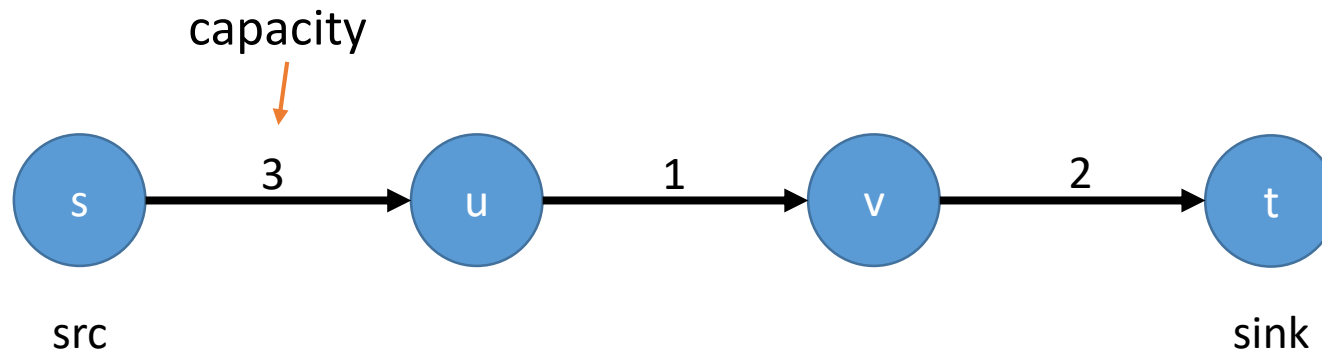


Push-Relabel Algorithm

COMP5112/MSBD5009 assignments tutorial

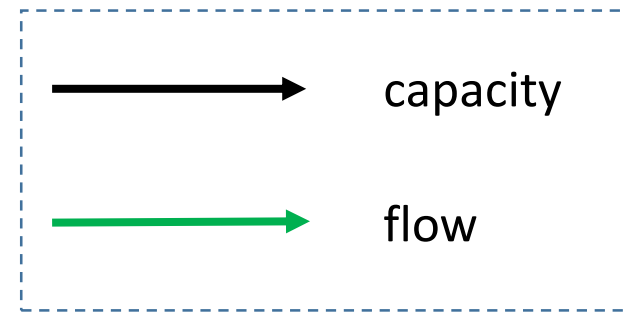
Maximum Flow

- A flow network is a directed graph, where each edge has a capacity and each edge receives a flow. The amount of flow on an edge cannot exceed the capacity of the edge.
- Maximum flow problems involve finding a feasible flow through a single-source (src) and a single-sink (sink) flow network that is maximum.

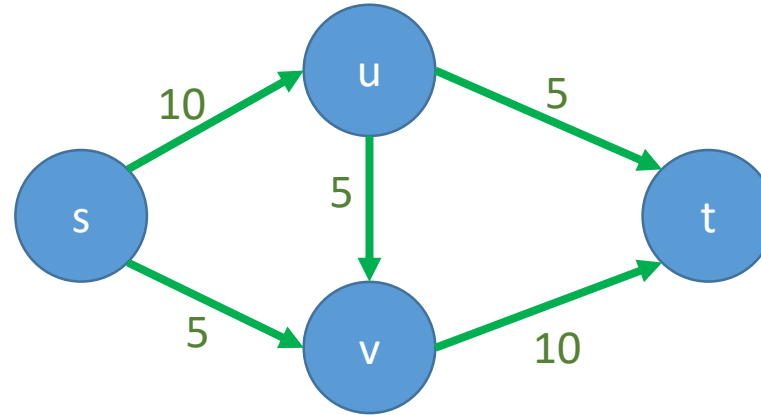
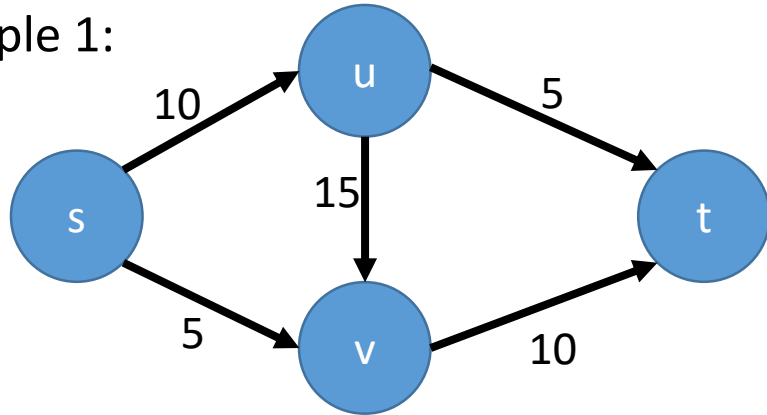


