Appendix 7

Some Special Plane Projective Transformations

Projective transformations (homographies) can be classified according to the algebraic and geometric multiplicity of their eigenvalues. The algebraic multiplicity of an eigenvalue is the number of times the root is repeated in the characteristic equation. The geometric multiplicity may be determined from the rank of the matrix $(\mathtt{H} - \lambda\mathtt{I})$, where $\mathtt{H}$ is the homography and λ the eigenvalue. A complete classification is given in projective geometry textbooks such as [Springer-64]. Here we mention several special cases which are important in practical situations, and occur at several points throughout this book. The description will be for plane transformations where $\mathtt{H}$ is a 3×3 matrix, but the generalization to 3-space transformations is straightforward.

The special forms are significant because $\mathtt{H}$ satisfies a number of relationships (remember the only restriction a general projective transformation is that it has full rank). Since $\mathtt{H}$ satisfies constraints it has fewer degrees of freedom, and consequently can be computed from fewer correspondences than a general projective transformation. The special transformations also have richer geometry and invariants than in the general case.

Note that unlike the special forms (affine etc.) discussed in chapter 2, which form subgroups, the following special projectivities do not form subgroups in general since they are not closed under multiplication. They do form a subgroup if all the elements have coincident fixed points and lines (i.e. differing only in their eigenvalues).

A7.1 Conjugate rotations

A rotation matrix $\mathtt{R}$ has eigenvalues $\{1, e^{i\theta}, e^{-i\theta}\}$, with corresponding eigenvectors $\{\mathbf{a}, \mathbf{I}, \mathbf{J}\}$, where $\mathbf{a}$ is the rotation axis, i.e. $\mathtt{R}\mathbf{a} = \mathbf{a}$, θ is the angle of rotation about the axis, and $\mathbf{I}$ and $\mathbf{J}$ (which are complex conjugate) are the circular points for the plane orthogonal to $\mathbf{a}$. Suppose a projective transformation between two planes has the form

$$\mathtt{H} = \mathtt{T}\,\mathtt{R}\,\mathtt{T}^{-1}$$

with $\mathtt{T}$ a general projective transformation; then $\mathtt{H}$ is a *conjugate rotation*. Eigenvalues are preserved under a conjugate relationship[1] so the eigenvalues of the projective transformation $\mathtt{H}$ are also $\{1, e^{i\theta}, e^{-i\theta}\}$ up to a common scale.

[1] Conjugacy is also referred to as a "similarity transformation". This meaning of "similarity" is unrelated to its use in this book as an isometry plus scaling transformation.

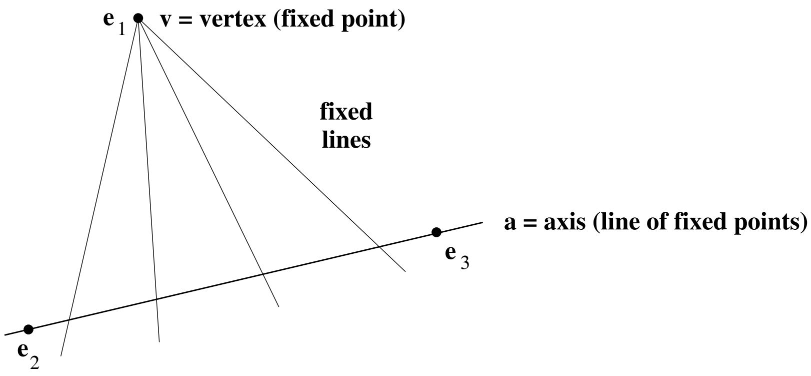

Fig. A7.1. **A planar homology**. *A planar homology is a plane projective transformation which has a line* $\mathbf{a}$ *of fixed points, called the* axis*, and a distinct fixed point* $\mathbf{v}$*, not on the line, called the* centre *or* vertex *of the homology. There is a pencil of fixed lines through the vertex. Algebraically, two of the eigenvalues of the transformation matrix are equal, and the fixed line corresponds to the 2D invariant space of the matrix (here the repeated eigenvalues are* λ_2 *and* λ_3*).*

