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Simulation and inference of stochastic differential equations

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Chapter 1

Discretization of stochastic differential equations

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Keywords 1.1

1.1 Gentle reminders: Gaussian vectors, Brownian motion, Stochastic differential equations

1.1.1 Gaussian random vectors

Definition 1.1. A random variable $X \in \mathbb{R}^n$ is a Gaussian vector if and only if, for all $a \in \mathbb{R}^n$, the random variable $\langle a\,;\,X\rangle$ is a Gaussian random variable.

For all random variable $X \in \mathbb{R}^n$, $X \sim \mathcal{N}(\mu, \Sigma)$ means that X is a Gaussian vector with mean $\mathbb{E}[X] = \mu \in \mathbb{R}^n$ and covariance matrix $\mathbb{V}[X] = \Sigma \in \mathbb{R}^{n\times n}$. The characteristic function of X is given, for all $t \in \mathbb{R}^n$, by

$$\mathbb{E}[\mathrm{e}^{i\langle t\,;\,X\rangle}] = \mathrm{e}^{i\langle t\,;\,\mu\rangle - t'\Sigma t/2}\,.$$

Therefore, the law of a Gaussian vector is uniquely defined by its mean vector and its covariance matrix. If the covariance matrix Σ is nonsingular, then the law of X has a probability density with respect to the Lebesgue measure on $\mathbb{R}^n$ given by :

$$x \mapsto \det(2\pi\Sigma)^{-1/2}\exp\left\{-(x-\mu)'\Sigma^{-1}(x-\mu)/2\right\}\ ,$$

where $\mu = \mathbb{E}[X]$.

Proposition 1.2 *Let $X \in \mathbb{R}^n$ be a Gaussian vector. Let $\{i_1,\ldots,i_d\}$ be a subset of $\{1,\ldots,n\}$, $d \geqslant 1$. If for all $1 \leqslant k \neq j \leqslant d$, $\mathrm{Cov}(X_{i_k},X_{i_j}) = 0$, then $(X_{i_1},\ldots,X_{i_d})$ are independent.*

PROOF. The random vector $(X_{i_1},\ldots,X_{i_d})'$ is a Gaussian vector with mean $(\mathbb{E}[X_{i_1}],\ldots,\mathbb{E}[X_{i_d}])'$ and diagonal covariance matrix $\mathrm{diag}(\mathbb{V}[X_{i_1}],\ldots,\mathbb{V}[X_{i_d}])$. Consider $(\xi_{i_1},\ldots,\xi_{i_d})$ i.i.d. random variables with distribution $\mathscr{N}(0,1)$ and define, for all $1 \leqslant j \leqslant d$,

$$Z_{i_j} = \mathbb{E}[\xi_{i_j}] + \sqrt{\mathbb{V}[X_{i_j}]}\xi_{i_j}\ .$$

Then, the random vector $(Z_{i_1},\ldots,Z_{i_d})'$ is a Gaussian vector with the same mean and the same covariance matrix as $(X_{i_1},\ldots,X_{i_d})'$. The two vectors have therefore the same characteristic function and the same law and $(X_{i_1},\ldots,X_{i_d})$ are independent as $(\xi_{i_1},\ldots,\xi_{i_d})$ are independent. ∎

Theorem 1.3 (Cochran). *Let $X \sim \mathscr{N}(0,I_n)$ be a Gaussian vector in $\mathbb{R}^n$, F be a vector subspace of $\mathbb{R}^n$ and $F^\perp$ its orthogonal. Denote by $\pi_F(X)$ (resp. $\pi_{F^\perp}(X)$) the orthogonal projection of X on F (resp. on $F^\perp$). Then, $\pi_F(X)$ and $\pi_{F^\perp}(X)$ are independent, $\|\pi_F(X)\|^2 \sim \chi^2(p)$ and $\|\pi_{F^\perp}(X)\|^2 \sim \chi^2(n-p)$, where p is the dimension of F.*

PROOF. Let $(u_1,\ldots,u_n)$ be an orthonormal basis of $\mathbb{R}^n$ where $(u_1,\ldots,u_p)$ is an orthonormal basis of F and $(u_{p+1},\ldots,u_n)$ and orthonormal basis of $F^\perp$. Consider the matrix $U \in \mathbb{R}^{n\times n}$ such that for all $1 \leqslant i \leqslant n$, the i-th column of U is u_i and $U_{(p)}$ (reps. $U^\perp_{(n-p)}$) the matrix made of the first p (resp. last $n-p$) columns of U. Note that

$$\pi_F(X) = \sum_{i=1}^{p}\langle X\,;\,u_i\rangle u_i\ ,$$

which can be written $\pi_F(X) = U_{(p)}U'_{(p)}X$. Similarly, $\pi_{F^\perp}(X) = U^\perp_{(n-p)}(U^\perp_{(n-p)})'X$ Therefore,

$$\begin{pmatrix}\pi_F(X)\\ \pi_{F^\perp}(X)\end{pmatrix} = \begin{pmatrix}U_{(p)}U'_{(p)}\\ U^\perp_{(n-p)}(U^\perp_{(n-p)})'\end{pmatrix}X$$

is a centered Gaussian vector with covariance matrix given by

$$\begin{pmatrix}U_{(p)}U'_{(p)} & 0\\ 0 & U^\perp_{(n-p)}(U^\perp_{(n-p)})'\end{pmatrix}\ .$$

