Part-I Condition Probability & Independence.

1. (Rolling a die):

Probability space:

$$\Omega := \{ w = (w_1, w_2, w_3 \dots w_N) : w_i \in \{1,2,3 \dots 6\},$$
$$i \in \{1, 2, \dots N\} \}$$

$A := 2^{\Omega}$, $P :=$ Uniform distribution

So event A_{ij} will be:

$$A_{ij} = \{ w \in \Omega : w_i = w_j \}, \text{ for all events: } 1 \leq i < j \leq N.$$

Prob. of pairs (dependent/independent):

$$P(A_{ij} \cap A_{i'j'})$$

$$= P(\{ w \in \Omega : w_i = w_j , w_{i'} = w_{j'} \}) \quad \because i \neq i' \; \& \; j \neq j' \; \& \text{ vice versa.}$$
$$= \frac{6^{N-4} \times 6 \times 1 \times 6 \times 1}{6^N} = \frac{1}{6^2}.$$

$$= P(\{ w \in \Omega : w_i = w_j = w_{j'} \}) \quad \because i = i' \; \& \; i \neq j', j \neq i', j \neq j'$$
$$= \frac{6^{N-3} \times 6 \times 1 \times 1}{6^N} = \frac{1}{6^2}$$

So, we see that in both cases Prob is same (all other cases). Hence we can say that A_{ij} are pairwise independent.

$$P(A_{ij} \cap A_{i'j'}) = P(A_{ij}) P(A_{i'j'}).$$

&

$$P_{A_{ij}} = \frac{6^{N-2} \times 6 \times 1}{6^N} = \frac{1}{6}.$$

For $A_{jj'}$ to be not independant:

let

$$i < j < j'$$

So

$$P(A_{ij} \cap A_{ij'} \cap A_{jj'}) = P(\{w \in \Omega ; w_i = w_j = w_{j'}\})$$

$$= \frac{6^{N-3} \times 6 \times 1 \times 1}{6^N} = \frac{1}{6^2}$$

$\frac{1}{6^2} \neq \frac{1}{6^3}$. So pairwise not independent.

$$P(A_{ij})P(A_{ij'})P(A_{jj'}) = \frac{1}{6^3}.$$

2. Friends & Random Numbers:

(a)

One Var:

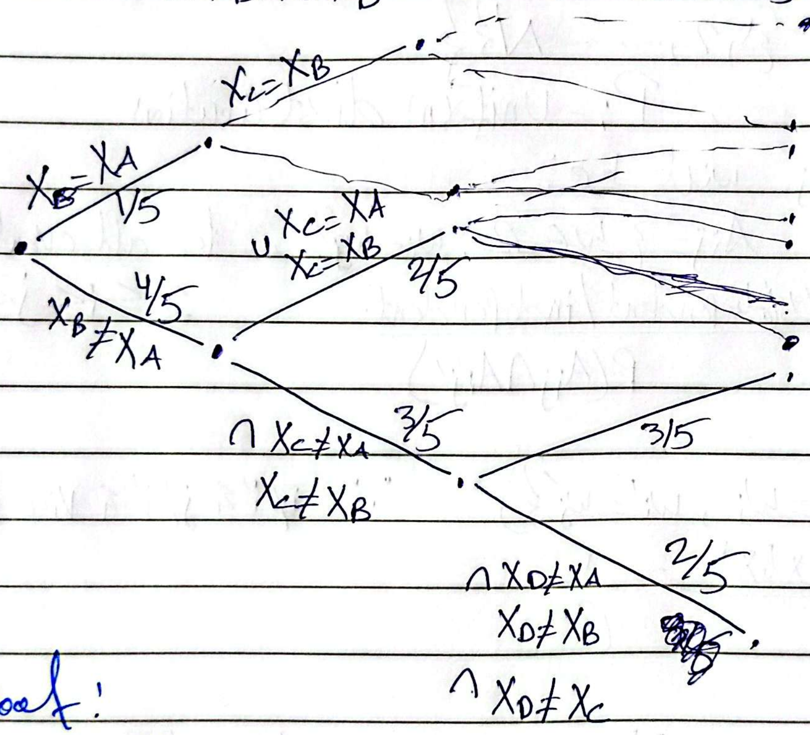
X_A for Alex, X_B for Blake

X_C for Chris, X_D for Dusty.