Consider two images obtained by a camera rotating about its centre (as in figure 2.5-(*p*36)b); then as shown in section 8.4.2(*p*204), the images are related by a conjugate rotation. In this case the complex eigenvalues determine the angle (θ) through which the camera rotates, and the eigenvector corresponding to the real eigenvalue is the vanishing point of the rotation axis. Note that θ – a metric invariant – can be measured directly from the projective transformation.

A7.2 Planar homologies

A plane projective transformation $\mathtt{H}$ is a planar homology if it has a line of fixed points (called the *axis*), together with a fixed point (called the *vertex*) not on the line, see figure A7.1. Algebraically, the matrix has two equal and one distinct eigenvalues and the eigenspace corresponding to the equal eigenvalues is two-dimensional. The axis is the line through the two eigenvectors (i.e. points) spanning this eigenspace. The vertex corresponds to the other eigenvector. The ratio of the distinct eigenvalue to the repeated one is a characteristic invariant μ of the homology (i.e. the eigenvalues are, up to a common scale factor, $\{\mu, 1, 1\}$).

Properties of a planar homology include:

- Lines joining corresponding points intersect at the vertex, and corresponding lines (i.e. lines through two pairs of corresponding points) intersect on the axis. This is an example of Desargues' Theorem. See figure A7.2a.
- The cross ratios defined by the vertex, a pair of corresponding points, and the intersection of the line joining these points with the line of fixed points, are the same for all points related by the homology. See figure A7.2b.
- For curves related by a planar homology, corresponding tangents (the limit of neighbouring points defining corresponding lines) intersect on the axis.
- The vertex (2 dof), axis (2 dof) and invariant (1 dof) are sufficient to define the homology completely. A planar homology thus has 5 degrees of freedom.

