0$
(c) The family $2x + By + 1 = 0$
(d) The family $y - 1 = m(x + 1)$

11. In each part, match the equation with one of the accompanying graphs.

(a) $y = \sqrt[5]{x}$ (b) $y = 2x^5$

y
y
$(x - 3)^5$
x
x
x
y

I II III

y
x
x
x
y
y

IV V VI

Figure Ex-11

(b) Sketch the graph of $y = \sqrt{x} + b$ for $b = \pm1, \pm2$, and ±3 in a single coordinate system.
(c) Sketch some typical members of the family of curves $y = a\sqrt{x} + b$.

17–18 Sketch the graph of the equation by making appropriate transformations to the graph of a basic power function. If you have a graphing utility, use it to check your work. ■

17. (a) $y = 2(x + 1)^2$ (b) $y = -3(x - 2)^3$ (c) $y = -\frac{3}{(x + 1)^2}$ (d) $y = \frac{1}{(x - 3)^5}$

18. (a) $y = 1 - \sqrt{x + 2}$ (b) $y = 1 - \sqrt[3]{x + 2}$ (c) $y = \frac{5}{(1 - x)^3}$ (d) $y = \frac{2}{(4 + x)^4}$

19. Sketch the graph of $y = x^2 + 2x$ by completing the square and making appropriate transformations to the graph of $y = x^2$.

20. (a) Use the graph of $y = \sqrt{x}$ to help sketch the graph of $y = \sqrt{|x|}$.
(b) Use the graph of $y = \sqrt[3]{x}$ to help sketch the graph of $y = \sqrt[3]{|x|}$.

12. The accompanying table gives approximate values of three functions: one of the form kx^2, one of the form kx^{-3}, and one of the form $kx^{3/2}$. Identify which is which, and estimate k in each case.

x	0.25	0.37	2.1	4.0	5.8	6.2	7.9	9.3
$f(x)$	640	197	1.08	0.156	0.0513	0.0[illegible]2[illegible]	0.0124	0.0
$g(x)$	0.03	0.06	2.20	8.00	16.8	19	43.2	12
$h(x)$	0.250	0.450	6.09	16.0	27.9	30.9	44.4	56.7

84

Table Ex-12

21. As discussed in this section, Boyle's law states that at a constant temperature the pressure P exerted by a gas is related to the volume V by the equation $PV = k$.
(a) Find the appropriate units for the constant k if pressure (which is force per unit area) is in newtons per square meter (N/m^2) and volume is in cubic meters (m^3).
(b) Find k if the gas exerts a pressure of 20,000 N/m^2 when the volume is 1 liter (0.001 m^3).
(c) Make a table that shows the pressures for volumes of 0.25, 0.5, 1.0, 1.5, and 2.0 liters.
(d) Make a graph of P versus V.

22. A manufacturer of cardboard drink containers wants to construct a closed rectangular container that has a square base and will hold $\frac{1}{10}$ liter (100 cm^3). Estimate the dimensions of the container that will require the least amount of material

13–14 Sketch the graph of the equation for $n = 1, 3$, and 5 in one coordinate system and for $n = 2, 4$, and 6 in another coordinate system. If you have a graphing utility, use it to check your work. ■

13. (a) $y = -x^n$ (b) $y = 2x^{-n}$ (c) $y = (x - 1)^{1/n}$

14. (a) $y = 2x^n$ (b) $y = -x^{-n}$ (c) $y = -3(x + 2)^{1/n}$

15. (a) Sketch the graph of $y = ax^2$ for $a = \pm1, \pm2$, and ±3 in a single coordinate system.
(b) Sketch the graph of $y = x^2 + b$ for $b = \pm1, \pm2$, and ±3 in a single coordinate system.
(c) Sketch some typical members of the family of curves $y = ax^2 + b$.

16. (a) Sketch the graph of $y = a\sqrt{x}$ for $a = \pm1, \pm2$, and ±3 in a

single coordinate system.
for its manufacture.

23–24 A variable y is said to be ***inversely proportional to the square of a variable*** x if y is related to x by an equation of the form $y = k/x^2$, where k is a nonzero constant, called the ***constant of proportionality***. This terminology is used in these exercises. ■

23. According to ***Coulomb's law***, the force F of attraction be tween positive and negative point charges is inversely pro portional to the square of the distance x between them. (a) Assuming that the force of attraction between two point charges is 0.0005 newton when the distance between them is 0.3 meter, find the constant of proportionality (with proper units).

(b) Find the force of attraction between the point charges when they are 3 meters apart.

(c) Make a graph of force versus distance for the two charges. *(cont.)*

(d) What happens to the force as the particles get closer and closer together? What happens as they get farther and farther apart?

24. It follows from Newton's Law of Universal Gravitation that the weight W of an object (relative to the Earth) is inversely proportional to the square of the distance x between the object and the center of the Earth, that is, $W = C/x^2$.

(a) Assuming that a weather satellite weighs 2000 pounds on the surface of the Earth and that the Earth is a sphere of radius 4000 miles, find the constant C.

25. Each curve in the family $y = 2x + b$ is parallel to the line $y = 2x$.

$= x^2 + bx + c$ is a translation

26. Each curve in the family y −3

of the graph of $y = x^2$.

27. If a curve passes through the point (2, 6) and y is inversely proportional to x, then the constant of proportionality is 3.

28. Curves in the family $y = -5 \sin(A\pi x)$ have amplitude 5 and period $2/|A|$.

29. In each part, match the equation with one of the accom

2

panying graphs, and give the equations for the horizontal and vertical asymptotes.

−1 o

(a) $y = x^2$

$x^2 - x - 2$ (b) $y = x - 1$

$x^2 - x - 6$

(c) $y = 2x^4$

$x^4 + 1$ (d) $y = 4\ (x + 2)^2$

33. In each part, find an equation for the graph that has

y y

form $y = y_0 + A \sin(Bx - C)$.

(b) Find the weight of the satellite when it is 1000 miles above the surface of the Earth.

(c) Make a graph of the satellite's weight versus its distance from the center of the Earth.

(d) Is there any distance from the center of the Earth at which the weight of the satellite is zero? Explain your reasoning.

25–28 True–False Determine whether the statement is true or false. Explain your answer. ■

30. Find an equation of the form $y = k/(x^2 + bx + c)$ whose graph is a reasonable match to that in the ac companying figure. If you have a graphing utility, use it to check your work.

y

x

−1

−1 1

Figure Ex-30

31–32 Find an equation of the form $y = D + A \sin Bx$ or $y = D + A \cos Bx$ for each graph. ■

31.

y y y

3 4 5

x x x

4 −4

6

−5

Not drawn to scale Not drawn to scale Not drawn to scale (a) (b) (c)

Figure Ex-31

32.

y y y

2

3

5

x x x

6

−5

Not drawn to scale Not drawn to scale Not drawn to scale (a) (b) (c)

Figure Ex-32

the

y y

6

x

x

3 y 1

x

o

−1

x

x

9c

e −1

$V = 120\sqrt{2}$ volts (V) at a frequency of 60 hertz (Hz). Write an equation that expresses V as a function of the time t, assuming that $V = 0$ if $t = 0$. [*Note:* 1 Hz = 1 cycle per second.]

