Effect of Quantum Repetition Code on Fidelity of Bell States in Bit Flip Channels

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***Abstract*—Quantum Entanglement is a key to establish secure communication between two nodes in a communication channel. Ensuring high fidelity entanglement has always been a challenging task owing to interaction with the hostile channel environment. Several methods have been proposed in this paper to mitigate the problem. It has been shown that fidelity can be improved by encoding bell pairs using Quantum Repetition Codes (QRCs). In this regard, various QRC-based techniques are also analyzed mathematically, and their performances have been compared.**

***Index Terms*—Bell pairs, Entanglement, Fidelity, Repetition Code, Shor Code, Quantum Channel.**

I. INTRODUCTION

Quantum entanglement is a strange and unique property of quantum mechanics, initially pointed out to oppose quantum mechanics in a famous paper by Einstein et al. [1] but was later proved to be a nonlocal property of quantum mechanics by Bell [2]. Quantum entanglement is a crucial part of quantum communication. For example, quantum teleportation [3], superdense coding [4] etc. rely on establishing an entanglement pair between two nodes.

Ensuring high fidelity quantum entanglement between two nodes separated by longer distance has always been very challenging. Entanglement distribution and swapping protocols along with entanglement purification have been proposed [5] to solve the fidelity problem in the 1st generation quantum repeaters [6]. However, in the above scheme, the communication rate decreases polynomially with the distance, thus becoming very slow for long distance classical communication. It also takes many rounds of purification to obtain high fidelity bell pairs in entanglement purification [5] [6]. On the other hand, use of quantum error correcting codes (QECC) in entangled states (bell states) have been proven to increase the fidelity much faster [7] [8].

In this paper, a novel technique has been developed to implement and analyze the effect of quantum repetition codes (QRCs) on the bell states using both the theoretical calculations and computer simulation using the quantum simulation tool, the **IBM QISKIT**, which is a "QASM Simulator'. The objective of the work is to investigate, whether the use quantum error correcting codes (QECCs) instead of entanglement purification, can achieve faster and more secure quantum communication.

The repetition code, which is an error correction code are used in the classical communication channel. Later, it was used for 3-qubit codes in quantum communication by A. Peres [9]. However, here the author did not apply the QRCs in bell states. In our work, we have made use of the repetition codes in bell states for the first time. We have modeled the bell states (2-qubit maximally entangled state) for 3-qubit and 5-qubit repetition codes. We have also designed the decoder circuit for 5-qubit repetition code, which works for bit flip or phase flip channels.

Section II of this article first describes the basic aspects of quantum communication. Then the formulation of the proposed model and simulation results are described in Section III. Finally, the conclusions are made in Section IV, followed by the Acknowledgement in Section V, and the list of references are included at the end of this article.

II. BASICS OF QUANTUM COMMUNICATIONS

Qubit is the fundamental block for quantum computing. A qubit is a superposition of up and down state, written as $|0\rangle$ and $|1\rangle$ respectively. In general, a single qubit state can be written as,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \ldots (1)$$

where, $(\theta \in [0,\pi]\,, \varphi \in [0,2\pi])$, $\{|0\rangle,|1\rangle\}$ are known as computational basis or Z-basis, and $\{|+\rangle,|-\rangle\}$ are known as diagonal basis or X-basis. Two qubit states are elements of the tensor product space of their corresponding Hilbert space. [10].

In this section, we shall briefly discuss quantum logic gates first, and then we shall have a quick overview of mixed state, density matrix, and fidelity. Furthermore, we shall briefly discuss various quantum channels to describe noise models. Finally, we shall describe the Knill-Laflamme Error correcting condition.

A. QUANTUM LOGIC GATES

Quantum logic gates are basically unitary operators (if U is a quantum logic gate, then $UU^{\dagger} = I$), hence these gates are reversible. According to Landauer's principle, there is no minimum energy dissipation for logic operations in reversible computing [11].

For single qubit logic gates, the gates basically indicate a rotation in the Bloch sphere. We shall briefly discuss some basic quantum logic gates here:

Bit Flip Gate (Pauli X Gate): This is basically NOT gate, it creates a 180° rotation about X-axis, as $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$.

The matrix representation of the X gate is given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ldots (2)$$

Phase Flip Gate (Pauli Z Gate): This is basically a bit flip gate in X-basis as $Z|+\rangle = |-\rangle, Z|-\rangle = |+\rangle$, and in Z-basis, $Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$.
The matrix representation of Z gate is given by

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \dots (3)$$

Hadamard Gate (H Gate): It creates equal superposition, as $H|0\rangle = |+\rangle, H|1\rangle = |-\rangle$.