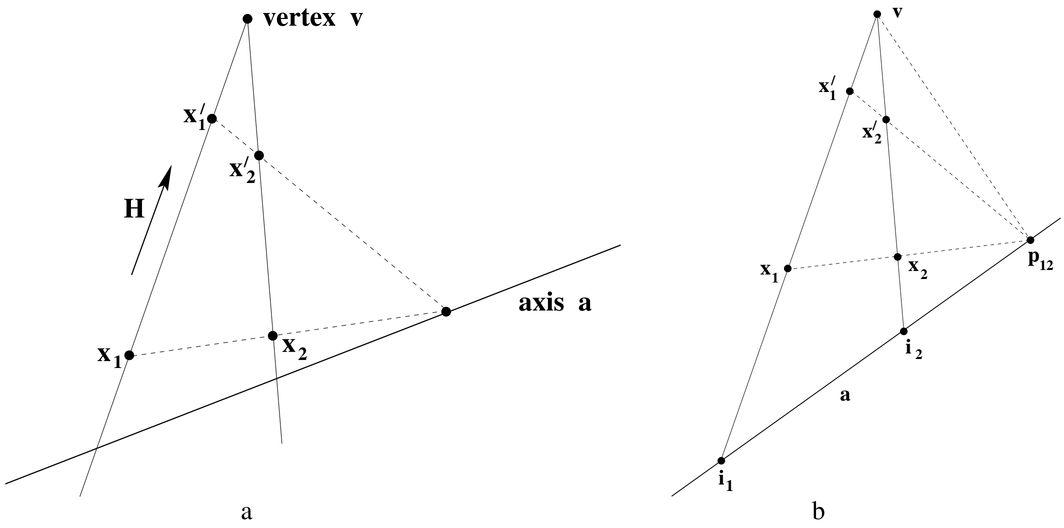

Fig. A7.2. **Homology transformation.** *(a) Under the transformation points on the axis are mapped to themselves; each point off the axis lies on a fixed line through* $\mathbf{v}$ *intersecting* $\mathbf{a}$ *and is mapped to another point on the line. Consequently, corresponding point pairs* $\mathbf{x} \leftrightarrow \mathbf{x}'$ *and the vertex of the homology are collinear. Corresponding lines – i.e. lines through pairs of corresponding points – intersect on the axis: for example, the lines* $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ *and* $\langle \mathbf{x}'_1, \mathbf{x}'_2 \rangle$. *(b) The cross ratio defined by the vertex* $\mathbf{v}$, *the corresponding points* $\mathbf{x}, \mathbf{x}'$ *and the intersection of their join with the axis* $\mathbf{i}$, *is the* characteristic invariant *of the homology, and is the same for all corresponding points. For example, the cross ratios of the four points* $\{\mathbf{v}, \mathbf{x}'_1, \mathbf{x}_1, \mathbf{i}_1\}$ *and* $\{\mathbf{v}, \mathbf{x}'_2, \mathbf{x}_2, \mathbf{i}_2\}$ *are equal since they are perspectively related by lines concurrent at* $\mathbf{p}_{12}$. *It follows that the cross ratio is the same for all points related by the homology.*

- 3 matched points are sufficient to compute a planar homology. The 6 degrees of freedom of the point matches over-constrain the 5 degrees of freedom of the homology.

A planar homology arises naturally in an image of two planes related by a perspectivity of 3-space (i.e. lines joining corresponding points on the two planes are concurrent). An example is the transformation between the image of a planar object and the image of its shadow on a plane. In this case the axis is the imaged intersection of the two planes, and the vertex is the image of the light source, see figure 2.5(p36)c.

Parametrization. The projective transformation representing the homology can be parametrized directly in terms of the 3-vectors representing the axis $\mathbf{a}$ and vertex $\mathbf{v}$, and the characteristic ratio μ, as

$$\mathtt{H} = \mathtt{I} + (\mu - 1)\frac{\mathbf{v}\mathbf{a}^{\mathsf{T}}}{\mathbf{v}^{\mathsf{T}}\mathbf{a}}$$

where $\mathtt{I}$ is the identity. It is straightforward to verify that the inverse transformation is given by

$$\mathtt{H}^{-1} = \mathtt{I} + \left(\frac{1}{\mu} - 1\right)\frac{\mathbf{v}\mathbf{a}^{\mathsf{T}}}{\mathbf{v}^{\mathsf{T}}\mathbf{a}}.$$

The eigenvectors are

$$\{\mathbf{e}_1 = \mathbf{v}, \mathbf{e}_2 = \mathbf{a}_1^{\perp}, \mathbf{e}_3 = \mathbf{a}_2^{\perp}\}$$

with corresponding eigenvalues

$$\{\lambda_1 = \mu, \lambda_2 = 1, \lambda_3 = 1\}$$

where $\mathbf{a}_i^\perp$ are two vectors that span the space orthogonal to the 3-vector $\mathbf{a}$, i.e. $\mathbf{a}^\mathsf{T}\mathbf{a}_i^\perp = 0$ and $\mathbf{a} = \mathbf{a}_1^\perp \times \mathbf{a}_2^\perp$.

If the axis or the vertex is at infinity then the homology is an affine transformation. Algebraically, if $\mathbf{a} = (0, 0, 1)^\mathsf{T}$, then the axis is at infinity; or if $\mathbf{v} = (v_1, v_2, 0)^\mathsf{T}$, then the vertex is at infinity; and in both cases the transformation matrix $\mathtt{H}$ has last row $(0, 0, 1)$.

Planar harmonic homology. A specialization of a planar homology is the case that the cross ratio is harmonic ($\mu = -1$). This planar homology is called a planar harmonic homology and has 4 degrees of freedom since the invariant is known. The transformation matrix $\mathtt{H}$ obeys $\mathtt{H}^2 = \mathtt{I}$, i.e. the transformation is a square root of the identity, which is called an involution (also a collineation of period 2). The eigenvalues are, up to a common scale factor, $\{-1, 1, 1\}$. Two pairs of point correspondences determine $\mathtt{H}$.

