Q1: a). Write $x_1 = x_1^+ - x_1^-$

$-y_3 = x_3.$

$$\begin{aligned}
\text{Max}\quad & -x_1^+ + x_1^- + 2x_2 - 2(-y_3) \\
\text{s.t.}\quad & x_1^+ - x_1^- + 2x_2 + y_3 \le 9 \\
& x_1^+ - x_1^- - x_2 - y_3 \ge 4 \\
& x_1^+ - x_1^- + x_2 = 7 \\
& x_1^+,\ x_1^-,\ x_2,\ y_3 \ge 0
\end{aligned}$$

Standard Form $\Rightarrow$

$$\begin{aligned}
\text{Max}\quad & -x_1^+ + x_1^- + 2x_2 + 2y_3 \\
\text{s.t.}\quad & x_1^+ - x_1^- + 2x_2 + y_3 \le 9 \\
& -x_1^+ + x_1^- + x_2 + y_3 \le -4 \\
& x_1^+ - x_1^- + x_2 \le 7 \\
& -x_1^+ + x_1^- - x_2 \le -7 \\
& x_1^+,\ x_1^-,\ x_2,\ y_3 \ge 0
\end{aligned}$$

b)

$$\begin{aligned}
\text{Max}\quad & -x_1^+ + x_1^- + 2x_2 + 2y_3 \\
\text{s.t.}\quad & x_1^+ - x_1^- + 2x_2 + y_3 + w_1 = 9 \\
& -x_1^+ + x_1^- + x_2 + y_3 + w_2 = -4 \\
& x_1^+ - x_1^- + x_2 + w_3 = 7 \\
& -x_1^+ + x_1^- - x_2 + w_4 = -7 \\
& x_1^+,\ x_1^-,\ x_2,\ y_3,\ w_1,\ w_2,\ w_3,\ w_4 \ge 0
\end{aligned}$$

Phase I:

$$\begin{aligned}
\text{Min}\quad & u. \\
\text{s.t.}\quad & x_1^+ - x_1^- + 2x_2 + y_3 + w_1 = 9 + u \\
& -x_1^+ + x_1^- + x_2 + y_3 + w_2 = -4 + u \\
& x_1^+ - x_1^- + x_2 + w_3 = 7 + u \\
& -x_1^+ + x_1^- - x_2 + w_4 = -7 + u \\
& x_1^+,\ x_1^-,\ x_2,\ y_3,\ w_1,\ w_2,\ w_3,\ w_4,\ u \ge 0
\end{aligned}$$

+1

-1

-1

-1

Most negative,

Basic Solution: $W_1 = 9 \quad W_2 = -4 \quad W_3 = 7 \quad W_4 = -7$

$x_1^+, x_1^-, x_2, y_3, u = 0$

Since $-x_1^+ + x_1^- + x_2 + y_3 = 0 > -4$,

the solution is infeasible.

$$
\begin{aligned}
\min\ & -x_1^+ + x_1^- - x_2 + W_4 = z - 7 && +1 \\
\text{s.t. } & 2x_1^+ - 2x_1^- + 3x_2 + y_3 + W_1 - W_4 = 16 && -1 \\
& 2x_2 + y_3 + W_2 - W_4 = 3 \\
& 2x_1^+ - 2x_1^- + 2x_2 + W_3 - W_4 = 14 && \times \tfrac{1}{2} \\
& x_1^+ - x_1^- + x_2 - W_4 + u = 7 && -1 \\
& x_1^+, x_1^-, x_2, y_3, W_1, W_2, W_3, W_4, u \geq 0
\end{aligned}
$$

Basic Solution: $W_1 = 16 \quad W_2 = 3 \quad W_3 = 14 \quad u = 7, \quad z = 7$

$x_1^+, x_1^-, x_2, W_4 \quad y_3 = 0$

Since $\min\{16/2, 14/2, 7/1\} = 7$

x_1^+ enter and Row 3 keep x_1^+, W_3 leaving.

$$
\begin{aligned}
\min\ & \tfrac{1}{2}W_3 + \tfrac{1}{2}W_4 = z \\
\text{s.t. } & x_2 + y_3 + W_1 - W_3 = 2 \\
& 2x_2 + y_3 + W_2 - W_4 = 3 \\
& x_1^+ - x_1^- + x_2 + \tfrac{1}{2}W_3 - \tfrac{1}{2}W_4 = 7 \\
& -\tfrac{1}{2}W_3 - \tfrac{1}{2}W_4 + u = 0 \\
& x_1^+, x_1^-, x_2, y_3, W_1, W_2, W_3, W_4, u \geq 0
\end{aligned}
$$

