HM III - Übungsblatt 4

Komplexes Kurvenintegral:

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt$$

Cauchy-Integralsatz:

$D \subset \mathbb{C}$ sternförmiges Gebiet, $f \in H(D)$. Dann gilt für jeden geschlossenen Integrationsweg γ:

$$\int_\gamma f(z)\, dz = 0$$

Cauchy-Integralsatz:

$D \subset \mathbb{C}$ Gebiet, $f \in H(D)$, $z_0 \in D$, $r > 0$, $\overline{U_r(z_0)} \subset D$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \quad \text{für alle } z \in U_r(z_0)$$

Cauchy-Integralformel für Ableitungen

Vorraussetzungen wie oben

$$\Rightarrow f^{(k)}(z) = \frac{k!}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta \quad \text{für alle } z_0 \in U_r(z_0) \text{ und } k \in \mathbb{N}.$$

Testat, Aufgabe 2, $\boxed{D}$:

$f(x) = e^x \cos(ay)$

Für welche $a \in \mathbb{R}$ ist f Realteil einer holomorphen Funktion?

Damit f Realteil einer holomorphen Funktion sein kann, muss f harmonisch sein:

$f_x(x,y) = e^x \cos(ay)$, $f_y(x,y) = -a e^x \sin(ay)$

$f_{xx}(x,y) = e^x \cos(ay)$, $f_{yy}(x,y) = -a^2 e^x \cos(ay)$

Also $\Delta f \overset{!}{=} 0$

$\Leftrightarrow e^x \cos(ay) + (-a^2) e^x \cos(ay) = 0$

$\Leftrightarrow \underbrace{e^x \cos(ay)}_{\neq 0} \cdot (1 - a^2) = 0$

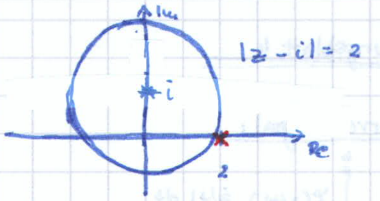
$\Leftrightarrow 1 - a^2 = 0$

$\Leftrightarrow |a| = 1$

$\Leftrightarrow \boxed{a = 1 \vee a = -1}$

Aufgabe 16

$$\int_{|z-i|=2} \frac{2e^{z}\sin(z)}{z(z-1)^2(z-2)}\,dz$$



PBZ des Integranden:

$$\frac{2}{z(z-1)^2(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{z-2}$$

$$\Rightarrow 2 = A(z-1)^2(z-2) + Bz(z-1)(z-2) + Cz(z-2) + Dz(z-1)^2$$

Einsetzen der Nullstellen des Nenners:

$$\left.\begin{array}{l} z=0 \Rightarrow A=-1 \\ z=1 \Rightarrow C=-2 \\ z=2 \Rightarrow D=1 \\ \text{Einsetzen} \Rightarrow B=0 \end{array}\right\} \Rightarrow \frac{2}{z(z-1)^2(z-2)} = \frac{-1}{z} + \frac{-2}{(z-1)^2} + \frac{1}{z-2}$$

$$\Rightarrow \int_{|z-i|=2} \frac{2e^{z}\sin(z)}{z(z-1)^2(z-2)}\,dz = -\int_{|z-i|=2} \frac{e^{z}\sin(z)}{z}\,dz - \int_{|z-i|=2} \frac{2e^{z}\sin(z)}{(z-1)^2}\,dz + \underbrace{\int_{|z-i|=2} \frac{e^{z}\sin(z)}{z-2}\,dz}_{\text{Cauchy-Integralsatz} \Rightarrow \int \ldots = 0}$$

[weil $z=2$ nicht in $|z-i|=2$ liegt]

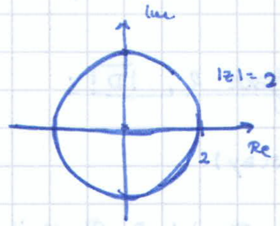
$f(z) = e^{z}\sin(z)$

$$= -2\pi i\cdot f(0) - 2\pi i\cdot 2\left(\frac{d}{dz} f(z)\Big|_{z=1}\right)$$

$$= -4\pi i\left(e^{z}\sin(z) + e^{z}\cos(z)\right)\Big|_{z=1} = -4\pi i\left(e\sin(1) + e\cos(1)\right)$$

Aufgabe 17

$$\int_{|z|=2} \frac{z^3+2z^2-3z+4}{(z-1)^n}\,dz, \quad \text{für } n\in\mathbb{N}_0$$



Setze $f(z) = z^3+2z^2-3z+4$, $f(1) = 4$

$f'(z) = 3z^2+4z-3$, $f'(1) = 4$

$f''(z) = 6z+4$, $f''(1) = 10$

$f'''(z) = 6$, $f'''(1) = 6$

$f^{(m)}(z) = 0 \quad \forall\, m > 3$

$n=0$: Integrand holomorph $\Rightarrow$ Integral gleich 0.

