$X_1, \dots, X_n \sim \mathrm{Poi}(\lambda), \lambda > 0$

iid.

Goal: Find test $\psi(X_1 \dots X_n) \in \{0,1\}$ with asymptotic level $\alpha$

① $H_0: \lambda = 2 \quad H_1: \lambda \neq 2$

$\Theta_0 = \{2\} \quad \Theta_1 = (0,\infty) \setminus \{2\}$

ⓐ Find test statistic / pivot $T_n$

$\psi = \mathbb{1}\{T_n > s\}$

ⓑ Determine $s$ to match level $\alpha$

# Hypothesis testing

→ type I error: $\alpha_\psi(\lambda) = \mathbb{P}_\lambda(\psi(X_1, \dots X_n) = 1), \lambda \in \Theta_0$

type II error: $\beta_\psi(\lambda) = \mathbb{P}_\lambda(\psi(X_1, \dots X_n) = 0), \lambda \in \Theta_1$

(asymptotic) level of test: $\sup_{\lambda \in \Theta_0} \limsup_{n \to \infty} \mathbb{P}_\lambda(\psi(X_1, \dots, X_n) = 1) = \alpha \; (\leq \alpha)$

Asymmetry: Can only "reject / refute" $H_0$, never "accept" it.

$X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Poi}(\lambda), \lambda > 0$

Goal: Find test $\psi$ with asymptotic level $\alpha$

(1) $H_0 : \lambda = 2 \quad H_1 : \lambda \neq 2$

(a) Find $T_n$

$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i \,; \quad T_n = \left| \sqrt{n}\, \frac{\hat{\lambda} - 2}{\sqrt{2}} \right|$

$\psi = \mathbb{1}\{T_n > s\}$

(b) Adjust $s$ to match level $\alpha$

$s = q_{\alpha/2}$

# Hypothesis testing (1) Two-sided test

(a) $\mathbb{E}[X_i] = \lambda, \quad \mathrm{Var}(X_i) = \lambda,$

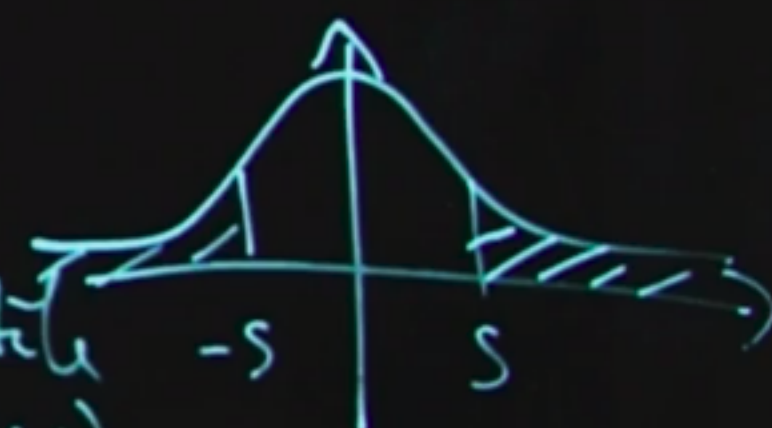
LLN: $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{\mathbb{P}_\lambda} \lambda$ ; CLT: $\sqrt{n}\,\frac{\hat{\lambda} - \lambda}{\sqrt{\lambda}} \xrightarrow[n\to\infty]{D} N(0,1)$

(b) $\alpha_\psi(2) = \mathbb{P}_2\left(\psi(X_1, \dots, X_n) = 1\right) = \mathbb{P}_2(T_n > s)$

$$= \mathbb{P}_2\Big( \Big| \underbrace{\sqrt{n}\, \frac{\hat{\lambda} - 2}{\sqrt{2}}}_{\xrightarrow[n\to\infty]{D} N(0,1)} \Big| > s \Big) \xrightarrow[n\to\infty]{} \mathbb{P}(|\overset{\sim N(0,1)}{Z}| > s)$$

