Universität Bielefeld
Tanja Pasurek
Ehsan Abedi WiSe 2024/25

Foundations of Statistics

Homework 5

Exercise 1 (Transformed density functions).

Let X be a continuous random variable. Define a new random variable $Y := g(X)$, where g is a measurable map. In this exercise, we answer the following **important question:** What is the distribution of Y?

(a) Transformation formula in **univariate** case:

Let $X : \Omega \to \mathcal{I} \subseteq \mathbb{R}$ be a continuous random variable taking values in some interval $\mathcal{I}$. Suppose that the map $g : \mathcal{I} \to \mathbb{R}$ is strictly monotone increasing or strictly monotone decreasing, so that it has an inverse $h = g^{-1} : \mathcal{J} \to \mathcal{I}$ defined on the image set $\mathcal{J} := g(\mathcal{I}) = \{y \in \mathbb{R} : y = g(x) \text{ for some } x \in \mathcal{I}\}$. Moreover, assume that there exist a continuous derivative $g'(x) \neq 0$ for all $x \in \mathcal{I}$, which in turn guarantees the existence of

$$h'(y) = \frac{\partial}{\partial y} g^{-1}(y) = \frac{1}{g'(h(y))} \quad \text{for all } y \in \mathcal{J}.$$

Then show that the continuous random variable $Y := g(X)$ has PDF

$$f_Y(y) = \left|h'(y)\right| \cdot f_X(h(y)), \quad y \in \mathcal{J}. \tag{1}$$

Hint: use the so-called **CDF method**.

(b) Formula (1) can be generalized to **multivariate** case:

Let $\mathbf{X} : \Omega \to \boldsymbol{\mathcal{X}} \subseteq \mathbb{R}^n$ be a continuous random variable with joint density function $f_{\mathbf{X}}$. Let $g : \mathcal{X} \to \mathbb{R}^n$ be differentiable bijection with non-zero derivative. Prove that $\mathbf{Y} := g(\mathbf{X})$ has joint density function

$$f_{\mathbf{Y}}(\boldsymbol{y}) = \left|\det(J_{g^{-1}}(\boldsymbol{y}))\right| \cdot f_{\mathbf{X}}(g^{-1}(\boldsymbol{y})), \qquad \boldsymbol{y} \in \boldsymbol{\mathcal{Y}} := g(\boldsymbol{\mathcal{X}}), \tag{2}$$

whenever $J_{g^{-1}}(\boldsymbol{y})$ is well-defined. Here $J_{g^{-1}}(\boldsymbol{y})$ is the Jacobian matrix (i.e. the matrix of partial derivatives) of the map g^{-1}.

Hint: For each (Borel) subset $A \subset \mathbb{R}^n$, find $\mathbb{P}(\boldsymbol{Y} \in A)$ and apply the multi-dimensional **change of variables formula** as in *Calculus I*.

Exercise 2 (Transformed density functions, 1D examples).

(a) Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with CDF F_X and PDF f_X, and consider the *linear transformation* $Y := rX + s$ for some $r, s \in \mathbb{R}$. Find the CDF F_Y and PDF f_Y of Y.

(b) Let a random variable X have PDF $f_X(x) = e^{-x}$ for $x > 0$. Define $Y := g(X) = \log X$. Check that

$$f_Y(y) = e^y e^{-e^y} \quad \text{for } y \in \mathbb{R}.$$

(*Warning:* Although $Y := g(X)$, in general $f_Y \neq g(f_X)$!)

(c) Let $X \sim \text{Unif}(0, 1)$. Find the distribution of the random variable $Y := \sqrt{X}$. Check your answer using simulation in `R`. To this end, simulate a large number (for instance, $n = 10^6$) of samples from the uniform distribution, square the values, make a histogram (with `freq=FALSE`) and superimpose the calculated density on top of the histogram. Compute $\mathbb{E}[Y]$ using both the LOTUS and the density f_Y that you have found.

Exercise 3 (Transformed density functions, 2D examples).

(a) Let X and Y be independent, continuous random variables with densities f_X and f_Y. Use formula (2) to find the density of $X + Y$ and compare the result with what we found in Exercise 2, HW 4.

(b) Let $X, Y \sim \mathcal{N}(0, 1)$ be independent. Show that $X+Y$ and $X-Y$ are independent as well (*Hint:* define $U := X+Y$ and $V := X-Y$ and compute their joint density $f_{U,V}(u, v)$ using formula (2)). Check the relationship between this result and what we found in Exercise 5, HW 3.

(c) Let $X_1, X_2 \sim \mathcal{N}(0, 1)$ be independent. Write the sample mean $\overline{X}$ and the sample variance S^2 in terms of X_1, X_2. Are $\overline{X}$ and S^2 independent?

Exercise 4 (A more general Law of Large Numbers (LLN)).

Let $(X_i)_{i \geq 1}$ be a sequence of independent random variables, with $\mathbb{E}(X_i) = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$ for all i. Suppose that $0 < \sigma_i^2 \leq M < \infty$ for all $i \geq 1$. Let a be an arbitrary positive number.

(a) Apply Chebyshev's inequality to show that for any $n \geq 1$

$$\mathbb{P}\left(\left|\overline{X}_n - \frac{1}{n}\sum_{i=1}^{n} \mu_i\right| > a\right) \leq \frac{\sum_{i=1}^{n} \sigma_i^2}{n^2 a^2}.$$

(b) Conclude from **(a)** that

$$\lim_{n \to \infty} \mathbb{P}\left(\left|\overline{X}_n - \frac{1}{n}\sum_{i=1}^{n} \mu_i\right| > a\right) = 0.$$

Check that the weak Law of Large Numbers (Th. 1 of Ch. 1.7) is a special case of this result.

Exercise 5 (Simulation of Law of Large Numbers (LLN) in `R`).

(a) To begin with, plot the densities of normal distribution with mean 2 and variance 1 (in blue) and Cauchy distribution with location parameter 2 and scale parameter 1 (in red) on the same plot. Which one has a heavier tail?

(b) Take a sample of $n = 5000$ realizations from $\mathcal{N}(2, 1)$. Calculate the cumulative arithmetic mean of your sample, that is the arithmetic mean of the first number, of the first two numbers, and so on (see ?`cumsum`). Plot the mean values obtained and overlaid them with a horisontal line corresponding to the actual mean value.

(c) Repeat the procedure in (b) with the Cauchy distribution with with location parameter 2 and scale parameter 1 (see ?`rcauchy`). Can we observe a similar convergence in this case? Justify your answer.

Hint: if you want to get a reproducible sequence of random numbers, use the command `set.seed` to start a random generator with any number of your choice (see e.g. [*Heumann et al.*], Appendix: Introduction to `R`, p. 418).

Exercise 6 (Simulation of Central Limit Theorem (CLT) in `R`).

Let $X_i, i \geq 1$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$ such that $\sigma^2 \in (0, \infty)$. CLT tells us that the distribution of standardized sum

$$Z_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma},$$

converges to the standard normal distribution $\mathcal{N}(0, 1)$.

To see this, you need to consider two natural numbers k and n. Fix $k = 1000$. At first, take an arbitrary n and generate $k = 1000$ random samples Z_n when we have i.i.d. Pois(0.5)–distributed random variables X_i, $1 \leq i \leq n$. Plot the corresponding histogram and overlaid it with the density of the normal distribution. Then increase n, while keep k fixed. Repeat the simulation when we have i.i.d. Exp(2)–distributed random variables X_i, $1 \leq i \leq n$. Does the result depend on distribution of X_i?