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ORIE 5530: Homework 3

1. This is a simulation question. Consider the Markov chain with the transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.38 & 0.22 \\ 0.12 & 0.7 & 0.18 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

   a. Simulate and plot a single realization $X_0, \ldots, X_n$ of this markov chain (start it with $X_0 = 1$). The plot should have time as the X-axis (going from 0 to n) and the y-axis should have 1,2,3. In each point in time the Markov chain will be in one of the states. Use the time horizon $n = 100$.

   b. Increase n to a 1000 and. Generate a single realization compute, over this realization

$$\frac{1}{n}\sum_{k=0}^{n-1} X_k,$$

   and, for $\beta = 0.9$

$$\sum_{k=0}^{n} \beta^k (X_k)^2.$$

   As before, use $X_0 = 1$ for your initial condition.

   c. Now repeat this over many realization and average to get approximations for the expectations

$$E[\frac{1}{n}\sum_{k=0}^{n-1} X_k | X_0 = 1] \text{ and } E[\sum_{k=0}^{n} \beta^k (X_k)^2 | X_0 = 1].$$

   d. Compute $E[\sum_{k=0}^{n} \beta^k (X_k)^2 | X_0 = 1]$ using what we learned in class and solving a system of equations. Compare to the simulation result.

2. Conditional Expectation.

   Two players A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have **won two more games than the other**.

   (a) Find the probability that A is the overall winner.

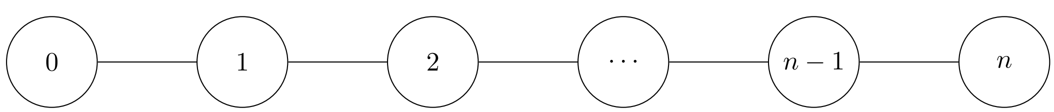
   (b) Find the expected number of games played.

   (Hint: What if you know the outcomes of the first several games?)

3. Conditional expectation.

   A particle moves along the following graph so that at each step it is equally likely to move to any of its neighbours. Starting at 0 show that the expected number of steps it takes to reach n is n^2.

   (Hint: Let T_i be the number of steps it takes to go from vertex $i-1$ to i, where $i = 1, \ldots, n$. Try to find $E[T_i]$ for all i recursively.)



4. Conditional expectation.

   A fair die is continually rolled until an even number has appeared on 10 distinct (not necessarily consecutive) rolls. Let X_i denote the number of rolls that land on size $i (i = 1, 2, 3, 4, 5, 6)$.

   (a) Find $E[X_2]$.

   (b) Find $E[X_1]$. (Hint: think about the number of ones between the $j - 1$st and jth even values) .

   Hint: condition on whether a 1 appeared before the first even number of after.

In [11]:
```
import numpy as np
import pandas as pd
from random import seed
from random import random
import matplotlib.pyplot as plt
```

Problem 1

(a)

In [33]:
```
P = np.array([[0.4, 0.38, 0.22],
              [0.12, 0.7, 0.18],
              [0.2, 0.5, 0.3]])
states_arr = np.array([1,2,3])
```

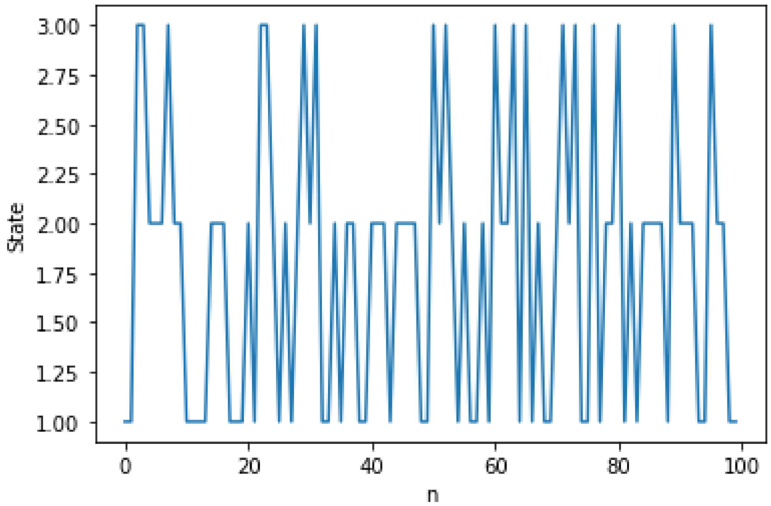
In [34]:
```
n = 100
X = np.zeros(100)
X[0] = 1
states.shape
```

