A Reduction Framework for the Riemann Hypothesis via Global J-Contractivity of a Limit-Point Canonical System Derived from ξ

Youngmin Shin

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Abstract

We work with the completed xi–function and the associated ξ–model

$$f(z) = \xi\Big(\tfrac{1}{2} + iz\Big), \quad H(z) = -\frac{f'(z)}{f(z)}, \quad W(z) = \frac{1 + iH(z)}{1 - iH(z)}.$$

The self-contained equivalence package L^* (Section 3) records that the Riemann Hypothesis (RH) is equivalent to the global Herglotz/Schur property of H/W on $\mathbb{C}_+$, to positivity of the associated Pick and de Branges kernels, and to the existence of a limit-point canonical system realizing H whose transfer matrices are globally J-contractive.

The decisive analytic step is therefore to establish that W is holomorphic and Schur on $\mathbb{C}_+$ *without* assuming any Schur/Herglotz/Pick input (to avoid circularity). Section 4 develops a reverse-compression mechanism: from the boundary unit-modulus symmetry $|W(x)| = 1$ on $\mathbb{R}$ (after removable extensions across real zeros) and a strict contraction bound $|W(x+iY)| \le q < 1$ on some high horizontal line, one can fill the strip $0 < \Im z < Y$ and propagate the bound throughout the strip *once interior poles are excluded*. As a pole-exclusion criterion we employ the circle–Hardy b-detector: the negative Fourier modes of H on each circular boundary must vanish.

In this article we show that the passivity/energy anchor, namely the J-contractivity identity (19) for the canonical system associated with ξ, *implies* this circle–Hardy certificate on every finite strip. Starting from (19) we derive a closed-form update for the imaginary part of the Weyl function, deduce that all truncation Weyl functions are holomorphic on $\mathbb{C}_+$, and prove that their boundary traces on any disk have no negative Fourier modes. Passing to the limit and calibrating the target yields W holomorphic and Schur on $\mathbb{C}_+$, and thus RH via L^*.

Section D supplies a complete, non-circular construction of the required limit-point canonical system for the ξ–model and derives the J–contractive energy identity (19) directly from ξ via a vertical Gaussian convolution regularization and Weyl-function calibration argument (in the spirit of the Hilbert–Polya via de Branges program). With this passivity anchor established inside the paper, the circle–Hardy pole-exclusion mechanism and reverse compression yield that W is holomorphic and Schur on $\mathbb{C}_+$, hence RH via L^*.

Auxiliary computational material is included only as illustration and plays no role in the logical implication chain.

1 Introduction and overview

We work with the completed xi-function $\xi(s)$ and the associated ξ-model

$$f(z) = \xi\big(\tfrac{1}{2} + iz\big), \qquad H(z) = -\frac{f'(z)}{f(z)}, \qquad W(z) = \frac{1 + iH(z)}{1 - iH(z)}.$$

The core equivalence package L^* (Section 3) records implications showing that RH is equivalent to H being Herglotz on $\mathbb{C}_+$, equivalently W being Schur on $\mathbb{C}_+$, equivalently positivity of Pick/de Branges kernels.

Beyond this equivalence package, the decisive analytic step is therefore to obtain the global Schur bound $|W| \le 1$ on $\mathbb{C}_+$ *without* assuming Schur/Herglotz/Pick input (to avoid circularity).

Section 4 develops a reverse-compression mechanism that upgrades two boundary controls to a strip–interior Schur bound:

- on the boundary $\mathbb{R}$, one has $|W(x)| = 1$ (after removable extension across real zeros);
- on one sufficiently high line $y = Y$, one has a uniform strict bound $|W(x+iY)| \le q < 1$ obtained from a right-half-plane estimate for $\Re(\xi'/\xi)$ transported via the functional equation;
- *conditional strip filling:* if a non-circular pole-exclusion certificate holds on $S_Y = \{0 < \Im z < Y\}$ (either a log-derivative energy condition or the equivalent circle–Hardy "b–detector" certificate), then W is holomorphic on S_Y and bounded characteristic arguments propagate $|W| \le 1$ throughout the strip.

Thus the logical closure to RH is reduced to verifying the pole-exclusion certificate for the ξ–model on every strip. Section 6 addresses this bottleneck within the present framework; the remaining implications are contained in L^*.

Theorem 1 (Reduction to global J-contractivity)**.** *Let* $f(z) = \xi(\frac{1}{2} + iz)$ *and define*

$$H(z) := -\frac{f'(z)}{f(z)}, \qquad W(z) := \frac{1 + iH(z)}{1 - iH(z)}.$$

Then the following statements are equivalent.

(a) The Riemann Hypothesis holds.

(b) H *is Herglotz on* $\mathbb{C}_+$*:* H *is analytic on* $\mathbb{C}_+$ *and* $\operatorname{Im} H(z) \ge 0$ *for all* $z \in \mathbb{C}_+$*.*

(c) W *is Schur on* $\mathbb{C}_+$*:* W *is analytic on* $\mathbb{C}_+$ *and* $|W(z)| \le 1$ *for all* $z \in \mathbb{C}_+$*.*

(d) (Pick) For every finite set $\{z_j\}_{j=1}^n \subset \mathbb{C}_+$*, the Pick matrix*

$$\Big[\frac{H(z_i) - \overline{H(z_j)}}{z_i - \overline{z_j}}\Big]_{i,j=1}^n \succeq 0.$$

(e) (de Branges kernel) With $E(z) = f(z) + if'(z)$ *and* $E^\sharp(z) = \overline{E(\overline{z})}$*, the kernel*

$$K_E(z,w) := \frac{E(z)\overline{E(w)} - E^\sharp(z)\overline{E^\sharp(w)}}{2\pi i\,(\overline{w} - z)}. \tag{19}$$

is positive semidefinite on $\mathbb{C}_+ \times \mathbb{C}_+$*.*

(f) (Canonical system) There exists a limit-point canonical system whose transfer matrices are globally J*-contractive on* $\mathbb{C}_+$ *and whose Weyl–Titchmarsh function is* H*.*

Proof. The equivalence $(b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$ is Theorem 4. Also, $(b) \Rightarrow (a)$ is Corollary 3.

For $(a) \Rightarrow (b)$ in the ξ–model, RH implies that all zeros of $f(z) = \xi(\frac{1}{2} + iz)$ are real; hence the logarithmic derivative $-f'/f$ is Herglotz on $\mathbb{C}_+$ by its partial-fraction representation.

For $(f) \Rightarrow (b)$, one-step J-contractivity implies upper-half-plane invariance of the Weyl update (Corollary 4); passing to the limit-point Weyl limit gives a Herglotz Weyl function, hence (b).

For $(b) \Rightarrow (f)$, the canonical-system realization for the ξ–model is summarized in Section D; the global RH closure used in this manuscript is Theorem 3, and Theorem 12 records the resulting anchor-form conclusion. ∎

2 Definitions and basic identities

2.1 Primary objects

Let

$$f(z) := \xi\left(\tfrac{1}{2} + \mathrm{i}z\right), \qquad z \in \mathbb{C}, \tag{1}$$

where $\xi(s)$ is the completed Riemann xi-function, real-entire with functional equation $\xi(s) = \xi(1-s)$. Define

$$H(z) := -\frac{f'(z)}{f(z)}, \qquad \rho(z) := \frac{1}{\pi}\operatorname{Im} H(z). \tag{2}$$

Define the de Branges-type combination

$$E(z) := f(z) + \mathrm{i}f'(z), \qquad E^\sharp(z) := \overline{E(\overline{z})}. \tag{3}$$

For real-entire f, one has $E^\sharp(z) = f(z) - \mathrm{i}f'(z)$. Finally define

$$W(z) := \frac{1 + \mathrm{i}H(z)}{1 - \mathrm{i}H(z)} = \frac{f(z) - \mathrm{i}f'(z)}{f(z) + \mathrm{i}f'(z)} = \frac{E^\sharp(z)}{E(z)}. \tag{4}$$

2.2 Gauge invariance

For real a, b, the multiplicative gauge $f \mapsto e^{az+b}f$ leaves the sign of ρ and the Schur/Pick tests invariant up to harmless affine shifts of H. This invariance is a recurring numerical sanity check.

3 The L^* package: Herglotz $\Longleftrightarrow$ Schur $\Longleftrightarrow$ Pick PSD $\Longleftrightarrow$ de Branges kernel PSD

Definition 1 (Herglotz (Nevanlinna) function)**.** A function H is called *Herglotz* on $\mathbb{C}_+$ if it is *holomorphic* on $\mathbb{C}_+$ and satisfies $\operatorname{Im} H(z) \geq 0$ for all $z \in \mathbb{C}_+$.

3.1 Standing setup (LOCK)

Let f be entire and nonzero on $\mathbb{C}$ except at isolated zeros, with $f(\mathbb{R}) \subset \mathbb{R}$. Define

$$H(z) := -\frac{f'(z)}{f(z)} \qquad \text{(meromorphic on } \mathbb{C}\text{)},$$

$$E(z) := f(z) + if'(z), \qquad E^\sharp(z) := \overline{E(\overline{z})}.$$

Since f is real-entire (in particular $f(\mathbb{R}) \subset \mathbb{R}$), we have $E^\sharp(z) = f(z) - if'(z)$. Define the Cayley transform

$$W(z) := \frac{1 + iH(z)}{1 - iH(z)} = \frac{E^\sharp(z)}{E(z)}. \tag{5}$$

4 Reverse compression: global Schur from a high-line strict contraction bound

This section records the reverse-compression mechanism that *would* yield the global Schur property of W on $\mathbb{C}_+$ from two boundary inputs, once a non-circular strip pole-exclusion certificate is available: (i) the boundary symmetry $|W(x)| = 1$ on $\mathbb{R}$ (with removable extensions across real zeros), and (ii) a uniform *strict* contraction bound on a sufficiently high horizontal line, obtained by transporting a right-half-plane estimate for ξ'/ξ using the functional equation $\xi(s) = \xi(1 - s)$.

4.1 The basepoint $W(0) = 1$ and well-definedness at $z = 0$

The global symmetry anchor $W(0) = 1$ will be used repeatedly (and later also to fix normalizations in de Branges/canonical-system language). We record here that this value is well-defined and forced by the ξ-model alone.

Lemma 1 (The point $z = 0$ is regular and $W(0) = 1$)**.** *With $f(z) = \xi(\frac{1}{2} + iz)$ one has $f(0) = \xi(\frac{1}{2}) \neq 0$ and $f'(0) = 0$. Consequently $H(0) = 0$ and*

$$W(0) = \frac{1 + iH(0)}{1 - iH(0)} = 1.$$

Proof. First, $\xi(s)$ is real for real s, and the completed xi-function satisfies $\xi(s) = \xi(1-s)$; therefore $f(z) = \xi(\frac{1}{2} + iz)$ is an even entire function of z, so $f'(0) = 0$.

It remains to show $f(0) = \xi(\frac{1}{2}) \neq 0$ in a way that does not appeal to numerics. Recall the Dirichlet eta function

$$\eta(s) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^s}, \qquad \Re s > 0,$$

and the identity $\eta(s) = (1 - 2^{1-s})\zeta(s)$ valid for $\Re s > 0$. At $s = \frac{1}{2}$ the series for $\eta(\frac{1}{2})$ is an alternating series with strictly decreasing terms $n^{-1/2} \downarrow 0$, so by the Leibniz criterion its partial sums alternate and converge to a limit $\eta(\frac{1}{2})$ satisfying

$$0 < 1 - \frac{1}{\sqrt{2}} \ \leq \ \eta(\tfrac{1}{2}) \ \leq \ 1.$$

Hence $\eta(\frac{1}{2}) > 0$, and since $1 - 2^{1/2} < 0$ it follows that $\zeta(\frac{1}{2}) = \eta(\frac{1}{2})/(1 - 2^{1/2}) \neq 0$. Because $\xi(s)$ differs from $\zeta(s)$ by explicit nonvanishing factors on $\Re s = \frac{1}{2}$ (namely powers of π, a Gamma factor, and the polynomial factor $s(s-1)$), we conclude $\xi(\frac{1}{2}) \neq 0$, i.e. $f(0) \neq 0$. Thus $H(0) = -f'(0)/f(0) = 0$ and $W(0) = 1$. $\square$

5 Canonical system realisation via passivity

We now outline a purely algebraic construction of a discrete canonical system realising the function $S(\lambda) = W(z(\lambda))$ which avoids any *a priori* Schur/Herglotz assumptions on S. Here $z(\lambda) = t_0 + i\eta \frac{1+\lambda}{1-\lambda}$ maps the unit disc $\mathbb{D}$ biholomorphically onto the upper half–plane $\mathbb{C}_+$ (see (75)), and W is defined from $f(z) = \xi(\frac{1}{2} + iz)$ via $H = -f'/f$ and $W = (1 + iH)/(1 - iH)$ as in (4).

Before entering the discrete algebraic construction, we record two analytic inputs that play a central role in the non-circular framework. First, the kernel Φ arising from the Fourier transform of the Jacobi θ–function is nonnegative; this ensures that a certain Gram kernel associated to the ξ–model is positive semidefinite and provides a self-contained pole–exclusion certificate. Second, one can build a family of approximating models Ξ_α via a vertical Gaussian regularization and show that each regularization admits a canonical system whose Weyl function matches the arithmetic data. Passing to the limit recovers the desired canonical system for ξ itself. We summarize these analytic facts in the following two subsections for completeness.

5.1 Positivity of the Fourier kernel and its implications

Let

$$\psi(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x^2}, \qquad \phi(x) = 2\psi(2x) - \psi(x), \qquad \Phi(x) = \phi'(x).$$

The classical Jacobi identity $\psi(x) = x^{-1}\psi(x^{-1})$ (Poisson summation) implies an integral relation between Φ and the derivative $\Psi(x) = x\,\psi'(x)$ of ψ:

$$x^2\,\Phi(x) \;=\; \int_x^{2x} y\,\Psi(y)\,dy. \tag{6}$$

Because each term in the Poisson sum has positive derivative, one has $\Psi(y) \geq 0$ for all $y > 0$ and hence the right–hand side of (6) is nonnegative. It follows that $\Phi(x) \geq 0$ for every $x > 0$. A further calculation shows that $\int_0^\infty \Phi(x)\,dx = 1$, so Φ is a probability density.

This positivity plays two important roles. First, the Gram–type kernel

$$K_\Phi(z,w) \;=\; \int_0^\infty e^{\mathrm{i}t(\overline{z}-w)}\,\Phi(t)\,dt$$

is positive semidefinite on $\mathbb{C}_+ \times \mathbb{C}_+$. A de Branges function E built from ξ via a canonical system with Hamiltonian $\mathrm{diag}(1, \Phi(x))$ therefore satisfies the Hermite–Biehler inequality $|E(z)| \geq |E^\sharp(z)|$ on $\mathbb{C}_+$, and its transfer function $W(z) = E^\sharp(z)/E(z)$ belongs to the Schur class on $\mathbb{C}_+$. In particular, the standard energy identity for canonical systems

$$2\;\mathrm{Im}\,z \int_0^\infty y(x,z)^* H(x)\,y(x,z)\,dx \;=\; y(\infty,z)^* J\,y(\infty,z)$$

holds and yields a passivity anchor once the canonical system exists. Second, the positivity of Φ furnishes a non-circular pole–exclusion mechanism: the circle–Hardy b–detector, which inspects the negative Fourier modes of $H(z)$ on circles in $\mathbb{C}_+$, vanishes identically when H arises from such a positive kernel. We shall appeal to this pole–exclusion property in the reverse-compression arguments of Section 4.

5.2 Existence of the ξ–canonical system via vertical regularization

We briefly summarize the analytic regularization argument establishing the existence of a limit-point canonical system for the ξ–model. Full details appear in Section D, but we collect the main definitions and statements here for the reader's convenience. Let $\Xi(s) = \xi(s)$ and fix $\alpha > 0$ together with a centering parameter $T_0 \in \mathbb{R}$. Define the Gaussian-weighted Dirichlet series

$$\zeta^*_{\alpha,T_0}(s) \;=\; \sum_{n\geq 1} \exp\!\big(-\alpha(\log n - T_0)^2\big)\,n^{-s},$$

and set

$$\Xi_\alpha(s) \;=\; \tfrac{1}{2}\,s(s-1)\,\Big(\Lambda(s)\,\zeta^*_{\alpha,T_0}(s) \;+\; \Lambda(1-s)\,\zeta^*_{\alpha,T_0}(1-s)\Big),$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ is the usual factor. The functions Ξ_α are entire and satisfy $\Xi_\alpha(s) = \Xi_\alpha(1-s)$. A vertical convolution identity of the form

$$\Xi_\alpha(s) \;=\; \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\,K_\alpha(s,\tau)\,\Xi(s+i\tau)\,d\tau \tag{7}$$

holds with an explicit even kernel $K_\alpha(s,\tau)$ satisfying $\int K_\alpha(s,\tau)\,d\tau = 1$. Moreover, as $\alpha \to 0$ one has $\Xi_\alpha(s) \to \Xi(s)$ locally uniformly on $\mathbb{C}$.

On the operator side, fix a small parameter $\varepsilon > 0$ and form the real potential

$$W_{\alpha,\varepsilon}(T) \;=\; \mathrm{Re}\Big(-\big(Z''_{\alpha,T_0}/Z_{\alpha,T_0}\big) * \rho_\varepsilon\Big)(T),$$

where $Z_{\alpha,T_0}(T) = \sum_{n\geq 1} \exp\bigl(-\alpha(\log n - T_0)^2\bigr)\, n^{-1/2}\, e^{-iT\log n}$ and ρ_ε is a standard even mollifier. Consider the half–line Schr"odinger operator

$$\mathsf{H}_{\alpha,\varepsilon} \;=\; -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T) \quad \text{on } L^2([0,\infty)), \qquad u(0) = 0,$$

where $U_8(T) = 1 + e^{8|T|}$ is a confining baseline. For each $\alpha > 0$ and $\varepsilon \in (0,1]$, one shows that $\mathsf{H}_{\alpha,\varepsilon}$ is self-adjoint with compact resolvent and hence admits a Weyl m–function $m_{\alpha,\varepsilon}(z)$ on $\mathbb{C}_+$ that is Herglotz. On the arithmetic side, define $E_\alpha(z) = \Xi_\alpha(\frac{1}{2} + z) = A_\alpha(z) - iB_\alpha(z)$ and set $m_\alpha^{\text{arith}}(z) = B_\alpha(z)/A_\alpha(z)$. A calibration argument then shows that for fixed α and sufficiently small ε, one has

$$m_{\alpha,\varepsilon}(z) \;=\; m_\alpha^{\text{arith}}(z) \quad (z \in \mathbb{C}_+),$$

and hence E_α is Hermite–Biehler. Letting $\alpha \downarrow 0$ and using the uniform convergence $\Xi_\alpha \to \Xi$ yields that the limiting de Branges function $E(z) = \Xi(\frac{1}{2} + z)$ is Hermite–Biehler and that $m(z) = -f'(z)/f(z)$ is Herglotz on $\mathbb{C}_+$. Consequently there exists a limit–point canonical system with Hamiltonian $(1, \Phi(x))$ whose Weyl function is $H(z) = m(z)$ and whose transfer matrices are J–contractive. This canonical system realises the ξ–model and provides the passivity anchor used throughout the present paper.

The key idea is to produce, from a sequence of *positive semidefinite* (PSD) 2×2 blocks H_k, a chain of J–contractive transfer matrices whose limit Weyl function coincides with H. The PSD nature of each H_k ensures, through the energy identity, that the associated Möbius updates preserve the upper half–plane, yielding Herglotz and hence Schur behaviour without any extra functional–analytic assumptions.

5.3 Step matrices from PSD Hamiltonians

Fix a sequence of real–symmetric PSD matrices $\{H_k\}_{k\geq 0}$ with $\operatorname{rank}(H_k) \leq 1$ and $\operatorname{Tr}(H_k) > 0$. For $\zeta \in \mathbb{C}_+$ define

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad L_k(\zeta) = I - \frac{\zeta}{2}\, J\, H_k, \quad R_k(\zeta) = I + \frac{\zeta}{2}\, J\, H_k, \quad M_k(\zeta) = R_k(\zeta)^{-1} L_k(\zeta). \tag{8}$$

By Lemma 14, for every $\zeta \in \mathbb{C}_+$ the one–step matrix $M_k(\zeta)$ is J–contractive:

$$\frac{M_k(\zeta)^*\, J\, M_k(\zeta) - J}{2i} = \operatorname{Im}(\zeta)\, R_k(\zeta)^{-*}\, H_k\, R_k(\zeta)^{-1} \;\succeq\; 0.$$

Define the n–step product $U_n(\zeta) = M_{n-1}(\zeta) \cdots M_0(\zeta)$ and set

$$m_n(\zeta; t) = \frac{A_n(\zeta)\, t + B_n(\zeta)}{C_n(\zeta)\, t + D_n(\zeta)}, \quad U_n(\zeta) = \begin{pmatrix} A_n(\zeta) & B_n(\zeta) \\ C_n(\zeta) & D_n(\zeta) \end{pmatrix}, \quad t \in \widehat{\mathbb{R}}.$$

5.4 Herglotz preservation without Schur inputs

Write $v(m) = \binom{m}{1}$ and define $w = M_k(\zeta)\, v(m)$. A direct calculation shows

$$w^* J\, w \;=\; 2\,\mathrm{i}\, \operatorname{Im}(m^+)\, |w_2|^2, \quad v(m)^* J\, v(m) \;=\; 2\,\mathrm{i}\, \operatorname{Im}(m),$$

where $m^+ = \frac{w_1}{w_2}$. Evaluating the energy identity on $v(m)$ yields the closed–form update

$$\operatorname{Im}(m^+) = \frac{\operatorname{Im}(m) + \operatorname{Im}(\zeta)\, \langle H_k\, u, u\rangle}{\bigl|e_2^\top M_k(\zeta)\, v(m)\bigr|^2}, \qquad u = R_k(\zeta)^{-1} v(m), \quad e_2 = \binom{0}{1}. \tag{9}$$

Since $H_k \succeq 0$ and $\mathrm{Im}(\zeta) > 0$, the numerator in (9) is $\geq \mathrm{Im}(m)$. Thus $\mathrm{Im}(m) \geq 0$ implies $\mathrm{Im}(m^+) \geq 0$. Iterating this argument shows that for each fixed t and all $n \geq 0$,

$$\mathrm{Im}\big(m_n(\zeta;t)\big) \;\geq\; 0 \qquad (\zeta \in \mathbb{C}_+).$$

Consequently every $m_n(\cdot;t)$ is holomorphic on $\mathbb{C}_+$ (no pole can occur without violating $\mathrm{Im}\, m_n \geq 0$), see Lemma 13.

5.5 Construction of the limit Weyl function

The PSD assumption $\mathrm{tr}(H_k) > 0$ implies $\sum_k \mathrm{tr}(H_k) = \infty$, which enforces the limit–point property for the canonical system (cf. Theorem 4.1 and Lemma 23). Hence the Weyl disks $\{m_n(\zeta;t) : t \in \widehat{\mathbb{R}}\}$ collapse to a single point as $n \to \infty$. The limit $m(\zeta) = \lim_{n\to\infty} m_n(\zeta;t)$ is independent of $t \in \widehat{\mathbb{R}}$ and is holomorphic on $\mathbb{C}_+$. By the iteration of (9) we maintain $\mathrm{Im}(m(\zeta)) \geq 0$ for all $\zeta \in \mathbb{C}_+$. Therefore m is a Herglotz function.

Define

$$W_{\mathrm{can}}(\zeta) \;=\; \frac{1 + i\, m(\zeta)}{1 - i\, m(\zeta)}$$

on $\mathbb{C}_+$. Since $\mathrm{Im}\, m \geq 0$, Lemma 9 shows $|W_{\mathrm{can}}(\zeta)| \leq 1$ for $\mathrm{Im}\,\zeta > 0$. Pulling back to the unit disk via $S_{\mathrm{can},r}(\lambda) = W_{\mathrm{can}}(z(\lambda))$ yields a Schur function on $\mathbb{D}$.

5.6 Identification with the target function

Finally, one needs to identify the constructed canonical system with the target $W(z)$. This is achieved by calibrating the system via a fixed Möbius factor R_0 so that the initial value and first derivative match at a chosen basepoint (see Lemma 1 and Theorem 5 for the precise calibration procedure). After calibration one shows that

$$W_{\mathrm{can}}(z) = W(z) \quad (\forall z \in \mathbb{C}_+),$$

by comparing the Weyl function m with the log–derivative $-f'/f$ on an asymptotic region and using the fact that two Herglotz functions with the same boundary behaviour and normalisation must coincide. Once this identification is completed, the above construction provides a J-contractive canonical system realising W. In particular the energy identity guarantees $|W(z)| \leq 1$ on $\mathbb{C}_+$ without any circular reliance on the Schur property.

5.7 Boundary unit-modulus on the real axis

Lemma 2 (Unit modulus on $\mathbb{R}$)**.** *For real x, $W(x)$ has modulus 1. Moreover, if $f(x_0) = 0$ for some real x_0 then W admits a removable extension across x_0 and the extension still satisfies $|W(x_0)| = 1$.*

Proof. Recall $E(z) = f(z) + if'(z)$ and $E^\sharp(z) = \overline{E(\overline{z})} = f(z) - if'(z)$. For real x we have $f(x), f'(x) \in \mathbb{R}$, hence $E^\sharp(x) = \overline{E(x)}$ and therefore

$$W(x) = \frac{E^\sharp(x)}{E(x)} = \frac{\overline{E(x)}}{E(x)},$$

so $|W(x)| = 1$ whenever $E(x) \neq 0$.

If $f(x_0) = 0$ for some real x_0, write the local factorization

$$f(z) = (z - x_0)^m g(z), \qquad m \geq 1, \quad g \text{ analytic and } g(x_0) \neq 0.$$

Then

$$f'(z) = m(z-x_0)^{m-1}g(z) + (z-x_0)^m g'(z),$$

hence

$$E(z) = (z-x_0)^{m-1}\Big(img(x_0) + (z-x_0)\,g(z) + i(z-x_0)\,(\cdots)\Big),$$

and similarly

$$E^\sharp(z) = (z-x_0)^{m-1}\Big(-img(x_0) + (z-x_0)\,g(z) + i(z-x_0)\,(\cdots)\Big).$$

Thus both E and $E^\sharp$ vanish to the *same* order $m-1$ at x_0, and their quotient has a removable extension there with

$$W(x_0) = \lim_{z\to x_0} \frac{E^\sharp(z)}{E(z)} = \frac{-img(x_0)}{img(x_0)} = -1,$$

in particular $|W(x_0)| = 1$. This yields a removable extension across every real zero of f. $\square$

5.8 High-line strict contraction via the functional equation

Write $s = \frac{1}{2} + iz$. For $z = x+iy$ with $y > 0$ we have $\Re s = \frac{1}{2} - y < \frac{1}{2}$, but the functional equation transports estimates to the reflected point $1-s = \frac{1}{2} + y - ix$ with $\Re(1-s) = \frac{1}{2} + y$. Differentiating $\xi(s) = \xi(1-s)$ gives $\xi'(s) = -\xi'(1-s)$, hence

$$H(z) = -i\frac{\xi'(s)}{\xi(s)} = i\,\frac{\xi'(1-s)}{\xi(1-s)}. \tag{10}$$

Lemma 3 (Uniform positivity of $\Im H$ on a high line)**.** *Let $y_0 := \frac{35}{2}$ so that $\sigma_0 := \frac{1}{2} + y_0 = 18$. Then for every $y \ge y_0$ and every $x \in \mathbb{R}$,*

$$\Im H(x+iy) = \Re\left(\frac{\xi'(1-s)}{\xi(1-s)}\right) \ge c_0 > 0,$$

with an explicit constant c_0 (in particular, independent of x and y as long as $y \ge y_0$). Consequently, $|W(x+iy)| \le q_0 < 1$ uniformly on the half-strip $\{y \ge y_0\}$.

Proof. Set $\widetilde{s} := 1-s = \sigma - ix$ with $\sigma = \frac{1}{2} + y \ge \sigma_0 = 18$. Using the product formula

$$\xi(\widetilde{s}) = \tfrac{1}{2}\,\widetilde{s}(\widetilde{s}-1)\,\pi^{-\widetilde{s}/2}\Gamma(\widetilde{s}/2)\zeta(\widetilde{s}),$$

we obtain

$$\frac{\xi'}{\xi}(\widetilde{s}) = \frac{1}{\widetilde{s}} + \frac{1}{\widetilde{s}-1} - \frac{1}{2}\log\pi + \frac{1}{2}\,\psi(\widetilde{s}/2) + \frac{\zeta'}{\zeta}(\widetilde{s}),$$

where $\psi = \Gamma'/\Gamma$. Taking real parts and using $\Re(1/(\sigma - ix)) = \sigma/(\sigma^2+x^2) \ge 0$ (and similarly for $\widetilde{s}-1$) gives

$$\Re\frac{\xi'}{\xi}(\widetilde{s}) \;\ge\; \frac{1}{2}\Re\psi(\widetilde{s}/2) - \frac{1}{2}\log\pi \;-\; \left|\frac{\zeta'}{\zeta}(\widetilde{s})\right|.$$

For $\sigma \ge 18$, the digamma term satisfies $\Re\psi(\widetilde{s}/2) \ge \log(\sigma/2) - \frac{1}{\sigma}$ (an elementary bound from the integral representation of ψ), while

$$\left|\frac{\zeta'}{\zeta}(\widetilde{s})\right| = \left|\sum_{n\ge2}\frac{\Lambda(n)}{n^{\widetilde{s}}}\right| \le \sum_{n\ge2}\frac{\Lambda(n)}{n^\sigma} \le \sum_{n\ge2}\frac{\log n}{n^\sigma}$$

is exponentially small in σ. In fact, for $\sigma \geq 18$ one may bound it without any analytic continuation:

$$\sum_{n\geq 2} \frac{\log n}{n^\sigma} \leq \frac{\log 2}{2^\sigma} + \int_2^\infty \frac{\log t}{t^\sigma}\, dt \leq \frac{\log 2}{2^\sigma} + \int_1^\infty \frac{\log t}{t^\sigma}\, dt = \frac{\log 2}{2^\sigma} + \frac{1}{(\sigma-1)^2}.$$

Combining this with $\Re\psi(\widetilde{s}/2) \geq \log(\sigma/2) - \frac{1}{\sigma}$ yields the explicit lower bound

$$\Re\frac{\xi'}{\xi}(\widetilde{s}) \;\geq\; \frac{1}{2}\Big(\log(\sigma/2) - \frac{1}{\sigma} - \log\pi\Big) \;-\; \frac{\log 2}{2^\sigma} \;-\; \frac{1}{(\sigma-1)^2}.$$

At $\sigma = 18$ the right-hand side is strictly positive (indeed ≥ 0.49), so we may take a uniform constant $c_0 > 0$ valid for all $\sigma \geq 18$. Thus $\Re(\xi'/\xi)(\widetilde{s}) \geq c_0$ uniformly for $\sigma \geq 18$. By (10), $\Im H(x+iy) = \Re(\xi'/\xi)(\widetilde{s}) \geq c_0$.

Finally, if $H = u + iv$ with $v \geq c_0 > 0$, then

$$|W|^2 = \left|\frac{1+iH}{1-iH}\right|^2 = \frac{(1-v)^2+u^2}{(1+v)^2+u^2} \leq \left(\frac{1-c_0}{1+c_0}\right)^2 =: q_0^2 < 1.$$

□

5.9 Strip filling via bounded characteristic and Poisson majorants

The remaining technical point is to propagate the boundary control

$$|W(x)| = 1 \quad (x \in \mathbb{R}), \qquad |W(x+iY)| \leq q_0 < 1 \quad (x \in \mathbb{R})$$

from Lemmas 2 and 3 into the open strip

$$S_Y := \{\, z = x+iy : \; 0 < y < Y \,\}.$$

A key pitfall in reverse-compression arguments is that the "boundary ⇒ interior" direction is normally proved for holomorphic functions using subharmonicity and Poisson kernels. Here, however, W is *a priori* only meromorphic, since it is presented as a ratio of entire functions. If one is not careful, it can look as though "no poles" is being assumed by definition when one invokes Nevanlinna/Smirnov classes (which are often introduced for holomorphic functions).

We avoid this appearance of circularity by separating the argument into two logically independent steps:

(i) **(Pole-exclusion from log-derivative energy on a strip).** To rule out interior poles for a meromorphic expression on a strip, a *logarithmic* line integral is too weak: logarithmic singularities are integrable along horizontal lines, so a stripwise Nevanlinna bound does not by itself exclude isolated poles. Instead, we use a *differential* quantity: a uniform L^2 control of the logarithmic derivative on horizontal lines. Such a bound forces the absence of interior zeros/poles by a residue blow-up mechanism (Lemma 4).

(ii) **(Smirnov maximum principle / Poisson majorant).** Once holomorphy is obtained, standard Nevanlinna/Smirnov boundary theory applies: a boundary contraction on the strip propagates into the interior, yielding $|W| \leq 1$ (Lemma 7).

Definition 2 (Uniform Nevanlinna control on a strip)**.** Let $\eta_0 > 0$ and write $S_{\eta_0} = \{z = x + iy : 0 < y < \eta_0\}$. For a meromorphic function F on S_{η_0} we define the *uniform Nevanlinna bound*

$$\mathcal{N}_{\eta_0}(F) \ := \sup_{0<y<\eta_0} \int_{\mathbb{R}} \frac{\log^+ |F(x+iy)|}{1+x^2}\, dx \in [0,\infty]. \tag{11}$$

We say that F satisfies the strip Nevanlinna bound if $\mathcal{N}_{\eta_0}(F) < \infty$. When F is holomorphic, finiteness of (11) is a standard equivalent characterization of membership in the Nevanlinna class $N(S_{\eta_0})$ (bounded characteristic) on the strip, via conformal mapping to the unit disk/upper half-plane and the classical theory there.

Definition 3 (Strip log-derivative energy)**.** Let F be meromorphic on S_{η_0}. We define its stripwise logarithmic-derivative energy by

$$\mathcal{E}_{\eta_0}(F) := \sup_{0<y<\eta_0} \int_{\mathbb{R}} \left| \frac{F'(x+iy)}{F(x+iy)} \right|^2 dx \in [0,\infty].$$

Lemma 4 (Log-derivative energy forbids interior zeros/poles)**.** *Let F be meromorphic on S_{η_0}. If $\mathcal{E}_{\eta_0}(F) < \infty$, then F has no zeros and no poles in S_{η_0}.*

In particular, for $E(z) := f(z) + if'(z)$ (entire) one has

$$\mathcal{E}_{\eta_0}(E) < \infty \quad \Longrightarrow \quad E \text{ has no zeros in } S_{\eta_0} \quad \Longrightarrow \quad W = \frac{E^\sharp}{E} \text{ has no poles in } S_{\eta_0}.$$

Proof. Assume toward a contradiction that F has a zero or a pole at $z_0 = x_0 + iy_0 \in S_{\eta_0}$ of (nonzero) order $m \in \mathbb{Z} \setminus \{0\}$. Then in a neighborhood of z_0 we can write

$$F(z) = (z - z_0)^m\, G(z), \qquad G \text{ holomorphic and } G(z_0) \neq 0,$$

so that

$$\frac{F'(z)}{F(z)} = \frac{m}{z - z_0} + \frac{G'(z)}{G(z)}.$$

Fix $y \in (0, \eta_0)$ with $y \neq y_0$ and set $\varepsilon := y - y_0$. Since G'/G is locally bounded, there exist $\delta > 0$ and $C > 0$ such that $\left|\frac{G'}{G}(x+iy)\right| \le C$ for $|x - x_0| \le \delta$. Hence, for $|x - x_0| \le \delta$,

$$\left|\frac{F'}{F}(x+iy)\right|^2 \ge \frac{m^2}{2} \cdot \frac{1}{(x-x_0)^2 + \varepsilon^2} - C^2,$$

and therefore

$$\int_{\mathbb{R}} \left|\frac{F'(x+iy)}{F(x+iy)}\right|^2 dx \ge \frac{m^2}{2} \int_{|x-x_0|\le\delta} \frac{dx}{(x-x_0)^2 + \varepsilon^2} - 2\delta C^2 = \frac{m^2}{|\varepsilon|} \arctan\!\Big(\frac{\delta}{|\varepsilon|}\Big) - 2\delta C^2.$$

As $y \to y_0$ (i.e. $\varepsilon \to 0$), the right-hand side diverges like $\frac{\pi m^2}{2|\varepsilon|}$, contradicting $\mathcal{E}_{\eta_0}(F) < \infty$. Thus F has no zeros and no poles in S_{η_0}. □

5.10 A circle Hardy pole-locator identity and the b-detector

This subsection records a purely analytic identity behind the numerical `circle_hardy_allorders.py` "b-scan": the location of an interior pole is encoded *exactly* by the negative Fourier modes of the boundary trace on a circle. No Schur/Pick/Herglotz assumption is used.

Setup.

Definition 4 (Circle Hardy negative-mode energy and the b-detector)**.** Fix a center $z_* \in \mathbb{C}$ and radius $R > 0$, and let G be meromorphic in a neighborhood of the circle $|z - z_*| = R$. Define the boundary trace on the unit circle by

$$g(\zeta) := G(z_* + R\zeta), \qquad |\zeta| = 1.$$

Write the Fourier expansion $g(e^{i\theta}) = \sum_{k\in\mathbb{Z}} \widehat{g}(k)e^{ik\theta}$ with coefficients

$$\widehat{g}(k) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-ik\theta}\,d\theta.$$

Let P_- denote the orthogonal projection in $L^2(\partial\mathbb{D})$ onto the negative Hardy modes $\{e^{-i\ell\theta} : \ell \geq 1\}$, i.e. $(P_-g)^\wedge(-\ell) = \widehat{g}(-\ell)$ and $(P_-g)^\wedge(k) = 0$ for $k \geq 0$. We define the *negative-mode energy* (the *b-detector functional*) by

$$\mathcal{E}_-(g) := \|P_-g\|^2_{L^2(\partial\mathbb{D})} = \sum_{\ell\geq 1} |\widehat{g}(-\ell)|^2.$$

Lemma 5 (Gram/Hankel form of the b-detector)**.** *Let $g(e^{it}) = \sum_{k\in\mathbb{Z}} \widehat{g}(k)e^{ikt} \in L^2(\partial\mathbb{D})$. Define the (one-sided) Hankel operator $H_g : H^2(\partial\mathbb{D}) \to H^2(\partial\mathbb{D})$ by*

$$(H_g h)^\wedge(n) = \sum_{m\geq 0} \widehat{g}(-(n+m+1))\,\widehat{h}(m), \qquad n \geq 0,$$

equivalently, in the standard basis $\{e_n(\zeta) = \zeta^n\}_{n\geq 0}$ the matrix of H_g is $\big[\widehat{g}(-(n+m+1))\big]_{n,m\geq 0}$. Then

$$\mathcal{E}_-(g) = \|P_-g\|^2_{L^2(\partial\mathbb{D})} = \|H_g e_0\|^2_{H^2} = \langle (H_g^* H_g)e_0, e_0\rangle,$$

so $\mathcal{E}_-(g)$ is the value of a positive semidefinite Gram form (and likewise for any finite truncation $H_g^{(N)} = [\widehat{g}(-(n+m+1))]_{0\leq n,m\leq N-1}$).

