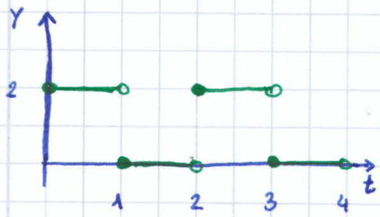
HöMa III - Übungsblatt 8

Aufgabe 86

$$f(t) = \begin{cases} 0 & : t < 0 \\ 2 & : 2n \le t < 2n+1 \\ 0 & : 2n+1 \le t < 2n+2 \end{cases}, \quad n \in \mathbb{N}_0$$

$y'' = f(t)$, $y(0) = y'(0) = 0$

Skizze von f:



(I) Die Lösungen der DGL ohne Anfangswerte

$y = t^2 + at + b$, falls $f(t) = 2$

$y = at + b$, falls $f(t) = 0$

Schreibe nun $y = (t-t_0)^2 + y'(t_0)(t-t_0) + y(t_0)$ bzw. $y = y'(t_0)(t-t_0) + y(t_0)$

t_0	Intervall	$y(t)$	Endwerte
0	$[0,1)$	t^2	$y(1) = 1$, $y'(1) = 1$
1	$[1,2)$	$2(t-1)+1 = 2t-1$	$y(2) = 3$, $y'(2) = 2$
2	$[2,3)$	$t^2 - 2t + 3$	$y(3) = 6$, $y'(3) = 4$
3	$[3,4)$	$4t - 6$	$y(4) = 10$, $y'(4) = 4$
4	$[4,5)$	$t^2 - 4t + 10$	$y(5) = 15$, $y'(5) = 6$

(II) Def. $f_0 : [0,2] \to \mathbb{R}$

$f_0(t) = 2(H(t) - H(t-1))$ Dann lässt sich f als 2-per. Fortsetzung von f_0 auffassen

DGL transformieren:

$$s^2 \mathcal{L}[y](s) = \left.\frac{\mathcal{L}[f_0](s)}{1-e^{-2s}}\right\} = \mathcal{L}[f](s)$$

Verschiebungssatz

$$= \frac{1 \cdot 2}{1-e^{-2s}}\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) = \frac{2}{s}\,\frac{1-e^{-s}}{1-e^{-2s}}$$

$$\Leftrightarrow \mathcal{L}[y](s) = \frac{2}{s^3}\,\frac{1+e^{-s}}{1-e^{-2s}}$$

$$\Rightarrow y = \sum_{k=0}^{\infty} y_0(t-2k) \quad \text{mit} \quad y_0(t) = H(t)\,t^2 - H(t-1)\,(t-1)^2$$

$$y_0'(t) = 2t\,H(t) - 2(t-1)\,H(t-1)$$

$$y_0''(t) = 2H(t) - 2H(t-1) = 2\left(H(t) - H(t-1)\right) = f_0(t)$$

$$\Rightarrow y(t) = \sum_{k=0}^{\infty} \left[H(t-2k)(t-2k)^2 - H(t-(2k+1))(t-(2k+1))^2 \right]$$

$$= t^2 - (t-1)^2 H(t-1) + (t-2)^2 H(t-2) - (t-3)^2 H(t-3) + (t-4)^2 H(t-4) - \dots$$

Aufgabe 37

$$f_1(t) = \begin{cases} \frac{t}{a} & : 0 \le t < a \\ 1 & : a \le t < b \\ 1 - \frac{t-b}{a} & : b \le t < a+b \\ 0 & : \text{sonst} \end{cases} \qquad 0 < a < b$$

$$\mathcal{L}[f_1](s) = \int_0^\infty f_1(t)e^{-st}\,dt = \int_0^a \frac{t}{a} e^{-st}\,dt + \int_a^b 1\,e^{-st}\,dt + \int_b^{a+b} \left(1 - \frac{t-b}{a}\right) e^{-st}\,dt$$

$$= \frac{1}{a}\int_0^a t\,e^{-st}\,dt + \left[-\frac{1}{s}e^{-st}\right]_a^b + \int_b^{a+b} 1\,e^{-st}\,dt - \frac{1}{a}\int_b^{a+b}(t-b)e^{-st}\,dt$$

$$= \frac{1}{a}\left[-t\frac{1}{s}e^{-st}\Big|_0^a + \frac{1}{s}\int_0^a e^{-st}\,dt\right] + \left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{as}\right] + \left[-\frac{1}{s}e^{-st}\right]_b^{a+b}$$