Out[34]:
```
(3,)
```

In [51]:
```
def state_dist(n,P):
    current = X[0]
    states = [X[0]]
    count = 1
    while count < n:
        if current == 1:
            next_state = np.random.choice(states_arr,replace=True,p=P[0])
            states.append(next_state)
        elif current == 2:
            next_state = np.random.choice(states_arr,replace=True,p=P[1])
            states.append(next_state)
        elif current == 3:
            next_state = np.random.choice(states_arr,replace=True,p=P[2])
            states.append(next_state)
        count += 1
    return states
```

In [52]:
```
plt.plot(np.arange(0,100) ,state_dist(100,P))
plt.xlabel('n')
plt.ylabel('State')
```

Out[52]:
```
Text(0, 0.5, 'State')
```



(b)

In [92]:
```
X = state_dist(1000,P)
print("Sum X: ", np.sum(X)/1000)
```

```
Sum X:  1.819
```

In [97]:
```
X = state_dist(1000,P)
X2 = np.square(X)
B = [0.9**i for i in range(1000)]
print("Sum B*X^2:" , np.dot(X2, B))
```

```
Sum B*X^2: 38.4147074904899
```

(c)

In [81]:
```
def average_state():
    result = []
    for i in range(1000):
        X = state_dist(1000,P)
        sums = np.sum(X)/1000
        result.append(sums)
    return np.mean(result)
```

In [98]:
```
print("Average state:", average_state())
```

```
Average state: 1.819986
```

In [84]:
```
def average_stateB():
    result = []
    B = [0.9**i for i in range(1000)]
    for i in range(1000):
        X = state_dist(1000,P)
        X2 = np.square(X)
        result.append(np.dot(X2, B))
    return np.mean(result)
```

In [99]:
```
print("Long run reward:", average_stateB())
```

```
Long run reward: 35.97793051594115
```

(d)

1. $P = \begin{bmatrix} 0.4 & 0.38 & 0.22 \\ 0.12 & 0.7 & 0.18 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$

$\pi_1 = 0.4\pi_1 + .12\pi_2 + .2\pi_3$

$\pi_2 = .38\pi_1 + .7\pi_2 + .5\pi_3$

$\pi_3 = .22\pi_1 + .18\pi_2 + .3\pi_3$

$1 = \pi_1 + \pi_2 + \pi_3$

(d) $E\left[\sum_{k=0}^{n} \beta^k (X_k)^2 \mid X_0 = 1\right] = V_k$

$\pi_j = \sum_i \pi_i P_{ij}$ & $\sum_j \pi_j = 1$

$V(S) = R(S) + \beta \sum_{S'} P(S'|S) V(S')$

$V(1) = X_1^2 + \beta(P_{12} V(2) + P_{13} V(3) + P_{11}(V_1))$

$V(2) = X_2^2 + \beta(P_{21} V(1) + P_{22} V(2) + P_{23} V(3))$

$V(3) = X_3^2 + \beta(P_{31} V(1) + P_{32} V(2) + P_{33} V(3))$

$V(1) = X_1^2 + (.9)(.38)V(2) + (.9)(.22)V(3) + (.9)(.4)V(1)$

$V(2) = X_2^2 + (.9)(.12)V(1) + (.9)(.7)V(2) + (.9)(.18)V(3)$

$V(3) = X_3^2 + (.9)(.2)V(1) + (.9)(.5)V(2) + (.9)(.3)V(3)$

$V(1) = X_1^{2}$ (1) $+ .36V(1) + .342V(2) + .198V(3)$

$V(2) = X_2^{2}$ (4) $+ .108V(1) + .63V(2) + .162V(3)$

$V(3) = X_3^{2}$ (9) $+ .18V(1) + .45V(2) + .27V(3)$

Solved in Python:

$V(1) = \frac{1693885}{41539} = 40.778$ $V(2) = \frac{1{,}849{,}885}{41{,}539} = 44.534$ $V(3) = \frac{2{,}070{,}135}{41{,}539} = 49.836$

In the simulation, we got 36.019 which is slightly lower than $V_1 = 40.778$.

