total grade: 11 / 12

Ex. 1
Normal Distribution

a) Show that:

$$Z := \frac{X-\mu}{\sigma} \text{ is } \mathcal{N}(0,1)$$

[2/2]

If X is normally distributed, any affine transformation of X will also be normally distributed.

Mean of Z:

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{E(X)-\mu}{\sigma} = 0$$

Var. of Z:

$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{Var(X)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1.$$

So,

$$F_X(x) = \Phi_{\mu,\sigma^2}(x) = \int_{-\infty}^{x} \phi_{\mu\sigma^2}(y)\,dy = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2\sigma^2}(y-\mu^2)\right) dy.$$

By def. of CDF:

$$P(Z \le x) = P(X \le \mu + \sigma x).$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp[-z^2/2]\,dz = \Phi_{0,1}(x).$$

b) $\Phi_{0,1}(x) + \Phi_{0,1}(-x) = 1$

As we know that

$$\Phi_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy$$

So, $$\Phi_{0,1}(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} \exp\left(-\frac{y^2}{2}\right) dy = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} \exp\left(-\frac{z^2}{2}\right) dz.$$

As they are normal so both are symmetric (reliable) & $\Phi_{0,1}(x)$, That's why.

$$\Phi_{0,1}(x) + \Phi_{0,1}(-x) = 1.$$

b) $P(X \le b) = \Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right)$

as by C.D.F, $\Phi_{\mu,\sigma^2}(x) = P(X \le x)$.

So,

$P(X \le b) = P\left(\frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$ ∵ by transformation $Z = \frac{X-\mu}{\sigma}$

$P(X \le b) = P\left(Z \le \frac{b-\mu}{\sigma}\right)$

we know Z is normally distributed

$\boxed{\text{So, } (P X \le b) = \Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right)}$ ✓

for $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$

$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$

$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \cancel{\le} \frac{a-\mu}{\sigma}\right)$ $<$

but $P\left(Z < \frac{a-\mu}{\sigma}\right)$

$= P\left(Z \le \frac{a-\mu}{\sigma}\right)$

as $P\left(Z = \frac{a-\mu}{\sigma}\right) = 0$

$\boxed{P(a \le X \le b) = \Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{a-\mu}{\sigma}\right)}$ — (b) ✓

d) $P(-b \le X \le b) = 0.8$ as from Part (c): $P(-b \le X \le b) = 2 \cdot P(X \le b) - 1 = 0.8$

So, $2 \cdot P(X \le b) - 1 = 0.8 \Rightarrow P(X \le b) = \frac{1.8}{2} = 0.9$, shows that

b is 90^{th} percentile of standard Normal Dist $= \boxed{1.281552}$ ✓

in R code.

e) $P(\mu - \sigma \le X \le \mu + \sigma)$

As from (b) we know: $P(\mu - \sigma \le X \le \mu + \sigma) =$

$= \Phi_{0,1}\left(\frac{\mu+\sigma-\mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{\mu-\sigma-\mu}{\sigma}\right) = \Phi_{0,1}(1) - \Phi_{0,1}(-1)$ ✓

As by symmetry: $\Phi(-1) = 1 - \Phi_{0,1}(1)$

$= \Phi_{0,1}(1) - (1 - \Phi_{0,1}(1)) = \Phi_{0,1}(1) - 1 + \Phi_{0,1}(1) = 2\Phi_{0,1}(1) - 1 = \boxed{0.68}$

in R code. ✓

```r
### EX-4##

#Q.1(d)
b <- qnorm(0.9)  # Since P(-b <= X <= b) = 0.8, we find the 90th percentile
b
#Q1(e)

p <- 2*pnorm(1) - 1
p
#Q1(f)
set.seed(123)
mu <- 5
sigma <- 2
X <- rnorm(10000, mean = mu, sd = sigma)
proportion <- mean(X >= (mu - sigma) & X <= (mu + sigma))
proportion