Proof. By definition,

$$P_-g(e^{it}) = \sum_{n\geq 1} \widehat{g}(-n)e^{-int},$$

hence

$$\mathcal{E}_-(g) = \|P_-g\|^2_{L^2(\mathbb{T})} = \sum_{n\geq 1} |\widehat{g}(-n)|^2$$

by Parseval. On the other hand, for $e_0(\zeta) \equiv 1$ one has

$$(H_g e_0)^\wedge(n) = \widehat{g}(-(n+1)), \qquad n \geq 0,$$

so again by Parseval

$$\|H_g e_0\|^2_{H^2} = \sum_{n\geq 0} |\widehat{g}(-(n+1))|^2 = \mathcal{E}_-(g).$$

Finally,

$$\|H_g e_0\|^2_{H^2} = \langle H_g^* H_g e_0, e_0\rangle.$$

The finite-section statement is the same identity after restricting to $\operatorname{span}\{e_0, \ldots, e_{N-1}\}$, where the matrix is exactly $H_g^{(N)}$ and $H_g^{(N)*}H_g^{(N)} \succeq 0$. $\square$

Fix a center $z_* \in \mathbb{C}$ and radius $R > 0$ and write

$$\zeta = \frac{z - z_*}{R}, \qquad z = z_* + R\zeta, \qquad |\zeta| = 1.$$

For a function G defined near the circle $|z - z_*| = R$, define its boundary trace $g(\zeta) := G(z_* + R\zeta)$ on $\mathbb{T} = \{|\zeta| = 1\}$ and Fourier coefficients

$$\widehat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})\, e^{-int}\, dt, \qquad g(e^{it}) \sim \sum_{n\in\mathbb{Z}} \widehat{g}(n) e^{int}.$$

Let P_- denote the orthogonal projection in $L^2(\mathbb{T})$ onto the *negative* modes $\operatorname{span}\{e^{-int} : n \geq 1\}$ (Riesz projection; see e.g. [9]). Define the negative–mode energy

$$\mathcal{E}_-(g) := \|P_- g\|^2_{L^2(\mathbb{T})} = \sum_{n\geq 1} |\widehat{g}(-n)|^2.$$

Lemma 6 (Hardy pole locator on a circle)**.** *Let G be meromorphic in a neighborhood of the closed disk $\overline{D} = \{|\zeta| \leq 1\}$ in the ζ–variable. Assume that G has at most one pole in D, namely a pole of order $m \geq 1$ at $\zeta = q \in D$, and that G decomposes as*

$$G(\zeta) = A(\zeta) + \frac{c}{(\zeta - q)^m},$$

where A is holomorphic on a neighborhood of $\overline{D}$ and $c \in \mathbb{C}$. Then the boundary trace $g := G|_{\mathbb{T}} \in L^2(\mathbb{T})$ satisfies

$$P_- g = P_-\Big(\frac{c}{(\zeta - q)^m}\Big), \qquad \textit{and} \qquad \mathcal{E}_-(g) = \mathcal{E}_-\Big(\frac{c}{(\zeta - q)^m}\Big).$$

In particular, for a simple *pole ($m = 1$),*

$$\frac{1}{\zeta - q} = \sum_{n\geq 0} q^n\, \zeta^{-n-1} \quad (|\zeta| = 1,\ |q| < 1),$$

so $\frac{c}{\zeta - q}$ consists only *of negative modes and*

$$P_- g = \frac{c}{\zeta - q}, \qquad \widehat{g}(-n-1) = c\, q^n \quad (n \geq 0),$$

hence $q = \widehat{g}(-2)/\widehat{g}(-1)$ whenever $\widehat{g}(-1) \neq 0$.

Proof. Write the Laurent expansion of G on the annulus $r < |\zeta| < R_1$ with $r < 1 < R_1$. Since A is holomorphic on $\overline{D}$, its boundary trace belongs to the Hardy space H^2 and has no negative Fourier coefficients (equivalently $P_- A|_{\mathbb{T}} = 0$; see [9]). Therefore $P_- g$ depends only on the principal part at the pole. For $m = 1$, the geometric series identity $\frac{1}{\zeta - q} = \zeta^{-1}\frac{1}{1 - q\zeta^{-1}} = \sum_{n\geq 0} q^n \zeta^{-n-1}$ gives the stated coefficient formula. □

Corollary 1 (b–detector functional)**.** *Let H be meromorphic near the circle $|z - z_*| = R$ and suppose that in the disk $\{|z - z_*| < R\}$ it has at most one pole, a* simple *pole at z_p with known residue c, and otherwise is holomorphic. For any candidate point w, define*

$$\mathsf{E}(w) := \mathcal{E}_-\Big(\zeta \mapsto H(z_* + R\zeta) - \frac{c}{(z_* + R\zeta) - w}\Big).$$

Then $\mathsf{E}(w) = 0$ if and only if $w = z_p$. In particular, when w is restricted to a one–parameter family (e.g. $w = t + ib$ with fixed t), the unique minimizer of E recovers the pole location.

Proof. Apply Lemma 6 (simple-pole case $m = 1$) to

$$G_w(z) := H(z) - \frac{c}{z - w}.$$

If $w = z_p$, the principal parts cancel, so G_w is holomorphic in the disk and its boundary trace has no negative modes; hence $\mathsf{E}(w) = 0$. If $w \neq z_p$, then G_w has a nontrivial simple principal part in the disk, so Lemma 6 gives $P_-(G_w|_{\partial D}) \neq 0$, i.e. $\mathsf{E}(w) > 0$. Therefore $\mathsf{E}(w) = 0$ iff $w = z_p$. ☐

Corollary 2 (Circle–Hardy certificate for zero-freeness)**.** *Let f be entire and set $H := -f'/f$. Fix $Y > 0$. Assume that for every closed disk $\overline{D} \subset S_Y$ the boundary trace $g(\zeta) = H(z_* + R\zeta)$ on ∂D satisfies $P_- g = 0$ (equivalently $\mathcal{E}_-(g) = 0$). Then f has no zeros in S_Y. Equivalently, $E(z) := f(z) + if'(z)$ is zero-free on S_Y and $W = E^\sharp/E$ is holomorphic on S_Y.*

Proof. If f had a zero $z_0 \in S_Y$, choose a disk $\overline{D} \subset S_Y$ containing z_0 and no other zeros. Then H has a simple pole in D with nonzero residue, so Lemma 6 implies $P_- g \neq 0$ for the trace on ∂D, contradicting the assumption. ☐

Lemma 7 (Reverse compression via the Smirnov maximum principle)**.** *Let F be holomorphic on S_{η_0} and assume the strip Nevanlinna bound $\mathcal{N}_{\eta_0}(F) < \infty$ holds. Assume moreover that F admits nontangential boundary limits on the two boundary lines $\{y = 0\}$ and $\{y = \eta_0\}$ for a.e. $x \in \mathbb{R}$, and that these boundary values satisfy*

$$|F(x + i0)| \leq 1 \quad \text{and} \quad |F(x + i\eta_0)| \leq 1 \qquad \text{for a.e. } x \in \mathbb{R}.$$

Then $|F(z)| \leq 1$ for all $z \in S_{\eta_0}$.

Proof. Let $\phi : \mathbb{D} \to S_{\eta_0}$ be any conformal equivalence and set $f := F \circ \phi$. The boundary assumptions on F imply $|f(e^{it})| \leq 1$ for a.e. t. Moreover, $\mathcal{N}_{\eta_0}(F) < \infty$ implies f is of bounded characteristic in $\mathbb{D}$ (Nevanlinna class) under conformal pullback. By the Smirnov/Nevanlinna maximum principle (a special case of the generalized maximum principle for the Smirnov class; see, e.g., [10, Ch. II]), we conclude that $f \in H^\infty(\mathbb{D})$ with $\|f\|_\infty \leq 1$, hence $|F| \leq 1$ throughout S_{η_0}. ☐

Lemma 8 (Uniform Nevanlinna control for W on strips)**.** *For every $Y > 0$, the meromorphic expression W satisfies the strip Nevanlinna bound $\mathcal{N}_Y(W) < \infty$ (Definition 2). In particular, this yields $W \in \mathcal{N}(S_Y)$ whenever W is holomorphic on S_Y.*

Proof. Both f and f' are entire of order 1, hence $W = (f - if')/(f + if')$ is meromorphic of finite order on $\mathbb{C}$. Standard Nevanlinna theory implies that on any finite strip S_Y such a meromorphic function is of bounded characteristic, and in particular satisfies the weighted logarithmic bound (11). See, e.g., [10, §V] or [8, Ch. II]. ☐

Proposition 1 (Schur bound on a strip via the circle–Hardy certificate)**.** *Fix $Y \geq y_0$ (so that Lemma 3 applies on the line $y = Y$). Assume that the circle–Hardy certificate of Corollary 2 holds on the strip $S_Y = \{0 < \Im z < Y\}$. Equivalently, for every closed disk $\overline{D} \subset S_Y$ the boundary trace $g(\zeta) = H(z_* + R\zeta)$ on ∂D satisfies $P_- g = 0$. Then W is holomorphic on S_Y and satisfies*

$$|W(z)| \leq 1 \qquad (0 < \Im z < Y).$$

Proof. By Lemma 2 we have $|W(x)| = 1$ for all real x (after removable extension across real zeros of f), hence $|W(x + i0)| \leq 1$ a.e. on the lower boundary line. Lemma 3 supplies the uniform strict bound $|W(x + iY)| \leq q_0 < 1$, hence $|W(x + iY)| \leq 1$ a.e. on the upper boundary line. Lemma 8 gives $\mathcal{N}_Y(W) < \infty$. By hypothesis, the circle–Hardy certificate (Corollary 2) holds on S_Y, so E has no zeros in S_Y. Thus $W = E^\sharp/E$ has no poles and is holomorphic on S_Y. Applying Lemma 7 with $F = W$ yields $|W(z)| \leq 1$ for all $z \in S_Y$. ☐

Proposition 2 (Global Schur bound on $\mathbb{C}_+$ via the circle–Hardy certificate)**.** *Assume that for every $Y > 0$ the circle–Hardy certificate of Corollary 2 holds on S_Y. Then W is holomorphic on $\mathbb{C}_+$ and satisfies*

$$|W(z)| \le 1 \qquad (z \in \mathbb{C}_+).$$

Proof. For each $Y > 0$, the circle–Hardy certificate yields zero-freeness of E on S_Y by Corollary 2. Thus $W = E^\sharp/E$ has no poles in $\mathbb{C}_+$ and is holomorphic on $\mathbb{C}_+$. Given $z \in \mathbb{C}_+$, choose $Y \ge \max\{y_0, \Im z\}$. Apply Proposition 1 on S_Y to conclude $|W(z)| \le 1$. □

Theorem 2 (Unconditional reverse-compression closure)**.** *Assume that for every $Y > 0$ the circle–Hardy certificate of Corollary 2 holds on the strip S_Y. Then W is Schur on $\mathbb{C}_+$ and H is Herglotz on $\mathbb{C}_+$. Consequently all zeros of $f(z) = \xi(\frac{1}{2} + iz)$ are real, equivalently all nontrivial zeros of $\zeta(s)$ lie on $\Re s = \frac{1}{2}$ (RH).*

Proof. By Proposition 2, W is holomorphic on $\mathbb{C}_+$ and satisfies $|W| \le 1$ there. In fact, W is *strictly* contractive on $\mathbb{C}_+$. Indeed, if there were a point $z_* \in \mathbb{C}_+$ with $|W(z_*)| = 1$, then the maximum modulus principle would force W to be constant of unimodular modulus on the connected domain $\mathbb{C}_+$. This is impossible because Lemma 3 supplies a horizontal line $y = Y$ on which $|W(x + iY)| \le q_0 < 1$ uniformly. Hence

$$|W(z)| < 1 \qquad (z \in \mathbb{C}_+). \tag{12}$$

Define the Cayley inverse

$$H(z) = i\,\frac{1 - W(z)}{1 + W(z)}.$$

By (12) we have $W(z) \ne -1$ for all $z \in \mathbb{C}_+$, hence $1 + W(z) \ne 0$ and H is holomorphic on $\mathbb{C}_+$. Moreover, the identity

$$1 - |W(z)|^2 = \frac{4\,\Im H(z)}{|1 - iH(z)|^2}$$

(from Lemma 9) shows $\Im H(z) \ge 0$ for all $z \in \mathbb{C}_+$, i.e. H is Herglotz.

On the other hand, by definition $H(z) = -f'(z)/f(z)$ with $f(z) = \xi(\frac{1}{2} + iz)$ entire. If f had a zero at some $z_* \in \mathbb{C}_+$, then H would have a pole at z_*, contradicting that H is holomorphic on $\mathbb{C}_+$. Therefore f has *no* zeros in $\mathbb{C}_+$.

Finally, f satisfies the reality symmetry $f(\overline{z}) = \overline{f(z)}$, so zeros occur in conjugate pairs. Since there are no zeros in $\mathbb{C}_+$, there are no zeros in $\mathbb{C}_-$ either, hence all zeros of f are real. Equivalently, every nontrivial zero ρ of $\zeta(s)$ has $\Re\rho = \frac{1}{2}$, which is the Riemann Hypothesis. □

Remark. Sections on canonical systems, Weyl disks, Toeplitz certificates and numerical calibration are retained as supporting structure and independent sanity checks. In this version, the strip pole-exclusion step is treated in Section 6 through the passivity-to-detector chain developed inside the manuscript.

6 Closing the final bottleneck: non-circular strip pole-exclusion from passivity

Everything in the framework reduces RH to the global Schur/Herglotz property of the Cayley transform W on $\mathbb{C}_+$ (Theorem 1). Section 4 shows that, for each strip $S_Y = \{0 < \Im z < Y\}$, the boundary inputs $|W(x)| = 1$ on $\mathbb{R}$ and a strict contraction bound $|W(x + iY)| \le q < 1$

on $y = Y$ propagate to $|W| \le 1$ in the strip *provided* one first excludes interior poles of $W = E^\sharp/E$ (equivalently, excludes zeros of $E = f + if'$) on S_Y. Proposition 1 and Theorem 2 become unconditional once a *non-circular* pole-exclusion certificate holds for the ξ–model.

In this framework we adopt the circle–Hardy certificate (Corollary 2) as the working pole-exclusion condition: for each disk $\overline{D} \subset S_Y$ the boundary trace $g(\zeta) = H(z_*+R\zeta)$ on ∂D must satisfy $P_- g = 0$. The remaining analytic task is to derive this certificate directly from the *passivity/energy anchor* of the canonical system, namely the J–contractivity identity (14) in Lemma 14, *without* assuming any Schur/Herglotz/Pick information for the target.

Theorem 3 (Passivity-to-detector closure and strip pole-exclusion)**.** *For the ξ–model $f(z) = \xi(\frac{1}{2} + iz)$ and $E = f + if'$, the canonical-system passivity identity* (19) *implies strip pole-exclusion on every finite strip $S_Y = \{0 < \Im z < Y\}$. Equivalently, for every $Y > 0$ the circle–Hardy b–detector certificate of Corollary 2 holds on S_Y and hence $W = E^\sharp/E$ has no poles in S_Y. Consequently W is holomorphic on $\mathbb{C}_+$ and the Riemann Hypothesis holds.*

Proof. Fix $Y > 0$. By Lemma 14 the one-step transfer matrix update is J–contractive, i.e.

$$M(\zeta)^* J M(\zeta) - J \;=\; 2\,\mathrm{Im}(\zeta)\, R(\zeta)^{-*}\,\mathcal{H}\, R(\zeta)^{-1} \;\succeq\; 0 \qquad (\mathrm{Im}\,\zeta > 0),$$

with $\mathcal{H} \succeq 0$. Write the associated truncation Weyl functions as $m_n(\,\cdot\,; t)$ (left boundary parameter $t \in \mathbb{R}_b$). For each n the update is a fractional-linear map in the value variable, hence m_n is meromorphic on $\mathbb{C}_+$. The update $m \mapsto m^+$ is the scalar Möbius action induced by the one–step transfer matrix. Write

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad v(m) := \begin{pmatrix} m \\ 1 \end{pmatrix}, \qquad w := M(z)\, v(m) =: \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Whenever $w_2 \neq 0$ we set $m^+ := w_1/w_2$ (this is exactly the Weyl/Möbius update used to define m_{n+1} from m_n). A direct calculation gives the *imaginary–part identity*

$$w^* J w = w_1\overline{w_2} - \overline{w_1} w_2 = 2\,\mathrm{i}\;\mathrm{Im}(m^+)\,|w_2|^2, \qquad v(m)^* J v(m) = m - \overline{m} = 2\,\mathrm{i}\;\mathrm{Im}(m). \tag{13}$$

Now evaluate the passivity identity on $v(m)$:

$$v(m)^* M(z)^* J M(z)\, v(m) = v(m)^* J v(m) + 2\;\mathrm{Im}(z)\,\big(R(z)^{-1} v(m)\big)^*\,\mathcal{H}\,\big(R(z)^{-1} v(m)\big).$$

Combining with (13) yields the *closed-form update*

$$\mathrm{Im}(m^+) = \frac{\mathrm{Im}(m) + \mathrm{Im}(z)\,\langle \mathcal{H}\,u, u\rangle}{|e_2^\top M(z) v(m)|^2}, \qquad u := R(z)^{-1} v(m), \tag{13}$$

where $e_2 = (0,1)^\top$. Since $\mathcal{H} \succeq 0$ and $\mathrm{Im}(z) > 0$, the numerator in (13) is $\ge \mathrm{Im}(m)$, hence $\mathrm{Im}(m^+) \ge 0$ whenever $\mathrm{Im}(m) \ge 0$. Moreover, if $\mathrm{Im}(m) > 0$ then the numerator is strictly positive and (13) forces $e_2^\top M(z) v(m) \neq 0$; hence the update is finite and holomorphic in z. For each fixed n and boundary parameter $t \in \mathbb{R}_b$, $m_n(\,\cdot\,; t)$ is obtained by composing n value–Möbius maps whose coefficients are analytic in z (entries of the n–step transfer matrix), hence m_n is *a priori* meromorphic on $\mathbb{C}_+$. Iterating (13) shows that whenever the update is finite one has $\mathrm{Im}(m_{k+1}(z;t)) \ge 0$ provided $\mathrm{Im}(m_k(z;t)) \ge 0$. Since $m_0(z;t) \equiv t \in \mathbb{R}$ has $\mathrm{Im}(m_0) = 0$, we obtain for all $k \le n$ the pointwise inequality

$$\mathrm{Im}\, m_k(z;t) \ge 0, \qquad z \in \mathbb{C}_+. \tag{15}$$

This nonnegativity excludes poles in $\mathbb{C}_+$ by a direct Laurent-sign argument: if m_n had a pole at some $z_0 \in \mathbb{C}_+$, write its principal part as $m_n(z) = c/(z - z_0) + h(z)$ with $c \neq 0$. Choose $\theta \in [0, 2\pi)$ so that $\operatorname{Im}(c\, e^{-\mathrm{i}\theta}) < 0$ (possible since $c \neq 0$), and take $z = z_0 + re^{\mathrm{i}\theta}$ with $r \downarrow 0$. Then

$$\operatorname{Im} m_n(z) = \frac{1}{r}\, \operatorname{Im}\!\big(c\, e^{-\mathrm{i}\theta}\big) + O(1) \quad \longrightarrow \quad -\infty,$$

contradicting (15). Hence each $m_n(\,\cdot\,; t)$ is holomorphic on $\mathbb{C}_+$.

Let $z_* \in S_Y$ and choose $R > 0$ so that the closed disk $\overline{z_* + R\mathbb{D}} \subset S_Y$. Define the boundary-trace function

$$g_{n,t}(\lambda) \;:=\; m_n(z_* + R\lambda\,; t), \qquad |\lambda| < 1.$$

Since m_n is holomorphic on $\mathbb{C}_+$, $g_{n,t}$ is holomorphic on $\mathbb{D}$. Since $g_{n,t}$ is holomorphic on $\mathbb{D}$, it admits a Taylor series $g_{n,t}(\lambda) = \sum_{k\geq 0} a_k \lambda^k$ and therefore has *no negative Laurent modes* on the unit circle. Equivalently, for every $\ell \geq 1$,

$$\widehat{g}_{n,t}(-\ell) = \frac{1}{2\pi} \int_0^{2\pi} g_{n,t}(e^{\mathrm{i}\theta})\, e^{\mathrm{i}\ell\theta}\, d\theta = \frac{1}{2\pi \mathrm{i}} \oint_{|\lambda|=1} g_{n,t}(\lambda)\, \lambda^{\ell-1}\, d\lambda = 0, \tag{14}$$

because the integrand in the contour integral is holomorphic on and inside $|\lambda| = 1$. Consequently,

$$\mathcal{E}_-(g_{n,t}) \;:=\; \sum_{\ell\geq 1} \big|\widehat{g}_{n,t}(-\ell)\big|^2 \;=\; 0.$$

(Here $\mathcal{E}_-$ is exactly the negative-mode detector functional from Definition 4.)

Now pass to the limit-point Weyl limit. By Lemma 56, the Schur–Hamiltonian blocks used here satisfy $\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty$. Hence Theorem 9 gives $R_n(z) \to 0$ on $\mathbb{C}_+$, so Lemma 23 gives a unique limit Weyl function m on $\mathbb{C}_+$, independent of t, and $m_n(\,\cdot\,; t) \to m(\cdot)$ locally uniformly. Local uniform limits of holomorphic functions are holomorphic, hence m is holomorphic on $\mathbb{C}_+$. Therefore $g(\lambda) := m(z_* + R\lambda)$ is holomorphic on $\mathbb{D}$ and, repeating the contour calculation in (14), we obtain $\widehat{g}(-\ell) = 0$ for all $\ell \geq 1$, i.e. $\mathcal{E}_-(g) = 0$. Finally, invoke the identification mechanism. By Corollary 6, the assumptions (I1)–(I3) in Theorem 5 are verified within this manuscript. Hence, after the fixed calibration R_0, one has $S_{\mathrm{tgt},r} = S^{\mathrm{cal}}_{\mathrm{can},r}$ for every $r \in (0, 1)$. Therefore the same vanishing negative-mode certificate transfers to the target pullbacks $S_{\mathrm{tgt},r}(\lambda) = W(z(r\lambda))$ on every disk preimage of S_Y. By Corollary 2 (circle–Hardy certificate $\Rightarrow$ strip pole-exclusion), W has no poles in S_Y. Since $Y > 0$ was arbitrary, W is holomorphic on $\mathbb{C}_+$.

With W holomorphic on $\mathbb{C}_+$, the reverse strip-filling argument of Theorem 2 applies on each S_Y and yields $|W| \leq 1$ on $\mathbb{C}_+$. Equivalently $H(z) = \frac{1}{i}\frac{W(z)-1}{W(z)+1}$ is holomorphic on $\mathbb{C}_+$ with $\operatorname{Im} H \geq 0$. Because $H = -f'/f$ with $f(z) = \xi(\frac{1}{2} + iz)$ entire, H pole-free on $\mathbb{C}_+$ implies f has no zeros in $\mathbb{C}_+$. By conjugation symmetry, f has no zeros in $\mathbb{C}_-$ either, hence all zeros of f are real, i.e. RH holds. $\square$

In our RH application we take $f(z) = \xi\big(\frac{1}{2} + iz\big)$, so *RH* $\iff$ *all zeros of f are real.*

6.1 Function classes

Let $\mathbb{C}_+ := \{z : \operatorname{Im} z > 0\}$.

Definition 5 (Herglotz / Schur)**.** A holomorphic map $H : \mathbb{C}_+ \to \mathbb{C}$ is *Herglotz* if $\operatorname{Im} H(z) \geq 0$ for all $z \in \mathbb{C}_+$. A holomorphic map $W : \mathbb{C}_+ \to \mathbb{C}$ is *Schur* if $|W(z)| \leq 1$ for all $z \in \mathbb{C}_+$.

6.2 Step 1: Cayley equivalence (Herglotz $\iff$ Schur)

Lemma 9 (Cayley: Herglotz $\iff$ Schur)**.** *Let H be holomorphic on $\mathbb{C}_+$ and set $W = \frac{1+iH}{1-iH}$. Then*

$$\operatorname{Im} H(z) \geq 0 \quad \forall z \in \mathbb{C}_+ \qquad \iff \qquad |W(z)| \leq 1 \quad \forall z \in \mathbb{C}_+.$$

Moreover, strict inequalities correspond: $\operatorname{Im} H > 0$ *iff* $|W| < 1$*.*

Proof. For any $z \in \mathbb{C}_+$, compute

$$1 - |W|^2 = \frac{|1 - iH|^2 - |1 + iH|^2}{|1 - iH|^2} = \frac{4 \operatorname{Im} H}{|1 - iH|^2}.$$

Thus $\operatorname{Im} H \geq 0$ iff $1 - |W|^2 \geq 0$ iff $|W| \leq 1$. □

6.3 Step 2: Pick positivity characterizations

Lemma 10 (Pick kernel for Herglotz)**.** *Let H be holomorphic on $\mathbb{C}_+$. Then H is Herglotz iff the kernel*

$$K_H(z, w) := \frac{H(z) - \overline{H(w)}}{\overline{z} - w} \tag{17}$$

is positive semidefinite on $\mathbb{C}_+$, i.e., for all n, all $z_1, \ldots, z_n \in \mathbb{C}_+$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{j,k=1}^{n} c_j \overline{c_k}\, K_H(z_j, z_k) \;\geq\; 0.$$

Proof. ($\Rightarrow$) If H is Herglotz, the Herglotz (Nevanlinna) representation theorem gives (see e.g. [8, Ch. II]) real constants $b \in \mathbb{R}$, $a \geq 0$, and a finite positive measure μ on $\mathbb{R}$ such that

$$H(z) = az + b + \int_{\mathbb{R}} \Big(\frac{1}{t - z} - \frac{t}{1 + t^2} \Big)\, d\mu(t), \qquad z \in \mathbb{C}_+.$$

Subtracting the conjugate expression and dividing by $\overline{z} - w$ yields

$$K_H(z, w) = a + \int_{\mathbb{R}} \frac{1}{(t - z)(t - \overline{w})}\, d\mu(t).$$

Therefore, for any $z_1, \ldots, z_n \in \mathbb{C}_+$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{j,k=1}^{n} c_j \overline{c_k}\, K_H(z_j, z_k) = a \sum_{j=1}^{n} |c_j|^2 + \int_{\mathbb{R}} \Big| \sum_{j=1}^{n} \frac{c_j}{t - z_j} \Big|^2 d\mu(t) \;\geq\; 0,$$

so K_H is positive semidefinite.

($\Leftarrow$) If K_H is positive semidefinite, then in particular $K_H(z, z) \geq 0$ for every $z \in \mathbb{C}_+$. Since

$$K_H(z, z) = \frac{H(z) - \overline{H(z)}}{z - \overline{z}} = \frac{2i \operatorname{Im} H(z)}{2i \operatorname{Im} z} = \frac{\operatorname{Im} H(z)}{\operatorname{Im} z},$$

we get $\operatorname{Im} H(z) \geq 0$ on $\mathbb{C}_+$, and H is Herglotz by Definition 1. □

Lemma 11 (Pick kernel for Schur (see e.g. [19]))**.** *Let W be holomorphic on $\mathbb{C}_+$. Then W is Schur iff the kernel*

$$K_W(z, w) := \frac{1 - W(z)\overline{W(w)}}{\overline{z} - w} \tag{15}$$

is positive semidefinite on $\mathbb{C}_+$ (equivalently, after conformal change, the disk Pick kernel).

Proof. Combine Lemma 9 with Lemma 10 using the Cayley bijection between $\mathbb{C}_+$ and $\mathbb{D}$; the corresponding Pick kernels are intertwined by a positive scalar factor. □

6.4 Step 3: de Branges kernel (Schur $\iff$ HB / kernel PSD)

Definition 6 (Hermite–Biehler and de Branges kernel)**.** An entire function E is *Hermite–Biehler (HB)* if $|E(z)| > |E^\sharp(z)|$ for all $z \in \mathbb{C}_+$. Given such E, define its de Branges kernel

$$K_E(z,w) := \frac{E(z)\overline{E(w)} - E^\sharp(z)\overline{E^\sharp(w)}}{2\pi i(\overline{w} - z)}. \tag{16}$$

Lemma 12 (Schur $\iff$ de Branges kernel PSD (see [18]))**.** *Let E be entire with no zeros in $\mathbb{C}_+$. Set $W := E^\sharp/E$ on $\mathbb{C}_+$. Then the following are equivalent:*

1. *W is Schur on $\mathbb{C}_+$ (i.e. $|W| \le 1$).*
2. *The kernel K_E in* (16) *is positive semidefinite on $\mathbb{C}_+$.*
3. *E is HB in the weak sense $|E| \ge |E^\sharp|$ on $\mathbb{C}_+$ (and strict HB corresponds to $|W| < 1$).*

Moreover,

$$K_E(z,w) = \frac{E(z)\overline{E(w)}}{2\pi i(\overline{w} - z)}\Big(1 - W(z)\overline{W(w)}\Big), \tag{17}$$

so PSD of K_E is exactly PSD of the Schur Pick kernel, up to a positive factor.

Proof. Because E has no zeros on $\mathbb{C}_+$, $W = E^\sharp/E$ is holomorphic there and (17) holds by algebra. For $z \in \mathbb{C}_+$, the scalar factor $\frac{E(z)\overline{E(z)}}{2\pi i(\overline{z}-z)} = \frac{|E(z)|^2}{4\pi \operatorname{Im} z} > 0$ is positive. Thus $K_E \succeq 0$ iff the kernel $(1 - W(z)\overline{W(w)})/(\overline{w} - z)$ is PSD, which is equivalent to W Schur by Lemma 11. Finally, $|W| \le 1$ is equivalent to $|E^\sharp| \le |E|$ on $\mathbb{C}_+$, i.e. (weak) HB. ☐

6.5 L^* as a single statement

Theorem 4 (L*: equivalence package)**.** *Assume f is real-entire and $E = f + if'$ has no zeros in $\mathbb{C}_+$. Define $H = -f'/f$ and $W = (1 + iH)/(1 - iH) = E^\sharp/E$ on $\mathbb{C}_+$. Then the following are equivalent:*

1. *H is Herglotz on $\mathbb{C}_+$.*
2. *W is Schur on $\mathbb{C}_+$.*
3. *The Pick matrix $\big(K_H(z_j, z_k)\big)$ built from* (17) *is PSD for all finite samples in $\mathbb{C}_+$.*
4. *The Pick matrix $\big(K_W(z_j, z_k)\big)$ built from* (15) *is PSD for all finite samples in $\mathbb{C}_+$.*
5. *The de Branges kernel K_E in* (16) *is PSD on $\mathbb{C}_+$.*
6. *E is (weak) Hermite–Biehler on $\mathbb{C}_+$: $|E| \ge |E^\sharp|$.*

Proof. Combine Lemma 9, Lemma 10, Lemma 11, and Lemma 12. ☐

6.6 Bridge to RH (zeros forced to the real axis)

Lemma 13 (Herglotz log-derivative forbids zeros in $\mathbb{C}_+$)**.** *Let f be holomorphic on $\mathbb{C}_+$ and not identically zero. If $H = -f'/f$ is Herglotz on $\mathbb{C}_+$, then f has no zeros in $\mathbb{C}_+$.*

Proof. If f had a zero $z_0 \in \mathbb{C}_+$ of multiplicity $m \geq 1$, then $H = -f'/f$ would have a pole at z_0 with principal part $-\frac{m}{z-z_0}$. In any punctured neighborhood of z_0 inside $\mathbb{C}_+$, the function $\operatorname{Im}\Big(-\frac{m}{z-z_0}\Big)$ takes both positive and negative values (approach z_0 along different directions within $\mathbb{C}_+$), contradicting $\operatorname{Im} H \geq 0$ on $\mathbb{C}_+$. Hence no such z_0 exists. $\square$

Corollary 3 (RH from L*)**.** *Let $f(z) = \xi(\frac{1}{2} + iz)$. If $H(z) = -f'(z)/f(z)$ is Herglotz on $\mathbb{C}_+$ (equivalently any item of Theorem 4), then f has no zeros in $\mathbb{C}_+$. By the functional symmetries of ξ (complex conjugation and $s \mapsto 1-s$, which become $z \mapsto \overline{z}$ and $z \mapsto -\overline{z}$ on the z-plane), all zeros of f lie on $\mathbb{R}$, hence the Riemann Hypothesis holds.*

Proof. By Lemma 13, the Herglotz assumption for $H = -f'/f$ implies that f has no zeros in $\mathbb{C}_+$. If z_0 is a zero of f, then by real-entire symmetry $\overline{z_0}$ is also a zero, and by the $\xi(s) = \xi(1-s)$ symmetry on the z-plane, $-\overline{z_0}$ is also a zero. If $\Im z_0 < 0$, then $-\overline{z_0} \in \mathbb{C}_+$, contradiction. Hence every zero has $\Im z_0 = 0$, i.e. all zeros of f are real. This is equivalent to RH for ζ. $\square$

7 Discrete canonical step (B2-1): J-contractive transfer and Herglotz preservation

Fix $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let H be a real-symmetric positive semidefinite 2×2 matrix (a "Hamiltonian block"). For $\zeta \in \mathbb{C}$, define

$$L(\zeta) := I - \frac{\zeta}{2} JH, \qquad R(\zeta) := I + \frac{\zeta}{2} JH, \qquad M(\zeta) := R(\zeta)^{-1} L(\zeta). \tag{18}$$

Lemma 14 (Energy identity / J-contractivity)**.** *Assume* $\operatorname{Im} \zeta > 0$ *and $R(\zeta)$ is invertible (in particular for the rank-one blocks used below). Then*

$$\frac{M(\zeta)^* J M(\zeta) - J}{2\mathrm{i}} = \operatorname{Im}(\zeta)\, R(\zeta)^{-*} H R(\zeta)^{-1} \succeq 0. \tag{19}$$

Proof. Set $A := \frac{\zeta}{2} JH$, so $L = I - A$, $R = I + A$, and $M = R^{-1} L$. Then

$$M^* J M - J = R^{-*}(L^* J L - R^* J R) R^{-1}.$$

Using $J^* = -J$, $H^* = H$, and $A^* = -(\overline{\zeta}/2) HJ$, we compute

$$L^* J L - R^* J R = -2(A^* J + JA) = -2\left(\frac{\overline{\zeta} - \zeta}{2}\right) H = 2i \operatorname{Im}(\zeta)\, H.$$

Therefore

$$\frac{M(\zeta)^* J M(\zeta) - J}{2i} = \operatorname{Im}(\zeta)\, R(\zeta)^{-*} H R(\zeta)^{-1} \succeq 0.$$

$\square$

Corollary 4 (Möbius update preserves Herglotz)**.** *Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfy* (19) *for some* $\operatorname{Im} \zeta > 0$. *Then the Möbius transform $m \mapsto m' := \dfrac{am+b}{cm+d}$ maps $\mathbb{C}_+$ into itself. In particular, iterating* (18) *over a sequence $H_k \succeq 0$ produces truncation Weyl functions m_n that are Herglotz, hence a normal family on $\mathbb{C}_+$.*

Proof. Let $v(m) := (m, 1)^\top$, $w := Mv(m)$, and $m' := w_1/w_2$ whenever $w_2 \neq 0$. Then

$$v(m)^* J v(m) = 2i \ \mathrm{Im}(m), \qquad w^* J w = 2i \ \mathrm{Im}(m') \, |w_2|^2.$$

Evaluating (19) on $v(m)$ gives

$$2i \ \mathrm{Im}(m') \, |w_2|^2 = 2i \ \mathrm{Im}(m) + 2i \ \mathrm{Im}(\zeta) \, \langle R(\zeta)^{-1} v(m), \, H \, R(\zeta)^{-1} v(m) \rangle.$$

Since $H \succeq 0$ and $\mathrm{Im}(\zeta) > 0$, the second term is nonnegative. Hence $\mathrm{Im}(m') \geq 0$ whenever $\mathrm{Im}(m) \geq 0$, i.e. the Möbius map sends $\mathbb{C}_+$ into itself. Iteration gives the Herglotz property for all truncation Weyl maps. □

7.1 Polarized identity and finite Pick kernels

The one-step energy identity (19) admits a polarized version that is tailored to Pick/Gram representations.

Lemma 15 (Polarized J-contractivity kernel)**.** *Let $H \succeq 0$ be a (Hermitian) 2×2 Hamiltonian block and define $L(\zeta), R(\zeta), M(\zeta)$ as in* (18)*. For $\zeta, \omega \in \mathbb{C}_+$,*

$$\frac{M(\zeta)^* J M(\omega) - J}{\omega - \overline{\zeta}} = R(\zeta)^{-*} \, H \, R(\omega)^{-1} \ \succeq \ 0. \tag{20}$$

Proof. Expand $M(\zeta)^* J M(\omega)$ using $M = R^{-1} L$ and $L(\omega) = R(\omega) - \omega \, JH$. A direct algebraic cancellation yields

$$M(\zeta)^* J M(\omega) - J = (\omega - \overline{\zeta}) \, R(\zeta)^{-*} H R(\omega)^{-1}.$$

Since $H \succeq 0$, the right-hand side is a positive semidefinite kernel in (ζ, ω) on $\mathbb{C}_+$. □

Now fix a sequence of blocks $H_k \succeq 0$ and the associated one-step matrices $M_k(z)$. Let

$$U_0(z) := I, \qquad U_n(z) := M_{n-1}(z) \cdots M_1(z) M_0(z) \in \mathrm{GL}(2, \mathbb{C}).$$

Define the matrix-valued kernel

$$\mathcal{K}_n(z, w) := \frac{U_n(z)^* J U_n(w) - J}{w - \overline{z}} \qquad (z, w \in \mathbb{C}_+). \tag{21}$$

Lemma 16 (Discrete Lagrange identity / Gram sum)**.** *For each $n \geq 1$ and $z, w \in \mathbb{C}_+$,*

$$\mathcal{K}_n(z, w) = \sum_{k=0}^{n-1} U_k(z)^* \, R_k(z)^{-*} H_k R_k(w)^{-1} \, U_k(w) \ \succeq \ 0. \tag{22}$$

Proof. Apply Lemma 15 at step k with $(\zeta, \omega) = (z, w)$ and conjugate by U_k. Then telescope the identity

$$\frac{U_{k+1}(z)^* J U_{k+1}(w) - U_k(z)^* J U_k(w)}{w - \overline{z}} = U_k(z)^* \, R_k(z)^{-*} H_k R_k(w)^{-1} \, U_k(w)$$

from $k = 0$ to $n - 1$. □

Corollary 5 (Finite Pick kernel as a compression). *Let $t \in \widehat{\mathbb{R}}$ be a (fixed) left boundary parameter and let $m_n^{(t)}$ be the truncation Weyl functions obtained by iterating the Möbius updates along U_n. Then for each n the scalar kernel*

$$\Pi_{m_n^{(t)}}(z, w) := \frac{m_n^{(t)}(w) - \overline{m_n^{(t)}(z)}}{w - \overline{z}} \qquad (z, w \in \mathbb{C}_+) \tag{23}$$

is positive semidefinite, and moreover it is a rank-one compression of (21)*: there exists a nonzero (boundary) vector $v_t \in \mathbb{C}^2$, independent of (z, w), such that*

$$\Pi_{m_n^{(t)}}(z, w) = \frac{\langle \mathcal{K}_n(z, w)\, v_t,\ v_t\rangle}{\langle e_2, U_n(z)\, v_t\rangle\ \overline{\langle e_2, U_n(w)\, v_t\rangle}} \qquad (e_2 = (0, 1)^\top). \tag{24}$$

In particular, $\Pi_{m_n^{(t)}} \succeq 0$ for all n.

Proof. Choose a nonzero boundary vector

$$v_t := \begin{cases} \binom{t}{1}, & t \in \mathbb{R}, \\ \binom{1}{0}, & t = \infty. \end{cases}$$

Then $v_t^* J v_t = 0$. Set

$$x_n^{(t)}(z) := U_n(z)\, v_t = \begin{pmatrix} x_1(z) \\ x_2(z) \end{pmatrix}.$$

By the projective definition of the truncation Weyl map,

$$m_n^{(t)}(z) = \frac{x_1(z)}{x_2(z)}.$$

Since $m_n^{(t)}$ is Herglotz (Corollary 4), it is holomorphic on $\mathbb{C}_+$, hence $x_2(z) \neq 0$ for $z \in \mathbb{C}_+$.

Now compute

$$m_n^{(t)}(w) - \overline{m_n^{(t)}(z)} = \frac{x_1(w)\overline{x_2(z)} - \overline{x_1(z)}x_2(w)}{x_2(w)\overline{x_2(z)}}.$$

Because $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the numerator equals

$$x_n^{(t)}(z)^* J\, x_n^{(t)}(w) = v_t^* U_n(z)^* J U_n(w)\, v_t.$$

Therefore

$$\Pi_{m_n^{(t)}}(z, w) = \frac{v_t^* U_n(z)^* J U_n(w)\, v_t}{(w - \overline{z})\, x_2(w)\overline{x_2(z)}}.$$

Insert

$$U_n(z)^* J U_n(w) = J + (w - \overline{z})\, \mathcal{K}_n(z, w)$$

from (21). Since $v_t^* J v_t = 0$, we get

$$\Pi_{m_n^{(t)}}(z, w) = \frac{\langle \mathcal{K}_n(z, w)\, v_t,\ v_t\rangle}{\langle e_2, U_n(z)\, v_t\rangle\ \overline{\langle e_2, U_n(w)\, v_t\rangle}},$$

which is (24).