No. between 1 — 5.

so

$$X_A, X_B, X_C, X_D : \Omega \to \{1,2,3,4,5\}$$



$\frac{1}{5} = \frac{25}{125}$

$\frac{4}{2} \times \frac{2}{5} = \frac{40}{125}$

$\frac{4}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{36}{125}$

$\frac{4}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{24}{125}$.

Proof:

$X_B = X_A$.

$$P(X_B = X_A) = P(X_B = X_A = 1) * P(X_B = X_A = 2) + \cdots P(X_B = X_A = 5).$$

$$= \frac{1}{5} \times \frac{1}{5} + \cdots \frac{1}{5}$$

$$= \frac{1}{25} \times 5 = \frac{1}{5}$$

Prob. of choosing same number (At least 2):

$$1 - \frac{24}{125} = \frac{101}{25} \quad \because 1 - \text{Prob of all choose unique number.}$$

Q2(b)

```r
#Set the number of simulations
rounds <- 1000

# Initialize counter for rounds with at least one match
match_count <- 0

set.seed(1)
# Run the simulation n times
for (i in 1:n) {
  # Generate random numbers for four friends
  choices <- sample(1:5, 4, replace = TRUE)
  # Check if there is a match among the numbers chosen by the friends
  if (any(duplicated(choices))) {
    match_count <- match_count + 1 # Increment if there's at least one match
  }
}
# Calculate the probability as a percentage
probability_of_match <- (match_count / rounds) * 100
#Estimated probability
print(probability_of_match)#
```

Q3 :

- we have $X : \Omega \longrightarrow \mathbb{N}$ is a random Variable that takes integer Values with $X \geq 0$ and $E(X)$ is expectation.

$\longrightarrow$ The Expectation $E(X)$ of a non negative integer valued Variable X is :

$$\boxed{E(X) = \sum_{K=0}^{\infty} K \cdot \mathbb{P}(X=K)}$$

$\longrightarrow$ we can rewrite $K \cdot \mathbb{P}(X=K)$ as K the sum of indicator events

$$K = \sum_{n=1}^{K} 1$$

So now we have :

$$K \cdot \mathbb{P}(X=K) = \sum_{n=1}^{K} \mathbb{P}(X=K)$$

we replace this in our formula :

$$\boxed{E(X) = \sum_{K=0}^{\infty} \sum_{n=1}^{K} \mathbb{P}(X=K)}$$

- right now we are summing over K first and then over n $\longrightarrow$ we can re arrange the summation to sum over n first and then over K

In the Original Order we have $n \leq K$, in the new order, we'll fix n first and then sum over all K $(K \geq n)$

So :
$$\boxed{E(X) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mathbb{P}(X=k)}$$

This means we are first summing over n starting from $n=1$ and going to ∞. And for each fixed n we sum over k (only $k \geq n$)

- $\boxed{\sum_{k=n}^{\infty} \mathbb{P}(X=k)} \Rightarrow$ That is $\mathbb{P}(X \geq n)$

So we can rewrite the formula as :

$$\Rightarrow \boxed{E(X) = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)}$$

Q4 :

a)

- $\boxed{I_A^2 = I_A}$: since I_A only takes values 0 and 1, squaring I_A will not change its value

$\Rightarrow$ If $I_A(\omega) = 1$ then $I_A^2(\omega) = 1$

If $I_A(\omega) = 0$ then $I_A^2(\omega) = 0$

So $\boxed{I_A^2 = I_A}$

- $\boxed{I_{A\cap B} = I_A I_B}$

- $I_A I_B = 1$ if both $I_A = 1$ and $I_B = 1$, which happens exactly when $w \in A \cap B$.

