Q1.

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 \\ \text{st} \quad & x_1 + 4x_2 \le 160 \\ & 2x_1 + 1x_2 \le 80 \\ & x_2 \le 30 \\ & x_1, x_2 \ge 0. \end{aligned}$$

a) Already in standard form:

$b = \begin{pmatrix} 160 \\ 80 \\ 30 \end{pmatrix}$, $c = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Dual: $\min b^T y$ s.t $A^T y \ge c$, $y \ge 0$. $\Rightarrow$

$$\boxed{\begin{aligned} \min \ & 160y_1 + 80y_2 + 30y_3 \\ \text{s.t} \ & y_1 + 2y_2 \ge 4 \\ & 4y_1 + y_2 + y_3 \ge 3 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}}$$

b) $\max \ 4x_1 + 3x_2 = u$

s.t
$$\begin{aligned} x_1 + 4x_2 + x_3 &= 160 \\ 2x_1 + 1x_2 + x_4 &= 80 \\ x_2 + x_5 &= 30 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

$\min(160/1, 80/2) = 40 \Rightarrow x_1$ entering, x_4 leaving.

$\max \quad x_2 - 2x_4 = u - 160$

s.t
$$\begin{aligned} \tfrac{7}{2}x_2 + x_3 - \tfrac{1}{2}x_4 &= 120 \\ x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4 &= 40 \\ x_2 + x_5 &= 30 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

$\min(120/\tfrac{7}{2}, 1/30) = 1/30 \Rightarrow x_2$ entering, x_5 leaving.

$\max \quad -2x_4 - x_5 = u - 190$

s.t
$$\begin{aligned} x_3 - \tfrac{1}{2}x_4 - \tfrac{7}{2}x_5 &= 15 \\ x_1 + \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 &= 25 \\ x_2 + x_5 &= 30 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

it results $x_1 = 25$, $x_2 = 30$ for original solution, and $x_1 = 25$, $x_2 = 30$, $x_3 = 15$, $x_4 = x_5 = 0$ for simplex method solution. Optimal value of objective $= 190$.

c). $y_1 = 0$, $y_2 = 2$, $y_3 = 1$ Since $y_1, y_2, y_3 =$ negative coefficient of x_3, x_4, x_5, and objective value $\Rightarrow Z - 190 = 0 \Rightarrow Z = 190$.

d). $x_1 = 25 \quad x_2 = 30$

$$\begin{aligned} \max.\ & 4x_1 + 3x_2 \\ \text{st.}\ & x_1 + 4x_2 \le 160 \\ & 2x_1 + 1x_2 \le 80 \\ & x_2 \le 30 \\ & x_1, x_2 \ge 0 \end{aligned} \Rightarrow \begin{aligned} obj &= 25\times 4 + 30\times 3 = 190 \\ & 25 + 4\times 30 = 145 \le 160 \ \checkmark \\ & 25\times 2 + 30 = 80 \le 80 \ \checkmark \\ & 30 \le 30 \ \checkmark \\ & 25,\ 30 \ge 0 \ \checkmark \end{aligned}$$

$$\begin{aligned} \min\ & 160y_1 + 80y_2 + 30y_3 \\ \text{s.t}\ & y_1 + 2y_2 \ge 4 \\ & 4y_1 + y_2 + y_3 \ge 3 \\ & y_1, y_2, y_3 \ge 0 \end{aligned} \Rightarrow \begin{aligned} obj &= 0 + 80\times 2 + 30 = 190 \ \checkmark \\ & \begin{cases} 0 + 2\times 2 = 4 \ge 4 \ \checkmark \\ 0 + 2 + 1 = 3 \ge 3 \ \checkmark \\ 0, 2, 1 \ge 0 \ \checkmark \end{cases} \end{aligned}$$

$\therefore$ it is feasible to Q c)'s answer.

e).
$$\begin{aligned} \max\ & 4x_1 + 3x_2 && = u \\ \text{s.t}\ & x_1 + 4x_2 + x_3 && = 160 \\ & 2x_1 + 1x_2 + x_4 && = 80 + \varepsilon \\ & x_2 + x_5 && = 30 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

$\min(160/1, 80/2) = 40 \Rightarrow x_1$ entering, x_4 leaving.

$$\begin{aligned} \max\ & x_2 - 2x_4 && = u - 160 - 2\varepsilon \\ \text{s.t}\ & \tfrac{7}{2}x_2 + x_3 - \tfrac{1}{2}x_4 && = 120 - \tfrac{1}{2}\varepsilon \\ & x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4 && = 40 + \tfrac{1}{2}\varepsilon \\ & x_2 + x_5 && = 30 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

$\min(120/\frac{7}{2}, 1/30) = 1/30 \Rightarrow x_2$ entering, x_5 leaving.

$$
\begin{array}{llr}
\max & & -2x_4 - x_5 = n - 190 - 2\varepsilon \\
\text{s.t} & & x_3 - \frac{1}{2}x_4 - \frac{7}{2}x_5 = 15 - \frac{1}{2}\varepsilon \\
& x_1 & + \frac{1}{2}x_4 - \frac{1}{2}x_5 = 25 + \frac{1}{2}\varepsilon \\
& \quad x_2 & + x_5 = 30 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 &
\end{array}
$$

Thus $x_1 = 25 + \frac{1}{2}\varepsilon$, $x_2 = 30$, obj for original question $= 190 + 2\varepsilon$.

(x_1 can increase by $\frac{1}{2}\varepsilon$, obj increase by 2ε, x_2 not change)

f). $\delta = 2$, from above implementation, x_2 will not be affected since it already reaches bondary. x_2 will only increase since 2nd constrain has positive coefficient of ε. only consider x_3 since it has negative coefficient of ε, since $x_3 \geq 0 \Rightarrow x_3 = 15 - \frac{1}{2}\varepsilon \geq 0$ $\varepsilon \leq 30$.