The matrix representation of a Hadamard gate is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \dots (4)$$

Note that, $X^2 = Z^2 = H^2 = I$, which can be thought of as quantum Interference.

Controlled NOT Gate (CNOT Gate): If $|x\rangle$ is the control qubit and $|y\rangle$ is the target qubit, then we define,

$$CNOT|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle \dots (5)$$

Toffoli Gate: Here two qubits are in control (say $|p\rangle, |q\rangle$), and one qubit is target (say $|r\rangle$). Then, $|r\rangle$ is flipped if both p, q equal to 1, else $|r\rangle$ is unchanged.

C. Density Matrix and Fidelity

Let us assume that we wish to get an entangled state of the $|\phi^+\rangle$ state. However, instead of having a pure state $|\phi^+\rangle$, we get a mixed state, an example of which is given in Eqn. (6) below.

$$\rho = p_1|\phi^+\rangle\langle\phi^+| + p_2|\phi^-\rangle\langle\phi^-| + p_3|\psi^+\rangle\langle\psi^+| + p_4|\psi^-\rangle\langle\psi^-| \dots (6)$$

Here, ρ is the density matrix, meaning we will have $|\phi^+\rangle$ with p_1 probability, $|\psi^-\rangle$ with p_4 probability and so on. Density matrices have non-negative eigenvalues, and the normalization condition is given by $tr[\rho] = 1$. We can define purity of density matrix as $tr[\rho^2]$, which becomes 1 for pure state [10]. We can define fidelity as, $F = \sqrt{\langle\phi^+|\rho|\phi^+\rangle} = \sqrt{p_1}$, which will be considered as a parameter to measure how close our result is to the theoretical one.

D. Quantum Channel

Quantum Channels are the most general quantum evolutions. Completely closed quantum system is not possible, so we describe the interaction with surrounding using Completely Positive Trace Preserving (CPTP) maps, which are considered as the most general quantum evolution. The CPTP maps are also known as the Krauss Operators. Noise Channels can also be described in a similar way. Here, we assume that noise acts locally and the channels are memoryless [12].

The Krauss Operators obey the normalization condition $\sum_i E_i E_i^\dagger = \mathbb{I}$, and the evolution of density matrix is given [10] [12] by

$$\rho' = \sum_i E_i \rho E_i^\dagger \dots (7)$$

The measurement channel, noise channel etc. can be described using the Krauss operators. For example, for measurement in Z basis, the Krauss operators are $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

E. Knill Laflamme Condition

In quantum communication, owing to the interaction of the qubits with the environment, the desired state gets evolved into an undesired state. By encoding first, one can correct these errors by meeting certain conditions.

Let us assume that ρ_c is a density matrix in coding subspace (say $\mathbb{H}_C$) and our error model is unitary (i.e., applies U_i error with p_i probability). After error, the evolved density matrix becomes $\rho' = \sum_i p_i U_i \rho U_i^\dagger$ which belongs to another subspace (say $\mathbb{H}_E$). Now, we can correct the error by applying the inverse unitary, if $\mathbb{H}_C$ and $\mathbb{H}_E$ are orthogonal [10].

In general noise models are not unitary, but in general, can be described by Krauss Operators E_i's. In such case, we can correct the error if the following condition is maintained [13]:

$$\rho_c E_k^\dagger E_r \rho_c = c_{kr}\rho_c \ \forall\ k, r, \rho_c \dots (8)$$

Here, c_{kr} is an element of any unitary matrix. This condition is known as Knill-Laflamme condition.

III. Simulation Results and Discussion

In the proposed work, at first, the quantum channel was theoretically modelled using the Krauss Operator. In the second stage, the quantum circuits for three and five qubit repetition codes were developed to carry out both the theoretical analysis and simulation. In the third stage, these models were simulated using the IBM QISKIT, to evaluate the performance of the proposed schemes. The Monte Carlo method has been implemented in Python using the above simulator to observe the effect of noise, with and without the QRCs.

In this work, the performance of the proposed scheme has been examined in a bit flip channel using 3-qubit and 5-qubit repetition codes. Each code has the following structure [12]-[14]:

(a) Encoding before sharing through the channel.
(b) Sharing through channel (Bit flip and phase flip channel models are assumed here).
(c) Syndrome detection and decoding.

In case of repetition, we apply the majority rules. For example, if we encode $|0\rangle$ as $|000\rangle$, then $|001\rangle, |010\rangle, |100\rangle$ will be decoded as $|0\rangle$ correctly. It should be noted that, a (2k+1,1) repetition code can correct k errors.

A. Simulation in a Bit Flip Channel

The Bell states are encoded first using various techniques (especially, repetition codes) and the $|\phi^+\rangle$ state may be encoded using Eq. (9) as follows:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle_L + |1,1\rangle_L) \dots (9)$$

We shall notice in the subsequent analyses that, by encoding $|\phi^+\rangle$ state according to Eq. (9) before applying distribution and swapping, the fidelity improves significantly.

