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Aufgabe 1

$$\omega(\vec{k}) = c|\vec{k}|^6 = ck^6$$

$$E(\vec{k}) = \hbar c k^6$$

in d-Dimensionen => d-Dimensionale Kugelkoordinaten

$$g(E) = \frac{V}{(2\pi)^3}\int d^3k \; \delta(E-E_k)$$

$$= \frac{V}{2\pi^2}\int_0^\infty k^2 \underbrace{\delta(E-\hbar c k^6)}_{g(k) \Rightarrow \text{ nicht einfach integrierbar}} dk$$

$$\int_{-\infty}^{\infty}\phi(x)\delta(g(x))dx = \sum_{i=1}^{n}\int_{-\infty}^{\infty}\phi(x)\frac{\delta(x-x_i)}{|g'(x_i)|}dx$$

mit x_i Nullstelle von $g(x)$ oder $\hbar c k^6 = E$ substituieren

NR: $E = \hbar c k^6 \iff \left(\frac{E}{\hbar c}\right)^{\frac{1}{6}} = k$

$$= \frac{V}{2\pi^2}\left(\frac{E}{\hbar c}\right)^{\frac{2}{6}}$$ ff.

b)

$$U(T) = \int \rho(E)\cdot E \cdot g(E)\, dE \qquad \text{mit } g(E) = \frac{1}{\exp\left(\frac{E}{k_B T}\right)-1}$$

Taylor: $g(E) \approx \frac{k_B T}{E}$

$$U(T) = \int \frac{V}{2\pi^2}\left(\frac{E}{\hbar c}\right)^{\frac{2}{6}}\cdot E \cdot \frac{k_B T}{E}\, dE$$

$$= \underbrace{\frac{V k_B}{2\pi^2}\cdot\left(\frac{1}{\hbar c}\right)^{\frac{2}{6}}}_{A}\cdot T\int E^{\frac{2}{3}}\, dE$$

$$= A\cdot T\cdot \frac{6}{2} E^{\frac{2}{6}+1} + C$$

$$= \frac{A6T}{2}E^{\frac{2}{6}+1} + C$$ ff.

c)

$$c_V = \left.\frac{\partial U}{\partial T}\right|_V = \frac{A6}{2}E^{\frac{2}{6}+1}$$ ff.

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A1	A2	A3	A4	Ges
2	3	2,5	3	10,5

A1)

$\omega(k) = c \cdot |\vec{k}|^b$ in d Dimensionen

a)

Zustandsdichte: $\rho(E) = \frac{1}{(2\pi)^d} \int d^dk \, \delta(E - E)$ mit $E = \hbar\omega(k)$

$\rightarrow \rho(E) \sim \int d^dk \, \delta(\hbar c |\vec{k}|^b - E) \sim \int_0^\infty dk \, k^{d-1} \Omega^d \delta(\hbar c k^b - E)$

NR: $\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{g'(x)}$ $\Rightarrow E - \hbar c k^b = 0$

$\Rightarrow k = \left(\frac{E}{\hbar c}\right)^{\frac{1}{b}}$

$\Rightarrow k$ für gerade b egal, da $\int_0^\infty$

$\int_0^\infty \delta(E - \hbar c k^b) \ldots = \int \delta\left(k - \left(\frac{E}{\hbar c}\right)^{\frac{1}{b}}\right) \frac{1}{-b \hbar c \left(\frac{E}{\hbar c}\right)^{\frac{b-1}{b}}}$

$\rho \sim E^{\frac{d}{b} - 1} \, \Theta(E)$

b)

$u(T) = \int \rho(E) \, b(E) \cdot E \, dE \sim \int_0^\infty E^{\frac{d}{b}} \frac{1}{e^{\beta(E-\mu)} - 1} dE$

$\underset{\mu=0}{\sim} \int_0^\infty x^{\frac{d}{b}} \beta^{-\frac{d}{b} - 1} \frac{1}{e^x - 1} \sim \beta^{-\frac{d}{b} - 1} \sim T^{\frac{d}{b} + 1}$

