Hongyi Nie hn327
Kaitlin Tubbs kt392

In [1]:
```
import numpy as np
import random
import matplotlib.pyplot as plt
```

Q1 a)

In [2]:
```
#PROBLEM 1
n = 3
P = np.array([[0, 0.6, 0.4],
              [0.3, 0, 0.7],
              [0.85, 0.15, 0]])

trajectory = np.zeros((6000,2))
sims = 1000
reward = []
current = 1
time = 0
i = 0
while time < 1000:
    if current == 1:
        t12 = np.random.exponential(1/.6)
        t13 = np.random.exponential(1/.4)
        t_out = min(t12,t13)
        if t12 < t13:
            next_state = 2
        else:
            next_state = 3


    elif current == 2:
        t21 = np.random.exponential(1/.6)
        t23 = np.random.exponential(1/1.4)
        t_out = min(t21,t23)
        if t21 < t23:
            next_state = 1
        else:
            next_state = 3

    elif current == 3:
        t31 = np.random.exponential(1/2.55)
        t32 = np.random.exponential(1/0.45)
        t_out = min(t31,t32)
        if t31 < t32:
            next_state = 1
        else:
```

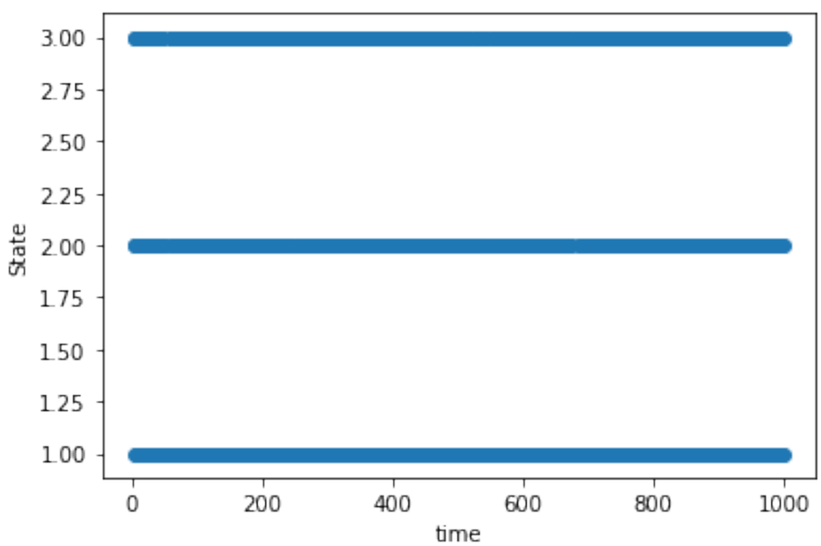
```
            next_state = 2

    time = time + t_out
    trajectory[i,0] = time
    trajectory[i,1] = next_state
    reward.append(next_state**2)
    current = next_state
    i += 1
```

In [3]:

```
times = trajectory[:,0]
times = times[times != 0]
X = trajectory[:,1]
X = X[X != 0]
plt.scatter(times,X)
plt.xlabel('time')
plt.ylabel('State')
```

Out[3]: Text(0, 0.5, 'State')



Q1 b)

Stimulation result = 7.075

The long run expectation we have here is 4.59578947368421. The absolute difference between stimulation result and formula result is 2.47921052631579, and the percentage difference is 53.9452588181402%.

In [4]:
```
total_rew = np.sum(reward)
lr_avg = total_rew/1000
lr_avg
```

Out[4]: 7.075

In [14]:
```
Math_lr_exp = (179/475) + 4*(132/475) + 9*(164/475)
print("The long run expectation we have here is {}.     The absolute d
```

```
The long run expectation we have here is 4.59578947368421.     The ab
solute difference between stimulation result and formula result is 2.
47921052631579, and the percentage difference is 53.9452588181402%.
```

In [ ]:

2. Cell Splitting: States: A, B

$N(t) = (N_A(t), N_B(t))$

- Rate $A \to B \sim \beta$ - Rate $B \to A \sim \alpha$ (splits into 2 cells)

$$Q = \begin{array}{c} \\ (0,1) \\ (1,0) \\ (1,1) \\ (0,2) \\ (2,0) \\ (1,2) \\ (2,1) \\ (2,2) \\ \vdots \end{array} \begin{array}{ccccccccccc} (0,1) & (1,0) & (1,1) & (0,2) & (2,0) & (1,2) & (2,1) & (2,2) & (3,0) & (0,3) & (3,1) \end{array}$$

$$\left[\begin{array}{ccccccccccc} -\alpha & 0 & 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & 0 & 0 \\ \beta & -\beta & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha+\beta) & \beta & 0 & 0 & \cdots & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 0 & 0 & \alpha & \cdots & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & -\beta & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\alpha+\beta) & 0 & \cdots & 0 & \beta & \alpha \\ & & & & & & \ddots & & & & \\ & & & & & & & & & & \end{array}\right]$$