$$- \frac{1}{a}\left[-\frac{1}{s}(t-b)e^{-st}\Big|_b^{a+b} + \frac{1}{s}\int_b^{a+b} e^{-st}\,dt\right]$$

$$= \frac{1}{a}\left[-a\frac{1}{s}e^{-as} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\Big|_0^a\right]\right] + \left[-\frac{1}{s}e^{-bs} + \frac{1}{s}e^{-as}\right]$$

$$+ \left[-\frac{1}{s}e^{-(a+b)s} + \frac{1}{s}e^{-bs}\right] - \frac{1}{a}\left[-\frac{1}{s}a e^{-(a+b)s} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_b^{a+b}\right.$$

$$= \frac{1}{a}\left[-\frac{a}{s}e^{-as} + \frac{1}{s}\left[-\frac{1}{s}e^{-as} + \frac{1}{s}e^{-0}\right]\right] + \left[-\frac{1}{s}e^{-bs} + \frac{1}{s}e^{-as}\right]$$

$$+ \left[-\frac{1}{s}e^{-(a+b)s} + \frac{1}{s}e^{-bs}\right] - \frac{1}{a}\left[-\frac{a}{s}e^{-(a+b)s} + \frac{1}{s}\left[-\frac{1}{s}e^{-(a+b)s} + \frac{1}{s}e^{-bs}\right]\right]$$

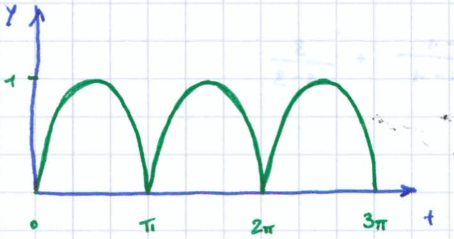
$$= -\frac{1}{s}e^{-as} - \frac{1}{as^2}e^{-as} + \frac{1}{as^2} - \frac{1}{s}e^{-bs} + \frac{1}{s}e^{-as} - \frac{1}{s}e^{-(a+b)s}$$

$$+ \frac{1}{s}e^{-bs} + \frac{1}{s}e^{-(a+b)s} + \frac{1}{as^2}e^{-(a+b)s} - \frac{1}{as^2}e^{-bs}$$

$$= \frac{1}{as^2}\left(1 - e^{-as} - e^{-bs} + e^{-(a+b)s}\right) = \frac{1}{as^2}\left(1 - e^{-as}\right)\left(1 - e^{-bs}\right)$$

(II)

$$f_2(t) = \begin{cases} |\sin(t)| & : t \geq 0 \\ 0 & : \text{sonst} \end{cases}$$



Fasse f_2 als π-periodische Fortsetzung der Funktion

$f_0 : [0, \pi] \to \mathbb{R}$, $f_0(x) = \sin(x) + H(x-\pi)\sin(x-\pi)$ auf

$$\Rightarrow \mathcal{L}[f_2](s) = \frac{\mathcal{L}[f_0](s)}{1-e^{-\pi s}} = \frac{1}{1-e^{-\pi s}} \left(\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \right)$$

$$= \frac{1+e^{-\pi s}}{1-e^{-\pi s}} \cdot \frac{1}{s^2+1}$$

Aufgabe 38

$$y' = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} y + e^{-t} \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Mit Laplace: $y = \begin{pmatrix} u \\ v \end{pmatrix}$

Übersetzen des DGL-Systems:

① $s\,\mathcal{L}[u](s) = -\mathcal{L}[u](s) + 4\mathcal{L}[v](s) + \frac{4}{s+1}$

② $s\,\mathcal{L}[v](s) - 1 = -2\mathcal{L}[u](s) + 5\mathcal{L}[v](s) + \frac{4}{s+1}$

Koeff. Matrix:

$$\begin{array}{cc} \mathcal{L}[u] & \mathcal{L}[v] \\ \end{array}$$
$$\left(\begin{array}{cc|c} s+1 & -4 & \frac{4}{s+1} \\ 2 & s-5 & \frac{4}{s+1} + 1 = \frac{s+5}{s+1} \end{array}\right)$$

