Aufgabe 47

a) $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \cdot (a^+ + a) \Rightarrow \hat{x}^4 = \left(\frac{\hbar}{2m\omega}\right)^2 (a+a^+)^4$

$\Rightarrow H_1 = \lambda x^4 = \lambda \left(\frac{\hbar}{2m\omega}\right)^2 (a^+ + a)^4$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

b) $\langle n | H_1 | n' \rangle \neq 0 ?$

$H_0 |n\rangle = E_n |n\rangle$

$X_{nn'} = \langle n | x | n' \rangle$

$= \langle n | a^+ + a | n' \rangle \sqrt{\frac{\hbar}{2m\omega}}$

$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n'}\,\delta_{nn'-1} + \sqrt{n+1}\,\delta_{nn'+1} \right)$

$$X = \begin{pmatrix} 0 & \sqrt{1} & 0 & \sqrt{2} & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ \vdots & & \ddots & & \ddots \end{pmatrix} \cdot \sqrt{\frac{\hbar}{2m\omega}}$$

$$x^2 = \frac{\hbar}{2m\omega} \begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 \\ 0 & 3 & 0 & \sqrt{2\cdot 3} \\ \sqrt{1\cdot 2} & 0 & 5 & 0 \\ 0 & \sqrt{2\cdot 3} & 0 & 7 \end{pmatrix}$$

$$x^4 = \left(\frac{\hbar}{2m\omega}\right)^2 \begin{pmatrix} 1 & 0 & \sqrt{1\cdot 2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & \sqrt{2\cdot 3} & 0 & 7 & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & \sqrt{6} & 0 & 7 & \ddots \end{pmatrix}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \begin{pmatrix} a(0) & 0 & b(0) & 0 & c(0) & 0 \\ 0 & a(1) & 0 & b(1) & 0 & c(1) \\ b(0) & 0 & a(2) & 0 & b(2) & 0 \\ 0 & b(1) & 0 & a(3) & 0 & b(3) \\ c(0) & 0 & b(2) & 0 & a(4) & 0 & \ddots \end{pmatrix}$$

$a(n) = n(6n+6) + 3 \qquad b(n) = (4n+6)\sqrt{(n+1)(n+2)}$

$c(n) = \sqrt{(n+1)(n+2)(n+3)(n+4)}$

$\Rightarrow$ Alle Matrixelemente mit $\delta_{nn'} = 1 \vee \delta_{nn'\pm 2} = 1 \vee \delta_{nn'\pm 4} = 1$ verschw. nicht

$$E_n^{(1)} = \langle n | H_1 | n \rangle = \lambda \langle n | x^4 | n \rangle = \frac{3}{4}\lambda \left(\frac{\hbar}{m\omega}\right)^2 (2n^2 + 2n + 1)$$

$$|0^{(1)}\rangle = N\left(|0\rangle + \sum_{k=1}^{\infty} c_{0k} |k\rangle\right), \quad c_{0k} = \frac{\langle k | H_1 | 0 \rangle}{\underbrace{E_0^{(0)}}_{\frac{\hbar\omega}{2}} - E_k^{(0)}}$$

und $E_n = \hbar\omega(n + \frac{1}{2})$

$$= N\left(|0\rangle - \left(\frac{b(0)}{2\hbar\omega}|2\rangle + \frac{c(0)}{4\hbar\omega}|4\rangle\right) \underbrace{\lambda\left(\frac{\hbar}{2m\omega}\right)^2}_{=\lambda'}\right)$$

$$= N\left(|0\rangle - \lambda'\left(\frac{6\sqrt{2}}{2\hbar\omega}|2\rangle + \frac{\sqrt{24}}{4\omega\hbar}|4\rangle\right)\right)$$

d) $\lambda < 0$?