Deep Hedging on Fixed Strike Asian Options

-Final Paper-

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❖ Introduction

As the world is innovating faster, financial transactions get more complex with the involvement of various actors in diverse situations. Individuals all become investors and borrowers, directly involved in the market, who have to go through trade-offs. Throughout the trade-offs that individuals face, individuals earn and lose money. The understanding of financial engineering helps individuals determine the outcome of their actions as an investor and a borrower; eventually, protecting from the potential risks and dangers.

Hedging is used a method to potentially reduce the risks of loss of an existing. By buying or selling investments or by taking opposite positions, investors can protect one's investments. When delta, a metric that estimates the price change, can be calculated explicitly, investors can use delta hedging. However, there are times in which the calculation of delta cannot be made. When delta cannot be calculated explicitly, deep hedging can be used.

Deep hedging uses machine learning and neutral networking to fill the gap of delta. The most representative case using deep hedging is with the Asian options. Unlike European options, Asian options do not have pricing formulas. Thus, Asian option makes it impossible to differentiate it or calculate delta. However, with the usage of deep hedging, the replication of options is possible which makes it possible to hedge Asian options without delta.

This paper sheds light on explanations about the concepts of binominal option pricing, Black-Scholes model, delta hedging, Asian options, and deep hedging. Ultimately, this paper focuses on the deep hedging method and its simulations and visualizations with fixed strike Asian options.

❖ Background

[1] Financial Transactions

For every financial activity, there is always a buyer and a seller. A financial transaction is an agreement between a buyer and a seller to exchange goods, services, or financial assets for monetary payment. There are four common types of financial transactions.

1) Spot Transactions. Spot transaction is an act of buying or selling that happens on a specified spot date with instant delivery. Unlike forward contracts, spot transactions do not take time value into account (such as interest rates or time to maturity), as it happens instantly.
2) Loans. Depending on the interest rate, the value of present money can differ from the value of future money. Loans allows one the get the product in the present and requires one to pay the seller in the future with the money that includes the interest.
3) Stocks. Europeans invested towards travelling the new lands in the 1400s, expecting higher returns from their investments. In the status quo, stocks play a similar role. Stocks allow an investor to pay the money in advance, and then get the product in the future.
4) Forward Contracts. In a forward contract, there are no cash flows or derivatives. There are only contracts and promises that state the product, the fixed price, and the maturity date. The transaction happens in the future. For instance, airports make a forward contract with oil companies to ensure the fixed price of oil for the future, so that they could open flight reservations at a certain price.

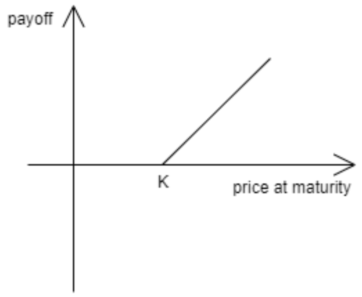
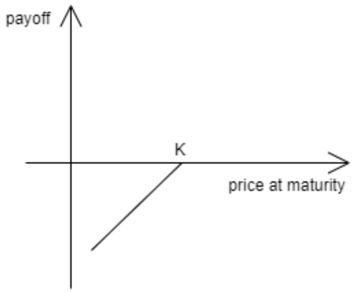
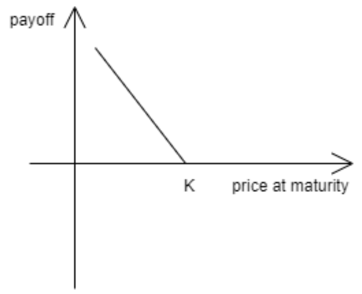
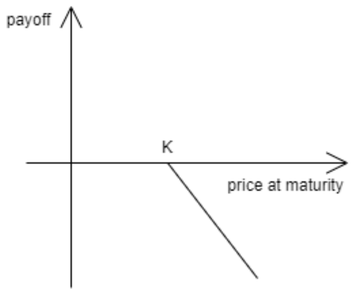
[2] Options

Option contracts are established to give sellers and buyers the opportunity to sell and buy. However, unlike forward contracts, option contracts do not require any obligation to the seller or buyer.

