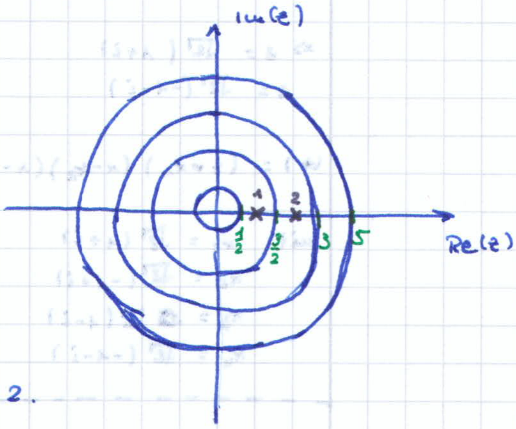
# HM 3 - Übungsblatt 6

## Aufgabe 26

(I) $$\int_{K_r(0)} \underbrace{\frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}}_{=: f(z)} \, dz \quad , \quad r \in \left\{ \frac{1}{2}, \frac{3}{2}, 3, 5 \right\}$$




f hat Singularitäten (genauer: Pole 1. Ordnung) in $z_1 = 1$, $z_2 = 2$.
Berechne daher hier die Residuen

$$\mathrm{Res}(f;1) = \lim_{z\to 1} \left( (z-1) f(z) \right) = \frac{0+(-1)}{-1} = \underline{1}$$

$$\mathrm{Res}(f;2) = \lim_{z\to 2} \left( (z-2) f(z) \right) = \frac{0+1}{1} = \underline{\underline{1}}$$

**Residuensatz:**

$$\int_K f(z)\,dz = 2\pi i \sum_k \mathrm{Res}(f; z_k)$$

$z_k$: Singularität von f in K

**Integralwerte berechnen:**

$$\int_{K_{\frac{1}{2}}(0)} f(z)\,dz = 0 \quad , \text{ da } f \text{ in } K_{\frac{1}{2}}(0) \text{ holomorph}$$

$$\int_{K_{\frac{3}{2}}(0)} f(z)\,dz = 2\pi i \,\mathrm{Res}(f;1) = \underline{2\pi i}$$

$$\int_{K_3(0)} f(z)\,dz = 2\pi i \left( \mathrm{Res}(f;1) + \mathrm{Res}(f;2) \right) = 2\pi i (1+1) = \underline{4\pi i}$$

$$\int_{K_5(0)} f(z)\,dz = \underline{4\pi i} \qquad \int_{K_\eta(0)} f(z)\,dz = 4\pi i \text{ für } \eta > 3$$

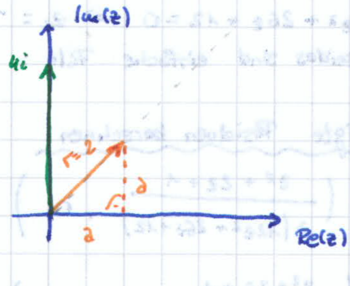
(II) $$\int_{|z|=2} \underbrace{\frac{e^{2z}}{(z+1)^4}}_{=: f(z)} \, dz \qquad f \text{ hat Pol 4. Ordnung in } z = -1$$

$$\Rightarrow \mathrm{Res}(f;-1) = \lim_{z\to -1} \frac{d^3}{dz^3} \left[ \frac{1}{3!} \left[ (z+1)^4 \cdot f(z) \right] \right] = \frac{1}{6} \lim_{z\to -1} 8 e^{2z} = \frac{1}{6} \cdot 8 e^{-2} = \frac{4}{3e^2}$$

$$\Rightarrow \int_{|z|=2} f(z)\,dz = 2\pi i \,\mathrm{Res}(f;-1) = 2\pi i \frac{4}{3\cdot e^2} = \frac{8\pi i}{3} e^{-2}$$

## Aufgabe 27

$$\int_{-\infty}^{\infty} \underbrace{\frac{x^2}{x^4+16}}_{=: f(x)} \, dx$$




Formel von Moivre:

$$(re^{i\varphi})^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

Es gilt $x^4 + 16 = (x^2 + 4i)(x^2 - 4i) = (*)$

$$r = 2 = \sqrt{4} = \sqrt{|4i|}$$

**Pythagoras:**

$$2a^2 = 4 \Leftrightarrow a^2 = 2 \Leftrightarrow a = \sqrt{2} \vee a = -\sqrt{2}$$

$\Rightarrow z = \sqrt{2}\,(1+i)$

$z = \sqrt{2}\,(-1-i)$

$(*) : (x-x_1)(x-x_2)(x-x_3)(x-x_4)$

mit $x_1 = \sqrt{2}\,(1+i)$

$x_2 = \sqrt{2}\,(-1+i)$

$x_3 = \sqrt{2}\,(1-i)$

$x_4 = \sqrt{2}\,(-1-i)$

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28.26 mit $H = \mathbb{H}_+$, $N = \{x_1, x_2, x_3, x_4\}$, $f : H \setminus N \to \mathbb{C}$ ist holomorph,

