Complexity Functionals on Sparse Random Graphs: Spectral Gap, Singular-Value Variance, and Ollivier-Ricci Curvature Contributions

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Abstract

We introduce a simple scalar functional on finite undirected graphs that linearly combines the spectral gap of the adjacency matrix, the variance of its singular values, and the average Ollivier–Ricci curvature. The functional is studied on Erdős–Rényi graphs in the sparse regime $p = 3/N$. Analytic bounds are derived for general graphs and exactly for regular graphs. Numerical experiments up to $N = 1000$ show mild scaling of the mean functional value and a small but persistent fraction of motifs that remain approximately stable under random edge perturbations. We discuss immediate extensions: ensemble universality, rigorous large-N asymptotics of the spectral term, curvature-scaling ablation, and the open question of continuum limits. All code, raw data, scaling plots, and a one-click Docker environment are publicly available at the companion repository.

1 Introduction

Graph functionals that combine spectral, variance, and geometric (curvature) information are useful for detecting structural robustness and community structure in complex networks. We define a minimal three-term functional and examine its behaviour on sparse random graphs. The work is purely computational and mathematical; no physical interpretation is claimed or implied.

2 Core Definitions and Setup

Let $G = (V, E)$ be a finite, simple, undirected graph with $|V| = N$. Let A be its adjacency matrix.

- Spectral gap: $\Delta\lambda = \lambda_{\max}(A) - \lambda_{\text{next}}$, where eigenvalues are sorted in descending order.
- Singular-value variance: $\text{Var}(\sigma_i) = \frac{1}{N}\sum_{i=1}^{N}(\sigma_i - \bar{\sigma})^2$, where σ_i are the singular values of A and $\bar{\sigma}$ is their mean.
- Average Ollivier–Ricci curvature: $\kappa_G = \frac{1}{|E|}\sum_{e\in E}\kappa(e)$, using the standard combinatorial definition.

We define the complexity functional

$$C(G) = w_1\Delta\lambda - w_2\text{Var}(\sigma_i) + w_3\kappa_G$$

with equal weights $w_1 = w_2 = w_3 = 1$ (fixed for all experiments).

A motif is declared *approximately stable* if, after 10–20 random single-edge flips (additions or deletions preserving simplicity), the change satisfies $|\delta C| < 0.01$.

3 Analytic Results

Proposition 1 (General bound). For any simple graph G,

$$C(G) \le \lambda_{\max}(A) - \frac{2|E|}{N} + \kappa_G.$$

Proof. $\Delta\lambda \le \lambda_{\max}(A)$ and $\mathrm{Var}(\sigma_i) \le \frac{1}{N}\sum_i \sigma_i^2 = 2|E|/N$. $\square$

Proposition 2 (Regular graphs). On any d-regular graph, $\lambda_{\max}(A) = d$ and $\sum_i \sigma_i^2 = Nd$, so the spectral and variance terms cancel exactly and

$$C(G) = \kappa_G.$$

Thus all regular graphs are exact extrema of $C(G)$ under this weighting.

In the sparse Erdős–Rényi regime $G(N, 3/N)$ the leading contribution to $C(G)$ is expected to approach an N-independent constant as $N \to \infty$ (standard spectral results for sparse ER graphs).

4 Computational Experiments

We generated ensembles of $G(N, 3/N)$ graphs for $N \in \{9, 19, 49, 81, 200, 500, 800, 1000\}$. For $N < 500$ we used 500 graphs per size; for $N \ge 500$ we used 50 graphs per size. For each graph we computed $C(G)$ and tested stability under 10 random edge flips.

Observed trends (finite-size):

- Mean $C(G)$ increases mildly with N.
- Fraction of approximately stable motifs remains small ($\sim$ 0.11–0.22) but non-zero at all sizes and decreases slowly.

5 Discussion and Immediate Extensions

The functional produces nontrivial structural differentiation even in purely random ensembles. Regular graphs are automatically preferred, and sparse random graphs admit a small population of robust motifs.

Four natural next steps are: 1. Universality: repeat the study on Barabási–Albert, Watts–Strogatz, random geometric, and configuration-model ensembles. 2. Rigorous asymptotics: prove concentration bounds for $\Delta\lambda$ and the full functional in the sparse regime. 3. Curvature ablation: set $w_3 = 0$ and quantify how the scaling exponents change. 4. Continuum behaviour: investigate whether convergent observables emerge for $N \gg 10^3$ and under what rewiring rules the functional admits a well-defined large-N limit.

No claim is made that the motifs correspond to any continuum geometric or physical objects; such questions are left for future work.

6 Code and Data Availability

All code (Python 3, NetworkX + cuGraph + NumPy), raw data, Jupyter notebook, scaling plots, and Docker environment are available at `https://github.com/syedrazaaftab/IET-Emergence/tree/computational-v1/computational-testbed`.

One-click reproduction instructions are included.