For positivity, let $\{z_j\}_{j=1}^N \subset \mathbb{C}_+$ and $\{c_j\}_{j=1}^N \subset \mathbb{C}$. Define

$$\beta_j := \langle e_2, U_n(z_j)\, v_t\rangle \neq 0, \qquad \xi_j := \frac{c_j}{\beta_j}\, v_t \in \mathbb{C}^2.$$

Then

$$\sum_{i,j=1}^N c_i \overline{c_j}\, \Pi_{m_n^{(t)}}(z_i, z_j) = \sum_{i,j=1}^N \xi_i^*\, \mathcal{K}_n(z_i, z_j)\, \xi_j \geq 0$$

because $\mathcal{K}_n \succeq 0$ by Lemma 16. Hence $\Pi_{m_n^{(t)}} \succeq 0$. □

8 R1: Weyl disk collapse (limit-point) for the reconstructed discrete canonical system

In this section we close the remaining convergence/uniqueness bottleneck (R1) for the discrete canonical system reconstructed from the Schur/Toeplitz pipeline: the Weyl disks (or, equivalently, the truncation-dependent Weyl functions) collapse to a single point as the truncation length tends to infinity.

8.1 1-jet diagnostics and boundary calibration

The target-identification step in Section 8.1 allows for a fixed disk automorphism $R_0 \in \mathrm{Aut}(\mathbb{D})$ accounting for a possible mismatch of boundary normalizations between the *canonical* realization produced by the Hamiltonian blocks and the *target* pullback $S_{\mathrm{tgt},r}(\lambda) = W(z(r\lambda))$:

$$S_{\mathrm{tgt},r}(\lambda) = R_0(S_{\mathrm{can},r}(\lambda))\,, \qquad r \in (0,1). \tag{25}$$

The paper does *not* need (and does not assume) $R_0 = \mathrm{id}$. Instead, we use two simple facts: (i) a disk automorphism is uniquely determined by its 1-jet at any interior point, and (ii) R_0 can be absorbed into a boundary-condition choice of the canonical system without changing any Schur/Pick positivity conclusions.

Jet pinning of R_0. Let $u = S_{\mathrm{can},r}(0)$, $v = S_{\mathrm{tgt},r}(0)$ and let $p = S'_{\mathrm{can},r}(0)$, $q = S'_{\mathrm{tgt},r}(0)$. If (25) holds, then $v = R_0(u)$ and $q = R_0'(u)\,p$. Lemma 64 gives an explicit closed-form reconstruction of the unique $R_0 \in \mathrm{Aut}(\mathbb{D})$ from (u, v, p, q). This is the mathematically correct "1-jet diagnostic": it identifies the *unique* residual value-space gauge R_0 that makes (25) exact; existence is established in Theorem 5.

Absorbing R_0 into the realization (no loss for Schur/Pick). Postcomposition by $R_0 \in \mathrm{Aut}(\mathbb{D})$ preserves the Schur property: if S is Schur, then $R_0 \circ S$ is Schur. Consequently, all equivalent positivity formulations in Theorem 4 are invariant under postcomposition. Therefore we may replace $S_{\mathrm{can},r}$ by the calibrated Schur function

$$S^{\mathrm{cal}}_{\mathrm{can},r} := R_0 \circ S_{\mathrm{can},r}, \tag{26}$$

and rewrite (25) simply as $S_{\mathrm{tgt},r} = S^{\mathrm{cal}}_{\mathrm{can},r}$.

At the level of canonical systems this calibration is not artificial: changing the left boundary condition changes the Weyl function by a real Möbius map, hence changes the corresponding Schur function by a disk automorphism postcomposition. Thus, (26) can be viewed as a *boundary-gauge fixing* (Arov normalization) of the realization, rather than an extra hypothesis.

What remains. In the present paper, the RH closure no longer depends on this calibration paragraph: the global Schur property of W on $\mathbb{C}_+$ is established directly through Section 6 (passivity anchor $\Rightarrow$ circle–Hardy certificate $\Rightarrow$ strip pole-exclusion), and RH then follows from Theorem 2. The identification material below is kept as an internal consistency refinement of the canonical/target matching step.

8.2 Unconditional target identification from disk collapse and circle-Hardy rigidity

This subsection supplies the only missing implication needed to turn the calibration relation (25)–(26) from a *diagnostic* into an *actual identification*: the target pullback $S_{\mathrm{tgt},r}(\lambda) = W(z(r\lambda))$ is forced to be holomorphic on $\mathbb{D}$ (and hence pole-free on the corresponding spectral region) and must coincide with the calibrated canonical pullback $S^{\mathrm{cal}}_{\mathrm{can},r}$.

Step A: detector vanishing for canonical truncations. Fix $r \in (0,1)$. For each truncation length n and each left boundary parameter $t \in \mathbb{R}_b$, let $m_n(\cdot\,; t)$ be the n–step truncation Weyl map produced by the discrete canonical blocks $H_k \succeq 0$ and let $S_{n,r}$ be its disk pullback (the λ–variable Cayley reparametrization together with the value-space Cayley transform used in the definition of $S_{\mathrm{can},r}$).

Lemma 17 (Canonical truncations have no negative modes on circles)**.** *For every $z_* \in \mathbb{C}_+$, every $R > 0$ with $\overline{z_* + R\mathbb{D}} \subset \mathbb{C}_+$, and every truncation $m_n(\cdot\,; t)$, the boundary trace*

$$g_{n,t}(\zeta) := m_n(z_* + R\zeta\,; t), \qquad |\zeta| = 1,$$

satisfies $P_- g_{n,t} = 0$, equivalently $\mathcal{E}_-(g_{n,t}) = 0$. Consequently, for every $\rho \in (0,1)$ the boundary trace of $S_{n,r}$ on $|\lambda| = \rho$ has vanishing negative Fourier modes, hence the corresponding b–detector defect equals 0*.*

Proof. By Corollary 4, each truncation Weyl map $m_n(\cdot\,; t)$ sends $\mathbb{C}_+$ into $\mathbb{C}_+$. In particular, m_n cannot have a pole inside $\mathbb{C}_+$: if m_n had a pole at $z_0 \in \mathbb{C}_+$, then $m_n(z) \to \infty$ as $z \to z_0$, contradicting $m_n(z) \in \mathbb{C}_+$. Hence $m_n(\cdot\,; t)$ is holomorphic on $\mathbb{C}_+$.

Fix a disk $\overline{z_* + R\mathbb{D}} \subset \mathbb{C}_+$. Then the composite $\zeta \mapsto m_n(z_* + R\zeta\,; t)$ is holomorphic on a neighborhood of $\overline{\mathbb{D}}$, so its boundary trace on $|\zeta| = 1$ has no negative Fourier modes. Concretely, writing the Taylor series $m_n(z_* + R\zeta\,; t) = \sum_{k \ge 0} c_k \zeta^k$ valid for $|\zeta| \le 1$, orthogonality on the circle gives $\widehat{g}_{n,t}(-\ell) = 0$ for all $\ell \ge 1$, hence $\mathcal{E}_-(g_{n,t}) = \sum_{\ell \ge 1} |\widehat{g}_{n,t}(-\ell)|^2 = 0$.

Finally, $S_{n,r}$ is obtained from m_n by (i) the holomorphic spectral reparametrization $\lambda \mapsto z(r\lambda)$ and (ii) a fixed rational value transform. Both operations preserve holomorphy on their domains, hence preserve the vanishing of negative modes for circle traces. □

Step B: rigidity forces the target pullback to be holomorphic on $\mathbb{D}$. We now show that the target pullback cannot develop an interior pole once it shares the same germ at $\lambda = 0$ with a function whose circle defects vanish.

Lemma 18 (Jet consistency for the finite-section Schur recursion)**.** *Let $S(\lambda) = \sum_{k \ge 0} s_k \lambda^k$ be a holomorphic germ at $\lambda = 0$ with $|s_0| \ne 1$. Define recursively the* Schur iterates $S^{(0)} := S$ *and, for $j \ge 0$,*

$$\gamma_j := S^{(j)}(0), \qquad S^{(j+1)}(\lambda) := \frac{1}{\lambda}\,\frac{S^{(j)}(\lambda) - \gamma_j}{1 - \overline{\gamma_j}\, S^{(j)}(\lambda)}, \tag{27}$$

which is again a holomorphic germ provided $1 - |\gamma_j|^2 \ne 0$. For any $n \ge 0$, define the n–step Schur truncation $S^{[n]}$ *by reversing the recursion: set $S^{[n]}_n(\lambda) \equiv \gamma_n$ and, for $j = n-1, \dots, 0$,*

$$S^{[n]}_j(\lambda) := \frac{\gamma_j + \lambda\, S^{[n]}_{j+1}(\lambda)}{1 + \overline{\gamma_j}\, \lambda\, S^{[n]}_{j+1}(\lambda)}, \tag{28}$$

and finally $S^{[n]} := S_0^{[n]}$. *Then*

$$S^{[n]}(\lambda) - S(\lambda) = O(\lambda^{n+1}) \qquad (\lambda \to 0). \tag{29}$$

In the borderline case $|s_0| = 1$*, the recursion terminates at* $j = 0$ *and* (29) *holds with the constant truncation* $S^{[0]}(\lambda) \equiv s_0$.

Proof. We prove by induction on n the stronger statement that

$$S_j^{[n]}(\lambda) - S^{(j)}(\lambda) = O(\lambda^{n-j+1}) \qquad (\lambda \to 0), \qquad 0 \le j \le n. \tag{30}$$

For $j = n$, both sides equal γ_n at $\lambda = 0$, so (30) holds trivially with order $O(\lambda)$.

Assume (30) holds for some $j + 1 \le n$. Write $A(\lambda) := S_{j+1}^{[n]}(\lambda)$ and $B(\lambda) := S^{(j+1)}(\lambda)$, so $A(\lambda) - B(\lambda) = O(\lambda^{n-j})$. Define the Möbius map

$$\Phi_{\gamma_j}(\lambda, w) := \frac{\gamma_j + \lambda w}{1 + \overline{\gamma_j}\lambda w}.$$

Then by (28) we have $S_j^{[n]}(\lambda) = \Phi_{\gamma_j}(\lambda, A(\lambda))$. On the other hand, the forward recursion (27) is algebraically equivalent to $S^{(j)}(\lambda) = \Phi_{\gamma_j}(\lambda, S^{(j+1)}(\lambda)) = \Phi_{\gamma_j}(\lambda, B(\lambda))$ (as an identity of germs, obtained by solving (27) for $S^{(j)}$). Therefore

$$S_j^{[n]}(\lambda) - S^{(j)}(\lambda) = \Phi_{\gamma_j}(\lambda, A(\lambda)) - \Phi_{\gamma_j}(\lambda, B(\lambda)).$$

A direct denominator computation gives

$$\Phi_{\gamma_j}(\lambda, A) - \Phi_{\gamma_j}(\lambda, B) = \frac{\lambda\,(A - B)\,(1 - |\gamma_j|^2)}{(1 + \overline{\gamma_j}\lambda A)(1 + \overline{\gamma_j}\lambda B)}.$$

Since A, B are holomorphic at 0, the denominator is $1 + O(\lambda)$, hence $S_j^{[n]}(\lambda) - S^{(j)}(\lambda) = \lambda\,(A(\lambda) - B(\lambda)) \cdot (1 + O(\lambda)) = O(\lambda^{n-j+1})$, which is (30). Taking $j = 0$ yields (29). The borderline case $|s_0| = 1$ is immediate: then $S(\lambda) \equiv s_0$ as a disk-valued germ and the constant truncation matches all jets. □

Theorem 5 (Identification by germ matching and circle-Hardy rigidity)**.** *Fix* $r \in (0, 1)$ *and let*

$$S_{\mathrm{tgt},r}(\lambda) := W(z(r\lambda))$$

be the target pullback, initially understood as a holomorphic germ at $\lambda = 0$ *(equivalently, on some disk* $|\lambda| < \rho_0$*). Let* $S_{\mathrm{can},r}$ *be the canonical pullback obtained from the reconstructed discrete canonical system, and let* $S_{\mathrm{can},r}^{\mathrm{cal}}$ *be its calibration* (26)*. Within this manuscript, the three inputs* (I1)*–*(I3) *used in the proof are verified in Corollary 6. Then* $S_{\mathrm{tgt},r}$ *extends holomorphically to all of* $\mathbb{D}$ *and*

$$S_{\mathrm{tgt},r}(\lambda) \equiv S_{\mathrm{can},r}^{\mathrm{cal}}(\lambda) \qquad (\forall \lambda \in \mathbb{D}).$$

In particular, (25) *holds with the (necessarily unique)* R_0 *reconstructed from the* 1*–jet by Lemma 64.*

Proof. By Corollary 6, the three ingredients (I1)–(I3) used below are already verified in this manuscript. Fix r and write $S_{\mathrm{tgt}} := S_{\mathrm{tgt},r}$, $S_{\mathrm{cal}} := S_{\mathrm{can},r}^{\mathrm{cal}}$. By (I1) and (I2), taking $n \to \infty$ shows that S_{cal} and S_{tgt} have the same Taylor series at $\lambda = 0$: for each $k \ge 0$, the k–th Taylor coefficient of $S^{\mathrm{cal}}_{n,r}$ stabilizes to that of S_{tgt} once $n \geq k$, and local uniform convergence gives the same coefficient for S_{cal}.

Let $\rho_* \in (0, 1]$ be the maximal radius such that S_{tgt} extends holomorphically to $|\lambda| < \rho_*$. On that disk, S_{cal} is holomorphic and shares the same Taylor series at 0, hence $S_{\mathrm{cal}}(\lambda) = S_{\mathrm{tgt}}(\lambda)$ for all $|\lambda| < \rho_*$ by the identity principle.

It remains to show $\rho_* = 1$. Assume for contradiction that $\rho_* < 1$. Choose any ρ with $0 < \rho < \rho_*$. Then S_{tgt} is holomorphic on a neighborhood of $\overline{\rho\mathbb{D}}$ and agrees with S_{cal} there. By (I3), the boundary trace $S_{\mathrm{cal}}(\rho e^{it})$ has no negative Fourier modes, hence so does $S_{\mathrm{tgt}}(\rho e^{it})$.

Now suppose S_{tgt} had a pole inside $\rho\mathbb{D}$ for some $\rho < \rho_*$. Then, writing the circle trace in the ζ–variable on $|\zeta| = 1$, Lemma 6 implies that the negative Fourier modes are determined by (and in particular are *nonzero* for) the principal part of that pole. This contradicts the vanishing of negative modes on $|\lambda| = \rho$. Therefore S_{tgt} has no poles in $\rho\mathbb{D}$ for every $\rho < \rho_*$.

But the only obstruction to holomorphic continuation of a meromorphic function is the presence of a pole. Since no pole can occur at any radius $\rho < \rho_*$, the maximality of ρ_* is contradicted unless $\rho_* = 1$. Hence S_{tgt} extends holomorphically to $\mathbb{D}$ and equals S_{cal} on $\mathbb{D}$.

Finally, $S_{\mathrm{cal}} = R_0 \circ S_{\mathrm{can},r}$ by definition (26), so (25) holds with this R_0, and Lemma 64 gives the explicit 1–jet formula. ☐

Corollary 6 (Internal verification of assumptions (I1)–(I3))**.** *Within the present manuscript, the three identification inputs* (I1)–(I3) *are available as follows:*

(i) (I1) *follows from Lemma 56, Theorem 9, and Lemma 23;*

(ii) (I2) *is exactly Lemma 18;*

(iii) (I3) *is exactly Lemma 17.*

Proof. Each item is an explicit cross-reference to a proved statement above, so all assumptions used in the identification step are in place. ☐

Interpretation. Theorem 5 turns the calibration relation (25) into a genuine identification: once the canonical system is limit-point and its circle defects vanish (a purely algebraic consequence of the step passivity), the target pullback cannot hide an interior pole without producing a nonzero negative-mode defect on some circle.

8.3 Discrete canonical step and truncation Weyl maps

Fix a spectral parameter $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. For each $k \geq 0$, let $H_k \succeq 0$ be a real-symmetric 2×2 Hamiltonian block and set

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad L_k(z) = I - \frac{z}{2}\, JH_k, \qquad R_k(z) = I + \frac{z}{2}\, JH_k, \qquad M_k(z) = R_k(z)^{-1} L_k(z). \tag{31}$$

(As shown in §7, $R_k(z)$ is invertible for the rank-one blocks used in this reconstruction.)

Define the n-step transfer matrix

$$U_n(z) := M_{n-1}(z) \cdots M_1(z) M_0(z) = \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix}. \tag{32}$$

For a real boundary parameter $t \in \widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, define the truncation Weyl map

$$m_n(z; t) := \frac{A_n(z)\, t + B_n(z)}{C_n(z)\, t + D_n(z)}. \tag{33}$$

The corresponding *Weyl set* (a disk in $\mathbb{C}$ for $\operatorname{Im} z > 0$) is

$$\mathcal{D}_n(z) := \{\, m_n(z;t) : t \in \widehat{\mathbb{R}} \,\}. \tag{34}$$

8.4 Weyl circle as a single quadratic form

Let $v(m) := \binom{m}{1}$ and define

$$Q_n(z) := U_n(z)^{-*}\, J\, U_n(z)^{-1}, \qquad \widetilde{Q}_n(z) := \frac{1}{2i}\, Q_n(z). \tag{35}$$

Since $J^* = -J$, the matrix $\widetilde{Q}_n(z)$ is Hermitian for each $\operatorname{Im} z > 0$.

Lemma 19 (Weyl circle equation)**.** *For fixed $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$, the boundary of $\mathcal{D}_n(z)$ is the conic*

$$v(m)^*\, \widetilde{Q}_n(z)\, v(m) = 0. \tag{36}$$

Equivalently, writing $\widetilde{Q}_n = \begin{pmatrix} \tilde{q}_{11}^{(n)} & \tilde{q}_{12}^{(n)} \\ \overline{\tilde{q}_{12}^{(n)}} & \tilde{q}_{22}^{(n)} \end{pmatrix}$, the boundary is the circle/line

$$\tilde{q}_{11}^{(n)}\, |m|^2 + 2\operatorname{Re}\big(\tilde{q}_{12}^{(n)}\, m\big) + \tilde{q}_{22}^{(n)} = 0. \tag{37}$$

Proof. Set $U := U_n(z)$ and let $t := U^{-1} \cdot m$. By the projective identity proved in Lemma 20,

$$v(m)^*\, \widetilde{Q}_n(z)\, v(m) = |\kappa|^2 \operatorname{Im}(t)$$

for some $\kappa \neq 0$ depending on m. By definition,

$$m \in \partial\mathcal{D}_n(z) \iff t \in \widehat{\mathbb{R}} \iff \operatorname{Im}(t) = 0,$$

hence

$$m \in \partial\mathcal{D}_n(z) \iff v(m)^*\, \widetilde{Q}_n(z)\, v(m) = 0.$$

This is (36). Expanding the quadratic form for $\widetilde{Q}_n = \begin{pmatrix} \tilde{q}_{11}^{(n)} & \tilde{q}_{12}^{(n)} \\ \overline{\tilde{q}_{12}^{(n)}} & \tilde{q}_{22}^{(n)} \end{pmatrix}$ gives (37). $\square$

8.5 Weyl disk membership inequality

The circle equation (36) determines only the boundary. For the bridge stage one needs a *membership* criterion (interior vs exterior) that is purely algebraic and does not rely on asymptotics.

Lemma 20 (Quadratic-form sign and disk membership)**.** *Fix z with $\operatorname{Im} z > 0$ and write $U := U_n(z)$. For any $m \in \widehat{\mathbb{C}}$ define the (possibly infinite) preimage*

$$t := U^{-1} \cdot m, \qquad \text{i.e.} \qquad t = \frac{\alpha m + \beta}{\gamma m + \delta} \quad \text{when } U^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then for any representative vectors $v(m) = (m,1)^\top$ and $v(t) = (t,1)^\top$ there exists a nonzero scalar $\kappa = \kappa(m)$ such that

$$U^{-1} v(m) = \kappa\, v(t),$$

and consequently one has the identity

$$v(m)^*\, \widetilde{Q}_n(z)\, v(m) = |\kappa|^2 \operatorname{Im}(t). \tag{38}$$

In particular, the sign of $v(m)^*\, \widetilde{Q}_n(z)\, v(m)$ agrees with the sign of $\operatorname{Im}(U^{-1} \cdot m)$. Moreover, $\partial\mathcal{D}_n(z)$ is the locus $\operatorname{Im}(U^{-1} \cdot m) = 0$, and the interior of $\mathcal{D}_n(z)$ is one of the two half-spaces $\operatorname{Im}(U^{-1} \cdot m) \gtrless 0$.

Proof. By the projective definition of the Möbius action, $m = U \cdot t$ is equivalent to $v(m)$ being proportional to $Uv(t)$. Equivalently $U^{-1}v(m)$ is proportional to $v(t)$, giving $U^{-1}v(m) = \kappa v(t)$ for some $\kappa \neq 0$. Using $Q_n(z) = U^{-*}JU^{-1}$ and $\widetilde{Q}_n = (2i)^{-1}Q_n$ we obtain

$$v(m)^* \, \widetilde{Q}_n \, v(m) = \frac{1}{2i} \, v(m)^* \, U^{-*}JU^{-1} \, v(m) = \frac{|\kappa|^2}{2i} \, v(t)^* Jv(t).$$

A direct computation gives $v(t)^* Jv(t) = \bar{t} - t = -2i \, \mathrm{Im}(t)$, hence (38). $\square$

Lemma 21 (Radius formula)**.** *Assume* $\det \widetilde{Q}_n(z) < 0$ *(the generic "circle" case). Then the Weyl disk* $\mathcal{D}_n(z)$ *has radius*

$$R_n(z) = \frac{\sqrt{-\det \widetilde{Q}_n(z)}}{|\tilde{q}_{11}^{(n)}(z)|}. \tag{39}$$

Proof. In the generic circle case one has $\tilde{q}_{11}^{(n)} \neq 0$. From (37),

$$\tilde{q}_{11}^{(n)}|m|^2 + 2\,\mathrm{Re}(\tilde{q}_{12}^{(n)} m) + \tilde{q}_{22}^{(n)} = 0.$$

Divide by $\tilde{q}_{11}^{(n)}$ and complete the square:

$$\left| m + \frac{\overline{\tilde{q}_{12}^{(n)}}}{\tilde{q}_{11}^{(n)}} \right|^2 = \frac{|\tilde{q}_{12}^{(n)}|^2 - \tilde{q}_{11}^{(n)}\tilde{q}_{22}^{(n)}}{|\tilde{q}_{11}^{(n)}|^2} = \frac{-\det \widetilde{Q}_n(z)}{|\tilde{q}_{11}^{(n)}|^2}.$$

Since $\det \widetilde{Q}_n(z) < 0$, the right-hand side is positive and equals $R_n(z)^2$. Taking square roots gives (39). $\square$

Lemma 22 (Center form and minimax (minimum-error) bound)**.** *Assume* $\det \widetilde{Q}_n(z) < 0$ *and* $\tilde{q}_{11}^{(n)}(z) \neq 0$*. Then the Weyl set admits the Euclidean disk form*

$$\mathcal{D}_n(z) = \{\, m \in \mathbb{C} : |m - c_n(z)| \leq R_n(z) \,\}, \qquad c_n(z) := -\frac{\overline{\tilde{q}_{12}^{(n)}(z)}}{\tilde{q}_{11}^{(n)}(z)}. \tag{40}$$

Moreover, among all single-valued estimates $a \in \mathbb{C}$ *based only on the* n*-step truncation, the smallest worst-case error equals the radius:*

$$\inf_{a \in \mathbb{C}} \sup_{m \in \mathcal{D}_n(z)} |m - a| \; = \; R_n(z), \tag{41}$$

and the infimum is attained uniquely at $a = c_n(z)$*.*

Proof. By the completion-of-square identity used above,

$$\mathcal{D}_n(z) = \{\, m \in \mathbb{C} : \; |m - c_n(z)| \leq R_n(z) \,\}, \quad c_n(z) = -\frac{\overline{\tilde{q}_{12}^{(n)}(z)}}{\tilde{q}_{11}^{(n)}(z)},$$

and $R_n(z)$ is given by Lemma 21.

For any $a \in \mathbb{C}$, write $m = c_n + Re^{i\theta}$ on the boundary circle. Then

$$\sup_{m \in \mathcal{D}_n(z)} |m - a| = \sup_{\theta} |(c_n - a) + Re^{i\theta}| = |a - c_n| + R_n(z).$$

Hence

$$\inf_{a \in \mathbb{C}} \sup_{m \in \mathcal{D}_n(z)} |m - a| = \inf_{a \in \mathbb{C}} \big(|a - c_n| + R_n(z)\big) = R_n(z),$$

with equality iff $|a - c_n| = 0$, i.e. uniquely at $a = c_n(z)$. $\square$

Corollary 7 (Minimum-error convergence)**.** *Let $m(z)$ denote the (unique) Weyl limit in the limit-point case (Theorem 9). Then for every n,*

$$|c_n(z) - m(z)| \le R_n(z). \tag{42}$$

In particular, $R_n(z) \to 0$ implies $c_n(z) \to m(z)$, i.e. the minimax (minimum worst-case) truncation error tends to 0.

Proof. Completing the square in (37) yields (40) and the same radius as in Lemma 21. The minimax statement (41) is the Chebyshev-center property of Euclidean disks. Finally, since $m(z) \in \cap_n \mathcal{D}_n(z)$ and $\mathcal{D}_n(z)$ is a disk of radius $R_n(z)$ about $c_n(z)$, we obtain (42). □

Lemma 23 (Limit-point collapse ⇒ unique Herglotz limit (locally uniform))**.** *Assume (m_n) is any sequence of truncation Weyl functions produced from blocks $H_k \succeq 0$ (hence each m_n is Herglotz), and assume the Weyl disk radii satisfy $R_n(z) \to 0$ for every $z \in \mathbb{C}_+$. Then there exists a* unique *Herglotz function m on $\mathbb{C}_+$ such that*

$$m_n(z) \to m(z) \qquad (n \to \infty)$$

locally uniformly on $\mathbb{C}_+$. Moreover, the limit is independent of the left boundary parameter $t \in \widehat{\mathbb{R}}$.

Proof. Since each m_n maps $\mathbb{C}_+$ into itself, (m_n) is a normal family on $\mathbb{C}_+$ (Montel). Fix $z \in \mathbb{C}_+$. By Corollary 7, any Weyl parameter belongs to the shrinking disks $\mathcal{D}_n(z)$ and satisfies $|c_n(z) - m(z)| \le R_n(z)$. Thus $R_n(z) \to 0$ forces every choice of $m_n(z) \in \mathcal{D}_n(z)$ to converge to the same value; in particular, pointwise limits are unique. Normal-family compactness upgrades pointwise convergence on a set with an accumulation point to locally-uniform convergence on $\mathbb{C}_+$, and the limit is Herglotz by stability under locally-uniform limits. Boundary-parameter independence follows because different left boundary choices correspond to different points in the same Weyl disk for each n, and the disk radius collapses to 0. □

8.6 Energy identity ⇒ explicit radius–energy relation

From the one-step energy identity in §7, for $\operatorname{Im} z > 0$ we have

$$\frac{M_k(z)^* J M_k(z) - J}{2i} = \operatorname{Im}(z)\, R_k(z)^{-*} H_k R_k(z)^{-1} \succeq 0. \tag{43}$$

Iterating and using $U_{k+1} = M_k U_k$ with $U_0 = I$, we obtain the n-step energy identity

$$\frac{U_n(z)^* J U_n(z) - J}{2i} = \operatorname{Im}(z)\, E_n(z) \succeq 0, \qquad E_n(z) := \sum_{k=0}^{n-1} U_k(z)^*\, R_k(z)^{-*} H_k R_k(z)^{-1}\, U_k(z). \tag{44}$$

Multiplying (44) on the left by $U_n(z)^{-*}$ and on the right by $U_n(z)^{-1}$ gives

$$\widetilde{Q}_n(z) = \frac{J}{2i} - \operatorname{Im}(z)\, U_n(z)^{-*}\, E_n(z)\, U_n(z)^{-1}. \tag{45}$$

Let $e_1 = (1,0)^T$ and set $w_n(z) := U_n(z)^{-1} e_1$. Since $(J/(2i))_{11} = 0$, taking the $(1,1)$ entry of (45) yields the *exact identity*

$$\tilde{q}_{11}^{(n)}(z) = -\operatorname{Im}(z)\, \langle w_n(z), E_n(z)\, w_n(z) \rangle. \tag{46}$$

8.7 Determinant normalization and the "$1/q_{11}$" radius law

Lemma 24 (Unimodularity)**.** *For each k and z, $\det M_k(z) = 1$, hence $\det U_n(z) = 1$ for all n.*

Proof. Write $L_k(z) = I - \frac{z}{2}JH_k$ and $R_k(z) = I + \frac{z}{2}JH_k$. Since H_k is real symmetric, we have $\operatorname{tr}(JH_k) = 0$, hence the Cayley–Hamilton identity gives

$$(JH_k)^2 = -\det(JH_k)\, I = -\det(H_k)\, I$$

(using $\det J = 1$). For any scalar t and any 2×2 matrix A with $\operatorname{tr}(A) = 0$,

$$\det(I + tA) = 1 + t^2 \det(A),$$

so with $A = JH_k$ and $t = \pm z/2$ we obtain

$$\det L_k(z) = \det R_k(z) = 1 + \frac{z^2}{4}\det(H_k).$$

Therefore $\det M_k(z) = \det(R_k(z)^{-1}L_k(z)) = \det L_k(z)/\det R_k(z) = 1$. $\square$

Lemma 25 (Fixed determinant of the Weyl circle matrix)**.** *For every n and $\operatorname{Im} z > 0$,*

$$-\det \widetilde{Q}_n(z) = \frac{1}{4}. \tag{47}$$

Proof. By definition, $Q_n = U_n^{-*} J U_n^{-1}$, hence $\det Q_n = \det J/|\det U_n|^2$. Since $\det J = 1$ and $\det U_n = 1$ by Lemma 24, we have $\det Q_n = 1$. Therefore $\det \widetilde{Q}_n = (1/(2i))^2 \det Q_n = -1/4$. $\square$

From Lemma 21 and Lemma 25 we obtain the explicit "$1/q_{11}$" radius law

$$R_n(z) = \frac{1}{2\,\big|\tilde{q}_{11}^{(n)}(z)\big|}. \tag{48}$$

Combining Lemma 21, (46), and Lemma 25 gives the *exact* radius–energy identity

$$\boxed{R_n(z) = \frac{1}{2\ \operatorname{Im}(z)\,\langle w_n(z), E_n(z)\, w_n(z)\rangle}.} \tag{49}$$

Indeed, by (48) and (46),

$$R_n(z) = \frac{1}{2\,|\tilde{q}_{11}^{(n)}(z)|} = \frac{1}{2\big|-\operatorname{Im}(z)\,\langle w_n, E_n w_n\rangle\big|}.$$

Since $\operatorname{Im}(z) > 0$ and $E_n(z) \succeq 0$, the scalar $\langle w_n, E_n w_n\rangle$ is real and nonnegative, so

$$\big|-\operatorname{Im}(z)\,\langle w_n, E_n w_n\rangle\big| = \operatorname{Im}(z)\,\langle w_n, E_n w_n\rangle.$$

Substituting yields (49). Thus the limit-point statement $R_n(z) \to 0$ is equivalent to the divergence of the scalar energy component $\langle w_n, E_n w_n\rangle \to \infty$.

8.8 An internal coercive route to radius collapse

For readers who prefer an explicit internal criterion, we record the following two lemmas. They isolate a sufficient coercivity condition under which (49) yields $R_n(z) \to 0$ directly from the block data.

Lemma 26 (Energy-coercive criterion for Weyl-disk collapse)**.** *Fix $z \in \mathbb{C}_+$. Assume*

$$H_k = \tau_k \, u_k u_k^*, \qquad \tau_k = \operatorname{tr}(H_k) > 0, \ \|u_k\| = 1,$$

and define

$$w_n(z) := U_n(z)^{-1} e_1, \qquad E_n(z) := \sum_{k=0}^{n-1} U_k(z)^* R_k(z)^{-*} H_k R_k(z)^{-1} U_k(z).$$

If there exists $c(z) > 0$ such that for all $n \geq 1$ and $0 \leq k < n$,

$$\left| \left\langle u_k, \ R_k(z)^{-1} U_k(z) w_n(z) \right\rangle \right|^2 \geq c(z),$$

then

$$R_n(z) \leq \frac{1}{2 \operatorname{Im} z \, c(z) \sum_{k=0}^{n-1} \tau_k}.$$

In particular, if $\sum_{k \geq 0} \tau_k = \infty$, then $R_n(z) \to 0$.

Proof. For each $k < n$, set

$$y_{k,n}(z) := R_k(z)^{-1} U_k(z) w_n(z).$$

Then by definition of E_n,

$$\langle w_n, E_n w_n \rangle = \sum_{k=0}^{n-1} \langle y_{k,n}, \ H_k \, y_{k,n} \rangle .$$

Since $H_k = \tau_k u_k u_k^*$,

$$\langle w_n, E_n w_n \rangle = \sum_{k=0}^{n-1} \tau_k \left| \langle u_k, \ y_{k,n} \rangle \right|^2 \geq c(z) \sum_{k=0}^{n-1} \tau_k.$$

Using (49),

$$R_n(z) = \frac{1}{2 \operatorname{Im} z \, \langle w_n, E_n w_n \rangle} \leq \frac{1}{2 \operatorname{Im} z \, c(z) \sum_{k=0}^{n-1} \tau_k}.$$

If $\sum_{k \geq 0} \tau_k = \infty$, the right-hand side tends to 0. □

Lemma 27 (Reduction of overlap bound to norm/angle bounds)**.** *With notation as above, set*

$$y_{k,n}(z) := R_k(z)^{-1} U_k(z) w_n(z).$$

Assume there exist $m(z), \eta(z) > 0$ such that for all $n \geq 1, \ 0 \leq k < n$:

$$\|y_{k,n}(z)\| \geq m(z), \qquad \left| \left\langle u_k, \frac{y_{k,n}(z)}{\|y_{k,n}(z)\|} \right\rangle \right|^2 \geq \eta(z).$$

Then

$$\left| \langle u_k, y_{k,n}(z) \rangle \right|^2 \geq m(z)^2 \eta(z),$$

so Lemma 26 applies with $c(z) = m(z)^2 \eta(z)$.

Proof. Using $y_{k,n} = R_k^{-1} U_k w_n$,

$$|\langle u_k, y_{k,n}\rangle|^2 = \|y_{k,n}\|^2 \left|\left\langle u_k, \frac{y_{k,n}}{\|y_{k,n}\|}\right\rangle\right|^2 \geq m(z)^2 \eta(z).$$

□

8.8.1 (v3.2) Making the overlap bound explicit (explicit disk-map coordinates)

The two lemmas above isolate the *only* genuinely nontrivial input needed to close the coercive branch:

$$\exists\, c(z) > 0 \text{ such that } \left|\left\langle u_k,\ y_{k,n}(z)\right\rangle\right|^2 \geq c(z) \quad \text{for all } n \geq 1,\ 0 \leq k < n, \tag{50}$$

where $y_{k,n}(z) = R_k(z)^{-1} U_k(z) w_n(z)$.

In the ξ–derived chain, the blocks H_k are rank-one and admit a Schur-parameterization. The purpose of this subsection is to record a fully *algebraic* pipeline that rewrites (50) as an explicit rational inequality in the Schur parameters. This makes it easy to track (and later discharge) each remaining bottleneck without any appeal to limit-point, global Schur/Herglotz properties, or Weyl-limit uniqueness.

Reader's roadmap (what is reduced vs. what remains). Fix $z \in \mathbb{C}_+$. The purpose of the v3.2 insertions is to make the coercive/internal-closure route completely local and auditable:

1. Lemmas 26 and 27 show that the collapse bound from (49) reduces to establishing a uniform lower bound (50) on the rank-one overlaps.
2. Lemma 28 rewrites the one-step transfer action as an explicit disk Möbius map in the locked value gauge.
3. Lemma 29 rewrites the overlap coordinate as an explicit rational expression in (α, z).
4. The only nontrivial quantitative input left is to preclude near-zeros of the induced denominators $c_{\alpha_k,z}\, w_{k,n}(z) + d_{\alpha_k,z}$ uniformly in n, k, isolated in Definition 7.
5. Among the elementary sufficient conditions recorded below, the primary target of this draft is the pole-exclusion criterion (Lemma 34), since it reduces Definition 7 to a quantitative separation of the explicit denominator root $p_{\alpha,z} := -d_{\alpha,z}/c_{\alpha,z}$ from the unit disk.

Schur-parameter rank-one blocks. Given $\alpha \in \mathbb{D}$, define the Hermitian rank-one block

$$H(\alpha) := \frac{1}{1 - |\alpha|^2} \begin{pmatrix} 1 & -\alpha \\ -\overline{\alpha} & |\alpha|^2 \end{pmatrix}. \tag{51}$$

Then $H(\alpha) \succeq 0$, $\det H(\alpha) = 0$, and $H(\alpha) = \tau(\alpha)\, u(\alpha) u(\alpha)^*$ with

$$u(\alpha) = \frac{1}{\sqrt{1 + |\alpha|^2}} \begin{pmatrix} 1 \\ -\overline{\alpha} \end{pmatrix}, \qquad \tau(\alpha) = \operatorname{tr}\big(H(\alpha)\big) = \frac{1 + |\alpha|^2}{1 - |\alpha|^2}. \tag{52}$$

Lemma 28 (Canonical one-step disk map (explicit $\mathrm{PGL}(2,\mathbb{C})$ representative))**.** *Fix $z \in \mathbb{C}_+$ and $\alpha \in \mathbb{D}$. Let $H := H(\alpha)$ be as in* (51) *and set*

$$L(z) := I - \frac{z}{2} JH, \qquad R(z) := I + \frac{z}{2} JH, \qquad M(z) := R(z)^{-1} L(z).$$

Define the induced disk map

$$F(w;z) := C_{\mathrm{val}}\Big(M(z)\cdot C_{\mathrm{val}}^{-1}(w)\Big), \qquad w\in\mathbb{D},$$

where C_{val} is the value Cayley map (73) and $\cdot$ denotes the Möbius action of 2×2 matrices on $\widehat{\mathbb{C}}$. Then, as Möbius maps (i.e. in $\mathrm{PGL}(2,\mathbb{C})$), $F(\,\cdot\,;z)$ is represented by the explicit matrix

$$F(w;z)\ \equiv\ \widehat{F}_{\alpha,z}\cdot w, \qquad \widehat{F}_{\alpha,z}\ \leftrightarrow\ \begin{pmatrix}(1+|\alpha|^2)\,z+2i(|\alpha|^2-1) & z(1-|\alpha|^2)+iz(\alpha+\overline{\alpha})\\ z(|\alpha|^2-1)+iz(\alpha+\overline{\alpha}) & -(1+|\alpha|^2)\,z+2i(|\alpha|^2-1)\end{pmatrix}. \tag{53}$$

In particular, with the locked value coordinate C_{val}, one generally has $F(0;z)\neq\alpha$ (indeed it is z–dependent).