$n=1$: $\displaystyle\int_{|z|=2} \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = 8\pi i$

$n=2$: $\displaystyle\int_{|z|=2} \frac{f(z)}{(z-1)^2}\,dz = \frac{2\pi i}{1!} f'(1) = 8\pi i$

$n=3$: $\displaystyle\int_{|z|=2} \frac{f(z)}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(1) = 10\pi i$

$n=4$: $\displaystyle\int_{|z|=2} \frac{f(z)}{(z-1)^4}\,dz = \frac{2\pi i}{3!} f'''(1) = 2\pi i$

$n\geq 5$: $\displaystyle\int_{|z|=2} \frac{f(z)}{(z-1)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(1) = 0$

Aufgabe 18

(I) $\int\limits_{|z|=1} x \, dz$

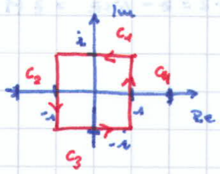
Der Kreis $|z|=1$ wird parametrisiert: $z(t) = e^{it} = \cos(t) + i\sin(t)$, $0 \le t \le 2\pi$

$\Rightarrow x(t) = \cos(t)$, $y(t) = \sin(t)$

$\Rightarrow z'(t) = -\sin(t) + i\cos(t)$

Def. $\Rightarrow \int_0^{2\pi} \cos(t)\left(-\sin(t) + i\cos(t)\right) dt = \int_0^{2\pi} -\sin(t)\cos(t)\,dt + i\int_0^{2\pi} \cos^2(t)\,dt = i\pi$ //

(II) $\int_C \bar{z} \, dz$



Parametrisierung der Teilkurven:

C_1: $z_1(t) = 1 + i - t$, $0 \le t \le 2$
$z_1'(t) = -1$, $\bar{z}_1(t) = 1 - i - t$

C_2: $z_2(t) = -1 + i - it$, $0 \le t \le 2$
$z_2'(t) = -i$, $\bar{z}_2(t) = -1 - i + it$

C_3: $z_3(t) = -1 - i + t$, $0 \le t \le 2$
$z_3'(t) = 1$, $\bar{z}_3(t) = -1 + i + t$

C_4: $z_4(t) = 1 - i + it$, $0 \le t \le$
$z_4'(t) = i$, $\bar{z}(t) = 1 + i - it$

$\Rightarrow \int_C \bar{z}\,dz = \int_0^2 \left(\bar{z}_1(t)\,z_1'(t) + \bar{z}_2(t)\,z_2'(t) + \bar{z}_3(t)\,z_3'(t) + \bar{z}_4(t)\,z_4'(t)\right) dt = \int_0^2 \sum_{j=1}^{4} \left(\bar{z}_j\, z_j'\right)(t)\,dt$

$= \ldots = \int_0^2 (-4 + 4i + 4t)\,dt = \left[-4t + 4it + 2t^2\right]_0^2 = -8 + 8i + 8 = 8i$

(III) $G \subset \mathbb{C}$ beschränkt mit den Vorraussetzungen des Gauß'schen Satzes ($\rightarrow$ 24.7 (I)).
(Trick: Identifiziere $\mathbb{C}$ mit $\mathbb{R}^2$)

C sei die Randkurve von G $\Rightarrow F_G = \frac{1}{2i} \cdot \int_C \bar{z}\,dz$.

Beweis: Schreibe $z(t) = x(t) + i\,y(t)$ $\Rightarrow \int_C \bar{z}\,dz = \int_C (x - iy)\,d(x + iy)$

$= \int_C \left[\underbrace{(x-iy)}_{=P}\,dx + i\,\underbrace{(x-iy)}_{=Q}\,dy\right] \underset{\text{Green}}{=} \iint_C \underbrace{(i+i)}_{Q_x - P_y}\,dx\,dy = 2i \iint_C 1\,dx\,dy = 2i \cdot F_G$

$\Rightarrow F_G = \frac{1}{2i}\int_C \bar{z}\,dz$

4.1: $\overset{(I)}{S(2) = 0}$, $\overset{(II)}{S(i) = \infty}$, $\overset{(III)}{S(0) = -2}$

Ansatz: $S(z) = \frac{az + b}{cz + d}$

(I) $\Rightarrow 2a + b = 0 \Leftrightarrow b = -2a$
(II) $\Rightarrow ci + d = 0 \rightsquigarrow d = -ic$

$\rightsquigarrow S(z) = \frac{az - 2a}{cz - ic} = \frac{a}{c}\,\frac{z-2}{z-i}$

S ist M.T. $\Rightarrow ad - bc \ne 0$
$\Rightarrow -aci + 2ac \ne 0$
$\Leftrightarrow c(2a - ia) \ne 0$
$\Rightarrow c \ne 0$

Setze $k = \frac{a}{c}$

(III)
$\Rightarrow S(0) = -2 \Rightarrow k \frac{-2}{-i} = -2$

$\Leftrightarrow k = -i$

$\Rightarrow S(z) = -i \frac{z-2}{z-i} = \frac{2i - iz}{z-i} = \frac{z-2}{iz+1}$

$w = \frac{z-2}{iz+1} \Leftrightarrow w(iz+1) = z-2 \Leftrightarrow w+2 = z - wiz = z(1-iw)$

$\Leftrightarrow z = \frac{w+2}{1-iw}$.