By Proposition 1.2, $\pi_F(X)$ and $\pi_{F^\perp}(X)$ are independent. On the other hand,

$$\|\pi_F(X)\|^2 = \sum_{i=1}^{p}\langle X\,;\,u_i\rangle^2 \quad \text{and} \quad \|\pi_{F^\perp}(X)\|^2 = \sum_{i=p+1}^{n}\langle X\,;\,u_i\rangle^2\ .$$

The random vector $(\langle X\,;\,u_i\rangle)_{1\leqslant i\leqslant n}$ is given by $U'X$: it is a Gaussian random vector with mean 0 and covariance matrix I_n. The random variables $(\langle X\,;\,u_i\rangle)_{1\leqslant i\leqslant n}$ are therefore i.i.d. with distribution $\mathscr{N}(0,1)$, which concludes the proof. ∎

1.1.2 Brownian motion

Definition 1.4. A continuous time process $(W_t)_{t>0}$ is a Brownian motion started at 0 if and only if:

i) $W_0 = 0$.
ii) $(W_t)_{t>0}$ is a Gaussian process.
iii) For all $(s,t) \in \mathbb{R}_+^2$, $W_t - W_s \sim \mathscr{N}(0, t-s)$.
iv) For all $(s,t) \in \mathbb{R}_+^2$, $s \leqslant t$, $W_t - W_s$ is independent of $\sigma((W_u)_{0 \leqslant u \leqslant s})$.
v) The trajectory $t \mapsto W_t$ is continuous.

When the trajectory $t \mapsto W_t$ is not assumed to be continuous, it can be shown that assumptions i) to iii) imply that it is almost surely continuous.

Proposition 1.5 *A Gaussian process $(W_t)_{t>0}$ with continuous trajectories and started at 0 is a Brownian motion if and only if the following properties hold.*

- *For all $t \geqslant 0$, $\mathbb{E}[W_t] = 0$.*
- *For all $(s,t) \in \mathbb{R}_+^2$, $\mathbb{E}[W_s W_t] = \min(s,t) = s \wedge t$.*

PROOF. Assume that$(W_t)_{t>0}$ is a Brownian motion.

- $\forall t \geqslant 0$, $\mathbb{E}[W_t] = \mathbb{E}[W_t - W_0] = 0$ since $W_t - W_0 \sim \mathscr{N}(0,t)$.
- For all $(s,t) \in \mathbb{R}_+^2$ such that $s \leqslant t$,

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s(W_s + W_t - W_s)] = \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] = \mathbb{E}[(W_s - W_0)^2] + \mathbb{E}[W_s]\mathbb{E}[W_t - W_s] = s + 0 = s$$

Assume now that $\mathbb{E}[W_t] = 0$ and that for all $(s,t) \in \mathbb{R}^2$, $\mathbb{E}[W_s W_t] = s \wedge t$.

- Let $(s,t) \in \mathbb{R}_+^2$ with $s < t$. To prove that $W_t - W_s$ is independent of $\sigma((W_u)_{0 \leqslant u \leqslant t})$, it is enough to prove that for all $0 \leqslant u \leqslant s$, $\mathrm{Cov}(W_t - W_s, W_u) = 0$. Note that,

$$\mathrm{Cov}(W_t - W_s, W_u) = \mathbb{E}[W_u W_t] - \mathbb{E}[W_u W_t] = u \wedge t - u \wedge s = u - u = 0\,.$$

For all $(s,t) \in \mathbb{R}_+^2$, $s \leqslant t$, $W_t - W_s$ is Gaussian and centerer. In addition,

$$\mathbb{V}[W_t - W_s] = \mathrm{e}[(W_t - W_s)^2] = \mathbb{E}[W_t^2] + \mathbb{E}[W_s^2] - 2\mathbb{E}[W_t W_s] = t + s - 2s \wedge t = t - s\,.$$

■

Corollary 1.6 *Let $(W_t)_{t>0}$ be a Brownian motion. Then, the following processes are also Brownian motions.*

- $(W_{t+t_0} - W_{t_0})_{t>0}$ *for all $t_0 \in \mathbb{R}_+$.*
- $(tW_{1/t})_{t>0}$.
- $(\alpha W_{t/\alpha^2})_{t>0}$ *for all $\alpha > 0$.*

PROOF. See exercises. ■

Proposition 1.7 *Let $(W_t)_{t>0}$ be a Brownian motion. Then,*

- $\limsup_{t\to+\infty} \frac{W_t}{\sqrt{t}} = +\infty$ *(and then* $\limsup_{t\to+\infty} W_t = +\infty$*) almost surely.*
- *The Brownian motion takes almost surely each real value infinitely many often.*

1.1.3 Stochastic differential equations

1.1.3.1 Construction

In the case of Riemann integrals, for all $T > 0$ and all continuous function $f : [0,T] \to \mathbb{R}$, define

$$I_{n,T}(f) = \sum_{i=0}^{n} f(t_i^n)\left(t_{i+1}^n - t_i^n\right) ,$$

where $(t_i^n)_{0\leqslant i\leqslant n+1}$ is a subdivision of $[0,T]$, $t_0^n = 0 < t_1^n < \ldots < t_{n+1}^n = T$. If $\sup_{0\leqslant i\leqslant n+1}(t_{i+1}^n - t_i^n) \to 0$, then $I_{n,T}(f)$ converges as n grows to infinity to a quantity denoted $\int_0^T f(u)\mathrm{d}u$.

