The Mathematical Magic of the Fibonacci Numbers

This page looks at some patterns in the Fibonacci numbers themselves, from the digits in the numbers to their factors and multiples and which are prime numbers. There is an unexpected pattern in the initial digits too. We also relate Fibonacci numbers to Pascal's triangle via the original rabbit problem that Fibonacci used to introduce the series we now call by his name. We can also make the Fibonacci numbers appear in a decimal fraction, introduce you to an easily learned number magic trick that only works with Fibonacci-like series numbers, see how Pythagoras' Theorem and right-angled triangles such as 3-4-5 have connections with the Fibonacci numbers and then give you lots of hints and suggestions for finding more number patterns of your own.

Take a look at the [Fibonacci Numbers List] or, better, see this list in [another browser window], then you can refer to this page and the list together.

Contents of this page

The icon means there is a You do the maths... section of questions to start your own investigations.
The calculator icon indicates that there is a live interactive calculator in that section.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

1 Patterns in the Fibonacci Numbers

1.1 The Final Digits

Here are some patterns people have already noticed in the final digits of the Fibonacci numbers:

- Look at the final digit in each Fibonacci number - the **units digit**:

  0, **1**, **1**, **2**, **3**, **5**, **8**, 1**3**, 2**1**, 3**4**, 5**5**, 8**9**, 14**4**, 23**3**, 37**7**, 61**0**, 98**7**, ...

  Is there a pattern in the final digits?

  0, **1**, **1**, **2**, **3**, **5**, **8**, **3**, **1**, **4**, **5**, **9**, **4**, **3**, **7**, **0**, **7**, ...

  Yes!

  It takes a while before it is noticeable. In fact, the series is just 60 numbers long and then it repeats the same sequence again and again all the way through the Fibonacci series - for ever. We say the series of final digits repeats *with a cycle length of* ***60***.
- Suppose we look at the final **two digits** in the Fibonacci numbers. Do they have a pattern?

0, **1**, **1**, **2**, **3**, **5**, **8**, **13**, **21**, **34**, **55**, **89**, 1**44**, 2**33**, 3**77**, 6**10**, 9**87**, ...

Yes, there is a pattern here too. After Fib(300) the last two digits repeat the same sequence again and again. The cycle length is **300** this time.

So what about the **last three digits**?
and the last **four** digits?
and so on??

- For the last **three** digits, the cycle length is **1,500**
- for the last **four** digits,the cycle length is **15,000** and
- for the last **five** digits the cycle length is **150,000**
- and so on...

1.2 Digit Sums

Michael Semprevivo suggests this investigation for you to try.
If we add all the digits of a number we get its **digit sum**.

Find Fibonacci numbers for which the *sum of the digits* of Fib(*n*) is equal to its *index number n*:
For example:-

Fib(10)=55

the tenth Fibonacci number is Fib(10) = 55.
The sum of its digits is 5+5 or **10** and that is also the index number of 55 (10-th in the list of Fibonacci numbers). So the index number of Fib(10) is equal to its digit sum.

Fib(11)=89

This time the digit sum is 8+9 = **17**.
But 89 is **not** the 17th Fibonacci number, it is the 11th (its index number is 11) so the digit sum of 89 is not equal to its index number.

Can you find other Fibonacci numbers with a digit sum equal to its index number?

Here are two more examples of the numbers we seek: Fib(1)=1 and Fib(5)=5.
There is also one more whose index number is less than 10 - what is it?

Can you find any more in this table of Fibonacci numbers up to Fib(300)?
As a check, you should be able to find TEN (including those above) up to Fib(200).
How many are there up to Fib(300)?
This makes a nice exercise in computer programming so the computer does the hard work.

A more difficult question is **Does this series (of Fibonacci numbers which have a digit sum equal to their index number) go on for ever?**

Robert Dawson of Saint Mary's University, Nova Scotia, Canada summarises a simple statistical argument (originally in the article referred to below by David Terr) that suggests there may be only a finite number (in fact, just 20 numbers) in this series:

> "The number of decimal digits in Fib(N) can be shown to be about 0.2 N, and the average value of a decimal digit is (0+1+...+8+9)/10 = 4·5. Thus, unless the digits of Fibonacci numbers have some so-far undiscovered pattern, we would expect the digit sum to be about 0.9 N. This falls further behind N as N gets larger. Fib(2222) (with 465 digits) is the largest known Fibonacci number with this property. There are no others with N<5000, and it seems likely that Fib(2222) is actually the largest one. However, no proof exists!"

1.2.1 A new research question for you to try

If you want to try *a new investigation*, how about converting the Fibonacci numbers to a base other than 10 (binary is base 2 or undecimal is base 11, for example) and seeing what you get for the digit sums in different bases. Are there any bases where the Fibonacci numbers with a sum of their base B digits equal to their index numbers form an infinite series? In which bases is it a finite series?

References

On the Sums of Digits of Fibonacci Numbers David Terr, *Fibonacci Quarterly*, vol. 34, August 1996, pages 349-355.
Two series in Sloane's Encyclopedia of Integer Sequences are relevant here: A020995 for the index numbers and A067515 for the Fibonacci numbers themselves

1.3 Fibonacci Number Digit Sums Calculator

C A L C U L A T O R of Digit Sums of Fibonacci Numbers

Find the digit sum of Fib(i) in base 10 for i= up to as a list

RESULTS CLEAR

: Digit Sums | Factors | Remainders | Quincunx | Fractions | Number trick | Pythag Tris

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

2 Factors of Fibonacci Numbers

There are some fascinating and simple patterns in the Fibonacci numbers when we consider their factors. You might like to click here to open a new browser window which shows the first 300 Fibonacci numbers and their factors. It will be helpful in the following investigations:

2.1 You do the maths...

1. Where are the **even Fibonacci Numbers**?
   Write down the *index numbers* i where Fib(i) is even.
   Do you notice a pattern?
   Write down the pattern you find as clearly as you can *first* in words and *then* in mathematics. Notice that 2=F(3) also.
2. Now find where there are Fibonacci numbers which are **multiples of 3**.
   and again write down the pattern you find in words and then in mathematics. Again notice that 3=F(4).
3. What about the **multiples of 5**? These are easy to spot because they end with **0** or **5**.
   Again, write down the pattern you find.

4. You can try and spot the multiples of 8, if you like now.
   Why 8? Because we have found the multiples of 2, then 3, then 5 and now 8 is the next Fibonacci number!
5. Do you think your patterns also have a pattern? That is, for *any Fibonacci Number F* can you tell me where you think all its multiples will appear in the whole list of Fibonacci Numbers?

So *every **Fibonacci** number is a factor of (an infinite number of) Fibonacci numbers*, that is:

Fibonacci numbers as Factors of Fibonacci numbers

	i	3	4	5	6	7	8	9	10	11	12	...
	Fib(i)	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**55**	**89**	**144**	**...**
Factors	**2=Fib(3)**	✔	✘	✘	✔	✘	✘	✔	✘	✘	✔	Every 3^{rd} Fib number
	3=Fib(4)	✘	✔	✘	✘	✘	✔	✘	✘	✘	✔	Every 4^{th} Fib number
	5=Fib(5)	✘	✘	✔	✘	✘	✘	✘	✔	✘	✘	Every 5^{th} Fib number
	8=Fib(6)	✘	✘	✘	✔	✘	✘	✘	✘	✘	✔	Every 6^{th} Fib number
	F(k)	...										F(all multiples of k)

Putting this into words we have:
Every *3*-rd Fibonacci number is a multiple of 2 i.e. a multiple of F(*3*)
Every *4*-th Fibonacci number is a multiple of 3 i.e. a multiple of F(*4*)
Every *5*-th Fibonacci number is a multiple of 5 i.e. a multiple of F(*5*)
Every *6*-th Fibonacci number is a multiple of 8 i.e. a multiple of F(*6*)

which suggests **the general rule**:

Every *k*-th Fibonacci number is a multiple of F(*k*)

or, expressed mathematically,

F(*nk*) is a multiple of F(*k*) for all values of *n* and *k* from 1 up.

References

A Primer For the Fibonacci Numbers: Part IX M Bicknell and V E Hoggatt Jr in *The Fibonacci Quarterly* Vol 9 (1971) pages 529 - 536 has several proofs that F(k) always divides exactly into F(nk): using the Binet Formula; by mathematical induction and using generating functions.

A free book with the whole collection of parts of the Primer is available online as a PDF or as separate parts from the Fibonacci Association.

3 Every number is a factor of some Fibonacci number

But what about numbers that are not Fibonacci numbers?
Which other numbers exactly divide into (are factors of) Fibonacci numbers?
The surprising answer is that **there are an infinite number of Fibonacci numbers with *any* given number as a factor!**

For instance, here is a table of the *smallest* Fibonacci numbers that have each of the integers from 1 to 13 as a factor:
This index number for n is called the **Fibonacci Entry Point of n**.

The smallest i for which Fib(i) is divisible by n

n	1	2	3	4	5	6	7	8	9	10	11	12	13	...	
Fib(i)	1	2	3	8	5	144	21	8	144	610	55	144	13	...	A047930
i	1	3	4	6	5	12	8	6	12	15	10	12	7	...	A001177

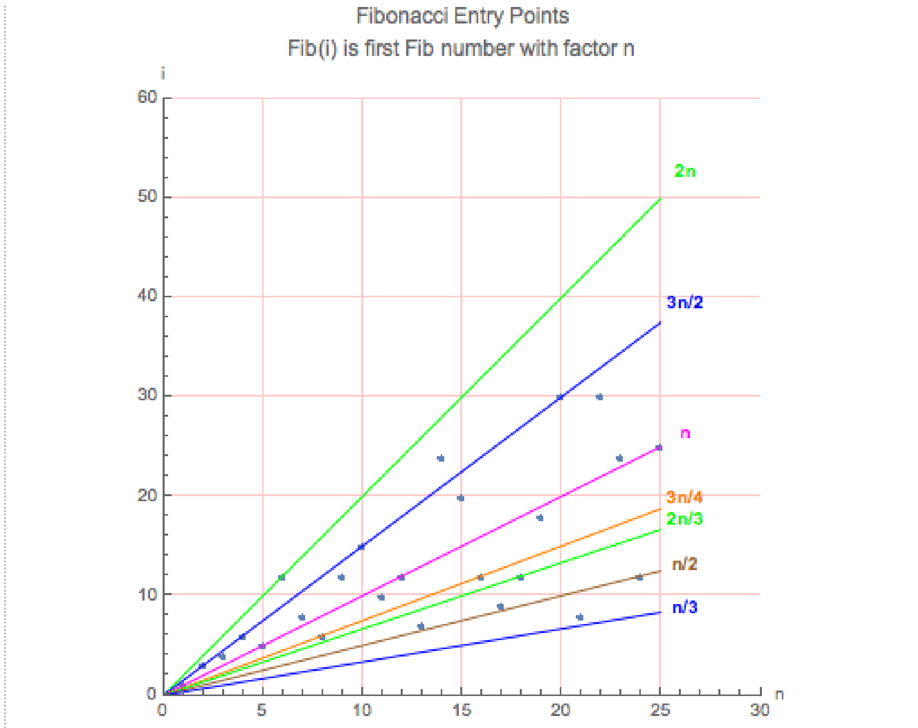
Since Fib(15) is the smallest Fibonacci number with 10 as a factor, then, using the result of the previous section, we then know that

> Fib(any multiple of 15) also has 10 as a factor

thus Fib(15), Fib(30), Fib(45), Fib(60), ..., Fib(15k), ... all have 10 as a factor. This applies to all numbers n as the factor of some Fibonacci number.

Here is a graph of the series, the index number **i** of the first Fibonacci number that has a factor of **n** for n up to various values:

Show the graph up to n= 25 100 250 500 1000 3000



The Fibonacci Series beginning 0,1,1,... is the *only* (general) Fibonacci sequence beginning a,b,a+b,... which has all the primes as factors of some number in the series. This was proved by Brother U Alfred in the reference.

References

Primes which are factors of all Fibonacci sequences,Brother U Alfred, *Fib Quart*, 2 (1964), pages 33-38. PDF

If we take each number n = 1,2,3,... and find the smallest i for which Fib(i) has n as a factor, we obtain this series of index (i) values:
1,3,4,6,5,12,8,6,12,15,10,12,7,24,20,12,9,12,18,30,8,30,... is Sloane's A001177

An early reference for the theorem that every number is a factor of some Fibonacci number seems to be N N Vorob'ev in his **Fibonacci Numbers** booklet, published by Pergamon in 1961 where a proof is also given. Also, he showed that the first Fibonacci number with a factor i will be found within the first i^2 Fibonacci numbers and thereafter at double, treble etc.. that index number. The original version is in Russian, *Chisla fibonachchi*, 1951 and it is now again in print as Fibonacci Numbers, N N Vorob'ev, Birkhauser (Jan 2003).

TheFibonacci Series Modulo m D. D. Wall, *The American Mathematical Monthly* (1960) pages 525-532
is the earliest and most comprehensive paper on the subject of Fibonacci factors and Pisano periods.
Wall uses:

- u_n for our Fib(n)
- v_n for our Lucas(n)
- f_n for our General Fibonacci Series beginning with a and b: G(a,b,n)
- *k(m)* for the Pisano period of the Fibonacci numbers modulo m
- *h(a,b,m)* for the Pisano period of the General Fibonacci sequence G(a,b,n) modulo m

In the following sections we use the name $\mathrm{FEP}(n)$ for the Fibonacci Entry Point of n, the smallest index of a Fibonacci number that has n as a factor and look at some patterns and special cases.

3.1 Are there any numbers n where $\mathrm{FEP}(n) = n$?

There are some numbers such as 5 that are also their own entry points since $\mathrm{Fib}(5) = 5$ is the first Fibonacci number with an factor of 5.

Are there others?

Yes! For instance $\mathrm{Fib}(12) = 144$, the twelfth Fibonacci number is the first one with a factor of 12.

A list of those values of n where $\mathrm{FEP}(n) = n$ with n up to 1000 is

1, 5, 12, 25, 60, 125, 300, 625

This is a combination of two series which becomes clear if you factorize each of these numbers.

Can you find the next two numbers? Show the answer

3.2 FEP(n) = n − 1

Fib(10) = 55 = 11 × 5 so the tenth is the first Fibonacci number with 11 as a factor: FEP(11) = 10.

Fib(18) = 2584 = 19 × 136 so the 18^{th} Fibonacci number is the first with 19 as a factor: FEP(19) = 18

The list of the first n where FEP(n) = n − 1 is:

11, 19, 31, 59, 71, 79, 131, 179, 191, ... A106535

Is there a formula for these values?

Brother U Alfred showed that all these terms end in 1 or 9:

References

Primes which are factors of all Fibonacci sequences,Brother U Alfred, *Fib Quart*, 2 (1964), pages 33-38.

