Höhere Mathematik III – Übungsblatt 7

Aufgabe 31

$$y''' + 2y' = \cos(t) + 1 \quad , \; y(0) = y'(0) = 0 \; , \; y''(0) = 0$$

① Transformieren der DGL:

29.4

$$\Rightarrow \left(s^3 \mathcal{L}[y](s) - s^2 y(0) - s y'(0) - y''(0)\right) + 2\left(s \mathcal{L}[y](s) - y(0)\right) = \frac{s}{s^2+1} + \frac{1}{s}$$

$$\Leftrightarrow s^3 \mathcal{L}[y](s) - 1 + 2s \mathcal{L}[y](s) = \frac{s}{s^2+1} + \frac{1}{s}$$

$$\Leftrightarrow \mathcal{L}[y](s)\left(s^3 + 2s\right) = \frac{s}{s^2+1} + \frac{1}{s} + 1$$

$$\Leftrightarrow \mathcal{L}[y](s) = \frac{s^3 + 2s^2 + s + 1}{s^2(s^2+1)(s^2+2)}$$

② PBZ:

$$\frac{s^3+2s^2+s+1}{s^2(s^2+1)(s^2+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} + \frac{Es+F}{s^2+2}$$

$$= \frac{1}{2s} + \frac{1}{2s^2} + \frac{1}{s^2+1} + \frac{-\frac{1}{2}s - \frac{3}{2\sqrt{2}}\sqrt{2}}{s^2+2}$$

$$-\frac{1}{2}\frac{s}{s^2+2} = -\frac{1}{2}\frac{s}{s^2+(\sqrt{2})^2} \;\bullet\!\!-\!\!\circ\; -\frac{1}{2}\cos(\sqrt{2}t)$$

$$\sin(\omega t) \;\bullet\!\!-\!\!\circ\; \frac{\omega}{s^2+\omega^2}$$

③ Rücktransformation:

$$\mathcal{L}^{-1}\left[\mathcal{L}[y]\right](t) = y(t) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot t + \sin(t) - \frac{1}{2}\cos(\sqrt{2}t) - \frac{3}{2\sqrt{2}}\sin(\sqrt{2}t)$$

Aufgabe 32

$$\boxed{t^n e^{ct} \;\circ\!\!-\!\!\bullet\; \frac{n!}{(s-c)^{n+1}}}$$

(I) $\mathcal{L}[f](s) = \frac{1}{s(s+1)}$

PBZ $\Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$

$\Rightarrow f(t) = 1 - e^{-t}$

(II) $\mathcal{L}[f](s) = \frac{1}{(s^2+1)(s^2+1)}$

$\Rightarrow f(t) = \sin(t) - \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$

(III) $\mathcal{L}[f](s) = \frac{s}{(s^2+4)^2} = -\frac{1}{2}\frac{d}{ds}\frac{1}{s^2+4} \overset{29.6}{=} \frac{1}{2}\mathcal{L}\left[t\cdot\frac{1}{2}\sin(2t)\right](s)$

↳ $\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$

↳ Aus der Ableitung

$\Rightarrow f(t) = \frac{1}{4}t\sin(2t)$

$$\boxed{t e^{ct}\sin(\omega t) \;\circ\!\!-\!\!\bullet\; \frac{2\omega(s-c)}{((s-c)^2+\omega^2)^2}}$$

(IV) $f(t) = \frac{1}{\sqrt{t}}$

$$\Rightarrow \mathcal{L}[f](s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty \frac{1}{\sqrt{t}}\, e^{-st}\, dt$$

$t = u^2$
$\frac{dt}{du} = 2u$
$dt = 2u\, du$
$t(0) = 0$
„$t(\infty) = \infty$"

$$= \int_0^\infty \frac{1}{u}\, e^{-su^2} \cdot 2u\, du = 2\int_0^\infty e^{-su^2}\, du$$

$v = \sqrt{s}\, u$
$dv = \sqrt{s}\, du$
$v(0) = 0$
„$v(\infty) = \infty$"

$$= 2\int_0^\infty e^{-s\left(\frac{v}{\sqrt{s}}\right)^2} \frac{dv}{\sqrt{s}}$$

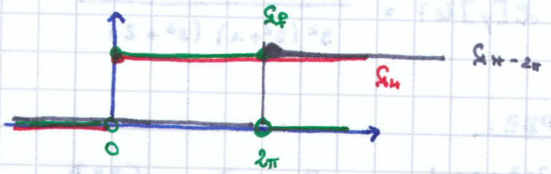
$$= \frac{2}{\sqrt{s}} \int_0^\infty e^{-v^2}\, dv$$

$$= \frac{2}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}}$$

$$\boxed{\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}}$$

Aufgabe 33

$$f(t) = \begin{cases} 1, & 0 \le t \le 2\pi \\ 0, & \text{sonst} \end{cases}$$



$y'' + y' = f(t)$, $y(0) = y'(0) = 0$

Schreibe $f(t)$ „clever" auf: $f(t) = \underline{H(t)} - \underline{H(t-2\pi)}$

$$\rightsquigarrow \mathcal{L}[f](s) = \frac{1}{s} - \frac{1}{s} e^{-2\pi s} = \frac{1}{s}\left(1 - e^{-2\pi s}\right)$$