Following this construction, consider a piecewise constant process on $[0,T]$, defined, for all $t \in [0,T]$ by

$$X_t = \sum_{i=0}^{n} X_{t_i^n} \mathbb{1}_{[t_i^n, t_{i+1}^n[}(t) ,$$

where $X_{t_i^n}$ is a $\mathscr{F}_{t_i^n}$-measurable random variable and $t_0^n = 0 < t_1^n < \ldots < t_{n+1}^n = T$. Then, define

$$\int_0^T X_s \mathrm{d}W_s = \sum_{i=0}^{n} X_{t_i^n}\left(W_{t_{i+1}^n} - W_{t_i^n}\right) .$$

If $(X_t)_{0\leqslant t\leqslant T}$ is a bounded contiuous process, the stochastic integral $\int_0^T X_s \mathrm{d}W_s$ is constructed as follows.

- For all $n \geqslant 1$, define
$$X_t^n = \sum_{k=0}^{n+1} X_{\frac{kT}{n}} \mathbb{1}_{[\frac{kT}{n}, \frac{(k+1)T}{n}[}(t)$$
and the associated stochastic integral $M_T^n = \int_0^T X_s^n \mathrm{d}W_s$.
- Then, it may be shown that $(M_T^n)_{n\geqslant 0}$ converges in $\mathscr{L}^2$ to a random variable denoted M_T : $\mathbb{E}[(M_T^n - M_T)^2] \to_{n\to\infty} 0$. This random variable is written $M_T = \int_0^T X_s \mathrm{d}W_s$.

In this chapter, the objective is to sample solutions to stochastic differential equations (SDE) of the form:

$$\mathrm{d}X_s = \alpha_\theta(X_s)\mathrm{d}s + \sigma_\theta(X_s)\mathrm{d}W_s , \tag{1.1}$$

with $\theta \in \mathbb{R}^q$ an unknown parameter to be estimated and

- $\alpha_\theta : \mathbb{R} \to \mathbb{R}$ and $\sigma_\theta : \mathbb{R} \to \mathbb{R}$ are two continuous functions.
- $(W_t)_{t\geqslant 0}$ is a Brownian motion associated with its filtration $\mathscr{F}_t = \sigma((W_t)_{0<u\leqslant t})$.

The process (X_t) is said to be a strong solution to (1.1) if and only if, almost surely, for all $t \geqslant 0$,

$$X_t = X_0 + \int_0^t \alpha_\theta(X_s)\mathrm{d}s + \int_0^t \sigma_\theta(X_s)\mathrm{d}W_s .$$

Example 1.8 (Movement ecology) *The process $(X_t)_{t\geqslant 0}$ is the 2-dimensional position of an individual:*

$$\mathrm{d}X_s = \nabla_\theta A_\theta(X_s)\mathrm{d}s + \sigma \mathrm{d}W_s \ ,$$

where $A_\theta : \mathbb{R}^2 \to \mathbb{R}$ is a potential function. In this framework, the movement is supposed to reflect the attractiveness of the environment, which is modeled using a real valued potential function defined on $\mathbb{R}^2$. The position may be observed only indirectly as follows:

$$Y_{t_k} = X_{t_k} + \varepsilon_k \ ,$$

where the $(\varepsilon_k)_{0\leqslant k\leqslant n}$ are i.i.d. $\mathcal{N}(0,\eta^2 I_2)$.

1.2 Simulation of the Brownian motion

1.2.1 Simulation of a skeleton

Assume that a time horizon $T > 0$ and n time steps $(t_1,\ldots,t_n)$ are defined such as $0 < t_1 < \ldots < t_n < T$. Consider the following algorithm to sample $(W_{t_1},\ldots,W_{t_n})$.

- Sample $(\varepsilon_1,\ldots,\varepsilon_n)$ i.i.d. with $\varepsilon_1 \sim \mathcal{N}(0,t_1)$ and $\varepsilon_i \sim \mathcal{N}(0,t_i - t_{i-1})$ for$1 < i \leqslant n$.
- Define $X_{t_1} = \varepsilon_1$ and for $i > 1$ $X_{t_i} = X_{t_{i-1}} + \varepsilon_i$.

Then, choosing, $X_0 = 0$ yields $(X_{t_1},\ldots,X_{t_n}) \overset{\mathcal{L}}{=} (W_{t_1},\ldots,W_{t_n})$.
PROOF. Note that,

$$(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}) = (\varepsilon_1,\ldots,\varepsilon_n) \overset{\mathcal{L}}{=} (W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$$

and, since $X_0 = W_0 = 0$,

$$(X_0, X_1, X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}) \overset{\mathcal{L}}{=} (W_0, W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}) \ .$$

By linear transformation, it yields

$$(X_0, X_1, \ldots, X_n) \overset{\mathcal{L}}{=} (W_0, W_1, \ldots, W_n) \ .$$

■

1.2.2 Completion of a Brownian trajectory

Assume that $(W_{t_1},\ldots,W_{t_n})$ is the same distribution as a Brownian motion at times $(t_1,\ldots,t_n)$. Conditionally to these random variables, the aim of this section is to sample sample the Brownian motion at other time steps.

