Macroeconomics C:
Problem Set 1

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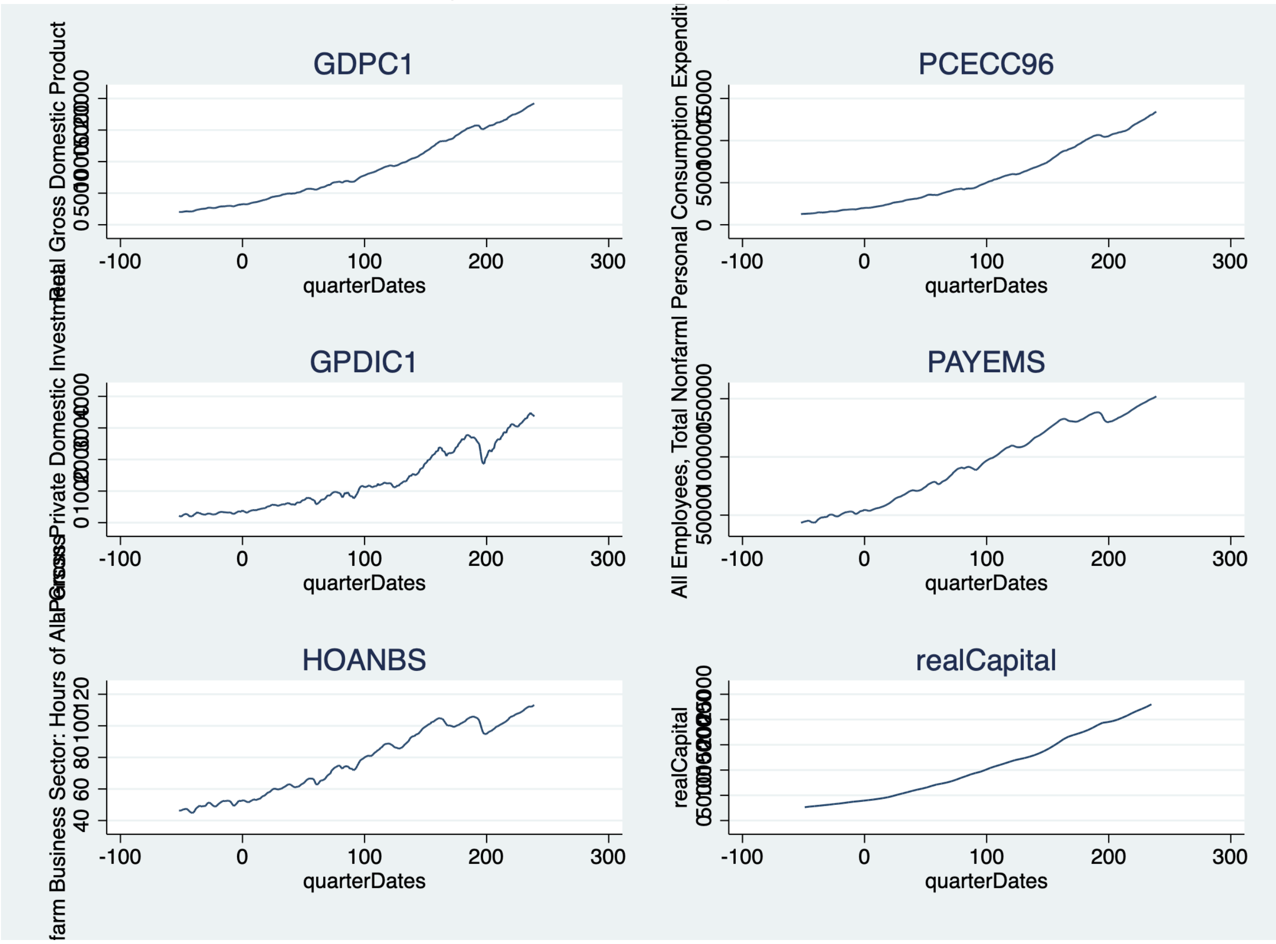
Question 1: Business Cycle Facts

Raw Series

Figure 1: Real Business Cycle Data



Logged Series

Figure 2: Real Business Cycle Data

HP Filtered Series

Figure 3: Filtered Consumption Data

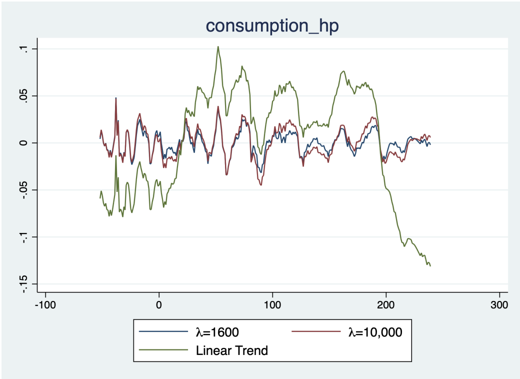


Figure 4: Filtered Employment Data

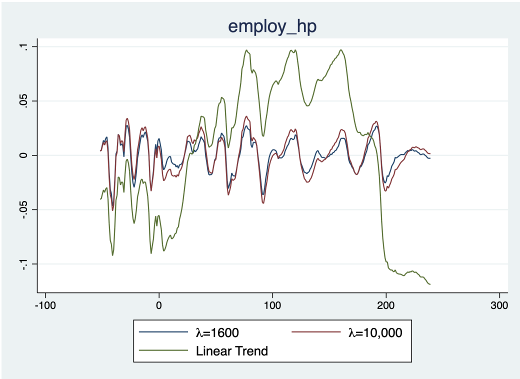


Figure 5: Filtered Output Data

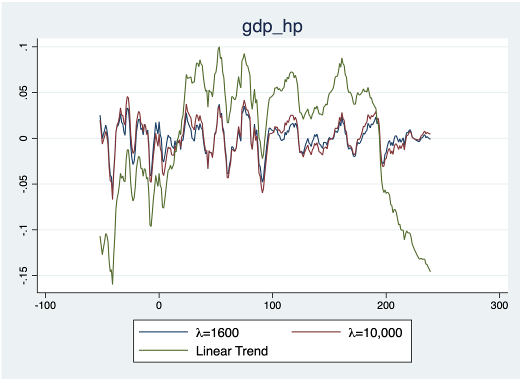


Figure 6: Filtered Employment Data

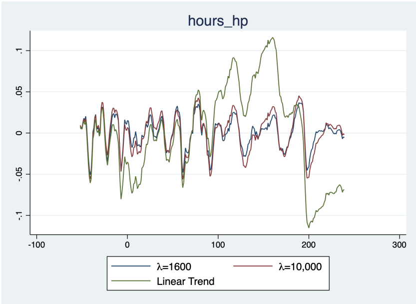


Figure 7: Filtered Capital Data

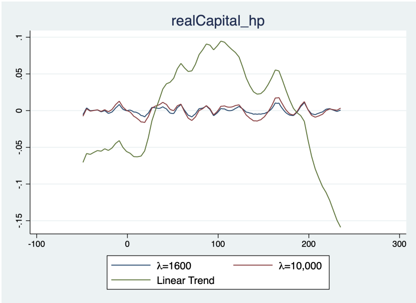


Figure 8: Filtered Investment Data

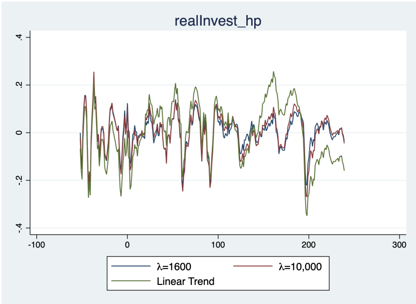


Figure 9: Filtered Solow Residual Data

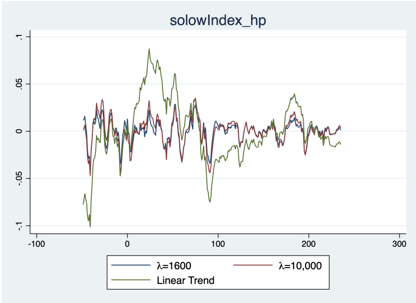


Figure 10: Standard Deviation and Correlation Tables

Lambda = 1600
Results Table

	Standard_Deviation	Relative_Standard_~n	First-Order_Autoco~n	Comtemporaneous_Co~h
Y	1.58	1.00	0.85	1.00
C	1.22	0.77	0.81	0.77
I	7.27	4.59	0.80	0.83
N	1.40	0.89	0.91	0.86
H	1.86	1.17	0.91	0.87
K	0.41	0.26	0.96	-0.05
SR	1.16	0.73	0.81	0.91

Lambda = 10000
Results Table

	Standard_Deviation	Relative_Standard_~n	First-Order_Autoco~n	Comtemporaneous_Co~h
Y	1.99	1.00	0.90	1.00
C	1.60	0.80	0.89	0.79
I	8.28	4.16	0.84	0.79
N	1.79	0.90	0.94	0.86
H	2.31	1.16	0.93	0.86
K	0.73	0.37	0.98	0.12
SR	1.45	0.73	0.87	0.89

Linear
Results Table

	Standard_Deviation	Relative_Standard_~n	First-Order_Autoco~n	Comtemporaneous_Co~h
Y	6.84	1.00	0.99	1.00
C	6.04	0.88	0.99	0.96
I	12.21	1.78	0.92	0.72
N	6.76	0.99	1.00	0.87
H	5.59	0.82	0.99	0.71
K	6.56	0.96	1.00	0.84
SR	3.34	0.49	0.97	0.48

Analysis

Regarding linear...

The linear filter yields the highest standard deviation. This is not surprising because the linear trend has no smoothing at all. It is effectively the deviation from a best fit line over the whole data set. The lambda parameter is a measure of how much "smoothing" the HP filter applies to achieve the trend. Linear is the smoothest a data series can get. The higher the lambda value, the more smoothing occurs. So comparing lambda = 1600 to lambda = 10000 to linear, it is no surprise that lambda = 1600 displays the least volatility. It may sound counter intuitive, but since there is less smoothing for the trend in this case, each data point is compared to the trend value averaged over a smaller amount of nearer data. When lambda = 10000, the trend value is created over a larger amount of data than when lamdba = 1600, but not over the entire data set like the linear case. What is smooth here, is the trend component (results of figure 10). What is jagged, is the cyclical component (figures 3 - 9).