3.3 FEP(n) = n + 1

There is also a sequence of numbers with an FEP just one more than the number itself: i.e. FEP(n) = n + 1:
for instance
3 is a factor of Fib(4) = 3 and
7 is a factor of Fib(8) = 21
and the list of the first few n where FEP(n) = n + 1 is:
2, 3, 7, 23, 43, 67, 83, 103, 127, 163, 167, ... A000057
All these values end in 3 or 7 but is there a formula for these?
Factoring shows that all these are prime.
Brother U Alfred showed that all these terms end in 3 or 7 - see the reference in the previous section.

3.4 FEP(n) = n + 5

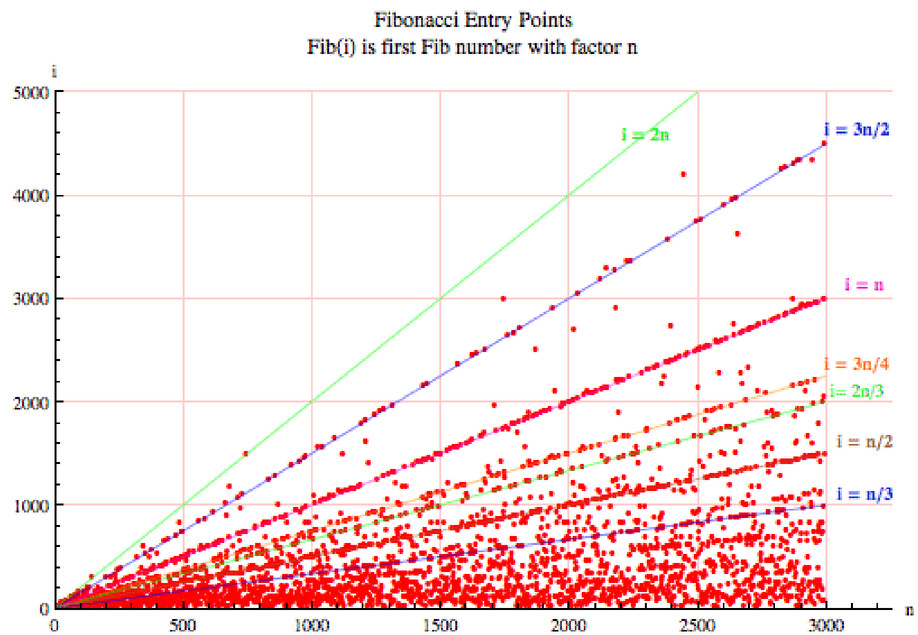
Benoit Cloitre noticed a connection between the n where FEP(n) = n + 1 and those n with FEP(n) = n + 5:

FEP(n) = n + 5: n=	10	15	35	115	215	335	415	515	635	...
FEP(n) = n + 1: n=	2	3	7	23	43	67	83	103	127	...

Can you find the connection? Show a hint

3.5 Other patterns in FEP(n)

There are also many other patterns in FEP(n). Here, for instance, are the equations of some of straight lines that lie on or near apparent lines of points in one of the graphs above:



It looks as if for any n, $\mathrm{FEP}(n) \le 2n$ but the only proof we have is that $\mathrm{FEP}(n) \le n^2$ in the Vorob'ev reference above.

3.6 Fibonacci Factors Calculator

C A L C U L A T O R Fibonacci Factors

Find the factors of i   for i=

Find the factors of Fib(i)   *up to*

R E S U L T S   CLEAR

+−×⁄: | Digit Sums | Factors | Remainders | Quincunx | Fractions | Number trick | Pythag Tris

3.7 Fibonacci Numbers with Index number factor

Earlier we saw that every number is a factor of some Fibonacci number, the first index number being called the FEP of that number.
Once we know the index number of the first Fibonacci number with n as a factor, FEP(n), then all the multiples of that index are also Fibonacci numbers with n as a factor. This since FEP(24) is 12, then Fib(12) has a factor of 24 and so does Fib(24), Fib(36), Fib(48), etc.
So now we can ask the question:

Which Fibonacci numbers Fib(n) have n as a factor?

For instance F(**12**)=144 which has **12** as a factor;
F(**25**)=75025 which has **25** as a factor;
but F(**15**)=610 which does not have **15** as a factor.

Here are the first few Fibonacci numbers F(n) with n as a factor:

n	**Fib(n) =**	**n**	**× m**
1	1 =	**1**	× 1
5	5 =	**5**	× 1
12	144 =	**12**	× 12
24	46368 =	**24**	× 1932
25	75025 =	**25**	× 3001
36	14930352 =	**36**	× 414732

The series of indices of such Fibonacci numbers is:

1, 5, 12, 24, 25, 36, 48, 60, 72, 96, ... A023172

Can you identify which numbers are in this sequence? Show the answer

3.8 Fibonacci numbers where i ± 1 is a factor of Fib(i)

i−1 is a factor of F(i)	
i	**Prime factors of F(i)**

	i−1	the rest
2	1	1
3	2	1
4	3	1
8	7	3
14	13	29
18	17	2^3×19
24	23	2^5×3^2×7
38	37	113×9349
44	43	3×89×199×307
48	47	2^6×3^2×7×23×1103
54	53	2^3×17×19×109×5779
68	67	3×1597×3571×63443
74	73	149×2221×54018521
84	83	2^4×3^2×13×29×211×281×421×1427
98	97	13×29×6168709×599786069

Fibonacci index numbers i where Fib(i) has i−1 as a factor:

2, 3, 4, 8, 14, 18, 24, 38, 44, 48, 54, 68, 74, 84, 98, 104, ... A100993

Though all the i listed here are one more than a prime (i-1 is prime), this is not true in general. The smallest such is 324, which has 323 as a factor of Fib(324) but 323=17×19. See the notes on A100993 in the link.

i+1 is a factor of F(i)		
i	Prime factors of F(i)	
	i+1	the rest
10	11	5
18	19	2^3×17
28	29	3×13×281
30	31	2^3×5×11×61
40	41	3×5×7×11×2161
58	59	19489×514229
60	61	2^4×3^2×5×11×31×41×2521
70	71	5×11×13×29×911×141961
78	79	2^3×233×521×859×135721
88	89	3×7×43×199×263×307×881×967
100	101	3×5^2×11×41×151×401×3001×570601

108	109	$2^4 \times 3^4 \times 17 \times 19 \times 53 \times 107 \times 5779 \times 11128427$
130	131	$5 \times 11 \times 233 \times 521 \times 2081 \times 24571 \times 14736206161$

Fibonacci index numbers i where Fib(i) has i+1 as a factor:
10,18,28,30,40,58,60,70,78,88,100,108,130,138,... A100992

All these i are one less than a prime (i+1 is prime) this is not always true. The smallest such is 441 since 442 is a factor of Fib(441) and 442 is not prime.

Are there any more patterns hidden here in the factors of Fibonacci numbers?

If so, let me know (click on **Dr Ron Knott** at the foot of this page for contact details) and I'll try to include some of them here.

3.8.1 You do the maths...

You might find this Table of Fibonacci Factors useful.

1. Another variation is to look at the remainder when we divide Fib(i) by i.
   Which index numbers, i, when we divide Fib(i) by i, have a remainder of 1?
   For example, the smallest is Fib(2)=1 which obviously has a reminder of 1 when we divide Fib(2) by 2;
   the next is Fib(11)=89 which, when we divide 11 leaves a remainder of 1.
   So your answer should start **2, 11, ...** Show the answer
2. What about those index numbers i where Fib(i) has a remainder of 1 less than i when Fib(i) is divided by i?
   This time we start again with Fib(2)=1 since it leaves a remainder of 2-1 when we divide it by i=2;
   Also Fib(3)=2 and Fib(4)=3 are also one less than their index numbers and the next is Fib(7)=13 which leaves a remainder of 7-1=6 when we divide 13 by 7.
   So this time your answer starts: **1, 2, 3, 4, 7, ...** Show the answer

3.9 The first Fibonacci number with a given prime as a factor: FEP(p) for prime p

Above we looked at *all* n and which was the first Fibonacci number for which that n was a factor.
In this section, we concentrate only on those n which are prime numbers.
If we look at the prime numbers and ask when they first appear as a factor of a Fibonacci number, we find they do so within the first p+1 Fibonacci numbers. In this table, i is the index of the first Fibonacci number that has the prime as a factor:

Prime p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71

first index i	3	4	5	8	10	7	9	18	24	14	30	19	20	44	16	27	58	15	68	70

The first-index numbers seem to be either equal to the prime (as for p=5) or one less (as p=11) or 1 more (p=7) in many cases. What about the others?
They are factors of p+1 (as in the case of p=17 where 9 is a factor of p+1=18) or of p–1 (e.g. p=29 where the index 14 is a factor of p-1=28).
This is true in general.

This series on the lower row: 3,4,5,8,10,7,9,18,24,14,30,19,20,44,16,27,58,15,68,70,... is Sloane's A001602. It is a subsequence of A001177 above, selecting the numbers at the prime positions.
The smallest Fibonacci number which has the n^{th} prime as a factor gives the series: 2, 3, 5, 21, 55, 13, 34, 2584, ... which is A051694. But as these Fibonacci numbers get large rapidly, it is easier to use the index numbers of such Fibonacci numbers to get the series above (A001602).
Vajda (reference at foot of this page), on page 84 states:

> Let F(*u*), *u* > 0, be the smallest Fibonacci number divisible by the prime *p*. The subscript *u* is called **the rank of apparition of *p***, and we know that it is a factor of, or equal to, *p*–1 or *p*+1.
> An alternative name is **the Fibonacci entry point** (FEP) and this applies to any number, not just the primes.

References

Clark Kimberling has a brief biography of Lucas

more information on Edouard Lucas (1842-1891) at the St. Andrews MacTutor site

On another of Ron Knott's Maths pages we look at the Lucas numbers, a series of numbers with the same rule as the Fibonacci numbers but starting with 2 and 1. In many senses Lucas numbers and Fibonacci numbers are twin series.

4 Fibonacci common factors

One of the fundamental divisibility properties of Fibonacci numbers concerns factors common to two Fibonacci numbers.
If a number is a factor of both F(n) and F(m) then it is also a factor of F(m+n).
This is a consequence of the formula:

$$F(n+m) = F(m-1)\,F(n) + F(m)\,F(n+1)$$

which is a special case of formula (8) in Vajda (reference at foot of this page)

But this is also part of a more general result about factors common to two Fibonacci numbers.
If g is the greatest divisor of **both** $F(m)$ and $F(n)$ then it is also a Fibonacci number. Which Fibonacci number? Its index number is the greatest divisor common to the two indices m and n!
If we use $gcd(a,b)$ to mean *the greatest common divisor (factor) of a and b* then we have:

gcd(F(m), F(n)) = F(gcd(m,n))

References
This result is Theorem 11 in Fibonacci Numbers, N N Vorob'ev (2003 translation of the 1951 Russian original).

This section was suggested by an email from Allyn Shell.

4.1 Neighbouring Fibonacci Numbers have no common factors

You might have noticed that no even Fibonacci number is next to another even Fibonacci number, or, no two neighbouring Fibonacci's have a common factor of 2.
In the last section we saw that Fib(3)=2 so we would expect the even Fibonacci numbers (with a factor of 2) to appear every at every *third* place in the list of Fibonacci numbers.

The same happens for a common factor of 3, since such Fibonacci's are at every 4-th place (Fib(4) is 3).

In fact, there will not be a *Fibonacci number* as a common factor between two neighbouring Fibonacci's for the same reason.

But what about other numbers as factors such as 6 or 7?

The answer is that *no number (bigger than 1) is a factor of two neighbouring Fibonacci numbers*.
Two numbers that have no common factors are called *relatively prime* (to each other).

There is a proof of this that Tom E Ace wrote to me about -- and it is so simple!

- If A and B have a common factor then it must also be a factor of A+B.
- If A and B have *no* common factor, then neither do B and A+B
  for if B and A+B had a common factor, then their *difference* would too but their difference is just A.

So in *any* Fibonacci-type series which starts with A and B, if A and B are relatively prime then so are *all pairs of consecutive numbers in the series*.
Alternatively, if A and B **have** a common factor then so do B and A+B (the next pair in the series) and so on, so that this factor is a factor of *all numbers in the series*.

Since F(1)=1 and F(2)=1 have no common factor, then no neighbouring pairs in the Fibonacci series have a common factor.
We have just shown that

F(n) and F(n+1) are relatively prime.

Now let's look at Fibonacci numbers that have no factors at all (apart from 1 and themselves of course), the prime Fibonacci numbers:

5 Fibonacci Numbers and Primes

We have seen from investigations above that F(*nk*) is a multiple of F(*k*) for all values of *n* and *k* = 1,2,...
This means that if the subscript has factors (i.e. is composite, is not a prime) then so is that Fibonacci number -- with one exception: can you find it?
So what about those Fibonacci numbers *with no factors* (apart from 1 and itself, of course)?
These are the **Fibonacci numbers that are primes**.
We can now deduce that

Any Fibonacci number that is a prime number must also have a subscript that is a prime number

again with one little exception - can you find it? Hint: you won't have to search far **for** it.

Unfortunately, the converse is **not** always true:
that is, it is **not** true that if a subscript is prime then so is that Fibonacci number.
The first case to show this is the 19th position (and 19 is prime) but
F(19)=4181 and F(19) is *not* prime because 4181=113x37.

In fact, a search using Maple finds that the list of index numbers, i, for which Fib(i) is prime begins as follows:

i	3	4	5	7	11	13	17	23	29	43	47	83
F(*i*)	2	3	5	13	89	233	1597	28657	514229	433494437	2971215073	99194853094755497

Now you should be able to spot the **odd one out**: that one number, i, which is *not a prime* in the list above, even though Fib(i) is.

The series continues (updated January 2007):

i	Fib(*i*)	Number of digits	Prime?
131	10663404174...72169	28	✔
137	19134702400...23917	29	✔
359	47542043773...76241	75	✔
431	52989271100...62369	90	✔
433	13872771278...68353	91	✔
449	30617199924...65949	94	✔
509	10597999265...29909	107	✔
569	36684474316...65869	119	✔
571	96041200618...74629	119	✔
2971	35710356064...16229	621	✔
4723	50019563612...91957	987	✔
5387	29304412869...55833	1126	✔
9311	34232086066...76289	1946	✔
9677	10565977873...70357	2023	✔
14431	35575535439...75869	3016	✔
25561	38334290314...14961	5342	✔
30757	30434499662...75737	6428	✔
35999	99214776140...24001	7523	✔
37511	96802910427...75089	7839	✔ Jun 2005
50833	13159270824...02753	10624	✔ Oct 2005
81839	97724940760...46561	17103	✔
104911	5660323637...84189	21925	?
130021	2706998033...75321	27173	?
148091	6904738850...74809	30949	?