```

73:1 (Top Level)

R 4.4.1 · ~/

```
> ### EX-4##
>
> #Q.1(d)
> b <- qnorm(0.9)  # Since P(-b <= X <= b) = 0.8, we find the 90th percentile
> b
[1] 1.281552
> #Q1(e)
>
> p <- 2*pnorm(1) - 1
> p
[1] 0.6826895
> #Q1(f)
> set.seed(123)
> mu <- 5
> sigma <- 2
> X <- rnorm(10000, mean = mu, sd = sigma)
> proportion <- mean(X >= (mu - sigma) & X <= (mu + sigma))
> proportion
[1] 0.6812
```

[2.5/3] + 0.5 for optional (*) d

plots good but I could not find the code

Ex. 1

Q.2(a). PMF of $Z = X+Y$

$$f_Z(z) = \sum_u f_X(z-u) f_Y(u)$$

By Def. of PMF:

$$f_Z(z) = P(Z=z).$$

for sum of 2 indep. variables $Z = X+Y$

$$f_Z(z) = P(X+Y=z)$$

By Law of total Prob.

$$f_Z(z) = \sum_u P(X=z-u \text{ and } Y=u)$$

$\because P(X) \cap P(Y)$

As X and Y are indep. so, Prob. of Joint event:

$$= P(X=z-u) \cdot P(Y=u) = f_X(z-u) f_Y(u)$$

Summing over all possible values:

$$\left[f_Z(z) = \sum_u f_X(z-u) f_Y(u) \right]$$

b) PDF of $Z = X+Y$ for continuous var. $f_Z(z) = \int_{-\infty}^{\infty} f_X(z-u) f_Y(u)\,du.$

Let $F_Z(z)$ is CDF of Z:

$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$

for Double integral: we consider all pairs (x,y) : $x+y \le z$.

As X & Y are continuous, so, Joint Prob. Density is $f_X(x) f_Y(y)$:

and indep.

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y)\,dx\,dy.$$

We'll differentiate $F_Z(z)$ to find PDF.

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y)\,dx\,d(y).$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy \Rightarrow f_Z(z) = (f_X * f_Y)(z).$$

Q.2

c) $X \sim Exp(\lambda)$ $Y \sim Exp(\lambda)$:

X & $Y \sim Exp(\lambda)$, so PDF is:

$$f_X(x) = \lambda e^{-\lambda x} \quad \because x \geq 0$$
$$f_Y(y) = \lambda e^{-\lambda y} \quad \because y \geq 0$$

by Q.2(b), PDF of $Z = X+Y$ is, & formula given (2):

$$f_Z(z) = \int_0^z f_X(z-u) f_Y(u) du,$$

substitute $f_X(z-u)$ $f_Y(u)$.

$$f_Z(z) = \int_0^z \lambda e^{-\lambda(z-u)} \lambda e^{-\lambda u} du.$$

$$= \lambda^2 \int_0^z e^{-\lambda z} du = \lambda^2 z e^{-\lambda z} \quad \text{for } z \geq 0$$

$$Z \sim Gamma(2, \lambda)$$

$X \sim N(\mu_1, \sigma_1^2)$ & $Y \sim N(\mu_2, \sigma_2^2)$.

As X & Y are normally distributed, so, PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1(x-\mu_1)^2}{2\sigma_1^2}\right).$$

Same for y with σ_2 & μ_2.

substituting this in formula (2).

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(z-u-\mu_1)^2}{2\sigma_1^2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(u-\mu_2)^2}{2\sigma_2^2}\right) du.$$

d) Cont...

