Supplemental Material to
Quantum Information Engines: Assessing Time, Cost and Performance Criteria

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In Section I, the dynamics of the coupled system-meter are investigated before delving into the expectation value of the meter state in Section II. Section III examines the energy change of the meter and the measurement energy required for the measurement protocol and the possible energy extraction after the measurement. Section IV presents the derivation of the IE power output for the cycle time $t_m \to 0$. Finally, Section V demonstrates the maximum IE efficiency over cycle time for various operation regimes.

I. DYNAMICS OF SYSTEM AND METER

We use an iterative numerical scheme to determine unitary time evolution of the density matrix for the coupled 2SS and meter given under the total Hamiltonian $\hat{H}$ given the initial density matrix (Eq. (2) in the main text) by

$$\begin{aligned}\hat{\rho}(t) &= e^{-i\hat{H}t/\hbar}\hat{\rho}(0)e^{i\hat{H}t/\hbar} \\ &= a\Big(e^{-i\hat{H}_0\Delta t/\hbar}e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}\Big)^N |0\rangle\langle 0| \otimes |D\rangle\langle D| \left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N \\ &+ b\Big(e^{-i\hat{H}_0\Delta t/\hbar}e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}\Big)^N |1\rangle\langle 1| \otimes |D\rangle\langle D| \left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N,\end{aligned} \tag{S.1}$$

while using the Trotter-splitting $e^{i(\hat{H}_0+\frac{\hat{p}^2}{2}+\hat{V})t/\hbar} = \left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N$ with $\Delta t = t/N$ for $N \to \infty$.

The joint probability $P_i(p,t)$ can be solved analytically using the Trotter splitting in Eq. (S.2) which results to

$$\begin{aligned}P_i(p,t) &= \sum_i \langle i|\langle p|\hat{\rho}(t)|p\rangle|i\rangle = \sum_i \langle i|\langle p|e^{-i\hat{H}t/\hbar}\hat{\rho}(0)e^{i\hat{H}t/\hbar}|p\rangle|i\rangle \qquad \text{(S.2)} \\ &= a\langle D|\langle 0|\left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N|0\rangle\langle 0| \\ &\otimes |p\rangle\langle p|\left(e^{-i\hat{H}_0\Delta t/\hbar}e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}\right)^N|0\rangle|D\rangle \\ &+ b\langle D|\langle 1|\left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N|1\rangle\langle 1| \\ &\otimes |p\rangle\langle p|\left(e^{-i\hat{H}_0\Delta t/\hbar}e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}\right)^N|1\rangle|D\rangle \\ &= a|\langle p|D\rangle|^2 \\ &+ b\int ds\int dm\int ds'\int dm'\int dx\,\langle D(s)|s\rangle e^{i\frac{s^2}{2}\Delta t/\hbar}\langle s|x\rangle e^{igx\Delta t/\hbar}\langle x|m\rangle \qquad \text{(S.3)} \\ &\langle m|\left(e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{ig\hat{x}\Delta t/\hbar}\right)^{N-1}|p\rangle\langle p|\left(e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-ig\hat{x}\Delta t/\hbar}\right)^{N-1}|m'\rangle \\ &\langle m'|e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}|s'\rangle\langle s'|e^{-ig\hat{x}\Delta t/\hbar}|D\rangle \\ &= a|\langle p|D\rangle|^2 \\ &+ b\langle D|p+gN\Delta t\rangle\,\Pi_{k=1}^N\left[e^{i(p+gk\Delta t)^2\Delta t/2\hbar}\right] \qquad \text{(S.4)} \\ &\Pi_{k=1}^N\left[e^{-i(p+gk\Delta t)^2\Delta t/2\hbar}\right]\langle p+gN\Delta t|D\rangle \\ &= a|D(p)|^2 + b|D(p+gN\Delta t)|^2 = a|D(p)|^2 + b|D(p+gt)|^2,\end{aligned}$$

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where we have exploit the completeness relation for the momentum eigenstates $\int dm \, |m\rangle \langle m| = \int dm' \, |m'\rangle \langle m'| = \int ds \, |s\rangle \langle s| = \int ds' \, |s'\rangle \langle s'| = \mathbb{I}$ and for the position eigenstates $\int dx \, |x\rangle \langle x| = \mathbb{I}$ in line (S.3). Using the relation $\langle x|m\rangle = e^{imxt/\hbar}/\sqrt{2\pi}$ and the identity $\int dx e^{i(q-a)x/\hbar} = 2\pi\delta(q-a)$, one arrives iteratively to the expression in line (S.4).