201107	3371962609...27913	42029	?
397379	8912712429...66921	83047	?
433781	3296782330...74981	90655	?
590041	8448035604...82641	123311	?
593689	2059052250...74289	124074	?
604711	5962634693...37389	126377	?

Those marked ✔ are **definitely prime**;
those marked**?** are **probably prime** but have not been proved so (explained here).

References

The largest known Fibonacci prime, F(81839) was reported in April 2001 by David Broadbent and Bouk de Water

The series of *index numbers i of the prime Fib(i)*: 3,4,5,7,11,13,17,23,29,43,47,83,... is Sloane's A001605

The glossary entry on Chris Caldwell's The Prime Pages under Fibonacci Prime has more information and references.

5.1 Fibonacci numbers and special prime factors

Every Fibonacci number is marked in a special way.
If we look at the prime factors of a Fibonacci number, there will be *at least one of them* that has never before appeared as a factor in any earlier Fibonacci number. This is known as **Carmichael's Theorem** and applies to all Fibonacci numbers except 4 special cases:

i. Fib(1)=1 (has no prime factors),
ii. Fib(2)=1 (has no prime factors),
iii. Fib(6)=8 which only has prime factor 2 which is also Fib(3),
iv. Fib(12)=144 which also only 2 and 3 as its prime factors and these have appeared earlier as Fib(3)=2 and Fib(4)=3.

Apart from these special cases, the theorem is true for all Fib(n).

Those prime factors that have never appeared earlier in the table are shown like this.

Here is the first part of a table of Fibonacci numbers and their prime factors:

The first 25 Fibonacci numbers Factored `n : F(n)=factorisation`
`0 : 0 =`

```
1 : 1 =
2 : 1 =
3 : 2 PRIME
4 : 3 PRIME
5 : 5 PRIME
6 : 8 = 2^3
7 : 13 PRIME
8 : 21 = 3 x 7
9 : 34 = 2 x 17
10 : 55 = 5 x 11
11 : 89 PRIME
12 : 144 = 2^4 x 3^2
13 : 233 PRIME
14 : 377 = 13 x 29
15 : 610 = 2 x 5 x 61
16 : 987 = 3 x 7 x 47
17 : 1597 PRIME
18 : 2584 = 2^3 x 17 x 19
19 : 4181 = 37 x 113
20 : 6765 = 3 x 5 x 11 x 41
21 : 10946 = 2 x 13 x 421
22 : 17711 = 89 x 199
23 : 28657 PRIME
24 : 46368 = 2^5 x 3^2 x 7 x 23
25 : 75025 = 5^2 x 3001
```

[See the whole list up to Fib(300) on the next web page in a new window.]

References

A Result about the Primes Dividing Fibonacci Numbers by M S Boase in *Fibonacci Quarterly* vol 39 (2001), pages 386-391 contains a proof of this result but does not refer to it as Carmichael's Theorem. The problem is traced back to an analogous result proved by K Zsigmondy in 1892.

A Simple Proof of Carmichael's Theorem on Primitive Divisors by M Yabuta in *Fibonacci Quarterly* vol 39 (2001), pages 439-443 also contains a proof and refers to the following article...

On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$ by R D Carmichael in *Annals of Maths* vol 15 (1913) pages 30-70, where Carmichael refers to such first-occurrence-prime-factors as *characteristic factors*

5.2 Fib(prime) and Carmichael's Theorem

Shane Findley of Dover, USA, points out that **all** the factors of Fib(p) when p is a prime number are characteristic prime factors. Let's have a look at what this means in terms of our Table of Fibonacci Factors.

This relies on two properties of Fib(i) that we have already seen (on this page):

1. In the Factors of Fibonacci Numbers section we saw that
   if *i* itself has a factor *k* (so that we can write *i* as *nk*)
   then Fib(*nk*) has Fib(*k*) as a factor also.
2. Also, if Fib(*i*) is a prime number then *i* itself must be prime - see the Fibonacci Primes section.

So if *i* is prime - and let's call it *p* here to remind ourselves we are considering the special case of Fib(*i*) for a *prime number i* - then *p* will have no factors and therefore Fib(*p*) also can have no earlier Fibonacci numbers as its factors.
Note that this does not mean Fib(*p*) itself must be prime, only that *no smaller **Fibonacci number** can be a factor*. We found an example of this in Fib(19) which is 4181 = 37 x 113, and, although 19 is a prime number, Fib(19) is not.
Carmichael's Theorem says that there are special prime factors of Fib(*p*) that have not occurred earlier in our list of Fibonacci numbers.
So, if *p* is a prime number, then

- either Fib(*p*) is prime - in which case this is the first time we have seen this prime number in the list of Fibonacci factors
- or else, if Fib(*p*) has factors, at least one of them is new - a *characteristic factor*.

But Shane's observation is that *all the prime factors of Fib(p) are characteristic factors*!

To put this more simply for a prime number *p*, Fib(*p*) is either

- a prime itself or
- is a product of prime factors that all appear to be characteristic (appear for the first time in our list of Fibonacci factors).

Here is a selection of lines from the Factors Table for those Fib(*i*) where *i* is a prime number. You will notice that either they are prime numbers or else their factors are all shown like this to show they are characteristic factors:

The Fibonacci Numbers Fib(*p*)
where *p* is a prime number less than 100

```
n :  F(n)=factorisation

2 : 1
3 : 2 PRIME
5 : 5 PRIME
7 : 13 PRIME
11 : 89 PRIME
13 : 233 PRIME
17 : 1597 PRIME
19 : 4181 = 37 x 113
23 : 28657 PRIME
29 : 514229 PRIME
```

```
31 : 1346269 = 557 x 2417
37 : 24157817 = 73 x 149 x 2221
41 : 165580141 = 2789 x 59369
43 : 433494437 PRIME
47 : 2971215073 PRIME
53 : 53316291173 = 953 x 55945741
59 : 956722026041 = 353 x 2710260697
61 : 2504730781961 = 4513 x 555003497
67 : 44945570212853 = 269 x 116849 x 1429913
71 : 308061521170129 = 6673 x 46165371073
73 : 806515533049393 = 9375829 x 86020717
79 : 14472334024676221 = 157 x 92180471494753
83 : 99194853094755497 PRIME
89 : 1779979416004714189 = 1069 x 1665088321800481
97 : 83621143489848422977 = 193 x 389 x 3084989 x 361040209
```

5.3 No primes next to Fibonacci's!

Let's look at the numbers next to each Fibonacci number...

n	3	4	5	6	7	8	9	10	11	12	13	...
Fib(n)	2	3	5	8	13	21	34	55	89	144	233	...
Fib(n)±1	1 **3**	**2** 4	4 6	**7** 9	12 14	20 22	33 35	54 56	88 90	143 145	232 234	...

We see there are a few small Fibonacci numbers which have a **prime neighbour** (shown **like this**) but then they seem to stop. (143=11x13 is *not* a prime.)
Is this just a fluke or a feature of all but a few initial Fibonacci numbers - that their neighbours are *never prime*?
Toby Gee, a student at John of Gaunt's School, Trowbridge proved this in the 1996/97 (see the references at the end of this section).

He gave formulae for the factors too:
For the Fibonacci numbers with even index numbers, that is F(2n), we have:-

$$F(2n) + (-1)^n = (F(n+2) + F(n)) F(n-1)$$
$$F(2n) - (-1)^n = (F(n) + F(n-2)) F(n+1)$$

and for the odd index numbers, F(2n+1), we have similarly:

$$F(2n+1) + (-1)^n = (F(n+1) + F(n-1)) F(n+1)$$
$$F(2n+1) - (-1)^n = (F(n+2) + F(n)) F(n)$$

All of these are derived from Vajda-15a and Vajda-15b (see my Fibonacci, Phi and Lucas Numbers Formulae page).

References

Letter from Toby Gee in *Mathematical Spectrum*, vol 29 (1996/1997), page 68.

Greatest Common Divisors in Altered Fibonacci Sequences U Dudley, B Tucker *Fibonacci Quarterly* 1971, pages 89-91 give these formulae too in an expanded form.

5.4 Almost no primes next to Fibonacci's powers either!!

So having seen that the Fibonacci numbers influence their neighbours so that no neighbour is prime, what about **neighbours of the squares of Fibonacci numbers**?

Two formulae answer our question immediately:

$$F(n)^2 + 1 = \begin{cases} F(n-2)F(n+2) \text{ if n is } \textit{odd} \\ F(n-1)F(n+1) \text{ if n is } \textit{even} \end{cases}$$

$$F(n)^2 - 1 = \begin{cases} F(n-1)F(n+1) \text{ if n is } \textit{odd} \\ F(n-2)F(n+2) \text{ if n is } \textit{even} \end{cases}$$

These two formulae tell us that

the neighbours of $F(n)^2$ are never prime,
in fact they are always *the product of two Fibonacci numbers*!

So we *could* now investigate **the neighbours of the cubes of Fibonacci Numbers** and indeed I will leave you to discover the formulae that apply in those cases.
You will find that they too are never prime!!
Spoiler: Show the factors

The general result was found by Vernon Hoggatt Jr and Marjorie Bicknell-Johnson in 1977.

*The smaller neighbour of **every power** of every Fibonacci number (beyond F(3)=4) is always composite.*
This is easy to see with a little algebra since $x^n - 1$ always has $x - 1$ as a factor no matter what number x is.

For the larger neighbour of a power of a Fibonacci number, all of them are again composite *except in one special case*: when the power is itself a power of 2 (that is, it is 4, 8, 16, 32, ...) AND also the Fibonacci index number is a multiple of 3. In this case the number *may* be prime!

For example, all these neighbours on the "+1" side of a power of a Fibonacci number are *prime*:

$F(9)^4 + 1 = 1336337$
$F(198)^4 + 1$ (165 digits)
$F(15)^8 + 1$ (23 digits)
$F(48)^8 + 1$ (78 digits)
$F(51)^8 + 1$ (83 digits)
$F(21)^{32} + 1$ (130 digits)

You will see that all the powers are themselves powers of 2 and all the indices are multiples of 3. It seems that such primes are quite rare though.

So Fibonacci numbers exert a *powerful* influence 🙂 in that they (almost always) make any number next to them or their powers factorize!

References

Composites and Primes Among Powers of Fibonacci Numbers increased or decreased by one V E Hoggatt Jr and M Bicknell-Johnson, *Fibonacci Quarterly* vol 15 (1977), page 2.

5.5 A Prime Curio

G. L. Honaker Jr. pointed me to a little curio about the Fibonacci and the prime numbers: that the number of primes less than 144, which is a Fibonacci number, is 34, also a Fibonacci number. He asks:

Can this happen with two larger Fibonacci numbers?

I pass this question on to you - can it? The link to the Prime Curio page uses the notation that

π(N) means ***the number of primes between 1 and N***

and includes N too if N is prime. (See also a graph of this function.)
Since the prime numbers begin

2, 3, 5, 7, 11, 13, 17, ...

then $\pi(8)=4$ (there are 4 primes between 1 and 8, namely 2, 3, 5 and 7) and $\pi(11)=5$.
Here are some smaller values that are also Fibonacci numbers:

$\pi(2) = 1$
$\pi(3) = 2$

$\pi(5) = 3$
$\pi(21) = 8$

5.6 Another Prime Curio

M J Zerger noticed that the four consecutive Fibonacci numbers: F(6)=13, F(7)=21, F(8)=34 and F(9)=55 have a product of **13x3x7x17x2x5x11** or rearranging the factors into order: **2x3x5x7x11x13x17** which is the product of the first seven prime numbers!

5.7 More Links and References on Prime Numbers

References

There is a complete list of all Fibonacci numbers and their factors up to the 1000-th Fibonacci and 1000-th Lucas numbers and partial results beyond that on Blair Kelly's Factorisation pages

Chris Caldwell's Prime Numbers site has a host of information.

For the real enthusiast, join the Yahoo group on the PrimeForm computer program and related matters to primes. Its **Files** folder has a section on Lucas and Fibonacci primes.

See Neil Sloane's Online Encyclopedia of Integer Sequences where series number A001605 is the series of i's for which Fib(i) is known to be prime: 3,4,5,7,11,13,17,23,29,... and contains fairly up-to-date information on the latest results.

Factorization of Fibonacci Numbers D E Daykin and L A G Dresel in *The Fibonacci Quarterly*, vol 7 (1969) pages 23 - 30 and 82 gives a method of factoring a Fib(n) for composite n using the "entry point" of a prime, that is, the index of the first Fibonacci number for which prime p is a factor.

Mathematics Teacher M J Zerger vol 89 (1996) page 26

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

6 Remainders after division or Fibonacci MOD n

Earlier on this page we looked at the final digits of a Fibonacci number

0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, ... A003893

which repeats after 60 digits.
These are the remainders of the Fibonacci numbers when we divide by 10, or, to use the mathematical term, **Fibonacci numbers modulo 10**.
x modulo n means **the remainder when we divide the whole number x by n** and it is also written as **x mod n** for short. You

may also see **x ≡ a (n)** meaning x mod n is a.
The final *two* digits are therefore the Fibonacci numbers mod 100:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 44, ... A105471

and this cycle repeats after 300 numbers.
The final *three* digits are the Fibonacci's mod 1000 with a cycle length of 1500, and so on.
If we look at the Fibonacci numbers mod 2, then we get either 0 or 1 as these are the only two remainders we can have on dividing by 2. However, if the number is *even* the remainder is 0 and if it is *odd* the remainder is 1. This is called the *parity* of a number x and so **x mod 2** tell us if x is even or odd.
For the Fibonacci numbers we have:

0	1	1	2	3	5	8	13	21	...
Even	Odd	Odd	Even	Odd	Odd	Even	Odd	Odd	...
0	1	1	0	1	1	0	1	1	... A011655

and the cycle repeats with a cycle length of 3.

So what about division by other values?

