Homework 2 : total grade : 8.5/12

1.(a) Given the increasing sequence (A_n), we define A as the union of all events A_n

[0/2]

$$\Rightarrow \boxed{A = \bigcup_{n=1}^{\infty} A_n}$$

- Because each Event (A_n) is contained within the next event (A_{n+1}), the sequence of probabilities is non decreasing

$$P(A_1) \leq P(A_2) \leq P(A_3) \leq P(A_4) \dots$$

- Since $P(A_n)$ is non decreasing, it must converge to a limit :

$$\boxed{\lim_{n \to \infty} P(A_n) = \ell}$$ (it is boundet above $P(\Omega) = 1$)

- $\boxed{\text{Continuity from below}}$: if (A_n) is an increasing sequence of events, then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$

→ we Know that :

$$\boxed{A_n \subset A_{n+1}} \Rightarrow \text{it means that } (A_n) \text{ is increasing}$$

$\Rightarrow$ By applying the continuity from below :

$$\boxed{P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)}$$

1.(b) Given the decreasing sequence (A_n), we define event (A) as the interaction of all events (A_n):

$$\boxed{A = \bigcap_{n=1}^{\infty} A_n}$$

- Because each event (A_n) contains the next event (A_{n+1}) the sequence of probabilities $P(A_n)$ ist non decreasing

$$P(A_1) \geqslant P(A_2) \geqslant P(A_3) \geqslant P(A_4) \dots$$

- Since $P(A_n)$ is non decreasing, it must converge to a limit ℓ

$$\boxed{\lim_{n\to\infty} P(A_n) = \ell} \qquad (\text{is bounded below by } 0)$$

- Continuity from above: if (A_n) is a decreasing sequence then

$$\boxed{P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)}$$

and we have $A_n \supset A_{n+1}$ so the sequence ist decreasing ✓

$\Rightarrow$ By applying the continuity from above, we have:

$$\boxed{P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)}$$ ✓

[1/2]

2. (a) we consider x and y two real numbers and $x > y$

the CDFs are: $F(x) = P(X \leq x)$ and $F(y) = P(X \leq y)$

the events are:

$$A = \{X \leq x\} \quad \text{and} \quad B = \{X \leq y\}.$$

• since $x < y$ it follows that $\boxed{A \subset B}$ because any outcome $w \in \Omega$ for which $X(w) \leq x$ also satisfies $X(w) \leq y$

• and since $A \subset B$ $\rightarrow$ we can apply the monotonicity property:

$$\boxed{\text{if } A \subset B \text{ then } P(A) \leq P(B)}$$

$\Rightarrow P(X \leq x) \leq P(X \leq y)$ this gives $\boxed{F(x) \leq F(y)}$ ✓

⚠ the inequality $F(x) \leq F(y)$ doesn't have to be strict. It's possible that we have $F(x) = F(y)$ if theres no outcomes in the probability between x and y. ✓

2. (b) we have $F(x) = P(X \leq x)$, since $F(x)$ is a probability, then it is bounded between 0 and 1.

$$0 \leq F(x) \leq 1 \quad \text{for all } x \in \mathbb{R}$$

• Consider the sequence $X \leq x$ as $x \rightarrow -\infty$

Since $X \leq x$ will be impossible for large negative x, the probability of this event will approach to 0 why?

$$\Rightarrow \boxed{\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} P(X \leq x) = 0}$$ ✓

• Same for $x \to +\infty$. the Event $X \leq x$ will cover almost all possible values of X and the probability will approach to 1

$$\Rightarrow \boxed{\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} P(X \leq x) = 1}$$ ✓

2. (c) the right limit $F(x^+)$ is:

$$F(x^+) = \lim_{z \to x^+} F(z)$$

it represents $F(z) = P(X \leq z)$ as z approaches x from the right

for $z > x$ we have $\{X \leq z\} \supset \{X \leq x\}$

$z \to x^+ \Rightarrow$ event $\{x \leq z\}$ goes down to $\{X \leq x\}$ from the right.

• Continuity from Above (prooved in exercice 1b)

the sequence of events $\{X \leq z\}$ ist decreasing for $z \to x^+$ as z gets closer to x

$$\Rightarrow \lim_{z \to x^+} P(X \leq z) = P\left(\bigcap_{z > x} \{X \leq z\}\right) = P(X \leq x)$$

?

$$\Rightarrow \quad F(x) = F(x^+) \qquad \text{right limit}$$ ✓

needs careful computation

2. (d). left limit is $F(x^-) = \lim_{z \to x^-} F(z)$

$\to F(z) = P(X \leq z)$; z approaches x from the left.