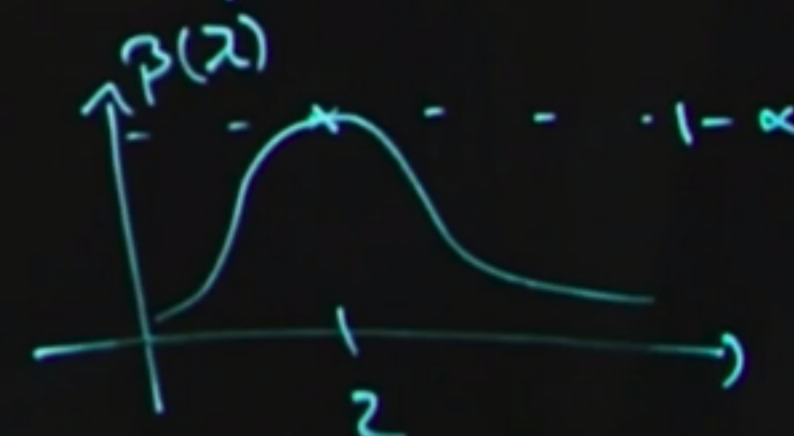
$$= 2(1 - \Phi(s)) \overset{!}{=} \alpha$$

$\Rightarrow \Phi(s) = 1 - \frac{\alpha}{2} \Rightarrow s = q_{\alpha/2},\ 1 - \frac{\alpha}{2}$ quantile of $N(0,1)$




type II error: $\lambda \neq 2$:

$$\mathbb{P}_\lambda(T_n \le s) = \mathbb{P}_\lambda\Big( \sqrt{n} \underbrace{\Big| \frac{\hat{\lambda} - 2}{\sqrt{2}} \Big|}_{\xrightarrow{\mathbb{P}_\lambda} \left|\frac{\lambda - 2}{\sqrt{2}}\right| \neq 0} \le s \Big) \xrightarrow[n\to\infty]{} 0$$



$X_1, \dots, X_n \overset{iid.}{\sim} \mathrm{Poi}(\lambda),\ \lambda > 0$

Goal: Find test $\psi$ with asymptotic level $\alpha$

② $H_0: \lambda \geq 2 \quad H_1: \lambda < 2$

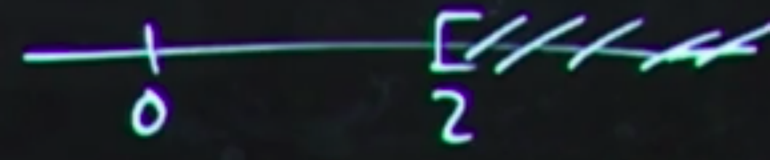
ⓐ Find $T_n$

$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad T_n = \frac{2-\hat{\lambda}}{\sqrt{2}} \cdot \sqrt{n}$

$\psi = \mathbb{1}\{T_n > s\}$

ⓑ Adjust $s = q_\alpha$

# Hypothesis testing ② One-sided test




ⓐ CLT: $\sqrt{n}\,\frac{\lambda - \hat{\lambda}}{\sqrt{\lambda}} \xrightarrow[n\to\infty]{D} N(0,1)$

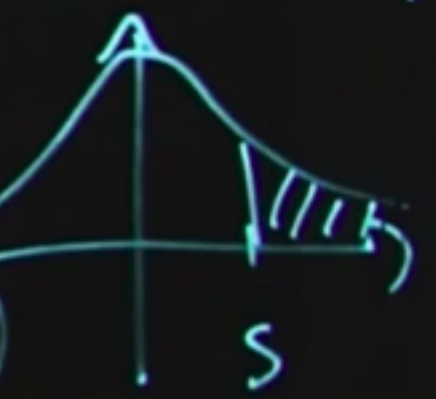
ⓑ $\underline{\lambda > 2}$: $\alpha_\psi(\lambda) = \mathbb{P}_\lambda(T_n > s) = \mathbb{P}_\lambda\Big(\underbrace{\sqrt{n}\,\frac{2-\hat{\lambda}}{\sqrt{2}}}_{\xrightarrow[n\to\infty]{\mathbb{P}_\lambda} \frac{2-\lambda}{\sqrt{2}} < 0} > s\Big)$