We will *always* get a cycle that repeats!
The reason is that there are only a finite number of remainders when we divide by N: the values 0, 1, 2, ..., N-1 (N of them) and therefore there are a finite number of *pairs* of remainders (N^2) so when we keep adding the latest two Fibonacci's to get the next, we *must* eventually get a pair of remainders that we have had earlier and the cycle will repeat from that point on.
Here are the Fibonacci numbers mod 3:

0, 1, 1, 2, 0, 2, 2, 1, (0, 1,...) A082115

so the cycle-length of the Fibonacci numbers mod 3 is 8.
For the divisors (moduli) 2, 3, 4, 5, and so on we have the following cycle lengths for the Fibonacci Numbers:

mod	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*...*
cycle length	3	8	6	20	24	16	12	24	60	10	24	... A001175

The cycle lengths are also called **the Pisano periods**. You can investigate the cycle-lengths for different divisors using this Calculator:

6.1 Fibonacci Number Remainder (mod) cycles Calculator

C A L C U L A T O R

Show the cycle of the Fibonacci numbers mod 2 up to mod 12

Show the cycle length

RESULTS CLEAR

+−×/: Digit Sums | Factors | Remainders | Quincunx | Fractions | Number trick | Pythag Tris

6.2 Some interesting facts about Pisano periods

- The list of "records" or the highest Pisano period so far, is given by

mod	2	3	5	6	**10**	25	30	**50**	98	125	150	206	243	**250**	490	566	590	A?
Pisano period	3	8	20	24	**60**	100	120	**300**	336	500	600	624	648	**1500**	1680	1704	1740	A?

- The Pisano periods can be at most 6 times the modulus with examples shown in **red** in the table above. This is only achieved for moduli that are twice a power of 5:
  - for 2×5 = 10, the cycle length is 60
  - 2×5^2 = 50 has cycle length 300
  - 2×5^3 = 250 with cycle length 1500
  - 10, 50, 250, 1250, 6250, ... A020699 (or A095687)
- Apart from modulo 2, all the cycle lengths are even.
  For mod 2 the cycle is 0,1,1, 0,1,1, ... of length 3
- There are some numbers n where the length of the cycle mod n is n itself, called *fixed points*,
  for example the Fibonacci numbers mod 24 are 0,1,1,2,3,5,8,13,21,10,7,17,0,17,17,10,3,13,16,5,21,2,23,1 which then repeats so the cycle has length 24: Pisano(24) = 24.
  The series of these *fixed points of the Pisano function* is

(1,) 24, 120, 600, 3000, 15000, 75000, 375000, ... A235702

The entries are all of the form 24×5^n so that starting with 24 each of the following is 5 times the previous one.

- A pattern in the primes:

prime p	3	7	13	17	23	37	43	53	... A071774
Pisano(p)	8	16	28	36	48	76	88	108	=2p+2

All these primes end in 3 or 7, but other such primes are not in this table, for example,

prime p	47	107	113	233	263	... A216067
Pisano(p)	32	72	76	52	176	≠2p+2

- Another primes pattern:

prime p	11	19	31	41	59	61	71	79	109	... A003147
Pisano(p)	10	18	30	40	58	60	70	78	108	p−1

All these primes end in 1 or 9, but again there are exceptions:

prime p	29	89	101	139	151	181	199
Pisano(p)	14	44	50	46	50	90	22

but you many have noticed that quite a few of the Pisano periods are *factors* of p-1.

6.3 A fast algorithm for computing Pisano periods?

The following results are from Marc Renault's page:

- Look at this table of Pisano periods of n, 2n, 3n, ... :

Pisano	**n**	**2n**	**3n**	**4n**	**5n**	**6n**	**7n**
n=2	3	6	24	12	60	24	48
n=3	8	24	24	24	40	24	16
n=4	6	12	24	24	60	24	48
n=5	20	60	40	60	100	120	80

Can you see that Pisano(kn) is a multiple of Pisano(n) for each of these n?

This can be proved true for all n - see Marc's page as mentioned above.

So if n divides m which we write as n | m then P(n) | P(m).

- Using the last result, if we find the prime factors of n, $n = p_1^{a_1}\, p_2^{a_2} \ldots$ then Pisano(n) will be the lowest common multiple (LCM) of all the Pisano($p_i^{a_i}$).
- What can we say about Pisano(p^a) for a prime p? Wall conjectured in 1960 (see references) that Pisano(p^a) = p^{a-1}P(p) for all primes p.

This still seems to be unproven.

If it is true, it means that we can find Pisano(n) for all n once we know Pisano(p) for all primes p that are (prime) factors of n. (see A060305)

Here is a table of the Pisano periods for the first 15 primes:

p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	...
Pisano(p)	3	8	20	16	10	28	36	18	48	14	30	76	40	88	32	108	58	60	136	70	148	78	168	44	196	A060305

Is there a formula for the cycle lengths of the Fibonacci Numbers?

Not quite! It seems to depend on the factorization of the divisor and the following article is an excellent summary:

References

Notes 88.52: Some Properties of finite Fibonacci Sequences D Vella, A Vella *Mathematical Gazette* 88 (2004) pages 494-499.

Dominic Vella's Mathematics page. Dominic and Alfred Vallas continue to do research on the Fibonacci numbers mod n and generalised Fibonacci numbers mod n.

Marc Renault has a list of the Pisano periods for 2 up to 2002 and his Master's Thesis on **Properties of the Fibonacci Sequence Under Various Moduli** is available on his website too. He also has a useful summary of his results and A formula for cycle length for almost all moduli.

That the remainders of the Fibonacci numbers when divided by a certain number is a cycle seems to have been known even to the French mathematician (though he was born in Italy in Turin) Joseph Louis Lagrange (1736-1813) as early as 1774: **Oeuvres de LaGrange**, published in Paris in 1877, pages 5-182.
However, the numbers were not then called the Fibonacci Numbers and their periods were not called Pisano periods!

Fibonacci Series Modulo m D D Wall, The American Mathematical Monthly, Vol. 67, No. 6 (Jun. - Jul., 1960), pages 525-532.

There are many interesting relationships in the Pisano Periods sequence (the series of cycle lengths for modulus 2 upwards). Here is a table of the cycle lengths (the Pisano periods) for some smaller moduli:
Add the **row label** to the **column label** to find N and the entry for that row and column is the Pisano period for modulus N:

6.4 Pisano Periods for moduli 2 - 299

+	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250	260	270	280	290
0	.	60	60	120	60	300	120	240	120	120	300	60	120	420	240	600	240	180	120	180	600	240	60	240	120	1500	420	360	240	420
1	.	10	16	30	40	72	60	70	216	112	50	152	110	130	32	50	48	72	90	190	136	42	252	80	240	250	168	270	56	392

2	3	24	30	48	48	84	30	24	120	48	72	48	60	120	210	36	216	264	336	96	150	108	456	84	330	48	390	72	96	444
3	8	28	48	40	88	108	48	148	168	120	208	76	40	144	140	72	328	348	120	388	112	280	448	52	648	240	176	112	568	588
4	6	48	24	36	30	72	96	228	48	96	84	72	30	408	24	240	120	168	48	588	72	72	48	168	60	768	120	276	210	336
5	20	40	100	80	120	20	140	200	180	180	80	240	500	360	140	60	40	400	380	280	40	440	600	160	560	360	540	100	360	580
6	24	24	84	24	48	48	120	18	264	48	108	42	48	36	444	168	168	120	120	336	624	72	228	174	120	384	144	48	420	228
7	16	36	72	76	32	72	136	80	56	196	72	168	256	276	112	316	336	232	180	396	48	240	456	312	252	516	88	556	80	360
8	12	24	48	18	24	42	36	168	60	336	72	174	192	48	228	78	48	132	96	120	168	108	72	144	60	264	408	138	48	444
9	24	18	14	56	112	58	48	78	44	120	108	144	88	46	148	216	364	178	144	22	90	296	114	238	168	304	268	120	612	336

Shrink the view

6.5 The relationship between the Pisano Period mod n and FEP(n)

If the initial digits of the Fibonacci series form a cycle of length 60 (the Pisano period of 10) then Fib(60) is the same as Fib(0), which is 0. So Fib(60) has the same remainder mod 10, namely 0, so 10 divides exactly into Fib(60).
The same is true for all Pisano periods mod n: if the Pisano period mod n is P then Fib(P) has n as a factor.
However, Pisano(n), the Pisano period mod n, may not be the first Fibonacci number which has n as a factor: for example the Fibonacci numbers mod 3 have a cycle of length 8, so that Fib(8)=Fib(0) mod 3, and in general Fib(n)=Fib(n+8) mod 3:

i	0	1	2	3	4	5	6	7	8	9	...
Fib(i)	0	1	1	2	3	5	8	13	21	34	...
Fib(i) mod 3	0	1	1	2	0	2	2	1	0	1	...

For 3, there are two 0s in each Pisano cycle of length 8.

Earlier on this page we looked at where a given number first appears as a factor of a Fibonacci number: the Fibonacci Entry Point for that number or FEP(n).
So the Pisano period Pisano(n) for n may be the index number of the first Fibonacci number to have n as a factor -- or it may be some multiple of it.
In fact, it can be proved that the Pisano period Pisano(n) is either

- equal to the Fibonacci entry point: Pisano(n)=FEP(n) or
- Pisano(n) is twice FEP(n) or
- Pisano(n) is four times FEP(n)

FEP(n) versus Pisano(n)

n	2	3	4	5	6	7	8	9	10	11	12	13	...

Pisano(n)	**3**	**8**	**6**	**20**	**24**	**16**	**12**	**24**	**60**	**10**	**24**	**28**	**...**
FEP(n)	**3**	**4**	**6**	**5**	**12**	**8**	**6**	**12**	**15**	**10**	**12**	**7**	**...**
Pisano(n) / FEP(n)	1	2	1	4	2	2	2	2	4	1	2	4	...

Since F(n) is a factor of F(kn) for all k, then the final row here tells us how many zeroes there are in the Pisano cycle mod n. See A001176.

If n itself is a Fibonacci number bigger than 2, as with 5, 8 and 13 in the table here, then this quotient is alternately 2 and 4, that is

$$\text{Pisano}(\text{Fib}(2n))=2 \text{ and } \text{Pisano}(\text{Fib}(2n+1))=4$$

with our usual system of indexing Fibonacci which begins Fib(0)=0 and Fib(1)=1. And here are some investigations to get you started discovering some of the Pisano period's fascinating properties:-

6.5.1 You do the maths...

1. What is special about the cycle lengths of moduli 2, 4, 8, 16, 32, 64, ...?
2. Does the same thing happen with the powers of 3: 3, 9, 27, 81, ...?
3. Take pairs of prime numbers such as 3 and 5 or 3 and 13 etc... Find the Pisano period of each in the pair. How is it related to the Pisano period of the product of your two primes?
4. Take any two numbers A and B where A is a factor of B.
   a. What are the Pisano periods of A and B?
   b. What is the Pisano period of their product AB?
   c. Is the Pisano period of A a factor of the Pisano period of B?
5. Look at the complete cycle for any modulus N. It always starts with 0, 1,... . Here is the complete cycle for 3:
   mod 3: 0 1 1 2 0 2 2 1 - it has **two** zeros.
   For 4 : 0 1 1 2 3 1 we have just **one** zero at the start
   and for 5: 0 1 1 2 3 0 3 3 1 4 0 4 4 3 2 0 2 2 4 1 - we have **four** zeros.
   Find a modulus N with some other number of zeros in its cycle.

References

The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence John Vinson, *Fibonacci Quarterly*, vol 1 (1963), pages 37-45.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

7 Benford's Law and initial digits

[With thanks to Robert Matthews of *The Sunday Telegraph* for suggesting this topic.]

Having looked at the *end digits* of Fibonacci numbers, we might ask

Are there any patterns in the initial digits of Fibonacci numbers?

What are the chances of a Fibonacci number beginning with "1", say? or "5"? We might be forgiven for thinking that they probably are all the same - each digit is equally likely to start a randomly chosen Fibonacci number. You only need to look at the Table of the First 100 Fibonacci numbers or use Fibonacci Calculator to see that this is not so. Fibonacci numbers seem far more likely to start with "1" than any other number. The next most popular digit is "2" and "9" is the least probable!

This law is called **Benford's Law** and appears in many tables of statistics. Other examples are a table of populations of countries, or lengths of rivers. About one-third of countries have a population size which begins with the digit "1" and very few have a population size beginning with "9".

Here is a table of the initial digits as produced by the Fibonacci Calculator:

```
Initial digit frequencies of fib(i) for i from 1 to 100:
    Digit:    1   2   3   4   5   6   7   8   9
Frequency:   30  18  13   9   8   6   5   7   4  100 values
  Percent:   30  18  13   9   8   6   5   7   4
```

What are the frequencies for the first 1000 Fibonacci numbers or the first 10,000? Are they settling down to fixed values (percentages)? Use the Fibonacci Calculator to collect the statistics. According to Benford's Law, large numbers of items lead to the following statistics for starting figures for the Fibonacci numbers as well as some natural phenomena

Digit:	1	2	3	4	5	6	7	8	9
Percentage:	30	18	13	10	8	7	6	5	5

7.1 You do the maths...

1. Look at a table of sizes of countries. How many countries areas begin with "1"? "2"? etc..
2. Use a table of population sizes (perhaps of cities in your country or of countries in the world). It doesn't matter if the figures are not the latest ones. Does Benford's Law apply to their initial digits?
3. Look at a table of sizes of lakes and find the frequencies of their initial digits.
4. Using the Fibonacci Calculator make a table of the first digits of powers of 2. Do they follow Benford's Law? What about powers of other numbers?
5. Some newspapers give lists of the prices of various stocks and shares, called "quotations". Select a hundred or so of the quotations (or try the first hundred on the page) and make a table of the distribution of the leading digits of the prices. Does it follow Benford's Law?
6. What other sets of statistics can you find which do show Benford's Law? What about the number of the house where the people in your class live? What about the initial digit of their home telephone number?
7. Generate some random numbers of your own and look at the leading digits.
   You can buy 10-sided dice (bi-pyramids) or else you can cut out a decagon (a 10-sided polygon with all sides the same length) from card and label the sides from 0 to 9. Put a small stick through the centre (a used matchstick or a cocktail stick or a small pencil or a ball-point pen) so that it can spin easily and falls on one of the sides at random. (See the footnote about dice and spinners on the "The Golden Geometry of the Solid Section or Phi in 3 dimensions" page, for picture and more details.)
   Are all digits equally likely or does this device show Benford's Law?
8. Use the random number generator on your calculator and make a table of leading-digit frequencies. Such functions will often generate a "random" number between 0 and 1, although some calculators generate a random value from 0 to the maximum size of number on the calculator. Or you can use the random number generator in the Fibonacci Calculator to both generate the values and count the initial digit frequencies, if you like.
   Do the frequencies of leading digits of random values conform to Benford's Law?
9. Measure the height of everyone in your class to the nearest centimetre. Plot a graph of their heights. Are all heights equally likely? Do their initial digits conform to Benford's Law? Suppose you did this for everyone in your school. Would you expect the same distribution of heights?
10. What about repeatedly tossing five coins all at once and counting the number of heads each time?
    What if you did this for 10 coins, or 20?
    What is the name of this distribution (the shape of the frequency graph)?

7.2 When does Benford's Law apply?

Random numbers are equally likely to begin with each of the digits 0 to 9. This applies to randomly chosen real numbers or randomly chosen integers.

