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Applied Optimization (WS 2024/2025)
Exercise Sheet No. 4

Upload Date: 2024-11-20.
Submission Deadline: In groups of three until **2024-12-04, 07:55** to the Moodle.
Return date: 2024-12-09/12 in the tutorials.

In case you have questions do not hesitate to ask your tutor or to contact the tutor team at apopt@techfak.uni-bielefeld.de.

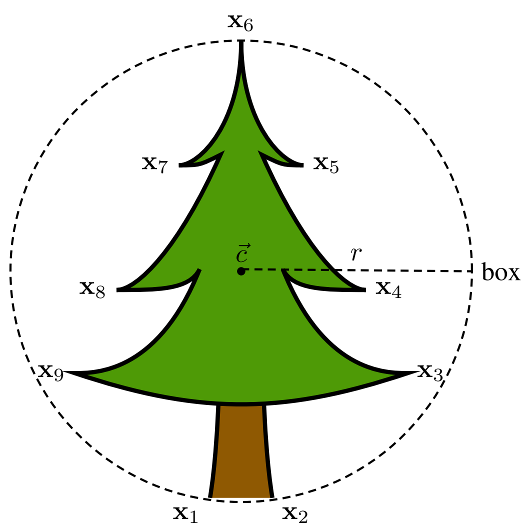
Remark: When handing in your solution, do not only provide the source code, but also all the other information we ask for (e.g. the output of the optimization, the runtime, etc.). We will *not* execute your code!

Remark: Note that you can score 24 points on this sheet. The additional 4 points do not count towards the 60% of all points you must achieve to pass the exercises.

1 Minimum Enclosing Gift Box

(*6.5 points*)

Assume you wish to put a christmas gift into a round box. The corners of your gift item are given by a set of two-dimensional points $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^2$. Now, you wish to find the smallest round box such that your gift still fits inside (also refer to the Figure below) [1].



We can formalize this problem as follows. Let $\mathbf{c}$ be the location of the center of our box and let r^2 be the squared radius. Then we wish to solve the problem:

$$\begin{aligned} \min_{\mathbf{c}\in\mathbb{R}^2, r^2\in\mathbb{R}} \quad & r^2 \\ \text{s.t.} \quad & (\mathbf{c}-\mathbf{x}_i)^T \cdot (\mathbf{c}-\mathbf{x}_i) \leqslant r^2 \qquad\qquad \forall i \in \{1,\ldots,m\} \end{aligned}$$

(a) *(1 pts.)* Write down the Lagrange dual of this problem.

Definition: For a convex primal problem

$$\begin{aligned} & \min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \geqslant 0 \end{aligned}$$

[1] Credits go to Allain Mathes for their tikz code.

the Wolfe dual is defined as

$$\begin{aligned} \sup_{\mathbf{x}\in\mathbb{R}^d,\lambda\in\mathbb{R}^m} \quad & f(\mathbf{x}) - \sum_i (\lambda)_i g_i(\mathbf{x}) \\ \text{s.t.} \quad & (\lambda)_i \geqslant 0 \\ & \nabla_{\mathbf{x}} f(\mathbf{x}) = \sum_{i=1}^m (\lambda)_i \nabla_{\mathbf{x}} g_i(\mathbf{x}). \end{aligned}$$

(b) *(2 pts.)* Write down the Wolfe dual of this problem.

(c) *(1 pts.)* Identify a feasible solution for $\mathbf{c}$, depending on λ, using the Wolfe dual.

(d) *(2.5 pts.)* Re-write the Wolfe dual, plugging your results for the side constraints into the objective function. Simplify as far as possible.

2 Reverse Modeling

(*10 points*)

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d},\ \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^N,\ N \geqslant 2, n \geqslant 1$ be real vectors. Consider the following optimization problems:

$$\begin{aligned} \min_{\alpha,\beta\in\mathbb{R}} \quad & \|\alpha\mathbf{a} - \beta\mathbf{c} + (1-\alpha)\mathbf{b} - (1-\beta)\mathbf{d}\| && (a) \\ \text{s.t.} \quad & 0 \leqslant \alpha, \beta \leqslant 1 \end{aligned}$$