Regarding the relative volatilities of the different data series...

The data here somewhat matches the results of Rebelo and King. Some ideas are consistent such as investment is much more volatile than output. This is clearly reflected in the relative standard deviations between output and investment. Although our data do not reflect it well, we expected to see consumption be much less volatile than output. It is only slightly less volatile than output in our data. It should be noted that our data includes more recent events than Rebelo and King including the early 2000's recession and the great recession (and subsequent strong recovery from the great recession). This all would lead us to expect higher volatility. But we would still expect output to be much more volatile than consumption. There is a lot that can be said about the data here. We also see that most terms are highly correlated with output and have fairly high levels of autocorrelation. The former is consistent with Rebelo and King. But it is interesting to note that our linear capital is actually highly correlated with output in the linear example. It seems like much of this can be attributed to the way we gathered data on capital. We used yearly data and then interpolated quarterly values. But the interpolation we used was linear. So our capital values are a little muddled by our data collection. Some other interesting ideas are hours worked is often very close to output in terms of volatility. The two are also highly correlated. This suggests that much of business cycles can be attributed to changes in hours worked.

Question 3: Practice Log Linearization

I'm sorry but some of my equation numbering is off. I attempted to latex for the first time.

(1) $Y = C + I + G + NX$ **Assume** $NX = 0$ **in steady state**

$$Y = C + I + G \tag{1}$$

Note this also means in steady state,

$$\overline{Y} = \overline{C} + \overline{I} + \overline{G} \tag{2}$$

Taking logs (a monotonic transformation) of equation (1) yields

$$\log Y = \log(C + I + G) \tag{3}$$

Applying a first order Taylor Approximation yields

$$\begin{aligned}\log \overline{Y} + \frac{Y - \overline{Y}}{\overline{Y}} &= \log\left(\overline{C} + \overline{I} + \overline{G}\right) + \frac{C - \overline{C}}{\overline{C} + \overline{I} + \overline{G}} \\ &+ \frac{I - \overline{I}}{\overline{C} + \overline{I} + \overline{G}} + \frac{G - \overline{G}}{\overline{C} + \overline{I} + \overline{G}}\end{aligned} \tag{4}$$

By equations (2) and (3),

$$\log \overline{Y} = \log\left(\overline{C} + \overline{I} + \overline{G}\right) \tag{5}$$

so (4) simplifies to

$$\frac{Y - \overline{Y}}{\overline{Y}} = \frac{C - \overline{C}}{\overline{C} + \overline{I} + \overline{G}} + \frac{I - \overline{I}}{\overline{C} + \overline{I} + \overline{G}} + \frac{G - \overline{G}}{\overline{C} + \overline{I} + \overline{G}} \tag{6}$$

We can define

$$\hat{y} = \frac{Y - \overline{Y}}{\overline{Y}} \tag{7}$$

so we get

$$\hat{y} = \frac{C - \overline{C}}{\overline{C} + \overline{I} + \overline{G}} + \frac{I - \overline{I}}{\overline{C} + \overline{I} + \overline{G}} + \frac{G - \overline{G}}{\overline{C} + \overline{I} + \overline{G}} \tag{8}$$

Now substituting in (2)

$$\hat{y} = \frac{C - \overline{C}}{\overline{Y}} + \frac{I - \overline{I}}{\overline{Y}} + \frac{G - \overline{G}}{\overline{Y}} \tag{9}$$

If we multiply each term on the RHS by 1, we get

$$\hat{y} = \frac{C - \overline{C}}{\overline{C}} \frac{\overline{C}}{\overline{Y}} + \frac{I - \overline{I}}{\overline{I}} \frac{\overline{I}}{\overline{Y}} + \frac{G - \overline{G}}{\overline{G}} \frac{\overline{G}}{\overline{Y}} \tag{10}$$

Applying equation 7 to the RHS yields

$$\hat{y} = \hat{c}\frac{\overline{C}}{\overline{Y}} + \hat{i}\frac{\overline{I}}{\overline{Y}} + \hat{g}\frac{\overline{G}}{\overline{Y}} \tag{11}$$

(2) $Y = (\alpha K^\rho + (1-\alpha)(AL)^\rho)^{\frac{1}{\rho}}$ **(CES production function; A is the level of technology; L is labor; K is capital; ρ, α are constants)**

$$Y = (\alpha K^\rho + (1-\alpha)(AL)^\rho)^{\frac{1}{\rho}} \tag{12}$$

Note this also means in steady state

$$\overline{Y} = (\alpha \overline{K}^\rho + (1-\alpha)(\overline{A}\overline{L})^\rho)^{\frac{1}{\rho}} \tag{13}$$

Taking logs (a monotonic transformation) of equation (1) yields

$$\log Y = \frac{1}{\rho} \log(\alpha K^\rho + (1-\alpha)(AL)^\rho) \tag{14}$$

Applying a first order Taylor Approximation yields

$$\begin{aligned}\log \overline{Y} + \frac{Y-\overline{Y}}{\overline{Y}} &= \frac{1}{\rho}\log\big(\alpha\overline{K}^\rho + (1-\alpha)(\overline{A}\overline{L})^\rho\big) + \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)(\overline{A}\overline{L})^\rho}(\alpha\rho\overline{K}^{\rho-1})(K-\overline{K}) \\ &+ \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)\rho(\overline{A}\overline{L})^\rho}(1-\alpha)\rho\overline{L}^\rho\overline{A}^{\rho-1}(A-\overline{A}) \\ &+ \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)\rho(\overline{A}\overline{L})^\rho}(1-\alpha)\rho\overline{A}^\rho\overline{L}^{\rho-1}(L-\overline{L})\end{aligned} \tag{15}$$

By equations (14) and (15),

$$\log \overline{Y} = \log\big(\alpha\overline{K}^\rho + (1-\alpha)(\overline{A}\overline{L})^\rho\big) \tag{16}$$

so (16) simplifies to

$$\begin{aligned}\frac{Y-\overline{Y}}{\overline{Y}} &= \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)(\overline{A}\overline{L})^\rho}(\alpha\rho\overline{K}^{\rho-1})(K-\overline{K}) \\ &+ \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)\rho(\overline{A}\overline{L})^\rho}(1-\alpha)\rho\overline{L}^\rho\overline{A}^{\rho-1}(A-\overline{A}) \\ &+ \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)\rho(\overline{A}\overline{L})^\rho}(1-\alpha)\rho\overline{A}^\rho\overline{L}^{\rho-1}(L-\overline{L})\end{aligned} \tag{17}$$

We can define

$$\hat{y} = \frac{Y-\overline{Y}}{\overline{Y}} \tag{18}$$

so we get

$$\begin{aligned}\hat{y} &= \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)(\overline{A}\overline{L})^\rho}(\alpha\rho\overline{K}^{\rho-1})(K-\overline{K}) \\ &+ \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)\rho(\overline{A}\overline{L})^\rho}(1-\alpha)\rho\overline{L}^\rho\overline{A}^{\rho-1}(A-\overline{A}) \\ &+ \frac{1}{\rho}\frac{1}{\alpha\overline{K}^\rho + (1-\alpha)\rho(\overline{A}\overline{L})^\rho}(1-\alpha)\rho\overline{A}^\rho\overline{L}^{\rho-1}(L-\overline{L})\end{aligned} \tag{19}$$

Now cancelling the ρ

$$\begin{aligned}\hat{y} = &\frac{1}{\alpha\overline{K}^{\rho} + (1-\alpha)(\overline{A}\overline{L})^{\rho}}(\alpha\overline{K}^{\rho-1})(K-\overline{K}) \\ &+ \frac{1}{\alpha\overline{K}^{\rho} + (1-\alpha)(\overline{A}\overline{L})^{\rho}}(1-\alpha)\overline{L}^{\rho}\overline{A}^{\rho-1}(A-\overline{A}) \\ &+ \frac{1}{\alpha\overline{K}^{\rho} + (1-\alpha)(\overline{A}\overline{L})^{\rho}}(1-\alpha)\overline{A}^{\rho}\overline{L}^{\rho-1}(L-\overline{L})\end{aligned} \tag{20}$$

Now substituting in equation (14)

$$\hat{y} = \frac{1}{\overline{Y}^{\rho}}(\alpha\overline{K}^{\rho-1})(K-\overline{K}) + \frac{1}{\overline{Y}^{\rho}}(1-\alpha)\overline{L}^{\rho}\overline{A}^{\rho-1}(A-\overline{A}) + \frac{1}{\overline{Y}^{\rho}}(1-\alpha)\overline{A}^{\rho}\overline{L}^{\rho-1}(L-\overline{L}) \tag{21}$$

If we multiply each term on the RHS by 1, we get

$$\hat{y} = \frac{1}{\overline{Y}^{\rho}}(\alpha\overline{K}^{\rho-1})(K-\overline{K})\frac{\overline{K}}{\overline{K}} + \frac{1}{\overline{Y}^{\rho}}(1-\alpha)\overline{L}^{\rho}\overline{A}^{\rho-1}(A-\overline{A})\frac{\overline{A}}{\overline{A}} + \frac{1}{\overline{Y}^{\rho}}(1-\alpha)\overline{A}^{\rho}\overline{L}^{\rho-1}(L-\overline{L})\frac{\overline{L}}{\overline{L}} \tag{22}$$

Applying equation (19) to the RHS yields

$$\hat{y} = \frac{\overline{K}}{\overline{Y}^{\rho}}(\alpha\overline{K}^{\rho-1})\hat{k} + \frac{\overline{A}}{\overline{Y}^{\rho}}(1-\alpha)\overline{L}^{\rho}\overline{A}^{\rho-1}\hat{a} + \frac{\overline{L}}{\overline{Y}^{\rho}}(1-\alpha)\overline{A}^{\rho}\overline{L}^{\rho-1}\hat{l} \tag{23}$$

Simplifying we get

$$\hat{y} = (\frac{\overline{K}}{\overline{Y}})^{\rho}(\alpha\overline{K}^{\rho-1})\hat{k} + (\frac{\overline{A}}{\overline{Y}})^{\rho}(1-\alpha)\overline{L}^{\rho}\overline{A}^{\rho-1}\hat{a} + (\frac{\overline{L}}{\overline{Y}})^{\rho}(1-\alpha)\overline{A}^{\rho}\overline{L}^{\rho-1}\hat{l} \tag{24}$$

$$\hat{y} = (\frac{\overline{K}}{\overline{Y}})^{\rho}\alpha\hat{k} + (\frac{\overline{A}}{\overline{Y}})^{\rho}(1-\alpha)\overline{L}^{\rho}\hat{a} + (\frac{\overline{L}}{\overline{Y}})^{\rho}(1-\alpha)\overline{A}^{\rho}\hat{l} \tag{25}$$

$$\hat{y} = (\frac{\overline{K}}{\overline{Y}})^{\rho}\alpha\hat{k} + (\frac{\overline{A}\overline{L}}{\overline{Y}})^{\rho}(1-\alpha)(\hat{a}+\hat{l}) \tag{26}$$

Note the first term on the RHS is like the MPK in steady state in percentage terms times the deviation of capital from steady state.