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_1^2}\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-u-\mu_1)^2}{2\sigma_1^2} - \frac{(u-\mu_2)^2}{2\sigma_2^2}\right) du.$$

Expanding quadratic terms:

$$(z-u-\mu_1)^2 = (z-\mu_1)^2 - 2(z-\mu_1)u + u^2$$
$$(u-\mu_2)^2 = u^2 - 2u\mu_2 + \mu_2^2$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_1^2}\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} -\frac{(z-\mu_1)^2 - 2(z-\mu_1)u + u^2}{2\sigma_1^2} - \frac{u^2 - 2u\mu_2 + \mu_2^2}{2\sigma_2^2} du.$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_1^2}\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} -\frac{(z-\mu_1)^2}{2\sigma_1^2} - \frac{\mu_2^2}{2\sigma_2^2} - \left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right)u^2 + u\left(\frac{z-\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2}\right) du.$$

Let $a = \frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}$ & $b = \frac{z-\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2}$.

$$-(au^2 - bu) = -a\left(u^2 - \frac{b}{a}u\right)$$

$$u^2 - \frac{b}{a}u = \left(u - \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}$$ ∵ by completing square.

becomes: $-\left(a\left(u - \frac{b}{2a}\right)^2 - \frac{b^2}{4a}\right)$

substituting back in integral:

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_1^2}\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(z-\mu_1)^2}{2\sigma_1^2} - \frac{\mu_2^2}{2\sigma_2^2} + \frac{b^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-a\left(u - \frac{b}{2a}\right)^2\right) du.$$

Integral at the end is Gaussian & evaluates to $\sqrt{\pi/a}$. So,

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left(-\frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_1^2)}\right)$$

Shows $Z = X+Y \sim \mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

d) ... Cont...

Expectation & Variance:

$$E(Z) = E(X+Y) = E(X) + E(Y).$$

as $E(X) = \mu_1$ & $E(Y) = \mu_2$

$$E(Z) = \mu_1 + \mu_2.$$

$$\mu_Z = \mu_1 + \mu_2.$$ ✓

$$Var(Z) = Var(X+Y) = Var(X) + Var(Y).$$

$Var(X) = \sigma_1^2$, & $Var(Y) = \sigma_2^2$.

So, $Var(Z)$:

$$\sigma_Z^2 = \sigma_1^2 + \sigma_2^2$$ ✓

exercice 3 : [2/2]

a • we have :

$X \sim Pois(\lambda)$: represents yearly number of car accidents.

• we define :

Y as the number of car accidents that result in casualities : $Y|X \sim Binom(X, p)$

• Distribution of X : $X \sim Pois(\lambda)$:

$$\boxed{P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}} , \quad x = 0, 1, 2 \ldots$$

• Distribution of $Y|x$:

$$Y|x = x \sim Binom(x, p)$$

with PMF :

$$P(Y=y|X=x) = \binom{x}{y} p^y (1-p)^{x-y} , \quad y = 0, 1, \ldots, x$$

✓

• Joint Distribution :

$$P(X=x, Y=y) = P(Y=y|X=x) \, P(X=x)$$

Substituting :

$$\Rightarrow P(X=x, Y=y) = \binom{x}{y} p^y (1-p)^{x-y} \cdot \frac{\lambda^x}{x!} \exp(-\lambda).$$

$y = 0, 1, \ldots, x$ ✓

and $P(X=x, Y=y) = 0$ if $y > x$.

(6) - $$P(Y=y) = \sum_{x=0}^{\infty} P(X=x, Y=y)$$

$$= \exp(-\lambda) \sum_{x=0}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} \frac{\lambda^x}{x!}$$

$$= \exp(-\lambda) \sum_{x=y}^{\infty} \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y} \frac{\lambda^x}{x!}$$

to simplify, we define $\underline{k = x-y}$:

so:
$$= \exp(-\lambda) \sum_{k=0}^{\infty} \frac{(y+k)!}{y!\,k!} p^y (1-p)^k \frac{\lambda^{y+k}}{(y+k)!}$$

$$= \exp(-\lambda) \frac{p^y \lambda^y}{y!} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (1-p)^k$$