Maximum flow is 1

Maximum Flow Examples

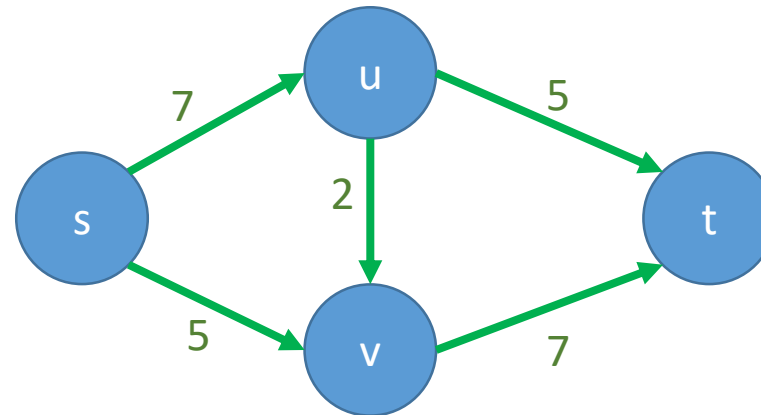
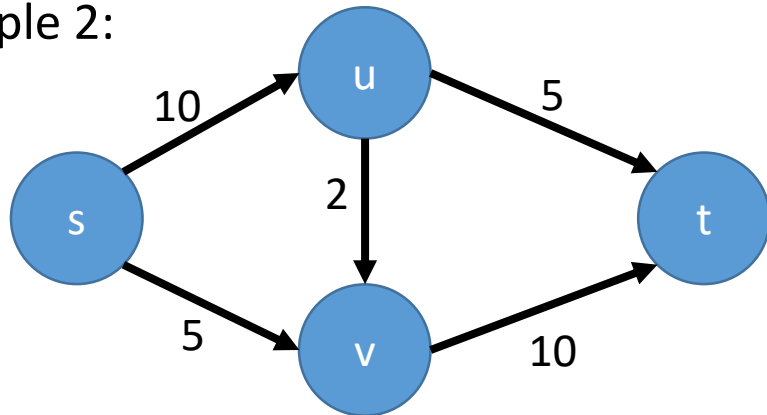


Example 1:



Maximum flow is 15

Example 2:

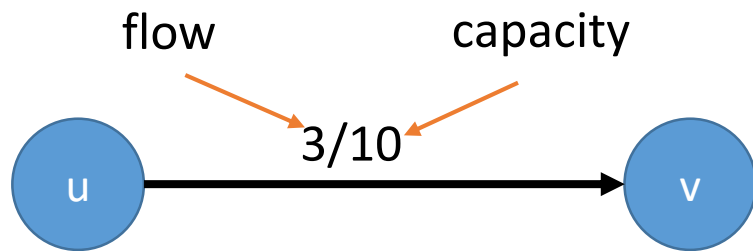


Maximum flow is 12

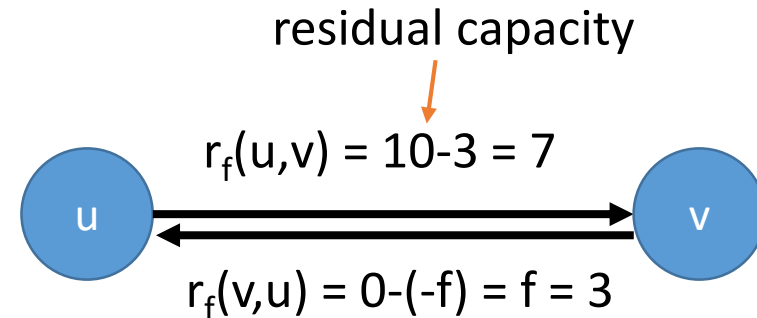
- Flow on an edge doesn't exceed the given capacity of the edge.
- Incoming flow is equal to outgoing flow for every vertex except s and t.