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(3) $K_t = (1-\delta)K_{t-1} + I_t - \psi(\frac{I_t}{K_{t-1}} - \delta)^2$

$$K_t = (1-\delta)K_{t-1} + I_t - \psi(\frac{I_t}{K_{t-1}} - \delta)^2 I_t \tag{27}$$

Imposing the steady state conditions $K_t = K_{t-1} = \overline{K}$ and $I_t = I_{t-1} = \bar{I}$

$$\overline{K} = (1-\delta)\overline{K} + \bar{I} - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2\bar{I} \tag{28}$$

Noticing $\frac{\bar{I}}{\bar{I}} = 1$, we can write

$$\overline{K} = (1-\delta)\overline{K} + \bar{I} - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2\bar{I} \tag{29}$$

Also in steady state, $\dfrac{\bar{I}}{\overline{K}} = \delta$, so we can write

$$\overline{K} = (1-\delta)\overline{K} + \bar{I} \tag{30}$$

Taking logs (a monotonic transformation) of equation (27) yields

$$\log K_t = \log\Big((1-\delta)K_{t-1} + I_t - \psi(\frac{I_t}{K_{t-1}} - \delta)^2 I_t\Big) \tag{31}$$

Applying this to the steady state yields

$$\log \overline{K} = \log\big((1-\delta)\overline{K} + \bar{I}\big) \tag{32}$$

Applying a first order Taylor Approximation yields

$$\log \overline{K} + \frac{K_t - \overline{K}}{\overline{K}} = \log\big((1-\delta)\overline{K} + \bar{I}\big) + \frac{1-\delta+\left(2\psi\frac{\bar{I}^2}{\overline{K}^2}\right)\left(\frac{\bar{I}}{\overline{K}} - \delta\right)}{(1-\delta)\overline{K}+\bar{I}}(K_{t-1} - \overline{K}) + \frac{1-\psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 - 2\psi\frac{\bar{I}}{\overline{K}}(\frac{\bar{I}}{\overline{K}} - \delta)}{(1-\delta)\overline{K}+\bar{I}}(I_t - \bar{I}) \tag{33}$$

By equation (32), $\log \overline{K} = \log\big((1-\delta)\overline{K} + \bar{I}\big)$ so we can remove the additive terms at the start of both the LHS and RHS to get

$$\frac{K_t - \overline{K}}{\overline{K}} = \frac{1-\delta+\left(2\psi\frac{\bar{I}^2}{\overline{K}^2}\right)\left(\frac{\bar{I}}{\overline{K}} - \delta\right)}{(1-\delta)\overline{K}+\bar{I}}(K_{t-1} - \overline{K}) + \frac{1-\psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 + 2\psi\frac{\bar{I}}{\overline{K}}(\frac{\bar{I}}{\overline{K}} - \delta)}{(1-\delta)\overline{K}+\bar{I}}(I_t - \bar{I}) \tag{34}$$

Let me now substitute in equation (30) on the RHS to get

$$\frac{K_t - \overline{K}}{\overline{K}} = \frac{1-\delta+\left(2\psi\frac{\bar{I}^2}{\overline{K}^2}\right)\left(\frac{\bar{I}}{\overline{K}} - \delta\right)}{\overline{K}}(K_{t-1} - \overline{K}) + \frac{1-\psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 + 2\psi\frac{\bar{I}}{\overline{K}}(\frac{\bar{I}}{\overline{K}} - \delta)}{\overline{K}}(I_t - \bar{I}) \tag{35}$$

But since in steady state, $\dfrac{\bar{I}}{\overline{K}} = \delta$, many of the terms on the RHS become zero in the numerator so we get

$$\frac{K_t - \overline{K}}{\overline{K}} = \frac{1-\delta}{\overline{K}}(K_{t-1} - \overline{K}) + \frac{1}{\overline{K}}(I_t - \bar{I}) \tag{36}$$

Defining the hatted terms of the form $\hat{k} = \frac{K - \overline{K}}{\overline{K}}$ and since $\dfrac{\bar{I}}{\overline{K}} = \delta$ we can rewrite equation (36) as

$$\hat{k} = (1-\delta)\hat{k} + \delta\hat{i} \tag{37}$$

(4) $K_t = (1-\delta)K_{t-1} + I_t - \psi(\frac{I_t}{K_{t-1}} - \delta)^2 I_t - \phi(\frac{I_t}{I_{t-1}} - 1)^2 I_t$

$$K_t = (1-\delta)K_{t-1} + I_t - \psi(\frac{I_t}{K_{t-1}} - \delta)^2 I_t - \phi(\frac{I_t}{I_{t-1}} - 1)^2 I_t \tag{38}$$

Imposing the steady state conditions $K_t = K_{t-1} = \overline{K}$ and $I_t = I_{t-1} = \bar{I}$

$$\overline{K} = (1-\delta)\overline{K} + \bar{I} - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 \bar{I} - \phi(\frac{\bar{I}}{\bar{I}} - 1)^2 \bar{I} \tag{39}$$

Noticing $\frac{\bar{I}}{\bar{I}} = 1$, we can write

$$\overline{K} = (1-\delta)\overline{K} + \bar{I} - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 \bar{I} \tag{40}$$

Also in steady state, $\frac{\bar{I}}{\overline{K}} = \delta$, so we can write

$$\overline{K} = (1-\delta)\overline{K} + \bar{I} \tag{41}$$

Taking logs (a monotonic transformation) of equation (41) yields

$$\log K_t = \log\Big((1-\delta)K_{t-1} + I_t - \psi(\frac{I_t}{K_{t-1}} - \delta)^2 I_t - \phi(\frac{I_t}{I_{t-1}} - 1)^2 I_t\Big) \tag{42}$$

Applying a first order Taylor Approximation yields

$$\begin{aligned}\log\overline{K} + \frac{K_t - \overline{K}}{\overline{K}} &= \log\big((1-\delta)\overline{K} + \bar{I}\big) + \frac{1 + \left(2\psi\frac{\bar{I}^2}{\overline{K}^2}\right)\left(\frac{\bar{I}}{\overline{K}} - \delta\right)}{(1-\delta)\overline{K} + \bar{I}}(K_{t-1} - \overline{K}) \\ &+ \frac{1 - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 - 2\psi\frac{\bar{I}}{\overline{K}}(\frac{\bar{I}}{\overline{K}} - \delta) + \phi(\frac{\bar{I}}{\bar{I}} - 1)^2 - 2\phi\frac{\bar{I}}{\bar{I}}(\frac{\bar{I}}{\bar{I}} - 1)}{(1-\delta)\overline{K} + \bar{I}}(I_t - \bar{I}) \\ &+ \frac{2\phi(\frac{\bar{I}}{\bar{I}})^2(\frac{\bar{I}}{\bar{I}} - 1)}{(1-\delta)\overline{K} + \bar{I}}(I_{t-1} - \bar{I})\end{aligned} \tag{43}$$

By equations (41) and (42), $\log\overline{K} = \log\big((1-\delta)\overline{K} + \bar{I}\big)$ so we can remove the additive terms at the start of both the LHS and RHS to get

$$\begin{aligned}\frac{K_t - \overline{K}}{\overline{K}} &= \frac{1 + \left(2\psi\frac{\bar{I}^2}{\overline{K}^2}\right)\left(\frac{\bar{I}}{\overline{K}} - \delta\right)}{(1-\delta)\overline{K} + \bar{I}}(K_{t-1} - \overline{K}) \\ &+ \frac{1 - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 + 2\psi\frac{\bar{I}}{\overline{K}}(\frac{\bar{I}}{\overline{K}} - \delta) + \phi(\frac{\bar{I}}{\bar{I}} - 1)^2 + 2\phi\frac{\bar{I}}{\bar{I}}(\frac{\bar{I}}{\bar{I}} - 1)}{(1-\delta)\overline{K} + \bar{I}}(I_t - \bar{I}) \\ &+ \frac{2\phi(\frac{\bar{I}}{\bar{I}})^2(\frac{\bar{I}}{\bar{I}} - 1)}{(1-\delta)\overline{K} + \bar{I}}(I_{t-1} - \bar{I})\end{aligned} \tag{44}$$

Let me now substitute in equation (41) on the RHS to get

$$\begin{aligned}\frac{K_t - \overline{K}}{\overline{K}} &= \frac{1 + \left(2\psi\frac{\bar{I}^2}{\overline{K}^2}\right)\left(\frac{\bar{I}}{\overline{K}} - \delta\right)}{\overline{K}}(K_{t-1} - \overline{K}) \\ &+ \frac{1 - \psi(\frac{\bar{I}}{\overline{K}} - \delta)^2 + 2\frac{\bar{I}}{\overline{K}}(\frac{\bar{I}}{\overline{K}} - \delta) + \phi(\frac{\bar{I}}{\bar{I}} - 1)^2 + 2\phi\frac{\bar{I}}{\bar{I}}(\frac{\bar{I}}{\bar{I}} - 1)}{\overline{K}}(I_t - \bar{I}) \\ &+ \frac{2\phi(\frac{\bar{I}}{\bar{I}})^2(\frac{\bar{I}}{\bar{I}} - 1)}{\overline{K}}(I_{t-1} - \bar{I})\end{aligned} \tag{45}$$

But since in steady state, $\frac{\bar{I}}{\bar{K}} = \delta$ and $I_{t-1} = I_t = \bar{I}$, many of the terms on the RHS become zero in the numerator so we get

$$\frac{K_t - \overline{K}}{\overline{K}} = \frac{1-\delta}{\overline{K}}(K_t - \overline{K}) + \frac{1}{\overline{K}}(I_t - \bar{I}) \tag{46}$$