$$= \exp(-\lambda) \frac{p^y \lambda^y}{y!} \sum_{k=0}^{\infty} \frac{(\lambda(1-p))^k}{k!}$$

$$= \exp(-\lambda) \frac{p^y \lambda^y}{y!} \exp(\lambda(1-p))$$

$$= \frac{(\lambda p)^y}{y!} \exp(-\lambda p)$$

$\Longrightarrow$ that means : $Y \sim \text{Pois}(\lambda p)$ ✓

(c) Since X and X' are both independant Poisson variables, we can apply formula (1) for $Z = X + X'$

$$P(Z=k) = \sum P(X=x)\, P(X'=k-x)$$

$$= \sum \frac{\lambda^x}{x!} \exp(-\lambda) \times \frac{\mu^{k-x}}{(x-k)!} \exp(-\mu)$$

$$= \exp(-(\lambda+\mu)) \frac{1}{k!} \sum \frac{k!}{x!(x-k)!} \cdot \lambda^x \mu^{k-x}$$

$$= \exp(-(\lambda+\mu)) \frac{1}{k!} (\lambda+\mu)^k.$$

this means:

$$\Rightarrow X + X' \sim \text{Pois}(\lambda+\mu)$$

(d) we know

$X \sim \text{Pois}(\lambda)$ and $X' \sim \text{Pois}(\mu)$ are independent.

$$P(X' = x \mid X + X' = k)$$

$$= \frac{P(X'=x, X+X'=k)}{P(X+X'=k)}$$

$$= \frac{P(X'=y, X=k-y)}{P(X+X'=k)}$$ why y? :)

$$= \frac{P(X'=y)\, P(X=k-y)}{P(X+X'=k)}$$

$$= \frac{\mu^y}{y!} \exp(-\mu) \cdot \frac{\lambda^{k-y}}{(k-y)!} \exp(-\lambda) \Big/ \left(\frac{(\lambda+\mu)^k}{k!} \exp(-(\lambda+\mu)) \right)$$

$$= \frac{k!}{y!\,(k-y)!} \cdot \frac{\mu^y \lambda^{k-y}}{(\lambda+\mu)^k}$$

$$= \binom{k}{y} \underbrace{\left(\frac{\mu}{\mu+\lambda}\right)^y}_{\ell} (1-q)^{k-y}$$

$$= \binom{k}{y} \ell^y (1-q)^{k-y}$$

$\Longrightarrow$ this means :

$$X' \mid (X + X') \sim \text{Binom}\left(X + X', \frac{\mu}{\mu+\lambda}\right) ✓$$

exercice 4 [1/1]

the definition of conditional Probability :

$$P(X > t+s \mid X > t) = \frac{P(X > t+s \text{ and } X > t)}{P(X > t)}$$

since $X > t+s$ implies $X > t$ we can simplify this to :

$$P(X > t+s \mid X > t) = \frac{P(X > t+s)}{P(X > t)}$$

for $X \sim Exp(\lambda)$, the CDF is:

$$F_X(x) = P(X \leq x) = 1 - e^{-\lambda x} \quad , x \geq 0$$

the complement is:

$$P(X > x) = 1 - F_X(x) = e^{-\lambda x}$$

using this, we find:

$$P(X > t+s) = e^{-\lambda(t+s)}$$ and

$$P(X > t) = e^{-\lambda t}$$

we substitute the expressions:

$$P(X > t+s \mid X > t) = \frac{P(X > t+s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

we simplify:

$$P(X > t+s \mid X > t) = e^{-\lambda s}$$

and $P(X > s) = e^{-\lambda s}$

So: $\boxed{P(X > t+s \mid X > t) = P(X > s)}$

exercice 5 : [2/2]

(a) Since the breakdowns occur randomly with a constant average rate, a Poisson distribution is appropriate for modeling X.