In a perspective image of a plane object with a bilateral symmetry, corresponding points in the image are related by a planar harmonic homology. The axis of the homology is the image of the symmetry axis. Algebraically, $\mathtt{H}$ is a conjugate reflection where the conjugating element is a plane projective transformation. In an affine image (generated by an affine camera) the resulting transformation is a *skewed symmetry*, and the conjugating element is a plane affine transformation. For a skewed symmetry the vertex is at infinity, and the lines joining corresponding points are parallel.

The harmonic homology can be parametrized as

$$\mathtt{H} = \mathtt{H}^{-1} = \mathtt{I} - 2\frac{\mathbf{v}\mathbf{a}^\mathsf{T}}{\mathbf{v}^\mathsf{T}\mathbf{a}}.$$

Again, if the axis or vertex is at infinity then the transformation is affine.

A7.3 Elations

An elation has a line of fixed points (the axis), and a pencil of fixed lines intersecting in a point (the vertex) on the axis. It may be thought of as the limit of a homology where the vertex is on the line of fixed points. Algebraically, the matrix has three equal eigenvalues, but the eigenspace is 2-dimensional. It may be parametrized as

$$\mathtt{H} = \mathtt{I} + \mu\mathbf{v}\mathbf{a}^\mathsf{T} \quad \text{with} \quad \mathbf{a}^\mathsf{T}\mathbf{v} = 0 \tag{A7.1}$$

where $\mathbf{a}$ is the axis, and $\mathbf{v}$ the vertex. The eigenvalues are all unity. The invariant space of $\mathtt{H}$ is spanned by $\mathbf{a}_1^\perp, \mathbf{a}_2^\perp$. This is a line (pencil) of fixed points (which includes $\mathbf{v}$ since $\mathbf{a}^\mathsf{T}\mathbf{v} = 0$). The invariant space of $\mathtt{H}^\mathsf{T}$ is spanned by vectors $\mathbf{v}_1^\perp, \mathbf{v}_2^\perp$ orthogonal to $\mathbf{v}$. This is a pencil of fixed lines, $\mathbf{l} = \alpha\mathbf{v}_1^\perp + \beta\mathbf{v}_2^\perp$, for which $\mathbf{l}^\mathsf{T}\mathbf{v} = 0$, i.e. all lines of the pencil are concurrent at the point $\mathbf{v}$.

An elation has 4 degrees of freedom: one less than a homology due to the constraint $\mathbf{a}^\mathsf{T}\mathbf{v} = 0$. It is defined by the axis $\mathbf{a}$ (2 dof), the vertex $\mathbf{v}$ on $\mathbf{a}$ (1 dof) and the parameter μ (1 dof). It can be determined from 2 point correspondences.

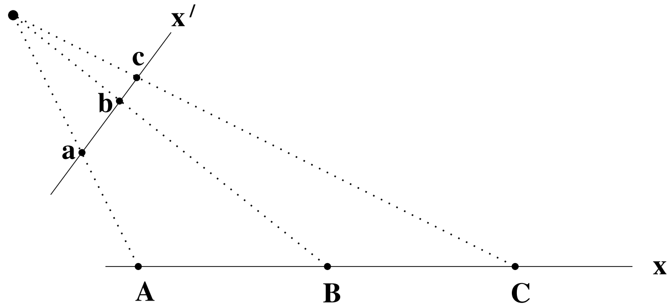

Fig. A7.3. **A line perspectivity**. *The lines joining corresponding points (*$\mathbf{a}, \mathbf{A}$ *etc.) are concurrent. Compare with figure A7.4.*