Defining the hatted terms of the form $\hat{k} = \frac{K-\overline{K}}{\overline{K}}$ we can rewrite equation (48) as

$$\hat{k} = (1-\delta)\hat{k} + \delta\hat{i} \tag{47}$$

(5) $\exp(i_t) = (\frac{P_t}{P_{t-1}})^{\phi_\pi}(\frac{Y_t}{Y_{t-1}})^{\phi_y}\exp(\rho i_{t-1})$

$$\exp(i_t) = (\frac{P_t}{P_{t-1}})^{\phi_\pi}(\frac{Y_t}{Y_{t-1}})^{\phi_y}\exp(\rho i_{t-1}) \tag{48}$$

Imposing the balanced growth path conditions $P_t = P_{t-1} = \gamma_p P$ where γ_p is the inflation rate and $Y_t = Y_{t-1} = \gamma_y$ where γ_y is the output growth rate. Note that if this is a steady state, these growth rates are just equal to 1 (indicating no growth).

$$\exp(\bar{i}) = \gamma_p^{\phi_\pi}\gamma_y^{\phi_y}\exp(\rho\bar{i}) \tag{49}$$

Taking logs of equation (49) yields

$$i_t = \phi_\pi log(\frac{P_t}{P_{t-1}}) + \phi_y log(\frac{Y_t}{Y_{t-1}}) + \rho i_{t-1} \tag{50}$$

Applying a first order Taylor Approximation yields

$$\begin{aligned}\bar{i} + (i_t - \bar{i}) = {} & [\phi_\pi log(\gamma_p) + \phi_y log(\gamma_y) + \rho\bar{i}] \qquad (51)\\ & + \frac{\phi_\pi}{\gamma_p}(\frac{P_t}{P_{t-1}} - \gamma_p)\\ & + \frac{\phi_y}{\gamma_y}(\frac{Y_t}{Y_{t-1}} - \gamma_y) + \rho(i_{t-1} - \bar{i}) \qquad (52)\end{aligned}$$

Substituting in equation (50) in the steady state on the RHS yields

$$\bar{i} + (i_t - \bar{i}) = \bar{i} + \frac{\phi_\pi}{\gamma_p}(\frac{P_t}{P_{t-1}} - \gamma_p) + \frac{\phi_y}{\gamma_y}(\frac{Y_t}{Y_{t-1}} - \gamma_y) + \rho(i_{t-1} - \bar{i}) \tag{53}$$

Defining the hatted terms of the form $\frac{\hat{p_t}}{p_{t-1}} = \frac{\frac{P_t}{P_{t-1}} - \gamma_p}{\gamma_p}$

$$(i_t - \bar{i}) = \phi_\pi\frac{\hat{p_t}}{p_{t-1}} + \phi_y\frac{\hat{y_t}}{y_{t-1}} + \rho(i_{t-1} - \bar{i}) \tag{54}$$

Note that if $\gamma_p = 1$ and $\gamma_y = 1$, we can rewrite equation (54) as

$$(i_t - \bar{i}) = \phi_\pi\frac{\hat{p_t}}{p_{t-1}} + \phi_y\frac{\hat{y_t}}{y_{t-1}} + \rho(i_{t-1} - \bar{i}) \tag{55}$$

(6) $A \cdot F(L) = \left(\frac{\partial U(C,1-L)}{\partial L}\right) / \left(\frac{\partial U(C,1-L)}{\partial C}\right)$

From here on out, I will define $MU_{\bar{L}} \equiv MU_L\Big|_{L=\bar{L}} \equiv \left(\frac{\partial U(C,1-L)}{\partial L}\Big|_{L=\bar{L}}\right)$ and $MU_{\bar{C}} \equiv MU_C\Big|_{C=\bar{C}} \equiv \left(\frac{\partial U(C,1-L)}{\partial C}\Big|_{C=\bar{C}}\right)$

$$A \cdot F(L) = MU_L/MU_C \tag{56}$$

We can write the steady state as

$$\overline{A} \cdot F(\bar{L}) = MU_{\bar{L}}/MU_{\bar{C}} \tag{57}$$

Taking logs (a monotonic transformation) yields

$$\log A + \log F(L) = \log MU_L - \log MU_C \tag{58}$$

Applying this to the steady state yields

$$\log \overline{A} + \log F(\bar{L}) = \log MU_{\bar{L}} - \log MU_{\bar{C}} \tag{59}$$

Applying a first order Taylor Approximation yields

$$\begin{aligned} \log \overline{A} + \frac{1}{\overline{A}}(A - \overline{A}) + \log F(\bar{L}) + \frac{F'(\bar{L})}{F(\bar{L})}(L - \bar{L}) = \log MU_{\bar{L}} + (\frac{MU_{\bar{L}\bar{L}}}{MU_{\bar{L}}} - \frac{MU_{\bar{L}\bar{C}}}{MU_{\bar{C}}})(L - \bar{L}) \\ - \log MU_{\bar{C}} - (\frac{MU_{\bar{C}\bar{C}}}{MU_{\bar{C}}} - \frac{MU_{\bar{L}\bar{C}}}{MU_{\bar{L}}})(C - \overline{C}) \end{aligned} \tag{60}$$

By equation (59), I can remove some of the additive terms to get

$$\begin{aligned} \frac{1}{\overline{A}}(A - \overline{A}) + \frac{F'(\bar{L})}{F(\bar{L})}(L - \bar{L}) = (\frac{MU_{\bar{L}\bar{L}}}{MU_{\bar{L}}} - \frac{MU_{\bar{C}\bar{L}}}{MU_{\bar{C}}})(L - \bar{L}) \\ -(\frac{MU_{\bar{C}\bar{C}}}{MU_{\bar{C}}} - \frac{MU_{\bar{L}\bar{C}}}{MU_{\bar{L}}})(C - \overline{C}) \end{aligned} \tag{61}$$

Defining the hatted terms of the form $\hat{L} = \frac{L-\bar{L}}{\bar{L}}$, and cleverly multiplying by 1, we can rewrite the above as

$$\begin{aligned} \hat{a} + (\frac{\bar{L} \cdot F'(\bar{L})}{F(\bar{L})})\hat{l} = (\frac{MU_{\bar{L}\bar{L}}}{MU_{\bar{L}}} - \frac{MU_{\bar{C}\bar{L}}}{MU_{\bar{C}}})\overline{L}\hat{l} \\ -(\frac{MU_{\bar{C}\bar{C}}}{MU_{\bar{C}}} - \frac{MU_{\bar{L}\bar{C}}}{MU_{\bar{L}}})\overline{C}\hat{c} \end{aligned} \tag{62}$$

Combining the $\hat{l}$ terms yields

$$\hat{a} = \left(\frac{MU_{\bar{L}\bar{L}}}{MU_{\bar{L}}} - \frac{MU_{\bar{C}\bar{L}}}{MU_{\bar{C}}} - (\frac{\bar{L} \cdot F'(\bar{L})}{F(\bar{L})\bar{L}})\right)\overline{L}\hat{l} - (\frac{MU_{\bar{C}\bar{C}}}{MU_{\bar{C}}} - \frac{MU_{\bar{L}\bar{C}}}{MU_{\bar{L}}})\overline{C}\hat{c}$$

(7) $Y_t = K_t^{\alpha_t} L_t^{1-\alpha_t}$

$$Y_t = K_t^{\alpha_t} L_t^{1-\alpha_t} \tag{63}$$

We can write the steady state as

$$Y = K^{\alpha} L^{1-\alpha} \tag{64}$$

Taking logs (a monotonic transformation) yields

$$\log(Y_t) = \alpha_t \log(K_t) + (1-\alpha_t)\log(L_t) \tag{65}$$

Now applying a first order Taylor Approximation

$$\log Y + \frac{Y_t - Y}{Y} = \alpha \log K + \alpha \frac{K_t - K}{K} + (1-\alpha)\log L + (1-\alpha)\frac{L_t - L}{L} + (\alpha_t - \alpha)\log K - (\alpha_t - \alpha)\log L \tag{66}$$

Subtracting out the steady state terms yields

$$\frac{Y_t - Y}{Y} = \alpha \frac{K_t - K}{K} + (1-\alpha)\frac{L_t - L}{L} + (\alpha_t - \alpha)\log K - (\alpha_t - \alpha)\log L \tag{67}$$

Rewriting in hatted terms....

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{l}_t + (\alpha_t - \alpha)\log K - (\alpha_t - \alpha)\log L \tag{68}$$

I will now define $\tilde{\alpha} = \alpha_t - \alpha$ so I get

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{l}_t + \tilde{\alpha}_t \log K - \tilde{\alpha}_t \log L \tag{69}$$

Question 2: RBC Model part 1

Write up the first order condition and market clearing conditions for the RBC model in King and Rebelo (2000)

Utility function is given by equation 4.2 from King and Rebelo:

$$u(c_t, L_t) = \log(c_t) + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) \tag{1}$$

Production function is given by equation 3.21 from King and Rebelo but dividing both sides by X_t:

$$y_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} \tag{2}$$

Exogenous productivity growth is given by equation 4.1 from King and Rebelo:

$$\log(A_t) = \rho \log(A_{t-1}) + \log(\epsilon_t) \tag{3}$$

So the market clearing condition is given by the resource constraint:

$$c_t + \gamma k_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} + (1-\delta)k_{t-1} \tag{4}$$

So the Planner problem is defined as

$$\max_{\{c_t, k_t, N_t\}} \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \frac{\theta}{1-\eta}\left((1-N_t)^{1-\eta} - 1\right)\right]\right) \tag{5}$$

subject to $c_t + \gamma k_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} + (1-\delta)k_{t-1}$, $\log(A_t) = \rho \log(A_{t-1}) + \log(\epsilon_t)$ and $L_t = 1 - N_t$

Taking FOCs for the planner problem: With respect to consumption:

$$\frac{\beta^t}{c_t} = \lambda_t \tag{6}$$

With respect to capital:

$$\mathbb{E}\left[\lambda_{t+1}\left((1-\alpha)A_{t+1}N_{t+1}^{\alpha}k_t^{-\alpha} + (1-\delta)\right)\right] = \lambda_t \gamma \tag{7}$$

