

$$\begin{aligned}
L &= \sum_{n=0}^{\infty} np_n \\
&= \sum_{n=1}^N n \left(\frac{\lambda}{\mu} \right)^n \left(\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \\
&= \left(\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \sum_{n=1}^N n \left(\frac{\lambda}{\mu} \right)^n
\end{aligned}$$

Développement de la somme vue en classe

$$= \left(\frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\frac{\left(\frac{\lambda}{\mu} - 1 \right) (N+1) \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right) \left(\left(\frac{\lambda}{\mu} \right)^{N+1} - 1 \right)}{\left(1 - \frac{\lambda}{\mu} \right)^2} \right)$$

Simplification

$$\begin{aligned}
&= \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\frac{\left(\frac{\lambda}{\mu} - 1 \right) (N+1) \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right) \left(\left(\frac{\lambda}{\mu} \right)^{N+1} - 1 \right)}{1 - \frac{\lambda}{\mu}} \right) \\
&= \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\frac{\left(\frac{\lambda}{\mu} - 1 \right) (N+1) \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right) \left(\left(\frac{\lambda}{\mu} \right)^{N+1} - 1 \right)}{\frac{\mu - \lambda}{\mu}} \right) \\
&= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\left(\frac{\lambda}{\mu} - 1 \right) (N+1) \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right) \left(\left(\frac{\lambda}{\mu} \right)^{N+1} - 1 \right) \right) \\
&= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\left(\frac{\lambda}{\mu} - 1 \right) (N+1) \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right) \left(\left(\frac{\lambda}{\mu} \right)^{N+1} - 1 \right) \right)
\end{aligned}$$

Développement des deux termes de la 3^e parenthèse

$$= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(N \left(\frac{\lambda}{\mu} \right)^{N+2} + \left(\frac{\lambda}{\mu} \right)^{N+2} - N \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right)^{N+2} + \left(\frac{\lambda}{\mu} \right) \right)$$

$$= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(N \left(\frac{\lambda}{\mu} \right)^{N+2} + \left(\frac{\lambda}{\mu} \right)^{N+2} - N \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right)^{N+2} + \left(\frac{\lambda}{\mu} \right)^N \right)$$

Ces deux termes (en jaune) s'annulent

$$= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(N \left(\frac{\lambda}{\mu} \right)^{N+2} - N \left(\frac{\lambda}{\mu} \right)^{N+1} - \left(\frac{\lambda}{\mu} \right)^{N+1} + \left(\frac{\lambda}{\mu} \right)^N \right)$$

Mise en évidence

$$= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\frac{\lambda}{\mu} \right) \left(N \left(\frac{\lambda}{\mu} \right)^{N+1} - N \left(\frac{\lambda}{\mu} \right)^N - \left(\frac{\lambda}{\mu} \right)^N + 1 \right)$$

Mise en évidence

$$= \left(\frac{\mu}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \left(\frac{\lambda}{\mu} \right) \left(N \left(\frac{\lambda}{\mu} \right)^{N+1} - (N+1) \left(\frac{\lambda}{\mu} \right)^N + 1 \right)$$

Ces deux termes (en vert) s'annulent

$$= \left(\frac{1}{\mu - \lambda} \right) \left(\frac{1}{1 - \left(\frac{\lambda}{\mu} \right)^{N+1}} \right) \lambda \left(N \left(\frac{\lambda}{\mu} \right)^{N+1} - (N+1) \left(\frac{\lambda}{\mu} \right)^N + 1 \right)$$

En réorganisant les termes

$$= \frac{\lambda \left(1 + N \left(\frac{\lambda}{\mu} \right)^{N+1} - (N+1) \left(\frac{\lambda}{\mu} \right)^N \right)}{(\mu - \lambda) \left(1 - \left(\frac{\lambda}{\mu} \right)^{N+1} \right)}$$