A Reduction Framework for the Riemann Hypothesis via Global J-Contractivity of a Limit-Point Canonical System Derived from ξ (Supplement)

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Abstract

This document is a supplement to `A_Reduction_Framework_RH_core`. It collects auxiliary modules (R2 bookkeeping/normal forms, A2 finite-section Toeplitz certificates, empirical/data snapshots, and local Schur diagnostics) that are not used in the logical implication chain to RH in the core document.

1 R2: Dynamic equilibrium of the Schur–canonical cocycle (holonomy + leakage)

This section pins down the *algebraic* content needed in R2. The goal is to compare, in $\mathrm{PGL}(2,\mathbb{C})$, the disk map induced by a one-step transfer matrix with the standard Schur-algorithm step.

Remark 1 (Role of the Schur normal form in this draft)*.* The coercive/internal-closure route in R1 is formulated in the locked value gauge, where the one-step disk map depends on the half-plane parameter z and is not normalized by $F(0) = \alpha$ (Lemma 29). Consequently, denominator control is not automatic and is isolated explicitly in Definition 7. The Schur step $\widehat{S}_{\alpha,\lambda}$ is introduced in R2 as a comparison normal form in $\mathrm{Aut}(\mathbb{D})$ and as a convenient language for gauge-invariant cocycle quantities (holonomy/leakage); it is not used to bypass the R1 denominator-separation bottleneck.

1.1 Möbius maps as PGL(2,C) elements

A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on the Riemann sphere by

$$A \cdot w := \frac{aw + b}{cw + d}, \qquad w \in \widehat{\mathbb{C}}. \tag{1}$$

Two matrices differing by a nonzero scalar define the same Möbius map. Thus the natural home is $\mathrm{PGL}(2,\mathbb{C})$. A convenient equality test (used in the code) is:

$$A \sim B \text{ in } \mathrm{PGL}(2,\mathbb{C}) \quad \Longleftrightarrow \quad A \cdot w_j = B \cdot w_j \text{ for three distinct } w_1, w_2, w_3 \in \widehat{\mathbb{C}}. \tag{2}$$

Lemma 1 (Cross-cancellation test in $\mathrm{PGL}(2,\mathbb{C})$)**.** *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and set $A \cdot w = (aw+b)/(cw+d)$. Fix $\alpha \in \mathbb{D}$ and $\lambda \in \mathbb{D}$. Then $A \cdot w \equiv \widehat{S}_{\alpha,\lambda}(w) = (\alpha + \lambda w)/(1 + \overline{\alpha}\,\lambda w)$ in $\mathrm{PGL}(2,\mathbb{C})$ iff the following polynomial identity holds in w:*

$$(aw + b)(1 + \overline{\alpha}\,\lambda w) - (cw + d)(\alpha + \lambda w) \equiv 0. \tag{3}$$

Equivalently, the coefficients of $1, w, w^2$ *vanish:*

$$\begin{aligned} b - d\alpha &= 0, \\ a + b\overline{\alpha}\,\lambda - c\alpha - d\lambda &= 0, \\ a\overline{\alpha}\,\lambda - c\lambda &= 0. \end{aligned}$$

Proof. The identity of Möbius maps

$$\frac{aw+b}{cw+d} = \frac{\alpha + \lambda w}{1 + \overline{\alpha}\,\lambda w}$$

is equivalent (on the open set where denominators do not vanish) to the cross-multiplied polynomial identity

$$(aw+b)(1+\overline{\alpha}\,\lambda w) - (cw+d)(\alpha + \lambda w) \equiv 0.$$

Since both sides are polynomials of degree at most 2, vanishing identically is equivalent to vanishing of the coefficients of $1, w, w^2$, yielding exactly the three scalar relations displayed above. Conversely, those three relations imply the polynomial identity, hence equality of the two Möbius maps in $\mathrm{PGL}(2, \mathbb{C})$. □

1.2 Cayley transforms and the Tier2 window map

We fix the Cayley maps (cf. §2):

$$C_{\mathrm{val}}(m) = \frac{m - \mathrm{i}}{m + \mathrm{i}}, \qquad C_{\mathrm{val}}^{-1}(w) = \mathrm{i}\,\frac{1+w}{1-w}, \tag{4}$$

$$C_{\mathrm{sp}}(\lambda) = \mathrm{i}\,\frac{1+\lambda}{1-\lambda}, \qquad C_{\mathrm{sp}}^{-1}(\zeta) = \frac{\zeta - \mathrm{i}}{\zeta + \mathrm{i}}. \tag{5}$$

The Tier2 disk-to-half-plane window is

$$z(\lambda) = t_0 + \eta\, C_{\mathrm{sp}}(\lambda) = t_0 + \mathrm{i}\eta\,\frac{1+\lambda}{1-\lambda}. \tag{6}$$

1.3 The Schur one-step map and its matrix

Given a Schur parameter $\alpha \in \mathbb{D}$ and $\lambda \in \mathbb{D}$, define the elementary Schur step

$$\widehat{S}_{\alpha,\lambda}(w) := \frac{\alpha + \lambda w}{1 + \overline{\alpha}\,\lambda w}, \qquad w \in \mathbb{D}. \tag{7}$$

In $\mathrm{PGL}(2, \mathbb{C})$ this is represented by

$$\widehat{S}_{\alpha,\lambda} \;\leftrightarrow\; \begin{pmatrix} \lambda & \alpha \\ \overline{\alpha}\,\lambda & 1 \end{pmatrix}. \tag{8}$$

It is classical that the sequence of Schur parameters uniquely determines the Schur function (obtained as the locally-uniform limit of its Schur iterates). See, e.g., Simon's OPUC notes.

1.4 Dynamic equilibrium viewpoint: cocycles, holonomy, and leakage

The discussion above isolates the *intrinsic* object used throughout the program: a family of disk automorphisms generated by the Schur one–step maps (7) with parameters $\{\alpha_k(r)\}$. Rather than insisting on a particular matrix *presentation* of each step (resolvent form, linear pencil form, or a Dirac/canonical transfer matrix), we treat every step as an element of $\mathrm{Aut}(\mathbb{D}) \simeq \mathrm{PSU}(1,1)$ and track the *gauge–invariant* quantities that survive all conjugations.

A convenient normal form is the "rotation–translation" factorization

$$g(w) = e^{i\theta}\,\tau_a(w), \qquad \tau_a(w) = \frac{a+w}{1+\overline{a}\,w}, \qquad a \in \mathbb{D},\ \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

which is unique for each $g \in \mathrm{Aut}(\mathbb{D})$. In particular, the composition of translations is *not* associative on the nose; its defect is a rotation (the *gyration*).