so we model X as :

$$X \sim Pois(\lambda) \quad \text{where } \lambda = 3$$

- For Poisson distributed random variable both the mean and variance are equal to λ

so : $E(X) = \lambda = 3$

and $Var(x) = \lambda = 3$

(b) we want to find $P(X \geq 6)$ when $X \sim Pois(3)$

$$P(X \geq 6) = 1 - P(X < 6)$$

PMF : $P(X=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, \ldots$

$P(X < 6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= 1 - (1 + 3 + 3^2/2 + 3^3/6 + 3^4/24 + 3^5/120)e^{-3}$$

$$\approx 0.083918 = 8.3918\ \%$$

using R:

```
(1- ppois (5, lambda 3)) *100
[1] 8.391794
```

c) In part a) we found that the mean $E(x)$ and $Var(x)$ of X are equal to 3.

we have: $P(X \geq 2\lambda) \leq P(X \geq 2\lambda) + P(X \leq 0)$

$$= P(X \geq 2\lambda \cup X \leq 0)$$

$$= P(X - \lambda \geq \lambda \cup X - \lambda \leq \lambda)$$

$$= P(|X - \lambda| \geq \lambda)$$

using Chebyshev's inequality:

$$P(|X - \lambda| \geq \lambda) = P(|X - E(x)| \geq \lambda) \leq \frac{Var(x)}{\lambda^2} = \frac{\lambda}{\lambda^2}$$

so:

$$= \frac{1}{\lambda} = .$$

$$P(X \geq 6) = P(X \geq 2\lambda) \leq \frac{1}{\lambda} = \frac{1}{3}$$

$\simeq 33.33\%$

comparison: we have in part b: $8.3918 < 33.33$

$\Rightarrow$ The probability from b is more accurate since it uses the specific distribution of X.

[1/2]

Q.6

As given in question Pearson correlation coefficient, Cauchy-Schwarz inequality & Var of X, Y.

As

$$|E(XY)| \leq \sqrt{E(X^2)} \cdot \sqrt{E(Y^2)}$$

If we write Covariance in terms of Expectation.

$$Cov(X,Y) = E((X-E(X))(Y-E(Y)))$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

By ~~Co~~ Pearson correlation co-efficient ρ & by Def.

$$E(X-E(X)^2) = Var(X) \quad , E(Y-E(Y)) = Var(Y)$$

not clear

So,

$$|Cov(X,Y)| \leq \sqrt{Var(X) \cdot Var(Y)}$$

So $\rho \leq 1$, This proves that $-1 \leq \rho(X,Y) \leq 1$.

C.S. applied to what?

$|\rho(X,Y)| = 1.$

This equality in Cauchy-Schwarz inequality holds if a constant exists:

$$X - E(X) = a(Y - E(Y))$$

$$X = aY + b. \qquad \because b = E(X) - aE(Y).$$

This holds if X,Y has linear relationship.