Options are bought and sold by the majority investors, leading to the development of ways to prevent the loss investors can receive. Methods of hedging can prevent the damage that investors could go through in the future.

A call option gives the holder the right to buy a product at a given price within a certain time. A put option gives the holder the right to sell a product at a given price within a certain time. Both gives the holder the right to claim one's profit, but there are yet no obligation that one has to risk a loss in order to follow for the transactions.

With two standards, the payoff diagram can be decomposed into four parts. The first standard is whether the option is a call or put option. The second standard is whether the investor took a short or long position.

Call option at a long position	Put option at a short position
$Max(S_T - K, 0)$	$-Max(K - S_T, 0)$
	
Put option at a long position	Call option at a short position
$Max(K - S_T, 0)$	$-Max(S_T - K, 0)$
	

[3] Binominal Option Pricing

Risk-neutral valuation uses the values derived from the calculations with the Arrow-Debreu Securities as discount factors and uses risk-neutral probabilities for the expected values. The risk-neutral valuation formula is as followed: $df \times E(X)$. The process of deriving this formula will be shown afterwards.

In the example,

$$10 \begin{array}{l} \nearrow 11 \\ \searrow 11 \end{array} \qquad 9 \begin{array}{l} \nearrow 22 \\ \searrow 0 \end{array} \qquad ? \begin{array}{l} \nearrow 22 \\ \searrow 11 \end{array}$$

$$df \times E(X) = (u_1 + u_2) \times \{22 \times p + 11 \times (1-p)\} = 14.5$$

$$u_1 = \frac{9}{22}, u_2 = \frac{11}{22}$$

$$p = \frac{\frac{9}{22}}{\frac{9}{22}+\frac{11}{22}}, 1-p = \frac{\frac{11}{22}}{\frac{9}{22}+\frac{11}{22}}$$

The purpose of deriving a binominal option pricing formula lies on the investor's wish to determine the exact value of a call. Without the taking the probability of the stock's fluctuation into account, the binominal option pricing formula allows investors to measure the exact value of a call only with its interest rate, underlying stock price, range of movement in the underlying stock price, and its striking price.

Throughout the process, we must assume that the stock price follows a multiplicative binominal process over discrete periods. We must also assume that the rate of interest stays constant, and investors can buy or lend limitlessly.

S stands for the present stock price and after the period has ended it would become either uS or dS, as the rate of return is either $u-1$ (with the probability of q) or $d-1$ (with the probability of $1-q$. We will also represent r as the riskless interest rate over one period plus one, and require it to be $u > r > d$.

$$S \begin{cases} uS \; with \; probability \; q \\ dS \; with \; probability \; 1-q \end{cases}$$

Let's have C as the present value of the call. If the stock price at the end of the period is uS, C_u is its value; if the stock price at the end of the period is dS, C_d is its value. When there is only one period until the expiration of the call, this means that:

$$C \begin{cases} C_u = \max[0, uS-K] \; with \; probability \; q \\ C_d = \max[0, dS-K] \; with \; probability \; 1-q \end{cases}$$

Let's say there is a portfolio that consists of Δ stocks and B dollars of riskless assets. At the end of the period, the portfolio would value as:

$$\Delta S + B \begin{cases} \Delta uS + rB \; with \; probability \; q \\ \Delta dS + rB \; with \; probability \; 1-q \end{cases}$$

Equalizing the value of the portfolio at the end of the period and value of the call at the end of the period,

$$\Delta uS + rB = C_u$$

$$\Delta dS + rB = C_d$$

This also means that

$$\Delta = \frac{C_u - C_d}{(u-d)S}$$

$$B = \frac{uC_d - dC_u}{(u-d)r}$$

Thus, the risk-neutral valuation formula can be derived as the following process.

