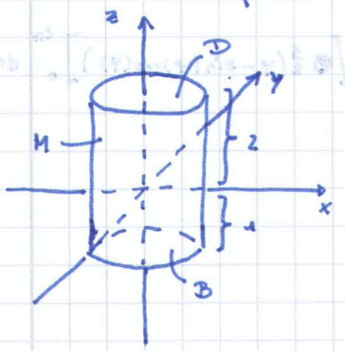
HöMa III – Blatt 1

Aufgabe 1

Z: Zylinder, Radius 2, Mittelachse ist z-Achse, Begrenzt durch $z=-1$ und $z=2$, Oberfläche sei O.

$$\vec{v}(x,y,z) = \begin{pmatrix} xy^2 \\ x^2+y^2+z^2 \\ 4x^2 \end{pmatrix}$$

Gesucht: Fluss durch O
(I) direkt
(II) mit Integralsatz



$O = D \cup M \cup B$

(I) D: $\vec{\Phi}(r,\varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ 2 \end{pmatrix}$, $0 \le r \le 2$, $0 \le \varphi \le 2\pi$

$$\Rightarrow \vec{\Phi}_r \times \vec{\Phi}_\varphi = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin(\varphi) \\ r\cos(\varphi) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} =: \vec{n}$$ zeigt nach außen wegen $r \ge 0$!

$$\Rightarrow \int_D \vec{v}\, d\vec{o} = \int_0^2\int_0^{2\pi} \begin{pmatrix} * \\ * \\ 4r^2\cos^2(\varphi) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} d\varphi\, dr = \int_0^2\int_0^{2\pi} 4r^3\cos^2(\varphi)\, d\varphi\, dr = \int_0^2 4r^3\, dr \int_0^{2\pi} \cos^2(\varphi)\, d\varphi$$

$$= r^4\Big|_0^2 \cdot \left[\tfrac{1}{2}\left(x + \sin(x)\cos(x)\right)\right]_0^{2\pi} = 16\pi$$

B: $\vec{\Phi}(r,\varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ -1 \end{pmatrix}$, $0 \le r \le 2$, $0 \le \varphi \le 2\pi$

$$\Rightarrow \vec{\Phi}_r \times \vec{\Phi}_\varphi = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$ falsch orientiert! $\Rightarrow$ Wähle $\vec{n} := \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$

$$\Rightarrow \int_B \vec{v}\, d\vec{o} = \int_0^2\int_0^{2\pi} \begin{pmatrix} * \\ * \\ 4r^2\cos^2(\varphi) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix} d\varphi\, dr = -16\pi$$

M: $\vec{\Phi}(\varphi, z) = \begin{pmatrix} 2\cos(\varphi) \\ 2\sin(\varphi) \\ z \end{pmatrix}$, $0 \le \varphi \le 2\pi$, $-1 \le z \le 2$

$$\Rightarrow \vec{\Phi}_\varphi \times \vec{\Phi}_z = \begin{pmatrix} -2\sin(\varphi) \\ 2\cos(\varphi) \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos(\varphi) \\ 2\sin(\varphi) \\ 0 \end{pmatrix}$$ Orientierung OK

$$\Rightarrow \int_M \vec{v}\, d\vec{o} = \int_{-1}^2\int_0^{2\pi} \begin{pmatrix} 8\cos(\varphi)\sin^2(\varphi) \\ 4+z^2 \\ 16\cos^2(\varphi) \end{pmatrix} \begin{pmatrix} 2\cos(\varphi) \\ 2\sin(\varphi) \\ 0 \end{pmatrix} d\varphi\, dz = \int_{-1}^2 \Big(16\sin^2(\varphi)\cos^2(\varphi) + \overbrace{8\sin(\varphi)}^{=0} + \overbrace{2z^2\sin(\varphi)}^{=0} \Big)\, d\varphi\, dz$$

$$= \int_{-1}^2\int_0^{2\pi} 16\sin^2(\varphi)\cos^2(\varphi)\, d\varphi\, dz = 3\cdot 16\int_0^{2\pi} \tfrac{1}{4}(\sin(2\varphi))^2\, d\varphi = 3\cdot 4\cdot\left[\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\left[2\varphi + \sin(2\varphi)\cos(2\varphi)\right]\right]_0^{2\pi} = 12\pi$$