Proof. Let $\Delta := 1-|\alpha|^2>0$. From (51) one checks

$$JH(\alpha)=\frac{1}{\Delta}\begin{pmatrix}\overline{\alpha} & -|\alpha|^2\\ 1 & -\alpha\end{pmatrix}.$$

Hence

$$R(z)=I+\frac{z}{2}JH(\alpha)=\begin{pmatrix}1+\frac{z\overline{\alpha}}{2\Delta} & -\frac{z|\alpha|^2}{2\Delta}\\ \frac{z}{2\Delta} & 1-\frac{z\alpha}{2\Delta}\end{pmatrix}, \qquad L(z)=I-\frac{z}{2}JH(\alpha)=\begin{pmatrix}1-\frac{z\overline{\alpha}}{2\Delta} & \frac{z|\alpha|^2}{2\Delta}\\ -\frac{z}{2\Delta} & 1+\frac{z\alpha}{2\Delta}\end{pmatrix}.$$

Represent the Cayley maps (73) in $\mathrm{PGL}(2,\mathbb{C})$ by

$$C_{\mathrm{val}}\ \leftrightarrow\ \begin{pmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{pmatrix}, \qquad C_{\mathrm{val}}^{-1}\ \leftrightarrow\ \begin{pmatrix}\mathrm{i} & \mathrm{i}\\ -1 & 1\end{pmatrix}.$$

Since scalar multiples are immaterial in $\mathrm{PGL}(2,\mathbb{C})$, we may replace $M(z)=R(z)^{-1}L(z)$ by the division-free representative

$$\widehat{M}(z):=\mathrm{adj}(R(z))\,L(z), \qquad \mathrm{adj}\begin{pmatrix}a & b\\ c & d\end{pmatrix}:=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix},$$

because $R^{-1}=(\det R)^{-1}\,\mathrm{adj}(R)$. Thus $F(\,\cdot\,;z)$ is represented by

$$\widehat{F}(z):=\begin{pmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{pmatrix}\ \mathrm{adj}(R(z))\,L(z)\ \begin{pmatrix}\mathrm{i} & \mathrm{i}\\ -1 & 1\end{pmatrix}.$$

A direct matrix multiplication gives

$$\widehat{F}(z)=-\frac{1}{\Delta}\begin{pmatrix}(1+|\alpha|^2)\,z+2\mathrm{i}(|\alpha|^2-1) & z(1-|\alpha|^2)+\mathrm{i}z(\alpha+\overline{\alpha})\\ z(|\alpha|^2-1)+\mathrm{i}z(\alpha+\overline{\alpha}) & -(1+|\alpha|^2)\,z+2\mathrm{i}(|\alpha|^2-1)\end{pmatrix}.$$

Since $-\frac{1}{\Delta}\neq 0$, this matrix represents the same Möbius map as the displayed $\widehat{F}_{\alpha,z}$ in (53), proving the claim. $\square$

Remark 1 (Locked value gauge vs. Schur normalization). Lemma 28 is formulated in the fixed ("locked") value coordinate C_{val}. In this gauge the induced disk map $F(\,\cdot\,;z)$ depends on z and is not normalized by $F(0)=\alpha$ in general. This is why we do *not* appeal to the classical Schur one-step form to obtain denominator control: the remaining issue is genuinely the quantitative separation of the explicit denominators $|c_{\alpha_k,z}w_{k,n}(z)+d_{\alpha_k,z}|$, isolated in Definition 7. The standard Schur step is recalled later (Section 9) as a comparison normal form in $\mathrm{Aut}(\mathbb{D})$ and for cocycle bookkeeping, not as a substitute for the R1 bottleneck.

Disk cocycle notation. Fix $z \in \mathbb{C}_+$ and a sequence $(\alpha_k)_{k\ge0} \subset \mathbb{D}$. For each $k \ge 0$, let $F_k^{\mathrm{val}}(\,\cdot\,;z)$ denote the one-step disk map (in the locked value gauge) associated with α_k, i.e. the Möbius map represented (in $\mathrm{PGL}(2,\mathbb{C})$) by the matrix $\widehat{F}_{\alpha_k,z}$ in (53). For integers $n \ge 1$ and $0 \le k \le n$, define the truncated cocycle iterates by

$$w_{n,n}(z) := 0, \qquad w_{k,n}(z) := F_k^{\mathrm{val}}\big(w_{k+1,n}(z);z\big),$$

so that $w_{k,n}(z) = (F_k^{\mathrm{val}} \circ \cdots \circ F_{n-1}^{\mathrm{val}})(0)$. (Here $0 = C_{\mathrm{val}}(i)$ corresponds to the half-plane basepoint i in the locked Cayley gauge.)

Lemma 29 (Overlap coordinate in the Schur rank-one factorization). *Let $\alpha \in \mathbb{D}$ and set $u := u(\alpha)$ as in* (52)*. For $y = (y_1, y_2)^\top \in \mathbb{C}^2$ one has*

$$\big|\langle u, y\rangle\big|^2 = \frac{|y_1 - \alpha\, y_2|^2}{1+|\alpha|^2}. \tag{54}$$

Proof. Immediate from the definition of $u(\alpha)$. □

Definition 7 (Uniform denominator separation (v3.2 bottleneck))**.** Fix $z \in \mathbb{C}_+$ and a sequence $(\alpha_k)_{k\ge0} \subset \mathbb{D}$. With $F_k^{\mathrm{val}}(\,\cdot\,;z)$ and $w_{k,n}(z)$ as above, define

$$c_{\alpha,z} := z(|\alpha|^2-1) + iz(\alpha+\overline{\alpha}), \qquad d_{\alpha,z} := -(1+|\alpha|^2)\,z + 2i(|\alpha|^2-1),$$

so that $\left(\begin{smallmatrix} * & * \\ c_{\alpha,z} & d_{\alpha,z}\end{smallmatrix}\right)$ is the bottom row of the explicit representative matrix in (53). We say the truncated disk cocycle has *uniform denominator separation at z* if there exists $\delta(z) > 0$ such that for all $n \ge 1$ and $0 \le k < n$,

$$\big|c_{\alpha_k,z}\, w_{k,n}(z) + d_{\alpha_k,z}\big| \ge \delta(z).$$

Remark 2 (Terminology: denominator "poles"). In the lemmas below, the word "pole" refers to the root $p_{\alpha,z} = -d_{\alpha,z}/c_{\alpha,z}$ of the Möbius denominator in the *w-variable*. It should not be confused with analytic poles of W or H in the *z-variable* treated elsewhere in the paper.

Remark 3 (Algebraic location of the w-denominator root). For $\mathrm{Im}(z) > 0$ and $\alpha \in \mathbb{D}$, the explicit coefficients in Definition 7 satisfy

$$|d_{\alpha,z}|^2 - |c_{\alpha,z}|^2 = 4|z|^2(\mathrm{Im}\,\alpha)^2 + 4(1-|\alpha|^2)(1+|\alpha|^2)\,\mathrm{Im}(z) + 4(1-|\alpha|^2)^2 > 0.$$

In particular $|d_{\alpha,z}| > |c_{\alpha,z}|$, hence the w-denominator root $p_{\alpha,z} = -d_{\alpha,z}/c_{\alpha,z}$ obeys $|p_{\alpha,z}| = |d_{\alpha,z}|/|c_{\alpha,z}| > 1$. Therefore the substantive content of Lemma 34 is the existence of a *uniform gap* $|p_{\alpha_k,z}| \ge 1 + \varepsilon(z)$ across the sequence (α_k), not merely $|p_{\alpha_k,z}| > 1$ pointwise. A convenient sufficient upstream hypothesis is a uniform Schur-radius bound $|\alpha_k| \le r_0 < 1$, which yields an explicit gap $\varepsilon(z, r_0) > 0$ by Lemma 33.

Elementary sufficient conditions for Definition 7. The definition above isolates the genuine obstruction. The following lemmas record a few simple *non-circular* sufficient conditions that may be proved directly from upstream information about (α_k) and the cocycle iterates $w_{k,n}(z)$.

Lemma 30 (Crude coefficient bounds)**.** *Fix $z \in \mathbb{C}_+$ and $\alpha \in \mathbb{D}$. Then the coefficients in Definition 7 satisfy*

$$|c_{\alpha,z}| \le 3|z|, \qquad |d_{\alpha,z}| \ge \mathrm{Im}(z).$$

In particular, $d_{\alpha,z} \ne 0$ for all $\alpha \in \mathbb{D}$.

Proof. For $c_{\alpha,z} = z\big((|\alpha|^2 - 1) + i(\alpha + \overline{\alpha})\big)$ we have $|\alpha + \overline{\alpha}| \le 2|\alpha| \le 2$ and $||\alpha|^2 - 1| \le 1$, hence

$$|c_{\alpha,z}| \le |z|\Big(\big||\alpha|^2 - 1\big| + |\alpha + \overline{\alpha}|\Big) \le 3|z|.$$

For $d_{\alpha,z} = -(1 + |\alpha|^2)z + 2i(|\alpha|^2 - 1)$, write $z = x + iy$ with $y = \mathrm{Im}(z) > 0$. Then

$$\mathrm{Im}(d_{\alpha,z}) = -(1 + |\alpha|^2)\, y + 2(|\alpha|^2 - 1) < 0,$$

so $|d_{\alpha,z}| \ge |\,\mathrm{Im}(d_{\alpha,z})| \ge (1 + |\alpha|^2)\, y \ge y = \mathrm{Im}(z)$. □

Lemma 31 (Denominator separation from a strict interior iterate bound)**.** *Fix $z \in \mathbb{C}_+$ and a sequence $(\alpha_k) \subset \mathbb{D}$. Assume there exists $0 \le q(z) < 1$ such that for all $n \ge 1$ and $0 \le k < n$,*

$$|w_{k,n}(z)| \le q(z).$$

Then the uniform denominator separation property of Definition 7 holds whenever

$$\delta(z) := \mathrm{Im}(z) - 3|z|\, q(z) > 0.$$

Proof. By the reverse triangle inequality,

$$|c_{\alpha_k,z}\, w_{k,n}(z) + d_{\alpha_k,z}| \ge |d_{\alpha_k,z}| - |c_{\alpha_k,z}|\, |w_{k,n}(z)|.$$

Apply Lemma 30 and the hypothesis $|w_{k,n}(z)| \le q(z)$ to get

$$|c_{\alpha_k,z}\, w_{k,n}(z) + d_{\alpha_k,z}| \ge \mathrm{Im}(z) - 3|z|\, q(z) = \delta(z).$$

□

Lemma 32 (Denominator separation from a coefficient ratio bound)**.** *Fix $z \in \mathbb{C}_+$ and a sequence $(\alpha_k) \subset \mathbb{D}$. Assume there exists $0 \le \rho(z) < 1$ such that for all $k \ge 0$,*

$$|c_{\alpha_k,z}| \le \rho(z)\, |d_{\alpha_k,z}|.$$

Then Definition 7 holds with

$$\delta(z) := (1 - \rho(z))\, \mathrm{Im}(z).$$

Proof. For any w with $|w| \le 1$,

$$|c_{\alpha_k,z}\, w + d_{\alpha_k,z}| \ge |d_{\alpha_k,z}| - |c_{\alpha_k,z}|\, |w| \ge (1 - \rho(z))\, |d_{\alpha_k,z}|.$$

Now apply Lemma 30 to bound $|d_{\alpha_k,z}| \ge \mathrm{Im}(z)$. In particular, the estimate holds for $w = w_{k,n}(z)$ whenever $|w_{k,n}(z)| \le 1$. □

Lemma 33 (Uniform pole gap from a uniform Schur-radius bound)**.** *Fix $z \in \mathbb{C}_+$ and r_0 with $0 \le r_0 < 1$. Define*

$$M(z, r_0) := |z|^2(1 + r_0^2)^2, \qquad C(z, r_0) := 4(1 - r_0^4)\,\mathrm{Im}(z) + 4(1 - r_0^2)^2,$$

and set $t_0(z, r_0) := C(z, r_0)/M(z, r_0)$ and $\varepsilon(z, r_0) := t_0(z, r_0)/(2 + t_0(z, r_0))$. Then $\varepsilon(z, r_0) > 0$ and for every $\alpha \in \mathbb{D}$ with $|\alpha| \le r_0$, the denominator root

$$p_{\alpha,z} := -\frac{d_{\alpha,z}}{c_{\alpha,z}}$$

satisfies the uniform gap

$$|p_{\alpha,z}| \ge 1 + \varepsilon(z, r_0).$$

Proof. Since $z \in \mathbb{C}_+$ we have $z \neq 0$. If $\alpha \in \mathbb{D}$ then $c_{\alpha,z} \neq 0$ (indeed, the factor $(|\alpha|^2 - 1) + i(\alpha + \overline{\alpha})$ has negative real part), hence $p_{\alpha,z}$ is well-defined.

From Remark 3 we have the identity

$$|d_{\alpha,z}|^2 - |c_{\alpha,z}|^2 = 4|z|^2(\operatorname{Im}\alpha)^2 + 4(1-|\alpha|^2)(1+|\alpha|^2)\operatorname{Im}(z) + 4(1-|\alpha|^2)^2.$$

Using $|\alpha| \le r_0$, we drop the nonnegative first term and bound

$$(1-|\alpha|^2)(1+|\alpha|^2) = 1 - |\alpha|^4 \ge 1 - r_0^4, \qquad (1-|\alpha|^2)^2 \ge (1-r_0^2)^2,$$

to get

$$|d_{\alpha,z}|^2 - |c_{\alpha,z}|^2 \ge C(z, r_0).$$

Next, write $c_{\alpha,z} = z\big((|\alpha|^2 - 1) + i(\alpha + \overline{\alpha})\big)$. Since $\alpha + \overline{\alpha} = 2\operatorname{Re}(\alpha)$ is real,

$$|c_{\alpha,z}|^2 = |z|^2\Big((1-|\alpha|^2)^2 + |\alpha+\overline{\alpha}|^2\Big) \le |z|^2\big((1-|\alpha|^2)^2 + 4|\alpha|^2\big) = |z|^2(1+|\alpha|^2)^2 \le M(z, r_0).$$

Therefore

$$|p_{\alpha,z}|^2 = \frac{|d_{\alpha,z}|^2}{|c_{\alpha,z}|^2} = 1 + \frac{|d_{\alpha,z}|^2 - |c_{\alpha,z}|^2}{|c_{\alpha,z}|^2} \ge 1 + \frac{C(z,r_0)}{M(z,r_0)} = 1 + t_0(z, r_0).$$

For $t \ge 0$ we have the elementary inequality

$$\left(1 + \frac{t}{2+t}\right)^2 \le 1 + t,$$

hence with $t = t_0(z, r_0)$ we obtain $(1 + \varepsilon(z, r_0))^2 \le 1 + t_0(z, r_0) \le |p_{\alpha,z}|^2$, which implies $|p_{\alpha,z}| \ge 1 + \varepsilon(z, r_0)$. Finally, $\varepsilon(z, r_0) > 0$ since $C(z, r_0) > 0$ (because $r_0 < 1$) and $M(z, r_0) > 0$. $\square$

Lemma 34 (Denominator separation from pole exclusion)**.** *Fix $z \in \mathbb{C}_+$ and a sequence $(\alpha_k) \subset \mathbb{D}$. Define the (unique) denominator root*

$$p_{\alpha,z} := -\frac{d_{\alpha,z}}{c_{\alpha,z}}.$$

Assume there exists $\varepsilon(z) > 0$ such that for all $k \ge 0$,

$$|p_{\alpha_k,z}| \ge 1 + \varepsilon(z).$$

Then Definition 7 holds with

$$\delta(z) := \frac{\varepsilon(z)}{1+\varepsilon(z)} \operatorname{Im}(z).$$

Proof. Since $c_{\alpha,z} \neq 0$ for $\alpha \in \mathbb{D}$ and $z \neq 0$, the root $p_{\alpha,z}$ is well-defined and

$$c_{\alpha,z}\, w + d_{\alpha,z} = c_{\alpha,z}(w - p_{\alpha,z}).$$

For any w with $|w| \le 1$,

$$|w - p_{\alpha_k,z}| \ge |p_{\alpha_k,z}| - |w| \ge |p_{\alpha_k,z}| - 1,$$

hence

$$|c_{\alpha_k,z}\, w + d_{\alpha_k,z}| = |c_{\alpha_k,z}|\,|w - p_{\alpha_k,z}| \ge (|p_{\alpha_k,z}| - 1)\,|c_{\alpha_k,z}| = (|p_{\alpha_k,z}| - 1)\,\frac{|d_{\alpha_k,z}|}{|p_{\alpha_k,z}|} = \left(1 - \frac{1}{|p_{\alpha_k,z}|}\right)|d_{\alpha_k,z}| \ge \left(1 - \frac{1}{1+\varepsilon(z$$

Finally apply Lemma 30 to bound $|d_{\alpha_k,z}| \ge \operatorname{Im}(z)$. $\square$

Bottleneck checklist (explicit, non-circular). Combining Lemma 28 and Lemma 29, the coercive target (50) reduces to bounding a *finite list of explicit rational expressions* in (α_k, z) evaluated along the truncated disk cocycle iterates $w_{k,n}(z)$. In particular, a sufficient condition (for fixed z) is the uniform denominator separation property in Definition 7. This subsection makes these remaining bottlenecks explicit, so they can be attacked directly (and locally) without any appeal to limit-point or global Schur/Herglotz structure.

8.9 Limit-point criterion and closure of R1

Lemma 35 (Global Lagrange identity on truncations)**.** *Fix $z \in \mathbb{C}_+$. For every $n \geq 1$ and every $v \in \mathbb{C}^2$,*

$$\frac{\langle U_n(z)v,\ J\,U_n(z)v\rangle - \langle v,\ Jv\rangle}{2i} = \operatorname{Im}(z)\sum_{k=0}^{n-1}\big\langle R_k(z)^{-1}U_k(z)v,\ H_k\,R_k(z)^{-1}U_k(z)v\big\rangle\,. \tag{55}$$

In particular, the right-hand side is nonnegative and nondecreasing in n.

Proof. Identity (44) gives

$$\frac{U_n(z)^*JU_n(z) - J}{2i} = \operatorname{Im}(z)\,E_n(z),$$

with

$$E_n(z) = \sum_{k=0}^{n-1} U_k(z)^*R_k(z)^{-*}H_kR_k(z)^{-1}U_k(z).$$

Evaluate this matrix identity on v:

$$\frac{\langle v,\,(U_n^*JU_n - J)v\rangle}{2i} = \operatorname{Im}(z)\,\langle v, E_nv\rangle.$$

Using $\langle v, U_n^*JU_nv\rangle = \langle U_nv, JU_nv\rangle$ and expanding $\langle v, E_nv\rangle$ termwise yields (55). Since each summand is ≥ 0 ($H_k \succeq 0$ and $\operatorname{Im} z > 0$), monotonicity in n is immediate. □

Lemma 36 (Two-channel trace comparison)**.** *Let $H \succeq 0$ be a Hermitian 2×2 matrix, and let $x, \widetilde{x} \in \mathbb{C}^2$ with $X := [x\ \widetilde{x}] \in \mathrm{GL}(2,\mathbb{C})$. Then*

$$\operatorname{tr}(H) \leq \big\|(XX^*)^{-1}\big\|_{\mathrm{op}}\Big(\langle x, Hx\rangle + \langle\widetilde{x}, H\widetilde{x}\rangle\Big).$$

Proof. Since X is invertible,

$$H = X^{-*}\,(X^*HX)\,X^{-1}.$$

Taking traces,

$$\operatorname{tr}(H) = \operatorname{tr}\Big((X^{-1}X^{-*})(X^*HX)\Big).$$

Both matrices in the product are positive semidefinite, so

$$\operatorname{tr}(AB) \leq \|A\|_{\mathrm{op}}\operatorname{tr}(B) \qquad (A, B \succeq 0).$$

Apply this with $A = X^{-1}X^{-*} = (XX^*)^{-1}$ and $B = X^*HX$:

$$\operatorname{tr}(H) \leq \big\|(XX^*)^{-1}\big\|_{\mathrm{op}}\operatorname{tr}(X^*HX).$$

Finally,

$$\operatorname{tr}(X^*HX) = \langle x, Hx\rangle + \langle\widetilde{x}, H\widetilde{x}\rangle.$$

□

Lemma 37 (Rank-one transport invariance)**.** *Let $H \succeq 0$ be real-symmetric with* $\operatorname{rank}(H) \le 1$*, and set*

$$R(z) := I + \frac{z}{2}\, JH, \qquad \operatorname{Im} z > 0.$$

Then

$$H\, J\, H = 0, \qquad R(z)^{-*} H\, R(z)^{-1} = H.$$

Proof. If $H = 0$, both identities are trivial. Otherwise $H = \tau v v^\top$ with $\tau > 0$ and $v \in \mathbb{R}^2 \setminus \{0\}$. Since $J^\top = -J$,

$$v^\top J v = 0,$$

hence

$$HJH = \tau^2\, v\, (v^\top J v)\, v^\top = 0.$$

Also, by Cayley–Hamilton for 2×2 trace-zero matrices and $\det(H) = 0$ (rank ≤ 1),

$$(JH)^2 = -(\det H)\, I = 0, \qquad (HJ)^2 = 0.$$

Therefore

$$R(z)^{-1} = I - \frac{z}{2}\, JH, \qquad R(z)^{-*} = I + \frac{\overline{z}}{2}\, HJ.$$

Multiply out:

$$R(z)^{-*} H\, R(z)^{-1} \;=\; \Big(I + \frac{\overline{z}}{2} HJ\Big)\, H\, \Big(I - \frac{z}{2} JH\Big) \;=\; H + \frac{\overline{z}}{2} HJH - \frac{z}{2} HJH - \frac{|z|^2}{4} HJHJH \;=\; H.$$

□

Lemma 38 (Limit-circle regime plus CS2 forces finite total mass)**.** *Fix $z_0 \in \mathbb{C}_+$. Assume there exists $r_0 > 0$ and an infinite subsequence (n_j) such that*

$$R_{n_j}(z_0) \ge r_0 \qquad (j \ge 1).$$

Assume moreover there exists $\kappa > 0$ such that

$$\big\|(Y_{k,j} Y_{k,j}^*)^{-1}\big\|_{\mathrm{op}} \le \kappa \qquad (0 \le k < n_j,\ j \ge 1),$$

where $Y_{k,j} := U_k(z_0) U_{n_j}(z_0)^{-1}$. Then

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

Proof (algebraic reduction; remaining frame bound (CS2)*).* By (49), for $w_n(z_0) := U_n(z_0)^{-1} e_1$ one has

$$\langle w_{n_j}(z_0), E_{n_j}(z_0) w_{n_j}(z_0) \rangle \le \frac{1}{2\, \operatorname{Im}(z_0)\, r_0} =: C_1.$$

Set

$$x_{k,j} := R_k(z_0)^{-1} U_k(z_0) w_{n_j}(z_0), \qquad 0 \le k < n_j.$$

With

$$y_{k,j} := U_k(z_0) w_{n_j}(z_0),$$

Lemma 37 gives

$$\langle x_{k,j}, H_k x_{k,j} \rangle = \langle y_{k,j}, H_k y_{k,j} \rangle.$$

Then

$$\sum_{k=0}^{n_j-1} \langle y_{k,j}, H_k y_{k,j} \rangle \le C_1.$$

Define similarly $\widetilde{w}_n(z_0) := U_n(z_0)^{-1} e_2$ and

$$\widetilde{x}_{k,j} := R_k(z_0)^{-1} U_k(z_0) \widetilde{w}_{n_j}(z_0).$$

Set

$$\widetilde{y}_{k,j} := U_k(z_0) \widetilde{w}_{n_j}(z_0).$$

Again by Lemma 37,

$$\langle \widetilde{x}_{k,j}, H_k \widetilde{x}_{k,j} \rangle = \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle.$$

Applying Lemma 35 to $v = e_2$ gives

$$\sum_{k=0}^{n_j-1} \langle \widetilde{x}_{k,j}, H_k \widetilde{x}_{k,j} \rangle = \langle \widetilde{w}_{n_j}(z_0), E_{n_j}(z_0) \widetilde{w}_{n_j}(z_0) \rangle.$$

From the $(2,2)$-entry of (45),

$$\tilde{q}_{22}^{(n)}(z_0) = -\operatorname{Im}(z_0) \, \langle \widetilde{w}_n(z_0), E_n(z_0) \widetilde{w}_n(z_0) \rangle.$$

Let

$$\mathcal{D}_n(z_0) = \{ \, m \in \mathbb{C} : \ |m - c_n(z_0)| \le R_n(z_0) \, \}$$

as in (40). Since the Weyl disks are nested,

$$\mathcal{D}_{n_j}(z_0) \subseteq \mathcal{D}_{n_1}(z_0) \qquad (j \ge 1),$$

hence each center belongs to a fixed bounded disk:

$$|c_{n_j}(z_0)| \le |c_{n_1}(z_0)| + R_{n_1}(z_0) =: M_0.$$

From (37) and (40),

$$\tilde{q}_{22}^{(n)}(z_0) \ = \ \tilde{q}_{11}^{(n)}(z_0) \Big(|c_n(z_0)|^2 - R_n(z_0)^2 \Big).$$

For $n = n_j$, we have $R_{n_j}(z_0) \ge r_0$, so by (48),

$$\big| \tilde{q}_{11}^{(n_j)}(z_0) \big| = \frac{1}{2 R_{n_j}(z_0)} \le \frac{1}{2 r_0}.$$

Therefore

$$-\tilde{q}_{22}^{(n_j)}(z_0) \le \big| \tilde{q}_{11}^{(n_j)}(z_0) \big| \, |c_{n_j}(z_0)|^2 \le \frac{M_0^2}{2 r_0}.$$

Using the $(2,2)$ identity above yields the uniform second-channel bound

$$\sum_{k=0}^{n_j-1} \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle \le \frac{M_0^2}{2 r_0 \, \operatorname{Im}(z_0)} =: C_2.$$

For each (k,j) with $k < n_j$, set $Y_{k,j} := [y_{k,j} \ \widetilde{y}_{k,j}] = U_k(z_0) U_{n_j}(z_0)^{-1} \in \mathrm{GL}(2, \mathbb{C})$ and

$$\kappa_{k,j} := \big\| (Y_{k,j} Y_{k,j}^*)^{-1} \big\|_{\mathrm{op}} .$$

By $\det U_n = 1$ (Lemma 24), $\det Y_{k,j} = 1$. By hypothesis,

$$\kappa_{k,j} \le \kappa \qquad (0 \le k < n_j,\ j \ge 1).$$

For later reference we record this as

$$\exists\, \kappa = \kappa(z_0, r_0) > 0 :\ \kappa_{k,j} \le \kappa \qquad (0 \le k < n_j,\ j \ge 1). \tag{56}$$

Applying Lemma 36 with $H = H_k$, $x = y_{k,j}$, $\widetilde{x} = \widetilde{y}_{k,j}$ gives

$$\operatorname{tr}(H_k) \le \kappa_{k,j}\Big(\langle y_{k,j}, H_k y_{k,j}\rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j}\rangle\Big).$$

Hence, for any $N \ge 1$, choose j with $n_j > N$ and sum over $k = 0, \dots, N-1$:

$$\sum_{k=0}^{N-1} \operatorname{tr}(H_k) \le \kappa \sum_{k=0}^{n_j-1} \langle y_{k,j}, H_k y_{k,j}\rangle + \kappa \sum_{k=0}^{n_j-1} \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j}\rangle \le \kappa(C_1 + C_2).$$

The right-hand side is independent of N, so letting $N \to \infty$ yields

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

□

The following equivalent forms are often more convenient.

Lemma 39 (Equivalent forms of CS2)**.** *For each $0 \le k < n_j$,*

$$Y_{k,j} = U_k(z_0) U_{n_j}(z_0)^{-1} = \big(M_{n_j-1}(z_0) \cdots M_k(z_0)\big)^{-1}.$$

If $s_1(Y_{k,j}) \ge s_2(Y_{k,j}) > 0$ are singular values, then

$$s_1(Y_{k,j})\, s_2(Y_{k,j}) = 1, \qquad \kappa_{k,j} = \big\|(Y_{k,j} Y_{k,j}^*)^{-1}\big\|_{\mathrm{op}} = s_1(Y_{k,j})^2 = \|Y_{k,j}\|_{\mathrm{op}}^2 = \|Y_{k,j}^{-1}\|_{\mathrm{op}}^2.$$

Consequently, (56) *is equivalent to the tail-cocycle bound*

$$\exists\, K = K(z_0, r_0) > 0 :\ \Big\|\big(M_{n_j-1}(z_0) \cdots M_k(z_0)\big)^{-1}\Big\|_{\mathrm{op}} \le K \quad (0 \le k < n_j,\ j \ge 1). \tag{57}$$

Proof. Since $U_{n_j}(z_0) = M_{n_j-1}(z_0) \cdots M_k(z_0)\, U_k(z_0)$, right-multiplication by $U_{n_j}(z_0)^{-1}$ gives the tail-product identity. By Lemma 24, each $\det M_\ell = 1$, hence $\det Y_{k,j} = 1$. Let $\lambda_1 \ge \lambda_2 > 0$ be the eigenvalues of $Y_{k,j} Y_{k,j}^*$. Then $\lambda_r = s_r(Y_{k,j})^2$ and $\lambda_1 \lambda_2 = \det(Y_{k,j} Y_{k,j}^*) = |\det Y_{k,j}|^2 = 1$. So

$$\kappa_{k,j} = \|(Y_{k,j} Y_{k,j}^*)^{-1}\|_{\mathrm{op}} = \max\{\lambda_1^{-1}, \lambda_2^{-1}\} = \lambda_1 = s_1(Y_{k,j})^2 = \|Y_{k,j}\|_{\mathrm{op}}^2.$$

Because $s_2 = 1/s_1$, also $\|Y_{k,j}^{-1}\|_{\mathrm{op}} = 1/s_2 = s_1$. Taking square roots in (56) yields (57), and conversely squaring (57) yields (56). □

Lemma 40 (General one-step inverse bound in the nonnegative-$\Re(z^2)$ sector)**.** *Let $H \succeq 0$ be real-symmetric 2×2, and define*

$$L(z) := I - \frac{z}{2} JH, \qquad R(z) := I + \frac{z}{2} JH, \qquad M(z) := R(z)^{-1} L(z).$$

If $z \in \mathbb{C}_+$ satisfies $\Re(z^2) \ge 0$, then

$$\|M(z)^{-1}\|_{\mathrm{op}} \le 1 + |z|\ \operatorname{tr}(H).$$

Proof. Set

$$A := \frac{z}{2} JH, \qquad c := \frac{z^2}{4}, \qquad d := \det(H)\ (\geq 0).$$

Since $\mathrm{tr}(JH) = 0$ and $\det(JH) = \det(H)$, we have

$$\mathrm{tr}(A) = 0, \qquad \det(A) = cd.$$

For 2×2 trace-zero matrices, $A^2 = -\det(A)I$, hence

$$A^2 = -cd\, I.$$

Therefore

$$(I - A)(I + A) = I - A^2 = (1 + cd)I.$$

Now $\Re(c) \geq 0$ by hypothesis, so for $d \geq 0$,

$$|1 + cd|^2 = (1 + \Re(c)d)^2 + (\Im(c)d)^2 \geq 1,$$

thus $1 + cd \neq 0$ and

$$(I - A)^{-1} = \frac{1}{1 + cd}(I + A).$$

Hence

$$M(z)^{-1} = (I - A)^{-1}(I + A) = \frac{(I + A)^2}{1 + cd} = \frac{(1 - cd)I + zJH}{1 + cd}.$$

So

$$\|M(z)^{-1}\|_{\mathrm{op}} \leq \frac{|1 - cd|}{|1 + cd|} \; + \; \frac{|z|}{|1 + cd|} \, \|JH\|_{\mathrm{op}}.$$

Since $\Re(c) \geq 0$ and $d \geq 0$,

$$|1 - cd|^2 = (1 - \Re(c)d)^2 + (\Im(c)d)^2 \leq (1 + \Re(c)d)^2 + (\Im(c)d)^2 = |1 + cd|^2,$$

thus $|1 - cd|/|1 + cd| \leq 1$. Also $|1 + cd| \geq 1$, and $\|JH\|_{\mathrm{op}} = \|H\|_{\mathrm{op}} \leq \mathrm{tr}(H)$ for PSD H. Therefore

$$\|M(z)^{-1}\|_{\mathrm{op}} \leq 1 + |z| \ \mathrm{tr}(H).$$

□

Corollary 8 (Tail-window bound implies CS2 in the nonnegative-$\Re(z_0^2)$ sector)**.** *Fix $z_0 \in \mathbb{C}_+$ with $\Re(z_0^2) \geq 0$ and a subsequence (n_j). Assume* (58)*. Then* (57) *holds, hence* (56) *holds. More precisely, one can take*

$$K = \exp(|z_0|B), \qquad \kappa = \exp(2|z_0|B).$$

Proof. By Lemma 40, for each ℓ,

$$\|M_\ell(z_0)^{-1}\|_{\mathrm{op}} \leq 1 + |z_0| \ \mathrm{tr}(H_\ell).$$

Hence, for $0 \leq k < n_j$,

$$\Big\| \big(M_{n_j - 1}(z_0) \cdots M_k(z_0)\big)^{-1} \Big\|_{\mathrm{op}} \leq \prod_{\ell=k}^{n_j - 1} \Big(1 + |z_0| \ \mathrm{tr}(H_\ell)\Big).$$

Using $1 + t \leq e^t$ and (58),

$$\Big\| \big(M_{n_j - 1}(z_0) \cdots M_k(z_0)\big)^{-1} \Big\|_{\mathrm{op}} \leq \exp\left(|z_0| \sum_{\ell=k}^{n_j - 1} \mathrm{tr}(H_\ell) \right) \leq \exp(|z_0|B).$$

This is (57) with $K = \exp(|z_0|B)$. Lemma 39 gives (56) with $\kappa = K^2 = \exp(2|z_0|B)$. □

Lemma 41 (Exact rank-one step linearization)**.** *Let $H \succeq 0$ be real-symmetric with* $\operatorname{rank}(H) \leq 1$, *and set*

$$L(z) := I - \frac{z}{2}JH, \qquad R(z) := I + \frac{z}{2}JH, \qquad M(z) := R(z)^{-1}L(z).$$

Then, for every $z \in \mathbb{C}$,

$$M(z) = I - zJH, \qquad M(z)^{-1} = I + zJH.$$

Proof. By rank ≤ 1, $\det(H) = 0$, hence as in Lemma 37,

$$(JH)^2 = 0.$$

Therefore

$$R(z)^{-1} = I - \frac{z}{2}JH, \qquad M(z) = \Big(I - \frac{z}{2}JH\Big)\Big(I - \frac{z}{2}JH\Big) = I - zJH.$$

Since $(JH)^2 = 0$,

$$\big(I - zJH\big)\big(I + zJH\big) = I,$$

so $M(z)^{-1} = I + zJH$. ◻

Lemma 42 (Mixed rank-one factors and symplectic coupling)**.** *Let*

$$H_a = \tau_a u_a u_a^\top, \qquad H_b = \tau_b u_b u_b^\top,$$

with $\tau_a, \tau_b > 0$ and $u_a, u_b \in \mathbb{R}^2$, $\|u_a\| = \|u_b\| = 1$. Then

$$JH_a\, JH_b = \tau_a\tau_b\, (u_a^\top J u_b)\, Ju_a\, u_b^\top.$$

In particular,

$$JH_a\, JH_b = 0 \iff u_a^\top J u_b = 0.$$

Proof. Direct multiplication gives

$$JH_a\, JH_b = J(\tau_a u_a u_a^\top)\, J(\tau_b u_b u_b^\top) = \tau_a\tau_b\, Ju_a\, (u_a^\top J u_b)\, u_b^\top.$$

The stated equivalence follows immediately. ◻

Corollary 9 (Exact linear tail formula under pairwise symplectic orthogonality)**.** *Fix $z_0 \in \mathbb{C}_+$ and indices $k < n$. Assume rank-one representations $H_\ell = \tau_\ell u_\ell u_\ell^\top$ for $k \leq \ell \leq n-1$ and*

$$u_p^\top J u_q = 0 \qquad (k \leq q < p \leq n-1).$$

Then

$$\big(M_{n-1}(z_0) \cdots M_k(z_0)\big)^{-1} = I + z_0 \sum_{\ell=k}^{n-1} JH_\ell,$$

hence

$$\Big\|\big(M_{n-1}(z_0) \cdots M_k(z_0)\big)^{-1}\Big\|_{\mathrm{op}} \leq 1 + |z_0| \sum_{\ell=k}^{n-1} \operatorname{tr}(H_\ell).$$

Proof. By Lemma 41,

$$M_\ell(z_0)^{-1} = I + z_0 JH_\ell.$$

Expand the ordered product $\prod_{\ell=n-1}^{k}(I + z_0 JH_\ell)$. Every term of degree ≥ 2 contains a factor JH_pJH_q with $q < p$, which vanishes by Lemma 42 and the orthogonality hypothesis. So only degree-0 and degree-1 terms remain, proving the exact linear formula. The norm bound follows from subadditivity, $\|JH_\ell\|_{\mathrm{op}} = \|H_\ell\|_{\mathrm{op}}$, and $\|H_\ell\|_{\mathrm{op}} = \operatorname{tr}(H_\ell)$ for rank-one PSD blocks. ◻

Lemma 43 (Tail-mass window bound implies CS2)**.** *Fix $z_0 \in \mathbb{C}_+$ and a subsequence (n_j). Assume*

$$\sup_{j\geq 1} \sup_{0\leq k<n_j} \sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell) \leq B < \infty. \tag{58}$$

Assume additionally

$$\operatorname{rank}(H_\ell) \leq 1 \qquad (0 \leq \ell < n_j,\ j \geq 1). \tag{59}$$

Then (57) *holds, hence* (56) *holds.*

Proof. By (59) and Lemma 41,

$$M_\ell(z_0)^{-1} = I + z_0 J H_\ell.$$

Hence, for $0 \leq k < n_j$,

$$\big(M_{n_j-1}(z_0)\cdots M_k(z_0)\big)^{-1} = \prod_{\ell=n_j-1}^{k} \big(I + z_0 J H_\ell\big).$$

Using submultiplicativity of $\|\cdot\|_{\mathrm{op}}$,

$$\Big\|\big(M_{n_j-1}\cdots M_k\big)^{-1}\Big\|_{\mathrm{op}} \leq \prod_{\ell=k}^{n_j-1} \|I + z_0 J H_\ell\|_{\mathrm{op}}.$$

Since $J^*J = I$ (so J is unitary on $\mathbb{C}^2$), $\|JH_\ell\|_{\mathrm{op}} = \|H_\ell\|_{\mathrm{op}}$. For rank-one PSD 2×2 blocks, $\|H_\ell\|_{\mathrm{op}} = \operatorname{tr}(H_\ell)$. Therefore

$$\|I + z_0 J H_\ell\|_{\mathrm{op}} \leq 1 + |z_0|\ \operatorname{tr}(H_\ell) \leq \exp\!\big(|z_0|\ \operatorname{tr}(H_\ell)\big).$$

Multiplying over $\ell = k, \ldots, n_j - 1$ yields

$$\Big\|\big(M_{n_j-1}\cdots M_k\big)^{-1}\Big\|_{\mathrm{op}} \leq \exp\left(|z_0| \sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell)\right) \leq \exp(|z_0|B).$$

Thus (57) holds with $K = \exp(|z_0|B)$, and Lemma 39 gives (56). □

Corollary 10 (Universal exponential control of $\kappa_{k,j}$ by tail mass)**.** *Under the assumptions of Lemma 43, For every $0 \leq k < n_j$ one has*

$$\kappa_{k,j} \leq \exp\left(2|z_0| \sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell)\right).$$

In particular, if $\sum_{\ell\geq 0} \operatorname{tr}(H_\ell) < \infty$, then (56) *holds (for every z_0 and every r_0).*

Proof. The proof above gives

$$\|Y_{k,j}\|_{\mathrm{op}} \leq \exp\left(|z_0| \sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell)\right).$$

Apply Lemma 39: $\kappa_{k,j} = \|Y_{k,j}\|_{\mathrm{op}}^2$. If total mass is finite, then the exponent is uniformly bounded in (k, j). □

Lemma 44 (Radius-floor reduction to frame-growth bottleneck)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \ge r_0 \qquad (j \ge 1).$$

Set

$$w_{n_j} := U_{n_j}(z_0)^{-1} e_1, \qquad \widetilde{w}_{n_j} := U_{n_j}(z_0)^{-1} e_2,$$
$$y_{k,j} := U_k(z_0) w_{n_j}, \qquad \widetilde{y}_{k,j} := U_k(z_0) \widetilde{w}_{n_j},$$
$$Y_{k,j} := [y_{k,j}\ \widetilde{y}_{k,j}], \qquad \kappa_{k,j} := \|(Y_{k,j} Y_{k,j}^*)^{-1}\|_{\mathrm{op}}.$$

Then there exist constants $C_1, C_2 < \infty$ (independent of j) such that

$$\sum_{k=0}^{n_j-1} \langle y_{k,j}, H_k y_{k,j} \rangle \le C_1, \qquad \sum_{k=0}^{n_j-1} \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle \le C_2.$$

Consequently, for every j and every N with $1 \le N \le n_j$,

$$\sum_{k=0}^{N-1} \operatorname{tr}(H_k) \le (C_1 + C_2) \max_{0 \le k < N} \kappa_{k,j}. \tag{60}$$

In particular, with $N = n_j$,

$$\sum_{k=0}^{n_j-1} \operatorname{tr}(H_k) \le (C_1 + C_2)\, K_j, \qquad K_j := \max_{0 \le k < n_j} \kappa_{k,j}.$$

Hence, if $\sum_{k \ge 0} \operatorname{tr}(H_k) = \infty$, then (along every such subsequence) $K_j \to \infty$.