Theorem 1 (Dynamic equilibrium normal form)**.** *Fix $0 < r < 1$ and $\lambda \in \mathbb{D}$. Let $\{\alpha_k(r)\}_{k\geq 0} \subset \mathbb{D}$ be the Schur parameters of S_r and let*

$$F_k(w) \;:=\; S_{\alpha_k(r),\lambda}(w) = \frac{\alpha_k(r) + \lambda w}{1 + \overline{\alpha_k(r)}\,\lambda w} \qquad (k \geq 0)$$

be the associated elementary steps in $\mathrm{Aut}(\mathbb{D})$*. For each $N \geq 1$ define the N–step cocycle $G_N := F_{N-1} \circ \cdots \circ F_0$.*

Then there exist uniquely determined sequences $(A_N, \Theta_N) \in \mathbb{D} \times (\mathbb{R}/2\pi\mathbb{Z})$ such that

$$G_N(w) = e^{i\Theta_N}\,\tau_{A_N}(w) \qquad (w \in \mathbb{D}),$$

and the updates (A_{N+1}, Θ_{N+1}) are governed by the gyro–addition/gyration identities of $\mathrm{Aut}(\mathbb{D})$ *(cf. Lemma 5 and Lemma 6).*

Moreover, the leakage *functional*

$$L_N(r) := \sum_{k=0}^{N-1} -\log\bigl(1 - |\alpha_k(r)|^2\bigr)$$

is additive along the cocycle and controls the hyperbolic size of G_N; in particular, any two matrix realizations of the same disk maps produce the same *(A_N, Θ_N) and the same leakage $L_N(r)$.*

Proof. For each k, $F_k \in \mathrm{Aut}(\mathbb{D})$, hence $G_N \in \mathrm{Aut}(\mathbb{D})$. Every $g \in \mathrm{Aut}(\mathbb{D})$ has a unique rotation–translation form

$$g = \rho_u \circ \tau_a, \qquad u \in U(1),\ a \in \mathbb{D}$$

(equivalently $g(w) = e^{i\Theta}\tau_A(w)$). Uniqueness follows from $g(0) = ua$ and $g'(0) = u(1 - |a|^2)$, which determine u and then a. Applying this to $g = G_N$ gives unique (A_N, Θ_N).

For the recursion, write $u_N := e^{i\Theta_N}$. Since $F_N = \tau_{\alpha_N(r)} \circ \rho_\lambda$, we have

$$G_{N+1} = F_N \circ G_N = \tau_{\alpha_N(r)} \circ \rho_{\lambda u_N} \circ \tau_{A_N}.$$

Using $\rho_u \circ \tau_a = \tau_{ua} \circ \rho_u$, this becomes

$$G_{N+1} = \tau_{\alpha_N(r)} \circ \tau_{(\lambda u_N)A_N} \circ \rho_{\lambda u_N}.$$

Now apply Lemma 5:

$$\tau_a \circ \tau_b = \tau_{a \oplus b} \circ \rho_{\mathrm{gyr}[a,b]}.$$

Hence

$$G_{N+1} = \tau_{\alpha_N(r) \oplus ((\lambda u_N) A_N)} \circ \rho_{\mathrm{gyr}[\alpha_N(r), (\lambda u_N) A_N]\, \lambda u_N}.$$

Rewriting again in the canonical rotation–translation order yields the next pair (A_{N+1}, Θ_{N+1}); therefore the updates are governed exactly by the gyro-addition/gyration identities. The phase increment is the gyration angle, with explicit series/tail control from Lemma 6.

For leakage, define $\ell_k := -\log(1 - |\alpha_k(r)|^2) \geq 0$. Then

$$L_N(r) = \sum_{k=0}^{N-1} \ell_k, \qquad L_{N+1}(r) = L_N(r) + \ell_N,$$

so additivity is immediate. Also $d_{\mathbb{D}}(0, F_k(0)) = 2 \operatorname{artanh} |\alpha_k(r)|$, and automorphism invariance of $d_{\mathbb{D}}$ plus the triangle inequality gives

$$d_{\mathbb{D}}(0, G_N(0)) \leq \sum_{k=0}^{N-1} 2 \operatorname{artanh} |\alpha_k(r)|,$$

so cumulative leakage controls hyperbolic size of the cocycle.

Finally, if two matrix realizations induce the same disk maps F_k, then they produce the same $G_N \in \mathrm{Aut}(\mathbb{D})$ for each N, hence the same (A_N, Θ_N) by uniqueness of the factorization. Since $|\alpha_k(r)| = |F_k(0)|$ is map-intrinsic, both realizations yield the same $L_N(r)$. □

Lemma 2 (Limit passage for a Schur family as $r \uparrow 1$)**.** *Let $\{S_r\}_{0<r<1}$ be analytic functions on the unit disk $\mathbb{D}$ such that*

$$|S_r(\lambda)| \leq 1 \qquad (\forall\, \lambda \in \mathbb{D},\ \forall\, 0 < r < 1).$$

Assume moreover that for each $\lambda \in \mathbb{D}$ the limit

$$S(\lambda) := \lim_{r \uparrow 1} S_r(\lambda)$$

exists (as a complex number). Then:

1. *S is analytic on $\mathbb{D}$ and $S_r \to S$ uniformly on compact subsets of $\mathbb{D}$.*
2. *S is Schur, i.e. $|S(\lambda)| \leq 1$ for all $\lambda \in \mathbb{D}$.*
3. *In particular, S admits radial boundary values $S(e^{it})$ for a.e. t, and $|S(e^{it})| \leq 1$ a.e. on $\partial\mathbb{D}$.*

Proof. Since $|S_r| \leq 1$ on $\mathbb{D}$, the family $\{S_r\}$ is uniformly bounded; hence it is a normal family. Therefore, for any sequence $r_n \uparrow 1$ there exists a subsequence r_{n_k} and an analytic function g on $\mathbb{D}$ such that $S_{r_{n_k}} \to g$ locally uniformly on $\mathbb{D}$. By the assumed pointwise existence of $S(\lambda) = \lim_{r\uparrow 1} S_r(\lambda)$, we have $g(\lambda) = S(\lambda)$ for every $\lambda \in \mathbb{D}$, hence $g \equiv S$.

This shows that every subsequential local-uniform limit is the same function S; consequently $S_r \to S$ locally uniformly on $\mathbb{D}$ as $r \uparrow 1$ (otherwise one could find a compact $K \Subset \mathbb{D}$, $\varepsilon > 0$, and $r_n \uparrow 1$ with $\sup_K |S_{r_n} - S| \geq \varepsilon$, contradicting normality). Local-uniform convergence implies that S is analytic on $\mathbb{D}$.

Finally, for each fixed $\lambda \in \mathbb{D}$,

$$|S(\lambda)| = \lim_{r \uparrow 1} |S_r(\lambda)| \leq 1,$$

so S is Schur. The a.e. existence and bound of radial limits in (3) follow from Fatou's theorem for bounded analytic functions (equivalently $S \in H^\infty$). □

Remark 2 (Status of the r
$uparrow1$ passage)*.* The limit passage r
$uparrow1$ for the radius-regularized Schur family $\{S_r\}_{0<r<1}$ is *fully closed* by Lemma 2: once the uniform Schur bound $|S_r| \le 1$ is established for each fixed $r < 1$, the limit $S = \lim_{ruparrow1} S_r$ exists as a Schur function on $\mathbb{D}$ with locally uniform convergence. No further analytic regularity input (Szegő, A_2, BMO, etc.) is used elsewhere in the logical chain beyond invoking this lemma.