Randomly chosen real numbers

If you stick a pin at random on a ruler which is 10cm long and it will fall in each of the 10 sections 0cm-1cm, 1cm-2cm, etc. with the same probability. Also, if you look at the initial digits of the points chosen (so that the initial digit of 0.02cm is 2 even though the point is in the 0-1cm section) then each of the 9 values from 1 to 9 is as likely as any other value.

Randomly chosen integers

This also applies if we choose random integers.
Take a pack of playing cards and remove the jokers, tens, jacks and queens, leaving in all aces up to 9 and the kings. Each card will represent a different digit, with a king representing zero. Shuffle the pack and put the first 4 cards in a row to represent a 4 digit integer. Suppose we have King, Five, King, Nine. This will represent "0509" or the integer 509 whose first digit is 5. The integer is as likely to begin with 0 (a king) as 1 (an ace) or 2 or any other digit up to 9.
But if our "integer" began with a king (0), then we look at the next "digit".
These have the same distribution as if we had chosen to put down just 3 cards in a row instead of 4. The first digits all have the same probability again. If our first two cards had been 0, then we look at the third digit, and the same applies again.
So if we ignore the integer 0, any randomly chosen (4 digit) integer begins with 1 to 9 with equal probability. (This is not quite true of a row of 5 or more cards if we use an ordinary pack of cards - why?)

So the question is, why does this *all-digits-equally-likely* property **not** apply to the first digits of each of the following:

- the Fibonacci numbers,
- the Lucas numbers,
- populations of countries or towns
- sizes of lakes
- prices of shares on the Stock Exchange

Whether we measure the size of a country or a lake in square kilometres or square miles (or square anything), does not matter - Benford's Law will still apply.
So when is a number *random*? We often meant that we cannot predict the next value. If we toss a coin, we can never predict if it will be Heads or Tails if we give it a reasonably high flip in the air. Similarly, with throwing a dice - "1" is as likely as "6". Physical methods such as tossing coins or throwing dice or picking numbered balls from a rotating drum as in Lottery games are always unpredictable.

The answer is that the Fibonacci and Lucas Numbers are governed by a **Power Law**.
We have seen that Fib(i) is round($\mathrm{Phi}^i/\sqrt{5}$) and Lucas(i) is round(Phi^i). Dividing by sqrt(5) will merely adjust the scale - which does not matter. Similarly, rounding will not affect the overall distribution of the digits in a large sample.

Basically, Fibonacci and Lucas numbers are **powers of Phi**. Many natural statistics are also governed by a power law - the values are related to B^i for some base value B. Such data would seem to include the sizes of lakes and populations of towns as well as non-natural data such as the collection of prices of stocks and shares at any one time. In terms of natural phenomena (like lake sizes or heights of mountains) the larger values are rare and smaller sizes are more common. So there are very few large lakes, quite a few medium sized lakes and very many little lakes. We can see this with the Fibonacci numbers too: there are 11 Fibonacci numbers in the range 1-100, but only one in the next 3 ranges of 100 (101-200, 201-300, 301-400) and they get increasingly rarer for large ranges of size 100. The same is true for any other size of range (1000 or 1000000 or whatever).

7.2.1 You do the maths...

1. Type a power expression in the *Eval(i)*= box, such as **pow(1.2,i)** and give a range of i values from i=1 to i=100. Clicking the *Initial digits* button will print the leading digit distribution.
   Change 1.2 to any other value. Does Benford's Law apply here?
2. Using *Eval(i)*=**randint(1,100000)** with an i range from 1 to 1000 (so that 1000 separate random integers are generated in the range 1 to 100000) shows that the leading digits are all equally likely.

References

Benford's Law for Fibonacci and Lucas Numbers, L. C. Washington, *The Fibonacci Quarterly* vol. 19, 1981, pages 175-177.

The original reference: **The Law of Anomalous Numbers** F Benford, (1938) *Proceedings of the American Philosophical Society* vol 78, pages 551-572.

The Math Forum's archives of the History of Mathematics discussion group have an email from Ralph A. Raimi (July 2000) about his research into Benford's Law. It seems that Simon Newcomb had written about it much earlier, in 1881, in **American Journal of Mathematics** volume 4, pages 39-40. The name **Benford** is, however, the one that is commonly used today for this law.

MathTrek by Ivars Peterson (author of **The Mathematical Tourist** and **Islands of Truth**) the editor of Science News Online has produced this very good, short and readable introduction to Benford's Law.

M Schroeder Fractals, Chaos and Power Laws, Freeman, 1991, ISBN 0-7167-2357-3. This is an interesting book but some of the mathematics is at first year university level (mathematics or physics degrees), unfortunately, and the rest will need sixth form or college level mathematics beyond age 16. However, it is still good to browse through. It has only a passing reference to Benford's Law: *The Peculiar Distribution of the Leading Digit* on page 116.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

8 Every number starts some Fibonacci Number

We found that every number is a factor of some Fibonacci number above but it is also true that we can always find a Fibonacci number that *begins* with a given number as its initial digits.
The first few Fibonacci numbers are 0,1,1,2,3,5,8. We have to go up to Fib(19)=4181 to find one beginning with 4 and Fib(15)=641 for 6. The index numbers (ranks) of the Fibonacci numbers that begin with 1 up to 20 are:

Fibonaccis starting with the number n

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Fib(i) starting with n	1	2	3	4181	5	610	75025	8	987	10946	1134903170	121393	13	144	1597	165580141	17711	1836311903	196418	20365011074
rank i	1	3	4	19	5	15	25	6	16	21	45	26	7	12	17	41	22	46	27	51

Ross Honsberger (see the references at the end of this section) gives the proof that we can always find a power of 2 starting with any given number and, in fact, this works for any other base number as well as 2. The problem goes back to a Hungarian Mathematical competition problem of 1928 (see references).

On another page we look at the Lucas numbers Lucas(n) = Fib(n-1) + Fib(n+1) and find that Lucas(i) is Round(Phi^i) so the initial-digits-of-powers applies to the Lucas numbers also.

Lucas numbers starting with the number n

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Lucas(i) starting with n	1	2	3	4	521	64079	7	843	9349	103682	11	123	1364	141422324	15127	167761	17393796001	18	199	20633239
rank i	1	0	2	3	13	23	4	14	19	24	5	10	15	39	20	25	49	6	11	35

References

Ingenuity in Mathematics Ross Honsberger, (Mathematical Association of America), 1975, *Essay Six; A Property of a^n* pages 38-45

[Challenging Mathematical Problems with Elementary Solutions, Vol 1] A M Yaglon and I M Yaglom, (Dover paperback edition 1987 of the 1964 original).
There is also a second volume:
[Challenging Mathematical Problems with Elementary Solutions, Vol 2] A M Yaglon and I M Yaglom, (Dover paperback edition 1987 of the 1967 original)
The original problem and its solution appear in
[Hungarian Problem Book II: Based on the Eötvös Competitions 1906-1928] J Kurschak (compiler) and E Rapaport (translator), Mathematical Association of America (1963)

8.1 Is there a Fibonacci number that ends with any given number?

It seems we can start off, with the aid of a Calculator, and find Fibonacci numbers ending with all values from 1 to 99. The list starts...

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Fib(i) ending with n	**1**	**2**	**3**	34	**5**	1094**6**	37**7**	**8**	8**9**	6**10**	177**11**	61922 0451666590 1352286753 8786329787 4269396**12**	**13**	704 9252476708 91258141**14**	6590346 2158763004 19824982**15**	57602 1322354247 5588620619 86853652**16**	241578**17**	1964**18**
rank i	1	3	4	9	5	21	14	6	11	15	22	216	7	111	130	168	37	27

The list of ranks of these Fibonaccis-ending-with-n (their Fibonacci index numbers) is 1, 3, 4, 9, 5, 21, 14, ... A023183
Jason Earls comments that there seem to be none that end with 100, 102, 108, 110, 116, 118, ... or in fact any number beyond 99 that has a remainder of 4 or 6 when divided by 8.
Can you prove he is right?

Show the proof

9 The Fibonacci Numbers in Pascal's Triangle

```
    1
   1 1
  1 2 1
 1 3 3 1
1 4 6 4 1
```

	0	**1**	**2**	**3**	**4**	**...**
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	

... ...

Each entry in the triangle on the left is the sum of the two numbers above it.
If we re-align the table to look the one on the right then each number is the sum of the one above it and the one to the left of that one where a blank space can be taken as "0". Note that each row starts and ends with "1".

Pascal's Triangle has lots of uses including

Calculating probabilities.
If you throw n coins randomly onto a table then the chance of getting H heads among them is the entry in row N, col H divided by 2^n:
for instance, for 3 coins, n=3 so we use row 3:
3 heads: H=3 is found in **1** way (HHH)
2 heads: H=2 can be got in **3** ways (HHT, HTH and THH)
1 head: H=1 is also found in **3** possible ways (HTT, THT, TTH)
0 heads: H=0 (i.e. all Tails) is also possible in just **1** way: TTT

Finding terms in a Binomial expansion: $(a+b)^n$
EG. $(a+b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3$

Can we find the Fibonacci Numbers in Pascal's Triangle? Yes! The answer lies in the diagonals in the triangle:

	1							
1	1	1						
1	1	2	1					
2	1	3	3	1				
3	1	4	6	4	1			
5	1	5	10	10	5	1		
8	1	6	15	20	15	6	1	
13	1	7	21	35	35	21	7	1
21	...							

9.1 Why do the Diagonals sum to Fibonacci numbers?

It is easy to see that the diagonal sums really are the Fibonacci numbers if we remember that each number in Pascal's triangle is the sum of two numbers in the row above it (blank spaces count as zero), so that 6 here is the sum of the two 3's on the row above.

The numbers in any diagonal row are therefore formed from adding numbers in the previous two diagonal rows as we see here where all the blank spaces are zeroes and where we have introduced an extra column of zeros which we will use later:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

The **green diagonal** sums to 5;
the **blue diagonal** sums to 8;
the **red diagonal** sums to 13

Each **red number** is the sum of a **blue** and a **green** number on the row above.

Notice that the **GREEN** numbers are on one diagonal and the **BLUE** ones on the next. The sum of all the green numbers is 5 and all the blue numbers add up to 8.

Because all the numbers in Pascal's Triangle are made the same way - by adding the two numbers *above and to the left* ***on the row above***, then we can see that each red number is just the sum of a green number and a blue number and we use up all the blue and green numbers to make all the red ones.

The sum of all the red numbers is therefore the same as the sum of all the blues and all the greens: 5+8=13!

The general principle that we have just illustrated is:

> The *sum* of the numbers on one diagonal is the sum of the numbers on the previous two diagonals.

If we let D(i) stand for the sum of the numbers on the Diagonal that starts with one of the extra zeros at the beginning of row i, then

$$D(0)=0 \text{ and } D(1)=1$$

are the two initial diagonals shown in the table above. The green diagonal sum is D(5)=5 (since its extra initial zero is in row 5) and the blue diagonal sum is D(6) which is 8. Our red diagonal is D(7) = 13 = D(6)+D(5).

We also have shown that this is always true: one diagonals sum id the sum of the previous two diagonal sums, or, in terms of our D series of numbers:

$$D(i) = D(i-1) + D(i-2)$$

But...

$$D(0) = 1$$
$$D(1) = 1$$
$$D(i) = D(i-1) + D(i-2)$$

is exactly the definition of the Fibonacci numbers! So $D(i)$ is just $F(i)$ and

the sums of the diagonals in Pascal's Triangle are the Fibonacci numbers!

9.2 Another arrangement of Pascal's Triangle

By drawing Pascal's Triangle with all the rows moved over by 1 place, we have a clearer arrangement which shows the Fibonacci numbers as sums of columns:

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	.	.	.	.	.	.	.	.	.
.	1	1	.	.	.	.	.	.	.
.	.	1	2	1	.	.	.	.	.
.	.	.	1	3	3	1	.	.	.
.	.	.	.	1	4	6	4	1	.
.	.	.	.	.	1	5	10	10	5
.	.	.	.	.	.	1	6	15	20
.	.	.	.	.	.	.	1	7	21
.	.	.	.	.	.	.	.	1	8
.	.	.	.	.	.	.	.	.	1
1	***1***	***2***	***3***	***5***	***8***	***13***	***21***	***34***	***55***

This table can be explained by referring to one of the (Easier) Fibonacci Puzzles - the one about Fibonacci for a Change. It asks how many ways you can pay n pence (in the UK) using only 1 pence and 2 pence coins. The order of the coins matters, so that 1p+2p will pay for a 3p item and 2p+1p is counted as a different answer. [We now have a new **two pound coin** that is increasing in circulation too!]

Here are the answers for paying up to 5p using only 1p and 2p coins:

1p	2p	3p	4p	5p
1p	2p 1p+1p	1p+2p 2p+1p 1p+1p+1p	2p+2p 1p+1p+2p 1p+2p+1p 2p+1p+1p 1p+1p+1p+1p	1p+2p+2p 2p+1p+2p 2p+2p+1p 1p+1p+1p+2p 1p+1p+2p+1p 1p+2p+1p+1p 2p+1p+1p+1p 1p+1p+1p+1p+1p

1 way	2 ways	3 ways	5 ways	8 ways

Let's look at this another way - arranging our answers according to **the number of 1p and 2p coins we use**. Columns will represent all the ways of paying the amount at the head of the column, as before, but now the rows represent **the number of coins in the solutions**:

cost:	**1p**	**2p**	**3p**	**4p**	**5p**
1 coin:	1p	2p			
2 coins:		1p+1p	1p+2p 2p+1p	2p+2p	
3 coins:			1p+1p+1p	1p+1p+2p 1p+2p+1p 2p+1p+1p	1p+2p+2p 2p+1p+2p 2p+2p+1p
4 coins:				1p+1p+1p+1p	2p+1p+1p+1p 1p+1p+1p+2p 1p+1p+2p+1p 1p+2p+1p+1p
5 coins:					1p+1p+1p+1p+1p

If you count the number of solutions in each box, it will be exactly the form of Pascal's triangle that we showed above!

