The Lotka - Volterra system

Define the LV system

- First we define the symbols for functions of time and parameters used in the model. We will make all definitions valid for arbitrary dimension.

In[170]:=
```
n = 2;  (* dimensionality of the system *)
```

In[171]:=
```
coordSymb = Subscript[x, #] & /@ Range[n];   (* generates coordinate symbols *)
coord = coordSymb⟦#⟧[t] & /@ Range[n];
(* generates coordinate functions of time *)
parametersS = Table[Subscript[r, ToString[i]], {i, 1, n}];
(* generates scaling parameters symbols *)
parametersA =
  Table[If[i == j, 0, Subscript[a, ToString[i] <> ToString[j]]], {i, 1, n}, {j, 1, n}];
(* generates interaction parameters symbols *)
allParameters = DeleteCases[Flatten[Prepend[parametersA, parametersS]], 0];
(* the list of all parameters *)
```

In[176]:=
```
Print["Species are designated by = ",
 coordSymb, " and by ", coord, " as functions of time."]
Print["Scaling parameters = ", parametersS]
Print["Interaction matrix without self-interaction = ", parametersA]
Print["All parameters = ", allParameters]
```

Species are designated by = $\{x_1, x_2\}$ and by $\{x_1[t], x_2[t]\}$ as functions of time.

Scaling parameters = $\{r_1, r_2\}$

Interaction matrix without self-interaction = $\{\{0, a_{12}\}, \{a_{21}, 0\}\}$

All parameters = $\{r_1, r_2, a_{12}, a_{21}\}$

- Define the vector field corresponding to the system of differential equations

In[180]:=
```
fVector = Table[
   coord⟦i⟧ parametersS⟦i⟧ + Sum[coord⟦i⟧ parametersA⟦i, j⟧ coord⟦j⟧,
     {j, 1, n}], {i, 1, n}
  ];
LVSystem = Thread[D[coord, t] == fVector]
Print["The vector field in the species space = ", MatrixForm[fVector]]
Print["And the equations of the dynamical system = ", MatrixForm[LVSystem]]
```

Out[181]= $\{x_1'[t] == r_1\, x_1[t] + a_{12}\, x_1[t]\, x_2[t],\ x_2'[t] == r_2\, x_2[t] + a_{21}\, x_1[t]\, x_2[t]\}$

The vector field in the species space = $\begin{pmatrix} r_1\, x_1[t] + a_{12}\, x_1[t]\, x_2[t] \\ r_2\, x_2[t] + a_{21}\, x_1[t]\, x_2[t] \end{pmatrix}$

And the equations of the dynamical system = $\begin{pmatrix} x_1'[t] == r_1\, x_1[t] + a_{12}\, x_1[t]\, x_2[t] \\ x_2'[t] == r_2\, x_2[t] + a_{21}\, x_1[t]\, x_2[t] \end{pmatrix}$

- **stable points - ADD LATER**

Define the numerical solver and plotter

- It is useful to define a function that gives the **Rule** for implementing the chosen parameters into any expression

```
In[184]:= ChooseParameters[modelParameters_] :=
           Thread[Flatten[allParameters] → Flatten[modelParameters]]
           (* use this to give values to the parameters *)
```

- The following function numerically solves the system of equations, provided the system, initial conditions, parameters and the duration of simulation

```
In[185]:= LVSolution[coord_, system_, modelParameters_, initcond_, tEnd_] := NDSolve[
              Join[system, initcond] /. ChooseParameters[modelParameters],
              coord,
              {t, 0, tEnd}
            ];
```

- The following are the function for plotting the solution, and a helping function that displays the parameters in the text form, used in options of **Plot**.

```
In[186]:= DisplayParameters[allParameters_, modelParameters_] :=
           TextString[Row[Table[ToString[allParameters⟦i⟧ // TraditionalForm] <> "=" <>
               ToString[modelParameters⟦i⟧] <> "   ", {i, 1, Length[allParameters]}]]]

In[187]:= plt[params_, solution_, tend_, plottitle_] := Plot[
            Evaluate[solution⟦1, All, 1⟧ /. solution],
            {t, 0, tend},
            AxesOrigin → {0, 0}, (* RegionFunction→Function[{x,y},tail<x<tail+4] *)
            PlotLegends → Table[ToString[solution⟦1, i, 1⟧, TraditionalForm], {i, 1, n}],
            PlotLabel → plottitle <> DisplayParameters[allParameters, params]]
```