1.5 Infinity normalization and analytic pinning at $\Im z \to +\infty$

We now record an explicit asymptotic of the ξ-derived logarithmic derivative $H(z) = -f'(z)/f(z)$, $f(z) = \xi(\frac{1}{2} + iz)$, which provides an analytic boundary normalization at infinity. This pins the residual affine gauge in the half-plane (and therefore removes the need for a postcomposed disk automorphism R_0) without any numerical input.

Lemma 3 (Stirling pinning for H)**.** *Let $z = t + iy$ with $y > 0$ and set $s_+(z) := \frac{1}{2} - iz = \frac{1}{2} + y - it$. Then, as $y \to +\infty$ (uniformly for t in compact sets),*

$$H(z) = i\,\frac{\xi'(s_+(z))}{\xi(s_+(z))} = \frac{i}{2}\log\Bigl(\frac{s_+(z)}{2\pi}\Bigr) + O\Bigl(\frac{1}{y}\Bigr),$$

where log *is the principal branch. In particular,*

$$\Im H(t+iy) = \frac{1}{2}\log\Bigl(\frac{y}{2\pi}\Bigr) + O\Bigl(\frac{1}{y}\Bigr), \qquad \Re H(t+iy) = O\Bigl(\frac{1}{y}\Bigr).$$

Proof. By the functional equation $\xi(s) = \xi(1-s)$ we have $f(z) = \xi(\frac{1}{2}+iz) = \xi(\frac{1}{2}-iz) = \xi(s_+(z))$, hence $H(z) = i\,\xi'(s_+(z))/\xi(s_+(z))$. Write $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. For $\Re s_+ = \frac{1}{2} + y > 1$ we have $\zeta(s_+) = 1 + O(2^{-y})$ and $\zeta'(s_+)/\zeta(s_+) = O(2^{-y})$. Using the digamma function $\psi = \Gamma'/\Gamma$ and Stirling's asymptotic $\psi(w) = \log w + O(1/w)$ in the right half-plane (see [15]),

$$\frac{\xi'(s_+)}{\xi(s_+)} = \frac{1}{s_+} + \frac{1}{s_+ - 1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi(s_+/2) + O(2^{-y}) = \frac{1}{2}\log\Bigl(\frac{s_+}{2\pi}\Bigr) + O\Bigl(\frac{1}{y}\Bigr).$$

Multiplying by i yields the claim, and the real/imaginary parts follow from $\log s_+ = \log|s_+| + i\arg(s_+)$ with $|s_+| = y + O(1)$ and $\arg(s_+) = O(1/y)$. $\square$

Corollary 1 (Limit and rate for W)**.** *With $W(z) = \frac{1+iH(z)}{1-iH(z)}$ we have, as $y \to +\infty$,*

$$W(t+iy) = -1 + \frac{4}{\log(y/2\pi)} + o\Bigl(\frac{1}{\log y}\Bigr),$$

in particular $W(t+iy) \to -1$.

Proof. From Lemma 3, $H(t+iy) = iL + o(1)$ with $L \to +\infty$. Then $W = (1-L+o(1))/(1+L+o(1))$ and $1+W = 2/(1+L)+o(1/L)$. Since $L = \Im H = \frac{1}{2}\log(y/2\pi)+o(\log y)$, the stated rate follows. $\square$

In view of the R_0 normalization step, the subsequent R_0-discussion can be read as an *optional* computational diagnostic (useful when comparing finite-step truncations and convention choices), not as a logical bottleneck in the analytic argument.

1.6 Entropy sum rule and the "independent inequality" anchor

This subsection records the one place where the argument must become genuinely *global* (hence non-circular): an entropy/energy identity whose integrand is *pointwise nonnegative*. It plays the role of an *independent inequality* that can be invoked without any appeal to Schur–Pick realization, target identification, or inner/outer factor arguments.

Entropy functional. For a Schur spectral function w on $\mathbb{C}_+$, define the (normalized) entropy

$$\mathcal{I}(w) := \frac{1}{\pi} \int_{\mathbb{R}} \log\Big(\frac{1}{1-|w(x)|^2}\Big) \frac{dx}{1+x^2} \in [0, \infty]. \tag{9}$$

Finiteness of $\mathcal{I}(w)$ is the Szegő condition for canonical systems.

Lemma 4 (Pointwise nonnegativity and rigidity)**.** *For every* 2×2 *positive semidefinite matrix* $\mathcal{H} \succeq 0$*,*

$$\operatorname{tr} \mathcal{H} - 2\sqrt{\det \mathcal{H}} \;\geq\; 0, \tag{10}$$

with equality iff $\mathcal{H}$ *has equal eigenvalues (equivalently,* $\mathcal{H} = \lambda I$ *for some* $\lambda \geq 0$*).*

Proof. Let $\lambda_1, \lambda_2 \geq 0$ be eigenvalues of $\mathcal{H}$. Then $\operatorname{tr} \mathcal{H} - 2\sqrt{\det \mathcal{H}} = (\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 \geq 0$, and equality is equivalent to $\lambda_1 = \lambda_2$. □

1.7 Optional calibration by a fixed disk automorphism R_0

The left boundary condition of the underlying canonical/Dirac system induces a fixed conjugacy $R_0 \in \operatorname{Aut}(\mathbb{D})$ on the disk-valued state variable. Empirically (and in standard Weyl theory), the discrepancy concentrates in the first few parameters. Mathematically, one selects R_0 by matching the "initial" Weyl value (or equivalently the first Schur parameter) of the model to the normalization used in the witness.

Every disk automorphism has the standard form

$$R_0(w) = e^{\mathrm{i}\phi} \frac{w-a}{1-\overline{a}\, w}, \qquad a \in \mathbb{D},\ \phi \in \mathbb{R}, \tag{11}$$

and can be represented (up to nonzero scalar) by a matrix $R_{0,\mathrm{mat}} \in \mathrm{PGL}(2, \mathbb{C})$. In the intended closure, R_0 is *fixed once and for all* by the left boundary normalization; it is not a per-step gauge. A practical paper-level way to pin it down is: (i) match a single initial Weyl/disk value (fixing a), and (ii) match one more noncollinear point or derivative/phase condition (fixing ϕ).

Gyro-decomposition (disk translation + pure-phase defect). It is sometimes useful to make explicit the fact that the only "noncommutative / nonassociative defect" of disk translations is a *pure phase* (a $U(1)$-rotation). This is the unit-disk analogue of the Thomas–Wigner rotation in special relativity, and it is the precise algebraic content behind our "gyro-rotation ledger" diagnostics; see e.g. [16, 14].

Definition 1 (Disk translations, Möbius addition, and rotations)**.** For $a \in \mathbb{D}$, define the disk translation (an automorphism of $\mathbb{D}$)

$$\tau_a(w) := \frac{a+w}{1+\overline{a}\, w}, \qquad w \in \mathbb{D},$$

and define the corresponding Möbius addition on $\mathbb{D}$ by

$$a \oplus b \;:=\; \tau_a(b) = \frac{a+b}{1+\overline{a}\, b} \qquad (a, b \in \mathbb{D}).$$

For $u \in U(1)$, write $\rho_u(w) := u\, w$ for the disk rotation.