The mathematics is in this formula:
where the big brackets with two numbers vertically inside them
are a special mathematical notation for the entry in Pascal's triangle
on row **n-k-1** and column **k**

$$\mathrm{Fib}(n) = \sum_{k=0}^{n-1} \binom{n-k-1}{k}$$

Or, an equivalent formula is:

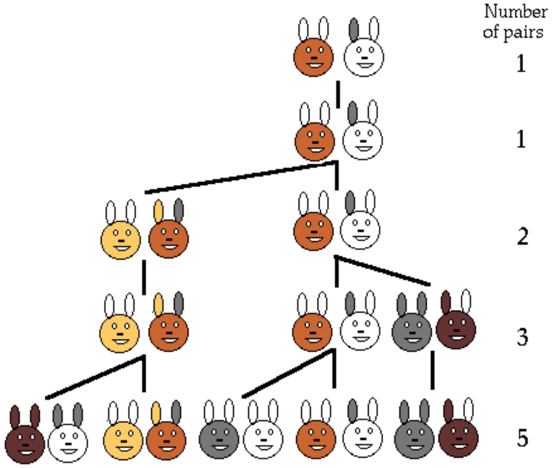
$$\mathrm{Fib}(n) = \sum_{k=1}^{n} \binom{n-k}{k-1}$$

9.3 Fibonacci's Rabbit Generations and Pascal's Triangle

Here's another explanation of how the Pascal triangle numbers sum to give the Fibonacci numbers, this time explained in terms of our original rabbit problem.

Let's return to Fibonacci's rabbit problem and look at it another way. We shall be returning to it several more times yet in these pages - and each time we will discover something different!

We shall make a family tree of the rabbits but this time we shall be interested only in the **females and ignore any males** in the population. If you like, in the diagram of the rabbit pairs shown here, assume that the rabbit on the left of each pair is male (say) and so the other is female. Now ignore the rabbit on the left in each pair!

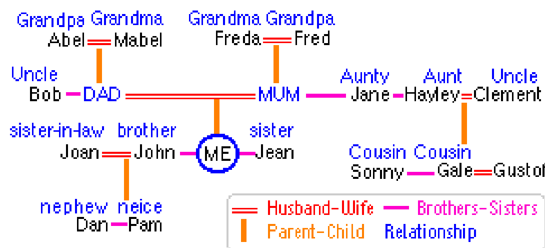


We will assume that **each mating produces exactly one female** and perhaps some males too but we only show the females in the diagram on the left. Also in the diagram on the left we see that each individual rabbit appears several times. For instance, the original brown female was mated with a while male and, since they never die, they both appear once on every line.

Now, in our new family tree diagram, **each female rabbit will appear *only once***. As more rabbits are born, so the Family tree grows adding a new entry for each newly born female.

As in an ordinary human family tree, we shall show parents above a line of all their children.

Here is a fictitious human family tree with the names of the relatives shown for a person marked as **ME**:



The diagram shows that:

- Grandpa Abel and Grandma Mabel are the parents of my Dad;
- Grandma Freda and Grandpa Fred are the parents of my Mum.
- Bob is my Dad's brother
- my Mum has two sisters, my aunts Hayley and Jane.
- Aunt Hayley became Hayley Weather when she married Clement Weather.
- They have two children, my cousins Sonny Weather and Gale Weather.
- Gale married Gustof Wind and so is now Gale Wind.
- My brother John and his wife Joan have two children, my nephew Dan and my niece Pam.

In this family tree of human relationships, the = joins people who are parents or signifies a marriage.

In our rabbit's family tree, rabbits don't marry of course, so we just have the vertical and horizontal lines:

The vertical line |

points from a mother (above) to the oldest daughter (below);

the horizontal line -

is drawn between sisters from the oldest on the left down to the youngest on the right;

the small letter r

represents a young female (a little rabbit) and

the large letter R

shows a mature female (a big Rabbit) who can and does mate every month, producing one new daughter each time.

As in Fibonacci's original problem (in its variant form that makes it a bit more realistic) we assume none die and that each month every mature *female* rabbit **R** always produces exactly one *female* rabbit **r** (we ignore males) each month.
So each month:

- each **r** will change to **R** (each matures after one month)
- each **R** produces a new baby rabbit and so adds another **r** at the end of the line below which are its direct descendants or children), that is produce a new baby rabbit each month

Here is the Rabbit Family tree as if grows month by month for the first 9 months and 5 generations:

KEY |:mother above with (first) daughter below ___:joins sisters
R:mother (produces a new female each month) **r**:new female
Month: 1 2 3 4 5 6 7 8 9

```
1   r
    ^Initially there is one immature rabbit (generation 1)

0 R + 1 r = 1
```

The Family tree is shown for the first 9 months as more females are added to it. We can see that our original female becomes a great-grandmother in month 7 when a fourth line is added to the Family tree diagram - a fourth generation!

Have you spotted the Pascal's triangle numbers in the Rabbit's Family Tree?

The numbers of rabbits in each generation, that is, along each level (line) of the tree, are the Pascal's triangle numbers that add up to give each Fibonacci number - the total number of (female) rabbits in the Tree. In month n there are a total of F(n) rabbits, a number made up from the entry in row (n-k) and column (k-1) of Pascal's triangle for each of the levels (generations) k from 1 to n. In other words, we are looking at this formula and explaining it in terms of generations, the original rabbit forming generation 1 and her daughters being generation 2 and so on:

$$\mathrm{Fib}(n) = \sum_{k=1}^{n} \binom{n-k}{k-1}$$

Remember that the rows and columns of Pascal's triangle in this formula begin at 0!
For example, in month 8, there are 4 levels and the number on each level is:

```
1   R
    |
2   R_______________R_________R_____R___R_r
    |               |         |     |
3   R_____R___R_r   R___R_r   R_r   r
    |     |         |
4   R_r   r         r
```

Generation 1: **1 rabbit which is Pascal's triangle row 7 (8-1), column 0 (1-1)**

Generation 2: **6 rabbits which is Pascal's triangle row 6 (8-2), column 1 (2-1)**

Generation 3: **10 rabbits which is Pascal's triangle row 5 (8-3), column 2 (3-1)**

Generation 4: **4 rabbits which is Pascal's triangle row 4 (8-4), column 3 (4-1)**

When k is bigger than 4, the column number exceeds the row number in Pascal's Triangle and all those entries are 0.

1+6+10+4 is F(8)=21

	col:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**...**
	0	1	0	0	0	0	0	0	**0**	0	...
	1	1	1	0	0	0	0	**0**	0	0	...
	2	1	2	1	0	0	**0**	0	0	0	...
	3	1	3	3	1	**0**	0	0	0	0	...
r	**4**	1	4	6	**4**	1	0	0	0	0	...
o	**5**	1	5	**10**	10	5	1	0	0	0	...
w	**6**	1	**6**	15	20	15	6	1	0	0	...
	7	**1**	7	21	35	35	21	7	1	0	...
	8	1	8	28	56	70	56	28	8	1	...
	...	...									

The general pattern for month n and level (generation) k is
Level k: is Pascal's triangle row n-k and column k-1 For month n we sum all the generations as k goes from 1 to n (but half of these will be zeros).

9.4 A Galton's Quincunx Simulator

Quincunx S I M U L A T O R

Drop 100 balls
falling into 10 boxes GO
Show pins?
Show ideal counts?

R E S U L T S CLEAR

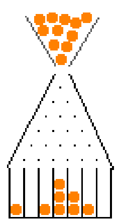
: Digit Sums | Factors | Remainders | Quincunx | Fractions | Number trick | Pythag Tris

There is a very nice Maths is Fun simulation of a quincunx showing each ball bouncing as it falls.

9.4.1 You do the maths...

1. Make a diagram of your own family tree. How far back can you go? You will probably have to ask your relatives to fill in the parts of the tree that you don't know, so take your tree with you on family visits and keep extending it as you learn about your ancestors!

2. Start again and draw the Female Rabbit Family tree, extending it month by month. Don't distinguish between r and R on the tree, but draw the newly born rabbits *using a new colour for each month* or, instead of using lots of colours, you could just put a number by each rabbit showing in which month it was born.
3. If you tossed a coin 10 times, how many possible sequences of Heads and Tails could there be in total (use Pascal's Triangle extending it to the row numbered 10)?
   In how many of these are there 5 heads (and so 5 tails)? What is the probability of tossing 10 coins and getting exactly 5 heads therefore - it is *not* 0·5! Draw up a table for each *even* number of coins from 2 to 10 and show the probability of getting exactly half heads and half tails for each case. What is happening to the probability as the number of coins gets larger?
4. Draw a *histogram* of the 10^{th} row of Pascal's triangle, that is, a bar chart, where each column on the row numbered 10 is shown as a bar whose height is the Pascal's triangle number. Try it again for tow 20 if you can (or use a Spreadsheet on your computer). The shape that you get as the row increases is called **a Bell curve** since it looks like a bell cut in half. It has many uses in *Statistics* and is a very important shape.
5. Make a **Galton Quincunx**.
   This is a device with lots of nails put in a regular hexagon arrangement. Its name derives from the Latin word *quincunx* for the X-like shape of the spots on the 5-face of a dice:



   The whole board is tilted forward slightly so that the top is raised off the table a little. When small balls are poured onto the network of nails at the top, they fall through, bouncing either to the right or to the left and so hit another nail on the row below. Eventually they fall off the bottom row of nails and are caught in containers.

   If you have a lot of nails and a lot of little balls (good sources for these are small steel ball-bearings from a bicycle shop or ping-pong balls for a large version or even dried peas or other cheap round seeds from the supermarket) then they end up forming a shape in the containers that is very much like the Bell curve of the previous exploration.
   You will need to space the nails so they are as far apart as about one and a half times the width of the balls you are using.
6. Let's see how the curve of the last two explorations, **the Bell curve might actually occur in some real data sets**.
   Measure the height of each person in your class and plot a graph similar to the containers above, labelled with heights to the nearest centimetre, each container containing one ball for each person with that height. What shape do you get? Try adding in the results from other classes to get one big graph.

This makes a good practical demonstration for a Science Fair or Parents' Exhibition or Open Day at your school or college. Measure the height of each person who passes your display and "add a ball" to the container which represents their height. What shape do you get at the end of the day?

7. **What else could you measure?**
    - The weight of each person to the nearest pound or nearest 500 grams;
    - their age last birthday;
      but remember some people do not like disclosing their age or knowing too accurately their own weight!
    - house or apartment number (what range of values should you allow for? In the USA this might be up to several thousands!)
    - the last 3 digits of their telephone number;

   or try these data sets using coins and dice:
    - the total number when you add the spots after throwing 5 dice at once;
    - the number of heads when you toss 20 coins at once.

   Do **all** of these give the Bell curve for large samples?
   If not, why do you think some do and some don't?
   Can you decide beforehand which will give the Bell curve and which won't? If a distribution is not a Bell curve, what shape do you think it will be? How can mathematics help?
8. Write out the first few powers of 11. Do they remind you of Pascal's triangle? Why? Why does the Pascal's triangle pattern break down after the first few powers?
   (Hint: consider $(a+b)^m$ where a=10 and b=1).
9. To finish, let's return to a **human family tree**. Suppose that the probability of each child being male is exactly 0.5 so that exactly half of all new babies will be male and half female.
    a. If a couple have 2 children, what are the four possible sequences of children they can have?
    b. What is it if they have 3 children?
    c. In what proportion of the couples that have 3 children will all 3 children be girls?
    d. Suppose a couple have 4 children, what is the probability now that all 4 will be girls?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

10 The Fibonacci Series as a Decimal Fraction

Have a look at this decimal fraction:

```
0·0112359550561...
```

It looks like it begins with the Fibonacci numbers, 0, 1, 1, 2, 3 and 5 and indeed it does if we express it as:

```
0·0                +
   1               +
    1              +
     2
      3
       5
        8
        13
         21
          34
           55
            89
            144
              ...
--------------------
0·011235955056179...
```

What is the value of this decimal fraction?

It can be expressed as

$$0/10 + 1/100 + 1/1000 + 2/10^4 + 3/10^5 + \dots$$

or, using powers of 10 and replacing the Fibonacci numbers by $F(i)$:

$$F(0)/10^1 + F(1)/10^2 + F(2)/10^3 + \dots + F(n\text{-}1)/10^n + \dots$$

or, if we use the negative powers of 10 to indicate the decimal fractions:

$$F(0)10^{-1} + F(1)10^{-2} + F(2)10^{-3} + \dots + F(n\text{-}1)10^{-n} + \dots$$

10.1 A Generating Function for the Fibonacci Numbers

To find the value of the decimal fraction we look at a generalisation, replacing 10 by x.

Let $P(x)$ be the polynomial in x whose coefficients are the Fibonacci numbers:

$$P(x)= 0 + 1\,x^2 + 1\,x^3 + 2\,x^4 + 3\,x^5 + 5\,x^6 + \dots$$

or

$$P(x)=F(0)\,x + F(1)\,x^2 + F(2)\,x^3 + \dots \dots +F(n-1)x^{\,n} + \dots$$

The decimal fraction 0.011235955... above is just

$$0\,(1/10) + 1\,(1/10)^2 + 1\,(1/10)^3 + 2\,(1/10)^4 + 3\,(1/10)^5 + \dots + F(n-1)\,(1/10)^n + \dots$$

which is just $P(x)$ with x taking the value 1/10, which we write as $P(1/10)$.

Now here is the interesting part of the technique!
We now write down $x\,P(x)$ and $x^2\,P(x)$ because these will "move the Fibonacci coefficients along":

$$\begin{aligned}
P(x) &= F(0)x + F(1)x^2 + F(2)x^3 + F(3)x^4 + \dots +F(n-1)x^n + \dots\\
x\,P(x) &= F(0)x^2 + F(1)x^3 + F(2)x^4 + \dots +F(n-2)x^n + \dots\\
x^2\,P(x) &= F(0)x^3 + F(1)x^4 + \dots +F(n-3)x^n + \dots
\end{aligned}$$

We can align these terms up so that all the same powers of x are in the same column (as we would do when doing ordinary decimal arithmetic on numbers) as follows:

$$\begin{array}{rcllllll}
P(x) &=& F(0)x + & F(1)x^2 + & F(2)x^3 + & F(3)x^4 + \dots & +F(n-1)x^n + \dots\\
x\,P(x) &=& & F(0)x^2 + & F(1)x^3 + & F(2)x^4 + \dots & +F(n-2)x^n + \dots\\
x^2\,P(x) &=& & & F(0)x^3 + & F(1)x^4 + \dots & +F(n-3)x^n + \dots
\end{array}$$

We have done this so that each Fibonacci number in $P(x)$ is aligned with the two previous Fibonacci numbers. Since the sum of the two previous numbers always equals the next in the Fibonacci series, then, when we take them away, the result will be zero - the terms will vanish!
So, if we take away the last two expressions (for $x\,P(x)$ and $x^2\,P(x)$) from the first equation for $P(x)$, the right-hand side will simplify since all but the first few terms vanish, as shown here:

$$\begin{array}{rllll}
P(x)= F(0)x + & F(1)x^2 + & F(2)x^3 + & F(3)x^4 + \dots & +F(n-1)x^n + \dots\\
x\,P(x) \;= & F(0)x^2 + & F(1)x^3 + & F(2)x^4 + \dots & +F(n-2)x^n + \dots\\
x^2\,P(x) \;= & & F(0)x^3 + & F(1)x^4 + \dots & +F(n-3)x^n + \dots\\
\hline
\end{array}$$

$$(1 - x - x^2)\ P(x) \ = \ F(0)x + (F(1)–F(0))x^2 + (F(2)–F(1)–F(0))x^3+...$$

Apart from the first two terms, the general term, which is just the coefficient of x^n, becomes $F(n)–F(n–1)–F(n–2)$ and, since $F(n) = F(n–1) + F(n–2)$ all but the first two terms become zero which is why we wrote down $x\ P(x)$ and $x^2\ P(x)$:

$$(1 - x - x^2)\ P(x) \ = \ x^2$$

$$P(x) \ = \ \frac{x^2}{1 - x - x^2} = \frac{1}{x^{-2} - x^{-1} - 1}$$

The polynomial $P(x)$ which has the Fibonacci numbers as the coefficients of its powers of x is called a **generating function for the Fibonacci numbers**.