I II

y

y

x

x

III IV

Figure Ex-29

Not drawn to scale Not drawn to scale Not drawn to scale (a) (b) (c)

Figure Ex-33

34. In the United States, a standard electrical outlet supplies sinusoidal electrical current with a maximum voltage of

35–36 Find the amplitude and period, and sketch at least two periods of the graph by hand. If you have a graphing utility, use it to check your work. ■

35. (a) $y = 3 \sin 4x$ (b) $y = -2 \cos \pi x$

(c) $y = 2 + \cos$

x 2

36. (a) $y = -1 - 4 \sin 2x$ (b) $y = \frac{1}{2}\cos(3x - \pi)$

(c) $y = -4 \sin$

x

3 + 2π

37. Equations of the form

$$x = A_1 \sin \omega t + A_2 \cos \omega t$$

arise in the study of vibrations and other periodic motion. Express the equation

$$x = 5\sqrt{3} \sin 2\pi t + \frac{5}{2} \cos 2\pi t$$

in the form $x = A \sin(\omega t + \theta)$, and use a graphing utility to confirm that both equations have the same graph.

38. Determine the number of solutions of $x = 2 \sin x$, and use a graphing or calculating utility to estimate them.

✔ QUICK CHECK ANSWERS 0.3

1. even; odd; negative **2.** $(-\ , +\ )$ **3.** (a) $[0, +\ )$ (b) $(-\ , +\ )$ (c) $(0, +\ )$ (d) $(-\ , 0) \cup (0, +\ )$ **4.** (a) algebraic (b) polynomial (c) not algebraic (d) rational (e) rational **5.** $|A|$; $2\pi/|B|$

INVERSE FUNCTIONS; INVERSE TRIGONOMETRIC FUNCTIONS

In everyday language the term "inversion" conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers, and in music a melodic inversion reverses an ascending interval to the corresponding descending interval. In mathematics the term ***inverse*** *is used to describe functions that reverse one another in the sense that each undoes the effect of the other. In this section we discuss this fundamental mathematical idea. In particular, we introduce inverse trigonometric functions to address the problem of recovering an angle that could produce a given trigonometric function value.*

y

$y = x^3 + 1$

$y = x^3 + 1$

INVERSE FUNCTIONS

The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics.

Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

x

x

$y = x^3 + 1$

can be solved for x as

a function of y: $x = \sqrt[3]{y - 1}$

$\boxed{y = f(x)}$

$\boxed{x = g(y)}$

y

$x = \sqrt[3]{y - 1}$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure 0.4.1).

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely.

y

For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y - 1}$ discussed above. When these functions are composed in either order, they cancel out the effect of one another

x

$x = \sqrt[3]{y - 1}$

in the sense that

$g(f(x)) = \sqrt[3]{f(x) - 1} = \sqrt[3]{(x^3 + 1) - 1} = x$

$f(g(y)) = [g(y)]^3 + 1 = (\sqrt[3]{y - 1})^3 + 1 = y$ (1)

Figure 0.4.1

Pairs of functions with these two properties are so important that there is special terminology for them.

WARNING

If f is a function, then the -1 in the symbol f^{-1} always denotes an inverse and *never* an exponent. That is, $f^{-1}(x)$ *never means* $\frac{1}{f(x)}$

0.4.1 definition If the functions f and g satisfy the two conditions $g(f(x)) = x$ for every x in the domain of f, $f(g(y)) = y$ for every y in the domain of g, then we say that ***f is an inverse of g*** and ***g is an inverse of f*** or that ***f and g are inverse functions***.

It can be shown (Exercise 62) that if a function f has an inverse, then that inverse is unique. Thus, if a function f has an inverse, then we are entitled to talk about "the" inverse of f, in which case we denote it by the symbol f^{-1}.

Example 1 The computations in (1) show that $g(y) = \sqrt[3]{y - 1}$ is the inverse of $f(x) = x^3 + 1$. Thus, we can express g in inverse notation as

$f^{-1}(y) = \sqrt[3]{y - 1}$

and we can express the equations in Definition 0.4.1 as

$f^{-1}(f(x)) = x$ for every x in the domain of f

$f(f^{-1}(y)) = y$ for every y in the domain of f^{-1} (2)

We will call these the ***cancellation equations*** for f and f^{-1}.

CHANGING THE INDEPENDENT VARIABLE

The formulas in (2) use x as the independent variable for f and y as the independent variable for f^{-1}. Although it is often convenient to use different independent variables for f and f^{-1}, there will be occasions on which it is desirable to use the same independent variable for both. For example, if we want to graph the functions f and f^{-1} together in the same xy-coordinate system, then we would want to use x as the independent variable and y as the dependent variable for both functions. Thus, to graph the functions $f(x) = x^3 + 1$ and $f^{-1}(y) = \sqrt[3]{y - 1}$ of Example 1 in the same xy-coordinate system, we would change the independent variable y to x, use

y as the dependent variable for both functions, and graph the equations $y = x^3 + 1$ and $y = \sqrt[3]{x - 1}$

We will talk more about graphs of inverse functions later in this section, but for reference we give the following reformulation of the cancellation equations in (2) using x as the independent variable for both f and f^{-1}:

$f^{-1}(f(x)) = x$ for every x in the domain of f

$f(f^{-1}(x)) = x$ for every x in the domain of f^{-1} (3)

Example 2 Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

Solution (a). $f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x$

$f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$

Solution (b). $f^{-1}(f(x)) = f^{-1}(x^3) = (x^3)^{1/3} = x$

$f(f^{-1}(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$

In general, if a function f has an inverse and $f(a) = b$, then the procedure in Example 3 shows that $a = f^{-1}(b)$; that is, f^{-1} maps each output of f back into the corresponding input (Figure 0.4.2).

b

f

a

f^{-1}

Figure 0.4.2 If f maps a to b, then f^{-1} maps b back to a.

Example 3 Given that the function f has an inverse and that $f(3) = 5$, find $f^{-1}(5)$.

Solution. Apply f^{-1} to both sides of the equation $f(3) = 5$ to obtain $f^{-1}(f(3)) = f^{-1}(5)$

and now apply the first equation in (3) to conclude that $f^{-1}(5) = 3$.

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (3) imply the following relationships between the domains and ranges of f and f^{-1}:

domain of f^{-1} = range of f

range of f^{-1} = domain of f (4)

One way to show that two sets are the same is to show that each is a subset of the other.

An alternative way to obtain a formula for $f^{-1}(x)$ with x as the independent variable is to reverse the roles of x and y at the outset and solve the equation $x = f(y)$ for y as a function of x.

Thus we can establish the first equality in (4) by showing that the domain of f^{-1} is a subset of the range of f and that the range of f is a subset of the domain of f^{-1}. We do this as follows: The first equation in (3) implies that f^{-1} is defined at $f(x)$ for all values of x in the domain of f, and this implies that the range of f is a subset of the domain of f^{-1}. Conversely, if x is in the domain of f^{-1}, then the second equation in (3) implies that x is in the range of f because it is the image of $f^{-1}(x)$. Thus, the domain of f^{-1} is a subset of the range of f. We leave the proof of the second equation in (4) as an exercise.

A METHOD FOR FINDING INVERSE FUNCTIONS

At the beginning of this section we observed that solving $y = f(x) = x^3 + 1$ for x as a function of y produces $x = f^{-1}(y) = \sqrt[3]{y-1}$. The following theorem shows that this is not accidental.

