1	2	3	4
3.5	5	5	4
4			

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warum ?

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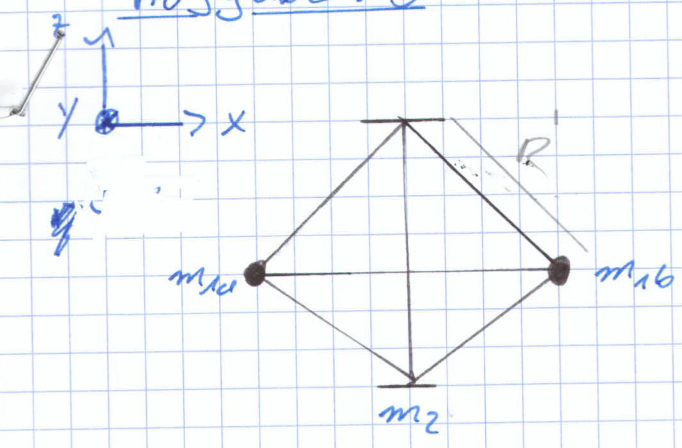
Aufgabe 13



Ohne ZB: 9 Freiheitsgrade

Drehung um die z-Achse

→ Drehmatrix D

$$D = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zwangsbedingungen: $z_{m_2} = 2R\cos\theta$ ✓ $x_{m_2} = y_{m_2} = 0$ ✓ m_2^{-1} ?

$x_{m_{1a}} = -x_{m_{1b}}$; ✓ $|\vec{r}_{m_{1a}} - \vec{r}_{m_2}| = R$ ✓

$z_{m_{1a}} = z_{m_{1b}}$ ✓ $|\vec{r}_{m_{1b}} - \vec{r}_{m_2}| = R$ ✓

$x^2+y^2+z^2-R^2 = 0$?

Für m_{1a}: $$\vec{r}_a = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R\sin\theta \\ 0 \\ R\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} R\cos(\omega t)\sin\theta \\ R\sin(\omega t)\sin\theta \\ R\cos\theta \end{pmatrix}$$

Für m_{1b}: $$\vec{r}_b = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -R\sin\theta \\ 0 \\ R\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} -R\cos(\omega t)\sin\theta \\ -R\sin(\omega t)\sin\theta \\ R\cos\theta \end{pmatrix}$$

b) $$\dot{\vec{r}}_a = \begin{pmatrix} -R\omega\sin(\omega t)\sin\theta + R\cos(\omega t)\cdot\dot{\theta}\cos(\theta) \\ R\omega\cos(\omega t)\sin\theta + R\sin(\omega t)\cdot\dot{\theta}\cos\theta \\ -R\dot{\theta}\sin\theta \end{pmatrix}$$

$$|\dot{\vec{r}}_a|^2 = R^2\omega^2\sin^2\theta + R^2\dot{\theta}^2\cos^2\theta + R^2\dot{\theta}\sin^2\theta$$

$$= R^2\omega^2\sin^2\theta + R^2\dot{\theta}^2$$

$$\dot{\vec{r}}_b = \begin{pmatrix} R\omega \sin(\omega t)\sin\theta - R\cos(\omega t)\,\dot\theta\cos(\theta) \\ -R\omega\cos(\omega t)\sin\theta - R\sin(\omega t)\,\dot\theta\cos(\theta) \\ -R\sin(\theta)\,\dot\theta \end{pmatrix}$$

$$|\dot{\vec{r}}_b|^2 = R^2\omega^2\sin^2\theta + R^2\dot\theta^2\cos^2(\theta) + R^2\sin^2(\theta)\dot\theta^2$$
$$= R^2\omega^2\sin^2\theta + R^2\dot\theta^2$$

für m_2: $|\dot{\vec{r}}_2|^2 = 2^2R^2\dot\theta^2\sin^2\theta$

$$L = T - V$$
$$= \frac{1}{2}m_1 2\left(R^2\omega^2\sin^2\theta + R^2\dot\theta^2\right) + \frac{1}{2}m_2 4R^2\dot\theta^2\sin^2\theta$$
$$+ 2Rg\cos\theta\,(m_1+m_2)$$ ✓

c) $m_2 = 0$

$$L = \frac{1}{2}m_1\left(R^2\omega^2\sin^2\theta + R^2\dot\theta^2\right) + 2Rg\cos\theta\, m_1$$ ✓

$$\frac{\partial L}{\partial\theta} = 2m_1R^2\sin\theta\cdot\left(\omega^2\cdot\cos\theta - \frac{g}{R}\right)$$

$$\frac{\partial L}{\partial\dot\theta} = 2m_1R^2\cdot\dot\theta \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = 2m_1R^2\ddot\theta$$ ✓

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0 = 2m_1R^2\ddot\theta - 2m_1R^2\sin\theta\left(\omega^2\cdot\cos\theta - \frac{g}{R}\right)$$
$$= \ddot\theta - \sin\theta\left(\omega^2\cdot\cos\theta - \frac{g}{R}\right)$$ ✓

d) $\theta \ll 1$

Das folgt nicht daraus.

$$\Rightarrow \quad \ddot\theta - \theta\left(\omega^2 - \frac{g}{R}\right) = 0 \quad \Rightarrow \omega^2 < \frac{g}{R}$$ (✓)

e) $$\ddot\theta - 2a\sin\theta\,\underbrace{\left(\omega^2\cdot\cos\theta - \frac{g}{R}\right)}_{=0} \quad \Rightarrow \theta = \arccos\left(\frac{g}{\omega^2 R}\right)$$ (✓)

das ist nur eine Lösung.