With respect to leisure:

$$\beta^t \theta (1-N_t)^{-\eta} = \lambda_t \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha-1} \tag{8}$$

Combining the consumption and capital FOCs:

$$\mathbb{E}\left[\frac{\beta^{t+1}}{c_{t+1}}\left((1-\alpha)A_{t+1}N_{t+1}^{\alpha}k_t^{-\alpha} + (1-\delta)\right)\right] = \frac{\beta^t}{c_t}\gamma \tag{9}$$

Because $U_{c,t} = \frac{1}{C_t}$, this is an Euler equation and dividing both sides by β^t:

$$\mathbb{E}\left[\beta U_{c,t+1}\left((1-\alpha)A_{t+1}N_{t+1}^{\alpha}k_t^{-\alpha} + (1-\delta)\right)\right] = \gamma U_{c,t} \tag{10}$$

Combining the consumption and leisure FOCs:

$$\beta^t \theta (1-N_t)^{-\eta} = \frac{\beta^t}{c_t} \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha-1} \tag{11}$$

$$\theta (1-N_t)^{-\eta} = \frac{1}{c_t} \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha-1} \tag{12}$$

Solving for the steady state

Now applying the steady state, I will drop all t subscripts and expectations.
Applying the steady state to the law of motion for the productivity trend and we know the expectation of $\epsilon = 0$ so we get

$$\log(A) = \rho \log(A) + \log(\epsilon) \log(A) \qquad = \rho \log(A) \tag{13}$$

Which implies

$$A = 1 \tag{14}$$

Here is the new Euler

$$\beta U_c \left((1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta)\right) = \gamma U_c \tag{15}$$
$$\beta \left((1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta)\right) = \gamma \tag{16}$$
$$(1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta) = \frac{\gamma}{\beta} \tag{17}$$
$$(1-\alpha)AN^\alpha k^{-\alpha} = \frac{\gamma}{\beta} - (1-\delta) \tag{18}$$
$$N^\alpha k^{-\alpha} = \left[\frac{\gamma}{\beta} - (1-\delta)\right] \frac{1}{(1-\alpha)A} \tag{19}$$
$$\frac{k}{N} = \left[\left(\frac{\gamma}{\beta} - (1-\delta)\right) \frac{1}{(1-\alpha)A}\right]^{\frac{-1}{\alpha}} \tag{20}$$
$$\frac{k}{N} = \left(\frac{(1-\alpha)A}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{21}$$
$$\frac{k}{N} = \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{22}$$

Notice the production function can be rewritten as

$$\frac{y}{N} = (\frac{k}{N})^{1-\alpha} \tag{23}$$
$$\frac{y}{N} = \left[\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}}\right]^{1-\alpha} \tag{24}$$

Now dividing resource constraint $c + \gamma k = k^{1-\alpha}N^\alpha + (1-\delta)k$ by k yields

$$\frac{c}{k} + \gamma = k^{-\alpha}N^\alpha + (1-\delta) \tag{25}$$
$$\frac{c}{k} = (\frac{k}{N})^{-\alpha} + (1-\delta) - \gamma \tag{26}$$

Plugging in the equation for $\frac{k}{N}$ yields

$$\frac{c}{k} = \left(\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \right)^{-\alpha} + (1-\delta) - \gamma \tag{27}$$

$$\frac{c}{k} = \left(\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right) \right)^{-1} + (1-\delta) - \gamma \tag{28}$$

$$\frac{c}{k} = \left(\frac{1-\alpha}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{-1} + (1-\delta) - \gamma \tag{29}$$

$$\frac{c}{k} = \left(\frac{\frac{\gamma}{\beta} - (1-\delta)}{1-\alpha} \right) + (1-\delta) - \gamma \tag{30}$$

Now dividing resource constraint $c + \gamma k = k^{1-\alpha} N^{\alpha} + (1-\delta)k$ by N yields:

$$\frac{c}{N} + \gamma \frac{k}{N} = k^{1-\alpha} N^{\alpha-1} + (1-\delta)\frac{k}{N} \tag{31}$$

$$\frac{c}{N} = \left(\frac{k}{N}\right)^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \frac{k}{N} \tag{32}$$

Plugging in the equation for $\frac{k}{N}$ yields

$$\frac{c}{N} = \left[\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \right]^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \tag{33}$$

Now doing the same for the labor leisure condition yields and plugging in that $\eta = 1$:

$$\theta (1-N)^{-1} = \frac{1}{c} \alpha A k^{1-\alpha} N^{\alpha-1} \tag{34}$$

$$\theta (1-N)^{-1} = \frac{\alpha}{c} \frac{y}{N} \tag{35}$$

Multiplying both sides by both sides by N

$$\theta N (1-N)^{-1} = \alpha \left(\frac{c}{N}\right)^{-1} \frac{y}{N} \tag{36}$$

$$N (1-N)^{-1} = \frac{\alpha}{\theta} \left(\frac{c}{N}\right)^{-1} \frac{y}{N} \tag{37}$$

$$\frac{1}{N} = 1 + \left[\frac{\alpha}{\theta} \left(\frac{c}{N}\right)^{-1} \frac{y}{N} \right]^{-1} \tag{38}$$

$$N = \left[1 + \left(\frac{\alpha}{\theta} \left(\frac{c}{N}\right)^{-1} \frac{y}{N} \right)^{-1} \right]^{-1} \tag{39}$$

Now that we've solved for N in terms of parameters, we can solve for k in $\frac{k}{N}$ and then we can solve for y in $\frac{y}{N}$. We can also solve for c in $\frac{c}{N}$. So now we've solved for all our variables in terms of parameters.

$$\frac{k}{N} = \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{40}$$

$$\frac{y}{N} = (\frac{k}{N})^{1-\alpha} \tag{41}$$

$$\frac{c}{N} = (\frac{k}{N})^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma\frac{k}{N} \tag{42}$$

$$N = \left[1 + \left(\frac{\alpha}{\theta}\left(\frac{c}{N}\right)^{-1}\frac{y}{N}\right)^{-1}\right]^{-1} \tag{43}$$

$$y_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} \tag{44}$$

MPK is the interest rate r so

$$r = (1-\alpha)k^{-\alpha}N^{\alpha} \tag{45}$$

$$r = (1-\alpha)\frac{y}{k} \tag{46}$$

$$r = \frac{\gamma}{\beta} - 1 + \delta \tag{47}$$

MPL is the wage rate w so

$$w = \alpha k^{1-\alpha}N^{\alpha-1} \tag{48}$$

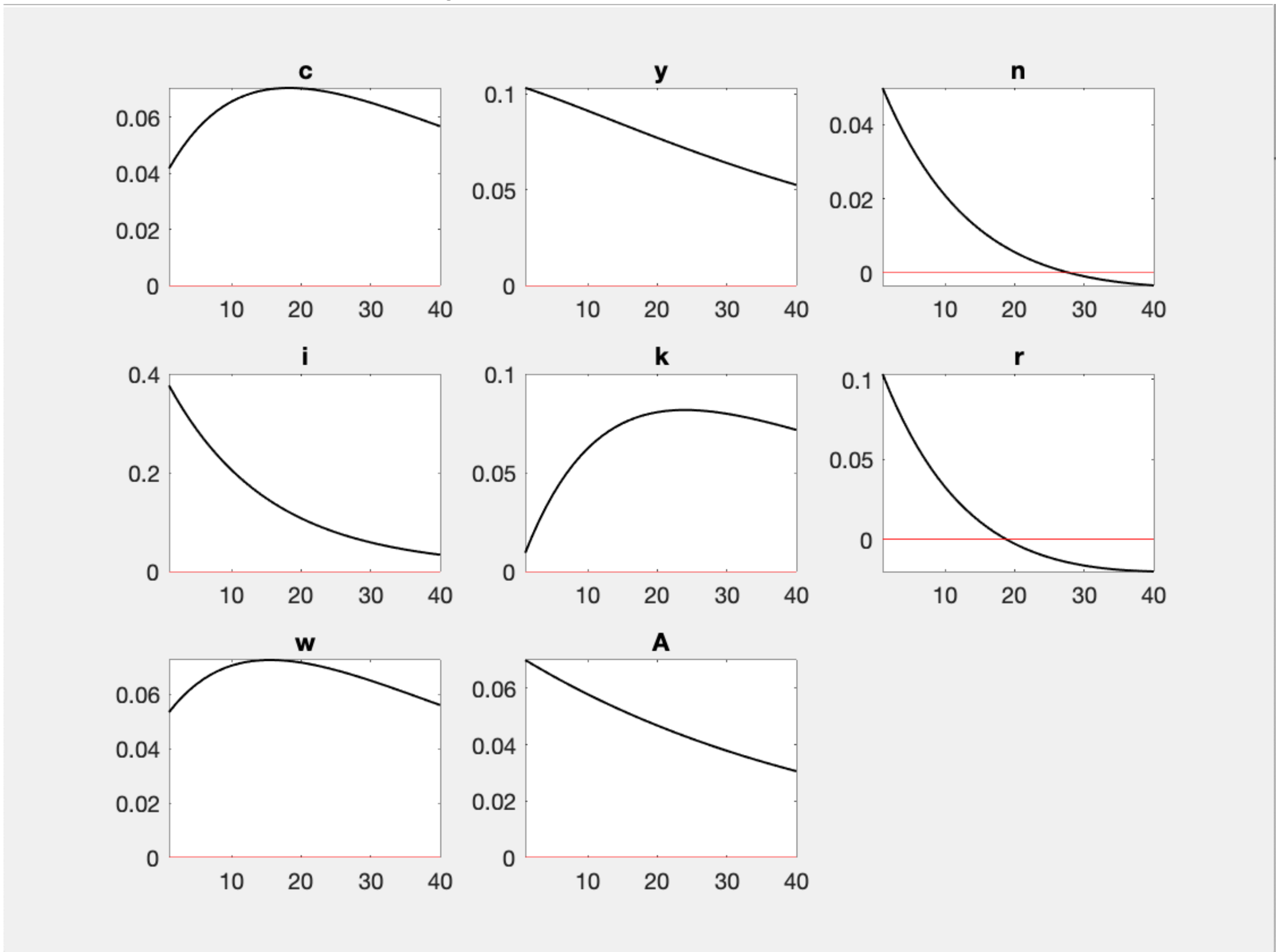
$$w = \alpha\left(\frac{k}{N}\right)^{1-\alpha} \tag{49}$$