$$C = \Delta S + B$$

$$= \frac{C_u - C_d}{(u-d)} + \frac{uC_d - dC_u}{(u-d)r}$$

$$= \left[\left(\frac{r-d}{u-d}\right) C_u + \left(\frac{u-r}{u-d}\right) C_d\right] / r$$

Having $\frac{1}{r}$ as the discounting factor and $\frac{r-d}{u-d}$ & $\frac{u-r}{u-d}$ as risk-neutral probabilities, we can define p as the following.

$$p \equiv \frac{r-d}{u-d}$$

$$1 - p \equiv \frac{u-r}{u-d}$$

Reorganizing the risk-neutral valuation formula, it can be shown as

$$C = \frac{[pC_u + (1-p)C_d]}{r}$$

[4] Monte Carlo Method

Monte Carlo methods uses random sampling and computational algorithms to earn numerical results, quantitative analysis, and decision making. It is used popularly in situations of numerical optimization. When random variables intervene, Monte Carlo methods can predict the probability of different outcomes. Despite of the interferences, as the Monte Carlo Method repeats with random samples, it tries to reduce the uncertainty.

[5] Black-Scholes Model

The Black-Scholes model provides the pricing formula of the European options. The Black-Scholes formula can be derived from the Black-Scholes equation. The formula estimates the price of European options and gives each option a risk of security and its expected return. By the information given by the Black-Scholes model, investors are able to reduce the potential risks by hedging the option by changing their positions or by buying or selling their options

The Black-Scholes equation is

$$\partial \mathrm{V}/\partial t + 1/2\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The Black-Scholes formula that calculates the European options follows three boundaries

$$C(0,t) = 0 \text{ for all } t$$

$$C(S,t) \rightarrow S - K \; as \; S \rightarrow \infty$$

$$C(S,T) = \max\{S-K, 0\}$$

Following these three boundaries,

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

Then d_1 and d_2 can be calculated as the following.

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2\sigma^2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Delta becomes the partial derivative of option price with respect to underlying assets and a call option's delta is $N(d_1)$.

❖ Delta Hedging and Deep Hedging

Using the delta calculated by the above formula, delta hedging is possible.

Following the Black-Scholes model, the value of delta converges to either 0 or 1. If the underlying asset price is lower than the strike price, delta heads towards 0, as the expiration date comes. If the underlying asset price is higher than the strike price, delta heads towards 1, as the expiration date comes.

If investors act according to the Black-Scholes model, if the stock prices rise above the strike price, the investors would own the stock. If stock prices fall below the strike price, the investors would not own the stock.

Let's visualize the result of delta hedging by using a histogram, a scatter plot, and a loss history plot.