0.4.2 theorem *If an equation* $y = f(x)$ *[...]* $= g(y)$*, then* f *has an inverse and that i[...]*

proof Substituting $y = f(x)$ into $x = g(y)$ yields $x = g(f(x))$, which confirms the first equation in Definition 0.4.1, and substituting $x = g(y)$ into $y = f(g(y))$ yields $y = f(g(y))$, which confirms the second equation in Definition 0.4.1. ■

Theorem 0.4.2 provides us with the following procedure for finding the inverse of a function.

A Procedure for Finding the Inverse of a Function f

Step 1. Write down the equation $y = f(x)$.

Step 2. If possible, solve this equation for x as a function of y.

Step 3. The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

Step 4. If y is acceptable as the independent variable for the inverse function, then you are done, but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

Example 4 Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the indepen dent variable, and state the domain of f^{-1}.

Solution. Following the procedure stated above, we first write

$$y = \sqrt{3x-2}$$

Then we solve this equation for x as a function of y:

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

which tells us that $f^{-1}(y) = \frac{1}{3}(y^2 + 2)$ (5) Since we want x to be the independent variable, we reverse x and y in (5) to produce the formula $f^{-1}(x) = \frac{1}{3}(x^2 + 2)$ (6) We know from (4) that the domain of f^{-1} is the range of f. In general, this need not be the same as the natural domain of the fomula for f^{-1}. Indeed, in this example the natural domain of (6) is $(-\infty, +\infty)$, whereas the range of $f(x) = \sqrt{3x-2}$ is $[0, +\infty)$. Thus, if we want to make the domain of f^{-1} clear, we must express it explicitly by rewriting (6) as $f^{-1}(x) =$

$$\tfrac{1}{3}(x^2 + 2),\ x \geq 0$$

EXISTENCE OF INVERSE FUNCTIONS

The procedure we gave above for finding the inverse of a function f was based on solving the equation $y = f(x)$ for x as a function of y. This procedure can fail for two reasons—the function f may not have an inverse, or it may have an inverse but the equation $y = f(x)$ cannot be solved explicitly for x as a function of y. Thus, it is important to establish conditions that ensure the existence of an inverse, even if it cannot be found explicitly.

If a function f has an inverse, then it must assign distinct outputs to distinct inputs. For example, the function $f(x) = x^2$ cannot have an inverse because it assigns the same value to $x = 2$ and $x = -2$, namely,

$$f(2) = f(-2) = 4$$

Thus, if $f(x) = x^2$ were to have an inverse, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. But this is impossible because $f^{-1}(4)$ cannot have two different values. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x as a function of y. We run into trouble immediately because the resulting equation $x = \pm\sqrt{y}$ does not express x as a *single* function of y.

A function that assigns distinct outputs to distinct inputs is said to be ***one-to-one*** or ***invertible***, so we know from the preceding discussion that if a function f has an inverse, then it must be one-to-one. The converse is also true, thereby establishing the following theorem.

> **0.4.3 theorem** *A function has an inverse if and only if it is one-to-one.*

Stated algebraically, a function f is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; stated geometrically, a function f is one-to-one if and only if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure 0.4.3). The latter statement together with Theorem 0.4.3 provides the following geometric test for determining whether a function has an inverse.

Figure 0.4.3

$y = f(x)$

$f(x_2)$ $f(x_1)$

y

x

x_1 x_2

One-to-one, since $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

$y = f(x)$

$f(x_1) = f(x_2)$

y

x

x_1 x_2

Not one-to-one, since $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

> **0.4.4 theorem** **(*The Horizontal Line Test*)** *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

Example 5 Use the horizontal line test to show that $f(x) = x^2$ has no inverse but that $f(x) = x^3$ does.

Solution. Figure 0.4.4 shows a horizontal line that cuts the graph of $y = x^2$ more than once, so $f(x) = x^2$ is not invertible. Figure 0.4.5 shows that the graph of $y = x^3$ is cut at most once by any horizontal line, so $f(x) = x^3$ is invertible. [Recall from Example 2 that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.]

y $y = x^3$

x

Figure 0.4.5

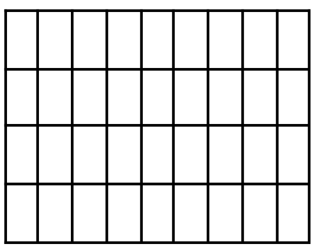
y
$y = x^2$

4

x
−2 2

Figure 0.4.4

y
4



3
2
1
x

Example 6 Explain why the function f that is graphed in Figure 0.4.6 has an inverse, and find $f^{-1}(3)$.

Solution. The function f has an inverse since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph

−3 −2 −1 1 0 23456**Figure 0.4.6**

The function $f(x) = x^3$ in Figure 0.4.5 is an example of an increasing function. Give an example of a decreasing func tion and compute its inverse.

we see that $f(2) = 3$, so $f^{-1}(3) = 2$.

INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

A function whose graph is always rising as it is traversed from left to right is said to be an ***increasing function***, and a function whose graph is always falling as it is traversed from left to right is said to be a ***decreasing function***. If x_1 and x_2 are points in the domain of a function f, then f is increasing if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$

and f is decreasing if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2$$

(Figure 0.4.7). It is evident geometrically that increasing and decreasing functions pass the horizontal line test and hence are invertible.

$f(x_1)$
y
Increasing

y

Decreasing $f(x_2)$

$f(x_1)$ x $f(x_2)$

x

x_1 x_2 x_1 x_2 $f(x_1) > f(x_2)$ if $x_1 < x_2$

$f(x_1) < f(x_2)$ if $x_1 < x_2$

Figure 0.4.7

y
(b, a)
y = x

b

a

b

a

GRAPHS OF INVERSE FUNCTIONS

(a, b)

Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions so we can compare the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

x

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$.

The points (a, b) and (b, a) are reflections about $y = x$.

coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure 0.4.8), and hence the graphs of

Figure 0.4.8

In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the

In summary, we have the following result.

y

$y = f^{-1}(x)$ (b, a)

$y = x$

x

$y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure 0.4.9).

$y = f(x)$

(a, b)

> **0.4.5 theorem** *If f has an*
> *reflections of one another*
> *image of the other with res*

The graphs of f and f^{-1} are reflections about $y = x$.

Figure 0.4.9

Example 7 Figure 0.4.10 shows the graphs of the inverse functions discussed in Examples 2 and 4.

$y = x$ $y = x$ $y = x$ $y = 2x$ y y y

$y = x^3$

$y = \frac{1}{3}(x^2 + 2)$

$y = x^{1/3}$

$y = \sqrt{3x - 2}$

$y = \frac{1}{2}x$

x x

x

Figure 0.4.10

RESTRICTING DOMAINS FOR INVERTIBILITY

If a function g is obtained from a function f by placing restrictions on the domain of f, then g is called a ***restriction*** of f. Thus, for example, the function

$$g(x) = x^3,\ x \geq 0$$

is a restriction of the function $f(x) = x^3$. More precisely, it is called the restriction of x^3 to the interval $[0, +\ )$.