$\lim_{R\to 0} R\,|f(Re^{i\varphi})| = 0$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(f; z) = 2\pi i\left(\operatorname{Res}(f; x_1) + \operatorname{Res}(f; x_2)\right)$$

$$\overset{*}{=} 2\pi i\left(\frac{(\sqrt{2}(i+1))^2}{4(\sqrt{2}(i+1))^3} + \frac{(\sqrt{2}(-1+i))^2}{4(\sqrt{2}(-1+i))^3}\right)$$

$$\boxed{* \quad R = \frac{f}{g},\ f,g \text{ holomorph},\ g(z_0) = 0,\ g'(z_0) \neq 0 \ \Rightarrow \operatorname{Res}(R(z); z_0) = \frac{f(z_0)}{g'(z_0)}}$$

$$= \frac{2\pi i}{4}\left(\frac{1}{\sqrt{2}(i+1)} + \frac{1}{\sqrt{2}(-1+i)}\right)$$

$$= \frac{2\pi i}{4\sqrt{2}}\left(\frac{1-i}{2} + \frac{-1-i}{2}\right) = \frac{\sqrt{2}\pi i}{4}\cdot\frac{(-2i)}{2} = \frac{\sqrt{2}\pi}{4} \quad //$$

## Aufgabe 28

$$\int_0^{2\pi} \frac{1+\cos(x)}{13+12\cos(x)}\,dx =: \alpha \qquad 28.25 \Rightarrow \cos(x) \mathrel{\hat{=}} \frac{x^2+1}{2x}$$

$$\Rightarrow \alpha = 2\pi \sum_{|z_k|<1} \operatorname{Res}\left(\frac{1}{z}\,\frac{1+\frac{z^2+1}{2z}}{13+12\,\frac{z^2+1}{2z}}; z_k\right) = 2\pi \sum_{|z_k|<1} \operatorname{Res}\left(\frac{z^2+2z+1}{z(12z^2+26z+12)}; z_k\right)$$

Singularitäten:

- $z_0 = 0$, einfacher Pol
- $12z^2+26z+12 = 0 \iff z_1 = -\frac{2}{3},\ z_2 = -\frac{3}{2}$

  Beides sind einfache Pole: $|z_1| < 1$, $|z_2| > 1$

Benötigte Residuen berechnen:

$$\operatorname{Res}\left(\frac{z^2+2z+1}{z(12z^2+26z+12)}, 0\right) = \lim_{z\to 0} \frac{z^2+2z+1}{12z^2+26z+12} = \frac{1}{12}$$

$$\operatorname{Res}\left(\frac{z^2+2z+1}{z(12z^2+26z+12)}, -\frac{2}{3}\right) = \lim_{z\to -\frac{2}{3}} \left(\left(z+\frac{2}{3}\right)\frac{z^2+2z+1}{z(z+\frac{2}{3})(z+\frac{3}{2})}\right)$$

$$= \lim_{z\to -\frac{2}{3}} \frac{z^2+2z+1}{z(z+\frac{3}{2})} \overset{[\dots]}{=} -\frac{1}{60}$$

$$\Rightarrow \alpha = \int_0^{2\pi} \frac{1+\cos(x)}{13+12\cos(x)}\,dx = 2\pi\left(\frac{1}{12} - \frac{1}{60}\right) = \frac{2\pi}{15} \quad //$$

## Aufgabe 5.1

$$\int_{|2z-2|=1} \left(\frac{z}{z+1}\right)^n dz \quad , n \in \mathbb{N}_0$$

$$= \int_{|2z-2|=1} \frac{z^n}{(z+1)^n}\, dz \overset{f(z)=z^n}{=} \int_{|2z-2|=1} \frac{f(z)}{(z+1)^n}\, dz = \frac{2\pi i}{(n-1)!}\, f^{(n-1)}(-1)$$

Es ist $f'(z) = n z^{n-1}$

$f''(z) = n(n-1) z^{n-2}$

$\vdots$

$f^{(n-2)}(z) = n(n-1)\cdot \ldots \cdot 3 \cdot z^2$

$f^{(n-1)}(z) = n(n-1)\cdot \ldots \cdot 3 \cdot 2 \cdot z^1$

$= n!\, z$

$$\Rightarrow \int_{|2z-2|=1} \frac{f(z)}{(z+1)^n}\, dz = \frac{2\pi i}{(n-1)!} \cdot n! \cdot (-1) = \underline{\underline{-2\pi n i}}$$