Gesamtfluss : 12π

(II) Satz von Gauß

Es ist $\operatorname{div}(\vec{v}) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \vec{v}_k(x,y,z) = y^2 + 2y$

$$\int_Z \operatorname{div}(\vec{v})\, d(x,y,z) = \int_Z (y^2+2y)\, d(x,y,z) \overset{ZK.}{=} \int_{-1}^{2}\int_0^2\int_0^{2\pi} (r^2\sin^2(\varphi) + 2r\sin(\varphi))\, r \, d\varphi\, dr\, dz$$

$$= \int_{-1}^{2}\int_0^2\int_0^{2\pi} r^3\sin^2(\varphi)\, d\varphi\, dr\, dz + \underbrace{\int_{-1}^{2}\int_0^2\int_0^{2\pi} 2r^2 \sin(\varphi)\, d\varphi\, dr\, dz}_{=0} = \int_{-1}^{2}\int_0^2 r^3 \left[\frac{1}{2}(\varphi - \sin(\varphi)\cos(\varphi))\right]_0^{2\pi} dr\, dz$$

$$= \pi \cdot \int_{-1}^{2} \frac{1}{4} r^4 \Big|_0^2\, dz = 4\pi \int_{-1}^{2} 1\, dz = 12\pi$$

Aufgabe 2

$\vec{v}: \mathbb{R}^3 \to \mathbb{R}^3 \qquad \vec{v}(x,y,z) = \begin{pmatrix} 2xy - z \\ x^2 + z - y^2 \\ -x + y \end{pmatrix}$

(I) $\operatorname{div}(\vec{v}) = 2y - 2y + 0 = 0$

(II) $\operatorname{rot}(\vec{v}) = \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ -1-(-1) \\ 2x - 2x \end{pmatrix} = \vec{0}$

(III) Da $\mathbb{R}^3$ sternförmig ist existiert ein Potential.

Sei $\phi(x,y,z)$ ein gesuchtes Potential.

1. Ansatz: $\phi_x \overset{!}{=} 2xy - z = v_1$

$\Rightarrow \phi = \int (2xy - z)\, dx = x^2y - zx + c(y,z)$

2. Ansatz: $\phi_y = x^2 + c_y(y,z) \overset{!}{=} x^2 + z - y^2 = v_2$

$\Rightarrow c_y(y,z) = z - y^2$

$\Rightarrow c(y,z) = \int c_y(y,z)\, dy = yz - \frac{y^3}{3} + c(z)$

$\Rightarrow \phi = x^2y - zx + yz - \frac{y^3}{3} + c(z)$

3. Ansatz: $\phi_z = -x + y + c_z(z) \overset{!}{=} -x + y = v_3$

$\Rightarrow c_z(z) = 0 \Rightarrow c(z) = c \quad , c \in \mathbb{R}$

$\Rightarrow \phi(x,y,z) = x^2y - zx + yz - \frac{y^3}{3} + c \quad , c \in \mathbb{R}$

(IV) Da $\mathbb{R}^3$ sternförmig ist und (I) gilt, existiert ein Vektorpotential.

VL $\Rightarrow w_1(x,y,z) = \int_{z_0}^{z} v_2(x,y,t)\,dt - \int_{y_0}^{y} v_3(x,t,z_0)\,dt$, $w_2 = -\int_{z_0}^{z} v_1(x,y,t)\,dt$, $w_3 = 0$

$y_0 = z_0 = 0$

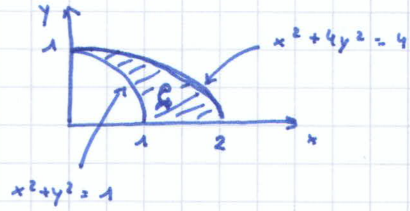
$$w_1 = \int_0^z (x^2-y^2+t)\,dt - \int_0^y (-x+t)\,dt = \Big[x^2t - y^2t + \frac{t^2}{2}\Big]_0^z - \Big[-xt + \frac{t^2}{2}\Big]_0^y$$

$$= x^2z - y^2z + \frac{z^2}{2} + xy - \frac{y^2}{2}$$

$$w_2 = -\int_0^z (2xy - t)\,dt = -\Big[2xyt - \frac{t^2}{2}\Big]_0^z = -2xyz + \frac{z^2}{2}$$

