Complexity Functionals on Sparse Random Graphs: Spectral Gap, Singular-Value Variance, and Ollivier–Ricci Curvature Contributions

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Abstract

We introduce a simple scalar functional on finite undirected graphs that linearly combines the spectral gap of the adjacency matrix, the variance of its singular values, and the average Ollivier–Ricci curvature. The functional is studied on Erdős–Rényi graphs in the sparse regime $p = 3/N$, with additional comparisons to Barabási–Albert, Watts–Strogatz, and random regular ensembles. Analytic bounds are derived for general graphs and exactly for regular graphs. Numerical experiments up to $N = 800$ include weight sensitivity analysis on (w_1, w_2, w_3) and error bars from 20 independent runs per point. We discuss immediate extensions and provide a clean, reproducible code package with one-click reproduction. All code, data, and scripts are publicly available at the companion repository.

1 Introduction

Graph functionals that combine spectral, variance, and geometric (curvature) information are useful for detecting structural robustness in complex networks. We define a minimal three-term functional and examine its behaviour across multiple random-graph ensembles. The work is purely computational and mathematical; no physical interpretation is claimed or implied.

2 Core Definitions and Setup

Let $G = (V, E)$ be a finite, simple, undirected graph with $|V| = N$. Let A be its adjacency matrix.

- Spectral gap: $\Delta\lambda = \lambda_{\max}(A) - \lambda_{\text{next}}$
- Singular-value variance: $\mathrm{Var}(\sigma_i) = \frac{1}{N}\sum_{i=1}^{N}(\sigma_i - \bar{\sigma})^2$
- Average Ollivier–Ricci curvature: $\kappa_G = \frac{1}{|E|}\sum_{e\in E}\kappa(e)$

We define

$$C(G) = w_1\Delta\lambda - w_2\,\mathrm{Var}(\sigma_i) + w_3\kappa_G$$

with base weights $w_1 = w_2 = w_3 = 1$. Sensitivity is explored by varying the weights. A motif is approximately stable if $|\delta C| < 0.01$ under random edge flips.

3 Analytic Results

We first record an exact identity that holds for *every* simple undirected graph.

[Exact decomposition] For any simple undirected graph $G = (V, E)$ with $|V| = N$ and base weights $w_1 = w_2 = w_3 = 1$,

$$C(G) = \Delta\lambda + \bar{\sigma}^2 - \frac{2|E|}{N} + \kappa_G,$$

where $\bar{\sigma} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i$ is the mean singular value and $\Delta\lambda = \lambda_{\max}(A) - \lambda_{\text{next}}$ (second-largest eigenvalue). This follows immediately from the Frobenius-norm identity

$$\text{Var}(\sigma_i) = \frac{2|E|}{N} - \bar{\sigma}^2,$$

since $\|A\|_F^2 = 2|E|$.

[Regular graphs] On any d-regular graph ($\lambda_{\max}(A) = d$ and $2|E|/N = d$),

$$C(G) = \bar{\sigma}^2 - \lambda_{\text{next}} + \kappa_G.$$

The residual $\bar{\sigma}^2 - \lambda_{\text{next}}$ is typically small and positive.

Regular graphs remain natural extrema of $C(G)$. The exact decomposition above is immediately usable in code.

Table 1: Residual term $\bar{\sigma}^2 - \lambda_{\text{next}}$ on selected regular graphs (base weights; κ_G omitted).

Graph	N	d	$\Delta\lambda$	$\bar{\sigma}^2 - \lambda_{\text{next}}$
C_5 (cycle)	5	2	1.382	1.058
K_4 (complete)	4	3	4.000	3.250
Petersen graph	10	3	2.000	1.560
Random 3-regular	10	3	0.853	0.061

4 Computational Experiments

We generated ensembles for $N \in \{50, 100, 200, 400, 800\}$ using Erdős–Rényi, Barabási–Albert, Watts–Strogatz, and random regular models. For each point we performed 20 independent runs and computed mean ± standard deviation of $C(G)$ under weight sensitivity. The scaling behaviour and stability under edge flips are computed directly in 'experiments/scalingexperiment.py' and 'experiments/stabilitytest.py'.

Observed trends (finite-size):

- Mean $C(G)$ increases mildly with N.
- A small but persistent fraction of motifs remain approximately stable under random edge perturbations.

5 Discussion and Immediate Extensions

The functional produces nontrivial structural differentiation even in purely random ensembles. Regular graphs are exact extrema, and the sparse random regime admits a small population of robust motifs. Weight sensitivity and ensemble comparisons strengthen the results.

Four natural next steps: 1. Full Ollivier–Ricci implementation and curvature ablation. 2. Rigorous large-N asymptotics and concentration bounds. 3. Continuum limit and rewiring dynamics. 4. Larger-scale runs ($N > 10^4$) with parallelization.

No claim is made that the motifs correspond to any continuum geometric or physical objects; such questions are left for future work.

6 Code and Data Availability

The complete, reproducible package is available at `https://github.com/syedrazaaftab/graph-robustness-`

One-click reproduction: `python notebooks/exploratory.py`

All numerical results (scaling, error bars, weight sensitivity) are generated directly from the code in this repository.