Residual Graph

- **Residual capacity** $r_f(v, w)$ of a vertex pair is $c(v, w) - f(v, w)$.
- Residual graph $G_f = (V, E_f)$ where E_f is the set of **residual edges** (v, w) with $r_f(v, w) > 0$.
- E.g, consider two vertices u and v :



Original graph



Residual graph

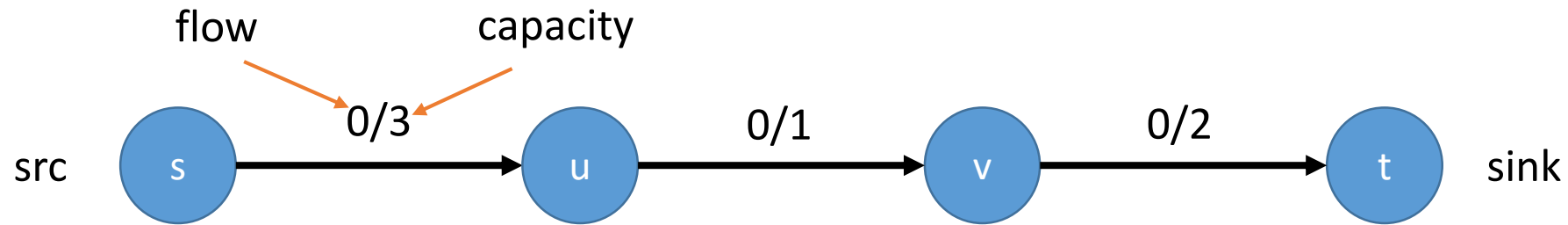
Push-Relabel Algorithm

- Push-Relabel algorithm works on the ***residual graphs***.
- The intuition behind the push-relabel algorithm is that we consider edges as water pipes and nodes are joints:
 - The source is considered to be at the ***highest level*** and it send water to all adjacent nodes;
 - Once a node has excess water ($\text{inflow} - \text{outflow} > 0$), it ***pushes*** water to a smaller height node;
 - If water gets locally trapped at a vertex, the vertex is ***relabeled*** which means its height is increased.

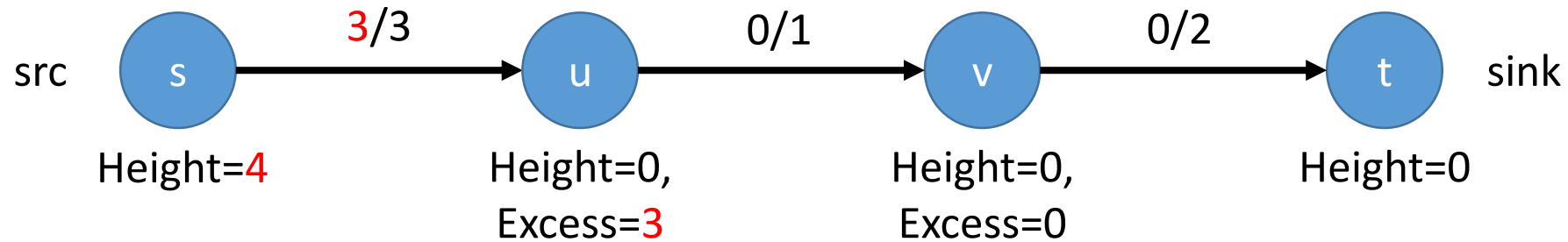
Push-Relabel Algorithm

- Excess Flow of u = Total inflow to u – Total outflow from u
- Three operations of Push-Relabel algorithm:
 - 1) **Initialize Preflow** : Initialize Flows and Heights
 - 2) **While** it is possible to perform a **Push()** or **Relabel()** on a vertex
// Or while there is a vertex that has excess flow
 Do Push() or Relabel()
// At this point all vertices have Excess Flow as 0 (Except source and sink)
 - 3) **Return** flow.

