A. Bosonized symmetry operations

SR: Added symmetry operations in the bosonized variables and fixed Klein factors

In this section we construct the Anti-Ferromagnetic Time-Reversal (AFTR) T_A and Glide G in the bosonized variables. For convenience we choose a new set of axes (see fig. 1). The wires are along the z-direction now, AFTR symmetry is along y-axis and Glide symmetry has the translation part along the x-axis.

Each wire has a chiral Dirac channel going in one direction, we denote the Dirac fermions on the x^{th} and y^{th} wire by $\psi_{x,y}$ and bosonize $\psi_{xy}(z) \sim e^{i\phi_{xy}(z)}$. The boson operators obey the equal-time commutation (ETCR)

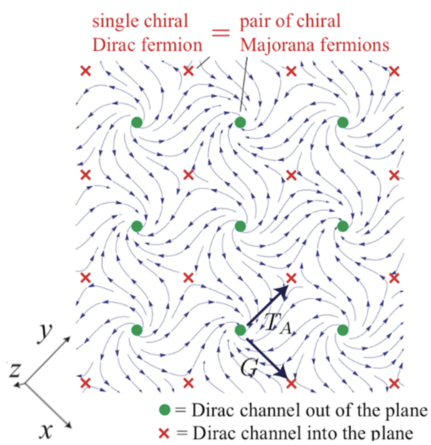

FIG. 1. Arrays of chiral Dirac channels symmetric under anti-ferromagnetic time reversal T_A and mirror glide G in rotated axes

$$\begin{aligned}[\phi_{xy}(z), \phi_{x'y'}(z')] =& (-1)^{x+y} i\pi[\delta^{xx'}\delta^{yy'} sgn(z-z') \\ &+ \delta^{xx'} sgn(y-y') + sgn(x-x')]\end{aligned} \tag{1}$$

where $sgn(s) = s/|s| = \pm 1$ when $s \neq 0$ and $sgn(0) = 0$. The first term in (1) is equivalent to commutation relation between conjugate bosonized fields given by

$$[\phi_{xy}(z), \partial_{z'}\phi_{x'y'}(z')] = 2\pi i(-1)^{x+y}\delta^{xx'}\delta^{yy'}\delta(z-z') \tag{2}$$

The alternating sign $(-1)^{x+y}$ in (1) changes the direction of propagation from wire to wire. The second and third term in (1) ensure correct anticommutation relations $\left\{e^{\pm i\phi_{xy}}, e^{\pm i\phi_{x'y'}}\right\} = 0$ between Dirac fermions on distint wires $x \neq x'$ and/or $y \neq y'$.

The anti-unitary AFTR symmetry T_A transforms the bosons as

$$T_A\phi_{xy}(z)T_A^{-1} = -\phi_{xy+1}(z) + \frac{1-(-1)^{x+y}}{2}\pi, \tag{3}$$

Glide symmetry G transforms the bosons as

$$G\phi_{xy}(z)G^{-1} = -\phi_{x+1y}(-z) + \frac{\pi}{2}. \tag{4}$$

Both these symmetries, T_A and G, commute with each other

$$\begin{aligned}GT_A\phi_{xy}(z)T_A^{-1}G^{-1} &= \phi_{x+1y+1}(-z) - (-1)^{x+y} \\ &= T_AG\phi_{xy}(z)G^{-1}T_A^{-1}.\end{aligned} \tag{5}$$

Other relations are

$$T_A^2\phi_{xy}(z)T_A^{-2} = \phi_{xy+2}(z) - \pi(-1)^{x+y} \tag{6}$$
$$G^2\phi_{xy}(z)G^{-2} = \phi_{x+2y}(z). \tag{7}$$

So $T_A^2 = (-1)translation(2e_y)$ and $G^2 = translation(2e_x)$.

Each Dirac mode can be split up in two Majorana modes and then the adjacent counter-propagating Majorana modes can be back-scattered in to each other and gap out.

The Dirac mode $\psi_{xy}(z) \sim e^{i\phi_{xy}(z)}$ can be split up in to two Majoranas $e^{i\phi_{xy}(z)} = \frac{1}{\sqrt{2}}(\gamma_{xy}(z) + i\delta_{xy}(z))$.

We write down a gapping term which preserves both the AFTR and Glide symmetries. The anti-unitary AFTR symmetry T_A transform the Dirac and Majoranas as

$$\begin{aligned}T_A\psi_{xy}(z)T_A^{-1} &= (-1)^{x+y}\psi_{xy+1}(z) \\ T_A\gamma_{xy}(z)T_A^{-1} &= (-1)^{x+y}\gamma_{xy+1}(z) \\ T_A\delta_{xy}(z)T_A^{-1} &= (-1)^{x+y+1}\delta_{xy+1}(z),\end{aligned} \tag{8}$$

and Glide G symmetry transforms them as

$$\begin{aligned}G\psi_{xy}(z)G^{-1} &= i\psi^{\dagger}_{x+1y}(-z) \\ G\gamma_{xy}(z)G^{-1} &= \delta_{x+1y}(-z) \\ G\delta_{xy}(z)G^{-1} &= \gamma_{x+1y}(-z)\end{aligned} \tag{9}$$

tranlation symmetry transforms them as

$$\begin{aligned}t_{11}\psi_{xy}t_{11}^{-1} &= \psi_{xy+1} \\ t_{1\bar{1}}\psi_{xy}t_{1\bar{1}}^{-1} &= \psi_{x+1y}\end{aligned} \tag{10}$$