2. Conditional Expectation.

Two players A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have **won two more games than the other**.

(a) Find the probability that A is the overall winner.

(b) Find the expected number of games played.

(Hint: What if you know the outcomes of the first several games?)

a) p = A wining Each game

$1-p$ = B wining Each game.

Let x = total # of wins in the first 2 rounds game (by A)

$$
\begin{aligned}
P(A \text{ wins}) &= P(A \text{ wins} \mid X=0)\cdot P(X=0) \\
&\quad + P(A \text{ wins} \mid X=1)\cdot P(X=1) \\
&\quad + P(A \text{ wins} \mid X=2)\cdot P(X=2) \\
&= 0 + P(A \text{ wins})\cdot p\cdot(1-p) + P(A \text{ wins})\cdot p\cdot(1-p) \\
&\quad + p^2
\end{aligned}
$$

$$\Rightarrow \quad P(A \text{ wins})\,(1-2p(1-p)) = p^2$$

$$P(A \text{ wins}) = \frac{p^2}{1-2p+2p^2}$$

b). Let Y = total umber of game that plays

$$
\begin{aligned}
E(Y) &= E(Y \mid X=0)\cdot P(X=0) \\
&\quad + E(Y \mid X=1)\cdot P(X=1) \\
&\quad + E(Y \mid X=2)\cdot P(X=2) \\
&= 2(1-p)^2 + 2(E(Y)+2)p(1-p) + 2p^2 \\
&= 2(1-p)^2 + 4p(1-p) + 2p(1-p)E(Y) + 2p^2
\end{aligned}
$$

$$E(Y)\,(1-2p(1-p)) = 2 - \cancel{4p} + \cancel{2p^2} + \cancel{2p^2} + \cancel{4p} - \cancel{4p^2} = 2$$

$$\Rightarrow \quad E(Y) = \frac{2}{1-2p+2p^2}$$

3.

- let T_i = expected # steps to reach n, starting at i

$$T_i = 1 + \frac{1}{2} T_{i+1} + \frac{1}{2} T_{i-1}$$

$$2T_i = 2 + T_{i+1} + T_{i-1}$$

$$2T_i - T_{i+1} = 2 + T_{i-1}$$

$$\rightarrow T_i - T_{i+1} = 2 + T_{i-1} - T_i$$

Base Case: $T_0 - T_1 = 1$, $T_n = 0$

$$T_i - T_{i+1} = 1 + 2i$$

$$\rightarrow T_0 = T_n + \sum_{i=0}^{n-1} T_i - T_{i+1}$$

$$= 0 + \sum_{i=0}^{n-1} 1 + 2i$$

$$= n + 2(n-1)\frac{n}{2}$$

$$= n + n^2 - n$$

$$\boxed{T_0 = n^2}$$ ✓

4. Conditional expectation.

A fair die is continually rolled until an even number has appeared on 10 distinct (not necessarily consecutive) rolls. Let X_i denote the number of rolls that land on size $i (i = 1, 2, 3, 4, 5, 6)$.

(a) Find $E[X_2]$.

(b) Find $E[X_1]$. (Hint: think about the number of ones between the $j-1$st and jth even values) .

Hint: condition on whether a 1 appeared before the first even number of after.

a). P(2 appear in even rolls) $= \frac{1}{6} \times 2 = \frac{1}{3}$

$E(X_2) = \frac{1}{3} \times 10 = \frac{10}{3}$

b). N = total number of roll to get 10 even number

$E(N) = N \cdot \frac{1}{p}$, $\sim$ Geometric dist ($p = \frac{1}{2}$)

$E(N) = 10 \cdot 2 = 20$

Consider all intervals, expected value to get 10 odd number

$= 20 - 10 = 10$.

P(1 appears in odd) $= \frac{1}{6} \times 2 = \frac{1}{3}$

$E(X_1) = \frac{1}{3} \times 10 = \frac{10}{3}$.