Numerical and visual analysis of the Lotka-Volterra system

Numerically solve the system for a given choice of parameters and initial conditions

- Let us choose some values of parameters (they all need to be positive since we require this system to describe realistic predator-prey population dynamics) and choose the initial conditions; we can always come back here to change them in order to explore the model

```
In[188]:= modelParameters = {1.5, -3., -1., 1.};
          initialConditions = Thread[(coord /. t → 0) == {12, 5}]
          (* give initial conditions here *)
          tend = 30;   (* time period over which to run the simulation *)
          ChooseParameters[modelParameters]

Out[189]= {x1[0] == 12, x2[0] == 5}

Out[191]= {r1 → 1.5, r2 → -3., a12 → -1., a21 → 1.}
```

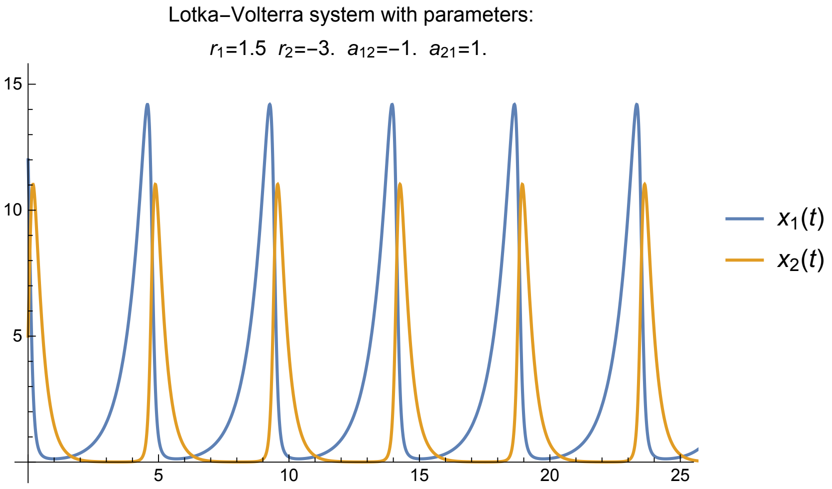
- Numerically solve the system using previously defined function **LVSolution**

```
In[192]:= sol = LVSolution[coord, LVSystem, modelParameters, initialConditions, tend];
```

- Plot the time evolution of the two populations

```
In[193]:= Show[plt[modelParameters, sol, tend, "Lotka-Volterra system with parameters: \n"],
           PlotRange → {{0, 25}, {0, 15}}]
```