- If $w \notin A \cap B$ then either (I_A or I_B) or (I_A and I_B) will be 0, so in this case $I_A \cdot I_B = 0$

$\Longrightarrow$ Conclusion $\boxed{I_{A\cap B} = I_A I_B}$

- $\boxed{I_{A\cup B} = I_A + I_B - I_A I_B}$

possible outcomes for w:

① $w \in A \cap B \Longrightarrow$ both A and B occur

$$I_A(w) = I_B(w) = 1$$

so: $I_A + I_B - I_A I_B = 2 - 1 = \boxed{1}$ reflects that $\boxed{w \in A \cup B}$

② only A occur $w \in A \setminus B$

we have $I_A = 1$ and $I_B = 0$

so $I_A + I_B - I_A I_B = 1 + 0 = 1$

reflects that $w \in A \cup B$ so $I_{A\cup B} = 1$.

③ only B occur: $w \in B \setminus A$

$I_A = 0$ and $I_B = 1$

so: $I_A + I_B - I_A I_B = 0 + 1 - (0 \cdot 1) = 1$ it matches $w \in A \cup B$

(a) Neither A nor B occur :

$$I_A = I_B = 0$$

so : $I_A + I_B - I_A I_B = 0$ → $\boxed{\text{reflects that } w \notin A \cup B \text{ so } I_{A\cup B} = 0}$

(b) $I_A \sim Ber(p)$; $p = \mathbb{P}(A)$

we have : $I_A \to 1$ if A occurs, $I_A \to 0$ if A don't occur

so : $\mathbb{P}(I_A = 1) = \mathbb{P}(A) = p$

$\mathbb{P}(I_A = 0) = \mathbb{P}(A^c) = 1 - p$

and this is exactly the probability mass function of a Bernoulli random variable with parameter p

$$\boxed{I_A \sim Bernoulli(p)} \text{ where } p = \mathbb{P}(A)$$

(c) relation $E(I_A) = \mathbb{P}(A)$:

we have $I_A \to 1$ if $w \in A$, $I_A \to 0$ if $w \notin A$

- Expectation of a discrete Random Variable X that take value x_1 and x_2 with $\mathbb{P}(X = x_1)$ and $P(X = x_2)$ is :

$$E(X) = \sum_k x_k \, \mathbb{P}(X = x_k)$$

so in our case I_A only takes 1 and 0 as values.

$$E(I_A) = 1 \times \mathbb{P}(I_A = 1) + 0 \times \mathbb{P}(I_A = 0)$$
$$= \mathbb{P}(I_A = 1)$$

$I_A = 1$ if $\omega \in A$ so:

$$\mathbb{P}(I_A = 1) = \mathbb{P}(\omega \in A) = \mathbb{P}(A)$$

so at the end we conclude that

$$E(I_A) = \mathbb{P}(A).$$

d) $X \sim Ber(p)$; $X = \mathbb{I}_A$

we have $X \begin{cases} \to 1 \ ; \ U < p \\ \to 0 \ ; \ U \geq p \end{cases}$

- since U is distributed on $[0,1]$, $\mathbb{P}(U < p) = p$

that means

$$\mathbb{P}(X = 1) = \mathbb{P}(U < p) = p$$

and $\mathbb{P}(X = 0) = \mathbb{P}(U \geq p) = 1 - p$

and this is the probability function of Bernoulli with parameter p $\boxed{X \sim Bernoulli(p)}$

now we have to show that $X = I_A$

$\rightarrow$ Def event $A = \{ \omega \in \Omega : U < p \}$ and

$$I_A = \begin{cases} 1 & \text{if } \omega \in A \quad (u < p) \\ 0 & \text{if } \omega \notin A \quad (u \geq p) \end{cases}$$

By construction :

$$X(\omega) = 1 \quad \text{if} \quad u < p$$
$$X(\omega) = 0 \quad \text{if} \quad u \geq p$$

that means $I_A = X$

Q5 :

we have: $Cov(Y, Z) = E(y - E[y])(z - E[z])$

using the linearity of expectation :

$$Cov(y, z) = E[YZ] - E[Y]E[z]$$

we know X_i has mean zero so Y and Z also have mean zero $\Rightarrow E[Y] = E[Z] = 0$