$$\begin{aligned} \min_{\xi_1,\ldots,\xi_n\in\mathbb{R}} \quad & \sum_{i=1}^n \left\| \mathbf{x}_i - \sum_{j=1}^n \xi_j \mathbf{x}_j \right\|^2 && (b) \\ \text{s.t.} \quad & \sum_{i=1}^n \xi_i = 1 \\ & \xi_i \geqslant 0 && \forall i = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max_{\xi_1,\ldots,\xi_n\in\mathbb{R}} \quad & \sum_{i=1}^n \xi_i \left\| \mathbf{x}_i - \frac{1}{n} \sum_{j=1}^n \mathbf{x}_j \right\| && (c) \\ \text{s.t.} \quad & \sum_{i=1}^n \xi_i = 1 \\ & \xi_i \geqslant 0 && \forall i = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{n}\in\mathbb{R}^N,\ d,\xi_1,\ldots,\xi_n\in\mathbb{R}} \quad & \sum_{i=1}^n \xi_i && (d) \\ \text{s.t.} \quad & \|\mathbf{n}\| = 1 \\ & \mathbf{n}^\top \mathbf{x}_i - \xi_i = d && \forall i = 1, \ldots, n \\ & \xi_i \geqslant 0 && \forall i = 1, \ldots, n \end{aligned}$$

$$\begin{aligned} \max_{\mathbf{c}\in\mathbb{R}^N} \min_{\mathbf{w},\mathbf{p}\in\mathbb{R}^N} \quad & \|\mathbf{p} - \mathbf{c}\| && (e) \\ \text{s.t.} \quad & \mathbf{w}^\top \mathbf{x}_i \geqslant 1 && \forall i = 1, \ldots, n \\ & \mathbf{w}^\top \mathbf{p} = 1 \end{aligned}$$

(a) *(5 × 1 pts.)* Give a geometric interpretation and its link to the formulation for each problem. E.g. for (a) this could be: (a) aims for the minimum distance in between the line segments $\overline{\mathbf{ab}}$ and $\overline{\mathbf{cd}}$; here $\alpha\mathbf{a} + (1-\alpha)\mathbf{b}$ refers to a point on the line segment from $\mathbf{a}$ to $\mathbf{b}$, and similarly for $\beta\mathbf{c} + (1-\beta)\mathbf{d}$ ...
Hint: Recall that the Hessian normal form describes a (hyper)plane by a normal vector $\mathbf{n}$, i.e., $\|\mathbf{n}\| = 1$, and a

distance d. The plane is given as the set of points $\mathbf{x}$ satisfying $\mathbf{n}^\top\mathbf{x} = d$. By dividing this by d this is equal to $\mathbf{w}^\top\mathbf{x} = 1$ with $\mathbf{w} = \mathbf{n}/d$.

(b) *(3 × 1 pts.)* Find an analytical, close form solution for problems (a)–(c).
Hint: It is not always necessary to use Lagrange or Wolfe duals.

(c) *(2 pts.)* Compute the Lagrange and the Wolfe dual for problem (d).

3 Hard to Optimize

(*7.5 points*)

Consider the following optimization problems:

$$\min_{(x,y)\in\mathbb{R}^2} \frac{1}{100}(x^2+y^2) - \frac{1}{2}(\cos(3x-3y)+\cos(3x+3y)) \tag{a}$$

$$\min_{(x,y)\in\mathbb{R}^2} (x-1)^2 + 100(x^2-y)^2 \tag{b}$$

$$\min_{(x,y)\in\mathbb{R}^2} xy \tag{c}$$

$$\text{s.t.} \qquad x^2+y^2 \leqslant 1 + \frac{1}{5}\cos\left(8\arctan\left(\frac{x}{y}\right)\right)$$

For each of the problems (a),(b), and (c)...

(a) *(3 × 0.5 pts.)* ... name at least two algorithms from the lecture that can be applied to the problem.

(b) *(3 × 1.5 pts.)* ... empirically evaluate the method. Answer the following questions (for different starting points):

- How well does the method identify the global optimum. (e.g. difference between found and global optimum in terms of function value or position in space; to do so first find the analytical solution[2])
- How long does it take to do so. (e.g. runtime, number of iterations, or number of function/gradient evaluations)

(c) *(3 × 0.5 pts.)* ... analyze your empirical findings. Provide explanations why you obtained the results you obtained.

[2]For (c) observe that if presented in polar coordinates, i.e. $(x,y) = (r\sin(\varphi), r\cos(\varphi))$, we have $\cos(8\arctan((r\sin(\varphi))/(r\cos(\varphi))) = \cos(8\varphi)$