HöMa III - Blatt 11

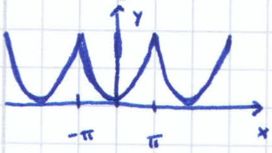
Aufgabe 51

(I) ges.: $f_1(x) = \cos(x)$, $f_2(x) = \cos^2(x)$
$\rightsquigarrow$ Fourierreihen!

$\cos(x)$ ist bereits F.R. mit $b_k = 0$ $\forall k \in \mathbb{N}$, $a_1 = 1$, $a_k = 0$ $\forall k \in \mathbb{N}\setminus\{0\}$

Es ist $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ mit $a_0 = 1$, $a_2 = \frac{1}{2}$, $a_k = b_k = 0$ sonst.

(II) $f_3(x) = x^2$, $x \in [-\pi, \pi]$ $\rightsquigarrow$ 2π-per.



f_3 ist reell und gerade

$\Rightarrow$ 1. $c_k = \overline{c_{-k}}$ ($\rightsquigarrow$ reell!)

2. $c_k = c_{-k}$ ($\rightsquigarrow$ gerade!)

Berechnung der Koeffizienten:

$\underline{k=0:}$ $\quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$

$\underline{k\neq 0:}$ $\quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-ikx}\,dx = \frac{1}{2\pi}\left[e^{-ikx}\left[\frac{x^2}{-ik} - \frac{2x}{(-ik)^2} + \frac{2}{(ik)^3}\right]\right]_{-\pi}^{\pi}$

$= \frac{1}{2\pi} \cdot 2\cos(k\pi) \cdot \frac{2\pi}{k^2}$

? $e^{ikx} = \sin(kx) + i\cos(kx)$

$= \frac{2\cos(k\pi)}{k^2} = \frac{2(-1)^k}{k^2}$

Die 2π-periodische Fortsetzung von x^2 ist stetig und stückweise stetig differenzierbar

$$\Rightarrow x^2 = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2(-1)^k}{k^2} e^{ikx} + \underbrace{\frac{\pi^2}{3}}_{=c_0}$$

Reelle F. Koeff.:

$a_k = c_k + c_{-k} \underset{2.}{=} 2c_k = \frac{4(-1)^k}{k^2}$, $k \neq 0$

$a_0 = 2c_0 = \frac{2\pi^2}{3}$

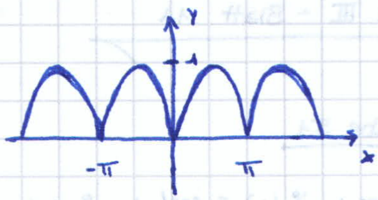
$b_k = i(c_k - c_{-k}) = 0$ $\forall k \in \mathbb{Z}$

$\rightsquigarrow$ reelle F.R.:

$$x^2 = \underbrace{\frac{\pi^2}{3}}_{\frac{a_0}{2}} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(kx)$$

Aufgabe 52

$f(x) = |\sin(x)|$, $T = 2\pi$, $x \in [-\pi, \pi]$



ges.: F.R. von f

Lösung: $b_k = 0 \ \forall k \in \mathbb{N}$, da f gerade

Berechnung der a_n:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin(x)| \cdot \cos(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx) \, dx$$

$$= \frac{2}{\pi} \cdot \begin{cases} \left[\frac{1}{2} \sin^2(x)\right]_0^{\pi} & , n = 1 \\ -\left[\frac{\cos((n+1)x)}{2(n+1)} - \frac{\cos((1-n)x)}{2(1-n)}\right]_0^{\pi} & , n \neq 1 \end{cases}$$

$$= \begin{cases} 0 & , n = 1 \\ \frac{2}{\pi} \left(\frac{1}{2(n+1)} + \frac{1}{2(1-n)} - \frac{\cos((n+1)\pi)}{2(n+1)} - \frac{\cos((1-n)\pi)}{2(1-n)} \right) & , n \neq 1 \end{cases}$$

Es ist $\cos(n\pi) = (-1)^n$ für alle $n \in \mathbb{Z}$

$$\Rightarrow \quad 1 - \cos((n+1)\pi) = 1 - \cos((1-n)\pi) = \begin{cases} 2, & n \text{ gerade} \\ 0, & n \text{ ungerade} \end{cases}$$

$$\Rightarrow \quad a_n = \begin{cases} 0 & , n \text{ ungerade} \\ -\frac{4}{\pi} \cdot \frac{1}{(n-1)(n+1)} & , n \text{ gerade} \end{cases}$$

$$\Rightarrow \quad a_0 = \frac{4}{\pi}$$

$$\rightsquigarrow \quad |\sin(x)| = \frac{2}{\pi} + \sum_{n=2}^{\infty} -\frac{4}{\pi} \frac{1}{n^2-1} \cos(nx)$$