Out[193]=



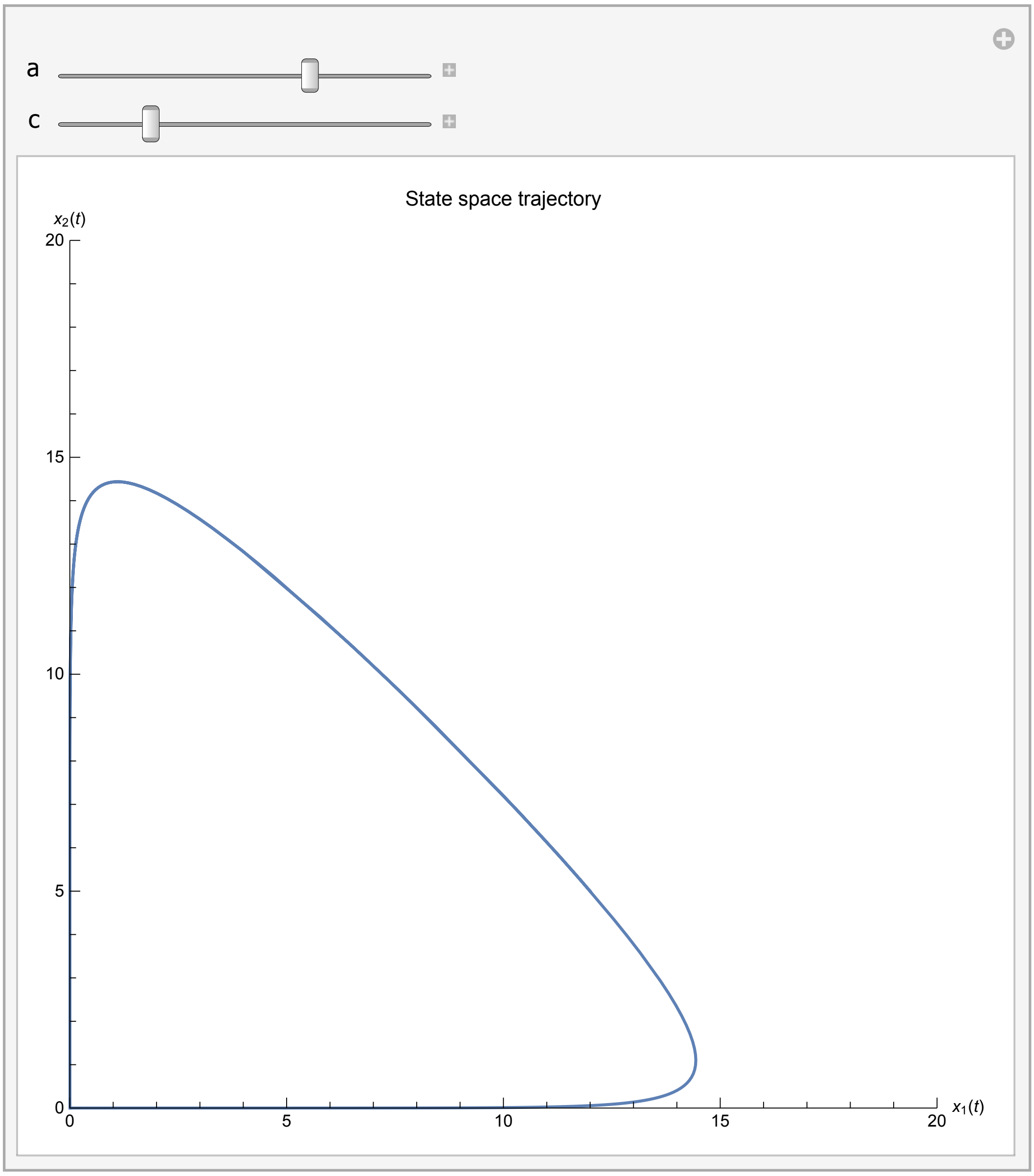
Numerically find dependence of the system on two varied parameters

- For the next example let's choose to vary the two scaling parameters and fix the interaction parameters. We interactively then change the parameters to display its influence on the trajectory in the state space

```
In[194]:= pfun = ParametricNDSolveValue[
             Join[LVSystem, initialConditions] /. ChooseParameters[{a, c, -1., 1.}],
             coordSymb, {t, 0, 25}, {a, c}];
```

```
In[195]:= Manipulate[
           ParametricPlot[Evaluate@{pfun[a, c]⟦1⟧[t], pfun[a, c]⟦2⟧[t]},
            {t, 0, 25}, PlotRange → {{0, 20}, {0, 20}}, ImageSize → 500,
            AxesLabel → Table[ToString[sol⟦1, i, 1⟧, TraditionalForm], {i, 1, n}],
            PlotLabel → "State space trajectory"],
           {a, 0.2, 1.5, 0.1}, {c, -2.0, 2.0, 0.1}]
```