Lemma 5 (Gyroassociativity and the gyration phase)**.** *For all $a, b, w \in \mathbb{D}$,*

$$a \oplus (b \oplus w) \;=\; (a \oplus b) \oplus \big(\mathrm{gyr}[a,b]\; w\big), \tag{12}$$

where the gyration *is the unimodular scalar*

$$\mathrm{gyr}[a,b] \;:=\; \frac{1 + a\bar{b}}{1 + \bar{a}\, b} \in U(1). \tag{13}$$

Equivalently, in $\mathrm{Aut}(\mathbb{D})$ *one has the factorization*

$$\tau_a \circ \tau_b \;=\; \tau_{a \oplus b} \circ \rho_{\mathrm{gyr}[a,b]}. \tag{14}$$

Proof. A direct calculation shows that the difference between the two sides of (12) is a rational function in w whose numerator is

$$-w\,(1-|a|^2)(1-|b|^2)\,\Big(-(a\bar{b}) + (\bar{a}\,b)\,\mathrm{gyr}[a,b] + \mathrm{gyr}[a,b] - 1\Big).$$

Setting the parenthesis to zero yields (13), and then (12) follows. The composition identity (14) is the same statement written in the map notation $\tau_a(w) = a \oplus w$ and $\rho_u(w) = uw$. □

Lemma 6 (Gyration angle: exact series and a tail bound)**.** *Let $\varepsilon := a\bar{b}$. Then*

$$\mathrm{gyr}[a,b] = \frac{1+\varepsilon}{1+\bar{\varepsilon}} = \frac{1+\varepsilon}{\overline{(1+\varepsilon)}} \in U(1), \qquad \phi(a,b) := \arg(\mathrm{gyr}[a,b]) = 2\ \mathrm{Arg}(1+\varepsilon).$$

For $|\varepsilon| < 1$ one has the convergent expansion

$$\log \mathrm{gyr}[a,b] = \log(1+\varepsilon) - \log(1+\bar{\varepsilon}) = 2\mathrm{i} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n}\, \mathrm{Im}(\varepsilon^n), \tag{15}$$

hence

$$\phi(a,b) = 2 \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\, \mathrm{Im}(\varepsilon^n) \;\approx\; 2\ \mathrm{Im}(a\bar{b}) - \mathrm{Im}\big((a\bar{b})^2\big) + \frac{2}{3}\,\mathrm{Im}\big((a\bar{b})^3\big) - \cdots. \tag{16}$$

Moreover,

$$|\phi(a,b)| = 2\big|\mathrm{Arg}(1+\varepsilon)\big| \leq \frac{2|\varepsilon|}{1-|\varepsilon|} \leq \frac{2|a||b|}{1-|a||b|}. \tag{17}$$

Proof. The identity $\mathrm{gyr}[a,b] = (1+\varepsilon)/\overline{(1+\varepsilon)}$ is immediate from (13), and it implies $\phi(a,b) = 2\ \mathrm{Arg}(1+\varepsilon)$. For $|\varepsilon| < 1$, apply $\log(1+z) = \sum_{n\geq 1}(-1)^{n+1} z^n/n$ to obtain (15); taking imaginary parts yields (16). Finally, $|\,\mathrm{Arg}(1+\varepsilon)| \leq \arctan\big(|\,\mathrm{Im}\,\varepsilon|/(1+\mathrm{Re}\,\varepsilon)\big) \leq |\varepsilon|/(1-|\varepsilon|)$ gives (17). □

Lemma 7 (Tail convergence of the cumulative gyration phase at fixed $r < 1$)**.** *Fix $r \in (0,1)$ and suppose $S_r : \mathbb{D} \to \mathbb{D}$ is holomorphic and extends holomorphically to $|z| < R$ for some $R > 1$ (for instance, $S_r(\lambda) = S(r\lambda)$ with S holomorphic on $\mathbb{D}$ gives $R = 1/r$). Let $\{\alpha_k(r)\}_{k\geq 0}$ be the Schur parameters of S_r. Then the tail is exponentially small: there exist $C > 0$ and $\gamma \in (0,1)$ such that*

$$|\alpha_k(r)| \leq C\,\gamma^k \qquad (k \geq 0), \tag{18}$$

and in particular $\sum_{k\geq 0} |\alpha_k(r)| < \infty$ and $\sum_{k \geq 0} |\alpha_k(r)|^2 < \infty$. Now define the per-step gyration scalar

$$u_k := \mathrm{gyr}[A_{k-1}, \alpha_k(r)] \in U(1), \qquad U_N := \prod_{k=0}^{N-1} u_k = e^{\mathrm{i}\Phi_N}. \tag{19}$$

If $\sup_k |A_k| \le 1$ *(automatic for the disk-automorphism recursion), then* Φ_N *converges as* $N \to \infty$ *and, for* N *large enough that* $|A_{k-1}\alpha_k(r)| \le \frac{1}{2}$ *for all* $k \ge N$*, the tail admits the quantitative bound*

$$|\Phi_\infty - \Phi_N| \le 4 \sum_{k \ge N} |A_{k-1}|\,|\alpha_k(r)| + O\Big(\sum_{k \ge N} |\alpha_k(r)|^2\Big). \tag{20}$$

Proof. The exponential decay (18) (equivalently: analytic continuation of the Szegő function / scattering function across $\partial\mathbb{D}$) is a classical Baxter-type phenomenon; see for example [2, 3] and the references therein. For the phase, write $u_k = (1+\overline{A_{k-1}\alpha_k})/(1+A_{k-1}\overline{\alpha_k})$ and set $\varepsilon_k := A_{k-1}\overline{\alpha_k}$. For $|\varepsilon_k| \le \frac{1}{2}$, Lemma 6 gives $|\arg u_k| \le 4|\varepsilon_k| + O(|\varepsilon_k|^2)$. Summability of $|\alpha_k|$ implies $\sum_{k\ge0} |\varepsilon_k| < \infty$ and $\sum_{k\ge0} |\varepsilon_k|^2 < \infty$, hence $\sum_k \arg u_k$ converges absolutely up to a harmless choice of branch, and (20) follows by summing the bound on $|\arg u_k|$ over $k \ge N$. □

Analytic pinning of R_0 from a 1-jet. The preceding lemmas explain why "rotation" effects can concentrate at the head when the tail has small amplitude. Independently, the global calibration R_0 can be pinned *deterministically* by an initial value and one derivative (= a first jet), without any multi-point fitting; see Lemma 17.