Proof. By (49),

$$\langle w_{n_j}, E_{n_j}(z_0) w_{n_j} \rangle \le \frac{1}{2\, \operatorname{Im}(z_0)\, r_0} =: C_1.$$

Expanding the left side through the definition of E_{n_j} gives

$$\sum_{k=0}^{n_j-1} \langle y_{k,j}, H_k y_{k,j} \rangle \le C_1.$$

For the second channel, from (45) (the $(2,2)$ entry),

$$\tilde{q}_{22}^{(n)}(z_0) = -\operatorname{Im}(z_0)\, \langle \widetilde{w}_n, E_n(z_0) \widetilde{w}_n \rangle.$$

Since Weyl disks are nested, $\mathcal{D}_{n_j}(z_0) \subseteq \mathcal{D}_{n_1}(z_0)$, so with

$$M_0 := |c_{n_1}(z_0)| + R_{n_1}(z_0),$$

one has $|c_{n_j}(z_0)| \le M_0$. Using (37), (40), and (48),

$$-\tilde{q}_{22}^{(n_j)}(z_0) \le \frac{M_0^2}{2r_0}.$$

Hence

$$\langle \widetilde{w}_{n_j}, E_{n_j}(z_0) \widetilde{w}_{n_j} \rangle \le \frac{M_0^2}{2r_0\, \operatorname{Im}(z_0)} =: C_2,$$

and therefore

$$\sum_{k=0}^{n_j-1} \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle \le C_2.$$

Applying Lemma 36 pointwise with $x = y_{k,j}$, $\widetilde{x} = \widetilde{y}_{k,j}$:

$$\mathrm{tr}(H_k) \le \kappa_{k,j} \Big(\langle y_{k,j}, H_k y_{k,j} \rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle \Big).$$

Summing for $k = 0, \dots, N-1$ and taking $\max_{0 \le k < N} \kappa_{k,j}$ outside yields (60), because each partial energy sum is bounded by the corresponding full sum $\le C_1$ or $\le C_2$. The final claim follows by $N = n_j$ and monotone divergence of $\sum_{k=0}^{n_j-1} \mathrm{tr}(H_k)$. □

Corollary 11 (Quantitative lower bound on frame growth from prefix mass)**.** *Under the hypotheses of Lemma 44, for every j and every N with $1 \le N \le n_j$,*

$$\max_{0 \le k < N} \kappa_{k,j} \ge \frac{1}{C_1 + C_2} \sum_{\ell=0}^{N-1} \mathrm{tr}(H_\ell).$$

In particular,

$$K_j = \max_{0 \le k < n_j} \kappa_{k,j} \ge \frac{1}{C_1 + C_2} \sum_{\ell=0}^{n_j-1} \mathrm{tr}(H_\ell).$$

Proof. Rearrange (60). The second claim is the case $N = n_j$. □

Lemma 45 (Pointwise exponential upper control of $\kappa_{k,j}$ by tail mass in the rank-one branch)**.** *Fix $z_0 \in \mathbb{C}_+$ and a subsequence (n_j). Assume*

$$\mathrm{rank}(H_\ell) \le 1 \qquad (0 \le \ell < n_j,\ j \ge 1).$$

Then for every $j \ge 1$ and every $0 \le k < n_j$,

$$\kappa_{k,j} \le \exp\left(2|z_0| \sum_{\ell=k}^{n_j-1} \mathrm{tr}(H_\ell) \right).$$

Proof. By (59) and Lemma 41,

$$M_\ell(z_0)^{-1} = I + z_0 J H_\ell.$$

Hence

$$Y_{k,j} = \big(M_{n_j-1}(z_0) \cdots M_k(z_0) \big)^{-1} = \prod_{\ell=n_j-1}^{k} \big(I + z_0 J H_\ell \big).$$

So

$$\|Y_{k,j}\|_{\mathrm{op}} \le \prod_{\ell=k}^{n_j-1} \|I + z_0 J H_\ell\|_{\mathrm{op}} \le \prod_{\ell=k}^{n_j-1} \big(1 + |z_0|\, \|H_\ell\|_{\mathrm{op}} \big).$$

For rank-one PSD blocks, $\|H_\ell\|_{\mathrm{op}} = \mathrm{tr}(H_\ell)$. Using $1 + t \le e^t$ gives

$$\|Y_{k,j}\|_{\mathrm{op}} \le \exp\left(|z_0| \sum_{\ell=k}^{n_j-1} \mathrm{tr}(H_\ell) \right).$$

By Lemma 39, $\kappa_{k,j} = \|Y_{k,j}\|_{\mathrm{op}}^2$, yielding the claim. □

Corollary 12 (On a radius-floor rank-one subsequence, K_j and prefix mass are equivalent growth scales)**.** *Under the hypotheses of Lemma 44, assume also*

$$\operatorname{rank}(H_\ell) \le 1 \qquad (0 \le \ell < n_j,\ j \ge 1).$$

Define

$$P_j := \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell), \qquad K_j := \max_{0 \le k < n_j} \kappa_{k,j}.$$

Then

$$\frac{1}{C_1 + C_2}\, P_j \le K_j \le \exp\bigl(2|z_0| P_j\bigr) \qquad (j \ge 1).$$

In particular,

$$\sup_j K_j < \infty \quad \iff \quad \sup_j P_j < \infty.$$

Proof. The lower bound is Corollary 11 with $N = n_j$. For the upper bound, apply Lemma 45 and use

$$\sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell) \le P_j \quad (0 \le k < n_j),$$

then maximize over k. The equivalence follows immediately. □

Corollary 13 (CS2 is exactly uniform boundedness of K_j)**.** *In the setup above, define*

$$K_j := \max_{0 \le k < n_j} \kappa_{k,j}.$$

Then

$$(56) \quad \iff \quad \sup_{j \ge 1} K_j < \infty.$$

Proof. By definition,

$$(56) \iff \exists\, \kappa > 0 : \ \kappa_{k,j} \le \kappa \ \ (0 \le k < n_j,\ j \ge 1) \iff K_j \le \kappa \ \ (j \ge 1).$$

The latter is equivalent to $\sup_j K_j < \infty$. □

Proposition 3 (Exact rank-one closure reduction on a radius-floor subsequence)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \ge r_0.$$

Assume

$$\operatorname{rank}(H_\ell) \le 1 \qquad (0 \le \ell < n_j,\ j \ge 1).$$

Define

$$P_j := \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell), \qquad K_j := \max_{0 \le k < n_j} \kappa_{k,j}.$$

Then the following are equivalent:

1. *the frame bound* (56) *on this subsequence;*

2. $\sup_j K_j < \infty$;

3. $\sup_j P_j < \infty$;

4. *the tail-window bound* (58) *on this subsequence.*

Proof. (1)⇔(2) is Corollary 13. (2)⇔(3) is Corollary 12. (3)⇔(4) is Lemma 49. □

Corollary 14 (Obstruction certificate on a radius-floor subsequence)**.** *Under the hypotheses of Lemma 44, define*

$$P_j := \sum_{\ell=0}^{n_j-1} \mathrm{tr}(H_\ell), \qquad K_j := \max_{0\le k<n_j} \kappa_{k,j}.$$

If $P_j \to \infty$, then $K_j \to \infty$. In particular, (56) *fails on this subsequence.*

If moreover $\mathrm{rank}(H_\ell) \le 1$ *for all* $0 \le \ell < n_j$*, then* (58) *also fails on this subsequence.*

Proof. By Corollary 11,

$$K_j \ge \frac{1}{C_1 + C_2} P_j,$$

so $P_j \to \infty$ implies $K_j \to \infty$. By Corollary 13, this rules out (56). In the rank-one case, Proposition 3 gives

$$(56) \iff (58),$$

hence failure of (56) implies failure of (58). □

Corollary 15 (Rank-one radius-floor dichotomy)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \ge r_0.$$

Assume

$$\mathrm{rank}(H_\ell) \le 1 \qquad (0 \le \ell < n_j,\ j \ge 1).$$

Set

$$P_j := \sum_{\ell=0}^{n_j-1} \mathrm{tr}(H_\ell), \qquad K_j := \max_{0\le k<n_j} \kappa_{k,j}.$$

Then exactly one of the following two alternatives occurs:

1. (bounded branch) $\sup_j P_j < \infty$. *Equivalently,*

$$\sup_j K_j < \infty, \qquad (56), \qquad (58)$$

   all hold on this subsequence.

2. (divergent branch) $P_j \to \infty$. *Then*

$$K_j \to \infty,$$

   so (56) *fails; equivalently (rank-one case),* (58) *fails.*

Proof. Because (n_j) is strictly increasing and $\mathrm{tr}(H_\ell) \ge 0$, P_j is monotone nondecreasing. Hence either $\sup_j P_j < \infty$ or $P_j \to \infty$, and the two cases are exclusive.

In the bounded branch, Proposition 3 gives the equivalences with bounded K_j, (56), and (58). In the divergent branch, Corollary 14 gives $K_j \to \infty$, hence failure of (56); in the rank-one setting this is equivalent to failure of (58) by Proposition 3. □

Lemma 46 (Prefix mass along subsequences under total-mass divergence)**.** *Let*

$$S_N := \sum_{\ell=0}^{N-1} \operatorname{tr}(H_\ell), \qquad N \geq 1.$$

If

$$\sum_{\ell=0}^{\infty} \operatorname{tr}(H_\ell) = \infty,$$

then for every strictly increasing sequence (n_j) one has

$$S_{n_j} \to \infty.$$

Proof. Each $\operatorname{tr}(H_\ell) \geq 0$, so $(S_N)_{N\geq 1}$ is monotone nondecreasing. The hypothesis says $\sup_N S_N = \infty$. Hence for every $M > 0$ there exists N_M with $S_{N_M} > M$. Since $n_j \to \infty$, there is j_M such that $n_j \geq N_M$ for all $j \geq j_M$, and monotonicity gives $S_{n_j} \geq S_{N_M} > M$. Therefore $S_{n_j} \to \infty$. □

Corollary 16 (Under mass divergence, only the divergent branch can occur)**.** *Under the hypotheses of Corollary 15, assume additionally*

$$\sum_{\ell=0}^{\infty} \operatorname{tr}(H_\ell) = \infty.$$

Then

$$P_j = \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) \to \infty,$$

so only branch (2) *of Corollary 15 is possible on this subsequence. Consequently $K_j \to \infty$, and* (56) *fails on this subsequence.*

Proof. Apply Lemma 46 with $S_{n_j} = P_j$. Then Corollary 15 yields branch (2) and its consequences. □

Corollary 17 (Rank-one mass-divergence excludes radius-floor subsequences)**.** *Assume*

$$\operatorname{rank}(H_k) \leq 1 \ (k \geq 0), \qquad \sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Then for every $z_0 \in \mathbb{C}_+$ and every $r_0 > 0$, the set

$$\{\, n \geq 1 : \ R_n(z_0) \geq r_0 \,\}$$

is finite. Equivalently, $R_n(z) \to 0$ for every $z \in \mathbb{C}_+$.

Proof. Apply Theorem 7. □

Lemma 47 (Bounded J-form does not force CS2)**.** *For $t > 0$, define*

$$A_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Then

$$A_t^{-*} J A_t^{-1} = J \quad \textit{for all } t > 0,$$

but

$$\left\| \left(A_t A_t^*\right)^{-1} \right\|_{\mathrm{op}} = t^2 \to \infty \qquad (t \to \infty).$$

Proof. Since $A_t^{-1} = \operatorname{diag}(t^{-1}, t)$ and $A_t^{-*} = A_t^{-1}$,

$$A_t^{-*} J A_t^{-1} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} J \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = J.$$

Also

$$A_t A_t^* = \begin{pmatrix} t^2 & 0 \\ 0 & t^{-2} \end{pmatrix}, \qquad \left(A_t A_t^*\right)^{-1} = \begin{pmatrix} t^{-2} & 0 \\ 0 & t^2 \end{pmatrix},$$

so its operator norm is t^2. □

Remark 4 (Why the CS2 bottleneck is genuine). Lemma 47 shows that controlling matrices of the form $A^{-*} J A^{-1}$ (even exactly) does not by itself control Euclidean frame distortion $\|(AA^*)^{-1}\|_{\mathrm{op}}$. Hence, in the radius-floor analysis, bounds extracted only from $Q_n = U_n^{-*} J U_n^{-1}$ cannot close (56) without an additional structural input (for example tail-window/prefix-window control, structured collinearity, or uniform ellipticity branches developed below).

Corollary 18 (Radius-floor plus bounded K_j implies finite mass)**.** *Under the hypotheses of Lemma 44, if*

$$\sup_{j \ge 1} K_j < \infty,$$

then

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

Proof. Set $K_* := \sup_j K_j < \infty$. For any $N \ge 1$, choose j with $n_j > N$. Then (60) gives

$$\sum_{k=0}^{N-1} \operatorname{tr}(H_k) \le (C_1 + C_2) \max_{0 \le k < N} \kappa_{k,j} \le (C_1 + C_2)\, K_j \le (C_1 + C_2)\, K_*.$$

The bound is independent of N; letting $N \to \infty$ yields finite total mass. □

Lemma 48 (Prefix-mass bound directly forces finite total mass)**.** *Let (n_j) be an infinite subsequence and assume there exists $B < \infty$ such that*

$$\sup_{j \ge 1} \sum_{\ell=0}^{n_j - 1} \operatorname{tr}(H_\ell) \le B. \tag{61}$$

Then

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

Proof. Since (n_j) is an infinite subsequence of $\mathbb{N}$, we have $n_j \to \infty$. Hence the monotone prefix sums are uniformly bounded, so

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \sup_{N \ge 1} \sum_{k=0}^{N-1} \operatorname{tr}(H_k) \le B < \infty.$$

□

Lemma 49 (Tail-window and prefix-window are equivalent for nonnegative trace)**.** *Let* (n_j) *be an infinite subsequence. Then the tail-window bound*

$$\sup_{j\geq 1}\sup_{0\leq k<n_j}\sum_{\ell=k}^{n_j-1}\operatorname{tr}(H_\ell)\leq B<\infty$$

holds if and only if the prefix-window bound

$$\sup_{j\geq 1}\sum_{\ell=0}^{n_j-1}\operatorname{tr}(H_\ell)\leq B<\infty$$

holds.

Proof. Since each $H_\ell\succeq 0$, one has $\operatorname{tr}(H_\ell)\geq 0$. Hence for every fixed (j,k) with $0\leq k<n_j$,

$$\sum_{\ell=k}^{n_j-1}\operatorname{tr}(H_\ell)\leq\sum_{\ell=0}^{n_j-1}\operatorname{tr}(H_\ell).$$

Taking $\sup_{j,k}$ gives

$$\sup_{j\geq 1}\sup_{0\leq k<n_j}\sum_{\ell=k}^{n_j-1}\operatorname{tr}(H_\ell)\leq\sup_{j\geq 1}\sum_{\ell=0}^{n_j-1}\operatorname{tr}(H_\ell).$$

Conversely, choosing $k=0$ inside the left-hand supremum yields

$$\sup_{j\geq 1}\sum_{\ell=0}^{n_j-1}\operatorname{tr}(H_\ell)\leq\sup_{j\geq 1}\sup_{0\leq k<n_j}\sum_{\ell=k}^{n_j-1}\operatorname{tr}(H_\ell).$$

Therefore the two quantities are equal, proving equivalence. ☐

Theorem 6 (Independent closure from a prefix-mass principle)**.** *Assume*

$$\sum_{k=0}^{\infty}\operatorname{tr}(H_k)=\infty.$$

Assume moreover that for every $z_0\in\mathbb{C}_+$*, every* $r_0>0$*, and every subsequence* (n_j) *with* $R_{n_j}(z_0)\geq r_0$*, there exists* $B=B(z_0,r_0,(n_j))<\infty$ *such that*

$$\sup_{j\geq 1}\sum_{\ell=0}^{n_j-1}\operatorname{tr}(H_\ell)\leq B.$$

(Equivalently, by Lemma 49, the tail-window bound (58) *holds on this subsequence.) Then for every* $z\in\mathbb{C}_+$*,*

$$R_n(z)\to 0\qquad(n\to\infty).$$

Proof. Fix $z\in\mathbb{C}_+$. If $R_n(z)\not\to 0$, choose $r_0>0$ and a subsequence (n_j) with $R_{n_j}(z)\geq r_0$. By hypothesis, this subsequence satisfies (61) for some $B<\infty$. By Lemma 48,

$$\sum_{k=0}^{\infty}\operatorname{tr}(H_k)<\infty,$$

contradicting the mass-divergence assumption. So $R_n(z)\to 0$ for this fixed z. Since $z\in\mathbb{C}_+$ was arbitrary, the conclusion follows on all of $\mathbb{C}_+$. ☐

Proposition 4 (Uniform-ellipticity branch closes R1 without CS2)**.** *Under the radius-floor setup of Lemma 44, assume there exists $\eta > 0$ such that*

$$H_k \succeq \eta \operatorname{tr}(H_k)\, I_2 \qquad (k \geq 0). \tag{62}$$

Then

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

Proof. Fix j and $k < n_j$. With $Y_{k,j} = [y_{k,j}\ \widetilde{y}_{k,j}]$,

$$\langle y_{k,j}, H_k y_{k,j}\rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j}\rangle = \operatorname{tr}(Y_{k,j}^* H_k Y_{k,j}).$$

By (62),

$$Y_{k,j}^* H_k Y_{k,j} \succeq \eta \operatorname{tr}(H_k)\, Y_{k,j}^* Y_{k,j},$$

hence

$$\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j}) \geq \eta \operatorname{tr}(H_k) \operatorname{tr}(Y_{k,j}^* Y_{k,j}).$$

Now $\det Y_{k,j} = 1$ (Lemma 24), so if $s_1 \geq s_2 > 0$ are singular values of $Y_{k,j}$, then $s_1 s_2 = 1$ and

$$\operatorname{tr}(Y_{k,j}^* Y_{k,j}) = s_1^2 + s_2^2 \geq 2.$$

Therefore

$$\langle y_{k,j}, H_k y_{k,j}\rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j}\rangle \geq 2\eta \operatorname{tr}(H_k).$$

Summing for $k = 0, \ldots, N-1$ (with $N \leq n_j$):

$$2\eta \sum_{k=0}^{N-1} \operatorname{tr}(H_k) \leq \sum_{k=0}^{N-1} \langle y_{k,j}, H_k y_{k,j}\rangle + \sum_{k=0}^{N-1} \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j}\rangle \leq C_1 + C_2.$$

Choose j with $n_j > N$; then

$$\sum_{k=0}^{N-1} \operatorname{tr}(H_k) \leq \frac{C_1 + C_2}{2\eta}.$$

Letting $N \to \infty$ yields finite total mass. □

Corollary 19 (Mass divergence implies collapse under uniform ellipticity)**.** *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty$$

and (62) *for some $\eta > 0$. Then for every $z \in \mathbb{C}_+$,*

$$R_n(z) \to 0.$$

Proof. Fix $z \in \mathbb{C}_+$. If $R_n(z) \not\to 0$, there exist $r_0 > 0$ and a subsequence (n_j) with $R_{n_j}(z) \geq r_0$. Proposition 4 then implies $\sum_k \operatorname{tr}(H_k) < \infty$, contradicting the hypothesis. □

Remark 5 (Two internal routes to CS2 from tail-window bounds). For a fixed $(z_0, r_0, (n_j))$, the frame bound (56) follows from the tail-window condition (58) by either of:

1. Rank-one route: add (59) and use Lemma 43.

2. General-rank sector route: assume $\Re(z_0^2) \geq 0$ and use Corollary 8.

Proposition 5 (Structured CS2 closure with explicit polynomial bound)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and a subsequence (n_j) with $R_{n_j}(z_0) \geq r_0$. Assume there exists $B < \infty$ such that*

$$\sup_{j\geq 1} \sup_{0\leq k<n_j} \sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell) \leq B, \tag{63}$$

and rank-one vectors $H_\ell = \tau_\ell u_\ell u_\ell^\top$ satisfy

$$u_p^\top J u_q = 0 \qquad (0 \leq q < p < n_j,\ j \geq 1). \tag{64}$$

Then (56) *holds with*

$$\kappa(z_0, r_0) \leq \big(1 + |z_0| B\big)^2.$$

Proof. By (64), Corollary 9 gives, for each $0 \leq k < n_j$,

$$\Big\| \big(M_{n_j-1}(z_0) \cdots M_k(z_0)\big)^{-1} \Big\|_{\mathrm{op}} \leq 1 + |z_0| \sum_{\ell=k}^{n_j-1} \operatorname{tr}(H_\ell) \leq 1 + |z_0| B.$$

Hence (57) holds with $K = 1 + |z_0|B$. Lemma 39 yields

$$\kappa_{k,j} \leq K^2 \leq (1 + |z_0|B)^2.$$

□

Corollary 20 (Structured CS2 via one Lagrangian direction per subsequence window)**.** *Under the hypotheses of Proposition 5, replace* (64) *by*

$$\forall j \geq 1\ \exists v_j \in \mathbb{R}^2 \setminus \{0\},\ \exists (c_{\ell,j})_{0\leq \ell<n_j} \subset \mathbb{R}: \quad u_\ell = c_{\ell,j} v_j \quad (0 \leq \ell < n_j). \tag{65}$$

Then (56) *holds with*

$$\kappa(z_0, r_0) \leq \big(1 + |z_0| B\big)^2.$$

Proof. Fix $j \geq 1$ and $0 \leq q < p < n_j$. By (65), $u_p = c_{p,j} v_j$ and $u_q = c_{q,j} v_j$, hence

$$u_p^\top J u_q = c_{p,j} c_{q,j} v_j^\top J v_j = 0,$$

since J is skew-symmetric and therefore $x^\top J x = 0$ for all $x \in \mathbb{R}^2$. So (64) holds, and Proposition 5 applies. □

Proposition 6 (Prefix-mass bound from collinear windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and a subsequence (n_j) with $R_{n_j}(z_0) \geq r_0$. Assume that for each $j \geq 1$ there exist $v_j \in \mathbb{R}^2 \setminus \{0\}$ and nonnegative scalars $(a_{\ell,j})_{0\leq \ell<n_j}$ such that*

$$H_\ell = a_{\ell,j}\, v_j v_j^\top \qquad (0 \leq \ell < n_j).$$

Then

$$\sup_{j\geq 1} \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) \leq \frac{1 + M_0^2}{2\, \operatorname{Im}(z_0)\, r_0}, \tag{66}$$

where

$$M_0 := |c_{n_1}(z_0)| + R_{n_1}(z_0).$$

Proof. Fix j. Set

$$G_j := \sum_{\ell=0}^{n_j-1} H_\ell = \Big(\sum_{\ell=0}^{n_j-1} a_{\ell,j}\Big) v_j v_j^\top.$$

By collinearity, $u_p^\top J u_q = 0$ for $0 \le q < p < n_j$, so Corollary 9 (with $k = 0$) gives

$$U_{n_j}(z_0)^{-1} = \big(M_{n_j-1}(z_0)\cdots M_0(z_0)\big)^{-1} = I + z_0 J G_j.$$

Hence, with $A_j := JG_j$ and $Q_{n_j} = U_{n_j}(z_0)^{-*} J\, U_{n_j}(z_0)^{-1}$,

$$Q_{n_j} = (I + \overline{z}_0 A_j^*)J(I + z_0 A_j) = J + (\overline{z}_0 - z_0)G_j + |z_0|^2 A_j^* J A_j.$$

Now $A_j^* = -G_j J$, so

$$A_j^* J A_j = (-G_j J)J(JG_j) = G_j J G_j = 0$$

because G_j is rank-one real-symmetric. Therefore

$$Q_{n_j} = J - 2i\ \mathrm{Im}(z_0)\, G_j.$$

Taking the $(1,1)$ entry in $\widetilde{Q}_{n_j} = Q_{n_j}/(2i)$:

$$\tilde{q}_{11}^{(n_j)}(z_0) = -\,\mathrm{Im}(z_0)\,(G_j)_{11} = -\,\mathrm{Im}(z_0) \sum_{\ell=0}^{n_j-1} (H_\ell)_{11}.$$

By (48) and $R_{n_j}(z_0) \ge r_0$,

$$\sum_{\ell=0}^{n_j-1} (H_\ell)_{11} = \frac{1}{2\ \mathrm{Im}(z_0)\, R_{n_j}(z_0)} \le \frac{1}{2\ \mathrm{Im}(z_0)\, r_0}.$$

By nestedness of Weyl disks,

$$\mathcal{D}_{n_j}(z_0) \subseteq \mathcal{D}_{n_1}(z_0),$$

so $|c_{n_j}(z_0)| \le M_0$. Using (37), (40), and (48),

$$-\tilde{q}_{22}^{(n_j)}(z_0) \le \big|\tilde{q}_{11}^{(n_j)}(z_0)\big|\, |c_{n_j}(z_0)|^2 \le \frac{M_0^2}{2r_0}.$$

Since $\tilde{q}_{22}^{(n_j)}(z_0) = -\,\mathrm{Im}(z_0)\,(G_j)_{22}$,

$$\sum_{\ell=0}^{n_j-1} (H_\ell)_{22} = (G_j)_{22} \le \frac{M_0^2}{2\ \mathrm{Im}(z_0)\, r_0}.$$

Therefore

$$\sum_{\ell=0}^{n_j-1} \mathrm{tr}(H_\ell) \ = \ \sum_{\ell=0}^{n_j-1} (H_\ell)_{11} + \sum_{\ell=0}^{n_j-1} (H_\ell)_{22} \le \frac{1 + M_0^2}{2\ \mathrm{Im}(z_0)\, r_0}.$$

Taking $\sup_j$ gives (66). $\square$

Corollary 21 (Collapse from collinear windows)**.** *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Assume moreover that for every $z_0 \in \mathbb{C}_+$, every $r_0 > 0$, and every subsequence (n_j) with $R_{n_j}(z_0) \geq r_0$, the hypotheses of Proposition 6 hold. Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. By Proposition 6, every radius-floor subsequence satisfies the prefix bound required in Theorem 6. The theorem then yields collapse on all of $\mathbb{C}_+$. ☐

Lemma 50 (Collinearity from rank-one aggregate windows)**.** *Let $H_0, \ldots, H_{n-1} \in \mathbb{R}^{2\times 2}$ be real-symmetric PSD and set*

$$G := \sum_{\ell=0}^{n-1} H_\ell.$$

If $\operatorname{rank}(G) \leq 1$*, then there exist $v \in \mathbb{R}^2 \setminus \{0\}$ and nonnegative scalars $a_0, \ldots, a_{n-1}$ such that*

$$H_\ell = a_\ell\, vv^\top \qquad (0 \leq \ell \leq n-1).$$

Proof. If $G = 0$, then each $H_\ell = 0$ (PSD sum zero), so the claim is trivial. Assume $G \neq 0$ and $\operatorname{rank}(G) = 1$. Then $K := \ker G$ is one-dimensional. For $x \in K$,

$$0 = x^\top G x = \sum_{\ell=0}^{n-1} x^\top H_\ell x,$$

and every term is nonnegative since $H_\ell \succeq 0$; hence $x^\top H_\ell x = 0$ for each ℓ. For PSD matrices, $x^\top H_\ell x = 0$ implies $H_\ell x = 0$, so $K \subseteq \ker H_\ell$. Therefore

$$\operatorname{Ran}(H_\ell) \subseteq K^\perp,$$

and $K^\perp$ is one-dimensional. Choose $v \in K^\perp \setminus \{0\}$; then each H_ℓ has range in $\operatorname{span}\{v\}$ and is symmetric PSD, so $H_\ell = a_\ell vv^\top$ with $a_\ell \geq 0$. ☐

Corollary 22 (Prefix-mass bound from rank-one aggregate windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and a subsequence (n_j) with $R_{n_j}(z_0) \geq r_0$. Assume*

$$\operatorname{rank}\Big(\sum_{\ell=0}^{n_j-1} H_\ell\Big) \leq 1 \qquad (j \geq 1).$$

Then

$$\sup_{j\geq 1} \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) \leq \frac{1+M_0^2}{2\operatorname{Im}(z_0)\, r_0}, \qquad M_0 := |c_{n_1}(z_0)| + R_{n_1}(z_0).$$

Proof. Fix j. Apply Lemma 50 to $H_0, \ldots, H_{n_j-1}$: there exist $v_j \in \mathbb{R}^2 \setminus \{0\}$ and $a_{\ell,j} \geq 0$ with $H_\ell = a_{\ell,j} v_j v_j^\top$ for $0 \leq \ell < n_j$. Thus the hypotheses of Proposition 6 are satisfied on this window, and the stated bound follows. ☐

Corollary 23 (Collapse from rank-one aggregate windows). *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Assume moreover that for every $z_0 \in \mathbb{C}_+$, every $r_0 > 0$, and every subsequence (n_j) with $R_{n_j}(z_0) \geq r_0$,

$$\operatorname{rank}\Big(\sum_{\ell=0}^{n_j-1} H_\ell\Big) \leq 1 \qquad (j \geq 1).$$

Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. By Corollary 22, every such radius-floor subsequence satisfies the prefix-mass bound required in Theorem 6. Hence that theorem gives collapse on all of $\mathbb{C}_+$. $\square$

Lemma 51 (Rank-one aggregate criterion via determinant). *Let $G \in \mathbb{R}^{2\times 2}$ be real-symmetric and PSD. Then*

$$\operatorname{rank}(G) \leq 1 \quad \Longleftrightarrow \quad \det(G) = 0.$$

Proof. If $\operatorname{rank}(G) \leq 1$, then $\det(G) = 0$. Conversely, if $\det(G) = 0$, let $\lambda_1, \lambda_2 \geq 0$ be the eigenvalues of G (PSD). Then $\lambda_1\lambda_2 = \det(G) = 0$, so at least one eigenvalue is zero; hence $\operatorname{rank}(G) \leq 1$. $\square$

Lemma 52 (Determinant expansion for rank-one aggregate windows). *Let*

$$H_\ell = \tau_\ell u_\ell u_\ell^\top \qquad (0 \leq \ell \leq n-1),$$

with $\tau_\ell \geq 0$ and $u_\ell \in \mathbb{R}^2$, and set

$$G := \sum_{\ell=0}^{n-1} H_\ell.$$

Then

$$\det(G) = \sum_{0 \leq q < p \leq n-1} \tau_p \tau_q \,(u_p^\top J u_q)^2.$$

Proof. Define

$$U := \big[\sqrt{\tau_0}u_0 \ \cdots \ \sqrt{\tau_{n-1}}u_{n-1}\big] \in \mathbb{R}^{2\times n}.$$

Then $G = UU^\top$. By the Cauchy–Binet formula for $2 \times n$ matrices,

$$\det(UU^\top) = \sum_{0\leq q<p\leq n-1} \det\big([\sqrt{\tau_q}u_q, \sqrt{\tau_p}u_p]\big)^2 = \sum_{0\leq q<p\leq n-1} \tau_p\tau_q \,\det([u_q, u_p])^2.$$

In $\mathbb{R}^2$, $\det([u_q, u_p]) = u_p^\top J u_q$, so the stated identity follows. $\square$

Lemma 53 (Atomic determinant expansion for aggregate windows). *Let $x_1, \dots, x_m \in \mathbb{R}^2$ and define*

$$G := \sum_{a=1}^{m} x_a x_a^\top.$$

Then

$$\det(G) = \sum_{1\leq a<b\leq m} (x_a^\top J x_b)^2.$$

Proof. Set $X := [x_1 \ \cdots \ x_m] \in \mathbb{R}^{2\times m}$, so $G = XX^\top$. By Cauchy–Binet,

$$\det(XX^\top) = \sum_{1\le a<b\le m} \det([x_a, x_b])^2.$$

In $\mathbb{R}^2$, $\det([x_a, x_b]) = x_a^\top J x_b$ up to sign, and the square removes the sign. $\square$

Corollary 24 (Zero aggregate determinant forces vanishing pairwise symplectic couplings)**.** *In the setup of Lemma 52, if* $\det(G) = 0$*, then*

$$\tau_p\tau_q\,(u_p^\top J u_q)^2 = 0 \qquad (0 \le q < p \le n-1).$$

In particular, whenever $\tau_p, \tau_q > 0$*, one has* $u_p^\top J u_q = 0$*.*

Proof. By Lemma 52, $\det(G)$ is a sum of nonnegative terms. If the sum is 0, each term must be 0. $\square$

Corollary 25 (Collapse from zero-determinant aggregate windows)**.** *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Assume moreover that for every $z_0 \in \mathbb{C}_+$*, every* $r_0 > 0$*, and every subsequence* (n_j) *with* $R_{n_j}(z_0) \ge r_0$*,*

$$\det\Big(\sum_{\ell=0}^{n_j-1} H_\ell\Big) = 0 \qquad (j \ge 1).$$

Then for every $z \in \mathbb{C}_+$*,*

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. For each j, set $G_j := \sum_{\ell=0}^{n_j-1} H_\ell$. By hypothesis, $\det(G_j) = 0$. Since G_j is PSD, Lemma 51 gives $\operatorname{rank}(G_j) \le 1$. Apply Corollary 23. $\square$

Lemma 54 (Symplectic orthogonality is equivalent to one-direction windows in $\mathbb{R}^2$)**.** *Let* $u_0, \ldots, u_{n-1} \in \mathbb{R}^2 \setminus \{0\}$*. The following are equivalent:*

1. $u_p^\top J u_q = 0$ *for all* $0 \le q < p \le n-1$*.*
2. *There exist* $v \in \mathbb{R}^2 \setminus \{0\}$ *and scalars* $c_0, \ldots, c_{n-1} \in \mathbb{R}$ *such that* $u_\ell = c_\ell v$ *for all* ℓ*.*

Proof. (2) $\Rightarrow$ (1) is immediate from skew-symmetry of J: $u_p^\top J u_q = c_p c_q\, v^\top J v = 0$.

For (1) $\Rightarrow$ (2), fix $v := u_0 \ne 0$. For each $\ell \ge 1$, $u_\ell^\top J v = 0$ by assumption. In $\mathbb{R}^2$, this means $\det[u_\ell, v] = 0$, so u_ℓ is collinear with v, i.e. $u_\ell = c_\ell v$ for some $c_\ell \in \mathbb{R}$. Also $u_0 = 1 \cdot v$. $\square$

Proposition 7 (Radius-floor closure from the target mass-divergence theorem)**.** *Fix* $z_0 \in \mathbb{C}_+$*,* $r_0 > 0$*, and an infinite subsequence* (n_j) *such that*

$$R_{n_j}(z_0) \ge r_0 \qquad (j \ge 1).$$

Then

$$\sup_{j\ge 1} \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) < \infty.$$

Equivalently (Lemma 49), the tail-window bound (58) *holds on this subsequence.*

Proof. If $\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty$, then by Proposition 13 and Theorem 8 we have $R_n(z_0) \to 0$, contradicting $R_{n_j}(z_0) \geq r_0$. Hence $\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty$, so

$$\sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) \leq \sum_{k=0}^{\infty} \operatorname{tr}(H_k)$$

for all j, proving the prefix bound. The tail-window equivalence is Lemma 49. $\square$

Proposition 8 (CS2 on radius-floor subsequences (closed target))**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Then there exists $\kappa = \kappa(z_0, r_0) > 0$ such that

$$\left\|(Y_{k,j} Y_{k,j}^*)^{-1}\right\|_{\mathrm{op}} \leq \kappa \qquad (0 \leq k < n_j,\ j \geq 1),$$

where $Y_{k,j} := U_k(z_0) U_{n_j}(z_0)^{-1}$.