Figure 11: Part 1 Steady States

```
STEADY-STATE RESULTS:

y                0.515539
k                3.79141
n                0.190104
c                0.420754
i                0.0947852
w                1.80792
r                0.0453252
A                1
```

Figure 12: Part 1 Impulse Response



Question 2: RBC Model part 2: non IES preferences

Write up the first order condition and market clearing conditions for the RBC model in King and Rebelo (2000)

The utility function where $\phi = 4$ is given by

$$u(c_t, L_t) = \frac{c_t^{1-\phi}}{1-\phi} + \theta \log(L_t) \tag{1}$$

Production function is given by equation 3.21 from King and Rebelo but dividing both sides by X_t:

$$y_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} \tag{2}$$

Exogenous productivity growth is given by equation 4.1 from King and Rebelo:

$$\log(A_t) = \rho \log(A_{t-1}) + \log(\epsilon_t) \tag{3}$$

So the market clearing condition is given by the resource constraint:

$$c_t + \gamma k_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} + (1-\delta)k_{t-1} \tag{4}$$

So the Planner problem is defined as

$$\max_{\{c_t, k_t, N_t\}} \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\phi}}{1-\phi} + \theta \log(1-N_t)\right) \tag{5}$$

subject to $c_t + \gamma k_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} + (1-\delta)k_{t-1}$, $\log(A_t) = \rho \log(A_{t-1}) + \log(\epsilon_t)$ and $L_t = 1 - N_t$

Taking FOCs for the planner problem: With respect to consumption:

$$\frac{\beta^t}{c_t^{\phi}} = \lambda_t \tag{6}$$

With respect to capital:

$$\mathbb{E}\left[\lambda_{t+1}\left((1-\alpha)A_{t+1}N_{t+1}^{\alpha}k_t^{-\alpha} + (1-\delta)\right)\right] = \lambda_t \gamma \tag{7}$$

With respect to leisure:

$$-\beta^t \frac{\theta}{1-N_t} = \lambda_t \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha-1} \tag{8}$$

Combining the consumption and capital FOCs:

$$\mathbb{E}\left[\frac{\beta^{t+1}}{c_{t+1}^{-\phi}}\left((1-\alpha)A_{t+1}N_{t+1}^{\alpha}k_t^{-\alpha} + (1-\delta)\right)\right] = \frac{\beta^t}{c_t^{\phi}}\gamma \tag{9}$$

Because $U_{c,t} = \frac{1}{C_t}$, this is an Euler equation and dividing both sides by β^t:

$$\mathbb{E}\left[\beta U_{c,t+1}\left((1-\alpha)A_{t+1}N_{t+1}^{\alpha}k_t^{-\alpha} + (1-\delta)\right)\right] = \gamma U_{c,t} \tag{10}$$

Combining the consumption and leisure FOCs:

$$\beta^t \frac{\theta}{1-N_t} = \frac{\beta^t}{c_t^{\phi}} \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha-1} \tag{11}$$

$$\frac{\theta}{1-N_t} = \frac{1}{c_t^{\phi}} \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha-1} \tag{12}$$

Solving for the steady state

Now applying the steady state, I will drop all t subscripts and expectations.
Applying the steady state to the law of motion for the productivity trend and we know the expectation of $\epsilon = 0$ so we get

$$\log(A) = \rho \log(A) + \log(\epsilon) \tag{13}$$

$$\log(A) = \rho \log(A) \tag{14}$$

Which implies

$$A = 1 \tag{15}$$

Here is the new Euler

$$\beta U_c \left((1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta)\right) = \gamma U_c \tag{16}$$
$$\beta \left((1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta)\right) = \gamma \tag{17}$$
$$(1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta) = \frac{\gamma}{\beta} \tag{18}$$
$$(1-\alpha)AN^\alpha k^{-\alpha} = \frac{\gamma}{\beta} - (1-\delta) \tag{19}$$
$$N^\alpha k^{-\alpha} = \left[\frac{\gamma}{\beta} - (1-\delta)\right] \frac{1}{(1-\alpha)A} \tag{20}$$
$$\frac{k}{N} = \left[\left(\frac{\gamma}{\beta} - (1-\delta)\right) \frac{1}{(1-\alpha)A}\right]^{\frac{-1}{\alpha}} \tag{21}$$
$$\frac{k}{N} = \left(\frac{(1-\alpha)A}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{22}$$
$$\frac{k}{N} = \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{23}$$

Notice the production function can be rewritten as

$$\frac{y}{N} = (\frac{k}{N})^{1-\alpha} \tag{24}$$
$$\frac{y}{N} = \left[\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}}\right]^{1-\alpha} \tag{25}$$

Now dividing resource constraint $c + \gamma k = k^{1-\alpha}N^\alpha + (1-\delta)k$ by k yields

$$\frac{c}{k} + \gamma = k^{-\alpha}N^\alpha + (1-\delta) \tag{26}$$
$$\frac{c}{k} = (\frac{k}{N})^{-\alpha} + (1-\delta) - \gamma \tag{27}$$

Plugging in the equation for $\frac{k}{N}$ yields

$$\frac{c}{k} = \left((\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)})^{\frac{1}{\alpha}} \right)^{-\alpha} + (1-\delta) - \gamma \tag{28}$$

$$\frac{c}{k} = \left((\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}) \right)^{-1} + (1-\delta) - \gamma \tag{29}$$

$$\frac{c}{k} = \left(\frac{1-\alpha}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{-1} + (1-\delta) - \gamma \tag{30}$$

$$\frac{c}{k} = \left(\frac{\frac{\gamma}{\beta} - (1-\delta)}{1-\alpha} \right) + (1-\delta) - \gamma \tag{31}$$

Now dividing resource constraint $c + \gamma k = k^{1-\alpha} N^{\alpha} + (1-\delta)k$ by N yields:

$$\frac{c}{N} + \gamma \frac{k}{N} = k^{1-\alpha} N^{\alpha-1} + (1-\delta)\frac{k}{N} \tag{32}$$

$$\frac{c}{N} = (\frac{k}{N})^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \frac{k}{N} \tag{33}$$

Plugging in the equation for $\frac{k}{N}$ yields

$$\frac{c}{N} = \left[\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \right]^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \tag{34}$$

Now doing the same for the labor leisure condition yields

$$\frac{\theta}{1-N} = \frac{1}{c_t^{\phi}} \alpha A_t k_{t-1}^{1-\alpha} N^{\alpha-1} \tag{35}$$

$$\frac{\theta}{1-N} = \frac{\alpha}{c^{\phi}} \frac{y}{N} \tag{36}$$

$$\theta = \frac{\alpha}{c^{\phi}} \frac{y}{N}(1-N) \tag{37}$$

Plugging in $N = 0.2$, we get $\theta = 43.629$.

Now that we've solved for N in terms of parameters, we can solve for k in $\frac{k}{N}$ and then we can solve for y in $\frac{y}{N}$. We can also solve for c in $\frac{c}{N}$. So now we've solved for all our variables in terms of parameters.

$$\frac{k}{N} = \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \tag{38}$$

$$\frac{y}{N} = (\frac{k}{N})^{1-\alpha} \tag{39}$$

$$\frac{c}{N} = (\frac{k}{N})^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \frac{k}{N} \tag{40}$$

$$N = 0.2 \tag{41}$$

$$y_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} \tag{42}$$

MPK is the interest rate r so

$$r = (1-\alpha)k^{-\alpha}N^{\alpha} \tag{43}$$

$$r = (1-\alpha)\frac{y}{k} \tag{44}$$

$$r = \frac{\gamma}{\beta} - 1 + \delta \tag{45}$$

MPL is the wage rate w so

$$w = \alpha k^{1-\alpha} N^{\alpha-1} \tag{46}$$

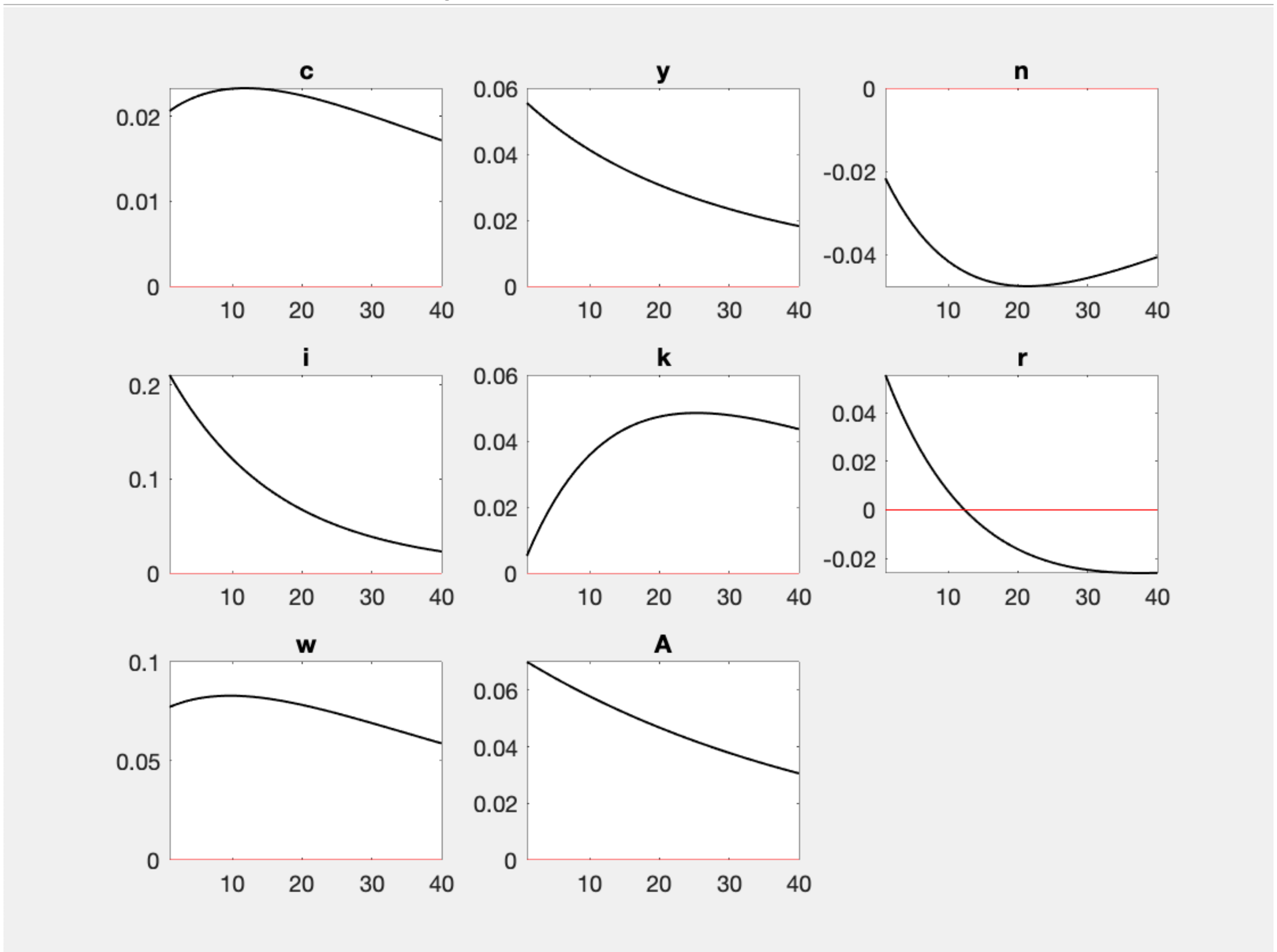
$$w = \alpha \left(\frac{k}{N}\right)^{1-\alpha} \tag{47}$$