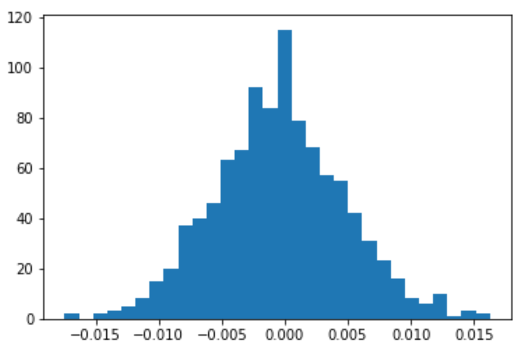


Figure 1. Histogram of Delta Hedging

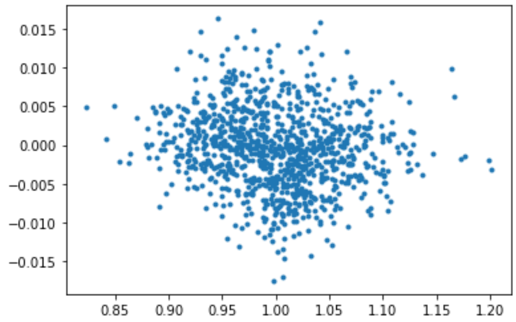


Figure 2. Scatter Plot of Delta Hedging

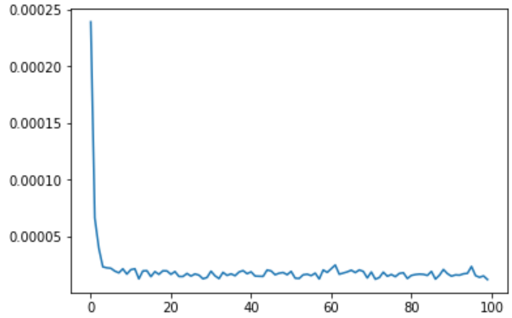


Figure 3. Loss History Plot of Delta Hedging

Figure 1 and figure 2 shows that hedging results are well concentrated to 0. Figure 3 also shows that the mean squared error reduces as the size of epoch increases.

On the other hand, deep hedging is used when the value of delta of an option cannot be explicitly shown or calculated because there is not a clear pricing formula existing. In these situations, neural networking and machine learning is needed to hedge the option.

Neural networks are algorithms that are used to learn a set of data to recognize an underlying pattern or relationship specifically in ways that follows the human brain. By adding layers of input, neural network would find the optimal form of the output. Increasing the number of nodes would increase the accuracy of deep hedging.

❖ Asian option

An Asian option is an example of an option that does not have a clear pricing formula; thus, an option that needs the help of neural networking to hedge.

European options or American options would calculate its payoff by taking into account the price of the underlying asset at exercise. Being a type of option contract, the Asian option is also known as the average value option. However, unlike the others, Asian options calculates the payoff of the option by calculating the average underlying price over some time period. Thus, this allows investors to buy the underlying asset for an average price, not a spot price.

These characteristics make Asian options an example of an exotic option and are used in different matters such as business problems compared to other types of options. Examples of some situations that Asian options are used can be when a firm is concerned about the average exchange rate, the manipulation of a spot price, or the volatility of an underlying asset.

There are two types of Asian options.

1) Fixed Strike: the average underlying price is calculated over a period of time and is used to replace the underlying price.
2) Fixed Price: the averaging price is used to replace the strike place.

There are several advantages of Asian options.

1) Asian options are cheaper than European or American options. Because Asian options used the average value, it can decrease its volatility of the option: making Asian options cheaper.
2) Asian options reduce the risks of market manipulation.

There are most basic two permutations of Asian options.

In the following, A is denoted for the average price in period $[0, T]$. K is denoted for the strike price. $S(T)$ is denoted for the price at maturity. k is denoted for a weighting.

1) Fixed strike Asian call option:

$$C(T) = \max(A(0,T) - K, 0)$$

   Fixed strike Asian put option:

$$P(T) = \max(K - A(0,T), 0)$$

2) Floating strike Asian call option:

$$C(T) = \max(S(T) - kA(0,T), 0)$$

   Floating strike Asian put option:

$$P(T) = \max(kA(0,T) - S(T), 0)$$

As Asian options use the average value, it is important to choose which method one will use in one's calculations. There are two representative ways of calculating A.

1) Arithmetic Average

- The arithmetic average in a continuous case is calculated as the following.

$$A(0,T) = \frac{1}{T}\int_0^T S(t)dt$$

- The arithmetic average in a discrete case is calculated as the following. At times of $0 = t_0, t_1, t_2, \dots, t_n = T\ and\ t_i = i \cdot \frac{T}{n}$, the average is calculated as