## Aufgabe 5.2

$$\int_\gamma \frac{1}{(1+z^2)(1-z)^2}\, dz \quad , \gamma \text{ pos. Parameter von } K_2(0).$$

Def. $f(z) = \dfrac{1}{(1+z^2)(1-z)^2}$

$\Rightarrow z_1 = i$, $z_2 = -i$ einfache Pole

$z_3 = 1$ doppelter Pol

$$\Rightarrow \operatorname{Res}(f; i) = \lim_{z\to i}\left[(z-i)\, f(z)\right] = \lim_{z\to i}\left[\frac{1}{(1+z^2)(1-z)^2}\right] = \frac{1}{2i(1-i)^2} = \frac{1}{2i(-2i)} = \frac{1}{4}$$

$$\Rightarrow \operatorname{Res}(f; -i) = \frac{1}{4}$$

$$\Rightarrow \operatorname{Res}(f; 1) = \lim_{z\to 1} \frac{1}{1!} \frac{d}{dz}\left[(z-1)^2 f(z)\right] = \lim_{z\to 1} \frac{d}{dz} \frac{1}{1+z^2} = \lim_{z\to 1}\left[\frac{-2z}{(1+z^2)^2}\right] = \frac{-2}{4} = \underline{\underline{-\frac{1}{2}}}$$

$$\Rightarrow \int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{3} \operatorname{Res}(f; z_j) = 2\pi i \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{2}\right) = \underline{\underline{0}}$$

## Aufgabe 6.1

$$\int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx$$

Es ist $x^4+4 = (x^2+2i)(x^2-2i) = \prod_{j=1}^{4} (x-x_j)$ mit $x_1 = 1+i$, $x_2 = -1-i$, $x_3 = -1+i$, $x_4 = 1-i$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{x^4+4} ; 1+i\right) + 2\pi i \cdot \operatorname{Res}\left(\frac{1}{x^4+4} , -1-i\right)$$

$$\operatorname{Res}\left(\frac{1}{x^4+4} , -1-i\right) = \frac{1}{4(i+1)^3} = \frac{1}{4} \frac{1}{i^3+3i^2\cdot 1+3i\cdot 1^2+1^3}$$

$$= \frac{1}{4} \cdot \frac{1}{-2+2i} = \frac{-8-8i}{64+64} = \frac{8(-1-i)}{128} = \frac{-1-i}{16} = -\frac{1}{16} - \frac{1}{16} i$$

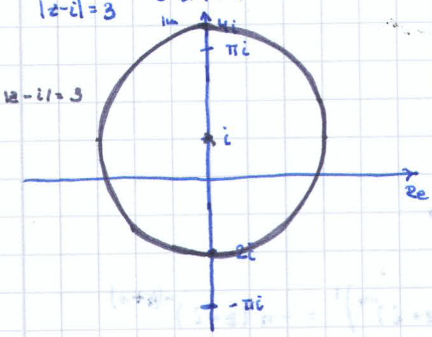
$$\operatorname{Res}\left(\frac{1}{x^4+4} ; -1+i\right) \overset{[\ldots]}{=} \frac{1}{16} - \frac{1}{16} i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx = 2\pi i \left( -\frac{1}{16} - \frac{1}{16} i + \frac{1}{16} - \frac{1}{16} i \right) = -2\pi i \, \frac{1}{8} i = \frac{\pi}{4}$$

# Globalübung - Blatt 6

## Aufgabe 29

$$\int_{|z-i|=3} \frac{2}{z\sinh(z)}\,dz$$




Singularitäten: $z=0 \to$ Pol 2. Ordnung

$\sinh(z)=0$

$\Rightarrow \frac{1}{2}(e^{z}-e^{-z})=0$

$\Leftrightarrow e^{z}=e^{-z}$

$\Leftrightarrow e^{2z}=1$

$\Rightarrow z=k\pi i\ ,\ k\in\mathbb{Z}$ da $e^{\omega}=1 \Leftrightarrow \omega=2k\pi i\quad k\in\mathbb{Z}$

$z=k\pi i\ ,\ k\in\mathbb{Z}\setminus\{0\} \to$ Pol 1. Ordnung

$$\int = 2\pi i\,\big(\operatorname{Res}(f,0)+\operatorname{Res}(f,i\pi)\big)$$

$$\operatorname{Res}\left(\frac{2}{z\sinh(z)},0\right)=\lim_{z\to0}\frac{d}{dz}\frac{2z}{\sinh(z)}=2\lim_{z\to0}\frac{\sinh(z)-z\cosh(z)}{\sinh^2(z)}$$

$$\overset{\text{„}\frac{0}{0}\text{"}}{=} 2\lim_{z\to0}\frac{-z\sinh(z)}{2\sinh(z)\cosh(z)} = 2\lim_{z\to0}\frac{-z}{2\cosh(z)}=0$$