Elations often arise in practice as conjugate translations. Consider a pattern on a plane that repeats by a translation $\mathbf{t} = (t_x, t_y)^\top$, for example identical windows on the wall of a building. This action is represented on the plane of the wall as

$$\mathtt{H}_{\mathrm{E}} = \begin{bmatrix} \mathtt{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

which is an elation where $\mathbf{v} = (t_x, t_y, 0)^\top$ is the translation direction of the repetition, and $\mathbf{a} = (0, 0, 1)^\top$ is the line at infinity. In an image of the wall the windows are related by a conjugate translation $\mathtt{H} = \mathtt{T}\,\mathtt{H}_{\mathrm{E}}\,\mathtt{T}^{-1}$, where $\mathtt{T}$ is the projectivity which maps the plane of the wall to the image. The image transformation $\mathtt{H}$ is also an elation. The vertex of this elation is the vanishing point of the translation direction, and the axis is the vanishing line of the wall plane.

A7.4 Perspectivities

One other special case of a projectivity is a *perspectivity*, which is shown in figure A7.3 for a 1D projectivity on the plane. The distinctive property of a perspectivity is that lines joining corresponding points are concurrent. The difference between a perspectivity and projectivity is made clear by considering the composition of two perspectivities. As shown in figure A7.4 the composition of two perspectivities is *not* in general a perspectivity. However, the composition *is* a projectivity because a perspectivity is a projectivity, and projectivities form a group (closed), so that the composition of two projectivities is a projectivity. To summarize:

- *The composition of two (or more) perspectivities is a projectivity, but not, in general, a perspectivity.*

A central projection image of a world plane, as in figure 2.3(*p*34), is an example of a 2D perspectivity between different planes. Notice that identifying the projectivity as a perspectivity requires the embedding of the planes in 3-space.

Finally, imagine that the planes and camera centre of figure 2.3(*p*34) are mapped (by another perspectivity) onto one of the planes. Then this imaged perspectivity is now a map between points on the same plane, and is seen to be a planar homology (section A7.2).

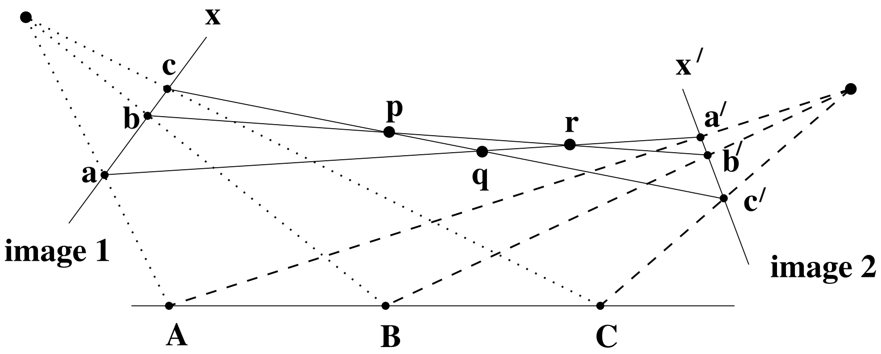

Fig. A7.4. **A line projectivity**. *Points* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *are related to points* $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ *by a line–to–line perspectivity. Points* $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ *are also related to points* $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ *by a perspectivity. However, points* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *are related to points* $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ *by a* projectivity*; they are* not *related by a perspectivity because lines joining corresponding points are* not *concurrent. In fact the pairwise intersections result in three distinct points* $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$.

Further reading. [Springer-64] classifies projectivities and covers special cases, e.g. planar homologies. Planar homologies appear in many guises: modelling imaged shadow relations in [VanGool-98]; modelling imaged extruded surfaces in [Zisserman-95a]; and modelling relations for planar pose recovery in [Basri-99]. The parametrization of planar homologies is given in Viéville and Lingrand [Vieville-95]. Elations appear in the grouping of imaged repeated patterns on a plane [Schaffalitzky-99, Schaffalitzky-00b] and in 3-space they appear in the generalized bas-relief ambiguity [Kriegman-98].