II. AVERAGE METER STATE

The expectation value of the meter outcome $\langle p(t_m)\rangle$ after the measurement of duration t_m, while using the Trotter-splitting $e^{i(\hat{H}_0+\frac{\hat{p}^2}{2}+\hat{V})t_m/\hbar} = \left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N$ with $\Delta t = t_m/N$ for $N \to \infty$, reads

$$\begin{aligned}
\langle p(t_m)\rangle &= \mathrm{tr}[\hat{\rho}(t_m)\hat{p}] \\
&= a \, \langle D| \, \langle 0| \left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N \hat{p}\left(e^{-i\hat{H}_0\Delta t/\hbar}e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}\right)^N |0\rangle \, |D\rangle \\
&+ b \, \langle D| \, \langle 1| \left(e^{i\hat{H}_0\Delta t/\hbar}e^{i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N \hat{p}\left(e^{-i\hat{H}_0\Delta t/\hbar}e^{-i\frac{\hat{p}^2}{2}\Delta t/\hbar}e^{-i\hat{V}\Delta t/\hbar}\right)^N |1\rangle \, |D\rangle \\
&= bg \int_{-\infty}^{\infty} dp D(p+gN\Delta t)\Pi_{k=1}^{N}\left[e^{i(p+gk\Delta t)^2\Delta t/2\hbar}\right] p \Pi_{k=1}^{N}\left[e^{-i(p+gk\Delta t)^2\Delta t/2\hbar}\right] D(p+gN\Delta t) \\
&= bg\left(\frac{2}{\pi\hbar^2 B}\right)^{1/2} \int_{-\infty}^{\infty} dp e^{-2(p+gt_m)^2/\hbar^2 B} p = -bgt_m.
\end{aligned} \tag{S.5}$$

III. ENERGY TRANSFER DURING MEASUREMENT

We determine now the expected energy of the system, meter and their mutual coupling after (unitary) time evolution t_m. We consider first the average energy change of the meter (change of kinetic energy of the free particle) $\langle \Delta W_M(t_m)\rangle = \frac{1}{2}(\langle \hat{p}^2(t_m)\rangle - \langle \hat{p}^2(0)\rangle)$ after the entangling system-meter evolution of time t_m by using the Trotter splitting as in Eq. (S.5) with $\Delta t = t_m/N$

$$\begin{aligned}
W_M(t_m) &= \frac{1}{2}\left[\mathrm{tr}[\hat{\rho}(t_m)\hat{p}^2] - \mathrm{tr}[\hat{\rho}(0)\hat{p}^2]\right] \\
&= \frac{b}{2}\Big[\langle D| \left(e^{i\hat{p}^2\Delta t/2\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N \hat{p}^2 \left(e^{-i\hat{p}^2\Delta t/2\hbar}e^{-i\hat{V}\Delta t/\hbar}\right)^N |D\rangle \\
&- \langle D| \, \hat{p}^2 \, |D\rangle \Big] \\
&= \frac{b}{2}\left(\frac{2}{\pi\hbar^2 B}\right)^{1/2}\left[\int_{-\infty}^{\infty} dp e^{-\frac{2(p+gN\Delta t)^2}{\hbar^2 B}} p^2 - \int_{-\infty}^{\infty} dp e^{-\frac{2p^2}{\hbar^2 B}} p^2\right] \\
&= \frac{bg^2 t_m^2}{2}.
\end{aligned} \tag{S.6}$$

Note that the $\langle \delta\hat{p}^2(t_m)\rangle = \langle \hat{p}^2(t_m)\rangle - \langle p(t_m)\rangle^2 \equiv 2\langle \Delta W_M(t_m)\rangle$. Consider next the change in expectation value of the system-meter coupling during evolution of time t_m by using the Trotter splitting as in Eq. (S.5) with $\Delta t = t_m/N$ and where $\hat{V} = g\hat{x} \otimes |1\rangle\langle 1|$,