Let us assume that the probability of bit flip is given by p. Then the Krauss operators of the bit flip channel for a single qubit is given by $\{\sqrt{p}X, \sqrt{1-p}\,I\}$.

A1. Results for No Repetition Code

Since we are using bipartite states, the Krauss Operators are $E_1 = pX_A \otimes X_B, E_2 = (1-p)I_A \otimes I_B, E_3 = \sqrt{p(1-p)}X_A \otimes I_B, E_4 = \sqrt{p(1-p)}I_A \otimes X_B$ and initial density matrix was given by $\rho_0 = |\phi^+\rangle_{AB}\langle\phi^+|_{AB}$. Then, we derive the expression for fidelity, for the case where no repetition code is used, as:

$$F = \sqrt{\langle\phi^+|_{AB}\rho_{AB}|\phi^+\rangle_{AB}} = \sqrt{p^2 + (1-p)^2} \ldots (10)$$

A2. Results for Three Qubit Repetition Code

For a 3-qubit repetition code, we encode $|0\rangle_L \Rightarrow |000\rangle$ and $|1\rangle_L \Rightarrow |111\rangle$, which can correct one bit-flip error since single bit flip errors forms subspaces orthogonal to each other. Thus, the bell state $|\phi^+\rangle$ can be encoded [8] as $\frac{1}{\sqrt{2}}(|000{,}000\rangle + |111{,}111\rangle)$, then distribute is accomplished using the bit-flip channel.

Figure-1 shows the distribution of a 3-qubit repetition code.

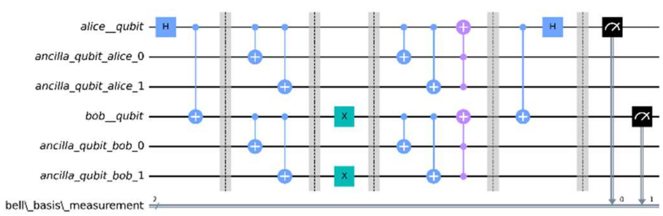

Figure 1: Distribution of the 3-qubit repetition code.

Here, two separate 3-qubit QRC-based encoding and decoding circuits are implemented in Alice and Bob's end block. After decoding both the qubits, a bell basis measurement is carried out to check the fidelity of the said implementation.

A probabilistic model has been built to simulate the bit flip channel (errors are put between encoding and the decoding parts), and Fidelity has been computed numerically using the QISKIT simulator with and without using the 3-qubit repetition code. The results of the analyses are shown in Figure 2. The details of the performance analysis for the 3-qubit repetition code are also included in Table 1.

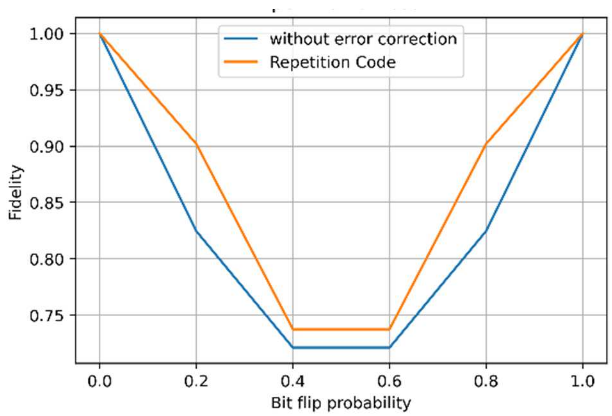

Figure 2: Fidelity improvement using 3-qubit repetition code.

Table 1. Performance of the 3-qubit repetition code.

#Bit flip	Probability	#Total	#Corrected
0	$(1-p)^6$	1	1
1	$p(1-p)^5$	6	6
2	$p^2(1-p)^4$	15	9
3	$p^3(1-p)^3$	20	0
4	$p^4(1-p)^2$	15	9
5	$p^5(1-p)$	6	6
6	p^6	1	1
Total		$2^6 = 64$	32

In this case, the theoretical Fidelity, F, which has been derived by us, is given by Eq. (11).

$$F = \sqrt{\sum_{i=0}^{6} a_i p^i (1-p)^{6-i}} \ldots (11)$$

where, $a_0 = 1, a_1 = 6, a_2 = 9, a_3 = 0, a_{6-i} = a_i$.

We note that, if the initial state were a product state, then a_i would be 0 for $i > 3$. Thus, it proves that use of QECCs in entangled states enhances the error correcting ability.