Out[195]=



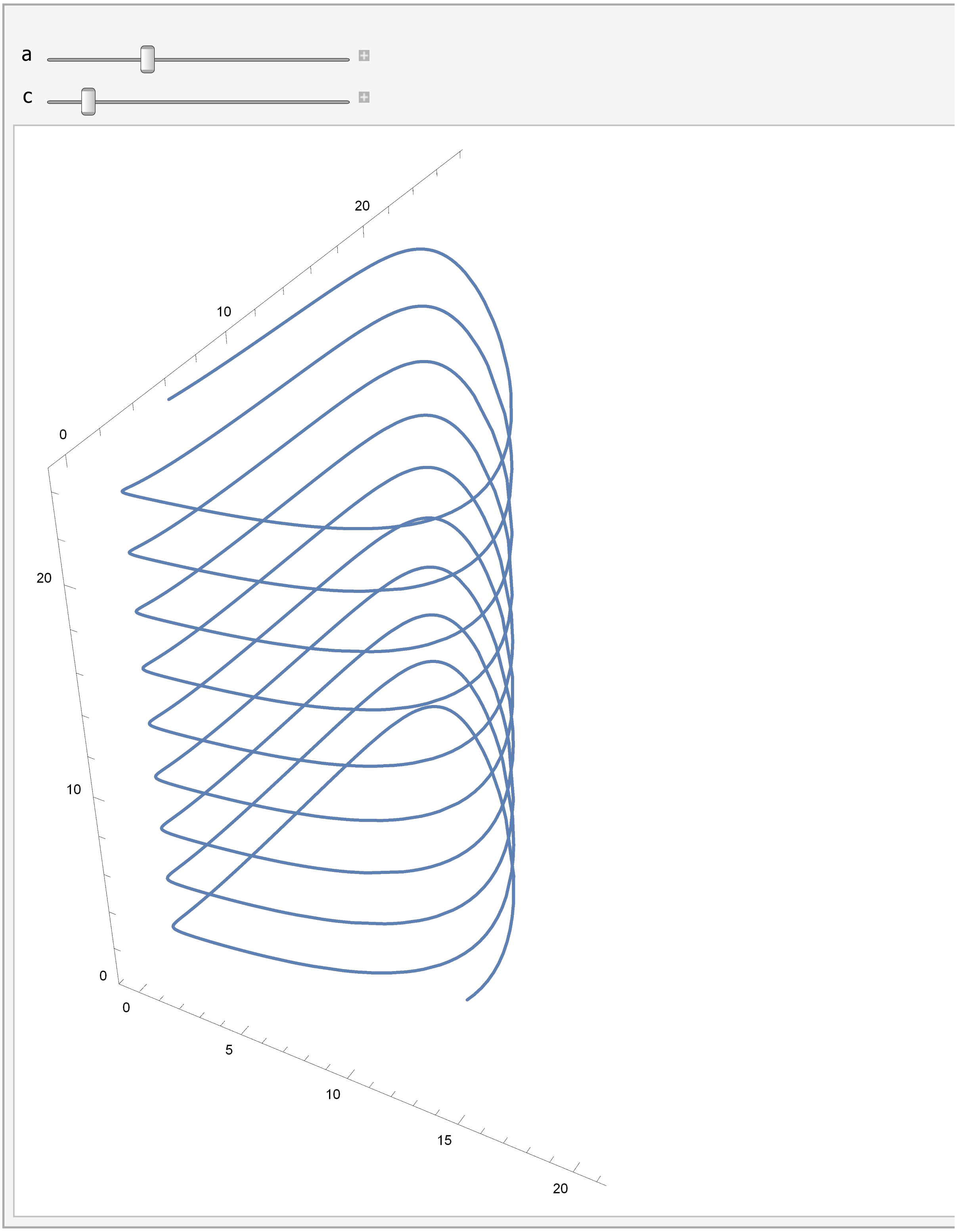
- Seems like the trajectory is always closed. This is the *limit cycle* of the Lotka-Volterra system.

3D plot of both coordinates against time

- We can also plot the state space at each moment in time (used as the third dimension), like a sequence of snapshots. **ADD LABELS**

```
In[196]:= Manipulate[
            ParametricPlot3D[
             Evaluate@{pfun[a, c]〚1〛[t], pfun[a, c]〚2〛[t], t},
             {t, 0, 25},
             PlotRange → {{-1, a + 15}, {-1, c + 30}, {0, 25}},
             ImageSize → 600,
             Boxed → False
            ],
            {a, 0.2, 20, 0.2},
            {c, -2.0, 0., 0.2}
          ]
```