$$A(0,T) = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$$

2) Geometric Average

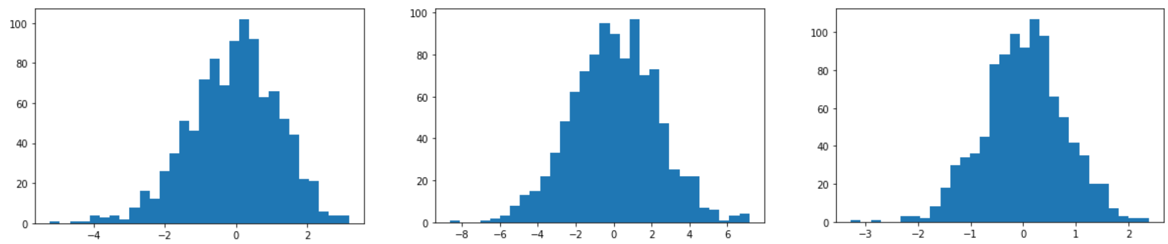
$$A(0,T) = \exp\left(\frac{1}{T}\int_0^T \ln\big(S(t)\big)\, dt\right)$$

❖ Deep Hedging with Asian Option

Neural networking and machine learning have allowed the deep hedging of options without pricing formula or a calculated explicit delta. Let's visualize how fixed strike Asian Options are hedged by neural networks. With other variables fixed, by changing the value of one variable at a time, simulations will be shown in the below. The sample size for every simulation below is equal.

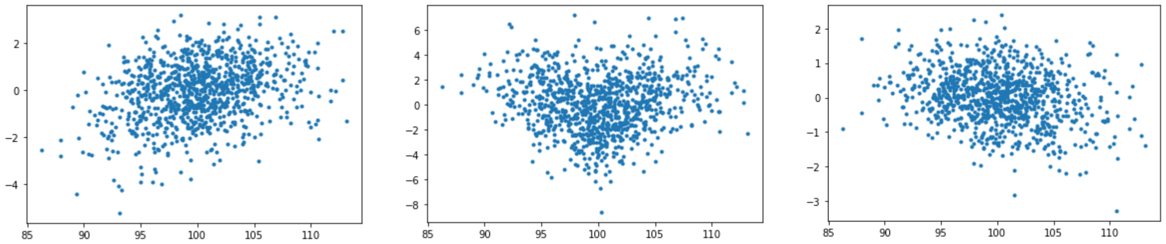
<u>1) Changing of Strike Price (K)</u>

Having other variable fixed, the below graphs, denoted as figure 4, figure 5, and figure 6, show the results of different strike prices (K): 90, 100, 110.