Corollary 2 (Schur-parameter form of the 1-jet calibration)**.** *Let* $S_{\rm can}, S_{\rm tgt} : \mathbb{D} \to \mathbb{D}$ *be holomorphic and assume* $S_{\rm tgt} = R_0 \circ S_{\rm can}$ *with*

$$R_0(w) = e^{i\phi}\,\frac{w-a}{1-\overline{a}\,w}, \qquad a \in \mathbb{D},\ \phi \in \mathbb{R}. \tag{21}$$

Define the first two Schur data at $\lambda = 0$ *by*

$$\alpha_0 := S_{\rm can}(0), \quad \alpha_1 := \frac{S'_{\rm can}(0)}{1-|\alpha_0|^2}, \qquad \beta_0 := S_{\rm tgt}(0), \quad \beta_1 := \frac{S'_{\rm tgt}(0)}{1-|\beta_0|^2}.$$

Then

$$\beta_0 = e^{i\phi}\,\frac{\alpha_0 - a}{1-\overline{a}\,\alpha_0}, \qquad \beta_1 = e^{i\phi}\,\alpha_1\,\frac{1-a\overline{\alpha_0}}{1-\overline{a}\,\alpha_0}. \tag{22}$$

In particular, $|\beta_1| = |\alpha_1|$*. If* $\alpha_1 \ne 0$ *and* $D := \alpha_1\beta_0\overline{\alpha_0} - \beta_1 \ne 0$*, then* a *and* $e^{i\phi}$ *are uniquely determined by*

$$a = \frac{N}{D}, \qquad N := \alpha_1\beta_0 - \beta_1\alpha_0, \qquad e^{i\phi} = \beta_0\frac{1-\overline{a}\,\alpha_0}{\alpha_0 - a} = \frac{\overline{\beta_1}\,D}{\overline{D}\,\alpha_1}. \tag{23}$$

If $\alpha_1 = 0$*, then* $\beta_1 = 0$ *and only the value constraint in* (22) *remains.*

Proof. The first relation in (22) is immediate from $\beta_0 = S_{\rm tgt}(0) = R_0(\alpha_0)$.

Differentiate (21):

$$R_0'(w) = e^{i\phi}\frac{1-|a|^2}{(1-\overline{a}\,w)^2}.$$

Since $S_{\rm tgt} = R_0 \circ S_{\rm can}$,

$$S'_{\rm tgt}(0) = R_0'(\alpha_0)\,S'_{\rm can}(0).$$

Also, for disk automorphisms,

$$1-|\beta_0|^2 = \frac{(1-|a|^2)(1-|\alpha_0|^2)}{|1-\overline{a}\,\alpha_0|^2}.$$

Divide the derivative identity by $1-|\beta_0|^2$ to obtain

$$\beta_1 = e^{i\phi}\,\alpha_1\,\frac{1-a\overline{\alpha_0}}{1-\overline{a}\,\alpha_0}.$$

Taking moduli gives $|\beta_1| = |\alpha_1|$.

Assume $\alpha_1 \neq 0$. Eliminating $e^{i\phi}$ from the two relations in (22) gives

$$\beta_1(\alpha_0 - a) = \alpha_1\beta_0(1-a\overline{\alpha_0}),$$

hence

$$a\big(\alpha_1\beta_0\overline{\alpha_0} - \beta_1\big) \;=\; \alpha_1\beta_0 - \beta_1\alpha_0.$$

If $D := \alpha_1\beta_0\overline{\alpha_0} - \beta_1 \neq 0$, this yields $a = N/D$ with $N := \alpha_1\beta_0 - \beta_1\alpha_0$. Then

$$e^{i\phi} = \beta_0\,\frac{1-\overline{a}\,\alpha_0}{\alpha_0 - a}$$

follows from the first relation.

For the second closed form, use $a = N/D$ to compute

$$1-a\overline{\alpha_0} = \frac{D-N\overline{\alpha_0}}{D} = -\frac{\beta_1(1-|\alpha_0|^2)}{D},$$

and similarly

$$1-\overline{a}\,\alpha_0 = -\frac{\overline{\beta_1}(1-|\alpha_0|^2)}{\overline{D}}.$$

Hence

$$\frac{1-\overline{a}\,\alpha_0}{1-a\overline{\alpha_0}} = \frac{\overline{\beta_1}\,D}{\beta_1\,\overline{D}},$$

and substituting into $e^{i\phi} = \frac{\beta_1}{\alpha_1}\frac{1-\overline{a}\,\alpha_0}{1-a\overline{\alpha_0}}$ gives

$$e^{i\phi} = \frac{\overline{\beta_1}\,D}{\overline{D}\,\alpha_1}.$$

If $\alpha_1 = 0$, then $S'_{\mathrm{can}}(0) = 0$, so by the chain rule $S'_{\mathrm{tgt}}(0) = R'_0(\alpha_0)S'_{\mathrm{can}}(0) = 0$, i.e. $\beta_1 = 0$. $\square$

Three-point test conventions and ζ-map stability. The step-comparison (optional) check should compare *induced disk maps* (Möbius actions) rather than raw matrix entries. If $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ acts by $A\cdot w = (aw+b)/(cw+d)$, and $C_{\mathrm{val}}(m) = (m-i)/(m+i)$ is the value-Cayley map, then the *canonical step* acts on the disk variable w by

$$F_k(w;\lambda) \;:=\; C_{\mathrm{val}}\Big(M_k(\zeta(\lambda))\cdot C_{\mathrm{val}}^{-1}(w)\Big), \qquad \zeta(\lambda) = i\,\frac{1+\lambda}{1-\lambda}.$$

The corresponding Schur elementary step is

$$S_{\alpha_k,\lambda}(w) \;:=\; \frac{\alpha_k + \lambda w}{1+\overline{\alpha_k}\,\lambda w}.$$

A robust "three-point" equality test for two disk maps F and G is: pick three distinct points $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ (e.g. $0, 1, \infty$), and check $F(w_j) = G(w_j)$ for $j = 1,2,3$; equivalently, verify the polynomial identity

$$(\widehat{A}w + \widehat{B})(1+\overline{\alpha}\lambda w) - (\widehat{C}w + \widehat{D})(\alpha + \lambda w) \equiv 0$$

for the 2×2 matrix $\begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{pmatrix}$ representing the conjugated canonical step.

Empirically, the auxiliary "ζ-map" extracted by this three-point checker is itself (to numerical precision) a Möbius transformation, and it is *stable* across the tail indices k. In the same $r = 0.9$ BASE run, scanning $\zeta_{\text{in}} \in \{0.07, 0.13, 0.19, 0.26, 0.31, 0.37\}$ with $\lambda = r\,\zeta_{\text{in}}$ and fitting

$$\zeta_{\text{ext}} \;\approx\; \frac{a\,\lambda + b}{c\,\lambda + 1}$$

returns max/mean fit errors $\lesssim 5 \times 10^{-13}$ across multiple holdout sizes and random seeds. For $k = 59$ one fit is

$$\begin{aligned} a &= 0.9989446722607414 - 0.04592930202847462\,i, \\ b &= -0.9989952105995181 + 0.0437306408910120\,i, \\ c &= -0.9999494610534518 + 0.0021986620182138875\,i, \end{aligned}$$

while for $k = 159$ it is

$$\begin{aligned} a &= 0.998944698511809 - 0.045928084988279536\,i, \\ b &= -0.9989952663608586 + 0.04372939870580888\,i, \\ c &= -0.9999494327013837 + 0.0021986884920096347\,i. \end{aligned}$$

Thus $|a_{59} - a_{159}| \sim 1.2 \times 10^{-6}$, $|b_{59} - b_{159}| \sim 1.2 \times 10^{-6}$, and $|c_{59} - c_{159}| \sim 4 \times 10^{-8}$. This k-independence is consistent with a *global* coordinate normalization (a fixed conjugacy), and it supports the paper's stance that step-comparison (optional) should be formulated via induced disk maps and a single global boundary calibration R_0.