Aufgabe 8

$$\vec{v} = \begin{pmatrix} x^4 \\ \frac{1}{4}x^3y^2 \end{pmatrix}$$



$$I := \int_{\partial G} \vec{v}\cdot\vec{n}\,ds \overset{\text{Gauß}}{=} \int_G \operatorname{div}(\vec{v})\,d(x,y) = \int_G \Big(4x^3 + \frac{1}{2}x^3y\Big)\,d(x,y)$$

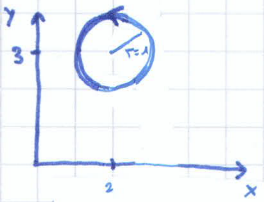
$$= \int_0^1 \int_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}} 4x^3 + \frac{1}{2}x^3y\,dx\,dy = \int_0^1 \int_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \frac{1}{2}x^3(8+y)\,dx\,dy = \int_0^1 \frac{1}{8}x^4\Big|_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \cdot (8+y)\,dy$$

$$= \frac{1}{8}\cdot\int_0^1 (8+y)\big(16(1-y^2)^2 - (1-y^2)^2\big)\,dy = \frac{15}{8}\int_0^1 \big(y^5+8y^4-2y^3-16y^2+y+8\big)\,dy = \frac{133}{16}$$

Präsenzübung - Blatt 1

Aufgabe 1.1

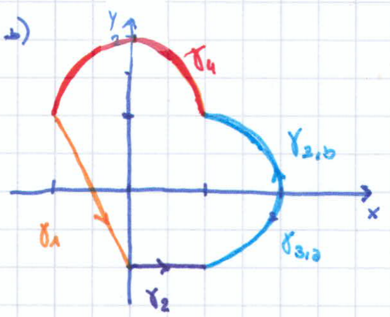
a) $R := \{(x,y)\in\mathbb{R}^2 : (x-2)^2 + (y-3)^3 \le 1\}$



GZS: $\vec{\Phi} : [0,2\pi] \to \mathbb{R}^2$, $\vec{\Phi}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

UZS: $\vec{\Phi} = [0,2\pi] \to \mathbb{R}^2$, $\vec{\Phi}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} \cos(2\pi-t) \\ \sin(2\pi-t) \end{pmatrix}$

b)



$\gamma_1 : [0,1] \to \mathbb{R}^2$, $\gamma_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\gamma_2 : [0,1] \to \mathbb{R}^2$, $\gamma_2(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\gamma_{3,a} : [\frac{3}{2}\pi, 2\pi] \to \mathbb{R}^2$, $\gamma_{3,a}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$\gamma_{3,b} : [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $\gamma_{3,b}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$\gamma_4 : [0,\pi] \to \mathbb{R}^2$, $\gamma_4(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

Globalübung - Blatt 1

4.) $\vec v : \mathbb{R}^3 \setminus \{0,0,0\} \to \mathbb{R}^3 \qquad \vec v = \vec\nabla (x^2+y^2+z^2)^{-1/2}$

(I) $v_1 = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$

$$\vec v = \frac{-1}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{\partial}{\partial x_1} v_1 = \frac{\partial}{\partial x}\left(-x(x^2+y^2+z^2)^{-3/2}\right) = -(x^2+y^2+z^2)^{-3/2} - x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot 2x$$
$$= (-x^2-y^2-z^2+3x^2)(x^2+y^2+z^2)^{-5/2} = (2x^2-y^2-z^2)(x^2+y^2+z^2)^{-5/2}$$

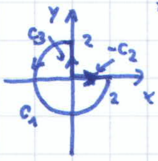
$$\begin{aligned} div(\vec v) = {} & (2x^2-y^2-z^2)(x^2+y^2+z^2)^{-5/2} \\ & + (-x^2+2y^2-z^2)(\quad)^{-5/2} \\ & + (-x^2-y^2+2z^2)(\quad)^{-5/2} = 0 \end{aligned}$$