Push-Relabel Algorithm Running Example



Preflow



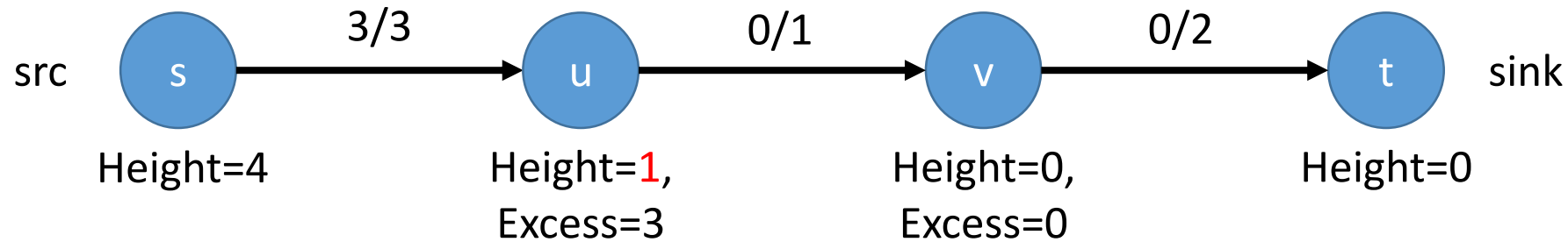
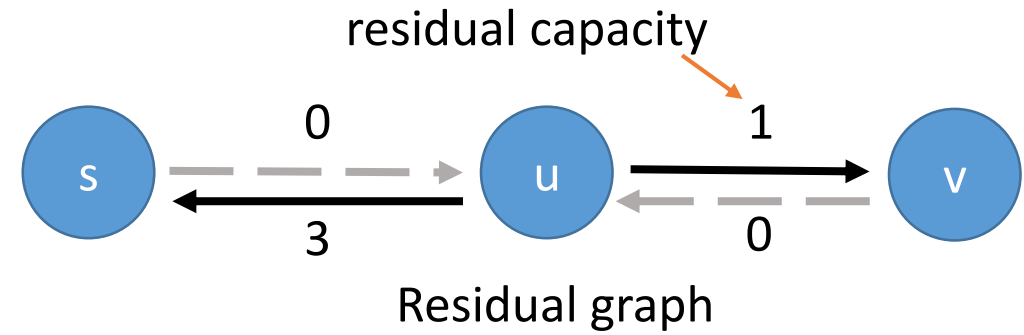
Initial Labeling:

- Initial height of *src* = Number of vertices = 4
- Initial height of other vertices = 0

Initial Flow:

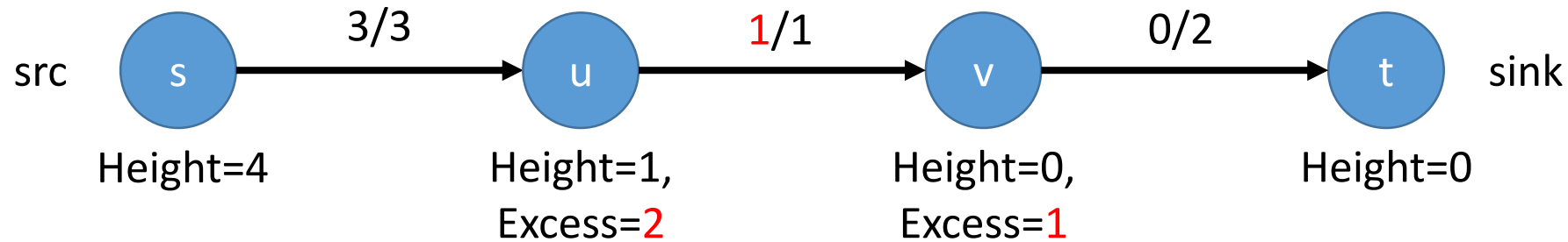
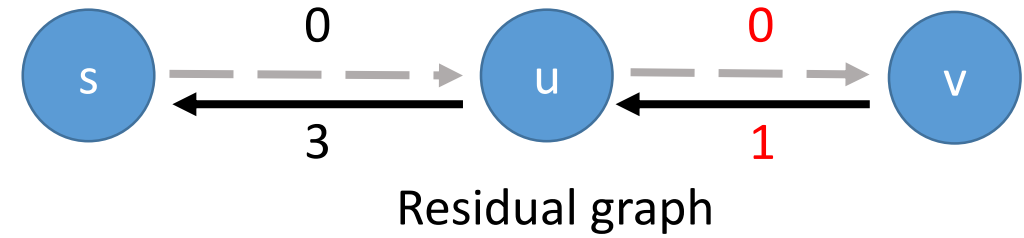
- Flow from *src* to its adjacents is equal to capacities of edges