$\Rightarrow S^{-1}(z) = \frac{z+2}{1-iz}$

Fixpunkte: $z = \pm (1+i)$

Globalübung - Blatt 4

Aufgabe 19

$f(z) = \log(z)$ um $z_0 = i$

$f(z_0) = i\frac{\pi}{2} [+ 2k\pi i]$, $k \in \mathbb{Z}$

$f'(z_0) = \frac{1}{z}$ $\quad f''(z) = -\frac{1}{z^2}$ $\quad f'''(z) = \frac{2}{z^3}$ $\quad f^{(4)}(z) = -\frac{6}{z^4}$ $\quad f^{(k)}(z) = (-1)^{k+1} \frac{(k-1)!}{z^k}$

$f^{(k)}(i) = (-1)^{k+1} \frac{(k-1)!}{i^k} = (-1)^{k+1} (-i)^k (k-1)! = -i^k (k-1)!$

$Tf(z) = f(i) + \sum_{k=1}^{\infty} \frac{f^{(k)}(i)}{k!} (z-i)^k = i\frac{\pi}{2} + 2\pi k i - \sum_{k=1}^{\infty} \frac{i^k}{k} (z-i)^k$

Aufgabe 20

$u(x,y) = e^x (x\cos(y) - y\sin(y))$ Ist u Realteil einer holomorphen Funktion f? Bestimmen Sie f ggf.

ja $\Leftrightarrow$ u harmonisch

$f = u + iv$ holomorph

$\Leftrightarrow$ $u_x = v_y$ $\Rightarrow u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = 0$
$u_y = -v_x$

u harmonisch $\Leftrightarrow u_{xx} + u_{yy} = 0$ $\quad$ zu $v_y = u_x$ und $v_x = -u_y$ suche v.
v existiert $\Leftrightarrow$ Integrabilitätsbedingungen erfüllt
$(v_y)_x = (v_x)_y \iff u_{xx} = -u_{yy}$

Hier: bestimme konjugiert harmonische Funktion v

$v_x = -u_y = -e^x(-x\sin(y) - \sin(y) - y\cos(y))$
$v_y = u_x = e^x(x\cos(y) - y\sin(y) + \cos(y))$

$\int xe^x\,dx = (x-1)e^x + C$
$\int y\sin(y)\,dy = \sin(y) - y\cos(y) + C$

$v = \int v_x\,dx = (x-1)e^x \sin(y) + e^x\sin(y) + ye^x\cos(y) = xe^x\sin(y) + ye^x\cos(y)$

Probe: $\frac{\partial}{\partial y}(v) = xe^x\cos(y) + e^x\cos(y) - ye^x\sin(y) \stackrel{!}{=} v_y$ ✓

$f(z) = f(x+iy) = e^x(x\cos(y) - y\sin(y)) + i e^x (x\sin(y) + y\cos(y))$
$= e^x(\cos(y)\cdot(x+iy) + i\sin(y)(x+iy)) = ze^{iy}\cdot e^x$

Sei $n \in \mathbb{N}$, $b \in \mathbb{R}$, $|b| < 1$

$\int_0^{2\pi} \frac{\cos(nx)}{1-2b\cos(x)+b^2}\,dx = \mathrm{Re} \int_0^{2\pi} \frac{e^{inx}}{1-2b\cos(x)+b^2}\,dx$

$\cos(nx) = \mathrm{Re}\, e^{inx} = \mathrm{Re}(e^{ix})^n$
$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$
$z = e^{ix} = \cos(x) + i\sin(x)$ $\quad 0 \le x \le 2\pi \hat{=} |z| = 1$
$\frac{dz}{dx} = ie^{ix} = iz \Leftrightarrow dx \hat{=} \frac{1}{iz}dz$

$= \mathrm{Re}\,\frac{1}{i} \int_{|z|=1} \frac{z^n}{1-b(z+\frac{1}{z})+b^2} \frac{1}{z}\,dz = \mathrm{Re}\,\frac{1}{i} \int_{|z|=1} \frac{z^n}{-bz^2 + (1+b^2)z - b}\,dz =: I$

NR: $-bz^2 + (1+b^2)z - b = -b(z^2 - (\frac{1}{b}+b)z + 1) = -b(z-b)(z-\frac{1}{b})$

$$* \int_{|z|} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$I = \operatorname{Re} \frac{1}{i} \int_{|z|=1} \frac{z^n}{-b(z-b)(z-\frac{1}{b})}\,dz = \operatorname{Re} \frac{1}{i} \int_{|z|=1} \frac{\frac{z^n}{-b(z-\frac{1}{b})}}{z-b}\,dz$$

$$\overset{*}{=} \operatorname{Re} \frac{1}{i}\, 2\pi i \,\frac{b^4}{-b(b-\frac{1}{b})} = 2\pi \frac{b^4}{1-b^2}$$