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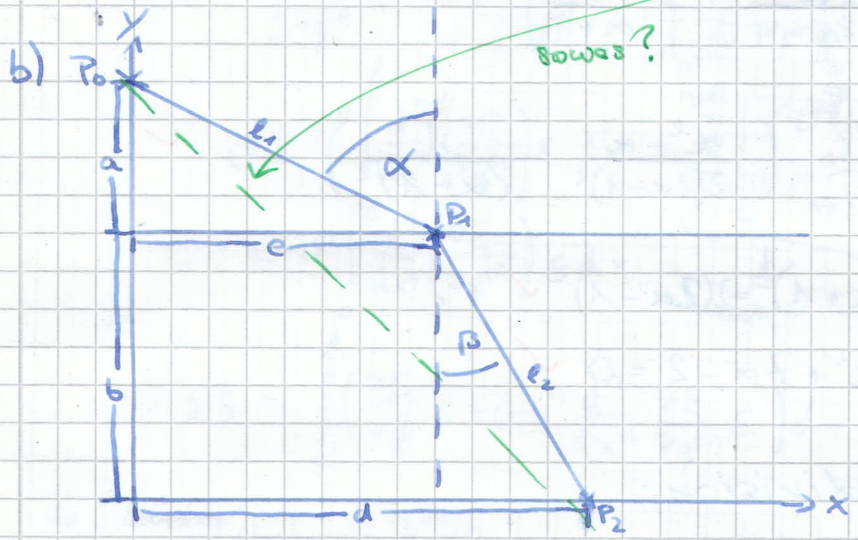
4a) $\delta\left(\int_{P_0}^{P_2} n(\vec{r})\,ds\right) = 0$

Das Licht nimmt zwischen den Punkten P_0 und P_2 den zeitlich kürzesten Weg, aber nicht unbedingt den geometrisch minimalen.

Was soll das sein?

b)



$t = t_1 + t_2 = \frac{\ell_1}{c_1} + \frac{\ell_2}{c_2}$

Zeit in den versch. Medien

$\ell_1 = \sqrt{e^2 + a^2}$

$\ell_2 = \sqrt{(d-e)^2 + b^2}$

$= \frac{\sqrt{e^2+a^2}}{c_1} + \frac{\sqrt{(d-e)^2+b^2}}{c_2}$

$\frac{dt}{de} = \frac{e}{c_1\sqrt{e^2+a^2}} - \frac{1}{c_2} \cdot \frac{d-e}{\sqrt{(d-e)^2+b^2}} = \frac{1}{c_1}\,\frac{e}{\ell_1} - \frac{1}{c_2}\,\frac{d-e}{\ell_2}$

$\frac{dt}{de} \overset{!}{=} 0$ Da nach dem Fermatschen Prinzip die minimale Zeit gebraucht wird.

$\Rightarrow \frac{1}{c_1}\,\frac{e}{\ell_1} = \frac{1}{c_2}\,\frac{d-e}{\ell_2}$ $\qquad \sin\alpha = \frac{e}{\ell_1} \qquad \sin\beta = \frac{d-e}{\ell_2}$

$\Rightarrow \frac{1}{c_1}\sin\alpha = \frac{1}{c_2}\sin\beta$

$\Leftrightarrow c_2 \sin\alpha = c_1 \sin\beta \rightarrow$ Gesetz von Snellius ✓

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15) $S = \int_0^{t_0} L\, dt = \int_0^{t_0} \frac{m}{2}\dot{x}^2 + mgx\, dt$

$$= \int_0^{t_0} \left(\frac{m}{2} \frac{\alpha^2 g^2 t_0^4}{4 t_0^{2\alpha}} t^{2\alpha-2} + \frac{mg^2}{2} \frac{t_0^2}{t_0^\alpha} t^\alpha \right) dt$$

$$= \frac{m}{8} \frac{\alpha^2 g^2 t_0^4}{t_0^{2\alpha}} \int_0^{t_0} t^{2\alpha-2}\, dt + \frac{mg^2 t_0^2}{2 t_0^\alpha} \int_0^{t_0} t^\alpha\, dt$$

$$= \frac{mg^2 t_0^3}{2} \left(\frac{\alpha^2}{8\alpha-4} + \frac{1}{\alpha+1} \right)$$

$$\frac{dS}{d\alpha} = \frac{mg^2 t_0^3}{2} \left(\frac{\alpha^2-\alpha}{2(2\alpha-1)^2} - \frac{1}{(\alpha+1)^2} \right) \stackrel{!}{=} 0$$

$$\Leftrightarrow \quad \alpha(\alpha-1)(\alpha+1)^2 = 2(2\alpha-1)^2$$

$$\Leftrightarrow \quad \alpha^4 + \alpha^3 - 9\alpha^2 + 7\alpha - 2 = 0$$

Polynomdivision:

	1	1	-9	7	-2
2	-	2	6	-6	2
	1	3	-3	1	0

$\Rightarrow \alpha_1 = 2$

Wolfram-Alpha liefert nur eine weitere reelle Lös:

$\alpha_2 = -1 - \sqrt[3]{2} - 2^{2/3}$, aber da $\alpha_2 < 1$ ist sie uninteressant,

$\frac{d^2 S}{d\alpha^2}(\alpha = 2) = \frac{5}{54} > 0 \Rightarrow$ Minimum

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Nr. 16 a) $\delta S = \delta \int_0^\ell E(q, q^{(n)}, x)\,dx = \int_0^\ell \delta E(q, q^{(n)}, x)\,dx$

$$\delta E = \frac{\partial E}{\partial q}\delta q + \frac{\partial E}{\partial q^{(n)}}\delta q^{(n)} + \underbrace{\frac{\partial E}{\partial x}\delta x}_{=0}$$

$$\Rightarrow \delta S = \int_0^\ell \left(\frac{\partial E}{\partial q}\varepsilon + \frac{\partial E}{\partial q^{(n)}}\varepsilon^{(n)}\right)dx$$

NR: $\int_0^\ell \frac{\partial E}{\partial q^{(n)}}\varepsilon^{(n)}\,dx = \left[\frac{\partial E}{\partial q^{(n)}}\varepsilon^{(n-1)}\right]_0^\ell - \left[\frac{d}{dt}\left(\frac{\partial E}{\partial q^{(n)}}\right)\varepsilon^{(n-2)}\right]_0^\ell + \dots + \left[(-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial E}{\partial q^{(n)}}\right)\varepsilon\right]_0^\ell$

$$+(-1)^n\int_0^\ell \frac{d^n}{dx^n}\left(\frac{\partial E}{\partial q^{(n)}}\right)\varepsilon\,dx = \underbrace{\sum_{i=1}^{n}\left[\frac{d^{n-i}}{dx^{n-i}}\left(\frac{\partial E}{\partial q^{(n)}}\right)\varepsilon^{(i-1)}(-1)^{n-i}\right]_0^\ell}_{\overset{!}{=}0\ (\text{Bedingung})}$$

(✓)

$$+(-1)^n\int_0^\ell \frac{d^n}{dx^n}\left(\frac{\partial E}{\partial q^{(n)}}\right)\varepsilon\,dx$$

$$\Rightarrow \delta S = \int_0^\ell \left(\frac{\partial E}{\partial q}\varepsilon + (-1)^n\frac{d^n}{dx^n}\frac{\partial E}{\partial q^{(n)}}\varepsilon\right)dx \overset{!}{=} 0$$

da ε beliebig

$$\Rightarrow \frac{\partial E}{\partial q} + (-1)^n\frac{d^n}{dx^n}\frac{\partial E}{\partial q^{(n)}} = 0$$ ✓ (sorry, dachte das " wäre ein n)

abe[r] ganz cool das mal gesehen zu haben.

mit $n=2$: $\frac{\partial E}{\partial q} + \frac{d^2}{dx^2}\frac{\partial E}{\partial q''} = 0$ ✓

$$\wedge\ \varepsilon\frac{dx}{dt}\left(\frac{\partial E}{\partial q''}\right)\Big|_0^\ell = \varepsilon'\frac{\partial E}{\partial q''}\Big|_0^\ell$$ ✓

b) $E = -\frac{k}{2}q''^2 + q f(x) \Rightarrow \frac{\partial E}{\partial q} = f(x),\ \frac{\partial E}{\partial q''} = -kq'' \Rightarrow \frac{d^2}{dx^2}\frac{\partial E}{\partial q''} = -kq^{(4)}(x)$

$$\Rightarrow f(x) - kq^{(4)}(x) = 0 \Rightarrow q^{(4)}(x) = \frac{f(x)}{k}$$ ✓

$$\wedge \left[\varepsilon k q^{(3)}(x)\right]_0^\ell = \left[-\varepsilon' k q^{(2)}(x)\right]_0^\ell \Leftrightarrow \left[\varepsilon q^{(3)}(x)\right]_0^\ell = \left[\varepsilon' q^{(2)}(x)\right]_0^\ell$$ ✓

c) $f(x) = -\rho g \Rightarrow q^{(4)}(x) = \underbrace{-\frac{\rho g}{k}}_{\lambda} \Rightarrow q(x) = \frac{\lambda x^4}{24} + c_1x^3 + c_2x^2 + c_3x + c_4$ ✓

mit $\underbrace{q(0) = q'(0) = 0}_{\Rightarrow\ \varepsilon(0) = \varepsilon'(0) = 0}$ ✓ $\Rightarrow q(x) = \frac{\lambda x^4}{24} + c_1x^3 + c_2x^2\quad (c_3 = c_4 = 0)$ ✓

hiermit kann man noch c_1 und c_2 bestimmen!

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