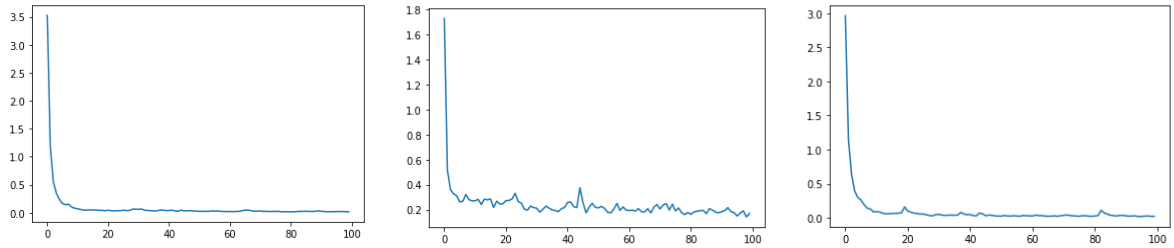
$K = 90$ $K = 100$ $K = 110$

Figure 4. Histograms for different strike prices



$K = 90$ $K = 100$ $K = 110$

Figure 5. Scatter plots for different strike prices



$K = 90$ $K = 100$ $K = 110$

Figure 6. Loss History Plots for different strike prices

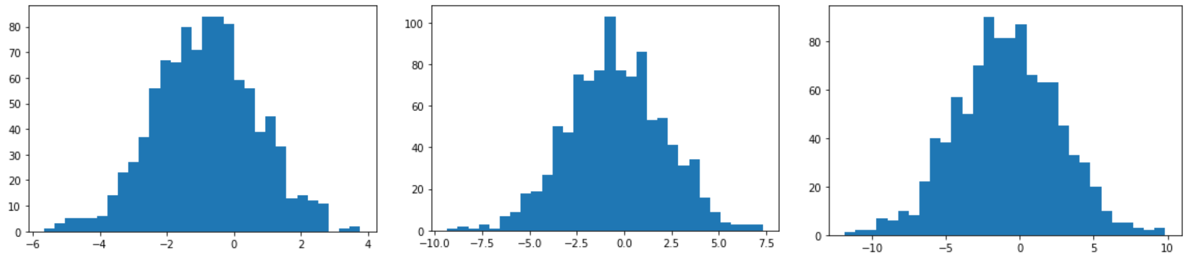
All nine graphs show that the Asian options are well hedged despite of the difference in strike price so that the hedging results are concentrated to 0 and the loss converges to 0.

Observations show that the closer K is set to S, the result of deep hedging is less accurate. In other words, the farther K gets from S, the result of deep hedging becomes more accurate. Because S was set as 100 throughout this simulation, the hedging results for $K = 90$ and $K = 110$ are more accurate.

The reason for this is as following: If K is much larger than S, it is highly possible that the option would not be exercised. Also, if S is much larger than K, it is highly possible that the option would be exercised. However, when K is similar with S, the possibility of the option being exercised or not both co-exists; therefore, making it harder to hedge.

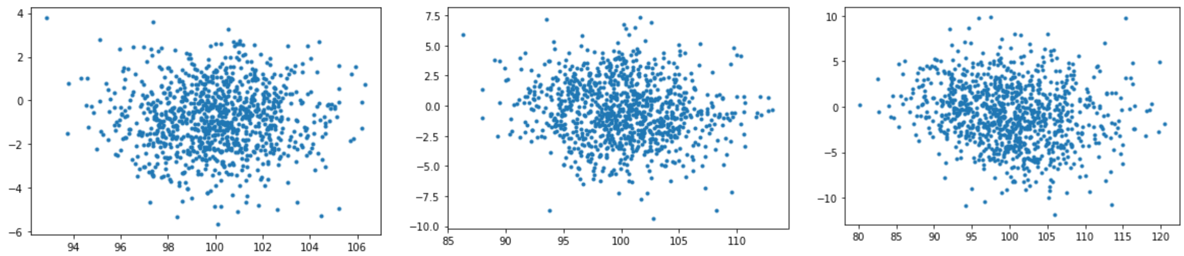
2) Changing of Value of Sigma (σ)

Having other variable fixed, the below graphs, denoted figure 7, figure 8, and figure 9 show the results of different values of sigma (σ): 0.1, 0.2, 0.3.



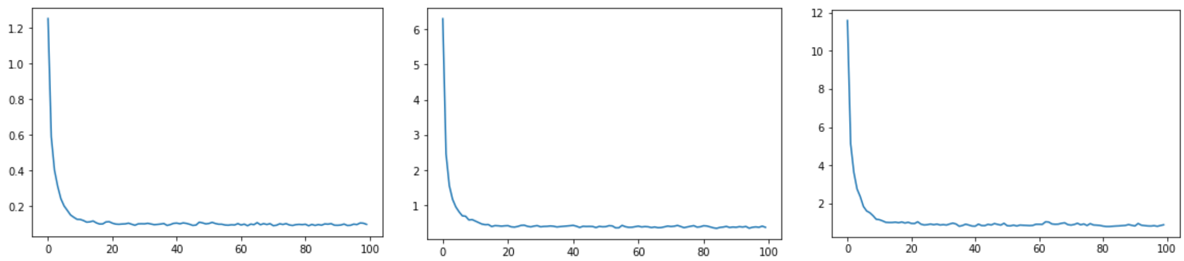
$\sigma = 0.1$ $\sigma = 0.2$ $\sigma = 0.3$