$\Rightarrow$ formula is $\boxed{cov(y, z) = E[YZ]}$

- now we have to calculate $E[YZ]$

$$Y = \sum_{i=1}^{n} x_i \quad \text{and} \quad Z = \sum_{j=1}^{n} a_j X_j$$

$$\Rightarrow \quad YZ = \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{j=1}^{n} a_j x_j \right)$$

$$YZ = \sum_{i=1}^{n} \sum_{j=1}^{n} a_j x_i x_j$$

$$E[YZ] = E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_j x_i x_j\right]$$

using linearity of expectation $= \sum_{i=1}^{n} \sum_{j=1}^{n} a_j \, E[\underbrace{x_i x_j}]$

since the variables are independent and have zero mean.

- if $i=j$ then $E[x_i x_j] = E[x_i^2]$
$= Var(x_i) = \sigma^2$

- if $x \neq j$ then $E(x_i x_j) = E(x_i] \, E(x_j]$
$= 0.0 = 0$

we have non zero terms when $i=j$
so the formula is:

$$E[YZ] = \sum_{i=1}^{n} a_i \, E[x_i^2] = \boxed{\sum_{i=1}^{n} a_i \sigma^2}$$

- we know $\sum_{i=1}^{n} a_i = 0$ so:

$$E[YZ] = \sigma^2 \sum_{i=1}^{n} a_i = \sigma^2 \times 0 = 0$$

and since $E[YZ]=0$ we have $Cov(Y,Z) = E[YZ] = 0$
so it means that Y and Z have zero covariance ✓

Q.6 Part IV. Simulation in R

(a) PMF $f(x) := P(X=x)$.

Variables including success on rolls, & loss. -1 is to indicate the lost bet.

$X: \Omega \to \{-1, 1, 2, 3\}$.

$S: \Omega \to \{0, 1, 2, 3\}$

As rolling dice for all scenarios will involve Binomial distribution.

(for 3 trials): $\sim \text{Binomial}(n=3, p=1/6)$.

$$P(S=k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

So,

$$P(X=-1) = P(S=0) = \frac{125}{216}$$

$$P(X=+1) = P(S=1) = \frac{75}{216}$$

$$P(X=+2) = P(S=2) = \frac{15}{216}$$

$$P(X=+3) = P(S=3) = \frac{1}{216}.$$

b) $E(X)$:

As: $f_x = \begin{cases} \frac{125}{216} & \text{for } x=-1, \\ 75/216 & \text{"} \quad \text{"} = 1, \\ 15/216 & \text{"} \quad \text{"} = 2, \\ 1/216 & \text{"} \quad \text{"} = 3. \end{cases}$

$$E(X) = (-1) \cdot \frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} = \frac{-17}{216}$$

$\simeq -0.078$, which is less than zero, shows Game isn't fair.

c) overall Profit : −704
average " : −0·0704

```r
set.seed(0)
nloop<-10000
a<-5
Win<-rep(NA,nloop)
for (k in 1:nloop){
  Dice<-sample(1:6,size=3,replace=TRUE)
  Count_a<-sum(Dice==a)
  Win[k]<-ifelse(Count_a==0,-1,Count_a)
}
sum(Win) ## overall
sum(Win)/nloop
```

40:19 (Top Level)