Relabel u



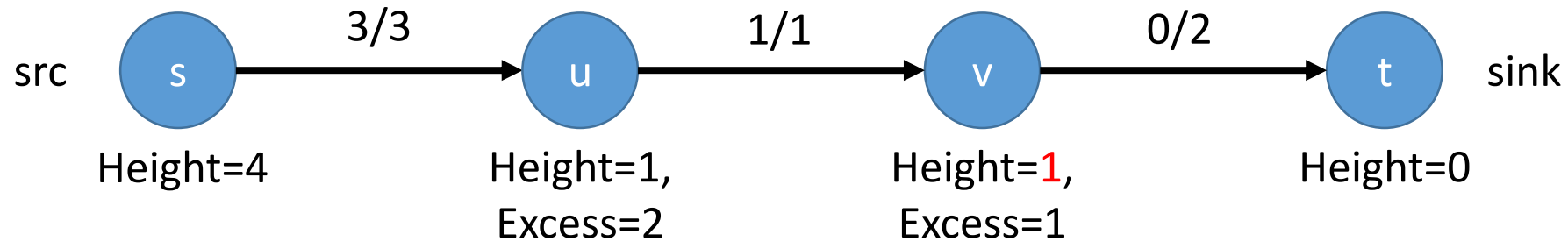
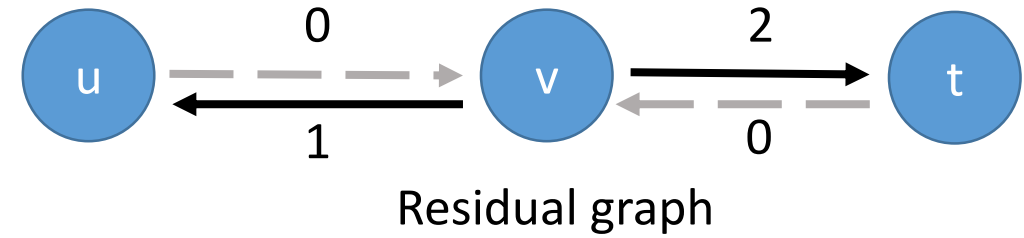
- The height of vertex u is relabeled (height becomes to 1) since it has excess flow and there is no adjacent with smaller height.
- The new height of u is equal to minimum of heights of its **adjacent vertex on the residual graph** ($r_f(v, w) > 0$) plus 1.

Push from u



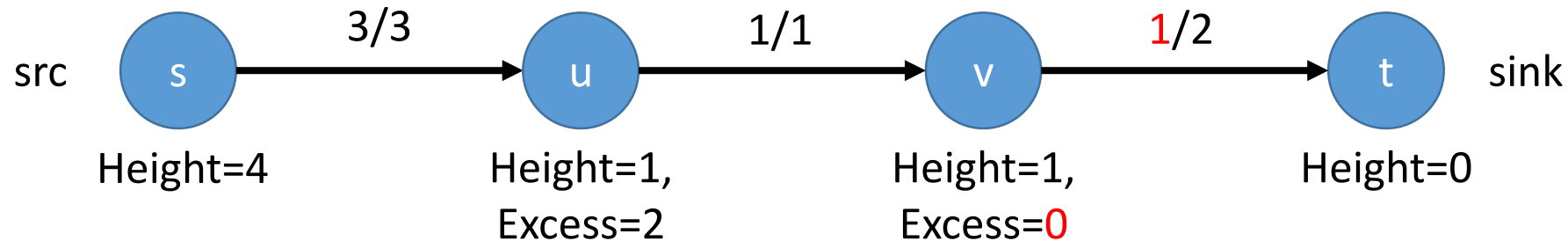
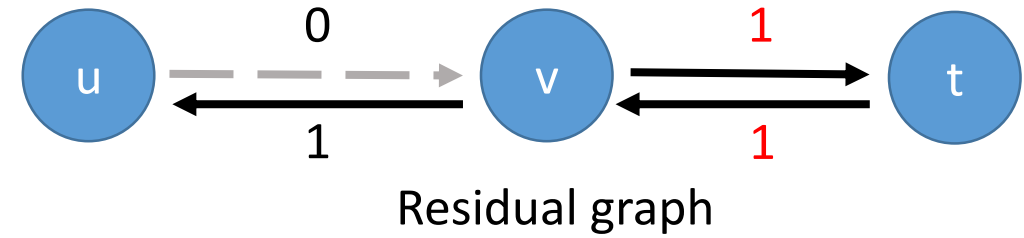
- The vertex u has excess flow and there is an adjacent with lower height, so **push** happens.
- Excess flow of vertex u is $(3 - 0 = 3)$ and the capacity of edge (u, v) is 1; thus, the amount of pushed flow is 1 (minimum of excess flow and the edge capacity).