$\sinh(i\pi)=0$

$$\operatorname{Res}\left(\frac{2}{z\sinh(z)},i\pi\right)=\frac{2}{\sinh(z)+z\cosh(z)}\bigg|_{z=i\pi}=\frac{2}{i\pi\cdot\frac{1}{2}(e^{i\pi}+e^{-i\pi})}=\frac{-2}{i\pi}$$

$$\int = 2\pi i\cdot\left(0-\frac{2}{i\pi}\right)=-4$$

## Aufgabe 30

$k\in\mathbb{N}_0 \qquad \displaystyle\int_0^{2\pi}\frac{\sin((k+1)t)}{\sin(t)}\,dt$

$z=e^{it} \qquad \dfrac{a^{k+1}-b^{k+1}}{a-b}=a^k+a^{k-1}b+a^{k-2}b^2+\ldots+b^k$

$$\frac{\sin((k+1)t)}{\sin(t)}=\frac{\frac{1}{2i}\left(e^{i(k+1)t}-e^{-i(k+1)t}\right)}{\frac{1}{2i}\left(e^{it}-e^{-it}\right)}$$

$e^{i(k+1)t}=z^{k+1}$

$e^{-i(k+1)t}=\left(\frac{1}{z}\right)^{k+1}$

$$=\frac{z^{k+1}-\left(\frac{1}{z}\right)^{k+1}}{z-\frac{1}{z}}=z^k+z^{k-2}+z^{k-4}+\left(\frac{1}{z}\right)^{k-2}+\left(\frac{1}{z}\right)^{k}=\sum_{j=0}^{k}z^{k-2j}$$

$$\Rightarrow \int_0^{2\pi}\frac{\sin((k+1)t)}{\sin(t)}\,dt=\int_{|z|=1}\sum_{j=0}^{k}z^{k-2j}\frac{1}{iz}\,dz=\frac{1}{i}\int_{|z|=1}\sum_{j=0}^{k}z^{k-2j-1}\,dz$$

$z=e^{it}$

$\frac{dz}{dt}=ie^{it}=iz$

$dt=\frac{1}{iz}\,dz$

$$=2\pi\operatorname{Res}\left(\sum_{j=0}^{k}z^{k-2j-1},0\right)$$

$0\le t\le 2\pi$

$\Leftrightarrow |z|=1$

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$$\sum_{j=0}^{k}z^{k-2j-1}=z^{k-1}+z^{k-3}+\ldots+z^{-1}+\ldots$$

$k$ ungerade $\Rightarrow k-1, k-3$, usw. gerade $\Rightarrow \frac{1}{z}$ kommt nicht vor $\operatorname{Res}(\ldots,0)=0$

$k$ gerade: $\frac{1}{z}$ kommt vor, $\operatorname{Res}(\ldots,0)=1$

Bestimmen Sie für $n = 1, 2, 3, 4, \ldots$

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^n}\,dx$$

$$\int = 2\pi i \sum_{\mathrm{Im}(z_k)>0} \mathrm{Res}\left(\frac{1}{(1+z^2)^n}, z_k\right)$$

$$= 2\pi i \;\mathrm{Res}\left(\frac{1}{(1+z^2)^n}, i\right) \quad \text{Pol } n\text{-ter Ordnung}$$

$$\mathrm{Res}\left(\frac{1}{(1+z^2)^n}, i\right) = \lim_{z\to i} \frac{1}{(n-1)!} \frac{d^{n-1}}{dt^{n-1}} \frac{1}{(z+i)^n}$$

$$= \frac{1}{(n-1)!} (-1)^{n-1} \frac{(2n-2)!}{(n-1)!} \frac{1}{(2i)^{2n-1}}$$

$$= \binom{2n-2}{n-1} \frac{1}{2^{2n-1}} \frac{(-1)^{n-1}}{i^{2n}} \quad i^{2n} = (-1)^n$$

$$= \binom{2n-2}{n-1} \frac{1}{2^{n-1}} (-i)$$

$$\left(\frac{1}{(z+i)^n}\right)' = \left((z+i)^{-n}\right)' = -n(z+i)^{-(n+1)}$$

$$\left(\frac{1}{(z+i)^n}\right)'' = (-1)^2\, n(n+1)(z+i)^{-(n+2)}$$

$$\left(\frac{1}{(z+i)^n}\right)^{(n-1)} = (-1)^{n-1}\, n(n+1)\ldots(2n-2)\,(z+i)^{-(2n-1)}$$

$$n(n+1)\ldots(2n-2) = \frac{1\cdot 2\cdot 3\cdots(n-1)\,n\cdots(2n-2)}{1\cdot 2\cdot 3\cdots(n-1)}$$

$$\int = 2\pi i\,(-i) \binom{2n-2}{n-1} \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-2}} \binom{2n-2}{n-1} \pi$$