Determinante des Systems:

$$\Delta = (s+1)(s-5) + 8 = (s-1)(s-3)$$

① $\Delta_u = \begin{vmatrix} \frac{4}{s+1} & -4 \\ \frac{s+5}{s+5} & s-5 \end{vmatrix} = \frac{8s}{s+1}$

$$\Rightarrow \mathcal{L}[u](s) = \frac{8s}{(s+1)(s-1)(s-3)} \overset{\text{PBZ}}{=} \frac{-1}{s+1} + \frac{-2}{s-1} + \frac{3}{s-3}$$

$\Rightarrow u(t) = -e^{-t} - 2e^{t} + 3e^{3t}$

② $\Delta_v = \dfrac{s^2+6s-3}{s+1}$

$\Rightarrow \mathcal{L}[v](s) = \dfrac{s^2+6s-3}{(s+1)(s-1)(s-3)} \overset{PBZ}{=} \dfrac{-1}{s+1} + \dfrac{-1}{s-1} + \dfrac{3}{s-3}$

$\Rightarrow v(t) = -e^{-t} - e^{t} + 3e^{3t}$

Aufgabe 8.2

$\mathcal{L}[f](s) = \dfrac{1}{s^2+1} = \dfrac{1}{(s+i)(s-i)}$

$\Rightarrow f(t) = \mathrm{Res}\left((s+i)\,\mathcal{L}[f](s)\, e^{st}; -i\right) + \mathrm{Res}\left((s-i)\,\mathcal{L}[f]s\, e^{st}; i\right)$

$= \lim\limits_{s\to -i}\left(\dfrac{e^{st}}{s-i}\right) + \lim\limits_{s\to i}\left(\dfrac{e^{st}}{s+i}\right) = \dfrac{1}{2i}\left(e^{it} - e^{-it}\right) = \sin(t)$

$\mathcal{L}[g](s) = \dfrac{s}{s^2+1} = \dfrac{s}{(s+i)(s-i)}$

$\Rightarrow g(t) = \mathrm{Res}\left((s+i)\,\mathcal{L}[g](s)\, e^{st}; -i\right) + \mathrm{Res}\left((s-i)\,\mathcal{L}[g](s)\, e^{st}; i\right)$

$= \lim\limits_{s\to -i}\left(\dfrac{s\,e^{st}}{s-i}\right) + \lim\limits_{s\to i}\left(\dfrac{s\,e^{st}}{s+i}\right) = \dfrac{-i}{-2i}e^{-it} + \dfrac{i}{2i}e^{it}$

$= \frac{1}{2}\left(e^{it} + e^{-it}\right) = \cos(t)$

Globalübung - Blatt 8

Aufgabe 40

$$y'' - 3y' + 2y = f(t)$$

---

y_1, y_2 Fundamentalsystem

„BEKANNT" AUS HÖMA II

Ansatz: $y_p = c_1(t)\, y_1(t) + c_2(t)\, y_2(t)$

Erinnerung: $$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$y_p(t) = -y_1(t) \int \frac{y_2(t) f(t)}{W(t)}\, dt + y_2(t) \int \frac{y_1(t) f(t)}{W(t)}\, dt$$

(I)

① FS: $p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

$y_1(t) = e^t \,,\; y_2(t) = e^{2t}$

② $$W(t) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$$

$$y_p(t) = -e^t \int_0^t \frac{e^{2s} f(s)}{e^{3s}}\, ds + e^{2t} \int_0^t \frac{e^s f(s)}{e^{3s}}\, ds$$

$$= -e^t \int_0^t e^{-s} f(s)\, ds + e^{2t} \int_0^t e^{-2s} f(s)\, ds$$

$$= \int_0^t (-e^{t-s} + e^{\underbrace{2t-2s}_{2(t-s)}})\, f(s)\, ds = (-e^t + e^{2t}) * f(t)$$

(II) Löse: $y'' - 3y' + 2y = 0 \,,\; y(0) = 0 \,,\; y'(0) = 1$

Ist ϕ die Lösung, so löst $\phi * f$ die inhomogene DGL $y'' - 3y' + 2y = f(t)$

$$s^2 Y - s \cdot 0 - 1 - 3sY - 0 + 2Y = 0$$

$$(s^2 - 3s + 2)\, Y = 1$$

$$Y = \frac{1}{s^2 - 3s + 2} = \frac{A}{s-1} + \frac{B}{s-2} = -\frac{1}{s-1} + \frac{1}{s-2}$$

$$\phi = -e^t + e^{2t}$$

Aufgabe 39

$$L\left[f(t)\right](s) = \frac{s}{(s^2-4)^2}$$

a) PBZ

$$(s^2-4)^2 = (s-2)^2(s+2)^2$$

$$\Leftrightarrow \frac{s}{(s-2)^2(s+2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2}$$