Figure 7. Histograms for different values of sigma



$\sigma = 0.1$ $\sigma = 0.2$ $\sigma = 0.3$

Figure 8. Scatter plots for different values of sigma



$\sigma = 0.1$ $\sigma = 0.2$ $\sigma = 0.3$

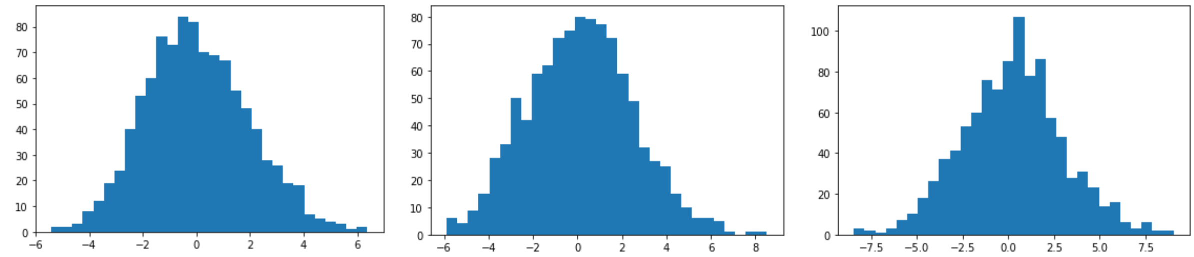
Figure 9. Loss History Plots for different values of sigma

All nine graphs show that the Asian options are well hedged despite of the difference in values of sigma so that the hedging results are concentrated to 0 and the loss converges to 0.

Observations show that as sigma becomes larger, the results of deep hedging become less accurate. This is because larger volatility is highly likely to make deep hedging more difficult, whereas smaller volatility can lead to stability and make deep hedging easier.

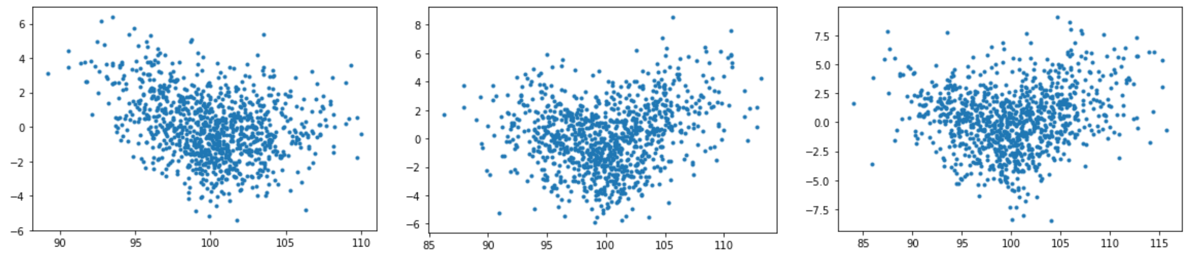
<u>3) Changing of Time Maturity (T)</u>

Having other variable fixed, the below graphs, denoted figure 10, figure 11, and figure 12, show the results of different time maturity (T): $30/365, 50/365, 70/365$.



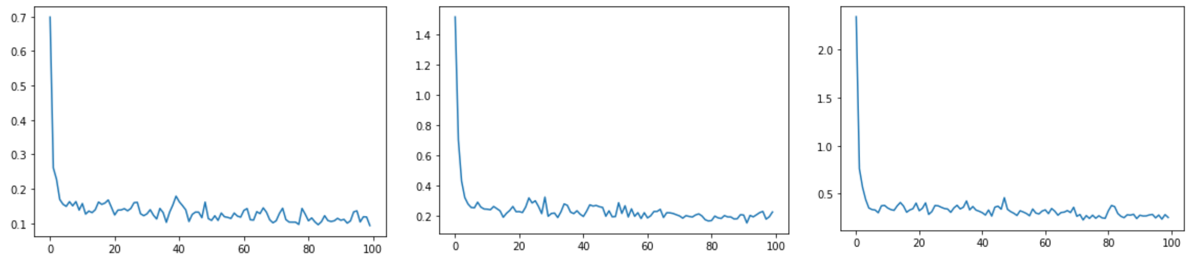
$T = 30/365$ $T = 50/365$ $T = 70/365$

Figure 10. Histograms for different time maturity



$T = 30/365$ $T = 50/365$ $T = 70/365$

Figure 11. Scatter plots for different time maturity



$T = 30/365$ $T = 50/365$ $T = 70/365$

Figure 12. Loss History Plots for different time maturity

All nine graphs show that the Asian options are well hedged despite of the difference in time maturity so that the hedging results are concentrated to 0 and the loss converges to 0. It is observable that the difference in time maturity does not hugely impact how well the option could be hedged.