Reihenwert für $x = 0$:

$(\sin(0) = 0 \quad , \cos(0) = 1)$

Einsetzen

$$\rightsquigarrow \quad 0 = \frac{2}{\pi} + \sum_{n=2}^{\infty} -\frac{4}{\pi} \frac{1}{n^2-1}$$

$$\Leftrightarrow \quad \sum_{n=2}^{\infty} \frac{1}{n^2-1} = -\frac{2}{\pi}\left(-\frac{\pi}{4}\right) = \frac{1}{2}$$

Reihenwert für $x = \frac{\pi}{2}$:

$(\sin(\frac{\pi}{2}) = 1 \quad , \cos(2n \frac{\pi}{2}) = (-1)^n)$

Einsetzen

$$\rightsquigarrow \quad 1 = \frac{2}{\pi} + \sum_{n=2}^{\infty} -\frac{4}{\pi} \frac{1}{n^2-1} \cos(n\pi) \overset{n=2k}{=} \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(2n\pi)$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2-\pi}{4} \quad //$$

Aufgabe 53

$f(t) = 1 - |t|$ auf $[-1, 1]$

$\check{f}$ sei die 2-periodische Forts. von f mit F.K.

$a_0 = 1$, $a_k = 0$ (gerades k)

$a_k = \frac{4}{k^2\pi^2}$ (ungerades k)

$b_k = 0$

(I) F.K. a_k', b_k' der 2π-periodischen Fortsetzung von $g: [-\pi, \pi] \to \mathbb{R}$, $t \mapsto 1 - \left|\frac{t}{\pi}\right|$

$$a_k' = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(kt)\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{t}{\pi}\right) \cos(kt)\, dt$$

$s = \frac{t}{\pi}$, $t = s\pi$

$t = \pm\pi$, $s = \pm 1$

$ds = \frac{1}{\pi} dt \Leftrightarrow dt = \pi\, ds$

$$= \frac{1}{\pi} \int_{-1}^{1} f(s) \cos(\underbrace{k}_{\in \mathbb{N}} s\pi)\, \pi\, ds = a_k$$

Theoretische Anmerkung:

Für eine 2-periodische Funktion ist $\omega = \frac{2\pi}{2} = \pi$ und

$$a_k \int_{t_0}^{t_0+2} f(t) \cos(\omega k t)\, dt = \int_{-1}^{1} f(s) \cos(k\pi s)\, ds$$

f ist gerade $\Rightarrow$ g ist gerade $\Rightarrow$ $b_k = b_k' = 0 \quad \forall k \in \mathbb{Z}$

(II) ges.: F.K. a_k'', b_k'' von $h: [-1, 1] \to \mathbb{R}$, $h(t) = \frac{1}{2} t (1 - |t|)$

Lösung: Es ist
$$h'(t) = \frac{d}{dt} \begin{cases} -\frac{1}{2} t^2 + \frac{1}{2} t, & t > 0 \\ \frac{1}{2} t^2 + \frac{1}{2} t, & t < 0 \end{cases}$$

$$= \begin{cases} -t + \frac{1}{2}, & t > 0 \\ t + \frac{1}{2}, & t < 0 \end{cases} = -|t| + \frac{1}{2}$$

$\Rightarrow h'(t) = \frac{1}{2} - |t| = f(t) - \frac{1}{2}$

$$\Rightarrow f(t) - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{4}{\pi^2 (2k-1)^2} \cos((2k-1)\pi) = h'(t)$$

$$\Rightarrow h(t) = \sum_{k=1}^{\infty} \frac{4}{\pi^3 (2k-1)^3} \sin((2k-1)\pi t) + C, \quad C \in \mathbb{R}$$

$h(0) = 0 \Rightarrow C = 0$

h ist ungerade $\Rightarrow a_k' = 0 \quad \forall k \in \mathbb{Z}$

$\Rightarrow b_k'' = 0$ für gerade $k \in \mathbb{N}$

$b_k'' = \frac{4}{\pi^3 k^3}$ für ungerade $k \in \mathbb{N}$

Aufgabe 11.1

$I = [a,b] \subset \mathbb{R}$

$y'' + cy' + dy = 0 \qquad c,d \in \mathbb{R}$

$y(a) = y(b) = 0$

$\mu := \frac{c^2}{4} - d \in \mathbb{R}$

Charakteristisches Polynom: $\lambda^2 + c\lambda + d = 0$

$$\Leftrightarrow \lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - d}$$

1. Fall: $\mu = 0 \Rightarrow \lambda = -\frac{c}{2}$

$\Rightarrow$ F.S.: $\{e^{\lambda x}, x e^{\lambda x}\}$

Randbedingungen: $\begin{pmatrix} e^{\lambda a} & a e^{\lambda a} \\ e^{\lambda b} & b e^{\lambda b} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = Rc$

mit $|R| = b e^{\lambda a + \lambda b} - a e^{\lambda a + \lambda b} = e^{\lambda a + \lambda b}(b-a) \neq 0$