Relabel v



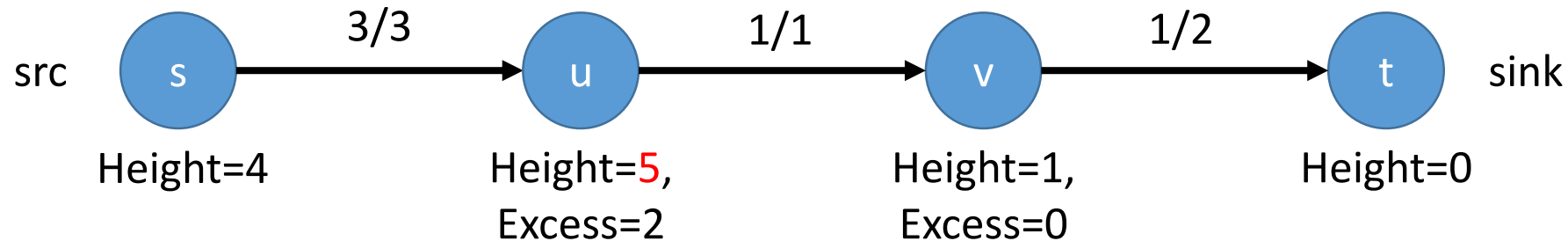
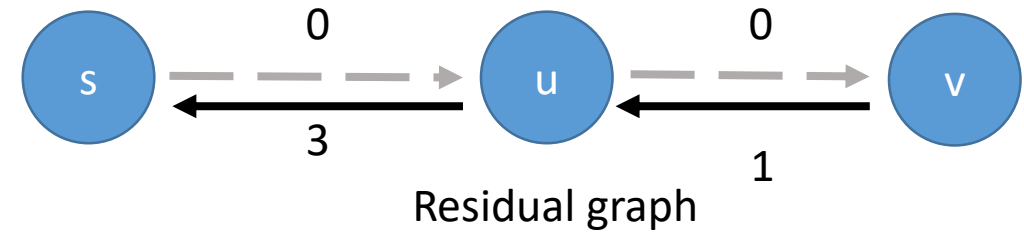
- The height of vertex v is relabeled (height becomes 1) as it has excess flow and there is no adjacent with smaller height.
- The new height of v is equal to minimum of heights of adjacent plus 1.