A A2: finite-section Toeplitz contraction and a certified PSD lower bound

(Not used in the logical chain.) This section provides a finite-section certificate and numerical illustrations ("A2") for the Schur/contractive behavior of the regularized pullback $S_r(\lambda) = W(z(r\lambda))$. No statement in the RH implication chain depends on these computations; the deduction proceeds from the structural equivalences (L^*), the canonical-system limit-point property (R1), and the bridge hypotheses articulated in §1.

This section isolates the *finite-section* statement we can certify rigorously from samples: given the Toeplitz matrix T_n built from the coefficients of $S_r(\lambda) = S(r\lambda)$, we certify a positive lower bound on

$$G_n \;:=\; I - T_n T_n^* \;\succeq\; 0,$$

without invoking any unstable global tail bound.

A.1 Why we use $S_r(\lambda) = S(r\lambda)$

Recall $S(\lambda) = W(z(\lambda))$ is analytic on $\mathbb{D}$ by construction (Tier2 pullback). In this section we use the standard r–regularization $S_r(\lambda) = S(r\lambda)$, $0 < r < 1$, so that S_r is analytic on a neighborhood of $\overline{\mathbb{D}}$ and its Taylor coefficients decay geometrically; this makes all Toeplitz and H^2 manipulations unconditional and numerically stable. The Schur/contractive property of the limit S is addressed elsewhere in the program; A2 only certifies the finite-section inequality for S_r. The operator-theoretic Toeplitz contraction lives on $H^2(\mathbb{D})$; the standard finite-section is formed from the Taylor coefficients of the *radius-shrunk* function $S_r(\lambda) = S(r\lambda)$, $0 < r < 1$, because S_r is bounded on $\overline{\mathbb{D}}$ and its coefficients decay geometrically. All finite Toeplitz constructions below use S_r, not raw S.

A.2 Cholesky residual certificate: a fully rigorous eigenvalue lower bound

Let G_n be the Hermitian matrix above. We want a certified $\delta > 0$ such that $G_n \succeq \delta I$, i.e. $\lambda_{\min}(G_n) \geq \delta$.

Lemma 8 (Residual-based PSD certification)**.** *Let A be Hermitian. Suppose we have an* approximate *Cholesky factor $\widetilde{L}$ (lower triangular) and define the residual*

$$R := A - \widetilde{L}\widetilde{L}^*.$$

If

$$\sigma_{\min}(\widetilde{L})^2 \; > \; \|R\|_2,$$

then $A \succ 0$ and moreover

$$\lambda_{\min}(A) \; \geq \; \sigma_{\min}(\widetilde{L})^2 - \|R\|_2 \; > 0.$$

Proof. Write $A = \widetilde{L}\widetilde{L}^* + R$. For any unit vector x,

$$x^* A x = \|\widetilde{L}^* x\|^2 + x^* R x \geq \sigma_{\min}(\widetilde{L})^2 - \|R\|_2.$$

Taking the infimum over $\|x\| = 1$ yields the claim. □

Certified δ construction. Set $A(\delta) := G_n - \delta I$. We choose an initial upper bound

$$\delta_{\text{hi}} := \min_i (G_n)_{ii} - \texttt{safety},$$

and then shrink δ (by bisection) until Lemma 8 certifies $A(\delta) \succ 0$. The resulting value δ_{cert} is a *rigorous* lower bound on $\lambda_{\min}(G_n)$ for that fixed (n, r).

A.3 Example: $r = 0.999$ finite-section certificate across n

At $t_0 = 109.099073$, $\eta = 0.12$, $r = 0.999$ (BASE sweep; $M = 2048$, $N = 320$, dps= 80), our v5 certificate returns strictly positive δ_{cert} for all tested n:

n	δ_{cert}	$\max_{\lvert\lambda\rvert=1} \lvert S_r(\lambda)\rvert$	$\lambda_{\min}(G_n)$ (float)
40	0.000144682136057	0.999935577779	0.000182274251789
80	0.000130287167722	0.999935577779	0.000182131047539
120	0.000123062494325	0.999935577779	0.000182104658436
160	0.000109394259632	0.999935577779	0.000182095428378
200	$9.61243788622e-05$	0.999935577779	0.00018209116094
240	0.000180900531113	0.999935577779	0.000182088843568
280	0.00017283980688	0.999935577779	0.000182087446752
320	0.000168831286709	0.999935577779	0.000182086540147

Interpretation. Even when older "global tail" bounds explode as $r \to 1$ (causing v4_strict to print `A2_pass=False`), the *finite-section* matrix G_n remains PSD with a certified positive gap. This is exactly what the A2 finite-section step is supposed to guarantee.

A.4 Operator and multiplier interpretation of the finite-section A2 certificate

This subsection records a standard operator-theoretic identity that clarifies what our finite-section matrix $G_n = I - T_{n,S_r} T_{n,S_r}^*$ is certifying, and why the radius-regularization $S_r(\lambda) = S(r\lambda)$ is natural.

Hardy space and de Branges–Rovnyak kernel. Let $H^2(\mathbb{D})$ be the Hardy space on the unit disk and let

$$k_w(z) := \frac{1}{1 - z\overline{w}} \qquad (z, w \in \mathbb{D})$$

be the Szegő kernel. For an analytic function S on $\mathbb{D}$, define the de Branges–Rovnyak / Pick kernel

$$K_S(z, w) := \frac{1 - S(z)\overline{S(w)}}{1 - z\overline{w}}. \tag{24}$$

It is classical that S is Schur on $\mathbb{D}$ (i.e. $\|S\|_\infty \le 1$) if and only if the kernel K_S is positive semidefinite (all finite Gram matrices $(K_S(z_j, z_k))_{j,k}$ are PSD).

Kernel identity as an operator equality. Let M_S be multiplication by S on $H^2(\mathbb{D})$: $(M_S f)(z) = S(z)f(z)$. Using the reproducing property $\langle f, k_w \rangle = f(w)$ and $M_S^* k_w = \overline{S(w)}\, k_w$, we have for all $z, w \in \mathbb{D}$,

$$\big\langle (I - M_S M_S^*)k_w,\, k_z \big\rangle = \frac{1 - S(z)\overline{S(w)}}{1 - z\overline{w}} = K_S(z, w). \tag{25}$$

Thus the positive kernel (24) is exactly the integral kernel (in the Szegő basis) of the positive operator $I - M_S M_S^*$.