$$Y = cX + d. \qquad \because d = E(X) - aE(Y).$$

$$Var(Y) = Var(cX + d) = c^2 Var(X)$$

It implies

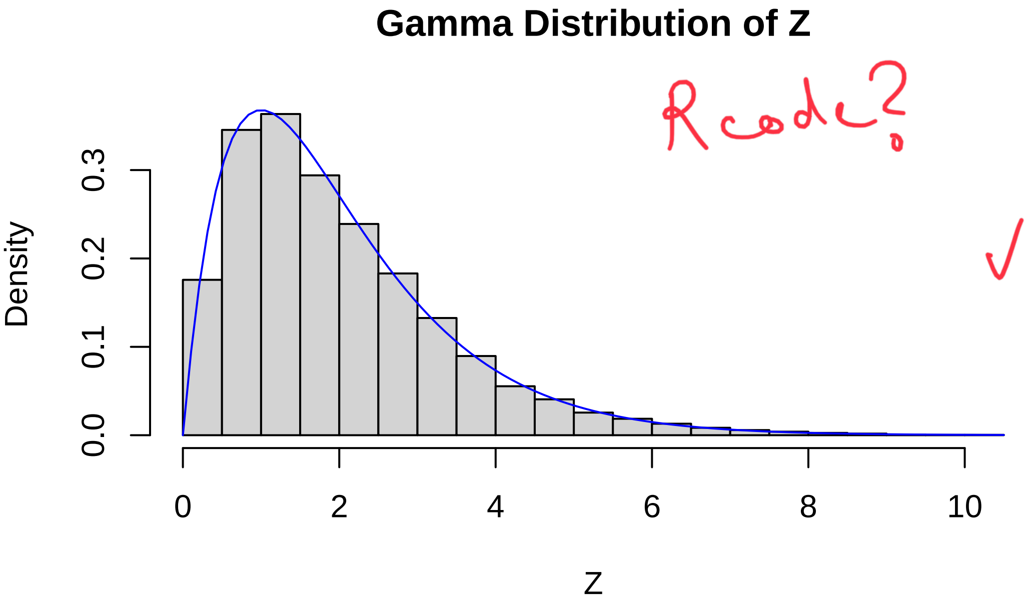
$$c = \pm\sqrt{\frac{Var(Y)}{Var(X)}} \qquad \begin{matrix} + \text{ if } \rho = 1 \\ - \text{ if } \rho = -1. \end{matrix}$$

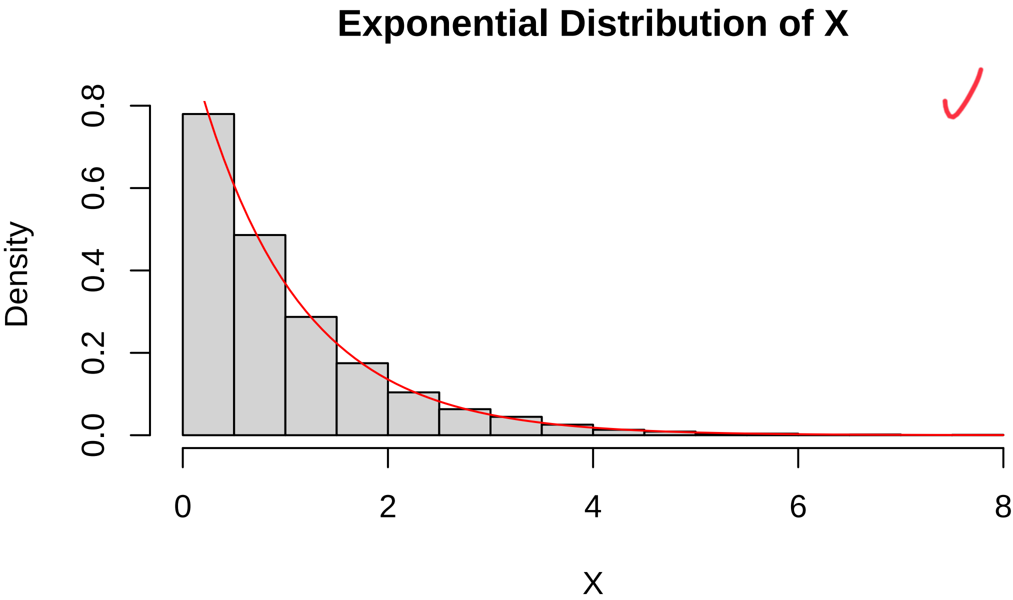
&

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))]$$

$$= cE(X - E(X))^2 = c\,Var(X) \quad \text{It shows/implies}$$

$\rho(X,Y) = \pm 1.$


Gamma Distribution of Z
Density
0.0
0.1
0.2
0.3
0
2
4
6
8
10
Z
Rcode?
✓


Exponential Distribution of X
Density
0.0
0.2
0.4
0.6
0.8
0
2
4
6
8
X