Sometimes it is possible to create an invertible function from a function that is not invertible by restricting the domain appropriately. For example, we showed earlier that $f(x) = x^2$ is not invertible. However, consider the restricted functions

$$f_1(x) = x^2,\ x \geq 0 \text{ and } f_2(x) = x^2,\ x \leq 0$$

the union of whose graphs is the complete graph of $f(x) = x^2$ (Figure 0.4.11). These restricted functions are each one-to-one (hence invertible), since their graphs pass the hor izontal line test. As illustrated in Figure 0.4.12, their inverses are

$$f^{-1}_1(x) = \sqrt{x} \text{ and } f^{-1}_2(x) = -\sqrt{x}$$

y $y = x^2$ $y = x$, $x \geq 0$ ^{2}, $x \leq 0$

5

y

4 3 2 1

$y = \sqrt{x}\ x$

$y = x^2$ $y = x$, $x \geq 0$ ^{2}, $x \leq 0$ x

Figure 0.4.11

−3 12345 −2 −1 −1

−2

Figure 0.4.12
$y = -\sqrt{x}$

INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle x using a known value of sin x, cos x, or some other trigonometric function. Recall that problems of this type involve the computation of "arc functions" such as arcsin x, arccos x, and so forth. We will conclude this section by studying these arc functions from the viewpoint of general inverse functions.

The six basic trigonometric functions do not have inverses because their graphs repeat periodically and hence do not pass the horizontal line test. To circumvent this problem we will restrict the domains of the trigonometric functions to produce one-to-one functions and then define the "inverse trigonometric functions" to be the inverses of these restricted functions. The top part of Figure 0.4.13 shows geometrically how these restrictions are made for sin x, cos x, tan x, and sec x, and the bottom part of the figure shows the graphs of the corresponding inverse functions

$$\sin^{-1} x,\ \cos^{-1} x,\ \tan^{-1} x,\ \sec^{-1} x$$

(also denoted by arcsin x, arccos x, arctan x, and arcsec x). Inverses of cot x and csc x are of lesser importance and will be considered in the exercises.

y 1 x y y 1 x 1 x

C ^ 6 c −1

C ^ 6 i ^{c} C ^ 6 c I C ^ 6 i c −1

−1

$y = \cos x$ $0 \leq x \leq$ c
$y = \tan x$

^ $< x <$ 6
$y = \sec x$

$0 \leq x \leq$ c, $x \neq$ 6

$y = \sin x$ ^ $\leq x \leq$ 6

y 6 c ^{c}

y 6 y y

x x x x

−1 1 ^ 6 −1 1 −1 1 ^

$y = \sin^{-1} x$ $y = \cos^{-1} x$ $y = \tan^{-1} x$ $y = \sec^{-1} x$ **Figure 0.4.13**

The following formal definitions summarize the preceding discussion.

If you have trouble visualizing the cor respondence between the top and bot tom parts of Figure 0.4.13, keep in mind that a reflection about $y = x$ converts vertical lines into horizontal lines, and vice versa; and it converts x-intercepts into y-intercepts, and vice versa.

0.4.7 definition The ***inverse cosine*** inverse of the restricted cosine func

co

0.4.8 definition The ***inverse tangen*** the inverse of the restricted tangent

tan

WARNING

The notations $\sin^{-1} x$, $\cos^{-1} x$,... are reserved exclusively for the inverse trigonometric functions and are not used for reciprocals of the trigonomet ric functions. If we want to express the reciprocal $1/\sin x$ using an exponent, we would write $(\sin x)^{-1}$ and *never* $\sin^{-1} x$.

0.4.9 definition* The ***inverse secan*** inverse of the restricted secant func

sec x

0.4.6 definition The ***inv*** inverse of the restricted

*There is no universal agreement on the definition of $\sec^{-1} x$, and some mathematicians prefer to restrict the domain of $\sec x$ so that $0 \le x<\pi/2$ or $\pi \le x < 3\pi/2$, which was the definition used in some earlier editions of this text. Each definition has advantages and disadvantages, but we will use the current definition to conform with the conventions used by the CAS programs *Mathematica*, *Maple*, and *Sage*.

Table 0.4.1 summarizes the basic properties of the inverse trigonometric functions we have considered. You should confirm that the domains and ranges listed in this table are consistent with the graphs shown in Figure 0.4.13.

Table 0.4.1
properties of inverse trigonometric functions

function	domain	range	basic relationships
$\sin^{-1}$ $\cos^{-1}$ $\tan^{-1}$ $\sec^{-1}$	$(-\infty, -1]$ $[1, +\infty)$	$[0, c/2)$ $(c/2, c]$	$\tan^{-1}(\tan x) = x$ if $-c/2 < x < c/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$[-1, 1]$	$[-c/2, c/2]$	$\sin^{-1}(\sin x) = x$ if $-c/2 \le x \le c/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le c$, $x \ne c/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$
$[-1, 1]$	$[0, c]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le c$	
$(-\infty, +\infty)$	$(-c/2, c/2)$	$\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$	

x

and, more generally, for a given value of y in the interval $-1 \le y \le 1$ you might want to

ME 2 a o −1

y

1

~~0.5~~

Figure 0.4.14

EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose sine is known. For example, you might want to find an angle x in radian measure such that

TECHNOLOGY MASTERY

Refer to the documentation for your calculating utility to determine how to calculate inverse sines, inverse cosines, and inverse tangents; and then confirm Equation (9) numerically by showing that

$\sin^{-1}(0.5) \approx 0.523598775598 \ldots \approx \pi/6$

$\sin x = \frac{1}{2}$ (7)

If $x = \cos^{-1} y$ is viewed as an angle in radian measure whose cosine is y, in what possible quadrants can x lie? An swer the same question for $x = \tan^{-1} y$ and $x = \sec^{-1} y$

solve the equation

$$\sin x = y \quad (8)$$

Because $\sin x$ repeats periodically, this equation has infinitely many solutions for x; how ever, if we solve this equation as

$$x = \sin^{-1} y$$

then we isolate the specific solution that lies in the interval $[-\pi/2, \pi/2]$, since this is the range of the inverse sine. For example, Figure 0.4.14 shows four solutions of Equation (7), namely, $-11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. Of these, $\pi/6$ is the solution in the interval $[-\pi/2, \pi/2]$, so

$$\sin^{-1} \tfrac{1}{2} = \pi/6 \quad (9)$$

In general, if we view $x = \sin^{-1} y$ as an angle in radian measure whose sine is y, then the restriction $-\pi/2 \leq x \leq \pi/2$ imposes the geometric requirement that the angle x in standard position terminate in either the first or fourth quadrant or on an axis adjacent to those quadrants.

Example 8 Find exact values of

(a) $\sin^{-1}(1/\sqrt{2})$ (b) $\sin^{-1}(-1)$

by inspection, and confirm your results numerically using a calculating utility. ***Solution (a).*** Because $\sin^{-1}(1/\sqrt{2}) > 0$, we can view $x = \sin^{-1}(1/\sqrt{2})$ as that angle in the first quadrant such that $\sin \theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin^{-1}(-1) < 0$, we can view $x = \sin^{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin x = -1$. Thus, $\sin^{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin^{-1}(-1) \approx -1.57 \approx -\pi/2$.

TECHNOLOGY MASTERY

There is little to be gained by memoriz

Most calculators do not provide a direct method for calculating inverse secants. In such situations the identity

$$\sec^{-1} x = \cos^{-1}(1/x) \quad (10)$$

is useful (Exercise 50). Use this formula to show that

$\sec^{-1}(2.25) \approx 1.11$ and $\sec^{-1}(-2.25) \approx 2.03$

If you have a calculating utility (such as a CAS) that can find $\sec^{-1} x$ directly, use it to check these values.