Two-variable analytic kernel and Schwarz reflection. Let S be analytic on $\mathbb{D}$ and define the coefficientwise conjugate

$$S^\#(z) := \overline{S(\overline{z})} \qquad (z \in \mathbb{D}).$$

Then $S^\#$ is analytic on $\mathbb{D}$ and satisfies $\overline{S(w)} = S^\#(\overline{w})$. For $0 < r < 1$ set $S_r(z) = S(rz)$ and define the *two-variable analytic kernel*

$$\Phi_r(z, \zeta) \;:=\; \frac{1 - S_r(z)\, S_r^\#(\zeta)}{1 - z\zeta} \qquad (z, \zeta \in \mathbb{D}). \tag{26}$$

This Φ_r is holomorphic in the pair (z, ζ) on a neighborhood of $\overline{\mathbb{D}}^2$ (because S_r is analytic on a neighborhood of $\overline{\mathbb{D}}$). Moreover the de Branges–Rovnyak kernel is obtained by the specialization

$$K_{S_r}(z, w) = \frac{1 - S_r(z)\overline{S_r(w)}}{1 - z\overline{w}} = \Phi_r(z, \overline{w}).$$

In particular, *holomorphy in two variables* is carried by Φ_r, while *positivity* is a property of the Hermitian specialization $K_{S_r}(z, w) = \Phi_r(z, \overline{w})$.

Membership Lemma (unconditional for $0 < r < 1$). Because Φ_r extends holomorphically to a neighborhood of the closed bidisk $\overline{\mathbb{D}}^2$, its Taylor series on $\mathbb{D}^2$ has absolutely summable coefficients; equivalently, Φ_r belongs to the bidisk Wiener algebra (or "$CA(\mathbb{D}^2)$" in the notation of [7]). By Theorem 4.1 of Aleksandrov–Peller [7], this implies that Φ_r is an *analytic Schur multiplier* (an element of $MA(\mathbb{D}^2)$), hence admits a Haagerup factorization

$$\Phi_r(z, \zeta) = \sum_{m \ge 1} \varphi_{r,m}(z)\, \psi_{r,m}(\zeta), \qquad \sum_{m \ge 1} \|\varphi_{r,m}\|_\infty \, \|\psi_{r,m}\|_\infty < \infty,$$

with analytic $\varphi_{r,m}, \psi_{r,m}$ on $\mathbb{D}$.

Operator square and Gram factorization (Schur case). If S_r is a Schur function (equivalently, $\|M_{S_r}\| \le 1$), then K_{S_r} is a positive kernel and (25) yields the positive operator

$$I - M_{S_r} M_{S_r}^* \succeq 0.$$

Let $\Gamma_r := (I - M_{S_r} M_{S_r}^*)^{1/2}$ be its positive square root. For any orthonormal basis $\{e_m\}_{m\ge1}$ of $H^2(\mathbb{D})$, the functions $g_{r,m} := \Gamma_r e_m \in H^2(\mathbb{D})$ are analytic and satisfy the Kolmogorov–Gram factorization

$$K_{S_r}(z,w) = \sum_{m\ge1} g_{r,m}(z)\,\overline{g_{r,m}(w)} \qquad (z,w \in \mathbb{D}), \tag{27}$$

and the operator identity

$$I - M_{S_r} M_{S_r}^* = \Gamma_r \Gamma_r^* \succeq 0. \tag{28}$$

We use [7] to emphasize that the *analytic* two-variable kernel Φ_r lies in the Haagerup tensor class (via $MA(\mathbb{D}^2)$), which is exactly the framework needed to interpret finite Toeplitz sections as compressions (next paragraph) and to connect the computational certificate (A2) to a standard analytic multiplier theory.

Finite Toeplitz sections are automatic compressions. Let $\mathcal{P}_{n-1} = \operatorname{span}\{1, z, \dots, z^{n-1}\} \subset H^2(\mathbb{D})$ and let P_n be orthogonal projection onto $\mathcal{P}_{n-1}$. Our finite Toeplitz section is the compression $T_{n,S_r} := P_n M_{S_r}\big|_{\mathcal{P}_{n-1}}$. Compressing (28) gives

$$I - T_{n,S_r} T_{n,S_r}^* = P_n (I - M_{S_r} M_{S_r}^*) P_n = (P_n\Gamma_r)(P_n\Gamma_r)^* \ \succeq\ 0. \tag{29}$$

Therefore the finite-section matrices G_n are PSD whenever S_r is Schur, and our Cholesky residual certificate is a numerical "PSD certificate" for the compression in (29).

What remains (and what does *not* remain) for A2. The finite-section step (A2) is a *computer-assisted certificate* about the truncated Toeplitz matrices built from numerically computed samples of S_r. It is therefore used only as a robustness/consistency check and plays no role in the formal implication "global Schur/Herglotz $\Rightarrow$ RH" (which is handled by the analytic package $L^\star$). In particular, we do *not* build any downstream logical step on A2. The genuinely analytic input is to establish the *global* Schur/Herglotz property for the specific ξ-derived pullback $S_r(\lambda) = W(z(r\lambda))$ (equivalently $\Im H \ge 0$ on $\mathbb{C}_+$), together with the normalization at infinity that fixes the boundary calibration; see §1.

B Empirical separation: BASE/CRIT pass vs OFFAXIS fail

A consistent numerical pattern across multiple windows is:

- **BASE (true ξ):** $\rho \ge 0$; Pick matrices are PSD; Toeplitz sections are contractive.
- **OFFAXIS injection:** ρ sign-flips and Pick/Toeplitz tests fail strongly.

The worst-tested Tier2 center is $t_0 \approx 109.099073$ at scale $\eta \approx 0.12$, where BASE remains strictly inside the Schur disk while OFFAXIS injections yield robust violations (e.g. $\max|S| \gg 1$, negative Toeplitz PSD eigenvalues).

C Reproducible parameters and data snapshots

Tier2 window map.

$$z(\lambda) = t_0 + \mathrm{i}\eta \frac{1+\lambda}{1-\lambda}, \qquad S(\lambda) = W(z(\lambda)).$$

Fixed parameters (current best-known).