$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ nur triviale Lösungen

2. Fall: $\mu > 0 \Rightarrow$ F.S.: $\{e^{\lambda_1 x}, e^{\lambda_2 x}\} \qquad \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2$

Allg. Lsg.: $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \qquad c_1, c_2 \in \mathbb{R}$

Randbedindungen: $\underbrace{\begin{pmatrix} e^{\lambda_1 a} & e^{\lambda_2 a} \\ e^{\lambda_1 b} & e^{\lambda_2 b} \end{pmatrix}}_{=:R} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

mit $|R| = e^{\lambda_1 a + \lambda_2 b} - e^{\lambda_1 b + \lambda_2 a} \neq 0$

da $\lambda_1 a + \lambda_2 b = \lambda_1 b + \lambda_2 a \Leftrightarrow \lambda_1 \underbrace{(a-b)}_{\neq 0} = \lambda_2 \underbrace{(a-b)}_{\neq 0} \Leftrightarrow \lambda_1 = \lambda_2$ ↯

$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ nur triviale Lsg.

3. Fall: $\mu < 0 \Rightarrow$ F.S.: $\{e^{-\frac{c}{2}x}\cos(x\sqrt{-\mu}),\ e^{-\frac{c}{2}x}\sin(x\sqrt{-\mu})\}$

Randbedingungen: $\underbrace{\begin{pmatrix} e^{-\frac{c}{2}a}\cos(a\sqrt{-\mu}) & e^{-\frac{c}{2}a}\sin(a\sqrt{-\mu}) \\ e^{-\frac{c}{2}b}\cos(b\sqrt{-\mu}) & e^{-\frac{c}{2}b}\sin(b\sqrt{-\mu}) \end{pmatrix}}_{=:R} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

mit $|R| = e^{-\frac{c}{2}(a+b)}\cos(a\sqrt{-\mu})\sin(b\sqrt{-\mu}) - e^{-\frac{c}{2}(a+b)}\cos(b\sqrt{-\mu})\sin(a\sqrt{-\mu})$

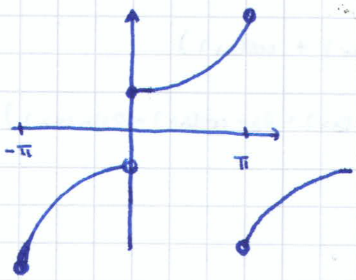
$= e^{-\frac{c}{2}(a+b)}\,(\cdot/\cdot) \underset{\text{Add. Theorem}}{=} e^{-\frac{c}{2}(a+b)}\sin(\sqrt{-\mu}(b-a))$

b) Fall 1, Fall 2 $\Rightarrow$ a) $\}\Rightarrow$ Beh.

Globalübung - Blatt 11

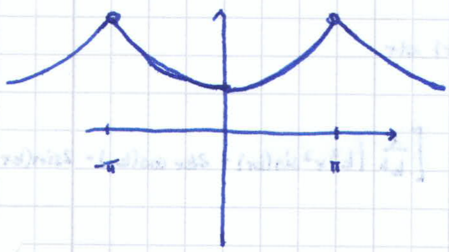
Aufgabe 54

$f(x) = e^x \quad , \quad 0 \leq x \leq \pi$



① Koeffizienten der Reihe der ungeraden Fortsetzung sind $\sim \frac{1}{k}$

$$\sum_{k=1}^{\infty} b_k \sin(kx) = \begin{cases} e^x & , \quad 0 < x < \pi \\ -e^{-x} & , \quad -\pi < x < 0 \\ 0 & , \quad x = 0, \pm\pi \end{cases}$$



② ... geraden ...