IDENTITIES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If we interpret $\sin^{-1} x$ as an angle in radian measure whose sine is x, and if that angle is *nonnegative*, then we can represent $\sin^{-1} x$ geometrically as an angle in a right triangle in which the hypotenuse has length 1 and the side opposite to the angle $\sin^{-1} x$ has length x (Figure 0.4.15*a*). Moreover, the unlabeled acute angle in Figure 0.4.15*a* is $\cos^{-1} x$, since

the cosine of that angle is x, and the unlabeled side in that figure has length $\sqrt{1 - x^2}$ by the Theorem of Pythagoras (Figure 0.4.15*b*). This triangle motivates a number of useful identities involving inverse trigonometric functions that are valid for $-1 \le x \le 1$; for example,

$$\cos(\sin^{-1} x) = 1 - x^2 \quad (12)$$

$$\sin(\cos^{-1} x) = 1 - x^2 \quad (13)$$

$$\tan(\sin^{-1} x) = \frac{x}{1 - x^2} \quad (14)$$

In a similar manner, $\tan^{-1} x$ and $\sec^{-1} x$ can be represented as angles in the right triangles shown in Figures 0.4.15*c* and 0.4.15*d* (verify). Those triangles reveal additional useful

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad (11)$$

ing these identities. What is important is the mastery of the *method* used to obtain them.

identities; for example,

$$\sec(\tan^{-1} x) = 1 + x^2 \quad (15)$$

$$\sin(\sec^{-1} x) = \frac{x^2 - 1}{x} \quad (x \ge 1) \quad (16)$$

x 1

$\cos^{-1} x$ x 1

$\sqrt{1 + x^2}$ $\tan^{-1} x$

$\sqrt{x^2 - 1}$

$\sec^{-1} x$

x

$\sin^{-1} x$

$\sin^{-1} x$

x

$\sqrt{1 - x}$ 1 1 2

(*a*) (*b*) (*c*) (*d*) **Figure 0.4.15**

REMARK The triangle technique does not always produce the most general form of an identity. For example, in Exercise 61 we will ask you to derive the following extension of Formula (16) that is valid for $x \le -1$ as well as $x \ge 1$:

$$\sin(\sec^{-1} x) = \frac{x^2 - 1}{|x|} \quad (|x| \ge 1) \quad (17)$$

Referring to Figure 0.4.13, observe that the inverse sine and inverse tangent are odd func tions; that is,

$\sin^{-1}(-x) = -\sin^{-1}(x)$ and $\tan^{-1}(-x) = -\tan^{-1}(x)$ (18–19)

Example 9 Figure 0.4.16 shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \le x \le \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by Figure 0.4.16). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ does not hold. For example, if the quantity x lies in the interval $\pi/2 \le x \le 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \le x \le \pi/2$, so

$$\sin^{-1}[\sin(x - \pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that $\sin^{-1}$ is an odd function, we can express $\sin^{-1}(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x - \pi)] = -\sin^{-1}[\sin(x - \pi)] = -(x - \pi)$$

This shows that on the interval $\pi/2 \le x \le 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with Figure 0.4.16.

y

c

x

− 4c

ÅO C c oå 4c

C

Figure 0.4.16

✔ QUICK CHECK EXERCISES 0.4 (*See page 52 for answers.*)

1. In each part, determine whether the function f is one-to-one. (a) $f(t)$ is the number of people in line at a movie theater at time t.
(b) $f(x)$ is the measured high temperature (rounded to the nearest °F) in a city on the xth day of the year.
(c) $f(v)$ is the weight of v cubic inches of lead. **2.** A student enters a number on a calculator, doubles it, adds 8 to the result, divides the sum by 2, subtracts 3 from the quo tient, and then cubes the difference. If the resulting number is x, then was the student's original number. **3.** If $(3, -2)$ is a point on the graph of an odd invertible func tion f, then and are points on the graph of f^{-1}.

(b) $\tan^{-1}(1) =$

(c) $\sin^{-1} \frac{1}{2}\sqrt{3} =$

(d) $\cos^{-1} \frac{1}{2} =$

(e) $\sec^{-1}(-2) =$

5. In each part, determine the exact value without using a cal culating utility.
(a) $\sin^{-1}(\sin \pi/7) =$
(b) $\sin^{-1}(\sin 5\pi/7) =$
(c) $\tan^{-1}(\tan 13\pi/6) =$
(d) $\cos^{-1}(\cos 12\pi/7) =$

EXERCISE SET 0.4 Graphing Utility

1. In (a)–(d), determine whether f and g are inverse functions. (a) $f(x) = 4x$, $g(x) = \frac{1}{4}x$
(b) $f(x) = 3x + 1$, $g(x) = 3x - 1$
(c) $f(x) = \sqrt[3]{x - 2}$, $g(x) = x^3 + 2$
(d) $f(x) = x^4$, $g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

4. In each part, determine the exact value without using a cal culating utility.
(a) $\sin^{-1}(-1) =$

3. In each part, use the horizontal line test to determine whether the function f is one-to-one.
(a) $f(x) = 3x + 2$ (b) $f(x) = \sqrt{x - 1}$ (c) $f(x) = |x|$
(d) $f(x) = x^3$
(e) $f(x) = x^2 - 2x + 2$ (f) $f(x) = \sin x$ **4.** In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one. (a) $f(x) = x^3 - 3x + 2$
(b) $f(x) = x^3 - 3x^2 + 3x - 1$

5. In each part, determine whether the function f defined by the table is one-to-one.

(a)

x	1	2	3	4	5	6
$f(x)$	2	−2	−1	0	1	3

(b)

x	1	2	3	4	5	6
$f(x)$	1	4	−7	6	−3	4

6. A face of a broken clock lies in the xy-plane with the cen ter of the clock at the origin and 3:00 in the direction of the positive x-axis. When the clock broke, the tip of the hour hand stopped on the graph of $y = f(x)$, where f is a function that satisfies $f(0) = 0$.

(a) Are there any times of the day that cannot appear in such a configuration? Explain.

(b) How does your answer to part (a) change if f must be an invertible function?

(c) How do your answers to parts (a) and (b) change if it was the tip of the minute hand that stopped on the graph of f?

7. (a) The accompanying figure shows the graph of a func tion f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.

(b) Find the domain and range of f^{-1}.

(c) Sketch the graph of f^{-1}.

y

2

1

x

0

−1

−2

−8 −7 −6 −5 −4 −3 −2 −1012345678

Figure Ex-7

8. (a) Explain why the function f graphed in the accompa nying figure has no inverse function on its domain $-3 \le x \le 4$.

(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.

y

x

−3 4

Figure Ex-8

9–16 Find a formula for $f^{-1}(x)$.

9. $f(x) = 7x - 6$ **10.** $f(x) = \dfrac{x+1}{x-1}$

11. $f(x) = 3x^3 - 5$ **12.** $f(x) = \sqrt[5]{4x+2}$ **13.** $f(x) = 3/x^2,\ x < 0$

14. $f(x) = 5/(x^2 + 1),\ x \ge 0$

15. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \ge 2 \end{cases}$ **16.** $f(x) = \begin{cases} 2x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

17–20 Find a formula for $f^{-1}(x)$, and state the domain of the function f^{-1}.

17. $f(x) = (x + 2)^4, x \geq 0$

18. $f(x) = \sqrt{x + 3}$ **19.** $f(x) = -\sqrt{3 - 2x}$ **20.** $f(x) = x - 5x^2, x \geq 1$

21. Let $f(x) = ax^2 + bx + c, a > 0$. Find f^{-1} if the domain of f is restricted to

(a) $x \geq -b/(2a)$ (b) $x \leq -b/(2a)$.