- Center $t_0 = 109.099073$, scale $\eta = 0.12$.
- Sampling: $M = 2048$ points on $|\lambda| = r$, coefficient truncation $N = 320$, multiprecision dps = 80.
- Rotation gauge: $\theta_* = 13.9804951740°$.
- Möbius/Blaschke correction parameters (used in the normalized Schur samples):

$$a = 0.16459943079112785 + 0.9759713935563084\,\mathrm{i},$$
$$k = 0.06396375824179397 - 0.9979522221186671\,\mathrm{i}.$$

- Disk automorphism calibration (golden $r = 0.9$): the fixed conjugacy used to align the canonical (Hamiltonian) output with the target normalization is

$$R_0(w) = e^{\mathrm{i}\phi}\,\frac{w - a_0}{1 - \overline{a_0}\,w},$$
$$(a_0, \phi) = \Big(-0.9990313692627192 + 0.04212179778532015\,\mathrm{i},\ 3.0728924217715075\Big) \quad \text{(scaled)},$$
$$(a_0, \phi) = \Big(-0.9996267280285634 + 0.0184460001274622\,\mathrm{i},\ 3.0728924217715075\Big) \quad \text{(RAW, } \zeta\text{-scan)}.$$

Ultimate R_0 splice/stability check (Arov–Dirac gauge). We ran the script `verify_R0_splice_and_stability_v5_ultimate.py` on the file `alphas_golden_R0_scaled_r0.9_M2048_K160_pad1024_theta13.9804951740.npy` with test points $\lambda \in \{0, 0.45, -0.45, 0.45i\}$ and a simple train/holdout split. Using *Arov–Dirac matrix generation* (no Cayley artifacts), the maximum holdout residual was 1.76×10^{-13} and the bulk gauge discrepancy was 5.09×10^{-6}, while the cross-ratio invariant checks remained 0 whenever well-defined. The reported large "drift_to_bulk" values at deep indices coincide with extreme conditioning (`condA` $\gg 1$) of the 3-point PGL solve and are therefore not a reliable gauge metric; the action residuals on the test points are the stable diagnostics.

Trace statistics from the r-sweep (K=160). The following are extracted from the stored sweep summaries:

r	$\sum_{k<160} \operatorname{tr} H_k$	$\max_{k<160} \operatorname{tr} H_k$	mean $\operatorname{tr} H_k$
0.90	174.918	9.088	1.093
0.93	178.744	12.898	1.117
0.95	183.620	17.746	1.148
0.97	194.076	28.121	1.213
0.98	205.570	39.505	1.285
0.99	232.343	65.940	1.452
0.995	265.765	98.733	1.661
0.998	308.744	140.074	1.930
0.999	329.083	158.823	2.057

D Diagnostic module: local Schur recursion expansions and trigger indices

This section is *diagnostic only*: it records local Taylor expansions of the Schur recursion that explain the numerical "blow-up / bounce-back" patterns observed in finite precision. No statement in this section is used in the formal implication chain to RH.

D.1 Local Taylor expansions of the Schur step

Let S_k be analytic in a neighborhood of 0 and write

$$S_k(z) = \alpha_k + b_k z + c_k z^2 + O(z^3), \qquad \alpha_k = S_k(0), \quad b_k = S_k'(0), \quad c_k = \tfrac{1}{2}S_k''(0).$$

Define the Schur iterate by

$$S_{k+1}(z) = \frac{S_k(z) - \alpha_k}{z\big(1 - \overline{\alpha_k}\, S_k(z)\big)}. \tag{30}$$

Set the local margin

$$\rho_k^2 := 1 - |\alpha_k|^2.$$

Lemma 9 (Exact one-line update for α_{k+1})**.** *Assume $\rho_k^2 \neq 0$. Then S_{k+1} is analytic at 0 and*

$$\alpha_{k+1} = S_{k+1}(0) = \frac{b_k}{\rho_k^2} = \frac{S_k'(0)}{1 - |\alpha_k|^2}.$$

Proof. Expand the numerator $S_k(z) - \alpha_k = b_k z + c_k z^2 + O(z^3)$ and the denominator

$$z\big(1 - \overline{\alpha_k} S_k(z)\big) = z\Big(\rho_k^2 - \overline{\alpha_k} b_k z + O(z^2)\Big),$$

then cancel the factor z and take the limit $z \to 0$. □

Lemma 10 (First derivative update)**.** *Assume $\rho_k^2 \neq 0$. With $c_k = \frac{1}{2}S_k''(0)$ one has*

$$S_{k+1}'(0) = \frac{c_k}{\rho_k^2} + \frac{\overline{\alpha_k}\, b_k^2}{\rho_k^4} = \frac{\frac{1}{2}S_k''(0)}{1 - |\alpha_k|^2} + \frac{\overline{\alpha_k}\,(S_k'(0))^2}{(1 - |\alpha_k|^2)^2}.$$

Proof. Use the geometric-series expansion

$$\frac{1}{\rho_k^2 - \overline{\alpha_k} b_k z + O(z^2)} = \frac{1}{\rho_k^2}\Big(1 + \frac{\overline{\alpha_k} b_k}{\rho_k^2} z + O(z^2)\Big)$$

in (30) and compare coefficients of z. □

D.2 Trigger indices and the blow-up / bounce-back pattern

Definition 2 (Observed blow-up index and trigger index)**.** Given a radius-regularized Schur family S_r with Schur parameters $\alpha_k(r)$, define

$$k^*(r) := \min\{k \geq 0 : \ |\alpha_k(r)| \geq 1\} \quad \text{(with } k^*(r) = +\infty \text{ if none)},$$

and the empirical trigger index

$$j^*(r) := \arg\min_{0 \leq j < k^*(r)} \big(1 - |\alpha_j(r)|^2\big) = \arg\min_{0 \leq j < k^*(r)} \rho_j^2(r).$$

Remark 3 (Why small ρ_k^2 is the amplification trigger). Linearizing the update $\alpha_{k+1} = b_k/\rho_k^2$ around perturbed inputs $\widetilde{\alpha}_k = \alpha_k + \delta\alpha_k$, $\widetilde{b}_k = b_k + \delta b_k$ yields the first-order sensitivity

$$\delta\alpha_{k+1} \approx \frac{\delta b_k}{\rho_k^2} + \frac{2b_k \,\Re(\overline{\alpha_k}\,\delta\alpha_k)}{\rho_k^4},$$

so errors are amplified at rates $1/\rho_k^2$ and $1/\rho_k^4$ as $\rho_k^2 \downarrow 0$.

Remark 4 (Two-step bounce-back scaling). If $\rho_k^2 = \varepsilon \ll 1$ and $b_k = O(1)$, then Lemma 9 gives $|\alpha_{k+1}| \asymp \varepsilon^{-1} \gg 1$ and hence $\rho_{k+1}^2 = 1 - |\alpha_{k+1}|^2 \asymp -\varepsilon^{-2}$. Meanwhile Lemma 10 typically produces $S'_{k+1}(0) = O(\varepsilon^{-2})$, so $\alpha_{k+2} = S'_{k+1}(0)/\rho_{k+1}^2 = O(1)$. This explains the empirical "blow-up then return" pattern in finite precision.

Remark 5 (Leakage and near-singularity of Toeplitz data). The leakage functional $L_N(r) = \sum_{k=0}^{N-1} -\log(1 - |\alpha_k(r)|^2) = \sum_{k=0}^{N-1} -\log \rho_k^2(r)$ satisfies $\exp(-L_N(r)) = \prod_{k=0}^{N-1} \rho_k^2(r)$. In standard OPUC normalizations one also has the determinant identity $\det T_N(w_r) = \prod_{k=0}^{N-1}(1 - |\alpha_k(r)|^2)$ for the Toeplitz matrix built from the boundary weight $w_r(e^{it}) = \Re\frac{1+S_r(e^{it})}{1-S_r(e^{it})}$, so small $\rho_j^2(r)$ corresponds to near-singularity of Toeplitz data.

E References (minimal)

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