Aufgabe 39

$$\vec{R} = \frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \gamma\frac{\hat{\vec{r}}}{\hat{r}}$$

$$\hat{\vec{R}}^{+} = \hat{\vec{R}} \quad , \quad [\hat{\vec{R}}, \hat{H}]$$

a) $\hat{\vec{R}} = a_r \hat{\vec{r}} + a_{pL}(\hat{\vec{p}}\times\hat{\vec{L}}) + a_p\vec{p}$

$$(\hat{\vec{L}}\times\hat{\vec{p}})_x = \hat{L}_y\hat{p}_z - \hat{L}_z\hat{p}_y = \hat{L}_y\hat{p}_z - \hat{p}_z\hat{L}_y + \hat{p}_z\hat{L}_y - \hat{L}_z\hat{p}_y$$

für y & z analog: $= +i\hbar p_x + \hat{p}_z\hat{L}_y - \hat{L}_z\hat{p}_y + \hat{p}_y\hat{L}_z - \hat{p}_y\hat{L}_y$

$$= +i\hbar p_x - (\hat{p}\times\hat{L})_x$$

$$\Rightarrow \hat{\vec{R}} = \frac{1}{2m}\left(2(\vec{p}\times\vec{L}) - 2i\hbar\hat{\vec{p}}\right) - \gamma\frac{\hat{\vec{r}}}{\hat{r}}$$

$$= \frac{1}{m}(\hat{\vec{p}}\times\hat{\vec{L}}) - \frac{i\hbar}{m}\hat{\vec{p}} - \gamma\frac{\vec{r}}{\hat{r}}$$

b) $\hat{\vec{L}}\cdot\hat{\vec{R}} = 0$

$$\hat{\vec{L}}\cdot\hat{\vec{R}} = \frac{1}{m}\{\vec{L}\underbrace{(\vec{p}\times\vec{L})}_{-\vec{L}\times\vec{p}+2i\hbar\vec{p}} - \frac{i\hbar}{m}\vec{L}\vec{p} - \underbrace{\frac{\gamma}{\hat{r}}\hat{\vec{L}}\cdot\hat{\vec{r}}}_{0}$$

$$= \frac{1}{m}\left(-\underbrace{(\vec{L}\times\vec{L})}_{i\hbar L}\hat{\vec{p}} + i\hbar\hat{\vec{L}}\hat{\vec{p}}\right) = 0$$

c) $$\hat{\vec{R}}^2 = \left(\frac{1}{m}(\hat{\vec{p}}\times\hat{\vec{L}}) - \frac{i\hbar}{m}\hat{\vec{p}} - \gamma\frac{\hat{\vec{r}}}{\hat{r}}\right)^2$$

$$= \underbrace{\frac{1}{m^2}(\hat{\vec{p}}\times\hat{\vec{L}})^2}_{(V)} - \underbrace{\frac{i\hbar}{m^2}(\hat{\vec{p}}\times\hat{\vec{L}})\hat{\vec{p}}}_{(IV)} - \underbrace{\frac{\gamma}{m}(\hat{\vec{p}}\times\hat{\vec{L}})\cdot\frac{\hat{\vec{r}}}{\hat{r}}}_{(II)}$$

$$- \frac{i\hbar}{m^2}\hat{\vec{p}}\cdot\underbrace{(\hat{\vec{p}}\times\hat{\vec{L}})}_{III} - \frac{\hbar^2}{m^2}\hat{\vec{p}}^2 + \frac{i\hbar\gamma}{m}\hat{\vec{p}}\cdot\frac{\hat{\vec{r}}}{\hat{r}}$$

$$- \frac{\gamma}{m}\frac{\hat{\vec{r}}}{\hat{r}}(\hat{\vec{p}}\times\hat{\vec{L}}) + \frac{i\hbar\gamma}{m}\frac{\hat{\vec{r}}}{\hat{r}}\hat{\vec{p}} + \frac{\hat{\vec{r}}^2}{\hat{r}^2}\gamma^2$$

$$(\mathrm{I})\quad \vec{\hat r}\cdot(\vec{\hat p}\times\vec{\hat L}) = (\vec{\hat r}\times\vec{\hat p})\vec{\hat L} = \vec{\hat L}^2$$