$$\begin{aligned}
W_{meas}(t_m) &= -\Bigg[\mathrm{tr}[\hat{\rho}(t_m)\hat{V}] - \mathrm{tr}[\hat{\rho}(0)\hat{V}]\Bigg] \qquad (S.7)\\
&= -bg\,\langle D|\left(e^{i\hat{p}^2\Delta t/2\hbar}e^{i\hat{V}\Delta t/\hbar}\right)^N \hat{x}\left(e^{-i\hat{p}^2\Delta t/2\hbar}e^{-i\hat{V}\Delta t/\hbar}\right)^N|D\rangle\\
&= -bg\left(\frac{2}{\pi\hbar^2 B}\right)^{1/2}\Bigg[\int_{-\infty}^{\infty} dp e^{-\frac{(p+gN\Delta t)^2}{\hbar^2 B}}\prod_{k=1}^{N}\Bigg[e^{i(p+gk\Delta t)^2\Delta t/2\hbar}\Bigg] i\hbar\frac{d}{dp}\\
&\Bigg\{\prod_{k=1}^{N}\Bigg[e^{-i(p+gk\Delta t)^2\Delta t/2\hbar}\Bigg]e^{-\frac{(p+gN\Delta t)^2}{\hbar^2 B}}\Bigg\}\Bigg]\\
&= -bg\left(\frac{2}{\pi\hbar^2 B}\right)^{1/2}\Bigg[\int_{-\infty}^{\infty} dp e^{-\frac{2(p+gN\Delta t)^2}{\hbar^2 B}}\sum_{k=1}^{N}(p+gk\Delta t)\Delta t/\Bigg]\\
&= -bg\left(\frac{2}{\pi\hbar^2 B}\right)^{1/2}\Bigg[\int_{-\infty}^{\infty} dp e^{-\frac{2(p+gN\Delta t)^2}{\hbar^2 B}}\big(pN\Delta t + \sum_{k=1}^{N} gk\Delta t^2\big)\Bigg]\\
&= -bg\left(\frac{2}{\pi\hbar^2 B}\right)^{1/2}\Bigg[\int_{-\infty}^{\infty} dp e^{-\frac{2(p+gN\Delta t)^2}{\hbar^2 B}}\big(pt_m + gN(N+1)(t_m/N)^2/2\big)\Bigg]\\
&= -bg\Big[\big(-gt_m^2/M + gt_m^2/2 + gt_m^2/2N\big)\Big]\\
&\underset{\lim N\to\infty}{=} \frac{bg^2t_m^2}{2}.
\end{aligned}$$

As expected from unitary evolution, $W_M(t_m) = W_{meas}(t_m)$. The output energy after the measurement of the meter outcome p reads according to Eq. (6) in the main text

$$G(p,t_m) = \Delta E[P_1(t_m|p) - P_1(0|p)] = \Delta E[P_1(t_m|p) - b]. \qquad (S.8)$$

We note that depending on the meter outcome p, $G(p,t_m)$ can be negative, while the average over all possible meter outcomes vanishes

$$W_{out}(t_m) = \int_{-\infty}^{\infty} dp Q(p,t_m)G(p,t_m) = 0. \qquad (S.9)$$

To see this, note that the integrand in Eq. (S.9) is the difference between the joint probabilities $P_1(p,t_m) - P_1(p,0)$ and the integral over all p just yields the constant probability that the 2SS is in state $i = 1$ irrespective of the meter outcome. As stated in the main text a productive use of the information engine is achieved by restricting the photon extraction attempts to events for which the measurement outcome p indicates that the 2SS probability to be in the excited state 1 is large enough relative to its thermal value. For our model these are events in which the meter outcome is smaller than some bound $p < p'$ where $-\infty < p' < 0$. In this case the averaged useful energy extracted per attempt is $\bar{W}_{out}(t_m,p') = \int_{-\infty}^{p'} dp\bar{Q}(p,t_m)G(p,t_m) > 0$ where $\bar{Q}(p,t_m) = Q(p,t_m)(\int_{-\infty}^{p'} dpQ(p,t_m))^{-1}$. However, not every engine cycle ends with an extraction attempt. We may define the average effective cycle time by $t_{eff} = (\int_{-\infty}^{p'} dpQ(p,t_m))^{-1}t_m$. We note that in making this identification we assume that the time needed for the photon extraction itself as well as the relaxation time associated with the restoration step discussed below can be disregarded. Otherwise these times should be added to t_m. The average useful energy extracted per cycle can be therefore written by

$$\begin{aligned}
W_{out}(t_m,p') &= \bar{W}_{out}(t_m,p')(t_m/t_{eff}) \qquad (S.10)\\
&= \int_{-\infty}^{p'} dpQ(p,t_m)G(p,t_m) > 0.
\end{aligned}$$

IV. POWER FOR $t_m \to 0$

We now determine the Power $\Pi(t_m, p')$ (Eq. (12) in the main text) in the limit $t_m \to 0$.

$$\Pi(t_m, p') = \frac{W_{out}(t_m, p')}{t_m} = \frac{\Delta E \int_{-\infty}^{p'} dpQ(p, t_m)[P_1(t_m|p) - P_1(0|p)]}{t_m} \tag{S.11}$$

$$= \sqrt{\frac{2}{\pi}} \frac{ab\Delta E\big[\int_{-\infty}^{p'+gt_m} dpe^{\frac{-p^2}{\hbar^2 B}} - \int_{-\infty}^{p'} dpe^{\frac{-p^2}{\hbar^2 B}}\big]}{t_m} = \sqrt{\frac{2}{\pi}} \frac{ab\Delta E\big[\int_{-\infty}^{(p'+gt_m)/\hbar\sqrt{B}} e^{\frac{-p^2}{\hbar^2 B}} - \int_{-\infty}^{p'/\hbar\sqrt{B}} e^{\frac{-p^2}{\hbar^2 B}}\big]}{t_m}$$

$$\lim_{t_m \to 0} = \sqrt{\frac{2}{\pi}} ab\Delta E e^{\frac{-p'^2}{\hbar^2 B}}.$$