2) a) bekannt für Bosonen

$$\Omega = k_B T \sum_i \ln\left(1 - e^{-\beta(\epsilon_i - \mu)}\right) \qquad i \text{ über alle Besetzungen}$$

$$\text{mit } \epsilon_i = \frac{p_i^2}{2m} \text{ und } \sum_i \to \frac{V}{(2\pi\hbar)^d}\int_{-\infty}^{\infty} d^d p \to \frac{V S_d}{(2\pi\hbar)^d}\int_0^{\infty} p^{d-1}\, dp$$

$$\to \Omega = \frac{k_B T V}{(2\pi\hbar)^d}\int_{-\infty}^{\infty} d^d p \ln\left(1 - e^{-\beta\left(\frac{p^2}{2m} - \mu\right)}\right)$$

$$= \frac{k_B T V}{(2\pi\hbar)^d}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^{\infty} dp\, p^{d-1} \ln\left(1 - e^{-\beta\left(\frac{p^2}{2m} - \mu\right)}\right)$$

$$\left.\begin{array}{l} x = \beta\frac{p^2}{2m} \quad \Leftrightarrow p = \sqrt{\frac{2m}{\beta}x} \\ dx = \frac{\beta}{2m} p\, dp \end{array}\right\} \text{Substitution}$$

$$\Omega = \frac{k_B T V}{(2\pi\hbar)^d}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^{\infty} dx\, \frac{2m}{\beta}\left(\frac{2m}{\beta}x\right)^{\frac{d-1}{2}} \ln\left(1 - e^{\beta\mu} e^{-x}\right)$$

$$= \frac{k_B T V\, 2\pi^{\frac{d}{2}}}{(2\pi\hbar)^d\,\Gamma\left(\frac{d}{2}\right)}\left(\frac{2m}{\beta}\right)^{\frac{d}{2}}\int_0^{\infty} dx\, x^{\frac{d}{2}-1} \ln\left(1 - z e^{-x}\right) \checkmark$$

$$\text{mit } z = e^{\beta\mu}$$

$$\frac{1}{\Gamma\left(\frac{d}{2}\right)}\int_0^{\infty} dx\, x^{\frac{d}{2}-1} \ln\left(1 - z e^{-x}\right)$$

$$= \underbrace{\frac{1}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{\frac{d}{2}} x^{\frac{d}{2}} \ln\left(1 - z e^{-x}\right)\Big|_0^{\infty}}_{=0} - \frac{1}{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)}\int_0^{\infty} x^{\frac{d}{2}}\frac{+z e^{-x}}{1 - z e^{-x}}\, dx$$

$$= \frac{-1}{\Gamma\left(\frac{d}{2}+1\right)}\int_0^{\infty} x^{\frac{d}{2}}\frac{1}{z^{-1}e^{x} - 1}\, dx = f^{+}_{\frac{d}{2}+1}(z)\cdot(-1)$$

$$\Rightarrow \Omega = -2 k_B T V\left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{d}{2}} f^{+}_{\frac{d}{2}+1}(z) = -\frac{V}{\lambda^d} k_B T f^{+}_{\frac{d}{2}+1}(z)$$

hier besser $n = \frac{\partial}{\partial(\mu\beta)} \ln \mathcal{Z}$

$$n = \frac{N}{V} = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu}\Big|_{T,V} = 2 k_B T\left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{d}{2}}\frac{\partial}{\partial\mu} f^{+}_{\frac{d}{2}+1}(z)$$

$$= 2 k_B T\left(\frac{m}{2\pi\beta\hbar^2}\right)^{\frac{d}{2}} \beta z \frac{\partial}{\partial z} f^{+}_{\frac{d}{2}+1}(z) = \frac{1}{\lambda^d} f^{+}_{\frac{d}{2}}(z)$$

b) $\Omega = -k_B T \ln \mathcal{Z}$

$$\Leftrightarrow \ln \mathcal{Z} = -\frac{\Omega}{k_B T} \overset{a)}{=} 2V \left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{d}{2}} f^{+}_{\frac{d}{2}+1}(z)$$

$$U = k_B T^2 \frac{\partial \ln \mathcal{Z}}{\partial T} = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} = -2V \frac{\partial}{\partial \beta}\left[\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{d}{2}} f^{+}_{\frac{d}{2}+1}(e^{\beta\mu})\right]$$

$$= -2V\left(\frac{m}{2\pi\hbar^2}\right)^{\frac{d}{2}}\left(-\frac{d}{2}\beta^{-\frac{d}{2}-1} f^{+}_{\frac{d}{2}+1}(z) + \beta^{-\frac{d}{2}} \mu z \frac{\partial}{\partial z} f^{+}_{\frac{d}{2}+1}(z)\right)$$