$(1,1) \to (0,2)$ time $= \exp(\beta)$

$\to (3,0)$ time $= \exp(\alpha)$

In General: $(i,j) \to (i-1, j+1) \sim \exp(\beta)$ $A \to B$

$(i,j) \to (i+2, j-1) \sim \exp(\alpha)$ $B \to 2A$

Q3. Customer come $= X \sim (\lambda = 1)$

Service time $= Y_i \sim (\lambda = \frac{1}{\frac{1}{3}} = 3)$

A. $X_1 \neq 0$

t_0: (X_1, x_2, x_3)

t_1

① : 1 customer is coming: (X_1+1, X_2, X_3)

by $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $P(X < Y_1) = \frac{1}{1+3} = \frac{1}{4}$

rate $= \lambda_x^* p = 1 * \frac{1}{4} = \frac{1}{4}$

② no customer is coming, customer go station 2

(X_1-1, X_2+1, X_3): $P(Y_1 < X) = \frac{3}{1+3} = \frac{3}{4}$

rate $= \lambda_{Y_1} p = 3 \times \frac{3}{4} = \frac{9}{4}$

more customer is coming $\rightarrow$ back to t_1

B. $X_1 = 0$

t_0: $(0, X_2, X_3)$

t_1

① : One customer is coming:

$(0+1, X_2, X_3) \Rightarrow (1, X_2, X_3)$

$P(X < \min(Y_1, Y_2)) = \frac{1}{3+3+1} = \frac{1}{7}$

rate $= \lambda p = 1 \times \frac{1}{7} = \frac{1}{7}$

② : the customer have finished station 2, no other customer is coming: $(0, X_2-1, X_3+1)$

$P(Y_2 < \min(X, Y_3)) = \frac{3}{3+3+1} = \frac{3}{7}$

rate $= \lambda_{Y_2} p = \frac{3}{7} \times 3 = \frac{9}{7}$

③ the customer have finished station 3
nother customer is coming.

$(0, X_2, X_3 - 1)$

$$P\{Y_3 < \min(X, Y_2)\} = \frac{3}{1+3+3} = \frac{3}{7}$$

$$\text{rate} = \lambda_{y_3} P = 3 \times \frac{3}{7} = \frac{9}{7}$$

4. Inventory Management

Demand $\sim$ Pois(μ) $\rightarrow$ cust request product every exp(μ) time

- Restock as soon as Inventory < 5
- Restock time $\sim$ exp(l) $\rightarrow$ Inventory back to 5

(a) 6 States: $S_i \equiv i$ units inventory, $i \in \{0,1,2,3,4,5\}$

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -l & 0 & 0 & 0 & 0 & l \\ \mu & -(\mu+l) & 0 & 0 & 0 & l \\ 0 & \mu & -(\mu+l) & 0 & 0 & l \\ 0 & 0 & \mu & -(\mu+l) & 0 & l \\ 0 & 0 & 0 & \mu & -(\mu+l) & l \\ 0 & 0 & 0 & 0 & \mu & -\mu \end{bmatrix} \end{array}$$

(b) $X_0 = 4$

Mean time to leave state 4; $T_1 \sim \exp(\mu+l)$

$$= \frac{1}{\mu+l}$$

(c) $X_0 = 5$

$$P_{ij} = \frac{\lambda_{ij}}{\lambda_i}, \qquad \lambda_i = \sum_{j \neq i} \lambda_{ij}$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\mu}{\mu+l} & 0 & 0 & 0 & 0 & l/\mu+l \\ 0 & \frac{\mu}{\mu+l} & 0 & 0 & 0 & l/\mu+l \\ 0 & 0 & \frac{\mu}{\mu+l} & 0 & 0 & l/\mu+l \\ 0 & 0 & 0 & \frac{\mu}{\mu+l} & 0 & l/\mu+l \\ 0 & 0 & 0 & 0 & 1 & l/\mu+l \end{bmatrix} \end{array}$$

4(c) Cont.

let T_i = expected time to hit state 0, starting from state i

$T_0 = 0$

$\rightarrow$ we want to solve for T_5

$T_1 = P_{10} + P_{15}(1+T_5)$

$\boxed{\text{let } a = \frac{\mu}{\mu+l}, \quad b = \frac{l}{\mu+l}}$

$T_2 = P_{21}(1+T_1) + P_{25}(1+T_5)$

$T_3 = P_{32}(1+T_2) + P_{35}(1+T_5)$

$T_4 = P_{43}(1+T_3) + P_{45}(1+T_5)$

$T_5 = P_{54}(1+T_4)$

Re-writing in terms of a & b we have:

$T_1 = a + b(1+T_5)$

$T_2 = a(1+T_1) + b(1+T_5)$

$T_3 = a(1+T_2) + b(1+T_5)$

$T_4 = a(1+T_3) + b(1+T_5)$

$T_5 = 1 + T_4$

$$
\begin{aligned}
T_5 &= 1 + a(1+T_3) + b(1+T_5) \\
&= 1 + a(1 + a(1+T_2) + b(1+T_5)) + b(1+T_5) \\
&= 1 + a + a^2 + aT_2 + ab + abT_5 + b(1+T_5) \\
&= 1 + a + a^2 + a(a(1+T_1) + b(1+T_5)) + ab + abT_5 + b(1+T_5) \\
&= \quad \text{ll} + a^2 + a^2T_1 + ab + abT_5 + ab + abT_5 + b(1+T_5) \\
&= 1 + a + 2a^2 + 2ab + a^2T_1 + 2abT_5 + b(1+T_5) \\
&= 1 + a + 2a^2 + 2ab + a^2(a + b(1+T_5)) + 2abT_5 + b(1+T_5) \\
&= \quad \text{ll} \quad + a^3 + a^2b + a^2bT_5 + 2abT_5 + b + bT_5
\end{aligned}
$$

$T_5 = 1 + a + 2a^2 + 2ab + a^3 + a^2b + b + T_5(a^2b + 2ab + b)$

$T_5(1 - a^2b - 2ab - b) = 1 + a + 2a^2 + 2ab + a^3 + a^2b + b$

$$\hookrightarrow \quad T_5 = \boxed{\frac{1 + a + b + 2a^2 + 2ab + a^2b + a^3}{1 - a^2b - 2ab - b}}$$

where $a = \frac{\mu}{\mu+l}$ & $b = \frac{l}{\mu+l}$

4(d) Long-run fraction with no inventory $= \Pi_0$

$$\sum_{j=0}^{5} \Pi_j = 1, \quad \Pi_j = \sum_i \Pi_i P_{ij}$$

$$\Pi_0 = \frac{\mu}{\mu+l}\Pi_1 = \left(\frac{\mu}{\mu+l}\right)^4 \Pi_5$$

$$\Pi_1 = \frac{\mu}{\mu+l}\Pi_2 = \left(\frac{\mu}{\mu+l}\right)^3 \Pi_5$$

$$\Pi_2 = \frac{\mu}{\mu+l}\Pi_3 = \left(\frac{\mu}{\mu+l}\right)^2 \Pi_5$$

$$\Pi_3 = \frac{\mu}{\mu+l}\Pi_4 = \left(\frac{\mu}{\mu+l}\right) \Pi_5$$

$$\Pi_4 = \Pi_5$$

$$\Pi_5 = \Pi_0 + \frac{l}{\mu+l}(\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4)$$

$$\sum_{j=0}^{5} \Pi_j = 1 \rightarrow \Pi_5 + \Pi_5 + \left(\frac{\mu}{\mu+l}\right)\Pi_5 + \left(\frac{\mu}{\mu+l}\right)^2\Pi_5 + \left(\frac{\mu}{\mu+l}\right)^3\Pi_5 + \left(\frac{\mu}{\mu+l}\right)^4\Pi_5 = 1$$

$$\Pi_5\left(2 + \left(\frac{\mu}{\mu+l}\right) + \left(\frac{\mu}{\mu+l}\right)^2 + \left(\frac{\mu}{\mu+l}\right)^3 + \left(\frac{\mu}{\mu+l}\right)^4\right) = 1$$

$$\Pi_5 = \frac{1}{2 + \left(\frac{\mu}{\mu+l}\right) + \left(\frac{\mu}{\mu+l}\right)^2 + \left(\frac{\mu}{\mu+l}\right)^3 + \left(\frac{\mu}{\mu+l}\right)^4}$$

$$\hookrightarrow \boxed{\Pi_0 = \left(\frac{\mu}{\mu+l}\right)^4 \Pi_5 = \frac{\left(\frac{\mu}{\mu+l}\right)^4}{2 + \left(\frac{\mu}{\mu+l}\right) + \left(\frac{\mu}{\mu+l}\right)^2 + \left(\frac{\mu}{\mu+l}\right)^3 + \left(\frac{\mu}{\mu+l}\right)^4}}$$