Lemma 1.9 *Assume that (X,Y,Z) is a centered Gaussian random vector. The, the conditional distribution of Y given (X,Z) is Gaussian with mean $\pi_{(X,Z)}(Y)$ and variance $\|Y - \pi_{(X,Z)}(Y)\|^2$ where $\pi_{X,Z}$ is the orthogonal projection on the vector space generated by (X,Z) for the scalar product $\langle U,V\rangle \mapsto \mathbb{E}[UV]$.*

PROOF. Soon. ■

The objective is now to sample W_u conditionally to $(W_{t_1},\ldots,W_{t_n})$ where $u \in (t_k,t_{k+1})$, $k \in \{1,\ldots,n-1\}$. This law is the law of W_u given $(W_{t_k}, W_{t_{k+1}})$. As $(W_{t_k}, W_u, W_{t_{k+1}})$ is a centered Gaussian random vector, it is enough to apply 1.9, ie. to compute $\pi_{(W_{t_k},W_{t_{k+1}})}(W_u)$ and $\|W_u - P_{(W_{t_k},W_{t_{k+1}})}(W_u)\|^2$. As $(W_{t_k}, W_{t_{k+1}} - W_{t_k})$ is an orthogonal basis of $\mathrm{Span}(W_{t_k}, W_{t_{k+1}})$ for $\langle\cdot,\cdot\rangle$. Then,

$$\begin{aligned}\pi_{(W_{t_k},W_{t_{k+1}})}(W_u) &= \left\langle W_u, \frac{W_{t_k}}{\|W_{t_k}\|} \right\rangle \frac{W_{t_k}}{\|W_{t_k}\|} + \left\langle W_u, \frac{W_{t_{k+1}-W_{t_k}}}{\|W_{t_{k+1}} - W_{t_k}\|} \right\rangle \frac{W_{t_{k+1}} - W_{t_k}}{\|W_{t_{k+1}} - W_{t_k}\|} \;, \\ &= \frac{\langle W_u, W_{t_k}\rangle}{\langle W_{t_k}, W_{t_k}\rangle} W_{t_k} + \frac{\langle W_u, W_{t_{k+1}} - W_{t_k}\rangle}{\langle W_{t_{k+1}} - W_{t_k}, W_{t_{k+1}} - W_{t_k}\rangle}\left(W_{t_{k+1}} - W_{t_k}\right) \;.\end{aligned}$$

Note that,

$$\langle W_{t_k}, W_{t_k}\rangle = \mathbb{E}[W_{t_k}^2] = t_k$$

and

$$\langle W_{t_k+1} - W_{t_k}, W_{t_k+1} - W_{t_k}\rangle = \mathbb{E}[\left(W_{t_k+1} - W_{t_k}\right)^2] = t_{k+1} - t_k \;.$$

In addition,

$$\langle W_u, W_{t_k}\rangle = \mathbb{E}[W_u W_{t_k}] = u \wedge t_k = t_k$$

and

$$\langle W_u, W_{t_{k+1}} - W_{t_k}\rangle = \langle W_u, W_{t_{k+1}}\rangle - \langle W_u, W_{t_k}\rangle = \mathbb{E}[W_u W_{t_{k+1}}] - \mathbb{E}[W_u W_{t_k}] = w \wedge t_{k+1} - w \wedge t_k = u - t_k$$

Conditionally to $(W_k, W_{t_{k+1}})$ the mean of W_u is$\mathbb{E}[W_u|W_{t_k}, W_{t_{k+1}}] = W_{t_k} + \frac{u-t_k}{t_{k+1}-t_k}(W_{t_{k+1}} - W_{t_k})$, i.e.,

$$\mathbb{E}[W_u|W_{t_k}, W_{t_{k+1}}] = \frac{t_{k+1} - u}{t_{k+1} - t_k} W_{t_k} + \frac{u - t_k}{t_{k+1} - t_k} W_{t_{k+1}} \;.$$

The conditional variance is

$$\begin{aligned}\mathbb{E}[(W_u - \mathbb{E}(W_u|W_{t_k}, W_{t_{k+1}}))^2] &= \left(\frac{t_{k+1} - u}{t_{k+1} - t_k}\right)^2 \underbrace{\mathbb{E}[(W_{t_k} - W_u)^2]}_{u - t_k} + \left(\frac{u - t_k}{t_{k+1} - t_k}\right)^2 \underbrace{\mathbb{E}[(W_u - W_{t_{k+1}})]}_{t_{k+1} - u} \;, \\ &= \frac{(t_{k+1} - u)^2 (u - t_k)}{(t_{k+1} - t_k)^2} + \frac{(u - t_k)^2 (t_{k+1} - u)}{(t_{k+1} - t_k)^2} \;, \\ &= \frac{(t_{k+1} - u)(u - t_k)}{t_{k+1} - t_k} \;.\end{aligned}$$

1.3 Discretization of SDE

In this section, $(X_t)_{0 \leqslant t \leqslant T}$ is solution to the following SDE:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \;, \tag{1.2}$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are continuous. Assume that σ and b are lipschitz: there exists $K \in \mathbb{R}_+$ such that for all $(x, y) \in \mathbb{R}^2$,