$$p = k_B T \frac{\partial \ln \mathcal{Z}}{\partial V} = 2\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{d}{2}} f^{+}_{\frac{d}{2}+1}(z)$$

$$\Rightarrow \frac{pV}{U} = \frac{f^{+}_{\frac{d}{2}+1}(z)}{\mu z \frac{\partial}{\partial z} f^{+}_{\frac{d}{2}+1}(z) - \frac{d}{2} f^{+}_{\frac{d}{2}+1}(z)} = \frac{1}{\mu z \left[\ln\left(f^{+}_{\frac{d}{2}+1}(z)\right)\right]^{-1} - \frac{d}{2}}$$

$$\frac{pV}{U} = \left(\mu z \left[\frac{\partial}{\partial z} \ln\left(f^{+}_{\frac{d}{2}+1}(z)\right)\right]^{-1} - \frac{d}{2}\right)^{-1}$$

$$U = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}_g = \frac{d}{2}\frac{\ln \mathcal{Z}_g}{\beta} = -\frac{d}{2}\Omega \quad \text{mit } pV = -\Omega$$

$$\Rightarrow \frac{pV}{U} = \frac{2}{d}$$

3/4

a) $U = \alpha |\vec{r}|^3 = \alpha r^3$

$\Rightarrow H = \sum_{i=0}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \alpha r_i^3$

$Z_k = \frac{1}{N!} (Z_{k_1})^N = \frac{1}{N!} (Z_{k,1,kin} \cdot Z_{k_1 pot})^N$

$Z_{k,1kin} = \underbrace{\frac{1}{(2\pi\hbar)^3}}_{h^3} \int \underbrace{d^3p}_{dp\,4\pi p^2} e^{-\frac{\beta}{2m}p^2} = \int_0^\infty dp\, p^2 e^{-\frac{\beta}{2m}p^2} \cdot \frac{1}{2\pi^2\hbar^3}$

$= \frac{1}{2\pi^2\hbar^3} \frac{\sqrt{\pi}}{4\beta^{3/2}} \cdot (2m)^{3/2} = \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} =: \left(\frac{1}{\lambda_T}\right)^3$

$Z_{k,1,pot} = \int d^3r\, e^{-\beta r^3 \alpha} = 4\pi \int_0^R dr\, r^2 e^{-\beta\alpha r^3} = \frac{4\pi}{3\alpha\beta} (\cancel{-e^{-\beta\alpha R^3} + 1})$ $1 - e^{-\beta\alpha R^3}$

$\Rightarrow Z_k = \frac{1}{N!} \left(\frac{m^{3/2}}{\sqrt{\pi}\hbar^3} \cdot \beta^{-5/2} \frac{2}{3\sqrt{2}} \left(\cancel{e^{-\beta\alpha R^3}} + 1\right)\right)^N$ $1 - e^{-\beta\alpha R^3}$

$\Rightarrow -\ln(Z_k) = \ln(N!) + N\left(\ln\left(\frac{2}{3\sqrt{2\pi}}\right) + \frac{\ln}{2}\left(\frac{\hbar^6 \beta^5}{m^3}\right) - \ln\left(e^{-\beta\alpha R^3} + 1\right)\right)$

b) $\frac{\partial}{\partial\beta}(-\ln(Z_k)) = \frac{5N}{2}\beta^{-1} + N \frac{\alpha R^3 e^{-\alpha\beta R^3}}{\cancel{e^{-\beta\alpha R^3} + 1}} =: U$ $1 - e^{-\beta\alpha R^3}$

$R \to \infty: \quad U = \frac{5N}{2}\beta^{-1} = \frac{5}{2} N k_B T$ $\qquad$ T groß $\to \beta$ klein $\to$ (2. Term $\ll$ 1 Term) $\Rightarrow \alpha \to 0$ $\Rightarrow U = \frac{3}{2} N k_B$