A3. Results for Five Qubit Repetition Code

A 5-qubit repetition code can correct up to 2 qubit bit-flip errors for encoding a single qubit. In this case, the encoding scheme was given by: $|0\rangle_L = |00000\rangle$ and $|1\rangle_L = |11111\rangle$. Thus, the bell state $|\phi^+\rangle$ can be encoded as $\frac{1}{\sqrt{2}}(|00000{,}00000\rangle + |11111{,}11111\rangle)$. Finally, the distribution is accomplished using the bit-flip channel. We designed the decoding part of five qubit QRC circuit for single qubit and then embedded this into bell state. The circuit implementing this error correcting code for entangled state is shown in Figure 3.

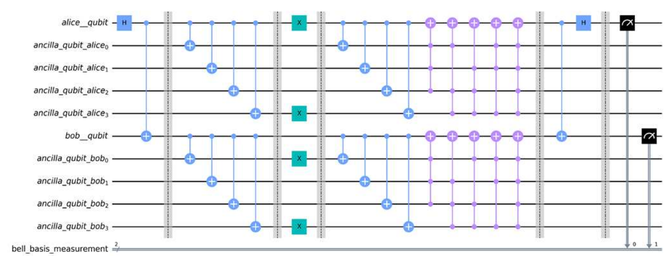

Figure 3: Distribution of the 5-qubit repetition code.

The result for this case has been analyzed theoretically with the help of QISKIT. Figure 4 shows a comparison with those with and without the 3-qubit repetition codes [Eq.s (10) and (11)].

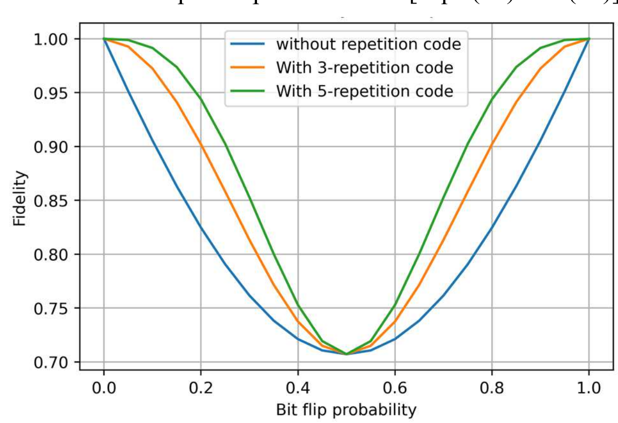

Figure 4: Theoretical Fidelity in a bit flip channel.

The fidelity in the 5-qubit case is seen to have improved more than that in the 3-qubit repetition code. The details of the result are shown in Table 2.

Table 2. Performance of the 5-qubit repetition code.

#Bit-flip	Probability	#Total	#Corrected
0	$(1-p)^{10}$	1	1
1	$p(1-p)^9$	10	10
2	$p^2(1-p)^8$	45	45
3	$p^3(1-p)^7$	120	100
4	$p^4(1-p)^6$	210	100
5	$p^5(1-p)^5$	252	0
6	$p^6(1-p)^4$	210	100
7	$p^7(1-p)^3$	120	100
8	$p^8(1-p)^2$	45	45
9	$p^9(1-p)$	10	10
10	p^{10}	1	1
Total		$2^{10}=1024$	512

Table 2 shows that the 5-qubit repetition code corrects all the errors, as has been expected. Additionally, it has also been able to correct some errors it was not supposed to correct at all; thus, it results in a much better fidelity than in the 3-qubit case.

The theoretical fidelity, for this case has been calculated by us using Eq. (12).

$$F = \sqrt{\sum_{i=0}^{10} b_i p^i (1-p)^{10-i}} \dots (12)$$

where, $b_0 = 1, b_1 = 10, b_2 = 45, b_3 = b_4 = 100, \; b_{10-i} = b_i$

IV. Conclusion and Future Prospects

In this paper, various quantum repetition codes (QRCs) have been applied in the bell states and their performance analyzed mathematically. The proposed general procedure to compute and compare various quantum error correcting codes (QECCs) in terms of fidelity, proves very helpful both in theoretical and experimental analyses. We also notice that the use of QECCs in entangled states increases the ability of the error correcting codes. This suggests that, instead of entanglement purification, use of the QECC can offer faster and more secure quantum communication.

The finding of this work will be proven useful in the design of second and third generation quantum repeaters [6] [8]. It is also expected that the theoretical and simulation models developed in this research can be extended for multiple applications in quantum noise and error correction techniques. In this work, only single qubit errors were handled, although there could be joint errors, measurement errors, memory errors etc. [15], which should have to be considered as well.

V. Acknowledgment

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All the codes are made available at:
https://www.github.com/syedshubha/QEC_Entanglement01

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