Figure 13: Part 2 Steady States

```
STEADY-STATE RESULTS:

y        0.523933
k        3.85314
n        0.1932
c        0.427605
i        0.0963285
w        1.80792
r        0.0453252
A        1
```

Figure 14: Part 2 Impulse Response



Here we have $\gamma = 4$. This means lower intertemporal elasticity of substitution. This means consumers are less willing to substitute consumption between time periods. In other words, they want smooth consumption relative to consumers with $\gamma = 1$ like we saw in the first part of this problem. So when there is a productivity shock, the wage needs to rise higher to induce workers to work more. Workers would prefer to consumption smooth by taking more leisure if they had a positive productivity shock. That is reflected in the negative response of n and in the higher response of wage relative to the impulse responses in the first problem. In line with this, we also see a lower response in terms of positive changes to output, capital, and investment here relative to part 1.

Question 2: RBC Model part 3

Write up the first order condition and market clearing conditions for the new utility function: Shock to disutility to labor

Utility function is given by equation 4.2 from King and Rebelo:

$$u(c_t, L_t) = \log(c_t) + \theta e^{\epsilon_t^L} \log(L_t) \tag{1}$$

Where ϵ_t^L is a shock to the disutility of labor, which follows an AR(1) process

$$\epsilon_t^L = \rho^L \epsilon_{t-1}^L + e_t^L \tag{2}$$

Production function is given by equation 3.21 from King and Rebelo but dividing both sides by X_t:

$$y_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} \tag{3}$$

Exogenous productivity growth is given by equation 4.1 from King and Rebelo:

$$\log(A_t) = \rho \log(A_{t-1}) + \log(\epsilon_t) \tag{4}$$

So the market clearing condition is given by the resource constraint:

$$c_t + \gamma k_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} + (1-\delta) k_{t-1} \tag{5}$$

So the Planner problem is defined as

$$\max_{\{c_t, k_t, N_t\}} \mathbb{E} \left(\sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \theta e^{\epsilon_t^L} \log(1 - N_t) \right] \right) \tag{6}$$

subject to $c_t + \gamma k_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} + (1-\delta) k_{t-1}$, $\log(A_t) = \rho \log(A_{t-1}) + \log(\epsilon_t)$, $\epsilon_t^L = \rho^L \epsilon_{t-1}^L + e_t^L$, and $L_t = 1 - N_t$

Taking FOCs for the planner problem: With respect to consumption:

$$\frac{\beta^t}{c_t} = \lambda_t \tag{7}$$

With respect to capital:

$$\mathbb{E}\left[\lambda_{t+1} \left((1-\alpha) A_{t+1} N_{t+1}^{\alpha} k_t^{-\alpha} + (1-\delta) \right) \right] = \lambda_t \gamma \tag{8}$$

With respect to leisure:

$$\beta^t \theta e^{\epsilon_t^L} (1 - N_t)^{-1} = \lambda_t \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha - 1} \tag{9}$$

Combining the consumption and capital FOCs:

$$\mathbb{E}\left[\frac{\beta^{t+1}}{c_{t+1}} \left((1-\alpha) A_{t+1} N_{t+1}^{\alpha} k_t^{-\alpha} + (1-\delta) \right) \right] = \frac{\beta^t}{c_t} \gamma \tag{10}$$

Because $U_{c,t} = \frac{1}{C_t}$, this is an Euler equation and dividing both sides by β^t:

$$\mathbb{E}\left[\beta U_{c,t+1} \left((1-\alpha) A_{t+1} N_{t+1}^{\alpha} k_t^{-\alpha} + (1-\delta) \right) \right] = \gamma U_{c,t} \tag{11}$$

Combining the consumption and leisure FOCs:

$$\beta^t \theta e^{\epsilon_t^L} (1 - N_t)^{-1} = \frac{\beta^t}{c_t} \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha - 1} \tag{12}$$

$$\theta e^{\epsilon_t^L} (1 - N_t)^{-1} = \frac{1}{c_t} \alpha A_t k_{t-1}^{1-\alpha} N_t^{\alpha - 1} \tag{13}$$

Solving for the steady state

Now applying the steady state, I will drop all t subscripts and expectations.
Applying the steady state to the law of motion for the productivity trend and we know the expectation of $\epsilon = 0$ so we get

$$\log(A) = \rho\log(A) + \log(\epsilon)\log(A) \qquad = \rho\log(A) \tag{14}$$

Which implies

$$A = 1 \tag{15}$$

Here is the new Euler

$$\beta U_c \big((1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta)\big) = \gamma U_c \tag{16}$$
$$\beta \big((1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta)\big) = \gamma \tag{17}$$
$$(1-\alpha)AN^\alpha k^{-\alpha} + (1-\delta) = \frac{\gamma}{\beta} \tag{18}$$
$$(1-\alpha)AN^\alpha k^{-\alpha} = \frac{\gamma}{\beta} - (1-\delta) \tag{19}$$
$$N^\alpha k^{-\alpha} = \left[\frac{\gamma}{\beta} - (1-\delta)\right]\frac{1}{(1-\alpha)A} \tag{20}$$
$$\frac{k}{N} = \left[\left(\frac{\gamma}{\beta} - (1-\delta)\right)\frac{1}{(1-\alpha)A}\right]^{\frac{-1}{\alpha}} \tag{21}$$
$$\frac{k}{N} = \left(\frac{(1-\alpha)A}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{22}$$
$$\frac{k}{N} = \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{23}$$

Notice the production function can be rewritten as

$$\frac{y}{N} = (\frac{k}{N})^{1-\alpha} \tag{24}$$
$$\frac{y}{N} = \left[\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}}\right]^{1-\alpha} \tag{25}$$

Now dividing resource constraint $c + \gamma k = k^{1-\alpha}N^\alpha + (1-\delta)k$ by k yields

$$\frac{c}{k} + \gamma = k^{-\alpha}N^\alpha + (1-\delta) \tag{26}$$
$$\frac{c}{k} = (\frac{k}{N})^{-\alpha} + (1-\delta) - \gamma \tag{27}$$

Plugging in the equation for $\frac{k}{N}$ yields

$$\frac{c}{k} = \left((\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)})^{\frac{1}{\alpha}} \right)^{-\alpha} + (1-\delta) - \gamma \tag{28}$$

$$\frac{c}{k} = \left((\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}) \right)^{-1} + (1-\delta) - \gamma \tag{29}$$

$$\frac{c}{k} = \left(\frac{1-\alpha}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{-1} + (1-\delta) - \gamma \tag{30}$$

$$\frac{c}{k} = \left(\frac{\frac{\gamma}{\beta} - (1-\delta)}{1-\alpha} \right) + (1-\delta) - \gamma \tag{31}$$

Now dividing resource constraint $c + \gamma k = k^{1-\alpha} N^{\alpha} + (1-\delta)k$ by N yields:

$$\frac{c}{N} + \gamma \frac{k}{N} = k^{1-\alpha} N^{\alpha-1} + (1-\delta)\frac{k}{N} \tag{32}$$

$$\frac{c}{N} = (\frac{k}{N})^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \frac{k}{N} \tag{33}$$

Plugging in the equation for $\frac{k}{N}$ yields

$$\frac{c}{N} = \left[\left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \right]^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)} \right)^{\frac{1}{\alpha}} \tag{34}$$

Now doing the same for the labor leisure condition yields and plugging in that $\eta = 1$:

$$\theta e^{\epsilon_t^L} (1-N)^{-1} = \frac{1}{c} \alpha A k^{1-\alpha} N^{\alpha-1} \tag{35}$$

$$\theta e^{\epsilon_t^L} (1-N)^{-1} = \frac{\alpha}{c} \frac{y}{N} \tag{36}$$

Multiplying both sides by both sides by N

$$\theta e^{\epsilon_t^L} N (1-N)^{-1} = \alpha \left(\frac{c}{N} \right)^{-1} \frac{y}{N} \tag{37}$$

$$N(1-N)^{-1} = \frac{\alpha}{\theta e^{\epsilon_t^L}} \left(\frac{c}{N} \right)^{-1} \frac{y}{N} \tag{38}$$

$$\frac{1}{N} = 1 + \left[\frac{\alpha}{\theta e^{\epsilon_t^L}} \left(\frac{c}{N} \right)^{-1} \frac{y}{N} \right]^{-1} \tag{39}$$

$$N = \left[1 + \left(\frac{\alpha}{\theta e^{\epsilon_t^L}} \left(\frac{c}{N} \right)^{-1} \frac{y}{N} \right)^{-1} \right]^{-1} \tag{40}$$

Now that we've solved for N in terms of parameters, we can solve for k in $\frac{k}{N}$ and then we can solve for y in $\frac{y}{N}$. We can also solve for c in $\frac{c}{N}$. So now we've solved for all our variables in terms of parameters.