- for $z < x$ we have $\{X \leq z\} \subset \{X \leq x\}$ meaning that $\{X \leq z\}$ expand toward $\{X \leq x\}$ as $z \to x^-$

- $\boxed{\text{Continuity from below}}$: (proved in exercice 1a)

$$\lim_{z \to x^-} P(X \leq z) = P\Big(\bigcup_{z<x} \{X \leq z\}\Big)$$

left limit $F(x^-)$ exists at each $x \in \mathbb{R}$.

- F is not necessarily left-continuous, it can also be $F(x) \neq F(x^-)$

we consider X a random variable that takes integer values

for any integer value the probability $P(X \leq x)$ changes at each integer value, creating that F ist not continuous.

for example : $F(3) = P(X \leq 3)$

but $F(3^-) = P(X \leq 0)$ and $F(3) \neq F(3^-)$

↳ F ist not necessarily left continuous

2(e) :

i)- $P(x < X \leq y) = P(X \leq y) - P(X \leq x) = \boxed{F(y) - F(x)}$

ii)- $P(x \leq X \leq y) \Rightarrow$ includes $X = x$ so we get

$P(x \leq X \leq y) = F(y) - F(x^-)$ (definition of left limit)

iii)- we have $F(x) = P(X \leq x)$ and $F(x^-) = P(X < x)$

difference $F(x) - F(x^-) = P(X \leq x) - P(X < x)$

- it captures only the probability of $X = x$

only Part not covered by $P(X < x)$ is $P(X = x)$

So : $\boxed{P(X = x) = F(x) - F(x^-)}$

[2/2]

Conditional Probability & Independence

3. $B \in \mathcal{A}$ be an event with $P(B) > 0$ ......

(a) $Q(\Omega) = 1$.

Sol: $Q(\Omega) = P(\Omega | B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$.

(b) For any ....

$$P\left(\bigcup_{n=1}^{\infty} A_n | B\right) = \sum_{n=1}^{\infty} P(A_n | B)$$

Sol: $P\left(\bigcup_{n=1}^{\infty} A_n | B\right) = \dfrac{P\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \cap B\right)}{P(B)}$

$$= P\left(\bigcup_{n=1}^{\infty} (A_n \cap B)\right) / P(B)$$

$$= \sum_{n=1}^{\infty} P(A_n \cap B) / P(B)$$

$$= \sum_{n=1}^{\infty} P(A_n \cap B) / P(B)$$

$$= \sum_{n=1}^{\infty} P(A_n | B)$$

4. (a) $P(A|B) + P(A^c|B) = 1$.

[2/2]

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$P(A^c|B) = \frac{P(A^c\cap B)}{P(B)}$$

$$= \frac{P(A\cap B)}{P(B)} + \frac{P(A^c\cap B)}{P(B)}$$

$$= \frac{P(A\cap B) + P(A^c\cap B)}{P(B)}$$

$\because P(A\cap B) + P(A^c\cap B) = P(A\cap B) \cup P(A^c\cap B) = B$

$$= \frac{P(B)}{P(B)} = 1$$

(b) Give examples to show following statements are false:

i) $P(A|B) + P(A|B^c) = 1$

Let $A = \Omega$

$P(A|B) + P(A|B^c) = P(\Omega|B) + P(\Omega|B^c)$ --- (1)

as from 3(a)

$P(\Omega|B) = 1$, So (1) becomes.

$= 1 + 1 = 2$, shows false (i) ✓

ii) $P(A|B) + P(A^c|B^c) = 1$.

Let $A = B$, $P(A|B) + P(A^c|B^c) = P(B|B) + P(B^c|B^c)$

$= 1 + 1 = 2$, shows (ii) is false ✓

- it captures only the probability of $X = x$

only Part not covered by $P(X < x)$ is $P(X = x)$

So: $\boxed{P(X=x) = F(x) - F(x^-)}$ ✓

[2/2]

5) a. Events A_1 and A_2 are independent when:

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$

So we need to show that $\boxed{P(A_1^c \cap A_2^c) = P(A_1^c) \times P(A_2^c)}$

- Event $\boxed{A_1^c \cap A_2^c}$ means that A_1 and A_2 don't occur,

that means: $A_1^c \cap A_2^c = (A_1 \cup A_2)^c$

- we have $P(A_1^c \cap P A_2^c)$

$$= P((A_1 \cup A_2)^c) = \boxed{1 - P(A_1 \cup A_2)} \quad (1)$$

and since A_1 and A_2 are independant:

$$\boxed{P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1) \times P(A_2)} \quad (2)$$

with (1) and (2) we have:

$$P(A_1^c \cap A_2^c) = 1 - (P(A_1) + P(A_2) - P(A_1) \times P(A_2))$$

- we have: $\begin{cases} P(A_1^c) = 1 - P(A_1) \quad \text{and} \\ P(A_2^c) = 1 - P(A_2) \end{cases}$

we multiply them: $P(A_1^c) \times P(A_2^c) = (1 - P(A_1)) \times (1 - P(A_2))$

$$= \underbrace{1 - P(A_1) - P(A_2) + P(A_1) \times P(A_2)}_{P(A_1^c \cap A_2^c)}$$

So: $P(A^c_1 \cap A^c_2) = P(A^c_1) \cdot P(A^c_2)$

and that means that A^c_1 and A^c_2 are independent ✓

5(b)- The event that none of $A_1, A_2, \dots, A_N$ occur is the interaction of their complements: $\bigcap_{n=1}^{N} A^c_N$

Since events $A_1, A_2, \dots A_N$ are independant, their complements $A_1^c, A_2^c, \dots A^c_N$ are also independent (as prooved in part(a))

so:

$$P\left(\bigcap_{n=1}^{N} A^c_n\right) = \prod_{n=1}^{N} P(A^c_n)$$

- we have $P(A^c_n) = 1 - P(A_n)$ so: $P\left(\bigcap_{n=1}^{N} A^c_n\right) = \prod_{n=1}^{N} (1 - P(A_n))$

- for any real number x, it holds that $1 - x \leq e^{-x}$

So: $\prod_{n-1}^{N} (1 - P((A_n)) \leq \prod_{n-1}^{N} e^{-P(A_n)}$

- simplifying using $\prod_{n=1}^{N} e^{-P(A_n)} = e^{-\sum_{n=1}^{N} P(A_n)}$

$$\boxed{P\left(\bigcap_{n=1}^{N} A^c_n\right) \leq \exp\left(-\sum_{n=1}^{N} P(A_n)\right)}$$ ✓

- The probability that none of the events $A_1, A_2 \dots, A_N$ occur is less than or equal to $\exp\left(-\sum_{n=1}^{N} P(A_n)\right)$.

part (h) missing [1.5/2]

6. Sensitivity = 96.52%
Specificity = 99.68%

Prob. of a false positive.

96.52% = 0.9652
99.68% = 0.9968

$$P(+|N) = 1 - P((+)^c|N)$$
$$= 1 - P(-|N)$$
$$= 1 - 0.9968$$
$$= 0.0032 \checkmark$$

(b) Prob. that a randomly selected person tests positive in Bielefeld in Oct. 2024

$$P(\text{Positive}) = P(+|C) \cdot P(C) + P(+|N) \cdot P(N)$$

P(C) = Prob. of being infected with covid.

$$P(C) = \frac{16}{100{,}000} = 0.00016$$

$$P(N) = 1 - P(C) = 1 - 0.00016 = 0.99984$$

$$P(\text{Positive}) = 0.9652 \cdot 0.00016 + 0.0032 \cdot 0.99984$$
$$= 0.00335392 \checkmark$$

(c) We can use Bayes theorem to find (P(C|+)).

$$P(C|+) = \frac{P(+|C) \cdot P(C)}{P(+)} = \frac{0.9652 \cdot (0.00016)}{0.00335392}$$
$$= 0.04640 = 4.64\% \checkmark$$

(d) Prob. of being infected despite a negative test.

$P(C|-) = ?$

$$P(C|-) = \frac{P(-|C) \cdot P(C)}{P(-)}$$

$P(-|C) = 1 - 0.9652 = 0.0348$

$P(-) = P(-|C) \cdot P(C) + P(-|N) \cdot P(N)$
$= 0.0348 \cdot 0.00016 + 0.9968 \cdot 0.99984$
$= 0.99952$

$$P(C|-) = \frac{0.0348 \cdot 0.00016}{0.99952} = 0.0000056$$
$= 0.00056\%$ ✓

(e) test from C:

People with covid = 16 $\because 100{,}000 \times 0.00016 = 16$

" without " = 99,984 $\because$ 1 − People with covid.