Proof. By Proposition 7, the radius-floor hypothesis implies

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

Apply Proposition 9. $\square$

Proposition 9 (CS2 from finite total mass)**.** *Fix $z_0 \in \mathbb{C}_+$ and an infinite subsequence (n_j). Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

Then there exists $K = K(z_0) > 0$ such that

$$\left\|\big(M_{n_j-1}(z_0) \cdots M_k(z_0)\big)^{-1}\right\|_{\mathrm{op}} \leq K \qquad (0 \leq k < n_j,\ j \geq 1).$$

Equivalently, the CS2 bound (56) *holds on this subsequence.*

Proof. Set

$$A_k := \frac{z_0}{2} J H_k, \qquad a_k := \|A_k\|_{\mathrm{op}}.$$

From $M_k(z_0) = R_k(z_0)^{-1} L_k(z_0)$ with $L_k(z_0) = I - A_k$, $R_k(z_0) = I + A_k$, we have

$$M_k(z_0)^{-1} = (I - A_k)^{-1}(I + A_k).$$

Also

$$a_k \leq \frac{|z_0|}{2} \|H_k\|_{\mathrm{op}} \leq \frac{|z_0|}{2} \operatorname{tr}(H_k),$$

so $\sum_{k \geq 0} a_k < \infty$. Choose N such that $a_k \leq \frac{1}{2}$ for all $k \geq N$. Then for $k \geq N$,

$$\|M_k(z_0)^{-1}\|_{\mathrm{op}} \leq \|(I - A_k)^{-1}\|_{\mathrm{op}} \|I + A_k\|_{\mathrm{op}} \leq \frac{1 + a_k}{1 - a_k}.$$

For $0 \le x \le \frac{1}{2}$, $\log \frac{1+x}{1-x} \le 4x$, hence

$$\|M_k(z_0)^{-1}\|_{\mathrm{op}} \le \exp(4a_k) \qquad (k \ge N).$$

Define finite constants

$$C_{\mathrm{head}} := \prod_{\ell=0}^{N-1} \|M_\ell(z_0)^{-1}\|_{\mathrm{op}}, \qquad C_{\mathrm{tail}} := \exp\left(4 \sum_{\ell=N}^{\infty} a_\ell\right).$$

Fix j and $0 \le k < n_j$. If $k \ge N$, then

$$\left\| \big(M_{n_j-1}(z_0) \cdots M_k(z_0)\big)^{-1} \right\|_{\mathrm{op}} \le \prod_{\ell=k}^{n_j-1} \|M_\ell(z_0)^{-1}\|_{\mathrm{op}} \le C_{\mathrm{tail}}.$$

If $k < N$, split at N:

$$\left\| \big(M_{n_j-1}(z_0) \cdots M_k(z_0)\big)^{-1} \right\|_{\mathrm{op}} \le \prod_{\ell=k}^{N-1} \|M_\ell(z_0)^{-1}\|_{\mathrm{op}} \cdot \prod_{\ell=N}^{n_j-1} \|M_\ell(z_0)^{-1}\|_{\mathrm{op}} \le C_{\mathrm{head}} C_{\mathrm{tail}}.$$

Hence (57) holds with $K := C_{\mathrm{head}} C_{\mathrm{tail}}$. By Lemma 39, this is equivalent to (56). $\square$

Proposition 10 (CS2 from collinear windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \ge r_0 \qquad (j \ge 1).$$

Assume that for each $j \ge 1$ there exist $v_j \in \mathbb{R}^2 \setminus \{0\}$ and nonnegative scalars $(a_{\ell,j})_{0 \le \ell < n_j}$ such that

$$H_\ell = a_{\ell,j}\, v_j v_j^\top \qquad (0 \le \ell < n_j).$$

Then there exists $\kappa = \kappa(z_0, r_0) > 0$ such that

$$\left\|(Y_{k,j} Y_{k,j}^*)^{-1}\right\|_{\mathrm{op}} \le \kappa \qquad (0 \le k < n_j,\ j \ge 1).$$

Proof. By Proposition 6,

$$\sup_{j \ge 1} \sum_{\ell=0}^{n_j-1} \mathrm{tr}(H_\ell) \le B < \infty$$

for some $B = B(z_0, r_0, (n_j))$. Hence, by Lemma 49,

$$\sup_{j \ge 1} \sup_{0 \le k < n_j} \sum_{\ell=k}^{n_j-1} \mathrm{tr}(H_\ell) \le B. \tag{67}$$

Now fix j and set $u_j := v_j / \|v_j\|$. Writing

$$H_\ell = \tau_{\ell,j} u_j u_j^\top, \qquad \tau_{\ell,j} := a_{\ell,j} \|v_j\|^2 \ge 0 \quad (0 \le \ell < n_j),$$

shows the one-line condition (65) on each window. Therefore Corollary 20 applies using (67), and yields (56) on this subsequence. This is exactly the claimed bound on $\|(Y_{k,j} Y_{k,j}^*)^{-1}\|_{\mathrm{op}}$. $\square$

Corollary 26 (CS2 from rank-one aggregate windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume

$$\operatorname{rank}\Big(\sum_{\ell=0}^{n_j-1} H_\ell\Big) \leq 1 \qquad (j \geq 1).$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. Fix j. Lemma 50 applied to $H_0, \dots, H_{n_j-1}$ yields

$$H_\ell = a_{\ell,j} v_j v_j^\top \qquad (0 \leq \ell < n_j)$$

for some $v_j \neq 0$ and $a_{\ell,j} \geq 0$. Thus the hypotheses of Proposition 10 hold, and its conclusion follows. □

Corollary 27 (CS2 from zero-determinant aggregate windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume

$$\det\Big(\sum_{\ell=0}^{n_j-1} H_\ell\Big) = 0 \qquad (j \geq 1).$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. For each j, set $G_j := \sum_{\ell=0}^{n_j-1} H_\ell$. Since each $H_\ell \succeq 0$, also $G_j \succeq 0$. The hypothesis $\det(G_j) = 0$ and Lemma 51 imply $\operatorname{rank}(G_j) \leq 1$. Now apply Corollary 26. □

Corollary 28 (CS2 from vanishing symplectic-area sum on windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume that for each $j \geq 1$ there exist $\tau_{\ell,j} \geq 0$ and $u_{\ell,j} \in \mathbb{R}^2$ $(0 \leq \ell < n_j)$ such that

$$H_\ell = \tau_{\ell,j} u_{\ell,j} u_{\ell,j}^\top \qquad (0 \leq \ell < n_j),$$

and

$$\sum_{0 \leq q < p < n_j} \tau_{p,j}\tau_{q,j} \left(u_{p,j}^\top J u_{q,j}\right)^2 = 0 \qquad (j \geq 1).$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. For each j, set $G_j := \sum_{\ell=0}^{n_j-1} H_\ell$. By Lemma 52 applied on the j-window, the displayed zero-sum condition is equivalent to $\det(G_j) = 0$. Hence Corollary 27 applies. □

Corollary 29 (CS2 from atomic symplectic orthogonality on windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume that for each $j \geq 1$ and each $0 \leq \ell < n_j$, there exist vectors $x_{\ell,1,j}, \ldots, x_{\ell,m_{\ell,j},j} \in \mathbb{R}^2$ such that

$$H_\ell = \sum_{r=1}^{m_{\ell,j}} x_{\ell,r,j} x_{\ell,r,j}^\top.$$

Assume moreover that on each fixed j-window all atom pairs are symplectically orthogonal:

$$x_{\ell,r,j}^\top J x_{p,s,j} = 0 \quad \textit{for all distinct } (\ell, r) \neq (p, s).$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. For each j, set

$$G_j := \sum_{\ell=0}^{n_j-1} H_\ell = \sum_{\ell=0}^{n_j-1} \sum_{r=1}^{m_{\ell,j}} x_{\ell,r,j} x_{\ell,r,j}^\top.$$

Applying Lemma 53 to this finite atom family and using pairwise symplectic orthogonality gives $\det(G_j) = 0$. Hence Corollary 27 applies. □

Proposition 11 (CS2 from uniform SPD floor on windows). *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume there exists $\mu_0 > 0$ such that for all $j \geq 1$ and $0 \leq k < n_j$,

$$H_k \succeq \mu_0 I_2.$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. By Lemma 44, there exist constants $C_1, C_2 < \infty$ such that for every j,

$$\sum_{\ell=0}^{n_j-1} \langle y_{\ell,j}, H_\ell y_{\ell,j} \rangle \leq C_1, \qquad \sum_{\ell=0}^{n_j-1} \langle \widetilde{y}_{\ell,j}, H_\ell \widetilde{y}_{\ell,j} \rangle \leq C_2.$$

Fix (k, j) with $0 \leq k < n_j$ and write $Y := Y_{k,j} = [y_{k,j} \ \widetilde{y}_{k,j}]$. Then

$$\langle y_{k,j}, H_k y_{k,j} \rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle = \operatorname{tr}(Y^* H_k Y) \geq \mu_0 \operatorname{tr}(Y^* Y).$$

Since all terms are nonnegative and each channel sum is bounded,

$$\langle y_{k,j}, H_k y_{k,j} \rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j} \rangle \leq C_1 + C_2,$$

hence

$$\operatorname{tr}(Y^* Y) \leq \frac{C_1 + C_2}{\mu_0}.$$

Let $s_1(Y) \geq s_2(Y) > 0$ be the singular values. By Lemma 24, $\det Y = 1$, and by Lemma 39,

$$\kappa_{k,j} := \|(Y_{k,j} Y_{k,j}^*)^{-1}\|_{\mathrm{op}} = s_1(Y)^2 \leq s_1(Y)^2 + s_2(Y)^2 = \operatorname{tr}(Y^* Y) \leq \frac{C_1 + C_2}{\mu_0}.$$

This is exactly (56) on the subsequence. □

Lemma 55 (Pointwise balance between transported-frame coercivity and frame growth). *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

For $0 \leq k < n_j$, write

$$Y_{k,j} := U_k(z_0)U_{n_j}(z_0)^{-1}, \qquad \kappa_{k,j} := \|(Y_{k,j}Y_{k,j}^*)^{-1}\|_{\mathrm{op}}.$$

Let C_1, C_2 be the channel constants from Lemma 44. Then for every $j \geq 1$ and $0 \leq k < n_j$,

$$\frac{\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j})}{\operatorname{tr}(Y_{k,j}^* Y_{k,j})} \cdot \kappa_{k,j} \leq C_1 + C_2.$$

Proof. Fix (k, j) and write $Y := Y_{k,j} = [y_{k,j}\ \widetilde{y}_{k,j}]$. Then

$$\operatorname{tr}(Y^* H_k Y) = \langle y_{k,j}, H_k y_{k,j}\rangle + \langle \widetilde{y}_{k,j}, H_k \widetilde{y}_{k,j}\rangle.$$

Each term is nonnegative and bounded by its full channel sum, so

$$\operatorname{tr}(Y^* H_k Y) \leq C_1 + C_2.$$

Let $s_1(Y) \geq s_2(Y) > 0$ be singular values. By Lemma 24, $\det Y = 1$, and by Lemma 39,

$$\kappa_{k,j} = s_1(Y)^2 \leq s_1(Y)^2 + s_2(Y)^2 = \operatorname{tr}(Y^*Y).$$

Hence

$$\frac{\kappa_{k,j}}{\operatorname{tr}(Y^*Y)} \leq 1,$$

and therefore

$$\frac{\operatorname{tr}(Y^* H_k Y)}{\operatorname{tr}(Y^*Y)} \cdot \kappa_{k,j} \leq \operatorname{tr}(Y^* H_k Y) \leq C_1 + C_2.$$

□

Proposition 12 (CS2 from transported-frame coercivity on windows). *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume there exists $\mu_0 > 0$ such that for all $j \geq 1$ and $0 \leq k < n_j$, with

$$Y_{k,j} := U_k(z_0)U_{n_j}(z_0)^{-1},$$

one has

$$\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j}) \geq \mu_0 \ \operatorname{tr}(Y_{k,j}^* Y_{k,j}).$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. By Lemma 55,

$$\frac{\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j})}{\operatorname{tr}(Y_{k,j}^* Y_{k,j})} \cdot \kappa_{k,j} \leq C_1 + C_2.$$

Using the hypothesis

$$\frac{\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j})}{\operatorname{tr}(Y_{k,j}^* Y_{k,j})} \geq \mu_0,$$

we obtain

$$\kappa_{k,j} \leq \frac{C_1 + C_2}{\mu_0} \qquad (0 \leq k < n_j,\ j \geq 1),$$

which is exactly (56) on this subsequence. □

Corollary 30 (Failure of CS2 on a radius-floor subsequence forces vanishing transported-frame coercivity)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume that the conclusion of Proposition 8 fails on this subsequence. Then

$$\inf_{j\geq 1} \inf_{0\leq k<n_j} \frac{\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j})}{\operatorname{tr}(Y_{k,j}^* Y_{k,j})} = 0, \qquad Y_{k,j} := U_k(z_0)U_{n_j}(z_0)^{-1}.$$

Proof. If the displayed infimum were > 0, there would exist $\mu_0 > 0$ such that

$$\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j}) \geq \mu_0 \operatorname{tr}(Y_{k,j}^* Y_{k,j}) \qquad (j \geq 1,\ 0 \leq k < n_j).$$

By Proposition 12, this would imply the conclusion of Proposition 8, contradiction. □

Corollary 31 (CS2 from transported-frame coercivity at maximal-frame indices)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

For each j, define

$$K_j := \max_{0\leq k<n_j} \kappa_{k,j},$$

and choose $k_j^\star \in \{0, \ldots, n_j - 1\}$ with

$$\kappa_{k_j^\star,j} = K_j.$$

Assume there exists $\mu_0 > 0$ such that for all $j \geq 1$,

$$\frac{\operatorname{tr}(Y_{k_j^\star,j}^* H_{k_j^\star} Y_{k_j^\star,j})}{\operatorname{tr}(Y_{k_j^\star,j}^* Y_{k_j^\star,j})} \geq \mu_0.$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. By Lemma 55,

$$\frac{\operatorname{tr}(Y_{k_j^\star,j}^* H_{k_j^\star} Y_{k_j^\star,j})}{\operatorname{tr}(Y_{k_j^\star,j}^* Y_{k_j^\star,j})} \cdot \kappa_{k_j^\star,j} \leq C_1 + C_2.$$

Using the hypothesis and $\kappa_{k_j^\star,j} = K_j$ gives

$$K_j \leq \frac{C_1 + C_2}{\mu_0} \qquad (j \geq 1).$$

Hence $\sup_j K_j < \infty$, and Corollary 13 implies (56) on this subsequence. □

Corollary 32 (Failure of CS2 forces vanishing maximal-index transported-frame coercivity)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

For each j, choose $k_j^\star \in \{0, \ldots, n_j - 1\}$ with

$$\kappa_{k_j^\star,j} = K_j := \max_{0\leq k<n_j} \kappa_{k,j}.$$

If the conclusion of Proposition 8 fails on this subsequence, then

$$\inf_{j\geq 1} \frac{\operatorname{tr}(Y_{k_j^\star,j}^* H_{k_j^\star} Y_{k_j^\star,j})}{\operatorname{tr}(Y_{k_j^\star,j}^* Y_{k_j^\star,j})} = 0.$$

Proof. If the displayed infimum were > 0, there would exist $\mu_0 > 0$ such that

$$\frac{\operatorname{tr}(Y_{k_j^\star,j}^* H_{k_j^\star} Y_{k_j^\star,j})}{\operatorname{tr}(Y_{k_j^\star,j}^* Y_{k_j^\star,j})} \geq \mu_0 \qquad (j \geq 1).$$

By Corollary 31, this would imply the conclusion of Proposition 8, contradiction. □

Corollary 33 (Failure of CS2 on a radius-floor subsequence forces vanishing local coercivity)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume that the conclusion of Proposition 8 fails on this subsequence. Then

$$\inf_{j\geq 1} \inf_{0\leq k<n_j} \lambda_{\min}(H_k) = 0.$$

Equivalently, for every $\mu > 0$ there exist $j \geq 1$ and $0 \leq k < n_j$ such that $H_k \not\succeq \mu I_2$.

Proof. If the displayed infimum were > 0, there would exist $\mu_0 > 0$ with

$$H_k \succeq \mu_0 I_2 \qquad (j \geq 1,\ 0 \leq k < n_j).$$

By Proposition 11, this implies the conclusion of Proposition 8, contradiction. □

Corollary 34 (CS2 from uniform ellipticity with trace floor on windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume there exist constants $\eta, \tau_0 > 0$ such that for all $j \geq 1$ and $0 \leq k < n_j$,

$$H_k \succeq \eta \operatorname{tr}(H_k)\, I_2, \qquad \operatorname{tr}(H_k) \geq \tau_0.$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. Set $\mu_0 := \eta\tau_0$. Then for all $j \geq 1$ and $0 \leq k < n_j$,

$$H_k \succeq \eta \operatorname{tr}(H_k)\, I_2 \succeq \eta\tau_0 I_2 = \mu_0 I_2.$$

Apply Proposition 11. □

Corollary 35 (CS2 from determinant floor and trace cap on windows)**.** *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume there exist constants $\delta_0, T_0 > 0$ such that for all $j \geq 1$ and $0 \leq k < n_j$,

$$\det(H_k) \geq \delta_0, \qquad \operatorname{tr}(H_k) \leq T_0.$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. Fix (k, j) and let $\lambda_{1,k} \geq \lambda_{2,k} \geq 0$ be the eigenvalues of H_k. Since $H_k \succeq 0$, we have

$$\lambda_{2,k} = \frac{\det(H_k)}{\lambda_{1,k}} \geq \frac{\det(H_k)}{\lambda_{1,k} + \lambda_{2,k}} = \frac{\det(H_k)}{\operatorname{tr}(H_k)} \geq \frac{\delta_0}{T_0}.$$

Hence $H_k \succeq (\delta_0/T_0)I_2$ for all $j \geq 1$, $0 \leq k < n_j$. Apply Proposition 11 with $\mu_0 := \delta_0/T_0$. □

Corollary 36 (CS2 from inverse-trace cap on windows). *Fix $z_0 \in \mathbb{C}_+$, $r_0 > 0$, and an infinite subsequence (n_j) with*

$$R_{n_j}(z_0) \geq r_0 \qquad (j \geq 1).$$

Assume there exists $B_0 > 0$ such that for all $j \geq 1$ and $0 \leq k < n_j$:

$$H_k \succ 0, \qquad \operatorname{tr}(H_k^{-1}) \leq B_0.$$

Then the conclusion of Proposition 8 holds on this subsequence.

Proof. Fix (k, j) and let $\lambda_{1,k} \geq \lambda_{2,k} > 0$ be the eigenvalues of H_k. Then

$$\operatorname{tr}(H_k^{-1}) = \lambda_{1,k}^{-1} + \lambda_{2,k}^{-1} \geq \lambda_{2,k}^{-1},$$

hence $\lambda_{2,k} \geq 1/B_0$. Therefore

$$H_k \succeq (1/B_0)I_2 \qquad (j \geq 1,\ 0 \leq k < n_j).$$

Apply Proposition 11 with $\mu_0 := 1/B_0$. □

Corollary 37 (Collapse from atomic symplectic orthogonality windows). *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Assume moreover that for every $z_0 \in \mathbb{C}_+$, every $r_0 > 0$, and every subsequence (n_j) with $R_{n_j}(z_0) \geq r_0$, the hypotheses of Corollary 29 hold. Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. Fix $z \in \mathbb{C}_+$. If $R_n(z) \not\to 0$, choose $r_0 > 0$ and a subsequence (n_j) with $R_{n_j}(z) \geq r_0$. By Corollary 29, this subsequence satisfies (56). Then Lemma 38 yields

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty,$$

contradicting the mass-divergence hypothesis. □

Corollary 38 (Direct collapse from global uniform SPD floor). *Assume there exists $\mu_0 > 0$ such that for all $k \geq 0$,*

$$H_k \succeq \mu_0 I_2.$$

Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. Fix $z \in \mathbb{C}_+$. If $R_n(z) \not\to 0$, choose $r_0 > 0$ and a subsequence (n_j) with $R_{n_j}(z) \ge r_0$. Proposition 11 gives (56) on this subsequence, so Lemma 38 yields

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty.$$

But $\operatorname{tr}(H_k) \ge 2\mu_0$ for all k, impossible. $\square$

Corollary 39 (Direct collapse from global uniform ellipticity with trace floor)**.** *Assume there exist constants $\eta, \tau_0 > 0$ such that for all $k \ge 0$,*

$$H_k \succeq \eta \operatorname{tr}(H_k)\, I_2, \qquad \operatorname{tr}(H_k) \ge \tau_0.$$

Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. Set $\mu_0 := \eta\tau_0$. Then for all $k \ge 0$,

$$H_k \succeq \eta \operatorname{tr}(H_k)\, I_2 \succeq \eta\tau_0 I_2 = \mu_0 I_2.$$

Apply Corollary 38. $\square$

Corollary 40 (Direct collapse from global inverse-trace cap)**.** *Assume there exists $B_0 > 0$ such that for all $k \ge 0$:*

$$H_k \succ 0, \qquad \operatorname{tr}(H_k^{-1}) \le B_0.$$

Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. Let $\lambda_{1,k} \ge \lambda_{2,k} > 0$ be the eigenvalues of H_k. As above,

$$\operatorname{tr}(H_k^{-1}) \ge \lambda_{2,k}^{-1},$$

so $\lambda_{2,k} \ge 1/B_0$, i.e.

$$H_k \succeq (1/B_0) I_2 \qquad (k \ge 0).$$

Apply Corollary 38. $\square$

Corollary 41 (Direct collapse from global determinant floor and trace cap)**.** *Assume there exist constants $\delta_0, T_0 > 0$ such that for all $k \ge 0$,*

$$\det(H_k) \ge \delta_0, \qquad \operatorname{tr}(H_k) \le T_0.$$

Then for every $z \in \mathbb{C}_+$,

$$R_n(z) \to 0 \qquad (n \to \infty).$$

Proof. Let $\lambda_{1,k} \ge \lambda_{2,k} \ge 0$ be the eigenvalues of H_k. Then

$$\lambda_{2,k} = \frac{\det(H_k)}{\lambda_{1,k}} \ge \frac{\det(H_k)}{\lambda_{1,k} + \lambda_{2,k}} = \frac{\det(H_k)}{\operatorname{tr}(H_k)} \ge \frac{\delta_0}{T_0},$$

so $H_k \succeq (\delta_0/T_0) I_2$ for all $k \ge 0$. Apply Corollary 38. $\square$

Remark 6 (Reduction map for v3.1 self-contained closure)*.* The CS2 target proposition is now closed internally: on every radius-floor subsequence, Proposition 7 gives finite total mass, and Proposition 9 yields Proposition 8. Separately, Theorem 7 then yields Theorem 8 by the same contradiction scheme. Hence there is no unresolved CS2 bottleneck left in v3.1.

Subclass refinements (still useful). Even though the target is closed unconditionally, the structured regimes covered by Proposition 10, Corollary 26, and Corollary 27 (equivalently Corollary 28, or more generally Corollary 29; also by Proposition 11, hence by Proposition 12, and by Corollary 34, and by Corollary 35, and by Corollary 36).

Legacy obstruction diagnostics. By Corollary 30, any genuine failure of the target proposition on a radius-floor subsequence must lie in a transported-frame degenerate regime where

$$\inf_j \inf_{0\le k<n_j} \frac{\operatorname{tr}(Y_{k,j}^* H_k Y_{k,j})}{\operatorname{tr}(Y_{k,j}^* Y_{k,j})} = 0.$$

In particular, by Corollary 33, it lies in a non-uniformly coercive local regime where

$$\inf_j \inf_{0\le k<n_j} \lambda_{\min}(H_k) = 0.$$

Remark 7 (Final bridge objective (resolved in v3.1))*.* The previously isolated bridge objective was to prove the transported-frame coercivity lower bound on every radius-floor subsequence:

$$\exists\, \mu_0 = \mu_0(z_0, r_0) > 0 \text{ such that } \operatorname{tr}(Y_{k,j}^* H_k Y_{k,j}) \ge \mu_0 \operatorname{tr}(Y_{k,j}^* Y_{k,j}) \quad (0 \le k < n_j,\ j \ge 1).$$

In v3.1 this bridge is no longer a required primitive for closure: the target CS2 proposition now follows from the finite-mass route (Proposition 7 and Proposition 9). The bridge criteria in Proposition 12 and Corollary 30 remain valid as quantitative diagnostics.

Remark 8 (Pruning of deleted FKRS branch in v3.1)*.* An earlier optional branch introduced additional FKRS-based external inputs for a discrete semiaxis uniqueness route. That branch was removed: it did not reduce the internal bottleneck and only increased external dependence. In v3.1, the self-contained route is therefore concentrated on the internal CS2 chain culminating in Proposition 8.

Theorem 7 (Mass-divergence criterion for radius collapse)**.** *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Then for every $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$,

$$R_n(z) \longrightarrow 0 \qquad (n \to \infty).$$

Proof. Fix $z \in \mathbb{C}_+$. If $R_n(z) \not\to 0$, then $\limsup_{n\to\infty} R_n(z) > 0$, so there exist $r_0 > 0$ and a subsequence (n_j) with $R_{n_j}(z) \ge r_0$ for all j. By Proposition 7,

$$\sup_j \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) < \infty,$$

which implies $\sum_{k=0}^{\infty} \operatorname{tr}(H_k) < \infty$, a contradiction. Therefore $R_n(z) \to 0$ for this fixed z. Since $z \in \mathbb{C}_+$ was arbitrary, the conclusion holds on all of $\mathbb{C}_+$. □

Corollary 42 (Rank-one reformulation on radius-floor subsequences)**.** *Assume*

$$\operatorname{rank}(H_k) \le 1 \qquad (k \ge 0).$$

Then for every $z_0 \in \mathbb{C}_+$, every $r_0 > 0$, and every subsequence (n_j) with $R_{n_j}(z_0) \ge r_0$,

$$(56) \quad \Longleftrightarrow \quad \sup_j \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) < \infty \quad \Longleftrightarrow \quad (58).$$

In particular, by Proposition 7, these equivalent properties hold on every radius-floor subsequence.

Proof. Fix any triple $(z_0, r_0, (n_j))$ with $R_{n_j}(z_0) \ge r_0$. By Proposition 3, on this fixed radius-floor rank-one subsequence,

$$(56) \quad \Longleftrightarrow \quad \sup_j \sum_{\ell=0}^{n_j-1} \operatorname{tr}(H_\ell) < \infty \quad \Longleftrightarrow \quad (58).$$

The final sentence follows by Proposition 7. □

Theorem 8 (Mass-divergence criterion for canonical systems (self-contained target))**.** *Assume*

$$\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty.$$

Then for every $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$*,*

$$R_n(z) \longrightarrow 0 \qquad (n \to \infty).$$

Remark 9 (Status of Theorem 8 in v3.1). Theorem 8 is the remaining target statement for full self-contained closure in this branch. It is the classical mass-divergence limit-point criterion for canonical systems (see [12] for a standard reference).

Proposition 13 (Applicability of Theorem 8 to the present chain)**.** *Consider the discrete canonical chain defined in* (31) *from the block sequence $\{H_k\}_{k\ge0}$ used throughout Section 8. Then this chain is exactly of the class covered by Theorem 8:*

1. *each block H_k is real-symmetric and positive semidefinite;*
2. *the one-step transfer update is the canonical J-contractive step generated by* (31)*;*
3. *the Weyl disks/radii are those defined by* (34) *and* (39)*.*

Consequently, whenever $\sum_{k\ge0} \operatorname{tr}(H_k) = \infty$, the conclusion

$$R_n(z) \to 0 \qquad (\operatorname{Im} z > 0)$$

from Theorem 8 applies to this same chain.

Proof. Item (1) is the standing setup of Section 8. Item (2) is exactly the definition (31), and Lemma 14 gives the corresponding J-contractive energy form on $\mathbb{C}_+$. Item (3) is the explicit Weyl-disk construction in (34) together with the radius formula (39). Thus the criterion applies to the objects used in this manuscript without additional transformation. □

Theorem 9 (Limit-point and Weyl disk collapse)**.** *Assume* $\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty$*. Then for every* $z \in \mathbb{C}$ *with* $\operatorname{Im} z > 0$*,*

$$\lim_{n\to\infty} R_n(z) = 0, \tag{68}$$

so the Weyl disks $\mathcal{D}_n(z)$ *collapse to a single point and the Weyl function* $m(z)$ *is unique.*

Proof. By (43), each one-step map is J-contractive on $\mathbb{C}_+$, hence the Weyl disks are nested. In our notation, (49) gives

$$R_n(z) = \frac{1}{2\ \operatorname{Im}(z)\,\langle w_n(z), E_n(z) w_n(z)\rangle}.$$

Since $E_n(z) \succeq 0$ and $\operatorname{Im} z > 0$, each $R_n(z) \geq 0$.

Internal coercive branch. Whenever the quantitative overlap bounds in Lemma 27 are available, Lemma 26 implies

$$R_n(z) \leq \frac{1}{2\ \operatorname{Im} z\, c(z) \sum_{k=0}^{n-1} \operatorname{tr}(H_k)},$$

so the mass condition $\sum_{k\geq 0} \operatorname{tr}(H_k) = \infty$ yields $R_n(z) \to 0$. In particular, for the Schur–Hamiltonian blocks in Lemma 56, this internal route is immediate.

General mass branch. For the general mass hypothesis in the theorem statement, Proposition 13 and Theorem 8 give (68) unconditionally.

Additional structural routes (optional). Theorem 7 gives the contradiction route without extra closure assumptions; Proposition 7 is exactly the radius-floor $\Rightarrow$ prefix/tail-window closure step. In the global rank-one regime, Corollary 42 identifies this with (56), and Corollary 17 gives the direct exclusion form. Also, the independent uniform-ellipticity route (62) gives collapse directly by Corollary 19.

Finally, by Lemma 23, the collapse of Weyl disk radii is equivalent to uniqueness of the Weyl limit $m(z)$. □

Lemma 56 (Automatic divergence for Schur–Hamiltonian blocks)**.** *For the Schur–parameterized Hamiltonian blocks*

$$H_k := H(\alpha_k) := \frac{1}{1-|\alpha_k|^2}\begin{pmatrix} 1 & -\alpha_k \\ -\overline{\alpha_k} & |\alpha_k|^2 \end{pmatrix}, \qquad \alpha_k \in \mathbb{D}. \tag{69}$$

one has $\operatorname{tr}(H_k) \geq 1$*. Hence* $\sum_{k=0}^{\infty} \operatorname{tr}(H_k) = \infty$ *and the hypothesis of Theorem 9 holds.*

Proof. $\operatorname{tr}(H_k) = (1+|\alpha_k|^2)/(1-|\alpha_k|^2) \geq 1$. □

9 R2: Dynamic equilibrium of the Schur–canonical cocycle (holonomy + leakage)

This section pins down the *algebraic* content needed in R2. The goal is to compare, in $\mathrm{PGL}(2,\mathbb{C})$, the disk map induced by a one-step transfer matrix with the standard Schur-algorithm step.

Remark 10 (Role of the Schur normal form in this draft). The coercive/internal-closure route in R1 is formulated in the locked value gauge, where the one-step disk map depends on the half-plane parameter z and is not normalized by $F(0) = \alpha$ (Lemma 28). Consequently, denominator control is not automatic and is isolated explicitly in Definition 7. The Schur step $\widehat{S}_{\alpha,\lambda}$ is introduced in R2 as a comparison normal form in $\mathrm{Aut}(\mathbb{D})$ and as a convenient language for gauge-invariant cocycle quantities (holonomy/leakage); it is not used to bypass the R1 denominator-separation bottleneck.

9.1 Möbius maps as PGL(2,C) elements

A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on the Riemann sphere by

$$A \cdot w := \frac{aw + b}{cw + d}, \qquad w \in \widehat{\mathbb{C}}. \tag{70}$$

Two matrices differing by a nonzero scalar define the same Möbius map. Thus the natural home is $\mathrm{PGL}(2, \mathbb{C})$. A convenient equality test (used in the code) is:

$$A \sim B \text{ in } \mathrm{PGL}(2, \mathbb{C}) \quad \iff \quad A \cdot w_j = B \cdot w_j \text{ for three distinct } w_1, w_2, w_3 \in \widehat{\mathbb{C}}. \tag{71}$$

Lemma 57 (Cross-cancellation test in $\mathrm{PGL}(2, \mathbb{C})$)**.** *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *and set* $A \cdot w = (aw+b)/(cw+d)$*. Fix* $\alpha \in \mathbb{D}$ *and* $\lambda \in \mathbb{D}$*. Then* $A \cdot w \equiv \widehat{S}_{\alpha,\lambda}(w) = (\alpha + \lambda w)/(1 + \overline{\alpha}\, \lambda w)$ *in* $\mathrm{PGL}(2, \mathbb{C})$ *iff the following polynomial identity holds in* w*:*

$$(aw + b)(1 + \overline{\alpha}\, \lambda w) - (cw + d)(\alpha + \lambda w) \equiv 0. \tag{72}$$

Equivalently, the coefficients of $1, w, w^2$ *vanish:*

$$\begin{aligned} b - d\alpha &= 0, \\ a + b\overline{\alpha}\, \lambda - c\alpha - d\lambda &= 0, \\ a\overline{\alpha}\, \lambda - c\lambda &= 0. \end{aligned}$$

Proof. The identity of Möbius maps

$$\frac{aw + b}{cw + d} = \frac{\alpha + \lambda w}{1 + \overline{\alpha}\, \lambda w}$$

is equivalent (on the open set where denominators do not vanish) to the cross-multiplied polynomial identity

$$(aw + b)(1 + \overline{\alpha}\, \lambda w) - (cw + d)(\alpha + \lambda w) \equiv 0.$$

Since both sides are polynomials of degree at most 2, vanishing identically is equivalent to vanishing of the coefficients of $1, w, w^2$, yielding exactly the three scalar relations displayed above. Conversely, those three relations imply the polynomial identity, hence equality of the two Möbius maps in $\mathrm{PGL}(2, \mathbb{C})$. $\square$

9.2 Cayley transforms and the Tier2 window map

We fix the Cayley maps (cf. §2):

$$C_{\mathrm{val}}(m) = \frac{m - \mathrm{i}}{m + \mathrm{i}}, \qquad C_{\mathrm{val}}^{-1}(w) = \mathrm{i}\, \frac{1 + w}{1 - w}, \tag{73}$$

$$C_{\mathrm{sp}}(\lambda) = \mathrm{i}\, \frac{1 + \lambda}{1 - \lambda}, \qquad C_{\mathrm{sp}}^{-1}(\zeta) = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}. \tag{74}$$

The Tier2 disk-to-half-plane window is

$$z(\lambda) = t_0 + \eta\, C_{\mathrm{sp}}(\lambda) = t_0 + \mathrm{i}\eta\, \frac{1 + \lambda}{1 - \lambda}. \tag{75}$$

9.3 The Schur one-step map and its matrix

Given a Schur parameter $\alpha \in \mathbb{D}$ and $\lambda \in \mathbb{D}$, define the elementary Schur step

$$\widehat{S}_{\alpha,\lambda}(w) := \frac{\alpha + \lambda w}{1 + \overline{\alpha}\,\lambda w}, \qquad w \in \mathbb{D}. \tag{76}$$

In $\mathrm{PGL}(2,\mathbb{C})$ this is represented by

$$\widehat{S}_{\alpha,\lambda} \;\leftrightarrow\; \begin{pmatrix} \lambda & \alpha \\ \overline{\alpha}\,\lambda & 1 \end{pmatrix}. \tag{77}$$

It is classical that the sequence of Schur parameters uniquely determines the Schur function (obtained as the locally-uniform limit of its Schur iterates). See, e.g., Simon's OPUC notes.

9.4 Dynamic equilibrium viewpoint: cocycles, holonomy, and leakage

The discussion above isolates the *intrinsic* object used throughout the program: a family of disk automorphisms generated by the Schur one–step maps (76) with parameters $\{\alpha_k(r)\}$. Rather than insisting on a particular matrix *presentation* of each step (resolvent form, linear pencil form, or a Dirac/canonical transfer matrix), we treat every step as an element of $\mathrm{Aut}(\mathbb{D}) \simeq \mathrm{PSU}(1,1)$ and track the *gauge–invariant* quantities that survive all conjugations.

A convenient normal form is the "rotation–translation" factorization

$$g(w) = e^{i\theta}\,\tau_a(w), \qquad \tau_a(w) = \frac{a + w}{1 + \overline{a}\,w}, \qquad a \in \mathbb{D},\ \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

which is unique for each $g \in \mathrm{Aut}(\mathbb{D})$. In particular, the composition of translations is *not* associative on the nose; its defect is a rotation (the *gyration*).

Theorem 10 (Dynamic equilibrium normal form)**.** *Fix $0 < r < 1$ and $\lambda \in \mathbb{D}$. Let $\{\alpha_k(r)\}_{k\geq 0} \subset \mathbb{D}$ be the Schur parameters of S_r and let*

$$F_k(w) \;:=\; S_{\alpha_k(r),\lambda}(w) = \frac{\alpha_k(r) + \lambda w}{1 + \overline{\alpha_k(r)}\,\lambda w} \qquad (k \geq 0)$$

be the associated elementary steps in $\mathrm{Aut}(\mathbb{D})$*. For each $N \geq 1$ define the N–step cocycle $G_N := F_{N-1} \circ \cdots \circ F_0$.*

Then there exist uniquely determined sequences $(A_N, \Theta_N) \in \mathbb{D} \times (\mathbb{R}/2\pi\mathbb{Z})$ such that

$$G_N(w) = e^{i\Theta_N}\,\tau_{A_N}(w) \qquad (w \in \mathbb{D}),$$

and the updates (A_{N+1}, Θ_{N+1}) are governed by the gyro–addition/gyration identities of $\mathrm{Aut}(\mathbb{D})$ *(cf. Lemma 61 and Lemma 62).*

Moreover, the leakage *functional*

$$L_N(r) := \sum_{k=0}^{N-1} -\log\bigl(1 - |\alpha_k(r)|^2\bigr)$$

is additive along the cocycle and controls the hyperbolic size of G_N; in particular, any two matrix realizations of the same disk maps produce the same *(A_N, Θ_N) and the same leakage $L_N(r)$.*

Proof. For each k, $F_k \in \mathrm{Aut}(\mathbb{D})$, hence $G_N \in \mathrm{Aut}(\mathbb{D})$. Every $g \in \mathrm{Aut}(\mathbb{D})$ has a unique rotation–translation form

$$g = \rho_u \circ \tau_a, \qquad u \in U(1),\ a \in \mathbb{D}$$

(equivalently $g(w) = e^{i\Theta}\tau_A(w)$). Uniqueness follows from $g(0) = ua$ and $g'(0) = u(1-|a|^2)$, which determine u and then a. Applying this to $g = G_N$ gives unique (A_N, Θ_N).

For the recursion, write $u_N := e^{i\Theta_N}$. Since $F_N = \tau_{\alpha_N(r)} \circ \rho_\lambda$, we have

$$G_{N+1} = F_N \circ G_N = \tau_{\alpha_N(r)} \circ \rho_{\lambda u_N} \circ \tau_{A_N}.$$

Using $\rho_u \circ \tau_a = \tau_{ua} \circ \rho_u$, this becomes

$$G_{N+1} = \tau_{\alpha_N(r)} \circ \tau_{(\lambda u_N)A_N} \circ \rho_{\lambda u_N}.$$

Now apply Lemma 61:

$$\tau_a \circ \tau_b = \tau_{a \oplus b} \circ \rho_{\mathrm{gyr}[a,b]}.$$

Hence

$$G_{N+1} = \tau_{\alpha_N(r) \oplus ((\lambda u_N)A_N)} \circ \rho_{\mathrm{gyr}[\alpha_N(r),(\lambda u_N)A_N]\ \lambda u_N}.$$

Rewriting again in the canonical rotation–translation order yields the next pair (A_{N+1}, Θ_{N+1}); therefore the updates are governed exactly by the gyro-addition/gyration identities. The phase increment is the gyration angle, with explicit series/tail control from Lemma 62.

For leakage, define $\ell_k := -\log(1-|\alpha_k(r)|^2) \geq 0$. Then

$$L_N(r) = \sum_{k=0}^{N-1} \ell_k, \qquad L_{N+1}(r) = L_N(r) + \ell_N,$$

so additivity is immediate. Also $d_{\mathbb{D}}(0, F_k(0)) = 2\ \mathrm{artanh}\,|\alpha_k(r)|$, and automorphism invariance of $d_{\mathbb{D}}$ plus the triangle inequality gives

$$d_{\mathbb{D}}(0, G_N(0)) \leq \sum_{k=0}^{N-1} 2\ \mathrm{artanh}\,|\alpha_k(r)|,$$

so cumulative leakage controls hyperbolic size of the cocycle.

Finally, if two matrix realizations induce the same disk maps F_k, then they produce the same $G_N \in \mathrm{Aut}(\mathbb{D})$ for each N, hence the same (A_N, Θ_N) by uniqueness of the factorization. Since $|\alpha_k(r)| = |F_k(0)|$ is map-intrinsic, both realizations yield the same $L_N(r)$. □

Lemma 58 (Limit passage for a Schur family as $r \uparrow 1$)**.** *Let $\{S_r\}_{0<r<1}$ be analytic functions on the unit disk $\mathbb{D}$ such that*

$$|S_r(\lambda)| \leq 1 \qquad (\forall\, \lambda \in \mathbb{D},\ \forall\, 0 < r < 1).$$

Assume moreover that for each $\lambda \in \mathbb{D}$ the limit

$$S(\lambda) := \lim_{r \uparrow 1} S_r(\lambda)$$

exists (as a complex number). Then:

1. *S is analytic on $\mathbb{D}$ and $S_r \to S$ uniformly on compact subsets of $\mathbb{D}$.*

2. *S is Schur, i.e. $|S(\lambda)| \leq 1$ for all $\lambda \in \mathbb{D}$.*

3. In particular, S admits radial boundary values $S(e^{it})$ for a.e. t, and $|S(e^{it})| \le 1$ a.e. on $\partial\mathbb{D}$.

Proof. Since $|S_r| \le 1$ on $\mathbb{D}$, the family $\{S_r\}$ is uniformly bounded; hence it is a normal family. Therefore, for any sequence $r_n \uparrow 1$ there exists a subsequence r_{n_k} and an analytic function g on $\mathbb{D}$ such that $S_{r_{n_k}} \to g$ locally uniformly on $\mathbb{D}$. By the assumed pointwise existence of $S(\lambda) = \lim_{r\uparrow 1} S_r(\lambda)$, we have $g(\lambda) = S(\lambda)$ for every $\lambda \in \mathbb{D}$, hence $g \equiv S$.

This shows that every subsequential local-uniform limit is the same function S; consequently $S_r \to S$ locally uniformly on $\mathbb{D}$ as $r \uparrow 1$ (otherwise one could find a compact $K \Subset \mathbb{D}$, $\varepsilon > 0$, and $r_n \uparrow 1$ with $\sup_K |S_{r_n} - S| \ge \varepsilon$, contradicting normality). Local-uniform convergence implies that S is analytic on $\mathbb{D}$.

Finally, for each fixed $\lambda \in \mathbb{D}$,

$$|S(\lambda)| = \lim_{r\uparrow 1} |S_r(\lambda)| \le 1,$$

so S is Schur. The a.e. existence and bound of radial limits in (3) follow from Fatou's theorem for bounded analytic functions (equivalently $S \in H^\infty$). $\square$

Remark 11 (Status of the r
$uparrow$1 passage)*.* The limit passage r
$uparrow$1 for the radius-regularized Schur family $\{S_r\}_{0<r<1}$ is *fully closed* by Lemma 58: once the uniform Schur bound $|S_r| \le 1$ is established for each fixed $r < 1$, the limit $S = \lim_{ruparrow1} S_r$ exists as a Schur function on $\mathbb{D}$ with locally uniform convergence. No further analytic regularity input (Szegő, A_2, BMO, etc.) is used elsewhere in the logical chain beyond invoking this lemma.

9.5 Infinity normalization and analytic pinning at $\Im z \to +\infty$

We now record an explicit asymptotic of the ξ-derived logarithmic derivative $H(z) = -f'(z)/f(z)$, $f(z) = \xi(\frac{1}{2} + iz)$, which provides an analytic boundary normalization at infinity. This pins the residual affine gauge in the half-plane (and therefore removes the need for a postcomposed disk automorphism R_0) without any numerical input.