V. MAXIMUM EFFICIENCY FOR DIFFERENT INITIAL THERMAL STATE

Fig. S.1 shows the efficiency $\eta_{max} \equiv \eta(t_m, p'_{max})$ against the measurement time t_m for different IE model parameter. The green curve corresponds to regimes of $W_{out} > W_{in}$ as discussed in the main text.

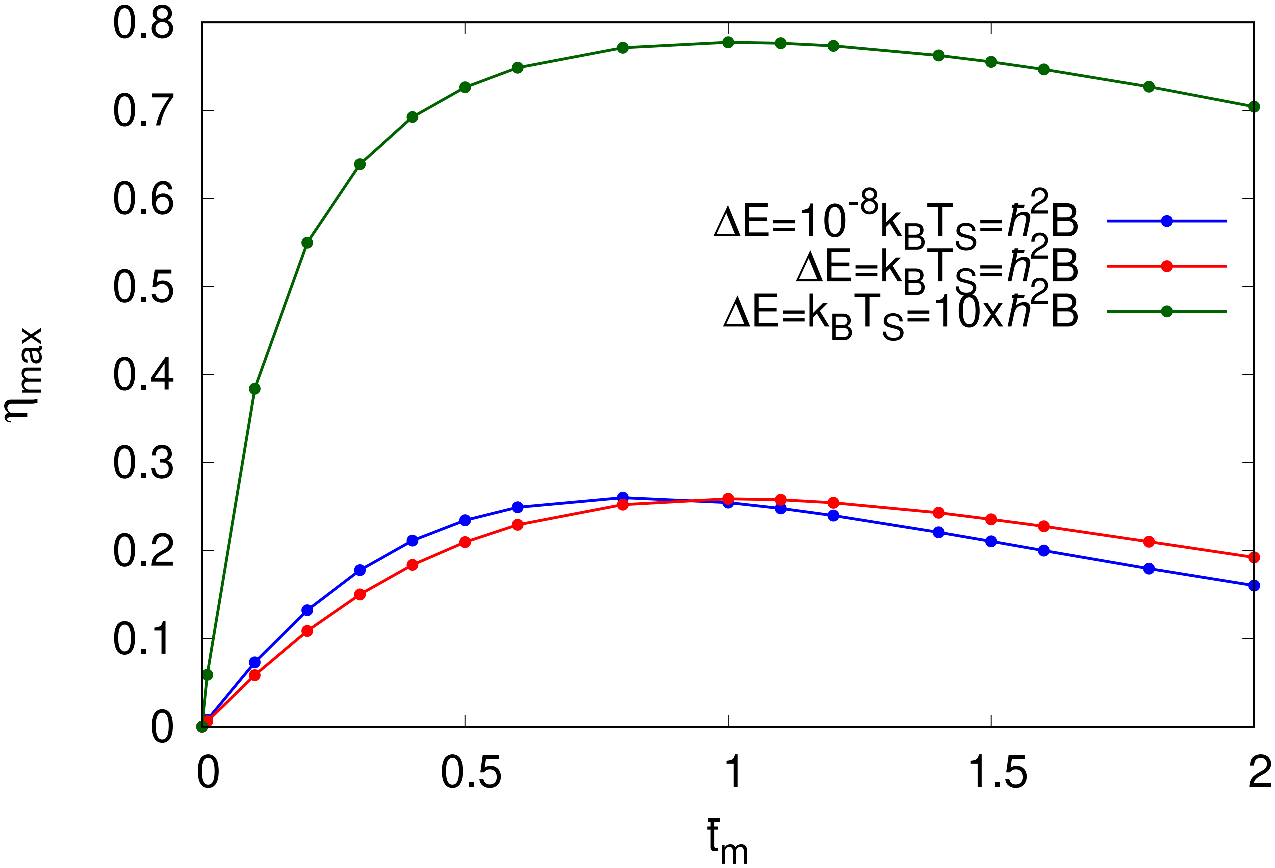

FIG. S.1. Maximal efficiency $\eta_{max}(t_m, p'_{max})$ (Eq. (11) in the main text) at outcome p'_{max} as function of system-meter interaction time t_m for different initial state of the 2SS defined by the temperature $a/b = e^{\Delta E/k_B T_S}$ where we choose $\Delta E = 25.58$meV (which corresponds to $k_B T$ for 300K).