$$(\mathrm{II})\quad (\vec{\hat p}\times\vec{\hat L})\cdot\vec{\hat r} = (-\vec{\hat L}\times\vec{\hat p}+2i\hbar\vec{\hat p})\cdot\vec{\hat r} = (-\vec{\hat L}\times\vec{\hat p})\cdot\vec{\hat r}$$
$$+2i\hbar\vec{\hat p}\vec{\hat r} = \vec{\hat L}^2+2i\hbar\,\vec{\hat p}\cdot\vec{\hat r}$$

$$(\mathrm{III})\quad \vec{\hat p}\cdot(\vec{\hat p}\times\vec{\hat L}) = (\vec{\hat p}\times\vec{\hat p})\vec{\hat L} = 0$$

$$(\mathrm{IV})\quad (\vec{\hat p}\times\vec{\hat L})\vec{\hat p} = (-\vec{\hat L}\times\vec{\hat p}+2i\hbar\vec{\hat p})\vec{\hat p} = 2i\hbar\vec{\hat p}^2$$

$$(\mathrm{V})\quad (\vec{p}\times\vec{L})^2 = \vec{p}^2\vec{L}^2$$

$$\frac{\vec{\hat r}\cdot\vec{\hat p}}{\hat r} = \vec{\hat r}\cdot\frac{1}{\hat r}\vec{\hat p} = \vec{\hat r}\left(\vec{\hat p}\frac{1}{\hat r}-i\hbar\frac{\vec{\hat r}}{\hat r^3}\right) = \vec{\hat r}\vec{\hat p}\frac{1}{\hat r}-i\hbar\frac{\vec{\hat r}^2}{\hat r^3}$$
$$= (\vec{\hat r}\vec{\hat p}-i\hbar)\frac{1}{\hat r} = (\vec{\hat p}\vec{\hat r}+2i\hbar)\frac{1}{\hat r} = \frac{\vec{\hat p}\vec{\hat r}}{\hat r}+\frac{2i\hbar}{\hat r}$$

$$\Rightarrow \vec{\hat R}^2 = \frac{1}{m^2}\vec{\hat p}^2\vec{\hat L}^2 - \frac{i\hbar}{m^2}2i\hbar\vec{\hat p}^2 - \frac{g}{m}\left(\vec{\hat L}^2+2i\hbar\vec{\hat p}\vec{\hat r}\right)\frac{1}{r}$$
$$-\frac{\hbar^2}{m^2}\vec{\hat p}^2+\frac{i\hbar g}{m}\frac{\vec{\hat p}\cdot\vec{\hat r}}{\hat r}-\frac{g}{m}\frac{\vec{\hat L}^2}{\hat r}+\frac{i\hbar g}{m}\frac{\vec{\hat r}\cdot\vec{\hat p}}{\hat r}+g^2$$
$$= \frac{1}{m^2}\vec{\hat p}^2(\vec{\hat L}^2+\hbar^2)-\frac{2g}{m}\frac{\vec{\hat L}^2}{\hat r}+g^2-\frac{i\hbar g}{m}\frac{\vec{\hat p}\vec{\hat r}}{\hat r}+\frac{i\hbar g}{m}\left(\frac{\vec{\hat p}\vec{\hat r}}{\hat r}+\frac{2i\hbar}{\hat r}\right)$$
$$= g^2+\frac{1}{m^2}\vec{\hat p}^2(\vec{\hat L}^2+\hbar^2)-\frac{2g}{m}\frac{\vec{\hat L}^2}{\hat r}-\frac{2\hbar^2 g}{m\hat r}$$
$$= g^2+\left(\frac{2H}{m}\right)(\vec{\hat L}^2+\hbar^2)$$

d) $$\hat A_\pm = \frac{1}{2}\left[\vec{\hat L}\pm\sqrt{\frac{m}{-2\hat H}}\,\vec{\hat R}\right]$$

$$\hat A_\pm^2 = \frac{1}{4}\left(\vec{\hat L}^2-\frac{m}{2\hat H}\vec{\hat R}^2\right) \overset{(c)}{=} \frac{1}{4}\left(\vec{\hat L}^2-\vec{\hat L}^2-\hbar^2-\frac{mg^2}{2\hat H}\right)$$
$$= -\frac{1}{4}\left(\hbar^2+\frac{mg^2}{2\hat H}\right) \Rightarrow \hat H = -\frac{mg^2}{2(4\hat A^2+\hbar^2)}$$

(III), (IV)

$$\Rightarrow E = \frac{-mg^2}{2(4\hbar^2 a_\pm(a_\pm+1))}$$