(II) S Kugeloberfläche, $R = 2$ $\qquad \vec n = \frac{1}{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\int_S \vec v \, d\vec o = \int_S \vec v \cdot \vec n \, do = \int_S -\frac{1}{4} \, do = -\frac{1}{4} \, 4\pi \, 2^2 = -4\pi$$

5.) $$\int_C \big(\underbrace{(x^2+y^2)}_{P} dx + \underbrace{4xy}_{Q} dy \big) = \iint_G (Q_x - P_y) \, d(x,y)$$

Satz von Green



C_1: $\vec\Phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2\cos(t) \\ 2\sin(t) \end{pmatrix}, \quad \frac{\pi}{2} \le t \le 2\pi$

$$\int_{\pi/2}^{2\pi} \left(4(-2\sin(t)) + 16\cos(t)\sin(t) \cdot 2\cos(t)\right) dt = \int_{\pi/2}^{2\pi} \left(-8\sin(t) + 32\cos^2(t)\sin(t)\right) dt = \left[8\cos(t) - \frac{32}{3}\cos^3(t)\right]_{\pi/2}^{2\pi}$$
$$= 8 - \frac{32}{3} = -\frac{8}{3}$$

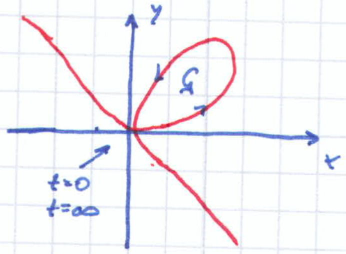
$-C_2$: $\vec\Phi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad 0 \le t \le 2$

$$\int_0^2 (t^2 \cdot 1 + 0) \, dt = \frac{t^3}{3}\Big|_0^2 = \frac{8}{3}$$

C_3: $\int_{C_3} (\dots) = 0 \qquad \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -\frac{16}{3}$

$Q_x - P_y = 4y - 2y = 2y \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(t) \\ r\sin(t) \end{pmatrix}, \quad 0 \le r \le 2, \quad \frac{\pi}{2} \le t \le 2\pi$

$$\iint_G (Q_x - P_y) \, d(x,y) = \int_0^2 \int_{\pi/2}^{2\pi} 2r\sin(t) \cdot r \, dt\, dr = 2 \cdot \left[-\cos(t)\right]_{\pi/2}^{2\pi} \left[\frac{r^3}{3}\right]_0^2 = 2 \cdot (-1) \cdot \frac{8}{3} = -\frac{16}{3}$$



$$x^3+y^3 = 3axy$$

$$x(t) = \frac{3at}{t^3+1} \qquad y(t) = \frac{3at^2}{t^3+1}$$

$t \neq -1$

$$x^3+y^3 = \frac{27a^3t^3 + 27a^3t^6}{(t^3+1)^3} = 27a^3t^3 \frac{1+t^3}{(t^3+1)^3} = 27a^3t^3 \cdot \frac{1}{(1+t^3)^2}$$

$$3axy = 27a^3t^3 \cdot \frac{1}{(1+t^3)^2} \qquad \square$$

$$\mathrm{vol}_2(G) = \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\, 9a^2 \int_0^\infty \frac{t(-t^4+2t) - t^2(1-2t^3)}{(t^3+1)^3}\,dt$$

$$\dot{x}(t) = 3a\,\frac{(t^3+1) - t\cdot 3t^2}{(t^3+1)^2} = 3a\,\frac{1-2t^3}{(t^3+1)^2}$$

$$= \frac{9}{2}a^2 \int_0^\infty \frac{t^2+t^5}{(t^3+1)^3}\,dt = \frac{3}{2}a^2 \int_0^\infty \frac{3t^2}{(t^3+1)^2}\,dt$$

$$= \frac{3}{2}a^2 \left.\frac{-1}{t^3+1}\right|_0^\infty = \frac{3}{2}a^2$$

$$\dot{y}(t) = 3a\,\frac{2t(t^3+1) - t^2\cdot 3t^2}{(t^2+1)^2}$$