for covid
True Positive:
$16 \times 0.9652 = 15.44$
False Negative: $16 \times 0.0348 = 0.5568$

for without covid:

False Positive = $99{,}984 \times 0.0032 = 319.94$
True Negative = $99{,}984 \times 0.9968 = 99{,}664$

Total No. of Positive = $15.44 + 319.94 = 335.38$

Prob. of +ve test being have covid in actual.

$$\frac{15.44}{335.38} = 0.046$$

$= 4.6\%$

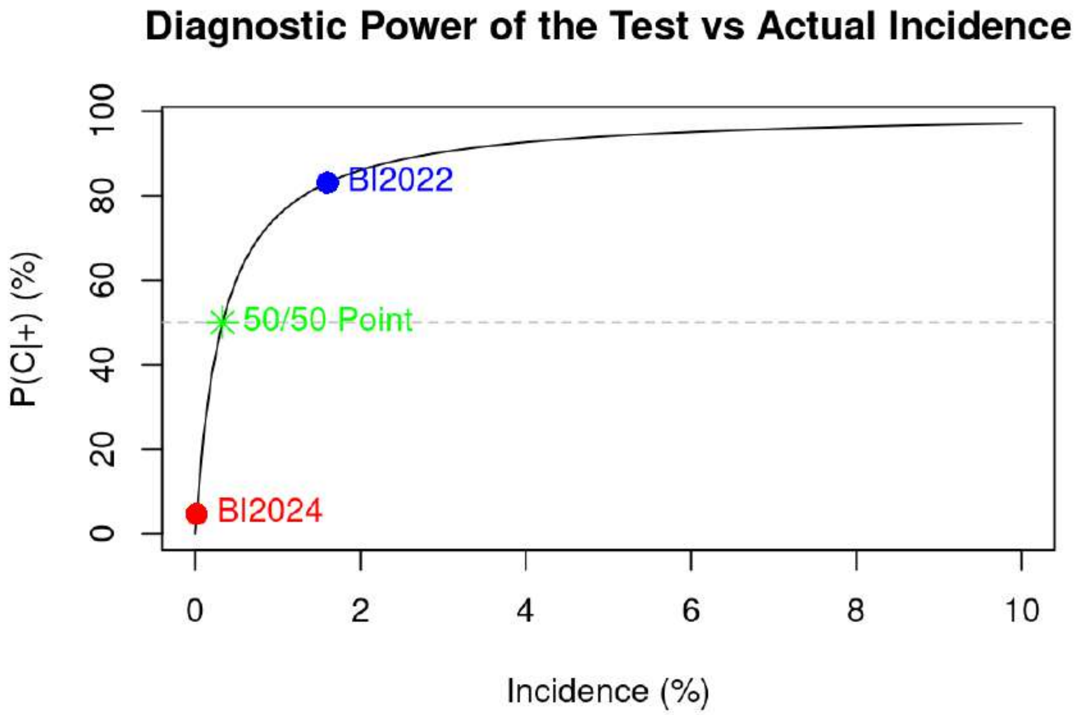
Ans 6.

```
# Sensitivity and Specificity
sensitivity <- 0.9652
specificity <- 0.9968

# Probability range for incidence P(C)
incidence <- seq(0, 0.2, by=0.001)

# Calculate P(C|+) for each incidence level
P_C_given_positive <- function(incidence) {
  P_positive <- sensitivity * incidence + (1 - specificity) * (1 - incidenc
  (sensitivity * incidence) / P_positive
}

# Apply function to incidence range
probabilities <- sapply(incidence, P_C_given_positive)
```



6) (f) We need to plot $P(C|+)$ as a function of actual incidence $P(C)$

$$P(C|+) = \frac{P(+|C) \cdot P(C)}{P(+)}$$

where $P(+) = P(+|C) \cdot P(C) + P(+|N) \cdot P(N)$.

$P(+|C) =$ sensitivity $= 0.9652$

$P(C) =$ infection rate in population

$P(+|N) = 1 -$ specificity $= 1 - 0.9968 = 0.0032$

(g) ~~In most~~ When the incidence is low the $P(C|+)$ is low. As incidence increases $P(C|+)$ increases.

$$P(C) = \frac{1600}{100000} = 0.016$$

$$P(C|+) = \frac{0.9652 \times 0.016}{(0.9652 \times 0.016) + (1 - 0.9968) \times (1 - 0.016)}$$

$$= 0.83 \simeq 83\%$$