The two-body gapping term H_{2-body}, a Dirac-Dirac back-scattering term between adjacent counter-propagating wires is given by

$$H_{2-body} = u\sum_{x,y}\psi^1_{xy}\psi^2_{xy+1}\psi^{1\dagger}_{xy}\psi^{2\dagger}_{xy+1} \tag{11}$$

The 2-body gapping term preserve anti-unitary AFTR T_A, glide G and the translation symmetries t_{11} and $t_{1\bar{1}}$.

We discussed above the $N = 2$. We now discuss the case with $N = 16$.

The sliding Luttinger liquid Lagrangian density is

$$\mathcal{L}_{layer} = \sum_{y=-\infty}^{\infty} \frac{(-1)^{x+y} K_{jk}}{2\pi} \partial_t \phi^j_{xy} \partial_z \phi^k_{xy} - V_{jk} \partial_z \phi^j_{xy} \partial_z \phi^k_{xy} \tag{12}$$

Each Dirac mode $\boldsymbol{\psi}_{x,y}$ at the x^{th} and y^{th} wire has eight species indexed by j, $\boldsymbol{\psi}_{x,y} = \left(\psi^1_{xy}, \psi^2_{xy}, ..., \psi^8_{xy}\right)$. They bosonize to $\psi^j_{xy}(z) \sim e^{i\phi^j_{xy}(z)}$ and the equal-time commutation relation (ETCR) of equation (1) is modified to

$$\begin{aligned}\left[\phi^j_{xy}(z), \phi^{j'}_{x'y'}(z')\right] =& (-1)^{x+y} i\pi [\delta^{jj'} \delta^{xx'} \delta^{yy'} sgn(z - z') \\ &+ \delta^{xx'} sgn(y - y') + sgn(x - x') \\ &+ \delta^{xx'} \delta^{yy'} sgn(j - j')]\end{aligned} \tag{13}$$

where $K = K_{E_8}$, V is some non-universal velocity matrix and j,k are repeated indices that are summed over. The symmetries relation equations (3), (4), (7) would remain the same except the bosons would be indexed by j now.

B. E8 state

refer to MJP notes

C. Gapping an even number of Majorana cones

We label the Dirac and Majorana fermions with 'a', $\psi^a_{xy} = \gamma^a_{xy} + i\delta^a_{xy}$, $a = 1, ..., N$. Here N is the number of Dirac modes running in each wire.

The AFTR and Glide symmetry operation remain the same and now just have an 'a' index. We can write down the $SO(N)_1$ current terms as

$$J^{ab}_{\gamma xy} = i\gamma^a_{xy}\gamma^b_{xy} \tag{14}$$
$$J^{ab}_{\delta xy} = i\delta^a_{xy}\delta^b_{xy} \tag{15}$$

where $1 \leq a < b \leq N$. They transform under the symmetry operations as

$$T_A J^{ab}_{xy} T_A^{-1} = -J^{ab}_{xy+1} \tag{16}$$
$$G J^{ab}_{xy} G^{-1} = J^{ab}_{x+1y} \tag{17}$$
$$t_{11} J^{ab}_{xy} t_{11}^{-1} = J^{ab}_{x+1y+1} \tag{18}$$
$$t_{1\bar{1}} J^{ab}_{xy} t_{1\bar{1}}^{-1} = J^{ab}_{x+1y-1} \tag{19}$$
$$\tag{20}$$

When N is even $N = 2r$, a first attempt would be to split it in to two Majorana sectors $SO(2r)_1 = SO(2r)^\delta_1 \times SO(2r)^\gamma_1$. An example of such a gapping term is

$$\sum_{x,y} \sum_{1 \leq a < b < N} J^{ab}_{\gamma xy} J^{ab}_{\delta xy+1} + J^{ab}_{\delta x+1y} J^{ab}_{\gamma x+1y+1} \tag{21}$$

The gapping term above preserves AFTR and Glide but breaks translation symmetries. We can try other similar combinations but we can't find a gapping term that preserves all the symmetries. This is because under symmetry transformations, Majorana species are exchanged between sectors and that beaks the symmetries of the gapping term as shown in fig. 2 a.