Basic Solution: $W_1 = 2, W_2 = 3, x_1^+ = 7, u = 0$,

$x_1^-, x_2, y_3, W_3, W_4 = 0$

It's the initial feasible solution!

c). To original objective: (since $u=0$, remove u)

$$
\begin{array}{llllllllll}
\text{Max} & -x_1^+ + x_1^- & +2x_2 & +2y_3 & & & & & = Z & \\
\text{S.t.} & & x_2 & +y_3 & +w_1 & & -w_3 & & = 2 & \\
& & 2x_2 & +y_3 & & +w_2 & & -w_4 & = 3 & \\
& x_1^+ - x_1^- & +x_2 & & & & +\frac{1}{2}w_3 & -\frac{1}{2}w_4 & = 7 & +1 \\
& & & & & & -\frac{1}{2}w_3 & -\frac{1}{2}w_4 & = 0 &
\end{array}
$$

$$
\begin{array}{llllllllll}
\text{Max} & & 3x_2 & +2y_3 & & & +\frac{1}{2}w_3 & -\frac{1}{2}w_4 & = Z+7 & -3 \\
\text{S.t.} & & x_2 & +y_3 & +w_1 & & -w_3 & & = 2 & -1 \\
& & 2x_2 & +y_3 & & +w_2 & & -w_4 & = 3 & \times\frac{1}{2} \\
& x_1^+ - x_1^- & +x_2 & & & & +\frac{1}{2}w_3 & -\frac{1}{2}w_4 & = 7 & -1 \\
& & & & & & -\frac{1}{2}w_3 & -\frac{1}{2}w_4 & = 0 &
\end{array}
$$

Since $\min\{2/1, 3/2, 7/1\} = 1.5$,

x_2 enter, w_2 leaving. row 2 keeps x_2.

$$
\begin{array}{llllllllll}
\text{Max} & & & \frac{1}{2}y_3 & & -\frac{3}{2}w_2 & +\frac{1}{2}w_3 & +w_4 & = Z+\frac{5}{2} & -1 \\
\text{S.t.} & & & \frac{1}{2}y_3 & +w_1 & -\frac{1}{2}w_2 & -w_3 & +\frac{1}{2}w_4 & = \frac{1}{2} & \times 2 \\
& & x_2 & +\frac{1}{2}y_3 & & +\frac{1}{2}w_2 & & -\frac{1}{2}w_4 & = \frac{3}{2} & -1 \quad -1 \\
& x_1^+ - x_1^- & & -\frac{1}{2}y_3 & & -\frac{1}{2}w_2 & +\frac{1}{2}w_3 & & = \frac{11}{2} & \\
& & & & & & -\frac{1}{2}w_3 & -\frac{1}{2}w_4 & = 0 &
\end{array}
$$

Basic Solution: $w_1 = \frac{1}{2}$, $x_2 = \frac{3}{2}$, $x_1^+ = \frac{11}{2}$,

$x_1^-, y_3, w_2, w_3, w_4 = 0$

Since $\min\{\frac{1}{2}/\frac{1}{2}, \frac{3}{2}/\frac{1}{2}\} = 1$.

y_3 entering, w_1 leaving while 1st row keeps y_3.

$$
\begin{array}{lrl}
\text{Max} & -w_1 - w_2 + \frac{3}{2}w_3 + \frac{1}{2}w_4 & = z + 2 \\
\text{s.t.} & y_3 + 2w_1 - w_2 - 2w_3 + w_4 & = 1 \\
x_2 & -w_1 + w_2 + w_3 - w_4 & = 1 \\
x_1^+ - x_1^- & +w_1 - w_2 - \frac{1}{2}w_3 + \frac{1}{2}w_4 & = 6 \\
 & -\frac{1}{2}w_3 - \frac{1}{2}w_4 & = 0
\end{array}
$$

Basic Solution: $x_1^+ = 6$ $x_2 = 1$ $y_3 = 1$

$x_1^-, w_1, w_2, w_3, w_4 = 0$

$\Rightarrow$ Now we have all x_1^+, x_2, y_3 variables have coefficient 1, thus we have arrived the optimal result.

Convert to original solution: $(x_1, x_2, x_3) = (6, 1, -1)$

and objective function = $\min x_1 - 2x_2 + 2x_3 = 6 - 2 - 2 = 2$