$$\Leftrightarrow s = A(s-2)(s+2)^2 + B(s+2)^2 + C(s+2)(s-2)^2 + D(s-2)^2$$

$s=2: \quad 2 = B \cdot 16 \Leftrightarrow B = \frac{1}{8}$

$s=-2: \quad -2 = D \cdot 16 \Leftrightarrow D = -\frac{1}{8}$

$s^3: \quad 0 = A + C$

$s=0: \quad 0 = -8A + \frac{1}{2} + 8C - \frac{1}{2}$

$\Leftrightarrow 0 = -8A + 8C$

$\Rightarrow A = C = 0$

$$L\left[f(t)\right](s) = \frac{1}{8}\left(\frac{1}{(s-2)^2} - \frac{1}{(s+2)^2}\right)$$

$$\Rightarrow f(t) = \frac{1}{8}\left(t e^{2t} - t e^{-2t}\right) = \frac{1}{4} t \sinh(2t)$$

b) $L\left[f(t)\right](s) = \frac{1}{2} \frac{2}{s^2-4} \cdot \frac{s}{s^2-4} = \frac{1}{2} L\left[\sinh(2t)\right] \cdot L\left[\cosh(2t)\right]$

$\Rightarrow f(t) = \frac{1}{2}\sinh(2t) * \cosh(2t)$

$\sinh(u) = \frac{1}{2}(e^u - e^{-u})$

$\cosh(u) = \frac{1}{2}(e^u + e^{-u})$

$$= \frac{1}{8}\int_0^t \left(e^{2(t-u)} - e^{-2(t-u)}\right)\left(e^{2u} + e^{-2u}\right) du$$

$$= \frac{1}{8}\int_0^t \left(e^{2t} + e^{2t-4u} - e^{-2t+4u} - e^{-2t}\right) du$$

$$= \frac{1}{8}\left(e^{2t} \frac{-1}{4} e^{-4u}\Big|_0^t - e^{-2t}\frac{1}{4}e^{4u}\Big|_0^t\right) + \frac{1}{8}t\left(e^{2t} - e^{-2t}\right)$$

$$= \frac{1}{32}\left(-e^{-2t} + e^{2t} - e^{2t} + e^{-2t}\right) + \frac{1}{8}t\left(e^{2t} - e^{-2t}\right) = \frac{1}{8}t\left(e^{2t} - e^{-2t}\right) = \frac{1}{4}t\sinh(2t)$$

c) $\dfrac{s}{(s^2-4)^2} = -\dfrac{1}{4}\dfrac{d}{ds}\dfrac{2}{s^2-4}$ $\qquad L\left[t\, f(t)\right](s) = -\dfrac{d}{ds} L\left[f(t)\right](s)$

$$= -\frac{1}{4}\frac{d}{ds} L\left[\sinh(2t)\right](s)$$

$$= \frac{1}{4} L\left[t\sinh(2t)\right](s)$$

$$= L\left[\frac{1}{4}t\sinh(2t)\right](s)$$

d) Residuensatz

$$f(t) = \sum \operatorname{Res}\left(\frac{z}{(z^2-4)^2}e^{tz}, z_k\right)$$

$$= \operatorname{Res}\left(\frac{ze^{tz}}{(z-2)^2(z+2)^2}, 2\right) + \operatorname{Res}\left(\frac{ze^{tz}}{(z-2)^2(z+2)^2}, -2\right)$$

$$= \lim_{z\to 2}\frac{d}{dz}\frac{ze^{tz}}{(z+2)^2} + \lim_{z\to -2}\frac{d}{dz}\frac{ze^{tz}}{(z-2)^2}$$

$$= \lim_{z\to 2}\frac{(e^{tz}+zte^{tz})(z+2)^2 - ze^{tz}\cdot 2(z+2)}{(z+2)^4} + \lim_{z\to -2}\frac{e^{tz}(1+tz)(z-2)^2 - ze^{tz}2(z-}{(z-2)^4}$$

$$= \frac{1}{256}\left(e^{2t}(1+2t)\cdot 16 - 16e^{2t} + e^{-2t}(1-2t)\cdot 16 - 16e^{-2t}\right)$$

$$= \frac{1}{8}t\left(e^{2t}-e^{-2t}\right) = \frac{1}{4}t\sinh(2t)$$

BRUNNEN