Bayesian Learning for Partially-Observed Dynamical Systems

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Tutorial 1 : Introduction to Markov chains.

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CHAPTER 1. MARKOV CHAINS, INVARIANT MEASURES

EXERCICE 1 Let

$$X_t = \sigma_t Z_t\ , \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2\ , \quad t \geq 1\ , \tag{1}$$

where the coefficients α_0, α_1 are positive and where $\{Z_t,\ t \in \mathbb{N}\}$ is an i.i.d. sequence of r.v. such that $\mathbb{E}[Z_0] = 0$, $\mathbb{E}[Z_0^2] = 1$, and $\{Z_t,\ t \in \mathbb{N}\}$ are independent of X_0

1. Assuming that Z_0 has the density q wrt the Lebesgue measure, show that $\{X_n,\ n \in \mathbb{N}\}$ is a Markov Chain with transition density

$$p(x,x') = \frac{1}{\sqrt{\alpha_0 + \alpha_1 x^2}} q\left(\frac{x'}{\sqrt{\alpha_0 + \alpha_1 x^2}}\right)\ .$$

EXERCICE 2 Consider a Gaussian AR(1) process, $X_t = \mu + \phi X_{t-1} + \sigma Z_t$, where $\{Z_t,\ t \in \mathbb{N}\}$ is an i.i.d. sequence of standard Gaussian random variables, independent of X_0. Assume that $|\phi| < 1$ and that X_0 is Gaussian with mean μ_0 and variance γ_0^2.

1. Show that if X_1 has the same distribution as X_0 then

$$\begin{cases} \mu + \phi \mu_0 = \mu_0 \\ \phi^2 \gamma_0^2 + \sigma^2 = \gamma_0^2 \end{cases}$$

2. Deduce an invariant distribution for $\{X_n,\ n \in \mathbb{N}\}$.

EXERCICE 3 Consider a Markov chain whose state space $\mathsf{X} = (0,1)$ is the open unit interval. If the chain is at x, it picks one of the two intervals $(0,\ x)$ or $(x,\ 1)$ with equal probability $1/2$, and then moves to a point y which is uniformly distributed in the chosen interval.

1. Show that this Markov chain has a transition density with respect to Lebesgue measure on the interval $(0,1)$, which is given by

$$k(x,y) = \frac{1}{2}\frac{1}{x}\mathbb{1}_{]0,x[}(y) + \frac{1}{2}\frac{1}{1-x}\mathbb{1}_{]x,1[}(y)\ . \tag{2}$$

2. Show that this Markov chain can be equivalently represented as an iterated random sequence.

$$X_t = \varepsilon_t\,[X_{t-1}U_t] + (1-\varepsilon_t)\,[X_{t-1} + U_t(1-X_{t-1})]\ , \tag{3}$$

where $\{U_n,\ n \in \mathbb{N}\}$ and $\{\varepsilon_n,\ n \in \mathbb{N}\}$ are i.i.d.random variables whose distribution should be given.

3. Assuming that the stationary distribution has a density p with respect to Lebesgue measure show that
$$p(y) = \frac{1}{2}\int_y^1 \frac{p(x)}{x}\mathrm{d}x + \frac{1}{2}\int_0^y \frac{p(x)}{1-x}\mathrm{d}x\,. \tag{4}$$
4. Deduce that
$$\int_0^z p(y)\mathrm{d}y = 2C\arcsin(\sqrt{z})\,, \tag{5}$$
for some constant C.
5. Conclude that $C = 1/\pi$.

EXERCICE 4 Consider for example an ARMA(1,1) model,
$$Y_t - \phi_1 Y_{t-1} = Z_t + \theta_1 Z_{t-1}\,, \quad t \geq 1\,, \tag{6}$$
where $\{Z_t,\, t \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with density q with respect to the Lebesgue measure on $\mathbb{R}$, and $\{Z_t,\, t \in \mathbb{N}\}$ is independent of Y_0, which has distribution χ. The process of interest, $\{Y_t,\, t \in \mathbb{N}\}$ is referred to as the *observations*.

1. Setting $X_t = \begin{pmatrix} Y_{t-1} \\ Z_{t-1} \end{pmatrix}$, show that the process (X_t, Y_t) is an Observation-Driven time series process.