22. The formula $F = \frac{9}{5}C + 32$, where $C \geq -273.15$ expresses the Fahrenheit temperature F as a function of the Celsius temperature C.

(a) Find a formula for the inverse function.

(b) In words, what does the inverse function tell you?

(c) Find the domain and range of the inverse function.

23. (a) One meter is about 6.214×10^{-4} miles. Find a for mula $y = f(x)$ that expresses a length y in meters as a function of the same length x in miles.

(b) Find a formula for the inverse of f.

(c) Describe what the formula $x = f^{-1}(y)$ tells you in practical terms.

24. Let $f(x) = x^2, x > 1$, and $g(x) = \sqrt{x}$.

(a) Show that $f(g(x)) = x, x > 1$, and $g(f(x)) = x, x > 1$.

(b) Show that f and g are *not* inverses by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflec tions of one another about $y = x$.

(c) Do parts (a) and (b) contradict one another? Ex plain.

25. (a) Show that $f(x) = (3 - x)/(1 - x)$ is its own in verse.

(b) What does the result in part (a) tell you about the graph of f?

26. Sketch the graph of a function that is one-to-one on $(- , +)$, yet not increasing on $(- , +)$ and not de creasing on $(- , +)$.

27. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$. **28.** Let $f(x) = \frac{x^3}{x^2 + 1}$. Find x if $f^{-1}(x) = 2$.

29. Prove that if $a^2 + bc = 0$, then the graph of

$$f(x) = \frac{ax + b}{cx - a}$$

is symmetric about the line $y = x$.

30. (a) Prove: If f and g are one-to-one, then so is the compo sition $f \circ g$.

(b) Prove: If f and g are one-to-one, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

31–34 True–False Determine whether the statement is true or false. Explain your answer. ■

31. If f is an invertible function such that $f(2) = 2$, then $f^{-1}(2) = \frac{1}{2}$.

32. If f and g are inverse functions, then f and g have the same domain.

33. A one-to-one function is invertible.

34. The range of the inverse tangent function is the interval $-\pi/2 \leq y \leq \pi/2$.

35. Given that $\theta = \tan^{-1} \frac{4}{3}$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

36. Given that $\theta = \sec^{-1} 2.6$, find the exact values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\csc\theta$.

37. For which values of x is it true that

(a) $\cos^{-1}(\cos x) = x$ (b) $\cos(\cos^{-1} x) = x$ (c) $\tan^{-1}(\tan x) = x$ (d) $\tan(\tan^{-1} x) = x$?

(b) $\cos\theta = 0.23,\ -90^\circ < \theta < 0^\circ$

44. The ***law of cosines*** states that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where a, b, and c are the lengths of the sides of a triangle and θ is the angle formed by sides a and b. Find θ, to the nearest degree, for the triangle with $a = 2$, $b = 3$, and $c = 4$.

45–46 Use a calculating utility to approximate the solution of each equation. Where radians are used, express your answer to four decimal places, and where degrees are used, express it to the nearest tenth of a degree. [*Note:* In each part, the solution is not in the range of the relevant inverse trigonometric function.] ■ **45.** (a) $\sin x = 0.37,\ \pi/2 < x < \pi$

(b) $\sin\theta = -0.61,\ 180^\circ < \theta < 270^\circ$

46. (a) $\cos x = -0.85,\ \pi < x < 3\pi/2$

38–39 Find the exact value of the given quantity. ■ **38.** $\sec\ \sin^{-1} -\frac{3}{4}$ **39.** $\sin\ 2\cos^{-1}\frac{3}{5}$

40–41 Complete the identities using the triangle method (Figure 0.4.15). ■

40. (a) $\sin(\cos^{-1}x) = ?$ (b) $\tan(\cos^{-1}x) = ?$ (c) $\csc(\tan^{-1}x) = ?$ (d) $\sin(\tan^{-1}x) = ?$ **41.** (a) $\cos(\tan^{-1}x) = ?$ (b) $\tan(\cos^{-1}x) = ?$ (c) $\sin(\sec^{-1}x) = ?$ (d) $\cot(\sec^{-1}x) = ?$

42. (a) Use a calculating utility set to radian measure to make tables of values of $y = \sin^{-1}x$ and $y = \cos^{-1}x$ for $x = -1, -0.8, -0.6, \ldots, 0, 0.2, \ldots, 1$. Round your answers to two decimal places.

(b) Plot the points obtained in part (a), and use the points to sketch the graphs of $y = \sin^{-1}x$ and $y = \cos^{-1}x$. Con firm that your sketches agree with those in Figure 0.4.13.

(c) Use your graphing utility to graph $y = \sin^{-1}x$ and $y = \cos^{-1}x$; confirm that the graphs agree with those in Figure 0.4.13.

43. In each part, sketch the graph and check your work with a graphing utility.

(a) $y = \sin^{-1}2x$ (b) $y = \tan^{-1}\frac{1}{2}x$

47. (a) Use a calculating utility to evaluate the expressions $\sin^{-1}(\sin^{-1}0.25)$ and $\sin^{-1}(\sin^{-1}0.9)$, and explain what you think is happening in the second calcula tion.

(b) For what values of x in the interval $-1 \le x \le 1$ will your calculating utility produce a real value for the function $\sin^{-1}(\sin^{-1}x)$?

48. A soccer player kicks a ball with an initial speed of 14 m/s at an angle θ with the horizontal (see the accom panying figure). The ball lands 18 m down the field. If air resistance is neglected, then the ball will have a parabolic trajectory and the horizontal range R will be given by

$$R = \frac{v^2}{g}\sin 2\theta$$

where v is the initial speed of the ball and g is the ac celeration due to gravity. Using $g = 9.8\ \text{m/s}^2$, approx imate two values of θ, to the nearest degree, at which the ball could have been kicked. Which angle results in the shorter time of flight? Why?

u

R

Figure Ex-48

49–50 The function $\cot^{-1}x$ is defined to be the inverse of the restricted cotangent function

$$\cot x,\ 0 < x < \pi$$

and the function $\csc^{-1}x$ is defined to be the inverse of the restricted cosecant function

$$\csc x,\ -\pi/2 < x < \pi/2,\ x = 0$$

Use these definitions in these and in all subsequent exercises that involve these functions. ■

49. (a) Sketch the graphs of $\cot^{-1}x$ and $\csc^{-1}x$. (b) Find the domain and range of $\cot^{-1}x$ and $\csc^{-1}x$.

(b) the minimum number of daylight hours at Fairbanks to one decimal place.

Source: This problem was adapted from *TEAM, A Path to Applied Mathematics*, The Mathematical Association of America, Washington, D.C., 1985.

50. Show that (a) $\cot^{-1}x = \tan^{-1}(1/x)$, if $x > 0$; $\pi + \tan^{-1}(1/x)$, if $x < 0$

(b) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$, if $|x| \ge 1$

(c) $\csc^{-1}x = \sin^{-1}\frac{1}{x}$, if $|x| \ge 1$.

51. Most scientific calculators have keys for the values of only $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$. The formulas in Exercise 50 show how a calculator can be used to obtain values of $\cot^{-1}x$, $\sec^{-1}x$, and $\csc^{-1}x$ for positive values of x. Use these formulas and a calculator to find numerical values for each of the following inverse trigonometric functions. Ex press your answers in degrees, rounded to the nearest tenth of a degree.