① Transformieren der DGL:

$$s^2 \mathcal{L}[y](s) + \mathcal{L}[y](s) = \frac{1}{s}\left(1 - e^{-2\pi s}\right)$$

$$\Leftrightarrow \mathcal{L}[y](s) \cdot (s^2+1) = \frac{1 - e^{-2\pi s}}{s}$$

$$\Leftrightarrow \mathcal{L}[y](s) = \frac{1 - e^{-2\pi s}}{s(s^2+1)}$$

② PBZ:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1} \Rightarrow A = 1,\ B = -1,\ C = 0$$

$$\Rightarrow \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

③ Rücktransformation

$$\mathcal{L}[y](s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)\left(1 - e^{-2\pi s}\right) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - \left(\frac{1}{s} - \frac{s}{s^2+1}\right) e^{-2\pi s}$$

29.3(III)

$$\Rightarrow y(t) = H(t)\left(1 - \cos(t)\right) - H(t-2\pi)\left(1 - \cos(t-2\pi)\right)$$

Für $0 \le t \le 2\pi$ ist $y(t) = 1 - \cos(t)$
für $t \ge 2\pi$ ist wegen $\cos(t-2\pi) = \cos(t)$ $\quad y(t) = 1 - \cos(t) - (1 - \cos(t)) = 0$

$$\Rightarrow y(t) = \begin{cases} 1 - \cos(t), & 0 \le t \le 2\pi \\ 0, & \text{sonst} \end{cases}$$

Alternative:

Betrachte: $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$

$\Rightarrow \varphi(t) = \sin(t)$

Dann ist die Lösung von $y'' + y = f(t)$, $y(0) = y'(0) = 0$ gegeben durch

$$y(t) = \varphi(t) * f(t) = \int_0^t \sin(u)\, f(t-u)\, du$$

1. Fall: $0 \le t \le 2\pi$

$$\Rightarrow y(t) = \int_0^t \sin(u)\, du = -\cos(u)\Big|_0^t = 1 - \cos(t)$$

2. Fall: $t \ge 2\pi$

$$\Rightarrow y(t) = \int_0^t 0\, du = 0$$

$$\Rightarrow y(t) = \begin{cases} 1-\cos(t) & , 0 \le t < 2\pi \\ 0 & , \text{sonst} \end{cases}$$

$a \in \mathbb{R}$ $\quad \mathcal{L}[f](s) = \dfrac{as}{(s^2+a^2)^2}$

Idee: Zerlege $\mathcal{L}[f]$ in Faktoren, von denen die inversen Laplacetransformationen bekannt sind, wende dann den Faltungssatz an:

$$\Rightarrow \mathcal{L}[f](s) = \frac{s}{s^2+a^2} \cdot \frac{a}{s^2+a^2} =: \mathcal{L}[f_1](s) \cdot \mathcal{L}[f_2](s)$$

$\mathcal{L}[f_1](s) \bullet\!\!-\!\!\circ \cos(at)$, $\mathcal{L}[f_2](s) \bullet\!\!-\!\!\circ \sin(at)$

$$\underset{\text{Faltungssatz}}{\Longrightarrow} f(t) = f_1(t) * f_2(t) = \int_0^t \cos(a\tau)\sin(at - a\tau)\, dt \qquad \Big|\ \sin(\alpha)\cos(\beta) = \tfrac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

Damit folgt für: $\alpha = at - a\tau$, $\beta = a\tau$

$$\Rightarrow f(t) = \int_0^t \tfrac{1}{2}\left(\sin(at) + \sin(at - 2a\tau)\right) d\tau = \tfrac{1}{2}\int_0^t \sin(at)\, d\tau + \tfrac{1}{2}\int_0^t \sin(at - 2a\tau)\, d\tau$$

$$= \tfrac{1}{2} t \sin(at) + \frac{1}{4a}\left[\cos(at - 2a\tau)\right]_0^t = \tfrac{1}{2} t \sin(at)$$

$$\Rightarrow \tfrac{1}{2} t \sin(at) \circ\!\!-\!\!\bullet \frac{as}{(s^2+a^2)^2}$$

Globalübung - Blatt 7

Aufgabe 34

(I) $f(t) = 1 + 2t - \underbrace{\int_0^t u\, f(t-u)\, du}_{t * f(t)}$ $\qquad L[f*g] = L[f]\cdot L[g]$

Laplace - Trans.