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leqslant K|x - y| \;,$$

then there exists a unique process $(X_t)_{0 \leqslant t \leqslant T}$ such that, almost surely, for all $0 \leqslant t \leqslant T$,

$$X_t - X_0 = \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s \;.$$

1.3.1 Euler-Maruyama scheme

Consider the evenly spaced partition of $[0,T]$ given by $(t_k^n = kT/N)_{0\leqslant k\leqslant n}$. To obtain a sample approximatively distributed as $(X_{t_0^n},\ldots,X_{t_n^n})$, the drift and diffusion of the SDE are assumed to be fixed on each interval (t_k^n,t_{k+1}^n) for $0\leqslant k\leqslant n-1$. Then, the approximate samples are defined as $\widetilde{X_0} = X_0$ and for all $k\in\{0,\ldots,n-1\}$,

$$\widetilde{X_{t_{k+1}^n}} = \widetilde{X_{t_k}^n} + \frac{T}{n}b(\widetilde{X_{t_k}^n}) + \sigma(\widetilde{X_{t_k}^n})\sqrt{\frac{T}{n}}\varepsilon_{k+1}\ ,$$

where $(\varepsilon_k)_{1\leqslant k\leqslant n}$ are i.i.d. with distribution $\mathscr{N}(0,1)$. A continuous approximation is given by $\overline{X_0} = X_0$ and for all $k\in\{0,\ldots,n-1\}$, and all $t\in[t_k^n,t_{k+1}^n)$,

$$\overline{X_t} - \overline{X_{t_k^n}} = b(\overline{X_{t_k^n}})(t-t_k^n) + \sigma(\overline{X_{t_k^n}})(W_t - W_{t_k^n})\ .$$

1.3.2 Approximation error for a Brownian motion

In the following, write for all $0\leqslant k\leqslant n-1$ and all $t\in[t_k^n,t_{k+1}^n)$, $\underline{t} = t_k^n$.

1.3.2.1 Lower bound

The objective here is to quantify, for $p\geqslant 2$, $\left\|\sup_{t\in[0,T]}|W_t - W_{\underline{t}}|\right\|_p$. Using elementary algebra yields:

$$\begin{aligned}
\left\|\sup_{t\in[0,T]}|W_t - W_{\underline{t}}|\right\|_p &= \left\|\max_{k\in\{1,\ldots,n\}}\sup_{t\in[t_{k-1}^n,t_k^n)}|W_t - W_{t_k^n}|\right\|_p\ ,\\
&= \sqrt{\frac{T}{n}}\left\|\max_{k\in\{1,\ldots,n\}}\sup_{t\in[t_{k-1}^n,t_k^n)}\sqrt{\frac{n}{T}}|W_t - W_{t_k^n}|\right\|_p\ ,\\
&= \sqrt{\frac{T}{n}}\left\|\max_{k\in\{1,\ldots,n\}}\sup_{t\in[k-1,k)}\sqrt{\frac{n}{T}}|W_{t\frac{T}{n}} - W_{(t-1)\frac{T}{n}}|\right\|_p\ ,\\
&= \sqrt{\frac{T}{n}}\left\|\max_{k\in\{1,\ldots,n\}}\sup_{t\in[k-1,k)}\sqrt{\frac{n}{T}}|W_t - W_{k-1}|\right\|_p\ ,\\
&= \sqrt{\frac{T}{n}}\left\|\max_{k\in\{1,\ldots,n\}}\varepsilon_k\right\|_p\ ,
\end{aligned}$$

where $\varepsilon_k = \sup_{t\in[k-1,k)}|W_t - W_{k-1}|$ and where we used that $(\sqrt{n/T}W_{tT/n})_{t\in[0,T]}$ is a Brownian motion. For all $k\in\{1,\ldots,n\}$, $\varepsilon_k\geqslant|W_k - W_{k-1}|\geqslant 0$, so that

$$
\begin{aligned}
\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \|_p &\geqslant \sqrt{\frac{T}{N}} \| \max_{k\in\{1,\dots,n\}} |W_k - W_{k-1}| \|_p \,, \\
&\geqslant \sqrt{\frac{T}{N}} \| \max_{k\in\{1,\dots,n\}} |W_k - W_{k-1}| \|_p \,, \\
&\geqslant \sqrt{\frac{T}{N}} \sqrt{\| \max_{k\in\{1,\dots,n\}} |W_k - W_{k-1}|^2 \|_{\frac{p}{2}}} \,, \\
&\geqslant \sqrt{\frac{T}{n}} c_p \sqrt{\log n} \,,
\end{aligned}
$$

where $c_p > 0$ does not depend on n and where we used that $(|W_k - W_{k-1}|)_{1\leqslant k\leqslant n}$ are i.i.d. with $W_k - W_{k-1} \sim \mathscr{N}(0,1)$. The last inequality is left to the reader.

