2002-B-1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

X_n : Nombre de lancers réussis sur n lancers.

On cherche $P(X_{100} = 50)$.

Exploration

Conjecture : $P(X_n = k) = \frac{1}{n-1}$ pour tout $n \in \{2, 3, \dots\}$ et $k \in \{1, \dots, n-1\}$.

Démonstration par récurrence

Base : $n = 2$. Directement, $P(X_2 = 1) = 1 = \frac{1}{2-1}$.

Hypothèse de récurrence : On suppose que $P(X_n = k) = \frac{1}{n-1}$ pour $n \geq 2$ et $k \in \{1, \dots, n-1\}$.

Pas de récurrence : On veut démontrer que $P(X_{n+1} = k) = \frac{1}{n}$ pour $k \in \{1, \dots, n\}$

Idée : conditionner le calcul de $P(X_{n+1} = k)$ par X_n.

$$P(X_{n+1} = k) = \sum_{i=1}^{n-1} P(X_{n+1} = k \mid X_n = i) \cdot P(X_n = i)$$

$$P(X_{n+1} = k) = \sum_{i=k-1}^{k} P(X_{n+1} = k \mid X_n = i) \cdot P(X_n = i)$$

$$P(X_{n+1} = k) = P(X_{n+1} = k \mid X_n = k-1) \cdot P(X_n = k-1) + P(X_{n+1} = k \mid X_n = k) \cdot P(X_n = k)$$

$$P(X_{n+1} = k) = \frac{k-1}{n} \cdot \frac{1}{n-1} + \left(1 - \frac{k}{n}\right) \cdot \frac{1}{n-1}$$

$$P(X_{n+1} = k) = \frac{k-1}{n(n-1)} + \frac{n-k}{n(n-1)}$$

$$P(X_{n+1} = k) = \frac{k-1+n-k}{n(n-1)}$$

$$P(X_{n+1} = k) = \frac{n-1}{n(n-1)}$$

$$P(X_{n+1} = k) = \frac{1}{n}.$$

Ainsi,

$$P(X_{100} = 50) = \frac{1}{100-1} = \frac{1}{99}.$$