❖ Conclusion

This paper ultimately shows that even without the explicit value of delta, deep hedging allows investors to hedge options. Therefore, also showing exotic options like the Asian option can be hedged by neural networks and investors can reduce the potential harm they could go through. Results show that first, the farther K gets from S, deep hedging becomes more accurate. Second, the decrease of sigma increases the accuracy of deep hedging. Lastly, the difference in time maturity did not largely influence the result of deep hedging.

More data learning and increased number of simulations would improve the accuracy of deep hedging. Increasing the number of nodes or the activation of neural networking could also improve the results of deep hedging.

❖ Appendix

The appendix shows the main code lines used for Asian Option Hedging.

[1] Delta Hedging

```
for i in range(M):
    cost = 0
    price = S[i,0]
    for j in range(N):
        d1 = (np.log(price/K)+(r+0.5*sig**2)*(T-j*dt))/(sig*np.sqrt(T-
j*dt))
        delta = norm.cdf(d1)
        cost = cost + delta*(price-S[i,j+1])
        price = S[i,j+1]

    cost = cost + np.maximum(S[i,N]-K, 0) - bscall(S0, K, T, r, sig)
```

[2] Calculation of S_{avg}

```
for i in range(M):

    S[i,0] = S0
    avg[i,0] = 0
    for j in range(N):
        S[i,j+1] = S[i,j] * (1 + rv[i,j])
        avg[i,j+1] = (avg[i,j]*j + S[i,j+1])/(j+1)
    S[i,N] = avg[i,N]
```

[3] Monte Carlo method used to calculate the premium of the Asian option

```
for i in range(M) :
  S_avg.append(np.mean(S[i,:]))
  Asian_payoff.append(np.maximum(S_avg[i] - K, 0))

asian_premium = np.mean(Asian_payoff) * np.exp(-r*T)
print(asian_premium)
```

[4] Neural Networking Process

```
for j in range(N):

    delta = tf.keras.layers.Dense(32, activation='tanh')(price)
    delta = tf.keras.layers.BatchNormalization()(delta)
    delta = tf.keras.layers.Dense(32, activation='relu')(delta)
    delta = tf.keras.layers.BatchNormalization()(delta)
    delta = tf.keras.layers.Dense(32, activation='leaky_relu')(delta)
    delta = tf.keras.layers.Dense(1, activation='sigmoid')(delta)

    new_price = tf.keras.layers.Input(shape=(1,), name='S'+str(j))
    my_input = my_input + [new_price]

    price_inc = tf.keras.layers.Subtract(name='price_inc_'+str(j))([pri
ce, new_price])
    cost = tf.keras.layers.Multiply(name="multiply_"+str(j))([delta, pr
ice_inc])
    hedge_cost = tf.keras.layers.Add(name='cost_'+str(j))([hedge_cost,
cost])
    price = new_price

payoff = tf.keras.layers.Lambda(lambda x : 0.5*(tf.abs(x-K)+x-
K))(price)
cum_cost = tf.keras.layers.Add(name="final")([hedge_cost, payoff])
cum_cost = tf.keras.layers.Subtract(name="final_")([cum_cost, premium])

model = tf.keras.Model(inputs=my_input, outputs=cum_cost)
```

[5] Dataset Used

```
p = asian_premium * np.ones([M,1])
c = np.zeros([M,1])
SS = [S[:,i].reshape(M,1) for i in range(N+1)]
x = [p]+[c]+[SS]
y = np.zeros([M,1])
```