Out[196]=

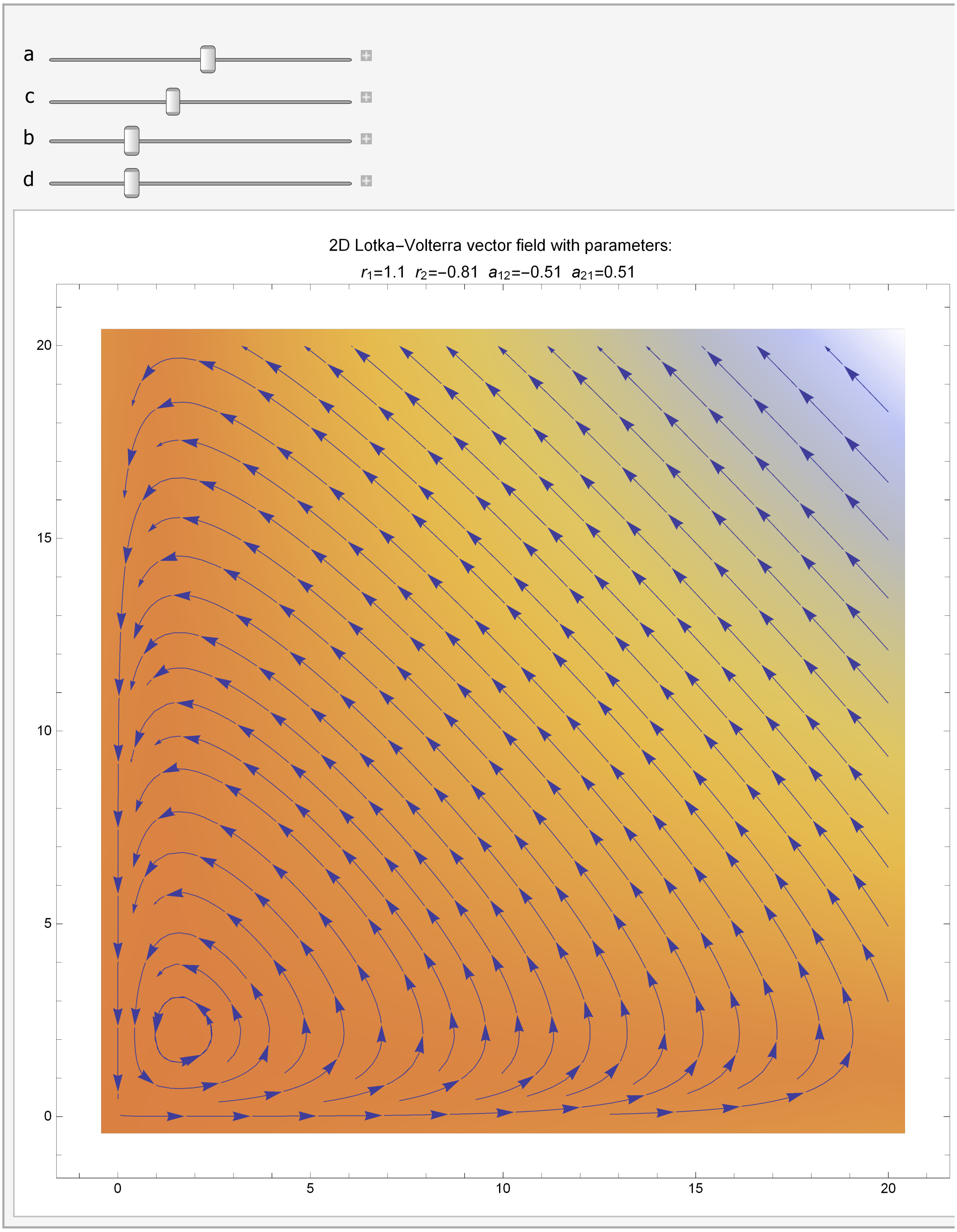


Plot the vector field and vary the parameters

- We can get insight into the system’s behavior without solving the differential equations. Simply examine the flow of the vector field in the state space. We do this interactively, leaving all four parameters arbitrary. **STABLE POINT - ADD LATER**

```
In[197]:= Manipulate[
            StreamDensityPlot[
             fVector /. Thread[coord → {x, y}] /. ChooseParameters[{a, -c, -b, d}],
             {x, 0, 20}, {y, 0, 20}, ColorFunction → "BeachColors", ImageSize → 600,
             PlotLabel → "2D Lotka-Volterra vector field with parameters: \n" <>
               DisplayParameters[allParameters, {a, -c, -b, d}]],
            {a, 0.1, 2., 0.1},
            {c, 0.01, 2., 0.1},
            {b, 0.01, 2., 0.1},
            {d, 0.01, 2., 0.1}
           ]
```

Out[197]=



2D Lotka–Volterra vector field with parameters:
r_1=1.1 r_2=−0.81 a_{12}=−0.51 a_{21}=0.51