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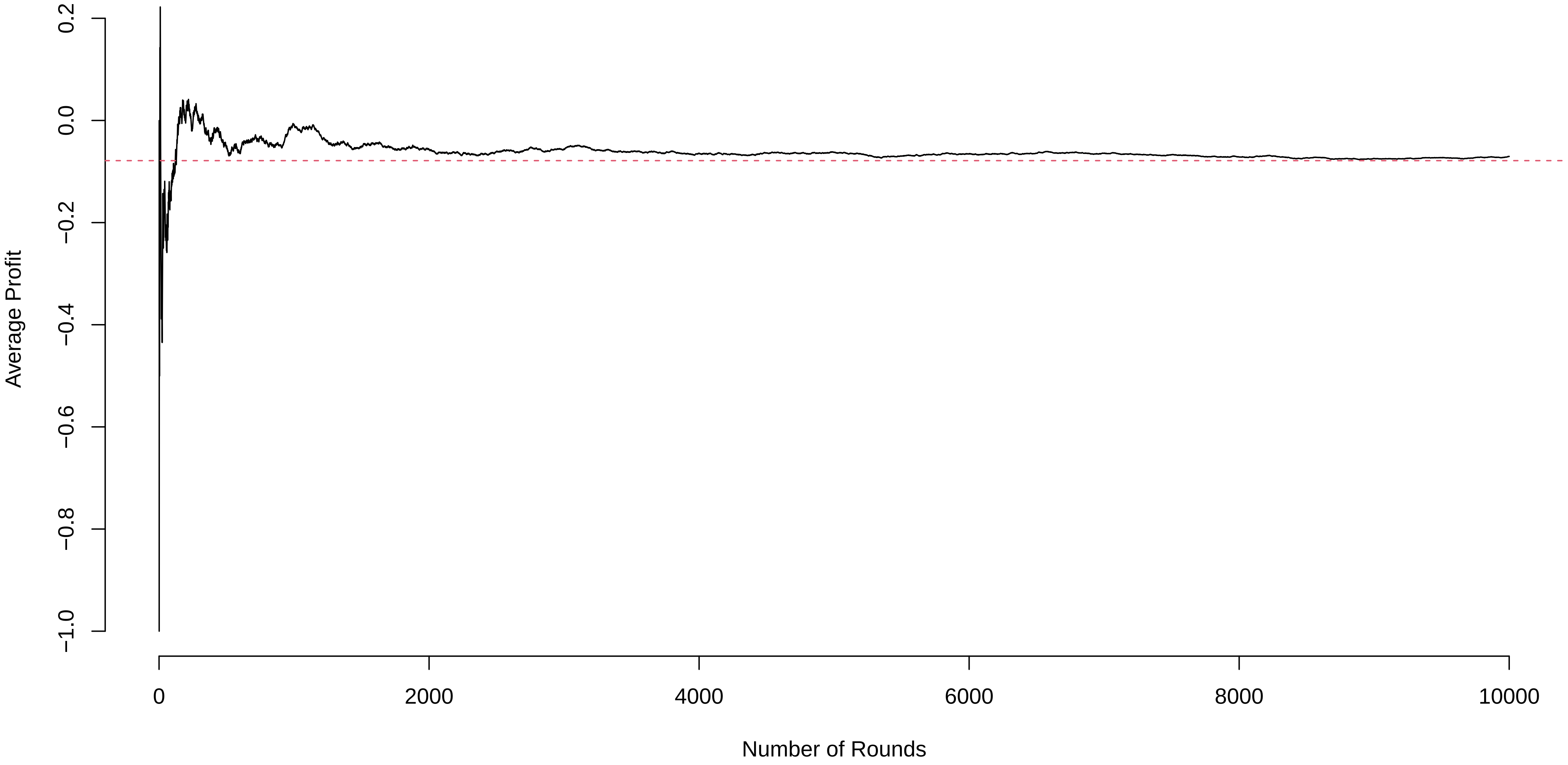
```
> nloop<-10000
> a<-5
> Win<-rep(NA,nloop)
> for (k in 1:nloop){
+   Dice<-sample(1:6,size=3,replace=TRUE)
+   Count_a<-sum(Dice==a)
+   Win[k]<-ifelse(Count_a==0,-1,Count_a)
+ }
> sum(Win) ## overall
[1] -704
> sum(Win)/nloop
[1] -0.0704
```

– 6d



Average Profit
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0
2000
4000
6000
8000
10000
Number of Rounds