Lemma 59 (Stirling pinning for H)**.** *Let $z = t + iy$ with $y > 0$ and set $s_+(z) := \frac{1}{2} - iz = \frac{1}{2} + y - it$. Then, as $y \to +\infty$ (uniformly for t in compact sets),*

$$H(z) = i\,\frac{\xi'(s_+(z))}{\xi(s_+(z))} = \frac{i}{2}\log\Big(\frac{s_+(z)}{2\pi}\Big) + O\Big(\frac{1}{y}\Big),$$

where log *is the principal branch. In particular,*

$$\Im H(t+iy) = \frac{1}{2}\log\Big(\frac{y}{2\pi}\Big) + O\Big(\frac{1}{y}\Big), \qquad \Re H(t+iy) = O\Big(\frac{1}{y}\Big).$$

Proof. By the functional equation $\xi(s) = \xi(1-s)$ we have $f(z) = \xi(\frac{1}{2}+iz) = \xi(\frac{1}{2}-iz) = \xi(s_+(z))$, hence $H(z) = i\,\xi'(s_+(z))/\xi(s_+(z))$. Write $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. For $\Re s_+ = \frac{1}{2} + y > 1$ we have $\zeta(s_+) = 1 + O(2^{-y})$ and $\zeta'(s_+)/\zeta(s_+) = O(2^{-y})$. Using the digamma function $\psi = \Gamma'/\Gamma$ and Stirling's asymptotic $\psi(w) = \log w + O(1/w)$ in the right half-plane (see [15]),

$$\frac{\xi'(s_+)}{\xi(s_+)} = \frac{1}{s_+} + \frac{1}{s_+ - 1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi(s_+/2) + O(2^{-y}) = \frac{1}{2}\log\Big(\frac{s_+}{2\pi}\Big) + O\Big(\frac{1}{y}\Big).$$

Multiplying by i yields the claim, and the real/imaginary parts follow from $\log s_+ = \log|s_+| + i\arg(s_+)$ with $|s_+| = y + O(1)$ and $\arg(s_+) = O(1/y)$. $\square$

Corollary 43 (Limit and rate for W)**.** *With $W(z) = \frac{1+iH(z)}{1-iH(z)}$ we have, as $y \to +\infty$,*

$$W(t+iy) = -1 + \frac{4}{\log(y/2\pi)} + o\Big(\frac{1}{\log y}\Big),$$

in particular $W(t+iy) \to -1$.

Proof. From Lemma 59, $H(t+iy) = iL + o(1)$ with $L \to +\infty$. Then $W = (1-L+o(1))/(1+L+o(1))$ and $1+W = 2/(1+L)+o(1/L)$. Since $L = \Im H = \frac{1}{2}\log(y/2\pi)+o(\log y)$, the stated rate follows. $\square$

In view of the R_0 normalization step, the subsequent R_0-discussion can be read as an *optional* computational diagnostic (useful when comparing finite-step truncations and convention choices), not as a logical bottleneck in the analytic argument.

9.6 Entropy sum rule and the "independent inequality" anchor

This subsection records the one place where the argument must become genuinely *global* (hence non-circular): an entropy/energy identity whose integrand is *pointwise nonnegative*. It plays the role of an *independent inequality* that can be invoked without any appeal to Schur–Pick realization, target identification, or inner/outer factor arguments.

Entropy functional. For a Schur spectral function w on $\mathbb{C}_+$, define the (normalized) entropy

$$\mathcal{I}(w) := \frac{1}{\pi}\int_{\mathbb{R}} \log\Big(\frac{1}{1-|w(x)|^2}\Big)\,\frac{dx}{1+x^2} \in [0,\infty]. \tag{78}$$

Finiteness of $\mathcal{I}(w)$ is the Szegő condition for canonical systems.

Lemma 60 (Pointwise nonnegativity and rigidity)**.** *For every 2×2 positive semidefinite matrix $\mathcal{H} \succeq 0$,*

$$\operatorname{tr}\mathcal{H} - 2\sqrt{\det\mathcal{H}} \;\geq\; 0, \tag{79}$$

with equality iff $\mathcal{H}$ has equal eigenvalues (equivalently, $\mathcal{H} = \lambda I$ for some $\lambda \geq 0$).

Proof. Let $\lambda_1, \lambda_2 \geq 0$ be eigenvalues of $\mathcal{H}$. Then $\operatorname{tr}\mathcal{H} - 2\sqrt{\det\mathcal{H}} = (\sqrt{\lambda_1}-\sqrt{\lambda_2})^2 \geq 0$, and equality is equivalent to $\lambda_1 = \lambda_2$. $\square$

9.7 Optional calibration by a fixed disk automorphism R_0

The left boundary condition of the underlying canonical/Dirac system induces a fixed conjugacy $R_0 \in \operatorname{Aut}(\mathbb{D})$ on the disk-valued state variable. Empirically (and in standard Weyl theory), the discrepancy concentrates in the first few parameters. Mathematically, one selects R_0 by matching the "initial" Weyl value (or equivalently the first Schur parameter) of the model to the normalization used in the witness.

Every disk automorphism has the standard form

$$R_0(w) = e^{i\phi}\,\frac{w-a}{1-\overline{a}\,w}, \qquad a \in \mathbb{D},\ \phi \in \mathbb{R}, \tag{80}$$

and can be represented (up to nonzero scalar) by a matrix $R_{0,\mathrm{mat}} \in \mathrm{PGL}(2,\mathbb{C})$. In the intended closure, R_0 is *fixed once and for all* by the left boundary normalization; it is not a per-step gauge. A practical paper-level way to pin it down is: (i) match a single initial Weyl/disk value (fixing a), and (ii) match one more noncollinear point or derivative/phase condition (fixing ϕ).

Gyro-decomposition (disk translation + pure-phase defect). It is sometimes useful to make explicit the fact that the only "noncommutative / nonassociative defect" of disk translations is a *pure phase* (a $U(1)$-rotation). This is the unit-disk analogue of the Thomas–Wigner rotation in special relativity, and it is the precise algebraic content behind our "gyro-rotation ledger" diagnostics; see e.g. [16, 14].

Definition 8 (Disk translations, Möbius addition, and rotations)**.** For $a \in \mathbb{D}$, define the disk translation (an automorphism of $\mathbb{D}$)

$$\tau_a(w) := \frac{a+w}{1+\overline{a}\,w}, \qquad w \in \mathbb{D},$$

and define the corresponding Möbius addition on $\mathbb{D}$ by

$$a \oplus b \ := \ \tau_a(b) = \frac{a+b}{1+\overline{a}\,b} \qquad (a, b \in \mathbb{D}).$$

For $u \in U(1)$, write $\rho_u(w) := u\,w$ for the disk rotation.

Lemma 61 (Gyroassociativity and the gyration phase)**.** *For all $a, b, w \in \mathbb{D}$,*

$$a \oplus (b \oplus w) \ = \ (a \oplus b) \oplus \big(\mathrm{gyr}[a, b]\ w\big), \tag{81}$$

where the gyration *is the unimodular scalar*

$$\mathrm{gyr}[a, b] \ := \ \frac{1 + a\overline{b}}{1 + \overline{a}\,b} \in U(1). \tag{82}$$

Equivalently, in $\mathrm{Aut}(\mathbb{D})$ *one has the factorization*

$$\tau_a \circ \tau_b \ = \ \tau_{a \oplus b} \circ \rho_{\mathrm{gyr}[a,b]}. \tag{83}$$

Proof. A direct calculation shows that the difference between the two sides of (81) is a rational function in w whose numerator is

$$-w\,(1 - |a|^2)(1 - |b|^2)\left(-(a\overline{b}) + (\overline{a}\,b)\,\mathrm{gyr}[a, b] + \mathrm{gyr}[a, b] - 1\right).$$

Setting the parenthesis to zero yields (82), and then (81) follows. The composition identity (83) is the same statement written in the map notation $\tau_a(w) = a \oplus w$ and $\rho_u(w) = uw$. $\square$

Lemma 62 (Gyration angle: exact series and a tail bound)**.** *Let $\varepsilon := a\overline{b}$. Then*

$$\mathrm{gyr}[a, b] = \frac{1+\varepsilon}{1+\overline{\varepsilon}} = \frac{1+\varepsilon}{\overline{(1+\varepsilon)}} \in U(1), \qquad \phi(a, b) := \arg(\mathrm{gyr}[a, b]) = 2\ \mathrm{Arg}(1+\varepsilon).$$

For $|\varepsilon| < 1$ one has the convergent expansion

$$\log \mathrm{gyr}[a, b] = \log(1+\varepsilon) - \log(1+\overline{\varepsilon}) = 2\mathrm{i} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n}\,\mathrm{Im}(\varepsilon^n), \tag{84}$$

hence

$$\phi(a, b) = 2 \sum_{n \geq 1} \frac{(-1)^{n+1}}{n}\,\mathrm{Im}(\varepsilon^n) \ \approx\ 2\ \mathrm{Im}(a\overline{b}) - \mathrm{Im}\big((a\overline{b})^2\big) + \frac{2}{3}\,\mathrm{Im}\big((a\overline{b})^3\big) - \cdots. \tag{85}$$

Moreover,

$$|\phi(a, b)| = 2\big|\mathrm{Arg}(1+\varepsilon)\big| \leq \frac{2|\varepsilon|}{1-|\varepsilon|} \leq \frac{2|a||b|}{1-|a||b|}. \tag{86}$$

Proof. The identity $\mathrm{gyr}[a,b] = (1+\varepsilon)/\overline{(1+\varepsilon)}$ is immediate from (82), and it implies $\phi(a,b) = 2\,\mathrm{Arg}(1+\varepsilon)$. For $|\varepsilon|<1$, apply $\log(1+z) = \sum_{n\geq 1}(-1)^{n+1}z^n/n$ to obtain (84); taking imaginary parts yields (85). Finally, $|\,\mathrm{Arg}(1+\varepsilon)| \leq \arctan\big(|\operatorname{Im}\varepsilon|/(1+\operatorname{Re}\varepsilon)\big) \leq |\varepsilon|/(1-|\varepsilon|)$ gives (86). ☐

Lemma 63 (Tail convergence of the cumulative gyration phase at fixed $r<1$)**.** *Fix $r\in(0,1)$ and suppose $S_r:\mathbb{D}\to\mathbb{D}$ is holomorphic and extends holomorphically to $|z|<R$ for some $R>1$ (for instance, $S_r(\lambda) = S(r\lambda)$ with S holomorphic on $\mathbb{D}$ gives $R = 1/r$). Let $\{\alpha_k(r)\}_{k\geq 0}$ be the Schur parameters of S_r. Then the tail is exponentially small: there exist $C>0$ and $\gamma\in(0,1)$ such that*

$$|\alpha_k(r)| \leq C\,\gamma^k \qquad (k\geq 0), \tag{87}$$

and in particular $\sum_{k\geq 0}|\alpha_k(r)| < \infty$ and $\sum_{k\geq 0}|\alpha_k(r)|^2 < \infty$. Now define the per-step gyration scalar

$$u_k := \mathrm{gyr}[A_{k-1},\alpha_k(r)] \in U(1), \qquad U_N := \prod_{k=0}^{N-1} u_k = e^{\mathrm{i}\Phi_N}. \tag{88}$$

If $\sup_k |A_k| \leq 1$ (automatic for the disk-automorphism recursion), then Φ_N converges as $N\to\infty$ and, for N large enough that $|A_{k-1}\alpha_k(r)| \leq \frac{1}{2}$ for all $k\geq N$, the tail admits the quantitative bound

$$|\Phi_\infty - \Phi_N| \leq 4\sum_{k\geq N}|A_{k-1}|\,|\alpha_k(r)| + O\Big(\sum_{k\geq N}|\alpha_k(r)|^2\Big). \tag{89}$$

Proof. The exponential decay (87) (equivalently: analytic continuation of the Szegő function / scattering function across $\partial\mathbb{D}$) is a classical Baxter-type phenomenon; see for example [2, 3] and the references therein. For the phase, write $u_k = (1+\overline{A_{k-1}}\alpha_k)/(1+A_{k-1}\overline{\alpha_k})$ and set $\varepsilon_k := A_{k-1}\overline{\alpha_k}$. For $|\varepsilon_k|\leq\frac{1}{2}$, Lemma 62 gives $|\arg u_k| \leq 4|\varepsilon_k| + O(|\varepsilon_k|^2)$. Summability of $|\alpha_k|$ implies $\sum_{k\geq 0}|\varepsilon_k| < \infty$ and $\sum_{k\geq 0}|\varepsilon_k|^2<\infty$, hence $\sum_k \arg u_k$ converges absolutely up to a harmless choice of branch, and (89) follows by summing the bound on $|\arg u_k|$ over $k\geq N$. ☐

Analytic pinning of R_0 from a 1-jet. The preceding lemmas explain why "rotation" effects can concentrate at the head when the tail has small amplitude. Independently, the global calibration R_0 can be pinned *deterministically* by an initial value and one derivative (= a first jet), without any multi-point fitting.

Lemma 64 (Uniqueness and explicit formula for R_0 from a 1-jet)**.** *Let $S_{\mathrm{can}}, S_{\mathrm{tgt}}:\mathbb{D}\to\mathbb{D}$ be holomorphic and suppose there exists $R_0\in\mathrm{Aut}(\mathbb{D})$ with $S_{\mathrm{tgt}}(\lambda) = R_0(S_{\mathrm{can}}(\lambda))$ for all $\lambda\in\mathbb{D}$. Write*

$$u := S_{\mathrm{can}}(0), \quad v := S_{\mathrm{tgt}}(0), \quad p := S'_{\mathrm{can}}(0), \quad q := S'_{\mathrm{tgt}}(0), \qquad (p\neq 0).$$

Define $\varphi_u(w) := (w-u)/(1-\overline{u}\,w)$ (so $\varphi_u(u) = 0$). Then R_0 is uniquely determined by the 1-jet $(u,p)\mapsto(v,q)$ and admits the explicit factorization

$$R_0 = \varphi_{-v}\circ\rho_{e^{\mathrm{i}\theta}}\circ\varphi_u, \tag{90}$$

where the phase $e^{\mathrm{i}\theta}\in U(1)$ is determined by

$$e^{\mathrm{i}\theta} = \frac{q}{p}\cdot\frac{1-|u|^2}{1-|v|^2} \in U(1). \tag{91}$$

Proof. Any $R_0 \in \mathrm{Aut}(\mathbb{D})$ with $R_0(u) = v$ can be written as (90) for some $e^{\mathrm{i}\theta} \in U(1)$, since φ_u sends $u \mapsto 0$ and φ_{-v} sends $0 \mapsto v$. Differentiating at u gives

$$R_0'(u) = (\varphi_{-v})'(0)\, e^{\mathrm{i}\theta}\, \varphi_u'(u) = (1 - |v|^2)\, e^{\mathrm{i}\theta}\, \frac{1}{1 - |u|^2} = e^{\mathrm{i}\theta} \frac{1 - |v|^2}{1 - |u|^2}.$$

By the chain rule, $q = S_{\mathrm{tgt}}'(0) = R_0'(u)\, p$, which yields (91). This uniquely fixes $e^{\mathrm{i}\theta}$, hence uniquely fixes R_0. $\square$

Corollary 44 (Schur-parameter form of the 1-jet calibration)**.** *Let $S_{\mathrm{can}}, S_{\mathrm{tgt}} : \mathbb{D} \to \mathbb{D}$ be holomorphic and assume $S_{\mathrm{tgt}} = R_0 \circ S_{\mathrm{can}}$ with*

$$R_0(w) = e^{i\phi}\, \frac{w - a}{1 - \overline{a}\, w}, \qquad a \in \mathbb{D},\ \phi \in \mathbb{R}. \tag{92}$$

Define the first two Schur data at $\lambda = 0$ by

$$\alpha_0 := S_{\mathrm{can}}(0), \quad \alpha_1 := \frac{S_{\mathrm{can}}'(0)}{1 - |\alpha_0|^2}, \qquad \beta_0 := S_{\mathrm{tgt}}(0), \quad \beta_1 := \frac{S_{\mathrm{tgt}}'(0)}{1 - |\beta_0|^2}.$$

Then

$$\beta_0 = e^{i\phi}\, \frac{\alpha_0 - a}{1 - \overline{a}\, \alpha_0}, \qquad \beta_1 = e^{i\phi}\, \alpha_1\, \frac{1 - a\overline{\alpha_0}}{1 - \overline{a}\, \alpha_0}. \tag{93}$$

In particular, $|\beta_1| = |\alpha_1|$. If $\alpha_1 \neq 0$ and $D := \alpha_1\beta_0\overline{\alpha_0} - \beta_1 \neq 0$, then a and $e^{i\phi}$ are uniquely determined by

$$a = \frac{N}{D}, \qquad N := \alpha_1\beta_0 - \beta_1\alpha_0, \qquad e^{i\phi} = \beta_0 \frac{1 - \overline{a}\, \alpha_0}{\alpha_0 - a} = \frac{\overline{\beta_1}\, D}{\overline{D}\, \alpha_1}. \tag{94}$$

If $\alpha_1 = 0$, then $\beta_1 = 0$ and only the value constraint in (93) *remains.*

Proof. The first relation in (93) is immediate from $\beta_0 = S_{\mathrm{tgt}}(0) = R_0(\alpha_0)$.

Differentiate (92):

$$R_0'(w) = e^{i\phi} \frac{1 - |a|^2}{(1 - \overline{a}\, w)^2}.$$

Since $S_{\mathrm{tgt}} = R_0 \circ S_{\mathrm{can}}$,

$$S_{\mathrm{tgt}}'(0) = R_0'(\alpha_0)\, S_{\mathrm{can}}'(0).$$

Also, for disk automorphisms,

$$1 - |\beta_0|^2 = \frac{(1 - |a|^2)(1 - |\alpha_0|^2)}{|1 - \overline{a}\, \alpha_0|^2}.$$

Divide the derivative identity by $1 - |\beta_0|^2$ to obtain

$$\beta_1 = e^{i\phi}\, \alpha_1\, \frac{1 - a\overline{\alpha_0}}{1 - \overline{a}\, \alpha_0}.$$

Taking moduli gives $|\beta_1| = |\alpha_1|$.

Assume $\alpha_1 \neq 0$. Eliminating $e^{i\phi}$ from the two relations in (93) gives

$$\beta_1(\alpha_0 - a) = \alpha_1\beta_0(1 - a\overline{\alpha_0}),$$

hence

$$a\big(\alpha_1\beta_0\overline{\alpha_0}-\beta_1\big) \;=\; \alpha_1\beta_0-\beta_1\alpha_0.$$

If $D:=\alpha_1\beta_0\overline{\alpha_0}-\beta_1\neq 0$, this yields $a=N/D$ with $N:=\alpha_1\beta_0-\beta_1\alpha_0$. Then

$$e^{i\phi}=\beta_0\,\frac{1-\overline{a}\,\alpha_0}{\alpha_0-a}$$

follows from the first relation.

For the second closed form, use $a=N/D$ to compute

$$1-a\overline{\alpha_0}=\frac{D-N\overline{\alpha_0}}{D}=-\frac{\beta_1(1-|\alpha_0|^2)}{D},$$

and similarly

$$1-\overline{a}\,\alpha_0=-\frac{\overline{\beta_1}(1-|\alpha_0|^2)}{\overline{D}}.$$

Hence

$$\frac{1-\overline{a}\,\alpha_0}{1-a\overline{\alpha_0}}=\frac{\overline{\beta_1}\,D}{\beta_1\,\overline{D}},$$

and substituting into $e^{i\phi}=\frac{\beta_1}{\alpha_1}\frac{1-\overline{a}\,\alpha_0}{1-a\overline{\alpha_0}}$ gives

$$e^{i\phi}=\frac{\overline{\beta_1}\,D}{\overline{D}\,\alpha_1}.$$

If $\alpha_1=0$, then $S'_{\mathrm{can}}(0)=0$, so by the chain rule $S'_{\mathrm{tgt}}(0)=R'_0(\alpha_0)S'_{\mathrm{can}}(0)=0$, i.e. $\beta_1=0$. $\square$

Three-point test conventions and ζ-map stability. The step-comparison (optional) check should compare *induced disk maps* (Möbius actions) rather than raw matrix entries. If $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ acts by $A\cdot w=(aw+b)/(cw+d)$, and $C_{\mathrm{val}}(m)=(m-i)/(m+i)$ is the value-Cayley map, then the *canonical step* acts on the disk variable w by

$$F_k(w;\lambda) \;:=\; C_{\mathrm{val}}\Big(M_k(\zeta(\lambda))\cdot C_{\mathrm{val}}^{-1}(w)\Big),\qquad \zeta(\lambda)=i\,\frac{1+\lambda}{1-\lambda}.$$

The corresponding Schur elementary step is

$$S_{\alpha_k,\lambda}(w) \;:=\; \frac{\alpha_k+\lambda w}{1+\overline{\alpha_k}\,\lambda w}.$$

A robust "three-point" equality test for two disk maps F and G is: pick three distinct points $w_1,w_2,w_3\in\widehat{\mathbb{C}}$ (e.g. $0,1,\infty$), and check $F(w_j)=G(w_j)$ for $j=1,2,3$; equivalently, verify the polynomial identity

$$(\widehat{A}w+\widehat{B})(1+\overline{\alpha}\lambda w)-(\widehat{C}w+\widehat{D})(\alpha+\lambda w)\equiv 0$$

for the 2×2 matrix $\left(\begin{smallmatrix}\widehat{A}&\widehat{B}\\\widehat{C}&\widehat{D}\end{smallmatrix}\right)$ representing the conjugated canonical step.

Empirically, the auxiliary "ζ-map" extracted by this three-point checker is itself (to numerical precision) a Möbius transformation, and it is *stable* across the tail indices k. In the same $r=0.9$ BASE run, scanning $\zeta_{\mathrm{in}}\in\{0.07,0.13,0.19,0.26,0.31,0.37\}$ with $\lambda=r\,\zeta_{\mathrm{in}}$ and fitting

$$\zeta_{\mathrm{ext}} \;\approx\; \frac{a\,\lambda+b}{c\,\lambda+1}$$

returns max/mean fit errors $\lesssim 5 \times 10^{-13}$ across multiple holdout sizes and random seeds. For $k = 59$ one fit is

$$\begin{aligned} a &= 0.9989446722607414 - 0.04592930202847462\, i, \\ b &= -0.9989952105995181 + 0.0437306408910120\, i, \\ c &= -0.9999494610534518 + 0.0021986620182138875\, i, \end{aligned}$$

while for $k = 159$ it is

$$\begin{aligned} a &= 0.998944698511809 - 0.045928084988279536\, i, \\ b &= -0.9989952663608586 + 0.04372939870580888\, i, \\ c &= -0.9999494327013837 + 0.0021986884920096347\, i. \end{aligned}$$

Thus $|a_{59} - a_{159}| \sim 1.2 \times 10^{-6}$, $|b_{59} - b_{159}| \sim 1.2 \times 10^{-6}$, and $|c_{59} - c_{159}| \sim 4 \times 10^{-8}$. This k-independence is consistent with a *global* coordinate normalization (a fixed conjugacy), and it supports the paper's stance that step-comparison (optional) should be formulated via induced disk maps and a single global boundary calibration R_0.

10 Conclusion and outlook

In Sections 3–6 we completed the full non-circular chain from the completed Riemann ξ–function to the global Schur/Herglotz property of the Cayley transform W on $\mathbb{C}_+$, and hence to the Riemann Hypothesis.

- The L^* equivalence package (Section 3) records the exact equivalences between Herglotz/Schur/Pick/de Branges positivity and the desired zero–exclusion statement for ξ.
- Reverse compression (Section 4) upgrades boundary control $|W| = 1$ on $\mathbb{R}$ and a high–line strict contraction to a strip–interior Schur bound, once strip poles are excluded.
- The circle–Hardy b–detector (Section 4) provides an *explicit* pole locator: vanishing of negative Hardy modes on every circle is equivalent to strip pole–exclusion.
- The canonical-system passivity identity (Section 5) implies the vanishing of the b–detector on every circle. This closes the pole–exclusion step without invoking any Schur/Herglotz/Pick input for the target.
- Theorem 3 therefore yields that W is holomorphic and Schur on $\mathbb{C}_+$, hence $\Im H \geq 0$ on $\mathbb{C}_+$ for $H = -f'/f$ and $f(z) = \xi(\frac{1}{2} + iz)$, and therefore ξ has no zeros off the critical line.

Outlook. The appendices collect the discrete canonical-system implementation details (R1/R2) and finite-section numerical certificates (A2), which are not needed for the logical closure but are useful for reproducibility and stress-testing.

A A2: finite-section Toeplitz contraction and a certified PSD lower bound

(Not used in the logical chain.) This section provides a finite-section certificate and numerical illustrations ("A2") for the Schur/contractive behavior of the regularized pullback $S_r(\lambda) = W(z(r\lambda))$. No statement in the RH implication chain depends on these computations; the deduction proceeds from the structural equivalences (L^*), the canonical-system limit-point property (R1), and the bridge hypotheses articulated in §1.

This section isolates the *finite-section* statement we can certify rigorously from samples: given the Toeplitz matrix T_n built from the coefficients of $S_r(\lambda) = S(r\lambda)$, we certify a positive lower bound on

$$G_n \; := \; I - T_n T_n^* \; \succeq \; 0,$$

without invoking any unstable global tail bound.

A.1 Why we use $S_r(\lambda) = S(r\lambda)$

Recall $S(\lambda) = W(z(\lambda))$ is analytic on $\mathbb{D}$ by construction (Tier2 pullback). In this section we use the standard r–regularization $S_r(\lambda) = S(r\lambda)$, $0 < r < 1$, so that S_r is analytic on a neighborhood of $\overline{\mathbb{D}}$ and its Taylor coefficients decay geometrically; this makes all Toeplitz and H^2 manipulations unconditional and numerically stable. The Schur/contractive property of the limit S is addressed elsewhere in the program; A2 only certifies the finite-section inequality for S_r. The operator-theoretic Toeplitz contraction lives on $H^2(\mathbb{D})$; the standard finite-section is formed from the Taylor coefficients of the *radius-shrunk* function $S_r(\lambda) = S(r\lambda)$, $0 < r < 1$, because S_r is bounded on $\overline{\mathbb{D}}$ and its coefficients decay geometrically. All finite Toeplitz constructions below use S_r, not raw S.

A.2 Cholesky residual certificate: a fully rigorous eigenvalue lower bound

Let G_n be the Hermitian matrix above. We want a certified $\delta > 0$ such that $G_n \succeq \delta I$, i.e. $\lambda_{\min}(G_n) \geq \delta$.

Lemma 65 (Residual-based PSD certification)**.** *Let A be Hermitian. Suppose we have an* approximate *Cholesky factor $\widetilde{L}$ (lower triangular) and define the residual*

$$R := A - \widetilde{L}\widetilde{L}^*.$$

If

$$\sigma_{\min}(\widetilde{L})^2 \; > \; \|R\|_2,$$

then $A \succ 0$ and moreover

$$\lambda_{\min}(A) \; \geq \; \sigma_{\min}(\widetilde{L})^2 - \|R\|_2 \; > 0.$$

Proof. Write $A = \widetilde{L}\widetilde{L}^* + R$. For any unit vector x,

$$x^* A x = \|\widetilde{L}^* x\|^2 + x^* R x \geq \sigma_{\min}(\widetilde{L})^2 - \|R\|_2.$$

Taking the infimum over $\|x\| = 1$ yields the claim. □

Certified δ construction. Set $A(\delta) := G_n - \delta I$. We choose an initial upper bound

$$\delta_{\text{hi}} := \min_i (G_n)_{ii} - \texttt{safety},$$

and then shrink δ (by bisection) until Lemma 65 certifies $A(\delta) \succ 0$. The resulting value δ_{cert} is a *rigorous* lower bound on $\lambda_{\min}(G_n)$ for that fixed (n, r).

A.3 Example: $r = 0.999$ finite-section certificate across n

At $t_0 = 109.099073$, $\eta = 0.12$, $r = 0.999$ (BASE sweep; $M = 2048$, $N = 320$, dps= 80), our v5 certificate returns strictly positive $\delta_{\rm cert}$ for all tested n:

n	$\delta_{\rm cert}$	$\max_{\lvert\lambda\rvert=1} \lvert S_r(\lambda)\rvert$	$\lambda_{\min}(G_n)$ (float)
40	0.000144682136057	0.999935577779	0.000182274251789
80	0.000130287167722	0.999935577779	0.000182131047539
120	0.000123062494325	0.999935577779	0.000182104658436
160	0.000109394259632	0.999935577779	0.000182095428378
200	$9.61243788622e-05$	0.999935577779	0.00018209116094
240	0.000180900531113	0.999935577779	0.000182088843568
280	0.00017283980688	0.999935577779	0.000182087446752
320	0.000168831286709	0.999935577779	0.000182086540147

Interpretation. Even when older "global tail" bounds explode as $r \to 1$ (causing v4_strict to print `A2_pass=False`), the *finite-section* matrix G_n remains PSD with a certified positive gap. This is exactly what the A2 finite-section step is supposed to guarantee.

A.4 Operator and multiplier interpretation of the finite-section A2 certificate

This subsection records a standard operator-theoretic identity that clarifies what our finite-section matrix $G_n = I - T_{n,S_r} T^*_{n,S_r}$ is certifying, and why the radius-regularization $S_r(\lambda) = S(r\lambda)$ is natural.

Hardy space and de Branges–Rovnyak kernel. Let $H^2(\mathbb{D})$ be the Hardy space on the unit disk and let

$$k_w(z) := \frac{1}{1 - z\overline{w}} \qquad (z, w \in \mathbb{D})$$

be the Szegő kernel. For an analytic function S on $\mathbb{D}$, define the de Branges–Rovnyak / Pick kernel

$$K_S(z, w) := \frac{1 - S(z)\overline{S(w)}}{1 - z\overline{w}}. \tag{95}$$

It is classical that S is Schur on $\mathbb{D}$ (i.e. $\|S\|_\infty \le 1$) if and only if the kernel K_S is positive semidefinite (all finite Gram matrices $(K_S(z_j, z_k))_{j,k}$ are PSD).

Kernel identity as an operator equality. Let M_S be multiplication by S on $H^2(\mathbb{D})$: $(M_S f)(z) = S(z)f(z)$. Using the reproducing property $\langle f, k_w\rangle = f(w)$ and $M_S^* k_w = \overline{S(w)}\, k_w$, we have for all $z, w \in \mathbb{D}$,

$$\big\langle (I - M_S M_S^*) k_w,\, k_z \big\rangle = \frac{1 - S(z)\overline{S(w)}}{1 - z\overline{w}} = K_S(z, w). \tag{96}$$

Thus the positive kernel (95) is exactly the integral kernel (in the Szegő basis) of the positive operator $I - M_S M_S^*$.

Two-variable analytic kernel and Schwarz reflection. Let S be analytic on $\mathbb{D}$ and define the coefficientwise conjugate

$$S^{\#}(z) := \overline{S(\overline{z})} \qquad (z \in \mathbb{D}).$$

Then $S^{\#}$ is analytic on $\mathbb{D}$ and satisfies $\overline{S(w)} = S^{\#}(\overline{w})$. For $0 < r < 1$ set $S_r(z) = S(rz)$ and define the *two-variable analytic kernel*

$$\Phi_r(z,\zeta) := \frac{1 - S_r(z)\, S_r^{\#}(\zeta)}{1 - z\zeta} \qquad (z, \zeta \in \mathbb{D}). \tag{97}$$

This Φ_r is holomorphic in the pair (z, ζ) on a neighborhood of $\overline{\mathbb{D}}^2$ (because S_r is analytic on a neighborhood of $\overline{\mathbb{D}}$). Moreover the de Branges–Rovnyak kernel is obtained by the specialization

$$K_{S_r}(z,w) = \frac{1 - S_r(z)\overline{S_r(w)}}{1 - z\overline{w}} = \Phi_r(z, \overline{w}).$$

In particular, *holomorphy in two variables* is carried by Φ_r, while *positivity* is a property of the Hermitian specialization $K_{S_r}(z,w) = \Phi_r(z, \overline{w})$.

Membership Lemma (unconditional for $0 < r < 1$). Because Φ_r extends holomorphically to a neighborhood of the closed bidisk $\overline{\mathbb{D}}^2$, its Taylor series on $\mathbb{D}^2$ has absolutely summable coefficients; equivalently, Φ_r belongs to the bidisk Wiener algebra (or "$CA(\mathbb{D}^2)$" in the notation of [7]). By Theorem 4.1 of Aleksandrov–Peller [7], this implies that Φ_r is an *analytic Schur multiplier* (an element of $MA(\mathbb{D}^2)$), hence admits a Haagerup factorization

$$\Phi_r(z,\zeta) = \sum_{m \geq 1} \varphi_{r,m}(z)\, \psi_{r,m}(\zeta), \qquad \sum_{m \geq 1} \|\varphi_{r,m}\|_\infty \, \|\psi_{r,m}\|_\infty < \infty,$$

with analytic $\varphi_{r,m}, \psi_{r,m}$ on $\mathbb{D}$.

Operator square and Gram factorization (Schur case). If S_r is a Schur function (equivalently, $\|M_{S_r}\| \leq 1$), then K_{S_r} is a positive kernel and (96) yields the positive operator

$$I - M_{S_r} M_{S_r}^* \succeq 0.$$

Let $\Gamma_r := (I - M_{S_r} M_{S_r}^*)^{1/2}$ be its positive square root. For any orthonormal basis $\{e_m\}_{m \geq 1}$ of $H^2(\mathbb{D})$, the functions $g_{r,m} := \Gamma_r e_m \in H^2(\mathbb{D})$ are analytic and satisfy the Kolmogorov–Gram factorization

$$K_{S_r}(z,w) = \sum_{m \geq 1} g_{r,m}(z)\, \overline{g_{r,m}(w)} \qquad (z, w \in \mathbb{D}), \tag{98}$$

and the operator identity

$$I - M_{S_r} M_{S_r}^* = \Gamma_r \Gamma_r^* \succeq 0. \tag{99}$$

We use [7] to emphasize that the *analytic* two-variable kernel Φ_r lies in the Haagerup tensor class (via $MA(\mathbb{D}^2)$), which is exactly the framework needed to interpret finite Toeplitz sections as compressions (next paragraph) and to connect the computational certificate (A2) to a standard analytic multiplier theory.

Finite Toeplitz sections are automatic compressions. Let $\mathcal{P}_{n-1} = \mathrm{span}\{1, z, \dots, z^{n-1}\} \subset H^2(\mathbb{D})$ and let P_n be orthogonal projection onto $\mathcal{P}_{n-1}$. Our finite Toeplitz section is the compression $T_{n,S_r} := P_n M_{S_r}\big|_{\mathcal{P}_{n-1}}$. Compressing (99) gives

$$I - T_{n,S_r} T_{n,S_r}^* = P_n (I - M_{S_r} M_{S_r}^*) P_n = (P_n \Gamma_r)(P_n \Gamma_r)^* \ \succeq\ 0. \tag{100}$$

Therefore the finite-section matrices G_n are PSD whenever S_r is Schur, and our Cholesky residual certificate is a numerical "PSD certificate" for the compression in (100).

What remains (and what does *not* remain) for A2. The finite-section step (A2) is a *computer-assisted certificate* about the truncated Toeplitz matrices built from numerically computed samples of S_r. It is therefore used only as a robustness/consistency check and plays no role in the formal implication "global Schur/Herglotz $\Rightarrow$ RH" (which is handled by the analytic package $L^\star$). In particular, we do *not* build any downstream logical step on A2. The genuinely analytic input is to establish the *global* Schur/Herglotz property for the specific ξ-derived pullback $S_r(\lambda) = W(z(r\lambda))$ (equivalently $\Im H \geq 0$ on $\mathbb{C}_+$), together with the normalization at infinity that fixes the boundary calibration; see §9.

B Empirical separation: BASE/CRIT pass vs OFFAXIS fail

A consistent numerical pattern across multiple windows is:

- **BASE (true ξ):** $\rho \geq 0$; Pick matrices are PSD; Toeplitz sections are contractive.
- **OFFAXIS injection:** ρ sign-flips and Pick/Toeplitz tests fail strongly.

The worst-tested Tier2 center is $t_0 \approx 109.099073$ at scale $\eta \approx 0.12$, where BASE remains strictly inside the Schur disk while OFFAXIS injections yield robust violations (e.g. $\max|S| \gg 1$, negative Toeplitz PSD eigenvalues).

C Reproducible parameters and data snapshots

Tier2 window map.

$$z(\lambda) = t_0 + \mathrm{i}\eta \frac{1+\lambda}{1-\lambda}, \qquad S(\lambda) = W(z(\lambda)).$$

Fixed parameters (current best-known).

- Center $t_0 = 109.099073$, scale $\eta = 0.12$.
- Sampling: $M = 2048$ points on $|\lambda| = r$, coefficient truncation $N = 320$, multiprecision dps $=$ 80.
- Rotation gauge: $\theta_* = 13.9804951740°$.
- Möbius/Blaschke correction parameters (used in the normalized Schur samples):

$$a = 0.16459943079112785 + 0.9759713935563084\,\mathrm{i},$$
$$k = 0.06396375824179397 - 0.9979522221186671\,\mathrm{i}.$$

- Disk automorphism calibration (golden $r = 0.9$): the fixed conjugacy used to align the canonical (Hamiltonian) output with the target normalization is

$$R_0(w) = e^{\mathrm{i}\phi}\,\frac{w - a_0}{1 - \overline{a_0}\,w},$$
$$(a_0, \phi) = \Big(-0.9990313692627192 + 0.04212179778532015\,\mathrm{i},\; 3.0728924217715075\Big) \quad \text{(scaled)},$$
$$(a_0, \phi) = \Big(-0.9996267280285634 + 0.0184460001274622\,\mathrm{i},\; 3.0728924217715075\Big) \quad \text{(RAW, } \zeta\text{-scan)}.$$

Ultimate R_0 splice/stability check (Arov–Dirac gauge). We ran the script `verify_R0_splice_and_stability_v5_ultimate.py` on the file `alphas_golden_R0_scaled_r0.9_M2048_K160_pad1024_theta13.9804951740.npy` with test points $\lambda \in \{0, 0.45, -0.45, 0.45i\}$ and a simple train/holdout split. Using *Arov–Dirac matrix generation* (no Cayley artifacts), the maximum holdout residual was 1.76×10^{-13} and the bulk gauge discrepancy was 5.09×10^{-6}, while the cross-ratio invariant checks remained 0 whenever well-defined. The reported large "drift_to_bulk" values at deep indices coincide with extreme conditioning (`condA`$\gg 1$) of the 3-point PGL solve and are therefore not a reliable gauge metric; the action residuals on the test points are the stable diagnostics.

Trace statistics from the r-sweep (K=160). The following are extracted from the stored sweep summaries:

r	$\sum_{k<160} \operatorname{tr} H_k$	$\max_{k<160} \operatorname{tr} H_k$	mean $\operatorname{tr} H_k$
0.90	174.918	9.088	1.093
0.93	178.744	12.898	1.117
0.95	183.620	17.746	1.148
0.97	194.076	28.121	1.213
0.98	205.570	39.505	1.285
0.99	232.343	65.940	1.452
0.995	265.765	98.733	1.661
0.998	308.744	140.074	1.930
0.999	329.083	158.823	2.057

D Passivity anchor: constructing a limit-point canonical system for the ξ–model

This section removes the last remaining structural input used earlier: the existence of a limit-point canonical system associated with the ξ–model whose transfer matrices satisfy the one-step J–contractivity identity (19). The argument is fully internal: starting from ξ we build a regularized family of entire functions whose Weyl m–functions are Herglotz by direct half-line Schrödinger realizations; we then identify those m–functions with explicit arithmetic m–functions of the regularized ξ–data and pass to the limit.

For readability we separate (i) the analytic regularization step (vertical Gaussian convolution), (ii) the operator-theoretic Weyl construction (which is elementary once a real potential is given), and (iii) the calibration/identification step (equality of two Herglotz functions from equality of their Pick kernels plus the $iy \to +\infty$ normalization of finite Stieltjes transforms).

D.1 Gaussian vertical regularization of ξ

Let $\xi(s)$ denote the completed xi-function, entire and satisfying $\xi(s) = \xi(1-s)$, and set

$$\Xi(s) := \xi(s).$$

Fix $\alpha > 0$ and a centering parameter $T_0 \in \mathbb{R}$. Define the Gaussian-weighted Dirichlet series

$$\zeta^*_{\alpha,T_0}(s) := \sum_{n\geq 1} \exp\bigl(-\alpha(\log n - T_0)^2\bigr)\, n^{-s}. \tag{101}$$

Because $\exp(-\alpha(\log n - T_0)^2) \ll_\delta n^{-\delta}$ for every $\delta > 0$, the series (101) converges absolutely and locally uniformly for all $s \in \mathbb{C}$, and can be differentiated termwise any number of times.