1.3.2.2 Upper bound

To obtain an upper bound, write

$$
\| \sup_{t\in[0,T]} |W_t - W_{\underline{t}}| \|_p = \sqrt{\frac{T}{n}} \sqrt{\| \max_{k\in\{1,\dots,n\}} \sup_{t\in[k-1,k[} |W_t - W_{k-1}|^2 \|_{\frac{p}{2}}} \,,
$$

where for all $1 \leqslant k \leqslant n$, $\sup_{t\in[k-1,k[} |W_t - W_{k-1}|^2 \stackrel{\mathscr{L}}{=} \sup_{t\in[0,1[} |W_t|^2 = \varepsilon$. Then, it can be proved that if for some $\lambda > 0$, $\mathbb{E}[\mathrm{e}^{\lambda\varepsilon}] < +\infty$,

$$
\| \max_{k\in\{1,\dots,n\}} \sup_{t\in[k-1,k[} |W_t - W_{k-1}|^2 \|_{\frac{p}{2}} \leqslant c_{p,\lambda} \log(n+1) \,,
$$

where $c_{p,\lambda}$ does not depend on n. For all $\lambda > 0$,

$$
\begin{aligned}
\mathbb{E}[e^{\lambda\varepsilon}] &= \mathbb{E}[e^{\lambda \sup_{t\in[0,1]} |W_t|^2}] \,, \\
&= \mathbb{E}[e^{\lambda \max(\sup_{t\in[0,1]}(W_t), \sup_{t\in[0,1]}(-W_t))^2}] \,, \\
&\leqslant \mathbb{E}[e^{\lambda \sup_{t\in[0,1]}(W_t)^2} + e^{\lambda \sup_{t\in[0,1]}(-W_t)^2}] \,, \\
&\leqslant 2\mathbb{E}[e^{\lambda (\sup_{t\in[0,1]}(W_t)^2}] \,,
\end{aligned}
$$

as $(-W_t)_{0\leqslant t\leqslant 1}$ has the same law as $(W_t)_{0\leqslant t\leqslant 1}$. By the reflection principle (see exercices), $\sup_{t\in[0,1]} W_t = |W_1|$. Then, as $W_1 \sim \mathscr{N}(0,1)$, if $\lambda \in [0, \frac{1}{2})$,

$$
\begin{aligned}
\mathbb{E}[e^{\lambda\varepsilon}] \leqslant 2\mathbb{E}[e^{\lambda W_1^2}] \leqslant 2\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\lambda x^2} e^{\frac{-x^2}{2}} \mathrm{d}x \leqslant 2\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(1-2\lambda)x^2} \mathrm{d}x \,, \\
\leqslant \frac{2}{\sqrt{1-\lambda}} \frac{1}{\sqrt{2\pi}} \sqrt{1-2\lambda} \int_{\mathbb{R}} e^{\frac{-x^2}{2\left(\frac{1}{\sqrt{1-2\lambda}}\right)^2}} \mathrm{d}x \,, \\
\leqslant \frac{2}{\sqrt{1-\lambda}} \,.
\end{aligned}
$$

Then, $\mathbb{E}[\mathrm{e}^{\lambda\varepsilon}] < +\infty$ which concludes the proof.

1.3.3 L_p*-mean error for general SDE*

Lemma 1.10 (Gronwald) *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous and locally bounded function and $\psi:\mathsf{R}_+\to\mathsf{R}_+$ be nondecreasing. If there exists $\alpha>0$ such that for all $t\geqslant 0$,*

$$f(t)\leq \alpha\int_0^t f(s)\mathrm{d}s+\psi(s)\;,$$

then,

$$\sup_{s\in[0,t]} f(s)\leqslant \mathrm{e}^{\alpha t}\psi(t)\;.$$

PROOF. Soon ■

Proposition 1.11 *Assume that b and σ are Lipschitz. Then, for all $p\geqslant 2$, there exists a constant c_p such that:*

$$\left\|\sup_{t\in[0,T]}\left|X_t-\overline{X_t}\right|\right\|_p\leq c_p\left(\sqrt{\frac{T}{n}}+1\right)\;.$$

PROOF. The proof is written in the case $p=2$. Note that

$$X_t-\overline{X_t}=\int_0^t (b(X_s)-b(\overline{X_{\underline{s}}}))\mathrm{d}s+\int_0^t(\sigma(X_s)-\sigma(\overline{X_{\underline{s}}}))\mathrm{d}W_s\;.$$

This yields, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|X_t-\overline{X_t}|^2\right]&=\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t b(X_s)-b(\overline{X_{\underline{s}}})\mathrm{d}s+\int_0^t\sigma(X_s)-\sigma(b(\overline{X_{\underline{s}}}))\mathrm{d}W_s\right)^2\right]\;,\\
&\leqslant 2\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t(b(X_s)-b(\overline{X_{\underline{s}}})\mathrm{d}s\right)^2\right]+2\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t(\sigma(X_s)-\sigma(\overline{X_{\underline{s}}}))\mathrm{d}W_s\right)^2\right]\;,\\
&\leqslant 2T\mathbb{E}[\int_0^T|b(X_s)-b(\overline{X_{\underline{s}}})|^2\mathrm{d}s]+2\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t(\sigma(X_s)-\sigma(\overline{X_{\underline{s}}}))\mathrm{d}W_s\right)^2\right]\;.
\end{aligned}$$