$\xrightarrow[n\to\infty]{} 0$

$\underline{\lambda = 2}$: $\alpha_\psi(2) = \mathbb{P}_2(T_n > s) = \mathbb{P}_2\left(\sqrt{n}\,\frac{2-\hat{\lambda}}{\sqrt{2}} > s\right)$

$\xrightarrow[n\to\infty]{} \mathbb{P}(Z > s) = 1 - \Phi(s) \overset{!}{=} \alpha$

$\Leftrightarrow \Phi(s) = 1-\alpha \Leftrightarrow s = q_\alpha$ $1-\alpha$ quantile of $N(0,1)$



$X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Poi}(\lambda), \lambda > 0$

Goal: Find test $\psi$ with asymptotic level $\alpha$

② $H_0: \lambda \geq 2 \quad H_1: \lambda < 2$

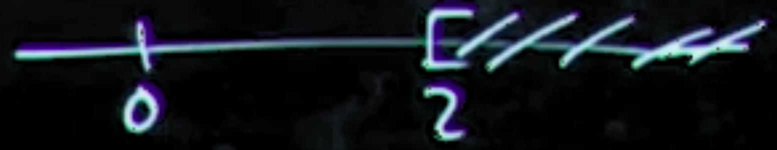
ⓐ Find $T_n$

$\hat\lambda = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad T_n = \frac{2-\hat\lambda}{\sqrt{2}}\sqrt{n}$

$\psi = \mathbb{1}\{T_n > s\}$

ⓑ Adjust $s = q_\alpha$

# Hypothesis testing ② One-sided test




ⓐ CLT: $\sqrt{n}\frac{\lambda - \hat\lambda}{\sqrt{\lambda}} \xrightarrow[n\to\infty]{D} N(0,1)$

ⓑ $\lambda > 2$: $\alpha_\psi(\lambda) = \mathbb{P}_\lambda(T_n > s) = \mathbb{P}_\lambda\Big(\sqrt{n}\underbrace{\frac{2-\hat\lambda}{\sqrt{2}}}_{} > s\Big)$

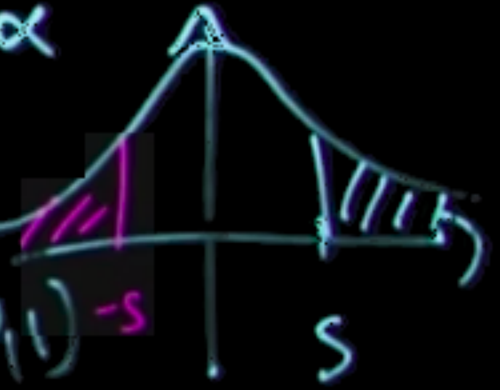
LLN: $\xrightarrow[n\to\infty]{\mathbb{P}_\lambda} \frac{2-\lambda}{\sqrt{2}} < 0$