$$\frac{k}{N} = \left(\frac{(1-\alpha)}{\frac{\gamma}{\beta} - (1-\delta)}\right)^{\frac{1}{\alpha}} \tag{41}$$

$$\frac{y}{N} = (\frac{k}{N})^{1-\alpha} \tag{42}$$

$$\frac{c}{N} = (\frac{k}{N})^{1-\alpha} + (1-\delta)\frac{k}{N} - \gamma\frac{k}{N} \tag{43}$$

$$N = \left[1 + \left(\frac{\alpha}{\theta e^{\epsilon_t^L}}\left(\frac{c}{N}\right)^{-1}\frac{y}{N}\right)^{-1}\right]^{-1} \tag{44}$$

$$y_t = A_t k_{t-1}^{1-\alpha} N_t^{\alpha} \tag{45}$$

MPK is the interest rate r so

$$r = (1-\alpha)k^{-\alpha}N^{\alpha} \tag{46}$$

$$r = (1-\alpha)\frac{y}{k} \tag{47}$$

$$r = \frac{\gamma}{\beta} - 1 + \delta \tag{48}$$

MPL is the wage rate w so

$$w = \alpha k^{1-\alpha}N^{\alpha-1} \tag{49}$$

$$w = \alpha\left(\frac{k}{N}\right)^{1-\alpha} \tag{50}$$

Figure 15: Part 3 Steady States

```
STEADY-STATE RESULTS:

y                 0.515539
k                 3.79141
n                 0.190104
c                 0.420754
i                 0.0947852
w                 1.80792
r                 0.0453252
A                 1
eepsl             1
```

Figure 16: Part 3 Impulse Response to Productivity Shock

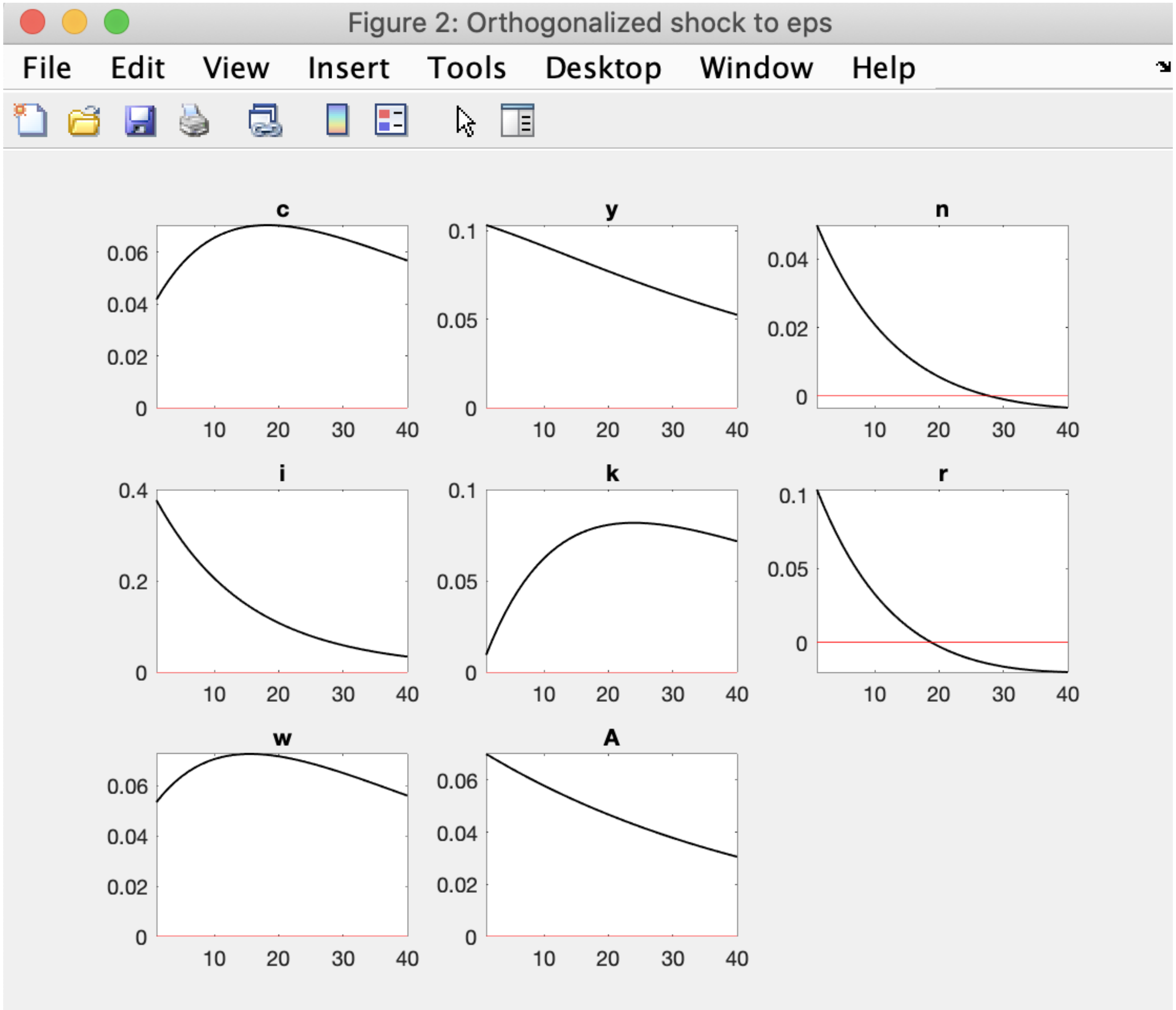
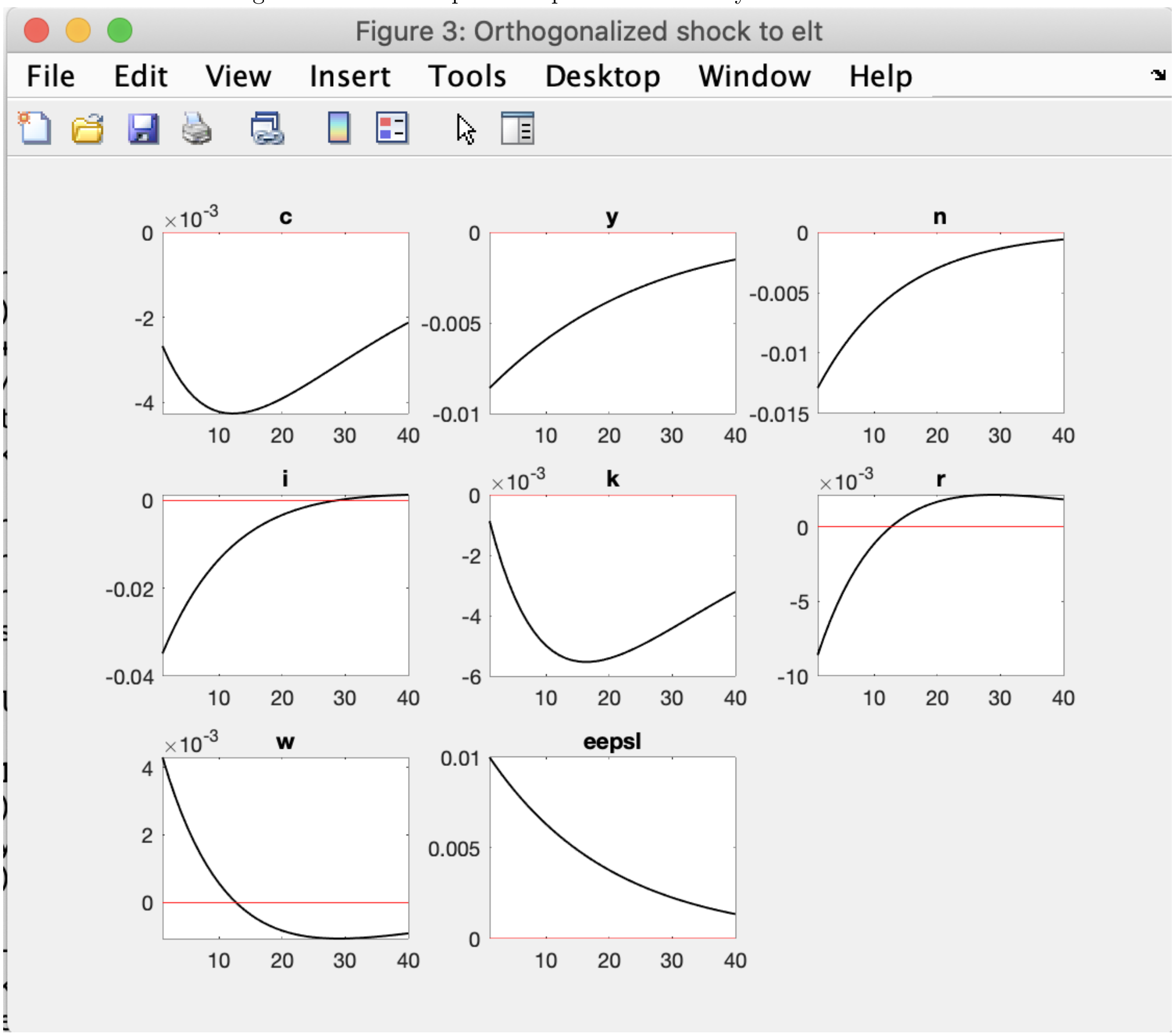


Figure 17: Part 3 Impulse Response to Disutility of Labor Shock



When there is a shock to disutility of labor, we see that workers cut back there hours. This makes sense because now they gain more utility from leisure. Since hours worked decreases, marginal product of labor increases so we see wage go up slightly. We also see investment,consumption, capital, and the interest rate all decrease as expected. What is interesting to note is that we see a delayed reaction to the shock in capital. This empirically makes sense as capital is largely set and there are adjustment costs. We are actually seeing a disutility of labor shock in some ways now with the coronavirus pandemic.

One thing that is interesting is the relative magnitude of these shocks. These impulse are almost a level of magnitude less than the responses to the actual productivity shocks. This suggests that while shocks to disutility of labor have an impact, they are unlikely to provide the volatility we need on their own to mimic the empirical results in the data. But because they are another valid source of volatility with economically sensible impulse responses, they could be included in an RBC model. So on one hand, they help, but on the other hand, they don't help much.