$\alpha \to 0: \quad \cancel{U = \frac{5N}{2}\beta^{-1} = \frac{5}{2} N k_B T}$ $\quad e^{-\beta\alpha R^3} \approx 1 - \beta\alpha R^3 \Rightarrow U = \frac{5}{2}N\beta^{-1} - \frac{N\alpha R^3}{\beta\alpha R^3} = \frac{3}{2}\frac{N}{\beta}$

c) $U = \frac{5N}{2} k_B T + \frac{N}{k_B T} \frac{3\alpha R^2 e^{-\alpha R^3/k_B T}}{e^{-\alpha R^3/k_B T} + 1} =: \frac{5}{2} N k_B T + \frac{3NC}{RT} \frac{e^{-\frac{C}{T}}}{e^{-\frac{C}{T}} + 1}$

$c_V = \frac{\partial U}{\partial T}\Big|_V = \frac{5 N k_B}{2} - \frac{3NC}{R} \frac{(T-C)e^{C/T} + T}{T^3 (e^{C/T} + 1)^2}$

$= \frac{5 N k_B}{2} - \frac{3N\alpha R^2}{k_B} \frac{\left(T - \frac{\alpha R^3}{k_B}\right) e^{\frac{\alpha R^3}{k_B T}} + T}{T^3 \left(e^{\frac{\alpha R^3}{k_B T}} + 1\right)^2}$ ff.

$T \to 0: \quad c_V = \frac{5 N k_B}{2}$

$T \to \infty: \quad c_V = \frac{5 N k_B}{2}$

2,5/4

$\epsilon_1 = -\epsilon, \epsilon_2 = \epsilon$

a)

i) unterscheidbar 8 Zustände EW: $1 \times -3\epsilon$, $1 \times 3\epsilon$, $3 \times -\epsilon$, $3 \times \epsilon$ ✓

ii) Bosonen $s=0$ 4 Zustände EW: $1 \times -3\epsilon$, $1 \times -\epsilon$, $1 \times \epsilon$, $1 \times 3\epsilon$ ✓

iii) Fermionen: Pauli → ~~2~~ 4 Zustände EW: $1 \times -\epsilon$, $1 \times \epsilon$ f

4 da das freie Fermion einmal Spin ↑ und ↓ haben kann.

b) $Z_k = \sum_\ell Z_m(\epsilon_\ell)\, e^{-\beta \epsilon_\ell}$

Entartung

Die Musterlösung ignoriert offenbar diese Entartung

Musterlösung geht von Ein-Teilchen Zustandssumme aus

I: $Z_k = e^{-3\beta\epsilon} + e^{3\beta\epsilon} + 3(e^{-\beta\epsilon} + e^{\beta\epsilon})$

Wolfram alpha

$= (\cosh(3\beta\epsilon) + 3\cosh(\beta\epsilon)) \cdot 2 \overset{!}{=} 4\cosh^3(\beta\epsilon) \cdot 2 = 8\cosh^3(\beta\epsilon)$ ✓

$U = -\frac{\partial}{\partial\beta}\left(\ln(2) + \ln\left(\cosh(3\beta\epsilon) + 3\cosh(\beta\epsilon)\right)\right)$

$= -\left(\frac{3\epsilon(\sinh(3\beta\epsilon) + \sinh(\beta\epsilon))}{\cosh(3\beta\epsilon) + 3\cosh(\beta\epsilon)}\right)$

$= -3\epsilon \tanh(\beta\epsilon)$ ✓

II: $Z_k = e^{-3\beta\epsilon} + e^{3\beta\epsilon} + e^{-\beta\epsilon} + e^{\beta\epsilon} = 2(\cosh(3\beta\epsilon) + \cosh(\beta\epsilon))$

$U = -\epsilon\,\frac{3\sinh(3\beta\epsilon) + \sinh(\beta\epsilon)}{\cosh(3\beta\epsilon) + \cosh(\beta\epsilon)}$ ✓

$= -\epsilon\,(\tanh(x) + 2\tanh(2x))$ ✓

III: $Z_k = 2(e^{-\beta\epsilon} + e^{\beta\epsilon}) = 4\cosh(\beta\epsilon)$ s.o.

$U = \tanh(\beta\epsilon) \cdot (-\epsilon)$ ✓

c) $T \to 0$: $U_1 = -3\epsilon$ ✓

$\beta \to \inf$ $\quad U_2 = -3\epsilon$ ✓ → Grundzustand

$U_3 = -\epsilon$ ✓

$T \to \infty$ $\quad U_1 = U_2 = U_3 = 0$ ✓ → gleich besetzt

$\beta \to 0$

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