Finding such a polynomial for other series of numbers is an important part of modern mathematics and has many applications. Abraham De Moivre (1667-1754) is the person who first wrote about this technique in his book on Probability *The Doctrine of Chances: Or, A Method of Calculating the Probability of Events in Play* in 1718. He used it exactly as we have done here for the Fibonacci numbers!

10.2 Fibonacci decimals

So now our fraction is just P(1/10), and the right hand side tells us its exact value:
1 / (100-10-1) = **1/89** = 0·0112358...
From our expression for P(x) we can also deduce the following:

```
10/89 = 0·112359550561...
```

If x=1/100, we have

```
P(1/100) = 0·00 01 01 02 03 05 08 13 21 34 55 ... = 1/(10000-100-1) = 1/9899
and
100/9899 = 0·01010203050813213455...
```

and so on.

10.3 An exact Fractions Calculator

The decimal expansions of fractions here in the Calculator are produced **to any number of decimal places**, unlike an ordinary calculator which only gives perhaps a maximum of 15 decimal places.

C A L C U L A T O R :

Fraction to Exact Decimal Converter

Select an example to try: Select:

Does it recur or terminate?

1

9801

Show the fixed and recurring parts up to 30 decimal places

Show all decimal places

R E S U L T S Clear the Results

:	Digit Sums	Factors	Remainders	Quincunx	Fractions	Number trick	Pythag Tris

A more extensive Fractions-Decimal Calculator, computing *all* digits even in long-period recurring decimals, is found here.

10.3.1 You do the maths...

1. Can you find exact fractions for the following where all continue with the Fibonacci series terms?
   a. 10102.0305081321... Show the answer
   b. 0.0010010020030050080013... Show the answer
   c. 1.001002003005008013... Show the answer
   d. 0.001002003005008013... Show the answer

e. 0.0001000100020003000500080013... Show the answer

2. Expand these fractions and say how they are related to the Fibonacci numbers:
   a. $\frac{10}{89}$ Show the answer
   b. $\frac{2}{995999}$ Show the answer
   c. $\frac{999}{995999}$ Show the answer
   d. $\frac{1001}{995999}$ Show the answer

References

The Decimal Expansion of 1/89 and related Results, *Fibonacci Quarterly*, Vol 19, (1981), pages 53-55.
Calvin Long solves the general problem for *all* Fibonacci-type sequences i.e. G(0)=c, G(1)=d are the two starting terms and G(i) = a G(i-1) + b G(i-1) defines all other values for integers a and b. For our "ordinary" Fibonacci sequence, a=b=1 and c=d=1. He gives the exact fractions for any base B (here B=10 for decimal fractions) and gives conditions when the fraction exists (i.e. when the series does not get too large too quickly so that we do have a genuine fraction).

A Complete Characterisation of the Decimal Fractions that can be Represented as SUM($10^{-k(i+1)}$F(ai)) where F(ai) is the aith Fibonacci number Richard H Hudson and C F Winans, *Fibonacci Quarterly*, 1981, Vol 19, pp 414 - 421.
This article examines all the decimal fractions where the terms are F(a), F(2a), F(3a) taken k digits at a time in the decimal fraction.

A Primer For the Fibonacci Numbers: Part VI, V E Hoggatt Jr, D A Lind in *Fibonacci Quarterly*, vol 5 (1967) pages 445 - 460
is a nice introduction to Generating Functions (a polynomial in x where the coefficients of the powers of x are the members of a particular series). The whole collection of articles is now available in book form by mail order from The Fibonacci Association. It is readable and not too technical. There is also a list of formulae for all kinds of generating functions, which, if we substitute a power of 10 for x, will give a large collection of fractions whose decimal expansion is , for example:

- the Lucas Numbers (see this page at this site) e.g. 1999/998999
- the squares of the Fibonacci numbers e.g. 999000/997998001
- the product of two neighbouring Fibonacci numbers e.g. 1000/997998001
- the cubes of the Fibonacci numbers e.g. 997999000/996994003001
- the product of three neighbouring Fibonacci numbers e.g. 2000000000/996994003001

- every k^{th} Fibonacci number e.g. 1000/997001 or 999000/997001
- etc.

Scott's Fibonacci Scrapbook, Allan Scott in *Fibonacci Quarterly* vol 6 number 2, (April 1968), page 176
is a follow-up article to the one above, extending the generating functions to Lucas cubes and Fibonacci fourth and fifth powers.
Note there are ***several corrections to these equations*** on page 70 of vol 6 number 3 (June 1968).

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

11 A Fibonacci Number Trick

Here is a little trick you can perform on friends which seems to show that you have amazing mathematical powers. We explain how it works after showing you the trick.

11.1 A Lightning Calculation

Here is Alice performing the trick on Bill:

Alice: Choose any two numbers you like, Bill, but not too big as you're going to have to do some adding yourself. Write them as if you are going to add them up and I'll, of course, be looking the other way!
Bill: OK, I've done that.

Bill chooses 16 and 21 and writes them one under the other: 16
21

Alice: Now add the first to the second and write the sum underneath to make the third entry in the column.
Bill: I don't think I'll need my calculator just yet.... Ok, I've done that.

Bill writes down 37 (=16+21) under the other two: 16
21
37

Alice: Right, now add up the second and your new number and again write their sum underneath. Keep on doing this, adding the number you have just written to the number before it and putting the new sum underneath. Stop when you have 10 numbers written down and draw a line under the tenth.

There is a sound of lots of buttons being tapped on Bill's calculator!

Bill: OK, the ten numbers are ready.

Bills column now looks like this: 16
21
37
58

```
  95
 153
 248
 401
 649
1050
```

Alice: Now I'll turn round and look at your numbers and write the sum of all ten numbers straight away!

She turns round and almost immediately writes underneath: 2728.

Bill taps away again on his calculator and is amazed that Alice got it right in so short a time [gasp!]

11.2 So how did Alice do it?

The sum of all ten numbers is just **eleven times the fourth number from the bottom**. Also, Alice knows the quick method of multiplying a number by eleven. The fourth number from the bottom is 248, and there is the quick and easy method of multiplying numbers by 11 that you can easily do in your head:

Starting at the right, just copy the last digit of the number as the last digit of your product. Here the last digit of 248 is 8 so the product also ends with 8 which Alice writes down:	... **248** 401 649 1050 8
Now, continuing in 248, keep adding up from the right each number and its neighbour, in pairs, writing down their sum as you go. If ever you get a sum bigger than 10, then write down the units digit of the sum and remember to carry anything over into your next pair to add. Here the pairs of 248 are (from the right) 4+8 and then 2+4. So, next to the 8 Alice thinks "4+8=12" so she writes 2 and *remembers there is an extra one to add on to the next pair*:	... **248** 401 649 1050 28
Then 2+4 is 6, **adding the one carried** makes 7, so she writes 7 on the left of those digits already written down:	... **248** 401 649 1050 728
Finally copy down the left hand digit (plus any carry). Alice sees that the left digit is 2 which, because there is nothing being carried from the previous pair, becomes the left-hand digit of the sum. The final sum is therefore 2728 = 11 x 248 .	... **248** 401 649 1050 2728

11.3 Why does it work?

You can see how it works using algebra and by starting with A and B as the two numbers that Bill chooses.
What does he write next? Just A+B in algebraic form.
The next sum is B added to A+B which is A+2B.
The other numbers in the column are 2A+3B, 3A+5B, ... up to 21A+34B.

```
  A
       B
  A +  B
  A + 2B
 2A + 3B
 3A + 5B
 5A + 8B
 8A +13B
13A +21B
21A +34B
--------
55A +88B
```

If you add these up you find the total sum of all ten is 55A+88B.
Now look at the fourth number up from the bottom. What is it?
How is it related to the final sum of 55A+88B?

So the trick works by a special property of adding up exactly ten numbers from a Fibonacci-like sequence and will work for any two starting values A and B!

Perhaps you noticed that the multiples of A and B were the Fibonacci numbers? This is part of a more general pattern which is the first investigation of several to spot new patterns in the Fibonacci sequence in the next section.

References

On a Fibonacci Arithmetical Trick C T Long, *Fibonacci Quarterly* vol 23 (1985), pages 221-231. This article introduces the above trick and generalises it.

11.4 Practice here with "Bill"

Here is your very own "Bill" to practice on.

- Click on the "Show Bill's list" button and he will think of two numbers and show you his list.
- Enter your answer in the **Sum:** box
- Click on **Am I right?** to see if you got it right!

Click on **Show Bill's list** as often as you like to get a new list.

Show Bill's list

Sum:

Am I right?

References

A Fibonacci Generalisation Brother Alfred Brousseau, *Fibonacci Quarterly* vol 5 (1967), pages 171-174. This article introduces the above trick and generalises it to sums of more numbers.

On a Fibonacci Arithmetical Trick C T Long, *Fibonacci Quarterly* vol 23 (1985), pages 221-231. Another generalisation.

12 Another Number Pattern

Dave Wood has found another number pattern that we can prove using the same method.

He notices that

```
f(10)-f(5)  is   55 -   5 which is  50  or   5 tens and  0;

f(11)-f(6)  is   89 -   8 which is  81  or   8 tens and  1;

f(12)-f(7)  is  144 -  13 which is  131 or  13 tens and  1.
```

It looks like the differences seem to be 'copying' the Fibonacci series in the tens and in the units columns.
If we continue the investigation we have:

```
f(13)-f(8)  is  233 -  21 which is  212 or  21 tens and  2;

f(14)-f(9)  is  377 -  34 which is  343 or  34 tens and  3;

f(15)-f(10) is  610 -  55 which is  555 or  55 tens and  5;

f(16)-f(11) is  987 -  89 which is  898 or  89 tens and  8;

f(17)-f(12) is 1597 - 144 which is 1453 or 144 tens and 13;
```

From this point on, we have to borrow a ten in order to make the 'units' have the 2 digits needed for the next Fibonacci number. Later we shall have to 'borrow' more, but the pattern still seems to hold.

In words we have:

For any Fibonacci number:
if we take away the Fibonacci number 5 before it
the result is ten times the number we took away
PLUS the Fibonacci number ten before it

In mathematical terms, we can write this as:

$$\text{Fib}(n) - \text{Fib}(n-5) = 10\ \text{Fib}(n-5) + \text{Fib}(n-10)$$

A Proof

That the pattern always holds is found by extending the table we used in the **Why does it work** section of the Number Trick above:

```
 A
      B
 A +  B
 A + 2B
2A + 3B
3A + 5B
5A + 8B
```

```
 8A +13B
13A +21B
21A +34B
34A +55B
```

We can always write *any* Fibonacci number Fib(n) as 34A+55B because, since the Fibonacci series extends backwards infinitely far, we just pick A and B as the two numbers that are 10 and 9 places before the one we want.

Now let's look at that last line: 34A +55B.
It is almost 11 times the number 5 rows before it:

```
11 x (3A+5B) = 33A+55B,
```

and it is equal to it if we add on an extra A, i.e. the number ten rows before the last one:

```
34A + 55B = 11 (3A+5B) + A
```

Putting this in terms of the Fibonacci numbers, where the 34A+55B is F(n) and 3A+5B is "the Fibonacci number 5 before it", or Fib(n-5) and A is "the Fibonacci number 10 before it" or Fib(n-10), we have:

```
34A + 55B = 11 (3A+5B) +  A
```

or

```
Fib(n)    = 11 Fib(n-5) + Fib(n-10)
```

We rearrange this now by taking Fib(n-5) from both sides and we have:

```
Fib(n) - Fib(n-5) = 10 Fib(n-5) + Fib(n-10)
```

which is just what Dave Wood observed!

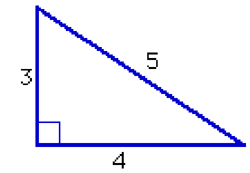
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

13 Fibonacci Numbers and Pythagorean Triangles

A **Pythagorean Triangle** is a right-angled triangle with sides which are whole numbers.

In any right-angled triangle with sides s and t and longest side (hypotenuse) h, **the Pythagoras Theorem** applies:

$$s^2 + t^2 = h^2$$



However, for a Pythagorean triangle, we also want the sides to be integers (whole numbers) too. A common example is a triangle with sides s=3, t=4 and h=5:
We can check Pythagoras theorem: $s^2 + t^2 = 3^2 + 4^2 = 9 + 16 = 25 = 5^2 = h^2$

Here is a list of some of the smaller Pythagorean Triangles:

s	t	h	*=primitive
3	4	5	*
6	8	10	2(3,4,5)
5	12	13	*
9	12	15	3(3,4,5)
8	15	17	*
12	16	20	4(3,4,5)
7	24	25	*
15	20	25	5(3,4,5)
10	24	26	2(5,12,13)
20	21	29	*
16	30	34	2(8,15,17)
18	24	36	6(3,4,5)

References

Here is [another longer list of Triples] generated using Autograph from Oundle School, Peterborough, UK.

You will see that some are just magnifications of smaller ones where all the sides have been doubled, or trebled for example. The others are "new" and are usually called **primitive Pythagorean triangles**.
Any Pythagorean triangle is either primitive or a multiple of a primitive and this is shown in the table above. Primitive Pythagorean triangles are a bit like *prime numbers* in that every Pythagorean triangle is either primitive (no number is `the 3 sides do not have a common factor) or is a multiple of a primitive Pythagorean triangle.

13.1 Using the Fibonacci Numbers to make Pythagorean Triangles

There is an easy way to generate Pythagorean triangles using 4 Fibonacci numbers. Take, for example, the 4 Fibonacci numbers:

1, 2, 3, 5

Let's call the first two a and b. Since they are from the Fibonacci series, the next is the sum of the previous two: a+b and the following one is b+(a+b) or a+2b:-

a	b	a+b	a+2b
1	2	3	5

You can now make a Pythagorean triangle as follows:

1. Multiply the two middle or inner numbers (here 2 and 3 giving 6);
2. Double the result (here twice 6 gives **12**). This is one side, s, of the Pythagorean Triangle.
3. Multiply together the two outer numbers (here 1 and 5 giving **5**). This is the second side, t, of the Pythagorean triangle.
4. The third side, the longest, is found by adding together the *squares* of the inner two numbers (here 2^2=4 and 3^2=9 and their sum is 4+9=**13**). This is the third side, h, of the Pythagorean triangle.