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) \qquad a_k \sim \frac{1}{k^2}$$

$$a_k = \frac{2}{(k^2+1)\pi}\left(e^{\pi}(-1)^k - 1\right) \qquad b_k = \frac{2}{(k^2+1)\pi}\left(k\, e^{\pi}(-1)^{k+1} - 1\right)$$

Aufgabe 55

$$f(x) = \frac{1}{5 + 3\cos(x)}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos(kx)}{5 + 3\cos(x)}\, dx \qquad \cos(kx) = \mathrm{Re}\left(e^{ikx}\right) \quad , \quad \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$= \mathrm{Re}\left(\frac{1}{\pi}\int \frac{e^{ikx}}{5 + 3\cos(x)}\, dx\right) = \mathrm{Re}\left(\frac{1}{\pi}\int \frac{\left(e^{ix}\right)^k}{5 + \frac{3}{2}\left(e^{ix} + \frac{1}{e^{ix}}\right)}\, dx\right)$$

$z = e^{ix}$
$\frac{dz}{dx} = i e^{ix} = iz$
$dx \hat{=} \frac{1}{iz}\, dz$

$$= \frac{1}{\pi}\mathrm{Re}\left(\int_{|z|=1} \frac{z^k}{5 + \frac{3}{2}\left(z + \frac{1}{z}\right)} \frac{1}{iz}\, dz\right) = \frac{1}{\pi}\mathrm{Re}\left(\int_{|z|=1} \frac{-2i z^k}{3z^2 + 10z + 3}\, dz\right)$$

$3z^2 + 10z + 3 = 0$

$\Rightarrow z^2 + \frac{10}{3}z + 1 = 0$

$\Rightarrow z_{1/2} = -\frac{5}{3} \pm \sqrt{\frac{25}{9} - 1} = -\frac{5}{3} \pm \frac{4}{3}$

$\Rightarrow z_1 = -3 \quad , \quad z_2 = -\frac{1}{3}$

$$= \frac{1}{\pi}\mathrm{Re}\left[2\pi i \,\mathrm{Res}\left(\frac{-2iz^k}{3z^2 + 10z + 3} ; -\frac{1}{3}\right)\right]$$

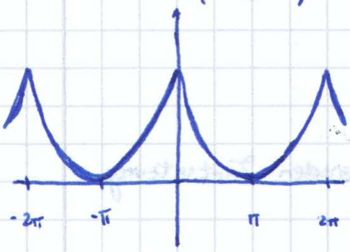
$$= 4\,\mathrm{Re}\left(\left.\frac{z^k}{6z + 10}\right|_{z = -\frac{1}{3}}\right)$$

$$= 4\,\frac{\left(-\frac{1}{3}\right)^k}{8} = \frac{1}{2}\left(-\frac{1}{3}\right)^k$$

$$\frac{1}{5 + 3\cos(x)} = \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k \cos(kx)$$

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \left(\frac{\pi - x}{2}\right)^2 \qquad x \in [0, 2\pi] \qquad f(x) = f(x - 2\pi)$$



$$a_k = \frac{2}{\pi} \int_0^{\pi} \frac{(\pi - x)^2}{4} \cos(kx)\, dx$$

$$\int x \cos(ax)\, dx = \frac{1}{a^2}\left(ax \sin(ax) + \cos(ax)\right)$$

$$\int x^2 \cos(ax)\, dx = \frac{1}{a^3}\left(a^2 x^2 \sin(ax) + 2ax \cos(ax) - 2 \sin(ax)\right)$$

$$= \frac{1}{2\pi} \int_0^{\pi} (\pi^2 - 2\pi x + x^2) \cos(kx)\, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \cos(kx)\, dx - \int_0^{\pi} x \cos(kx)\, dx + \frac{1}{2\pi} \int_0^{\pi} x^2 \cos(kx)\, dx$$

$$= \frac{\pi}{2} \left[\sin(kx)\right]_0^{\pi} - \left[\frac{1}{k^2}\left(kx \sin(kx) + \cos(kx)\right)\right]_0^{\pi} + \frac{1}{2\pi} \left[\frac{1}{k^3}\left(k^2 x^2 \sin(kx) + 2kx \cos(kx) - 2 \sin(kx)\right)\right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{\cos(kx)}{k^2} (-2\pi) + \frac{2x \cos(kx)}{k^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \frac{1}{k^2} \left[-\pi (-1)^k + \pi + \pi (-1)^k\right] = \frac{1}{k^2}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \frac{(\pi - x)^2}{4}\, dx = \frac{1}{2\pi} \int_0^{\pi} (x - \pi)^2\, dx = \frac{1}{2\pi} \frac{(x - \pi)^3}{3}\Big|_0^{\pi} = \frac{1}{2\pi} \frac{\pi^3}{3} = \frac{\pi^2}{6}$$

$$f(x) = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

$$f(0) = \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$f'(x) = \frac{\pi - x}{2}$$