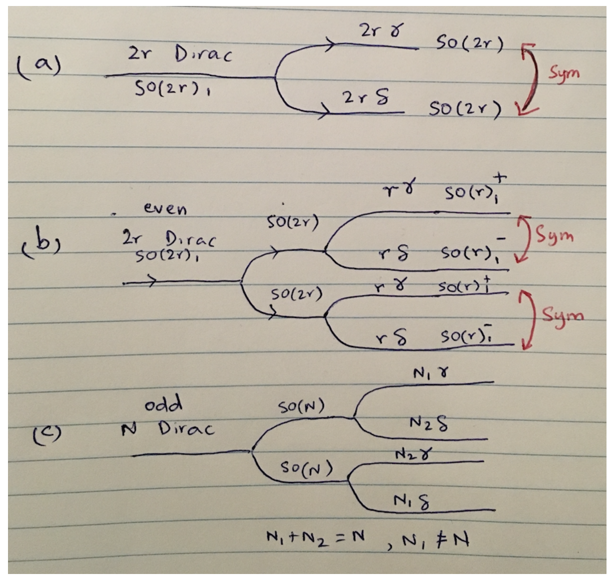

FIG. 2. a) splitting scheme that doesn't work, b) splitting even number of modes, c) splitting odd number of modes

To fix this issue we introduce a different splitting scheme $SO(2r)_1 = SO(2r) \times SO(2r) = (SO(r)^+_1 \times SO(r)^-_1) \times (SO(r)^+_1 \times SO(r)^-_1)$. As shown in fig. 2 b, now the symmetry transforms Majorana species within a sector and not between sectors. Such a splitting scheme preserves the gapping term symmetries.

We now write the gapping term explicitly. For even Dirac modes $\psi^a_{xy} \sim e^{i\phi^a_{xy}}$, $a = 1, ..., 2r$. Half of the modes $a = 1, ..., r$ are in the (+) sector and the other half in $a = r + 1, ..., 2r$ in the (-) sector. The gapping term is $H^a = \psi^{a\dagger}\psi^a = i\partial\phi^a$, we can define the vertex operator

$$E^{\vec{\alpha}} = \psi^{\alpha_1}\psi^{\alpha_2} = e^{i\vec{\alpha}.\vec{\phi}} \tag{22}$$

where $\vec{\alpha}$ is a vector with only two non-zero entries $\vec{\alpha} = (0, .., \pm1, .., \pm1, ..0)^T$. $Tr(\vec{\alpha}) = \vec{\alpha}.\vec{t} = \alpha_1 + \alpha_2$ where $t = (1, 1.., 1)^T$. These vertex operators transform under symmetries as

$$T_A E^{\vec{\alpha}}_{xy} T_A^{-1} = E^{\vec{\alpha}}_{xy+1} \tag{23}$$
$$G E^{\vec{\alpha}}_{xy} G^{-1} = E^{\vec{\alpha}}_{x+1y}, if\ Tr(\vec{\alpha}) = 0 \tag{24}$$
$$= -E^{-\vec{\alpha}}_{x+1y}, if\ Tr(\vec{\alpha}) = \pm 2 \tag{25}$$

The gapping term $\sum_{x,y}\sum_{\alpha} E^{\vec{\alpha}}_{+,xy} E^{-\vec{\alpha}}_{-,xy}$ preserves AFTR, Glide and translation symmetries.

D. Gapping an odd number of Majorana cones

We can't use the same splitting scheme as the previous section for odd number of Dirac modes. The issue is illustrated in fig 2 c, as $N_1 \neq N_2$, we can't exchange species under symmetry transformations within a sector. Here, we use a different splitting scheme to get around this problem (refer to Sahoo, Zhang, Teo paper).

We consider the case of N=9 Dirac modes (18 Majoranas), this can be extended to other odd number of modes too. $SO(18)_1 = SO(9)\times SO(9) = [SO(3)^{+,\gamma}_3 \times SO(3)^{+,\delta}_3]\times[SO(3)^{-,\gamma}_3 \times SO(3)^{-,\delta}_3]$. The currents $\vec{J} = (J_1, J_2, J_3)$ for $SO(3)^{+,\gamma}_3$ and $SO(3)^{-,\gamma}_3$ are given below

$$\begin{aligned} J_1^{+,\gamma} &= i(\gamma^{12}+\gamma^{45}+\gamma^{78}),\ J_1^{-,\gamma} = i(\gamma^{14}+\gamma^{25}+\gamma^{36}) \\ J_2^{-,\gamma} &= i(\gamma^{23}+\gamma^{56}+\gamma^{89}),\ J_2^{-,\gamma} = i(\gamma^{47}+\gamma^{58}+\gamma^{69}) \\ J_3^{+,\gamma} &= i(\gamma^{31}+\gamma^{64}+\gamma^{97}),\ J_1^{-,\gamma} = i(\gamma^{71}+\gamma^{82}+\gamma^{93}) \end{aligned} \tag{26}$$

The gapping term is then given by

$$H_{gap-SO(3)_3} = \sum_{xy} \vec{J}^{+,\gamma}_{xy}\vec{J}^{-,\gamma}_{xy+1} + \vec{J}^{+,\delta}_{xy}\vec{J}^{-,\delta}_{xy+1} \tag{27}$$

SR: End of edit