$\Rightarrow \quad F(s) = \frac{1}{s} + \frac{2}{s^2} - \frac{1}{s^2} F(s)$

$\Leftrightarrow \quad F(s) \underbrace{\left(1 + \frac{1}{s^2}\right)}_{\frac{s^2+1}{s^2}} = \frac{2+s}{s^2}$

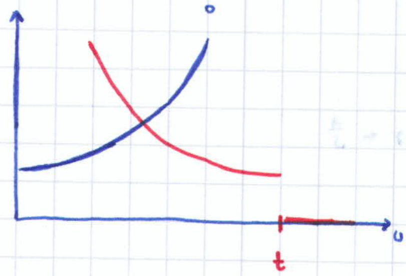
$\Rightarrow F(s) = \frac{2+s}{s^2+1} = \frac{2}{s^2+1} + \frac{s}{s^2+1}$

$\Rightarrow \quad f(t) = 2\sin(t) + \cos(t)$

(II) $f(t) = \begin{cases} e^t, & t \geq 0 \\ 0, & t < 0 \end{cases} = e^t \cdot H(t)$

$f * f: \quad L[f*f] = L[f]\cdot L[f] = \frac{1}{s-1} \cdot \frac{1}{s-1} = \frac{1}{(s-1)^2} = L[te^t]$

$(f*f)(t) = \int_0^t f(u)\, f(t-u)\, du = \int_0^t e^u\, e^{t-u}\, du = \int_0^t e^t\, du = e^t \int_0^t 1\, du = te^t$



Aufgabe 35

$L[y'](s) = sL[y] - y(0)$

$L[y''](s) = s^2 L[y] - s\cdot y(0) - y'(0)$

$ty'' + (1-2t)y' - 2y = 0 \quad , \; y(0) = 1, \; y'(0) = 2$

$ty'' + y' - 2ty' - 2y = 0$

29.6 $\quad L[t\, f(t)] = -\frac{d}{ds} L[f(t)]$

$L[ty''] = -\frac{d}{ds}\left(s^2 L[y] - sy(0) - y'(0)\right) = -s^2 \frac{d}{ds} L[y] - 2s\, L[y] + y(0)$

$L[ty'] = -\frac{d}{ds}\left(s L[y] - y(0)\right) = -s\frac{d}{ds} L[y] - L[y]$

Alles einsetzen:

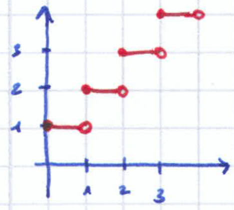
$-s^2\frac{d}{ds}L[y] - 2s\, L[y] \cancel{+1} + s\,L[y] \cancel{-1} + 2s\frac{d}{ds}L[y] + \cancel{2\,L[y]} - \cancel{2L[y]} = 0$

$\frac{d}{ds} L[y] \underbrace{(-s^2 + 2s)}_{s(2-s)} + L[y]\underbrace{(-2s+s)}_{-s} = 0$

$\frac{d}{ds} L[y] = \frac{s}{s(2-s)} L[y] = \frac{-1}{s-2} L[y]$

$L[y] = C\frac{1}{s-2} \Rightarrow y = Ce^{2t}$ $\quad$ Wegen $y(0) = 1$ ist $\boxed{y = e^{2t}}$

$f(t) = [t] + 1$ Gaußklammer



$$f(t) = \sum_{k=0}^{\infty} f_k(t) \quad , \quad f_k = \begin{cases} k+1 & , \ k \le t \le k+1 \\ 0 & , \ \text{sonst} \end{cases}$$

$$L[f](t) = \sum_{k=0}^{\infty} L[f_k(t)](s)$$

$$L[f_{k+1}] = \frac{k+1}{s}\left(e^{-ks} - e^{-(k+1)s}\right)$$

$$\Rightarrow L[f](t) = \frac{1}{s}\sum_{k=0}^{\infty} (k+1)\left(e^{-ks} - e^{-(k+1)s}\right)$$

$$= \frac{1}{s}\sum_{k=0}^{\infty} (k+1)e^{-ks}\left(1 - e^{-s}\right)$$

$$= \frac{1-e^{-s}}{s}\sum_{k=0}^{\infty} (k+1)\, e^{-ks} \qquad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$= \frac{1-e^{-s}}{s}\sum_{k=0}^{\infty} (k+1)(e^{-s})^k \qquad \left(\frac{1}{1-x}\right)^2 = \sum_{k=0}^{\infty} k\, x^{k-1}$$

$$= \frac{1-e^{-s}}{s} \cdot \frac{1}{(1-e^{-s})^2} \qquad = \sum_{k=0}^{\infty} (k+1)\, x^k$$

$$= \frac{1}{s(1-e^{-s})}$$

Man sieht: $f(t) = f(t-1) + 1$

$$\Rightarrow L[f](s) = L[f](s)\, e^{-s} + \frac{1}{s} \Leftrightarrow (1-e^{-s})\, L[f](s) = \frac{1}{s}$$