We have generated the 12, 5,13 Pythagorean triangle, or, putting the sides in order, the **5, 12, 13** triangle this time.

Try it with 2, 3, 5 and 8 and check that you get the Pythagorean triangle: 30, 16, 34.
Is this one primitive?

In fact, this process works for **any two numbers a and b**, not just Fibonacci numbers. The third and fourth numbers are found using **the Fibonacci rule**: add the latest two values to get the next.
Four such numbers are part of a *generalised Fibonacci series* which we could continue for as long as we liked, just as we did for the (real) Fibonacci series.

All primitive Pythagorean triangles can be generated in this way by choosing suitable starting numbers a and b but not all non-primitive ones are!
For the reason for this and lots more on Pythagorean triangles see my Pythagorean Triangles page.

Try the Fibonacci method for yourself and check with this Calculator:

13.2 Pythagorean Triples from Fibonacci-type Series Calculator

C A L C U L A T O R

Fibonacci starting at ______ , ______ Find the Pythagorean Trangle

Pythagorean triangle (, ,) Find the Fibonacci starting numbers

R E S U L T S CLEAR

+−×⁄: Digit Sums | Factors | Remainders | Quincunx | Fractions | Number trick | Pythag Tris

References

Connections in Mathematics: An Introduction to Fibonacci via Pythagoras E A Marchisotto in *Fibonacci Quarterly*, vol 31, 1993, pages 21 - 27.
This article explores many ways of introducing the Fibonacci numbers in class starting from the Pythagorean triples, with an extensive Appendix of references useful for the teacher and comparing different approaches. Highly recommended!

Pythagorean Triangles from the Fibonacci Series C W Raine in *Scripta Mathematica* vol 14 (1948) page 164.

13.3 Fibonacci Numbers as the sides of Pythagorean Triangles

Can we form a triangle (not necessarily right-angled) from three distinct Fibonacci numbers?

No, because of the following condition that must be true for any and all triangles:

in any triangle the longest side must be shorter than the sum of the other two sides

This is called the *Triangle Inequality*.
Since three *consecutive Fibonacci numbers* already have the third number equal to the sum of the other two, then the Triangle Inequality fails. Or, if you prefer, the two shorter sides collapse onto the third to form a straight line when you try to construct a triangle from these numbers.

If the smallest side is smaller, that makes it worse, as it does if the longer side gets longer!
So we have

No triangle has sides which are three distinct Fibonacci numbers

So can we have a triangle with three Fibonacci numbers as sides, but with two sides equal?

Yes: 3,3,5 will do. The longest side, 5, is now less than the sum of the other two, 3+3. But this triangle is not right-angled: $3^2 + 3^2$ is 18 whereas 5^2 is 25.

No Pythagorean triangle has two equal sides. If we ask for just two sides which are Fibonacci numbers, the third being any whole number, then there are at least *two* Pythagorean triangles with Fibonacci numbers on *two* sides:

3, 4, 5 and 5, 12, 13

It is still an unsolved problem as to whether there are any more right-angled (Pythagorean) triangles with just two Fibonacci numbers as sides.

*Can we have any other Pythagorean triangles with a Fibonacci number as the **hypotenuse** (the longest side)? Yes!*

We can make every odd-indexed Fibonacci number the hypotenuse of a Pythagorean triangle using the technique of the section above.
We take 4 consecutive Fibonacci numbers:

$$F(n-1)\ F(n)\ F(n+1)\ F(n+2)$$

and get the two sides of a Pythagorean Triangle:

$$2\,F(n)F(n+1) \text{ and } F(n-1)F(n+2)$$

The hypotenuse is the sum of the squares of the middle two numbers: $F(n)^2 + F(n+1)^2$ and Lucas showed in 1876 that this is just $F(2n+1)$.
So we have the following Pythagorean Triangle with an odd-indexed Fibonacci number as hypotenuse:

$$2F(n)F(n+1)\ ;\ F(n-1)F(n+2)\ ;\ F(2n+1)$$

Here are some examples:

n	2	3	4	5	...
2F(n)F(n + 1)	4	12	30	80	...
F(n – 1)F(n + 2)	3	5	16	39	...
F(2n + 1)	5	13	34	89	...

Is this the only series of Pythagorean Triangles with hypotenuses that are Fibonacci numbers?

Pythagorean Triples A F Horadam *Fibonacci Quarterly* vol 20 (1982) pages 121-122.

13.3.1 Square Fibonacci Numbers

Let's have a further look at that formula of Lucas from 1876:

$$F(n)^2 + F(n + 1)^2 = F(2n + 1)$$

We can make this into a Pythagorean triangle whenever F(2n+1) is a square number. So we now have the question

Which Fibonacci numbers are square numbers?

We only have to look at the first few Fibonacci numbers to spot these square numbers:

$$F(0)=0=0^2;\ F(1)=F(2)=1=1^2;\ F(12)=144=12^2$$

But a longer look does not reveal any more squares among the Fibonacci's. Are these the only Fibonacci squares? Yes, as was proved by Cohn in the article below. Therefore...

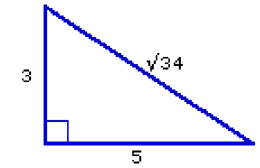
Two consecutive Fibonacci numbers **cannot** be the sides of a Pythagorean Triangle.

References

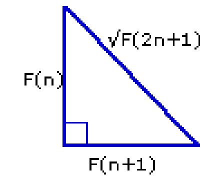
Square Fibonacci Numbers Etc. J H E Cohn in *Fibonacci Quarterly* vol 2 (1964) pages 109-113

13.4 Other right-angled triangles and the Fibonacci Numbers

Even if we don't insist that all three sides of a right-angled triangle are integers, Fibonacci numbers still have some interesting applications.
For instance, if we choose two *consecutive* Fibonacci numbers as the sides next to the right angle, then the third side *squared* is also a Fibonacci number.
For instance, if the sides are 3 and 5, by Pythagoras' Theorem we have that the hypotenuse, h, is given by:
$3^2 + 5^2 = h^2$
and 9 + 25 = 34, another Fibonacci number. So h is √34. If we look at the indices of the Fibonacci numbers, we can directly predict which Fibonacci number will be the square of the hypotenuse.



For our example here, F(*4*)=3 and F(*5*)=5 and the hypotenuse-squared is 34=F(*9*). *9*, the index of the h^2 Fibonacci number is the sum of the other two indices, *4* and *5*. This is also true in general, provided the two Fibonacci sides are consecutive Fibonacci numbers, say F(n) and F(n+1):



$$F(n)^2 + F(n + 1)^2 = F(2\ n + 1) \text{ .... Lucas (1876)}$$

This rule was known (and proved) by E Lucas in 1876.

I am grateful to Richard Van De Plasch for pointing out this application of Lucas's formula to right-angled triangles.

13.4.1 You do the maths...

1. There are just 3 other right-angled triangles with Fibonacci sides that are *not* consecutive Fibonacci numbers and also with a *hypotenuse whose square is a Fibonacci number*. What are they? (Hint: the sides and the hypotenuse-squared are all single digit numbers!)
2. With sides 1 and 3, a right-angled triangle has hypotenuse √10 and, although 10 is not a Fibonacci number it is *twice* a Fibonacci number.
   This is also true of 1, 5, √26 because 26 is twice 13 and F(7) = 13
   and 2, 8, √68 and 68 is twice 34 which is F(9).
   Are there any other such triangles? If so, is there a general rule?
3. Try replacing the factor 2 in the previous investigation by *another number*. What new rules can you find?

The investigations above will lead you to Catalan's Identity of 1879 which is on this site's Formula page. We can rearrange it as follows:

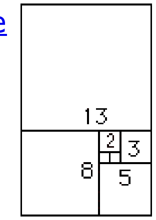
$$F(n)^2 - (-1)^{n-r}F(r)^2 = F(n + r)\ F(n - r) \text{ .... Catalan (1879)}$$

For the investigations above we use a special case where r is an odd number (2k + 1) more than n; i.e. (n − r) is (2k + 1) for any integer k.
This means that:

- the $-(-1)^{n-r}$ on the left-hand side becomes just a + sign since n − r is odd;
- the factor F(n − r) on the right is just F(2k + 1) which is why the factors above: 2, 5, 13, 34,... are alternate Fibonacci numbers - the ones with an odd index.
- the Fibonacci number F(n + r) in the hypotenuse has an index (n + r) which is the sum of the indices of the Fibonacci numbers on the other two sides of the triangle (n and r).

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

14 Maths from the Fibonacci Spiral diagram



Let's look again at the Fibonacci squares and spiral that we saw in the Fibonacci Spiral section of the Fibonacci in Nature page.
Wherever we stop, we will always get a rectangle, since the next square to add is determined by the longest edge on the current rectangle. Also, those longest edges are just the sum of the latest two sides-of-squares to be added. Since we start with squares of sides 1 and 1, this tells us why the squares sides are the Fibonacci numbers (the next is the sum of the previous 2).

Also, we see that each rectangle is a jigsaw puzzle made up of all the earlier squares to form a rectangle. All the squares and all the rectangles have sides which are Fibonacci numbers in length. What is the mathematical relationship that is shown by this pattern of squares and rectangles? We express each rectangle's area as a sum of its component square areas:
The diagram shows that

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 = 13\times21$$

and also, the smaller rectangles show:

$$\begin{aligned}
1^2 + 1^2 &= 1\times2 \\
1^2 + 1^2 + 2^2 &= 2\times3 \\
1^2 + 1^2 + 2^2 + 3^2 &= 3\times5 \\
1^2 + 1^2 + 2^2 + 3^2 + 5^2 &= 5\times8 \\
1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 &= 8\times13
\end{aligned}$$

This picture actually is a convincing proof that the pattern will work for any number of squares of Fibonacci numbers that we wish to sum. They always total to the largest Fibonacci number used in the squares multiplied by the next Fibonacci number.
That is a bit of a mouthful to say - and to understand - so it is better to express the relationship in the language of mathematics:

$$1^2 + 1^2 + 2^2 + 3^2 + \ldots + F(n)^2 = F(n)F(n+1)$$

and it is true for ANY n from 1 upwards.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

15 ..and now it's your turn!

Here are some series that use the Fibonacci numbers. Compute a few terms and see if you can spot the pattern, i.e. guess the formula for the general term and write it down mathematically:

15.1 You do the maths...

1. F(1), F(1)+F(2), F(1)+F(2)+F(3), ... = 1, 2, 4, 7, 12, 20, ...

   Keun Young Lee, a student at the Glenbrook North High School in Chicago, told me of a generalisation of this.
   Can you spot it too?

   What is F(k)+F(k+1)+...+F(n)?

   e.g. 5+8+13 (k=5 and n=7) is 26
   3+5+8+13+21 (k=4 and n=8) is 50.
   This problem will be the same as the first problem here if you let k=1 and this is a useful check on your formula.

2. F(1), F(1)+F(3), F(1)+F(3)+F(5), ... = 1, 3, 8, 21, ...
3. F(2), F(2)+F(4), F(2)+F(4)+F(6), ... = 1, 4, 12, 33, ...
4. F(1)+F(4), F(2)+F(5), F(3)+F(6), ... = 4, 6, 10, 16, ...
5. F(1)+F(5), F(2)+F(6), F(3)+F(7), ... = 6, 9, 15, 24, ...
6. $F(1)^2+F(2)^2$, $F(2)^2+F(3)^2$, $F(3)^2+F(4)^2$, ...= 2, 5, 13, 34, ...
7. Can you find a connection between the terms of:
   1x3, 2x5, 3x8, 5x13, ... , F(n-1)xF(n+1), ...
   and the following series?
   2x2, 3x3, 5x5, 8x8, ... , F(n)xF(n), ...
   The connection was first noted by Cassini (1625-1712) in 1680 and is called **Cassini's Relation** (see Knuth, **The Art of Computer Programming**, Volume 1:*Fundamental Algorithms*, section 1.2.8).
8. Try choosing different small values for a and b and finding some more Pythagorean triangles.
   Tick those triangles that are primitive and out a cross by those which are multiples (of a primitive triangle).
   Can you find the simple condition on a and b that tells us when the generated Pythagorean triangle is primitive? [Hint: the condition has two parts: i) what happens if both a and b have a common factor? ii) why are no primitive triangles generated if a and b are both odd?].
9. Find all 16 primitive Pythagorean triangles with all 3 sides less than 100.
   Use your list to generate *all* Pythagorean triangles with sides smaller than 100. How many are there in all?

   [Optional extra part: Can you devise a method to find which a and b generated a given Pythagorean triangle?
   Eg Given Pythagorean triangle 9,40,41 (and we can check that $9^2 + 40^2 = 41^2$), how do we calculate that it was generated from the values

a=1, b=4?]

If you don't know how to begin, or get stuck,
look at the Hints and Tips page to get you going!

So try them for yourself. This is where Mathematics becomes more of an Art than a Science, since you are relying on your intuition to "spot" the pattern. No one is quite sure where this ability in humans comes from. It is not easy to get a computer to do this (although Maple is quite good at it) - and it may be something specifically human that a computing machine can never really copy, but no one is sure at present. Here are two references if you want to explore further the arguments and ideas of why an electronic computer may or may not be able to mimic a human brain:

References

Prof Roger Penrose's book **Shadows of the Mind** published in 1994 by Oxford Press makes interesting reading on this subject.

Dr. Math has some interesting replies to questions about the Fibonacci series and the Golden section together with a few more formulae for you to check out.

Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, S Vajda, Dover reprint (2007).
This is a wonderful book - now back in print after many years - which is full of formulae on the Fibonacci numbers and Phi. Do try and find it in your local college or university library. It is well worth dipping in to if you are studying maths at age 16 or beyond! Most of Vajda's formulae are available on my Fibonacci, Phi and Lucas Numbers Formulae page too, with some corrections of Vajda's (rare) errors.

Mathematical Mystery Tour by Mark Wahl, 1989, is full of many mathematical investigations, illustrations, diagrams, tricks, facts, notes as well as guides for teachers using the material. It is a great resource for your own investigations.

Fibonacci Home Page

← The Puzzling World of the Fibonacci Numbers

This is the first page on this Topic.

WHERE TO NOW?

↓ The first 500 Fibonacci Numbers
↓ A Formula for Fibonacci Numbers

The next Topic is...
→ The Golden Section - the Number and Its Geometry