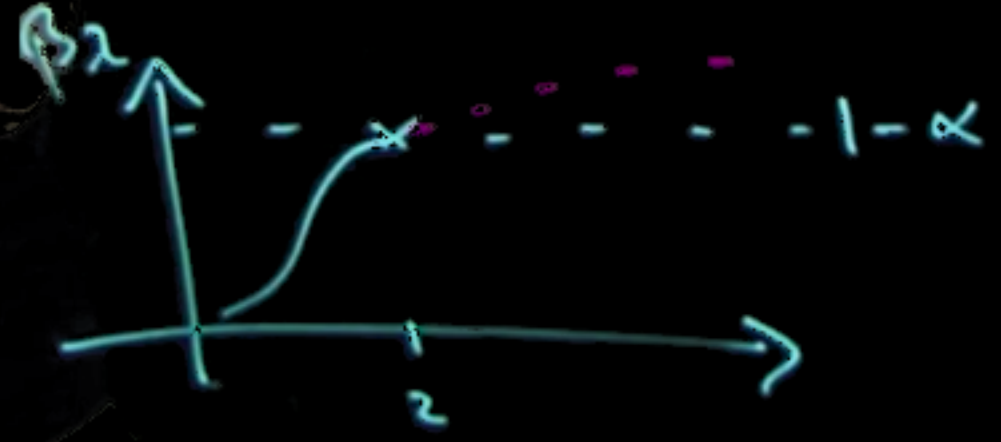
$\xrightarrow[n\to\infty]{} 0$

$\lambda = 2$: $\alpha_\psi(2) = \mathbb{P}_2(T_n > s) = \mathbb{P}_2\Big(\sqrt{n}\frac{2-\hat\lambda}{\sqrt{2}} > s\Big)$

$\xrightarrow[n\to\infty]{} \mathbb{P}(Z > s) = 1 - \Phi(s) \overset{!}{=} \alpha$

$\Leftrightarrow \Phi(s) = 1-\alpha \Leftrightarrow s = q_\alpha$ $1-\alpha$ quantile of $N(0,1)$



$X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Poi}(\lambda), \lambda > 0$

<u>Goal</u>: Find test $\psi$ with asymptotic level $\alpha$

③ $H_0: |\lambda - 2| \le 1$

$H_1: |\lambda - 2| > 1$

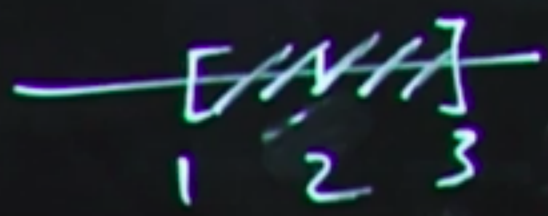
ⓐ Find $T_n^l, T_n^r$

$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^n X_i$

ⓑ Adjust $s_l, s_r$

$s_l = q_\alpha, \ s_r = q_\alpha$

# Hypothesis testing ③ Composite test




ⓐ $\psi = \mathbb{1}\{T_n^l > s_l \text{ or } T_n^r > s_r\}$

$= \mathbb{1}\left\{\max\left\{\frac{T_n^l}{s_l}, \frac{T_n^r}{s_r}\right\} > 1\right\}$

$T_n^l = \sqrt{n}\frac{1-\hat{\lambda}}{\sqrt{1}} = (1-\hat{\lambda})\sqrt{n}, \quad T_n^r = \frac{\hat{\lambda}-3}{\sqrt{3}}\sqrt{n}$

ⓑ $\underline{\lambda = 1}: \alpha_\psi(1) = \mathbb{P}_1(T_n^l > s_l \text{ or } T_n^r > s_r)$

$\le \mathbb{P}_1(T_n^l > s_l) + \mathbb{P}_1(T_n^r > s_r)$

- $\mathbb{P}_1(T_n^l > s_l) = \mathbb{P}_1(\sqrt{n}(1-\hat{\lambda}) > s_l) \xrightarrow[n\to\infty]{} \mathbb{P}(\overset{\sim N(0,1)}{Z} > s_l) \overset{!}{=} \alpha$
- $\mathbb{P}_1(T_n^r > s_r) = \mathbb{P}_1\left(\sqrt{n}\frac{\hat{\lambda}-3}{\sqrt{3}} > s_r\right) \xrightarrow[n\to\infty]{} 0$ by LLN

$\underline{\lambda = 3}: \mathbb{P}_3(T_n^r > s_r) = \mathbb{P}_3\left(\sqrt{n}\frac{\hat{\lambda}-3}{\sqrt{3}} > s_r\right) \xrightarrow[n\to\infty]{} \mathbb{P}(\overset{\sim N(0,1)}{Z} > s_r) \overset{!}{=} \alpha$

$\underline{\lambda \in (1,3)}: \mathbb{P}_\lambda(T_n^l > s_l) = \mathbb{P}(\sqrt{n}\underbrace{(1-\hat{\lambda})}_{\to 1-\lambda < 0} > s_l) \xrightarrow[n\to\infty]{} 0$