Aufgabe 22

$$\Psi(x,0) = \frac{16}{\sqrt{63a}} \cdot \sin^5\left(\frac{\pi x}{a}\right) = \frac{16}{\sqrt{63a}} \frac{1}{16}\Big(10\underbrace{\sin\left(\frac{\pi x}{a}\right)}_{\phi_1} - 5\underbrace{\sin\left(\frac{3\pi x}{a}\right)}_{\phi_3}$$

$$+\underbrace{\sin\left(\frac{5\pi x}{a}\right)}_{\phi_5}\Big) = c_1\phi_1 + c_2\phi_3 + c_3\phi_5$$

$$\phi_n = \sin\left(\frac{n\pi x}{a}\right)$$

$$H\phi_n = E_n\phi_n \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\phi_n = \underbrace{\frac{\hbar}{2m}\left(\frac{n\pi}{a}\right)^2}_{E_n}\phi_n$$

$$\Psi(x,t) = \frac{10}{\sqrt{63a}} e^{\frac{-iE_1 t}{\hbar}} \sin\left(\frac{\pi x}{a}\right) - \frac{5}{\sqrt{63a}} e^{-i\frac{E_3 t}{\hbar}} \sin\left(\frac{3\pi x}{a}\right)$$

$$+ \frac{1}{\sqrt{63a}} e^{\frac{-iE_5 t}{\hbar}} \sin\left(\frac{5\pi x}{a}\right)$$

$$P_1 = \int_0^{a/3} \Psi^*(x,t)\,\Psi(x,t)\,dx \qquad P_2 = \int_{a/3}^{a} \Psi^*(x,t)\,\Psi(x,t)$$

$$|\Psi(x,t)|^2 = \frac{1}{63a}\Big(100\sin^2\left(\frac{\pi x}{a}\right) + 25\sin^2\left(\frac{3\pi x}{a}\right) + \sin^2\left(\frac{5\pi x}{a}\right)$$

$$-100\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{2\pi\hbar}{ma^2}t\right)$$

$$-10\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{4\pi h}{ma^2}t\right)$$

$$+20\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{6\pi h}{ma^2}t\right)\Big)$$

$$P_1 = \frac{1}{63a}\Big(\frac{100}{24}\left(4 - \frac{3\sqrt{3}}{\pi}\right)a - 50e^{-\frac{i\hbar t E_2}{\hbar}}\frac{3\sqrt{3}a}{16\pi} - 10e^{-\frac{it(E_1-E_5)}{\hbar}}$$

$$\cdot\frac{3\sqrt{3}a}{16\pi}a - \frac{50}{16\pi}e^{\frac{itE_2}{\hbar}}3\sqrt{3}a + \frac{25}{6}a - \frac{5}{32\pi}e^{\frac{itE_4}{\hbar}}3\sqrt{3}a$$

$$+\frac{25}{6}a - \frac{5}{32a}e^{\frac{itE_4}{\hbar}}3\sqrt{3}a - \frac{10}{16\pi}e^{it(E_5-E_1)}\sqrt{3}a$$

$$-\frac{5}{32\pi}e^{\frac{itE_4}{a}}3\sqrt{3}a$$

$$P_2 = 1 - P_1$$

$$\vec{j} = \frac{i\hbar}{2m}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right)$$

$$\vec{\nabla}\Psi^* = \frac{1}{\sqrt{63a}}\Big(\frac{10\pi}{a}e^{\frac{iE_1 t}{\hbar}}\cos\left(\frac{\pi x}{a}\right) - \frac{15}{a}e^{\frac{iE_3 t}{\hbar}}\cos\left(\frac{3\pi x}{a}\right) +$$

$$\frac{5\pi}{a}e^{\frac{iE_5 t}{\hbar}}\cos\left(\frac{5\pi x}{a}\right)\Big) \quad ; \quad \vec{\nabla}\Psi = \hbar\, c$$

(hermitisch konjugiert)

$$\vec{j} = \frac{i\hbar}{126ma}\Big(-\frac{150\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)\left(-e^{\frac{it(E_1-E_3)}{\hbar}}+e^{\frac{it(E_3-E_1)}{\hbar}}\right)$$

$$+\frac{50\pi}{a}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right)\left(-e^{\frac{it(E_1-E_5)}{\hbar}}+e^{\frac{it(E_5-E_1)}{\hbar}}\right)-\frac{50}{a}\pi\sin\left(\frac{\pi x}{a}\right)$$

$$\cos\left(\frac{\pi x}{a}\right)\left(-e^{\frac{it(E_3-E_1)}{\hbar}}+e^{\frac{it(E_1-E_3)}{\hbar}}\right)-\frac{25}{a}\pi\sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right)$$

$$\left(-e^{it(E_3-E_5)\cdot\hbar^{-1}}+e^{\frac{it}{\hbar}(E_5-E_3)}\right)+\frac{10\pi}{a}\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)$$

$$\left(-e^{\frac{it(E_5-E_1)}{\hbar}}+e^{\frac{it(E_1-E_5)}{\hbar}}\right)-\frac{15\pi}{a}\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)$$

$$\left(-e^{\frac{it(E_5-E_3)}{\hbar}}+e^{\frac{it}{\hbar}(E_3-E_5)}\right)\Big)$$

$$=\frac{5\pi\hbar}{\sqrt{1323}\,ma^2}\cdot\left(60\sin\left(\frac{2\pi\hbar}{ma^2}t\right)+3\sin\left(\frac{4\pi\hbar}{ma^2}t\right)+4\sin\left(\frac{6\pi\hbar}{ma^2}t\right)\right)$$