(a) $\cot^{-1}0.7$ (b) $\sec^{-1}1.2$ (c) $\csc^{-1}2.3$ **52.** An Earth-observing satellite has horizon sensors that can measure the angle θ shown in the accompanying figure. Let R be the radius of the Earth (assumed spherical) and h the distance between the satellite and the Earth's surface. (a) Show that $\sin\theta = \frac{R}{R+h}$.

(b) Find θ, to the nearest degree, for a satellite that is 10,000 km from the Earth's surface (use $R = 6378$ km).

54. A camera is positioned x feet from the base of a missile

R

u

h

launching pad (see the accompanying figure). If a missile of length a feet is launched vertically, show that when the base of the missile is b feet above the camera lens, the angle θ subtended at the lens by the missile is

$$\theta = \cot^{-1} x$$

$$a + b - \cot^{-1} x_b$$

a

b u

Earth

Figure Ex-52

Camera Launchpad

Figure Ex-54

55. An airplane is flying at a constant height of 3000 ft above water at a speed of 400 ft/s. The pilot is to release a survival package so that it lands in the water at a sighted point P. If air resistance is neglected, then the package will follow a parabolic trajectory whose equation relative to the coordinate system in the accompanying figure is

$$y = 3000 - \frac{g}{2v^2} x^2$$

where g is the acceleration due to gravity and v is the speed of the airplane. Using $g = 32$ ft/s^2, find the "line of sight" angle θ, to the nearest degree, that will result in the package hitting the target point.

y

Parabolic trajectory of object

u

x

3000 ft

P

Line of sight

Figure Ex-55

53. The number of hours of daylight on a given day at a given point on the Earth's surface depends on the latitude λ of the point, the angle γ through which the Earth has moved in its orbital plane during the time period from the vernal equinox (March 21), and the angle of inclination φ of the Earth's axis of rotation measured from ecliptic north ($\varphi \approx 23.45^\circ$). The number of hours of daylight h can be approximated by the formula $h =$

$$\begin{cases} 24, & D \geq 1 \\ 12 + \frac{2}{15}\sin^{-1} D, & |D| < 1 \\ 0, & D \leq -1 \end{cases}$$

where

$$D = \frac{\sin\varphi \sin\gamma \tan\lambda}{\sqrt{1 - \sin^2\varphi \sin^2\gamma}}$$

and $\sin^{-1} D$ is in degree measure. Given that Fairbanks, Alaska, is located at a latitude of $\lambda = 65^\circ$ N and also that $\gamma = 90^\circ$ on June 20 and $\gamma = 270^\circ$ on December 20, approximate

(a) the maximum number of daylight hours at Fairbanks to one decimal place

56. Prove:

(a) $\sin^{-1}(-x) = -\sin^{-1} x$ (b) $\tan^{-1}(-x) = -\tan^{-1} x$.

57. Prove:

(a) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(b) $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

58. Prove:

(a) $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}$ $(|x| < 1)$ (b) $\cos^{-1} x = \frac{\pi}{2} - \tan^{-1} \frac{x}{\sqrt{1 - x^2}}$ $(|x| < 1)$.

59. Prove:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$$

provided $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$. [*Hint:* Use an identity for $\tan(\alpha + \beta)$.]

60. Use the result in Exercise 59 to show that

(a) $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \pi/4$

(b) $2\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \pi/4$.

61. Use identities (10) and (13) to obtain identity (17).

62. Prove: A one-to-one function f cannot have two different inverses.

✔ QUICK CHECK ANSWERS 0.4

1. (a) not one-to-one (b) not one-to-one (c) one-to-one **2.** $\sqrt[3]{x - 1}$ **3.** $(-2, 3)$; $(2, -3)$ **4.** (a) $-\pi/2$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/3$ (e) $2\pi/3$
5. (a) $\pi/7$ (b) $2\pi/7$ (c) $\pi/6$ (d) $2\pi/7$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

When logarithms were introduced in the seventeenth century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have replaced logarithm tables for numerical calculations, the logarithmic functions have wide-ranging applications in mathematics and science. In this section we will review some properties of exponents and logarithms and then use our work on inverse functions to develop results about exponential and logarithmic functions.

IRRATIONAL EXPONENTS

Recall from algebra that if b is a nonzero real number, then nonzero *integer* powers of b are defined by

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ factors}} \quad \text{and} \quad b^{-n} = \frac{1}{b^n}$$

and if $n = 0$, then $b^0 = 1$. Also, if p/q is a positive *rational* number expressed in lowest terms, then

$$b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p \quad \text{and} \quad b^{-p/q} = \frac{1}{b^{p/q}}$$

One approach is to define irrational powers of b via successive approximations using rational powers of b. For example, to define 2^π consider the decimal representation of π:

Table 0.5.1

2^x x8.000000

3

If b is negative, then some fractional powers of b will have imaginary values—the quantity $(-2)^{1/2} = \sqrt{-2}$, for example. To avoid this complication, we will assume throughout this section that $b > 0$, even if it is not stated explicitly.

There are various methods for defining *irrational* powers such as

3.1415926 ...

From this decimal we can form a sequence of rational numbers that gets closer and closer

$2^\pi, 3^{\sqrt{2}}, \pi^{-\sqrt{7}}$

3.1
3.14
3.141
3.1415
3.14159 3.141592 3.1415926
8.574188 8.815241 8.821353 8.824411
8.824962 8.824974 8.824977

to π, namely, 3.1, 3.14, 3.141, 3.1415, 3.14159

and from these we can form a sequence of *rational* powers of 2:

$2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, 2^{3.14159}$

Since the exponents of the terms in this sequence get successively closer to π, it seems plausible that the terms themselves will get successively closer to some number. It is that number that we *define* to be 2^π. This is illustrated in Table 0.5.1, which we generated using

TECHNOLOGY MASTERY

Use a calculating utility to verify the re sults in Table 0.5.1, and then verify (1) by using the utility to compute 2^π di rectly.

a calculator. The table suggests that to four decimal places the value of 2^{π} is $2^{\pi} \approx 8.8250$ (1)

With this notion for irrational powers, we remark without proof that the following familiar laws of exponents hold for all real values of p and q:

$$b^p b^q = b^{p+q}, \quad \frac{b^p}{b^q} = b^{p-q}, \quad (b^p)^q = b^{pq}$$

THE FAMILY OF EXPONENTIAL FUNCTIONS

A function of the form $f(x) = b^x$, where $b > 0$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \quad f(x) = \left(\frac{1}{2}\right)^x, \quad f(x) = \pi^x$$

Note that an exponential function has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^{\pi}$ would *not* be classified as exponential functions, since they have a variable base and a constant exponent.

Figure 0.5.1 illustrates that the graph of $y = b^x$ has one of three general forms, depending on the value of b. The graph of $y = b^x$ has the following properties:

- The graph passes through $(0, 1)$ because $b^0 = 1$.
- If $b > 1$, the value of b^x increases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x increase indefinitely. If you traverse the graph from right to left, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x.

$y = b^x$
$(0 < b < 1)$

- If $0 < b < 1$, the value of b^x decreases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x. If you traverse the graph from right to left, the values of b^x increase indefinitely.
- If $b = 1$, then the value of b^x is constant.