By assumption, there exist c_b and c_σ positive constants such that for all $(x,y)\in\mathbb{R}^2$, $|b(x)-b(y)|\leq c_b|x-y|$ and $|\sigma(x)-\sigma(y)|\leq c_\sigma|x-y|$. Then, by Doob's inequality and Itô isometry,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|X_t-\overline{X_t}|^2\right]&\leq 2Tc_b^2\mathbb{E}\left[\int_0^T|X_s-\overline{X_{\underline{s}}}|^2\mathrm{d}s\right]+2\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_0^t(\sigma(X_s)-\sigma(\overline{X_{\underline{s}}}))\mathrm{d}W_s\right)^2\right]\;,\\
&\leq 2Tc_b^2\mathbb{E}\left[\int_0^T|X_s-\overline{X_{\underline{s}}}|^2\mathrm{d}s\right]+8\mathbb{E}\left[\int_0^T(\sigma(X_s)-\sigma(\overline{X_{\underline{s}}}))^2\mathrm{d}s\right]\;.
\end{aligned}$$

Therefore, there exists $C>0$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|X_t-\overline{X_t}|^2\right]&\leqslant C\mathbb{E}\left[\int_0^T|X_s-\overline{X_{\underline{s}}}|^2\mathrm{d}s\right]\;,\\
&\leqslant C\left(\mathbb{E}\left[\int_0^T|X_s-\overline{X_s}|^2\mathrm{d}s\right]+\mathbb{E}\left[\int_0^T|\overline{X_s}-\overline{X_{\underline{s}}}|^2\mathrm{d}s\right]\right)\;,\\
&\leqslant C\int_0^T\mathbb{E}\left[\sup_{0\leq u\leq s}|X_u-\overline{X_u}|^2\mathrm{d}s\right]+\mathbb{E}\left[\int_0^T|\overline{X_s}-\overline{X_{\underline{s}}}|^2\mathrm{d}s\right]\;.
\end{aligned}$$

Then, the proof is concluded by Gronwald's lemma. ■

1.3.4 Higher order discretization scheme: the Milstein scheme

It is possible to obtain better rates of convergence by approximating $\int_0^t b(X_s)\mathrm{d}s$ et $\int_0^t \sigma(X_s)\mathrm{d}W_s$ more precisely. For instance, the Milstein scheme is given by $\widetilde{X_0} = x_0$ and for all $0 \leqslant k \leqslant n-1$,

$$\widetilde{X_{t_{k+1}^n}} = \widetilde{X_{t_k^n}} + b(\widetilde{X_{t_k^n}})\frac{T}{n} + \sigma(\widetilde{X_{t_k^n}})\underbrace{\sqrt{\frac{T}{n}}\varepsilon_k}_{W_{t_{k+1}^n}-W_{t_k^n}} + \frac{\sigma^2(\widetilde{X_{t_k^n}})}{2}\frac{T}{n}(\varepsilon_k^2-1)\;,$$

where the $(\varepsilon_k)_{0\leqslant k\leqslant n-1}$ are i.i.d. with $\varepsilon_1 \sim \mathcal{N}(0,1)$.

1.4 Exercices

Brownian motion

Let $(W_t)_{t\geqslant 0}$ be a Brownian motion started at 0.

1. Show that the following processes are Brownian motions.

   a. $(W_{t+t_0} - W_{t_0})_{t\geqslant 0}$ for all $t_0 \geqslant 0$.
   b. $(\alpha W_{\alpha^{-2}t})_{t\geqslant 0}$ pour tout $\alpha > 0$.

   For all $t \geqslant 0$, write $Z_t = W_{t+t_0} - W_{t_0}$. Then,

   $$\mathbb{E}[Z_t] = \mathbb{E}[W_{t+t_0} - W_{t_0}] = \mathbb{E}[W_{t+t_0}] - \mathbb{E}[W_{t_0}] = 0\;.$$

   On the other hand, $t \mapsto Z_t$ is continuous and $Z_0 = 0$. Finally, for all $0 \leqslant s \leqslant t$,

   $$\begin{aligned}\mathbb{E}[(W_{s+t_0} - W_{t_0})(W_{t+t_0} - W_{t_0})] &= \mathbb{E}[W_{s+t_0}W_{t+t_0} + W_{t_0}^2 - W_{s+t_0}W_{t_0} - W_{t_0}W_{t+t_0}]\;,\\ &= \mathbb{E}[W_{s+t_0}W_{t+t_0}] + \mathbb{E}[W_{t_0}^2] - \mathbb{E}[W_{s+t_0}W_{t_0}] - \mathbb{E}[W_{t_0}W_{t+t_0}]\;,\\ &= s + t_0 + t_0 - t_0 - t_0\;,\\ &= s\;.\end{aligned}$$

   By linearity, $(Z_t)_{t\geq 0}$is a Gaussian process. Then, $(Z_t)_{t\geqslant 0}$ is a Brownian motion.
   Write for all $t \geqslant 0$, $\widetilde{Z}_t = \alpha W_{\frac{t}{\alpha^2}}$. By linearity, it is a centered Gaussian process, continuous and such that $\widetilde{Z}_0 = 0$. Then, For all $(s,t) \in \mathbb{R}^2$ such that $s \leqslant t$,

   $$\mathbb{E}[\widetilde{Z}_s\widetilde{Z}_t] = \mathbb{E}\left[\left(\alpha W_{\frac{s}{\alpha^2}}\right)\left(\alpha W_{\frac{t}{\alpha^2}}\right)\right] = \alpha^2 \mathrm{e}\left[W_{\frac{s}{\alpha^2}}W_{\frac{t}{\alpha^2}}\right] = \alpha^2\frac{s}{\alpha^2} = s = s\wedge t\;,$$

   which concludes the proof.
2. For all $0 \leqslant s \leqslant t$, compute $\mathbb{E}[W_sW_t^2]$, $\mathbb{E}[W_t|W_s]$ and $\mathbb{E}[(W_t - W_s)^2Y]$ where Y is a bounded random variable measurable with respect to $\sigma(\{W_u\}_{0\leqslant u\leqslant s})$.