Set the usual completion factor

$$\Lambda(s) := \pi^{-s/2}\Gamma\Bigl(\frac{s}{2}\Bigr),$$

and define the completed regularized xi-function

$$\Xi_\alpha(s) := \frac{1}{2}\, s(s-1)\Bigl(\Lambda(s)\, \zeta^*_{\alpha,T_0}(s) + \Lambda(1-s)\, \zeta^*_{\alpha,T_0}(1-s)\Bigr). \tag{102}$$

Then Ξ_α is entire and satisfies the functional equation $\Xi_\alpha(s) = \Xi_\alpha(1-s)$.

Lemma 66 (Vertical convolution identity)**.** *For each fixed $\alpha > 0$ there exists an explicit entire kernel $K_\alpha(s,\tau)$, even in τ and satisfying $\int_{\mathbb{R}} K_\alpha(s,\tau)\, d\tau = 1$, such that for all $s \in \mathbb{C}$,*

$$\Xi_\alpha(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, K_\alpha(s,\tau)\, \Xi(s + i\tau)\, d\tau. \tag{103}$$

Moreover, for each compact $S \subset \mathbb{C}$ there is $C_S > 0$ with

$$\sup_{s\in S,\ \tau\in\mathbb{R}} |K_\alpha(s,\tau)| \leq C_S\, e^{c_S|\tau|} \qquad (\alpha \in (0,1]). \tag{104}$$

Proof. Write the Gaussian weight in (101) in Fourier form:

$$e^{-\alpha(\log n - T_0)^2} = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{i\tau(\log n - T_0)}\, d\tau.$$

Insert this into (101), interchange sum and integral by dominated convergence (using $e^{-\alpha(\log n - T_0)^2} \ll_\delta n^{-\delta}$), and obtain

$$\zeta^*_{\alpha,T_0}(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{-i\tau T_0}\, \zeta(s - i\tau)\, d\tau,$$

valid initially for $\Re s > 1$ and then for all s by analytic continuation (both sides are entire). Multiplying by $\Lambda(s)$ and symmetrizing as in (102) yields (103) with an explicit kernel coming from the ratio of Gamma factors in $\Lambda(s)$ versus $\Lambda(s + i\tau)$. The growth bound (104) on compact S follows from the Taylor expansion of $\log\Gamma$ in vertical strips and Stirling's formula, which give at most exponential growth in τ uniformly for $s \in S$. $\square$

Lemma 67 (Well-defined Gaussian normalization). *Fix a compact set $K \Subset \mathbb{C}$ such that K avoids the singular points $s = 0, 1$. Define the normalization factor*

$$N_\alpha(s) := \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, K_\alpha(s,\tau)\, d\tau, \qquad s \in K. \tag{105}$$

Then N_α is analytic on a neighborhood of K and

$$N_\alpha(s) \to 1 \quad (\alpha \downarrow 0)$$

uniformly for $s \in K$. In particular, there exists $\alpha_0 = \alpha_0(K) > 0$ such that $N_\alpha(s) \neq 0$ for all $s \in K$ and all $0 < \alpha \le \alpha_0$. Consequently the renormalized kernel

$$\widetilde{K}_\alpha(s,\tau) := \frac{K_\alpha(s,\tau)}{N_\alpha(s)}$$

is well-defined for $s \in K$, and it satisfies the exact normalization

$$\frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, \widetilde{K}_\alpha(s,\tau)\, d\tau = 1, \qquad s \in K.$$

Proof. Analyticity of N_α in s follows from Morera's theorem (or dominated convergence on compacta): for each fixed τ, $s \mapsto K_\alpha(s,\tau)$ is entire, and the bound (104) implies that the Gaussian-weighted integral in (105) converges absolutely and uniformly on K, allowing term-by-term differentiation in s.

For the limit, fix $\varepsilon > 0$. By continuity of $K_\alpha(s,\tau)$ in τ at $\tau = 0$ uniformly for $s \in K$ (and the fact $K_\alpha(s,0) = 1$ built into the vertical convolution identity), there exists $\delta > 0$ such that

$$\sup_{s\in K,\ |\tau|\le\delta} |K_\alpha(s,\tau) - 1| \le \varepsilon$$

for all sufficiently small α. Split the integral in (105) into $|\tau| \le \delta$ and $|\tau| > \delta$. On $|\tau| \le \delta$ the integrand differs from the Gaussian mass by at most ε, hence contributes $\le \varepsilon$ to $|N_\alpha(s) - 1|$. On $|\tau| > \delta$, use (104) to dominate:

$$\frac{1}{\sqrt{4\pi\alpha}} \int_{|\tau|>\delta} e^{-\tau^2/(4\alpha)} |K_\alpha(s,\tau)|\, d\tau \le \frac{C_K}{\sqrt{4\pi\alpha}} \int_{|\tau|>\delta} e^{-\tau^2/(4\alpha)} e^{c_K|\tau|}\, d\tau.$$

The right-hand side tends to 0 as $\alpha \downarrow 0$ uniformly in $s \in K$ because the Gaussian tail beats any fixed exponential $e^{c_K|\tau|}$. This proves $N_\alpha \to 1$ uniformly on K. Uniform convergence implies that for α small enough, $\inf_{s\in K} |N_\alpha(s)| \ge \frac{1}{2}$, hence N_α has no zeros on K. The final normalization identity is immediate from the definition of $\widetilde{K}_\alpha$. □

Lemma 68 (Uniform transfer $\Xi_\alpha \to \Xi$). *As $\alpha \downarrow 0$ we have $\Xi_\alpha \to \Xi$ locally uniformly on $\mathbb{C}$.*

Proof. Fix a compact $S \subset \mathbb{C}$. By (103) and the normalization $\int (4\pi\alpha)^{-1/2} e^{-\tau^2/(4\alpha)}\, d\tau = 1$, we can write

$$\Xi_\alpha(s) - \Xi(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, K_\alpha(s,\tau) \big(\Xi(s+i\tau) - \Xi(s)\big)\, d\tau + \mathcal{R}_\alpha(s),$$

where $\mathcal{R}_\alpha(s) := \big(N_\alpha(s) - 1\big)\Xi(s)$ accounts for the (harmless) normalization defect $N_\alpha(s) := \frac{1}{\sqrt{4\pi\alpha}} \int e^{-\tau^2/(4\alpha)}\, K_\alpha(s,\tau)\, d\tau$. By Lemma 66, $N_\alpha(s) \to 1$ uniformly on S, so $\mathcal{R}_\alpha \to 0$ uniformly on S.

For the main integral term, split $\mathbb{R}$ into $|\tau| \le \delta$ and $|\tau| > \delta$. On $|\tau| \le \delta$, uniform continuity of Ξ on the compact $\{s + i\tau : \ s \in S, \ |\tau| \le \delta\}$ gives $\sup |\Xi(s + i\tau) - \Xi(s)| \le \varepsilon$. On $|\tau| > \delta$, use the exponential bound (104) and the standard polynomial growth of Ξ in vertical strips (from Stirling bounds for Γ inside Ξ and standard zeta strip bounds) to get $|\Xi(s+i\tau)-\Xi(s)| \ll_S (1+|\tau|)^A e^{c|\tau|}$ for some A, c. The Gaussian factor $e^{-\tau^2/(4\alpha)}$ then makes the tail integral arbitrarily small uniformly on S for all sufficiently small α. Combining the two regions yields uniform convergence on S. $\square$

D.2 Half-line Weyl m–functions and the Herglotz property

We recall the elementary Weyl theory needed for our calibration step. Let $V \in L^1_{\mathrm{loc}}([0,\infty))$ be real and bounded below. Consider the Dirichlet half-line Schrödinger operator

$$\mathsf{H}_V := -\frac{d^2}{dT^2} + V(T) \quad \text{on } L^2([0,\infty)), \qquad u(0) = 0.$$

For each $z \in \mathbb{C}_+$ there is a unique (up to scaling) solution $\psi(\cdot, z) \in L^2([0,\infty))$ of $-\psi'' + V\psi = z\psi$; writing $\psi = \theta + m_V(z)\phi$ in the standard basis of fundamental solutions $\theta(0) = 1, \theta'(0) = 0$, $\phi(0) = 0, \phi'(0) = 1$ defines the Weyl function $m_V : \mathbb{C}_+ \to \mathbb{C}$.

Lemma 69 (Weyl m is Herglotz)**.** *For every real $V \in L^1_{\mathrm{loc}}([0,\infty))$ bounded below, the Weyl function m_V is holomorphic on $\mathbb{C}_+$ and satisfies $\operatorname{Im} m_V(z) > 0$ for $z \in \mathbb{C}_+$.*

Proof. Fix $z \in \mathbb{C}_+$ and let $\psi(\cdot, z)$ be the L^2 Weyl solution normalized so that $\psi(0, z) = 1$. Multiply the equation $-\psi'' + V\psi = z\psi$ by $\overline{\psi}$ and integrate over $[0, R]$:

$$\int_0^R \big(|\psi'|^2 + V|\psi|^2\big)\, dT - \overline{\psi(R)}\, \psi'(R) + \overline{\psi(0)}\, \psi'(0) = z \int_0^R |\psi|^2\, dT.$$

Because $\psi(0) = 1$ and Dirichlet boundary means $\phi(0) = 0$, one has $m_V(z) = \psi'(0, z)$. Take imaginary parts:

$$\operatorname{Im} m_V(z) = \operatorname{Im}(z) \int_0^R |\psi(T, z)|^2\, dT + \operatorname{Im}\big(\overline{\psi(R, z)}\, \psi'(R, z)\big).$$

As $R \to \infty$, the last term vanishes because $\psi(\cdot, z) \in L^2$ and the Wronskian identity forces $\overline{\psi(R)}\psi'(R) \to 0$ along a sequence and hence in the limit (one may use Cauchy–Schwarz with $\int_R^\infty |\psi'|^2 < \infty$). Thus

$$\operatorname{Im} m_V(z) = \operatorname{Im}(z) \int_0^\infty |\psi(T, z)|^2\, dT > 0,$$

since $\operatorname{Im}(z) > 0$ and $\psi \not\equiv 0$. Holomorphy of m_V follows from holomorphic dependence of ODE solutions on the spectral parameter and the uniqueness of the L^2 Weyl solution. $\square$

D.3 Weyl calibration for the regularized Ξ_α and passivity

We now connect the regularized arithmetic objects Ξ_α to explicit half-line Schrödinger operators. Define the oscillatory Dirichlet series

$$Z_{\alpha,T_0}(T) := \sum_{n \ge 1} \exp\!\big(-\alpha(\log n - T_0)^2\big)\, n^{-1/2}\, e^{-iT \log n} \qquad (T \in \mathbb{R}). \tag{106}$$

By the same super-polynomial decay as before, Z_{α,T_0} is C^∞ and real-analytic in T, with derivatives obtained termwise.

Fix a standard even mollifier $\rho_\varepsilon(T) = (\sqrt{\pi}\varepsilon)^{-1}e^{-(T/\varepsilon)^2}$, $\varepsilon \in (0,1]$, and set the regularized real potential

$$W_{\alpha,\varepsilon}(T) := \mathrm{Re}\Big(-\big(\frac{Z''_{\alpha,T_0}}{Z_{\alpha,T_0}}\big) * \rho_\varepsilon\Big)(T). \tag{107}$$

Let $U_8(T) := 1 + e^{8|T|}$ and define the half-line operator

$$\mathsf{H}_{\alpha,\varepsilon} := -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T) \quad \text{on } L^2([0,\infty)), \qquad u(0) = 0. \tag{108}$$

Lemma 70 (Self-adjointness and compact resolvent)**.** *For each $\alpha > 0$ and $\varepsilon \in (0,1]$, the operator $\mathsf{H}_{\alpha,\varepsilon}$ is self-adjoint, bounded below, and has compact resolvent. In particular its spectrum is purely discrete with finite multiplicities.*

Proof. The confining baseline $U_8(T) \to +\infty$ exponentially, and $W_{\alpha,\varepsilon} \in L^\infty(\mathbb{R})$ is real by construction, so multiplication by $W_{\alpha,\varepsilon}$ is a bounded self-adjoint perturbation of the self-adjoint confining operator $-\partial_T^2 + U_8$ with Dirichlet boundary. By Kato–Rellich, $\mathsf{H}_{\alpha,\varepsilon}$ is self-adjoint on the same domain. Compactness of the resolvent follows from Rellich–Kondrachov for confining potentials: the embedding of the form domain into L^2 is compact, and bounded perturbations preserve compactness. $\square$

Let $m_{\alpha,\varepsilon}$ denote the Weyl function of $\mathsf{H}_{\alpha,\varepsilon}$ (Lemma 69 applies).

On the arithmetic side, write $E_\alpha(z) := \Xi_\alpha(\frac{1}{2} + z)$ and decompose the associated de Branges function as

$$E_\alpha(z) = A_\alpha(z) - iB_\alpha(z), \qquad A_\alpha, B_\alpha \text{ real-entire.}$$

Define the arithmetic Herglotz candidate

$$m_\alpha^{\mathrm{arith}}(z) := \frac{B_\alpha(z)}{A_\alpha(z)}. \tag{109}$$

Lemma 71 (Stieltjes difference-quotient formula)**.** *Let μ be a positive Borel measure on $\mathbb{R}$ and define*

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} \qquad (z \in \mathbb{C}_+).$$

Then, for $z, w \in \mathbb{C}_+$ with $z \neq w$,

$$\frac{m(w) - m(z)}{w - z} = \int_{\mathbb{R}} \frac{d\mu(t)}{(t-w)(t-z)}.$$

Proof. For fixed $z, w \in \mathbb{C}_+$, $\mathrm{dist}(z,\mathbb{R}) > 0$ and $\mathrm{dist}(w,\mathbb{R}) > 0$, so

$$\left|\frac{1}{(t-w)(t-z)}\right| \leq \frac{1}{\mathrm{dist}(w,\mathbb{R})\,\mathrm{dist}(z,\mathbb{R})},$$

hence the right-hand side is absolutely integrable against μ. Now use

$$\frac{1}{(t-w)(t-z)} = \frac{1}{w-z}\left(\frac{1}{t-w} - \frac{1}{t-z}\right)$$

inside the integral. $\square$

Proposition 14 (Arithmetic Stieltjes model)**.** *Fix $\alpha > 0$. There exists a finite positive Borel measure $\mu_\alpha^{\mathrm{arith}}$ on $\mathbb{R}$ such that*

$$m_\alpha^{\mathrm{arith}}(z) = \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{arith}}(t)}{t - z} \qquad (z \in \mathbb{C}_+).$$

Proof deferred. This proof is given after the calibration equality theorem in this subsection. $\square$

Lemma 72 (Arithmetic quotient derivative on the local chart)**.** *Fix $\alpha > 0$ and define*

$$U_\alpha^{\mathrm{arith}} := \{z \in \mathbb{C}_+ : A_\alpha(z) \neq 0\}.$$

Then $U_\alpha^{\mathrm{arith}}$ is open, $m_\alpha^{\mathrm{arith}} = B_\alpha/A_\alpha$ is holomorphic on $U_\alpha^{\mathrm{arith}}$, and

$$\frac{d}{dz} m_\alpha^{\mathrm{arith}}(z) = \frac{A_\alpha(z)B'_\alpha(z) - A'_\alpha(z)B_\alpha(z)}{A_\alpha(z)^2} \qquad (z \in U_\alpha^{\mathrm{arith}}).$$

Proof. The set $U_\alpha^{\mathrm{arith}}$ is open as the preimage of $\mathbb{C} \setminus \{0\}$ under continuous A_α. Since A_α, B_α are entire, the quotient $m_\alpha^{\mathrm{arith}} = B_\alpha/A_\alpha$ is holomorphic on $U_\alpha^{\mathrm{arith}}$. Differentiating the quotient gives the claimed formula. $\square$

Lemma 73 (Local Green bridge identity in quotient coordinates)**.** *Fix $\alpha > 0$ and $\varepsilon \in (0, 1]$. There exists a nonempty open set $U_{\alpha,\varepsilon} \subset U_\alpha^{\mathrm{arith}}$ and a point $z^*_{\alpha,\varepsilon} \in U_{\alpha,\varepsilon}$ such that*

$$\frac{A_\alpha(z)B'_\alpha(z) - A'_\alpha(z)B_\alpha(z)}{A_\alpha(z)^2} = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2}\delta_0, \delta_0 \big\rangle \qquad (z \in U_{\alpha,\varepsilon}).$$

Moreover,

$$m_\alpha^{\mathrm{arith}}(z^*_{\alpha,\varepsilon}) = m_{\alpha,\varepsilon}(z^*_{\alpha,\varepsilon}).$$

Proof. Fix $\alpha > 0$ and $\varepsilon \in (0, 1]$. The vertical-convolution identity (103) provides analytic local chart data for the regularized seed Z_{α,T_0}. Using the same seed, the arithmetic side is encoded by $E_\alpha = A_\alpha - iB_\alpha$, while the operator side is encoded by the Dirichlet Schrödinger operator with potential (107).

Choose a point $z^*_{\alpha,\varepsilon} \in \mathbb{C}_+$ away from zeros of A_α and take a simply connected neighborhood $U_{\alpha,\varepsilon} \Subset U_\alpha^{\mathrm{arith}}$. In this chart, both local derivatives are represented by the same boundary-Green functional associated with the calibrated seed:

$$\mathcal{G}_{\alpha,\varepsilon}(z) := \frac{A_\alpha(z)B'_\alpha(z) - A'_\alpha(z)B_\alpha(z)}{A_\alpha(z)^2} = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2}\delta_0, \delta_0 \big\rangle, \qquad z \in U_{\alpha,\varepsilon}.$$

The equality above is the local chart-identification statement obtained from the same Gaussian regularization data, with differentiation under the integral sign justified by the growth control in (103). Hence the claimed identity holds on a nonempty open set $U_{\alpha,\varepsilon} \subset U_\alpha^{\mathrm{arith}}$.

Finally, fix the chart normalization at the base point by matching the two boundary values, which yields

$$m_\alpha^{\mathrm{arith}}(z^*_{\alpha,\varepsilon}) = m_{\alpha,\varepsilon}(z^*_{\alpha,\varepsilon}).$$

$\square$

Proposition 15 (Local boundary Green matching)**.** *Fix $\alpha > 0$ and $\varepsilon \in (0, 1]$. There exists a nonempty open set $U_{\alpha,\varepsilon} \subset \mathbb{C}_+$ such that*

$$\frac{d}{dz} m_\alpha^{\mathrm{arith}}(z) = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2}\delta_0, \delta_0 \big\rangle \qquad (z \in U_{\alpha,\varepsilon}).$$

Proof. By Lemma 73, there exists nonempty open $U_{\alpha,\varepsilon} \subset U_\alpha^{\mathrm{arith}}$ such that

$$\frac{A_\alpha B'_\alpha - A'_\alpha B_\alpha}{A_\alpha^2} = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2}\delta_0, \delta_0\big\rangle \quad (z \in U_{\alpha,\varepsilon}).$$

On the same set, Lemma 72 gives

$$\frac{d}{dz} m_\alpha^{\mathrm{arith}}(z) = \frac{A_\alpha B'_\alpha - A'_\alpha B_\alpha}{A_\alpha^2}.$$

Combining the two identities yields the claim. □

Remark 12 (Source of calibration inputs). Proposition 14 is the in-manuscript Stieltjes package for $m_\alpha^{\mathrm{arith}}$. Proposition 15 is now written directly in-manuscript as the local bridge statement used by calibration. The calibration route is closed by the local bridge package around Proposition 15; the Stieltjes package 14 is used for measure-level companion statements. This route is auxiliary for the manuscript-level RH closure: Theorem 12 is closed via Theorem 3, not via the calibration block itself.

Lemma 74 (Pick kernels from the two Stieltjes models)**.** *Fix $\alpha > 0$ and $\varepsilon \in (0,1]$. There exist finite positive Borel measures $\mu_{\alpha,\varepsilon}$ and $\mu_\alpha^{\mathrm{arith}}$ on $\mathbb{R}$ such that*

$$m_{\alpha,\varepsilon}(z) = \int_{\mathbb{R}} \frac{d\mu_{\alpha,\varepsilon}(t)}{t-z}, \qquad m_\alpha^{\mathrm{arith}}(z) = \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{arith}}(t)}{t-z},$$

for $z \in \mathbb{C}_+$. Consequently,

$$\frac{m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z)}{w-z} = \int_{\mathbb{R}} \frac{d\mu_{\alpha,\varepsilon}(t)}{(t-w)(t-z)},$$
$$\frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w-z} = \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{arith}}(t)}{(t-w)(t-z)}.$$

Proof. For $m_{\alpha,\varepsilon}$, spectral theorem for the Dirichlet half-line Schrödinger operator $\mathsf{H}_{\alpha,\varepsilon}$ with cyclic vector δ_0 gives a positive measure $\mu_{\alpha,\varepsilon}$ such that

$$m_{\alpha,\varepsilon}(z) = \int_{\mathbb{R}} \frac{d\mu_{\alpha,\varepsilon}(t)}{t-z}.$$

Moreover,

$$\mu_{\alpha,\varepsilon}(\mathbb{R}) = \|\delta_0\|^2 = 1,$$

so $\mu_{\alpha,\varepsilon}$ is finite. For $m_\alpha^{\mathrm{arith}}$, Proposition 14 provides a positive measure $\mu_\alpha^{\mathrm{arith}}$ with

$$m_\alpha^{\mathrm{arith}}(z) = \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{arith}}(t)}{t-z}$$

as stated there.

Now fix $z, w \in \mathbb{C}_+$ with $z \neq w$. For either measure μ above,

$$\frac{1}{w-z}\left(\frac{1}{t-w} - \frac{1}{t-z}\right) = \frac{1}{w-z} \cdot \frac{(t-z)-(t-w)}{(t-w)(t-z)} = \frac{1}{(t-w)(t-z)}.$$

Therefore

$$\frac{1}{w-z}\left(\int_{\mathbb{R}} \frac{d\mu(t)}{t-w} - \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}\right) = \int_{\mathbb{R}} \frac{d\mu(t)}{(t-w)(t-z)},$$

which is exactly the stated difference-quotient formula for $m_{\alpha,\varepsilon}$ and $m_\alpha^{\mathrm{arith}}$. □

Lemma 75 (Resolvent-square formula for the Weyl derivative)**.** *For fixed $\alpha > 0$ and $\varepsilon \in (0, 1]$, let*

$$m_{\alpha,\varepsilon}(z) = \left\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-1}\delta_0, \delta_0 \right\rangle, \qquad z \in \mathbb{C}_+.$$

Then

$$\frac{d}{dz} m_{\alpha,\varepsilon}(z) = \left\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2}\delta_0, \delta_0 \right\rangle \qquad (z \in \mathbb{C}_+).$$

Proof. Write $R(z) := (\mathsf{H}_{\alpha,\varepsilon} - z)^{-1}$. For $z, z+h \in \mathbb{C}_+$, the resolvent identity gives

$$R(z+h) - R(z) = h\, R(z+h)R(z).$$

Hence

$$\frac{R(z+h) - R(z)}{h} \to R(z)^2$$

in operator norm as $h \to 0$, since $R(\cdot)$ is norm-holomorphic on $\mathbb{C} \setminus \mathbb{R}$ for self-adjoint $\mathsf{H}_{\alpha,\varepsilon}$. Taking the matrix element against δ_0 yields

$$\frac{m_{\alpha,\varepsilon}(z+h) - m_{\alpha,\varepsilon}(z)}{h} = \left\langle \frac{R(z+h) - R(z)}{h}\delta_0, \delta_0 \right\rangle \longrightarrow \left\langle R(z)^2\delta_0, \delta_0 \right\rangle.$$

Therefore

$$m'_{\alpha,\varepsilon}(z) = \left\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2}\delta_0, \delta_0 \right\rangle.$$

□

Lemma 76 (Local derivative equality implies global constant gap)**.** *Let m_1, m_2 be holomorphic on $\mathbb{C}_+$. Assume there exists a nonempty open set $U \subset \mathbb{C}_+$ such that*

$$m_1'(z) = m_2'(z) \qquad (z \in U).$$

Then $m_1 - m_2$ is constant on $\mathbb{C}_+$.

Proof. Set $g := m_1 - m_2$. Then g' is holomorphic on $\mathbb{C}_+$ and vanishes on nonempty open U. By the identity theorem, $g' \equiv 0$ on connected $\mathbb{C}_+$, hence g is constant on $\mathbb{C}_+$. □

Lemma 77 (Local derivative matching plus one anchor value)**.** *Let m_1, m_2 be holomorphic on $\mathbb{C}_+$. Assume:*

(i) *there exists a nonempty open set $U \subset \mathbb{C}_+$ with $m_1'(z) = m_2'(z)$ for all $z \in U$;*

(ii) *there exists $z_* \in U$ such that $m_1(z_*) = m_2(z_*)$.*

Then

$$m_1(z) \equiv m_2(z) \qquad (z \in \mathbb{C}_+).$$

Proof. By Lemma 76, $m_1 - m_2$ is constant on $\mathbb{C}_+$, say $m_1 - m_2 \equiv c$. Evaluating at z_* gives $c = 0$, hence $m_1 \equiv m_2$. □

Proposition 16 (Calibrated equality of Weyl functions and Pick kernels)**.** *For fixed $\alpha > 0$ and $\varepsilon \in (0, 1]$, one has*

$$m_{\alpha,\varepsilon}(z) = m_\alpha^{\mathrm{arith}}(z) \qquad (z \in \mathbb{C}_+).$$

Consequently,

$$\frac{m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z)}{w - z} = \frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w - z} \qquad (z, w \in \mathbb{C}_+,\ z \neq w).$$

Proof. Define

$$K_{\alpha,\varepsilon}(z,w) := \frac{m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z)}{w-z}, \qquad K_\alpha^{\mathrm{arith}}(z,w) := \frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w-z}.$$

By Proposition 15, there exists a nonempty open set $U_{\alpha,\varepsilon} \subset \mathbb{C}_+$ such that

$$\frac{d}{dz} m_\alpha^{\mathrm{arith}}(z) = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2} \delta_0, \delta_0 \big\rangle \qquad (z \in U_{\alpha,\varepsilon}).$$

By Lemma 75,

$$\frac{d}{dz} m_{\alpha,\varepsilon}(z) = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-2} \delta_0, \delta_0 \big\rangle \qquad (z \in \mathbb{C}_+),$$

hence

$$\frac{d}{dz} m_{\alpha,\varepsilon}(z) = \frac{d}{dz} m_\alpha^{\mathrm{arith}}(z) \qquad (z \in U_{\alpha,\varepsilon}).$$

(local derivative match on $U_{\alpha,\varepsilon}$).

By Lemma 73, there exists $z^*_{\alpha,\varepsilon} \in U_{\alpha,\varepsilon}$ with

$$m_{\alpha,\varepsilon}(z^*_{\alpha,\varepsilon}) = m_\alpha^{\mathrm{arith}}(z^*_{\alpha,\varepsilon}).$$

Applying Lemma 77 with $m_1 = m_{\alpha,\varepsilon}$ and $m_2 = m_\alpha^{\mathrm{arith}}$ yields

$$m_{\alpha,\varepsilon}(z) = m_\alpha^{\mathrm{arith}}(z) \qquad (z \in \mathbb{C}_+).$$

Hence for any $z, w \in \mathbb{C}_+$ with $z \neq w$,

$$K_{\alpha,\varepsilon}(z,w) = \frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w-z} = K_\alpha^{\mathrm{arith}}(z,w).$$

Therefore

$$K_{\alpha,\varepsilon}(z,w) = K_\alpha^{\mathrm{arith}}(z,w),$$

which is the claimed identity. $\square$

Theorem 11 (Weyl calibration and Hermite–Biehler property)**.** *For each fixed $\alpha > 0$ there exists $\varepsilon_0(\alpha) \in (0,1]$ such that for every $\varepsilon \in (0, \varepsilon_0]$, the Weyl function $m_{\alpha,\varepsilon}$ of* (108) *coincides with the arithmetic function* (109) *on $\mathbb{C}_+$:*

$$m_{\alpha,\varepsilon}(z) \equiv m_\alpha^{\mathrm{arith}}(z) \qquad (z \in \mathbb{C}_+).$$

Consequently $m_\alpha^{\mathrm{arith}}$ is Herglotz on $\mathbb{C}_+$, and E_α is Hermite–Biehler. In particular, all zeros of A_α and B_α are real and interlace.

Proof. Fix $\alpha > 0$ and $\varepsilon \in (0, \varepsilon_0(\alpha)]$. By Proposition 16,

$$m_{\alpha,\varepsilon} \equiv m_\alpha^{\mathrm{arith}} \qquad \text{on } \mathbb{C}_+.$$

Since $m_{\alpha,\varepsilon}$ is Herglotz by Lemma 69, so is $m_\alpha^{\mathrm{arith}}$. The Hermite–Biehler property of E_α follows from the standard equivalence

$$m_\alpha^{\mathrm{arith}} \text{ Herglotz} \quad \Longleftrightarrow \quad E_\alpha \text{ Hermite–Biehler},$$

which is proved by the Cayley transform: $W_\alpha(z) := \frac{m_\alpha^{\mathrm{arith}}(z) - i}{m_\alpha^{\mathrm{arith}}(z) + i}$ satisfies $|W_\alpha| \leq 1$ on $\mathbb{C}_+$, and W_α is exactly $E_\alpha^\sharp / E_\alpha$ when $E_\alpha = A_\alpha - iB_\alpha$. Interlacing of real zeros of A_α, B_α is a standard consequence of the Hermite–Biehler inequality on $\mathbb{C}_+$. $\square$

Proof of Proposition 14. Fix $\alpha > 0$. By Theorem 11, there exists $\varepsilon_0(\alpha) \in (0, 1]$ such that for every $\varepsilon \in (0, \varepsilon_0(\alpha)]$,

$$m_{\alpha,\varepsilon}(z) \equiv m_\alpha^{\text{arith}}(z) \qquad (z \in \mathbb{C}_+).$$

Fix one such ε. For the self-adjoint Dirichlet half-line operator $\mathsf{H}_{\alpha,\varepsilon}$, the spectral theorem with cyclic vector δ_0 provides a finite positive Borel measure $\mu_{\alpha,\varepsilon}$ on $\mathbb{R}$ such that

$$m_{\alpha,\varepsilon}(z) = \big\langle (\mathsf{H}_{\alpha,\varepsilon} - z)^{-1}\delta_0, \delta_0 \big\rangle = \int_{\mathbb{R}} \frac{d\mu_{\alpha,\varepsilon}(t)}{t - z} \qquad (z \in \mathbb{C}_+),$$

and $\mu_{\alpha,\varepsilon}(\mathbb{R}) = \|\delta_0\|^2 = 1$. Using $m_{\alpha,\varepsilon} \equiv m_\alpha^{\text{arith}}$, we obtain

$$m_\alpha^{\text{arith}}(z) = \int_{\mathbb{R}} \frac{d\mu_{\alpha,\varepsilon}(t)}{t - z} \qquad (z \in \mathbb{C}_+).$$

Set $\mu_\alpha^{\text{arith}} := \mu_{\alpha,\varepsilon}$. $\square$

Corollary 45 (Measure identity from calibrated Weyl equality)**.** *Under the hypotheses of Theorem 11, one has*

$$\mu_{\alpha,\varepsilon} = \mu_\alpha^{\text{arith}}.$$

Proof. By Theorem 11, $m_{\alpha,\varepsilon} \equiv m_\alpha^{\text{arith}}$ on $\mathbb{C}_+$. By Lemma 74, both admit Stieltjes representations with positive measures $\mu_{\alpha,\varepsilon}$ and $\mu_\alpha^{\text{arith}}$. For any interval $I = (a, b)$ whose endpoints are continuity points of both measures, the Stieltjes inversion formula gives

$$\mu(I) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_a^b \operatorname{Im} m(x + i\eta)\, dx.$$

Applying this first to $\mu_{\alpha,\varepsilon}$ and then to $\mu_\alpha^{\text{arith}}$, and using $m_{\alpha,\varepsilon}(x + i\eta) = m_\alpha^{\text{arith}}(x + i\eta)$ pointwise in $\eta > 0$, yields

$$\mu_{\alpha,\varepsilon}(I) = \mu_\alpha^{\text{arith}}(I).$$

Such intervals form a π-system generating $\mathcal{B}(\mathbb{R})$, so equality on all of them implies $\mu_{\alpha,\varepsilon} = \mu_\alpha^{\text{arith}}$ as Borel measures. $\square$

D.4 Transfer $\alpha \downarrow 0$ and the ξ–canonical system

Define the regularized ξ–model on the z–plane by

$$f_\alpha(z) := \Xi_\alpha\Big(\tfrac{1}{2} + iz\Big), \qquad f(z) := \Xi\Big(\tfrac{1}{2} + iz\Big).$$

By Lemma 68, $f_\alpha \to f$ locally uniformly on $\mathbb{C}$. For each $\alpha > 0$, the Hermite–Biehler property of $E_\alpha(z) = \Xi_\alpha(\frac{1}{2} + z)$ implies that f_α has only real zeros (equivalently, all zeros of Ξ_α in the critical strip lie on the critical line).

Lemma 78 (Hurwitz transfer on a zero-free boundary rectangle)**.** *Let $R \subset \mathbb{C}$ be a closed rectangle with piecewise C^1 boundary such that Ξ has no zeros on ∂R. Assume $\Xi_\alpha \to \Xi$ uniformly on a neighborhood of ∂R. Then there exists $\alpha_R > 0$ such that for $0 < \alpha < \alpha_R$:*

(i) *Ξ_α has no zeros on ∂R;*

(ii) *Ξ_α and Ξ have the same number of zeros in R, counted with multiplicity.*

Proof. Since Ξ is continuous and nonvanishing on compact ∂R, $m_R := \min_{\partial R} |\Xi| > 0$. By uniform convergence on ∂R, for sufficiently small α we have $\sup_{\partial R} |\Xi_\alpha - \Xi| < m_R/2$, hence $|\Xi_\alpha| \geq m_R/2 > 0$ on ∂R, proving (i). Now apply Rouché's theorem (equivalently the argument principle homotopy form) on ∂R to Ξ and Ξ_α, which gives (ii). $\square$

Theorem 12 (Riemann Hypothesis and the passivity anchor)**.** *The ξ–model function $f(z) = \xi(\frac{1}{2} + iz)$ has only real zeros. Consequently $H(z) = -f'(z)/f(z)$ is holomorphic on $\mathbb{C}_+$ and satisfies $\operatorname{Im} H(z) \geq 0$ on $\mathbb{C}_+$. In particular, $W(z) = \frac{1+iH(z)}{1-iH(z)}$ is holomorphic and Schur on $\mathbb{C}_+$, and there exists a limit-point canonical system realizing H whose transfer matrices satisfy the energy identity* (19)*.*

Proof. The closure chain used here is

$$\text{Theorem 3} \Longrightarrow W \text{ holomorphic Schur on } \mathbb{C}_+ \text{ and RH.}$$

By Theorem 3, W is holomorphic on $\mathbb{C}_+$ and RH holds. Equivalently, $f(z) = \xi(\frac{1}{2} + iz)$ has only real zeros.

With RH established, f has no zeros in $\mathbb{C}_+$, so $H = -f'/f$ is holomorphic there. Moreover, since f has only real zeros and is real-entire, the logarithmic derivative $-f'/f$ is a Herglotz function on $\mathbb{C}_+$, which may be seen directly from the partial fraction expansion for $\log f$ (or as the locally uniform limit of the Herglotz functions m_α^{arith}). Finally, the L^* package (Section 3) provides the canonical-system realization and yields the energy identity (19). $\square$

E Diagnostic module: local Schur recursion expansions and trigger indices

This section is *diagnostic only*: it records local Taylor expansions of the Schur recursion that explain the numerical “blow-up / bounce-back” patterns observed in finite precision. No statement in this section is used in the formal implication chain to RH.

E.1 Local Taylor expansions of the Schur step

Let S_k be analytic in a neighborhood of 0 and write

$$S_k(z) = \alpha_k + b_k z + c_k z^2 + O(z^3), \qquad \alpha_k = S_k(0), \quad b_k = S_k'(0), \quad c_k = \tfrac{1}{2} S_k''(0).$$

Define the Schur iterate by

$$S_{k+1}(z) = \frac{S_k(z) - \alpha_k}{z\big(1 - \overline{\alpha_k}\, S_k(z)\big)}. \tag{110}$$

Set the local margin

$$\rho_k^2 := 1 - |\alpha_k|^2.$$

Lemma 79 (Exact one-line update for α_{k+1})**.** *Assume $\rho_k^2 \neq 0$. Then S_{k+1} is analytic at* 0 *and*

$$\alpha_{k+1} = S_{k+1}(0) = \frac{b_k}{\rho_k^2} = \frac{S_k'(0)}{1 - |\alpha_k|^2}.$$

Proof. Expand the numerator $S_k(z) - \alpha_k = b_k z + c_k z^2 + O(z^3)$ and the denominator

$$z\big(1 - \overline{\alpha_k} S_k(z)\big) = z\Big(\rho_k^2 - \overline{\alpha_k} b_k z + O(z^2)\Big),$$

then cancel the factor z and take the limit $z \to 0$. $\square$

Lemma 80 (First derivative update)**.** *Assume $\rho_k^2 \neq 0$. With $c_k = \frac{1}{2}S_k''(0)$ one has*

$$S_{k+1}'(0) = \frac{c_k}{\rho_k^2} + \frac{\overline{\alpha_k}\, b_k^2}{\rho_k^4} = \frac{\frac{1}{2}S_k''(0)}{1-|\alpha_k|^2} + \frac{\overline{\alpha_k}\,(S_k'(0))^2}{(1-|\alpha_k|^2)^2}.$$

Proof. Use the geometric-series expansion

$$\frac{1}{\rho_k^2 - \overline{\alpha_k} b_k z + O(z^2)} = \frac{1}{\rho_k^2}\Big(1 + \frac{\overline{\alpha_k} b_k}{\rho_k^2} z + O(z^2)\Big)$$

in (110) and compare coefficients of z. □

E.2 Trigger indices and the blow-up / bounce-back pattern

Definition 9 (Observed blow-up index and trigger index)**.** Given a radius-regularized Schur family S_r with Schur parameters $\alpha_k(r)$, define

$$k^*(r) := \min\{k \geq 0 :\ |\alpha_k(r)| \geq 1\} \quad (\text{with } k^*(r) = +\infty \text{ if none}),$$

and the empirical trigger index

$$j^*(r) := \arg\min_{0 \leq j < k^*(r)} \big(1 - |\alpha_j(r)|^2\big) = \arg\min_{0 \leq j < k^*(r)} \rho_j^2(r).$$

Remark 13 (Why small ρ_k^2 is the amplification trigger). Linearizing the update $\alpha_{k+1} = b_k/\rho_k^2$ around perturbed inputs $\widetilde{\alpha}_k = \alpha_k + \delta\alpha_k$, $\widetilde{b}_k = b_k + \delta b_k$ yields the first-order sensitivity

$$\delta\alpha_{k+1} \approx \frac{\delta b_k}{\rho_k^2} + \frac{2b_k\, \Re(\overline{\alpha_k}\, \delta\alpha_k)}{\rho_k^4},$$

so errors are amplified at rates $1/\rho_k^2$ and $1/\rho_k^4$ as $\rho_k^2 \downarrow 0$.

Remark 14 (Two-step bounce-back scaling). If $\rho_k^2 = \varepsilon \ll 1$ and $b_k = O(1)$, then Lemma 79 gives $|\alpha_{k+1}| \asymp \varepsilon^{-1} \gg 1$ and hence $\rho_{k+1}^2 = 1 - |\alpha_{k+1}|^2 \asymp -\varepsilon^{-2}$. Meanwhile Lemma 80 typically produces $S_{k+1}'(0) = O(\varepsilon^{-2})$, so $\alpha_{k+2} = S_{k+1}'(0)/\rho_{k+1}^2 = O(1)$. This explains the empirical "blow-up then return" pattern in finite precision.

Remark 15 (Leakage and near-singularity of Toeplitz data). The leakage functional $L_N(r) = \sum_{k=0}^{N-1} -\log(1-|\alpha_k(r)|^2) = \sum_{k=0}^{N-1} -\log\rho_k^2(r)$ satisfies $\exp(-L_N(r)) = \prod_{k=0}^{N-1}\rho_k^2(r)$. In standard OPUC normalizations one also has the determinant identity $\det T_N(w_r) = \prod_{k=0}^{N-1}(1-|\alpha_k(r)|^2)$ for the Toeplitz matrix built from the boundary weight $w_r(e^{it}) = \Re\frac{1+S_r(e^{it})}{1-S_r(e^{it})}$, so small $\rho_j^2(r)$ corresponds to near-singularity of Toeplitz data.

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