Some typical members of the family of exponential functions are graphed in Figure 0.5.2. This figure illustrates that the graph of $y = (1/b)^x$ is the reflection of the graph of $y = b^x$ about the y-axis. This is because replacing x by $-x$ in the equation $y = b^x$ yields

$$y = b^{-x} = (1/b)^x$$

The figure also conveys that for $b > 1$, the larger the base b, the more rapidly the function $f(x) = b^x$ increases for $x > 0$.

4

3

2

1

+

$y = b^x$

y

$(b > 1)$

x

Figure 0.5.1

$(b = 1)$

x

$y = 1^x = 1$

2^x 3^x 10^x $\frac{1}{2}^x$ $\frac{1}{3}^x$ $\frac{1}{10}^x$

x y

−2 −1 1 2

Figure 0.5.2 The family $y = b^x$ $(b > 0)$

The domain and range of the exponential function $f(x) = b^x$ can also be found by examining Figure 0.5.1:

- If $b > 0$, then $f(x) = b^x$ is defined and has a real value for every real value of x, so the natural domain of every exponential function is $(- \ , + \ )$.
- If $b > 0$ and $b \ = 1$, then as noted earlier the graph of $y = b^x$ increases indefinitely as it is traversed in one direction and decreases toward zero but never reaches zero as it is traversed in the other direction. This implies that the range of $f(x) = b^x$ is $(0, + \ )$.*

Example 1 Sketch the graph of the function $f(x) = 1 - 2^x$ and find its domain and range.

Solution. Start with a graph of $y = 2^x$. Reflect this graph across the x-axis to obtain the graph of $y = -2^x$, then translate that graph upward by 1 unit to obtain the graph of $y = 1 - 2^x$ (Figure 0.5.3). The dashed line in the third part of Figure 0.5.3 is a horizontal asymptote for the graph. You should be able to see from the graph that the domain of f is $(- \ , + \ )$ and the range is $(- \ , 1)$.

y y y

$y = 2^x$

x

$y = 1$

1 0

x −1

x

$y = 1 - 2^x$

$y = -2^x$

Figure 0.5.3

THE NATURAL EXPONENTIAL FUNCTION

y $y = e^x$

Slope = 1

(0, 1)

x

The use of the letter e is in honor of the Swiss mathematician Leonhard Eu ler (biography on p. 3) who is credited with recognizing the mathematical im portance of this constant.

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter e, is a certain irrational number whose value to six decimal places is

$$e \approx 2.718282 \qquad (2)$$

This base is important in calculus because, as we will prove later, $b = e$ is the only base for which the slope of the tangent line** to the curve $y = b^x$ at any point P on the curve is equal to the y-coordinate at P. Thus, for example, the tangent line to $y = e^x$ at $(0, 1)$ has slope 1 (Figure 0.5.4).

The function $f(x) = e^x$ is called the ***natural exponential function***. To simplify typog raphy, the natural exponential function is sometimes written as $\exp(x)$, in which case the relationship $e^{x_1+x_2} = e^{x_1}e^{x_2}$ would be expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

Figure 0.5.4 The tangent line to the graph of $y = e^x$ at $(0, 1)$ has slope 1.

*We are assuming without proof that the graph of $y = b^x$ is a curve without breaks, gaps, or holes.

**The precise definition of a tangent line will be discussed later. For now your intuition will suffice.

TECHNOLOGY MASTERY

Your technology utility should have keys or commands for approximating e and for graphing the natural exponential function. Read your documentation on how to do this and use your utility to confirm (2) and to generate the graphs in Figures 0.5.2 and 0.5.4.

The constant e also arises in the context of the graph of the equation

$$y = \left(1 + \frac{1}{x}\right)^x \qquad (3)$$

As shown in Figure 0.5.5, $y = e$ is a horizontal asymptote of this graph. As a result, the value of e can be approximated to any degree of accuracy by evaluating (3) for x sufficiently large in absolute value (Table 0.5.2).

y
6
5 4 3 2 1
$y = \left(1 + \frac{1}{x}\right)^x$
$y = e$
x
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7

Figure 0.5.5

Table 0.5.2
approximations of e by $(1 + 1/x)^x$ for increasing values of x

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

LOGARITHMIC FUNCTIONS

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x$$

(read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x. Thus, for example,

$$\log_{10} 100 = 2,\ \log_{10}(1/1000) = -3,\ \log_2 16 = 4,\ \log_b 1 = 0,$$

Logarithms with base 10 are called ***common logarithms*** and are often written without explicit reference to the

$\log_b b = 1$
base. Thus, the symbol
$\log x$ generally denotes
$\log_{10} x$.

$10^2 =$ 100	$10^{-3} = 1/1000$	$2^4 =$ 16	$b^0 =$ 1	$b^1 =$ b

We call the function $f(x) = \log_b x$ the ***logarithmic function with base b***.

Logarithmic functions can also be viewed as inverses of exponential functions. To see why this is so, observe from Figure 0.5.1 that if $b > 0$ and $b = 1$, then the graph of $f(x) = b^x$ passes the horizontal line test, so b^x has an inverse. We can find a formula for this inverse with x as the independent variable by solving the equation

$$x = b^y$$

for y as a function of x. But this equation states that y is the logarithm to the base b of x, so it can be rewritten as

$$y = \log_b x$$

Thus, we have established the following result.

0.5.1 theorem *If $b > 0$ and $b = 1$, then b^x and $\log_b x$ are inverse functions.*

y

b

1

$y = b^x$

$y = \log_b x$

x

1 b

Figure 0.5.6

It follows from this theorem that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$ (see Figure 0.5.6 for the case where $b > 1$). Figure 0.5.7 shows the graphs of $y = \log_b x$ for various values of b. Observe that they all pass through the point $(1, 0)$.

The most important logarithms in applications are those with base e. These are called ***natural logarithms*** because the function $\log_e x$ is the inverse of the natural exponential function e^x. It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than $\log_e x$. For example,

$\ln 1 = 0$, $\ln e = 1$, $\ln 1/e = -1$, $\ln(e^2) = 2$

Since ε e	Since $e^{-1} = 1$	Since $e^2 =$ e^2

In general,

Si

$y = \log_2 x$

y
4
3
2
1

−1

$y = \log_e x$

$y = \log_4 x$

$y = \log_{10} x$

x

1 2 3 4 5 6 7 8 9 10

Figure 0.5.7 The family $y = \log_b x$ $(b > 1)$

$$y = \ln x \text{ if and only if } x = e^y$$

As shown in Table 0.5.3, the inverse relationship between b^x and $\log_b x$ produces a correspondence between some basic properties of those functions.

Table 0.5.3

correspondence between properties of logarithmic and exponential functions

property of b^x	property of $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

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Use your graphing utility to generate the graphs of $y = \ln x$ and $y = \log x$.

It also follows from the cancellation properties of inverse functions [see (3) in Section 0.4] that

$\log_b(b^x) = x$ for all real values of x

$b^{\log_b x} = x$ for $x > 0$ (4)

In the special case where $b = e$, these equations become

$\ln(e^x) = x$ for all real values of x

$e^{\ln x} = x$ for $x > 0$ (5)

In words, the functions b^x and $\log_b x$ cancel out the effect of one another when composed in either order; for example,

$\log 10^x = x,\ 10^{\log x} = x,\ \ln e^x = x,\ e^{\ln x} = x,\ \ln e^5 = 5,\ e^{\ln \pi} = \pi$