Push from v



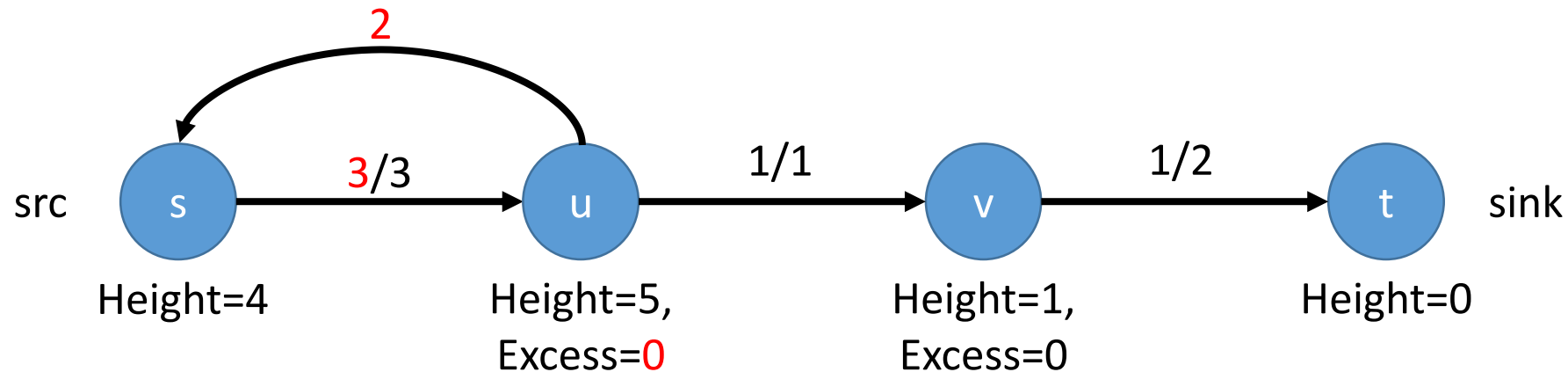
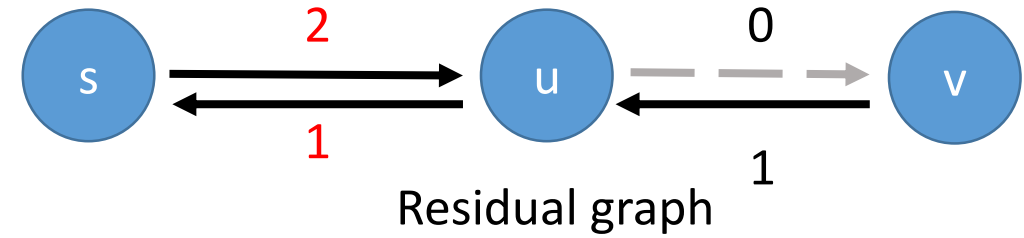
- The vertex v has excess flow and there is an adjacent with lower height, so **push** happens.
- Excess flow of vertex v is $(1 - 0 = 1)$ and the capacity of edge (v, t) is 2; thus, the amount of pushed flow is 1 (minimum of excess flow and the edge capacity).

Relabel u



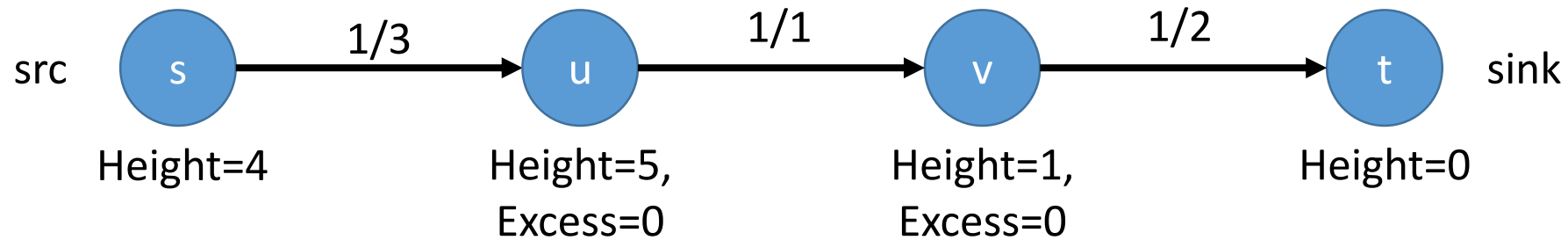
- The height of vertex u is relabeled (height becomes to 5), since it has excess flow and there is no adjacent with smaller height.
- The new height of u is equal to minimum of heights of adjacent plus 1.
- **Note vertex v is not an adjacent of u in the residual graph!**

Push from u



- The vertex u has excess flow and there is an adjacent with lower height, so **push** happens.
- Excess flow of vertex u is $(3 - 1 = 2)$ and the capacity of edge (u,s) is 3; thus, the amount of pushed flow is 2 (minimum of excess flow and the edge capacity).

End of push-relabel



- Maximum flow achieved since there is no relabel possible and no vertex has excess flow.
- Maximum flow is 1.

References

- https://en.wikipedia.org/wiki/Flow_network
- https://en.wikipedia.org/wiki/Maximum_flow_problem
- <https://www.geeksforgeeks.org/push-relabel-algorithm-set-1-introduction-and-illustration/>
- <https://www.geeksforgeeks.org/ford-fulkerson-algorithm-for-maximum-flow-problem/>

Thanks !
Q & A