   Let $0 \leqslant s \leqslant t$.

$$
\begin{aligned}
\mathbb{E}[W_sW_t^2] &= \mathbb{E}[W_s(W_t - W_s + W_s)^2] \,, \\
&= \mathbb{E}[W_s(W_t - W_s)^2] + 2\mathbb{E}[W_s^2(W_t - W_s)] + \mathbb{E}[W_s^3] \,, \\
&= \mathbb{E}[W_s]\mathbb{E}[(W_t - W_s)^2] + 2\mathbb{E}[W_s^2]\mathbb{E}[W_t - W_s] + \mathbb{E}[W_s^3] \,, \\
&= 0 + 0 + 0 \,, \\
&= 0 \,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathbb{E}[W_t|W_s] &= \mathbb{E}[W_t - W_s + W_s|W_s] \,, \\
&= \mathbb{E}[W_t - W_s|W_s] + \mathrm{e}[W_s|W_s] \,, \\
&= \mathbb{E}[W_t - W_s|W_s] + W_s \,, \\
&= W_s \,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathbb{E}[(W_t - W_s)^2Y] &= \mathbb{E}\left[\mathbb{E}[(W_t - W_s)^2Y|\sigma(\{W_u\}_{0\le u\le s})]\right] \,, \\
&= \mathbb{E}[Y\mathbb{E}[(W_t - W_s)^2|\sigma(\{W_u\}_{0\le u\le s})]] \,, \\
&= \mathbb{E}[Y\mathbb{E}[(W_t - W_s)^2]] \,, \\
&= \mathbb{E}[Y(t-s)] \,, \\
&= (t-s)\mathbb{E}[Y] \,.
\end{aligned}
$$

3. Let $(B_t)_{t\geqslant 0}$ be a Brownian motion started at 0 independent of W and $\rho \in (0,1)$. Show that $(Z_t)_{t\geqslant 0}$ is a Brownian motion, where for all $t \geqslant 0$, $Z_t = \rho W_t + \sqrt{1-\rho^2}B_t$.

   For all $(s,t) \in \mathbb{R}^2$ such that $s \leqslant t$,

$$
\begin{aligned}
\mathbb{E}[Z_sZ_t] &= \mathbb{E}[(\rho W_s + \sqrt{1-\rho^2}B_s)(\rho W_t + \sqrt{1-\rho^2}B_t)] \,, \\
&= \rho^2\mathbb{E}[W_sW_t] + (1-\rho^2)\mathbb{E}[B_tB_s] + \rho\sqrt{1-\rho^2}\mathbb{E}[W_sB_t + W_tB_s] \,, \\
&= \rho^2 s + (1-\rho^2)s + 0 \,, \\
&= s \,.
\end{aligned}
$$

Brownian bridge

Let $(B_t)_{0\leqslant t\leqslant 1}$ be a centered Gaussian process, centered, and such that for all $0 \leqslant s \leqslant t \leqslant 1$, $\mathrm{Cov}(B_s, B_t) = \min(s,t) - st$.

1. Prove that $(\tilde{B}_t)_{0\leqslant t\leqslant 1}$ has the same law as $(B_t)_{0\leqslant t\leqslant 1}$ where for all $0 \leqslant t \leqslant 1$, $\tilde{B}_t = B_{1-t}$.
2. Let $(W_t)_{t\leqslant 0}$ be a Brownian motion and define for all $t \geqslant 0$, $\tilde{W}_t = W_t - tW_1$. Prove that $\tilde{W}$ has the same law as B and is independent of W_1.

Reflection principle - simulation of a first passage time

Let $(W_t)_{t\leqslant 0}$ be a Brownian motion started at 0 and write $S_t = \sup_{0\leqslant s\leqslant t} W_s$. For all $a \leqslant b$, $b > 0$, show that

$$
\mathbb{P}(S_t \geqslant b\,; W_t \leqslant a) = \mathbb{P}(W_t \geqslant 2b - a) \,.
$$

Prove that S_t has the same law as $|W_t|$ and provide, for all $x > 0$, the probability density function of

$$\tau_x = \inf_{t \geqslant 0} \{W_t \geqslant x\} \; .$$

Simulation of the maximum of a Brownian motion

Let $(W_t)_{t \leqslant 0}$ be a Brownian motion started at 0.

1. Show that for all x, y in $\mathbb{R}$,

$$\mathbb{P}\left(\max_{0 \leqslant s \leqslant t} W_s \geqslant y \middle| W_t = x \right) = \exp\left(\frac{-2y(y-x)}{t} \right) ,$$

   when $y \geqslant \max(0, x)$.
2. Show that conditionally on $\{W_t = x\}$, $\max_{0 \leqslant s \leqslant t} W_s$ has the same distribution as

$$Z = \frac{x + \sqrt{x^2 - 2t \log U}}{2} ,$$

   where U is uniformly distributed on $(0,1)$.