3

Projective Geometry and Transformations of 3D

This chapter describes the properties and entities of projective 3-space, or $\mathbb{P}^3$. Many of these are straightforward generalizations of those of the projective plane, described in chapter 2. For example, in $\mathbb{P}^3$ Euclidean 3-space is augmented with a set of ideal points which are on a *plane* at infinity, $\boldsymbol{\pi}_\infty$. This is the analogue of $\mathbf{l}_\infty$ in $\mathbb{P}^2$. Parallel lines, and now parallel *planes*, intersect on $\boldsymbol{\pi}_\infty$. Not surprisingly, homogeneous coordinates again play an important role, here with all dimensions increased by one. However, additional properties appear by virtue of the extra dimension. For example, two lines always intersect on the projective plane, but they need not intersect in 3-space.

The reader should be familiar with the ideas and notation of chapter 2 before reading this chapter. We will concentrate here on the differences and additional geometry introduced by adding the extra dimension, and will not repeat the bulk of the material of the previous chapter.

3.1 Points and projective transformations

A point $\mathbf{X}$ in 3-space is represented in homogeneous coordinates as a 4-vector. Specifically, the homogeneous vector $\mathbf{X} = (\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4)^\mathsf{T}$ with $\mathrm{X}_4 \neq 0$ represents the point $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})^\mathsf{T}$ of $\mathbb{R}^3$ with inhomogeneous coordinates

$$\mathrm{X} = \mathrm{X}_1/\mathrm{X}_4,\ \ \mathrm{Y} = \mathrm{X}_2/\mathrm{X}_4,\ \ \mathrm{Z} = \mathrm{X}_3/\mathrm{X}_4.$$

For example, a homogeneous representation of $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})^\mathsf{T}$ is $\mathbf{X} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z}, 1)^\mathsf{T}$. Homogeneous points with $\mathrm{X}_4 = 0$ represent points at infinity.

A projective transformation acting on $\mathbb{P}^3$ is a linear transformation on homogeneous 4-vectors represented by a non-singular 4×4 matrix: $\mathbf{X}' = \mathtt{H}\mathbf{X}$. The matrix $\mathtt{H}$ representing the transformation is homogeneous and has 15 degrees of freedom. The degrees of freedom follow from the 16 elements of the matrix less one for overall scaling.

As in the case of planar projective transformations, the map is a collineation (lines are mapped to lines), which preserves incidence relations such as the intersection point of a line with a plane, and order of contact.

3.2 Representing and transforming planes, lines and quadrics

In $\mathbb{P}^3$ points and *planes* are dual, and their representation and development is analogous to the point–line duality in $\mathbb{P}^2$. Lines are self-dual in $\mathbb{P}^3$.

3.2.1 Planes

A plane in 3-space may be written as

$$\pi_1 \mathrm{X} + \pi_2 \mathrm{Y} + \pi_3 \mathrm{Z} + \pi_4 = 0. \tag{3.1}$$

Clearly this equation is unaffected by multiplication by a non-zero scalar, so only the three independent ratios $\{\pi_1 : \pi_2 : \pi_3 : \pi_4\}$ of the plane coefficients are significant. It follows that a plane has 3 degrees of freedom in 3-space. The homogeneous representation of the plane is the 4-vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^\mathsf{T}$.

Homogenizing (3.1) by the replacements $\mathrm{X} \mapsto \mathrm{X}_1/\mathrm{X}_4$, $\mathrm{Y} \mapsto \mathrm{X}_2/\mathrm{X}_4$, $\mathrm{Z} \mapsto \mathrm{X}_3/\mathrm{X}_4$ gives

$$\pi_1 \mathrm{X}_1 + \pi_2 \mathrm{X}_2 + \pi_3 \mathrm{X}_3 + \pi_4 \mathrm{X}_4 = 0$$

or more concisely

$$\boldsymbol{\pi}^\mathsf{T} \mathbf{X} = 0 \tag{3.2}$$

which expresses that the point $\mathbf{X}$ is on the plane $\boldsymbol{\pi}$.

The first 3 components of $\boldsymbol{\pi}$ correspond to the plane normal of Euclidean geometry – using inhomogeneous notation (3.2) becomes the familiar plane equation written in 3-vector notation as $\mathbf{n}.\widetilde{\mathbf{X}} + d = 0$, where $\mathbf{n} = (\pi_1, \pi_2, \pi_3)^\mathsf{T}$, $\widetilde{\mathbf{X}} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^\mathsf{T}$, $\mathrm{X}_4 = 1$ and $d = \pi_4$. In this form $d/\|\mathbf{n}\|$ is the distance of the plane from the origin.

Join and incidence relations. In $\mathbb{P}^3$ there are numerous geometric relations between planes and points and lines. For example,

(i) A plane is defined uniquely by the join of three points, or the join of a line and point, in general position (i.e. the points are not collinear or incident with the line in the latter case).
(ii) Two distinct planes intersect in a unique line.
(iii) Three distinct planes intersect in a unique point.

These relations have algebraic representations which will now be developed in the case of points and planes. The representations of the relations involving lines are not as simple as those arising from 3D vector algebra of $\mathbb{P}^2$ (e.g. $\mathbf{l} = \mathbf{x} \times \mathbf{y}$), and are postponed until line representations are introduced in section 3.2.2.

Three points define a plane. Suppose three points $\mathbf{X}_i$ are incident with the plane $\boldsymbol{\pi}$. Then each point satisfies (3.2) and thus $\boldsymbol{\pi}^\mathsf{T} \mathbf{X}_i = 0$, $i = 1, \ldots, 3$. Stacking these equations into a matrix gives

$$\begin{bmatrix} \mathbf{X}_1^\mathsf{T} \\ \mathbf{X}_2^\mathsf{T} \\ \mathbf{X}_3^\mathsf{T} \end{bmatrix} \boldsymbol{\pi} = \mathbf{0}. \tag{3.3}$$

Since three points $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ in general position are linearly independent, it follows that the 3×4 matrix composed of the points as rows has rank 3. The plane $\boldsymbol{\pi}$ defined by the points is thus obtained uniquely (up to scale) as the 1-dimensional (right) null-space. If the matrix has only a rank of 2, and consequently the null-space is 2-dimensional, then the points are collinear, and define a pencil of planes with the line of collinear points as axis.

In $\mathbb{P}^2$, where points are dual to lines, a line $\mathbf{l}$ through two points $\mathbf{x}, \mathbf{y}$ can similarly be obtained as the null-space of the 2×3 matrix with $\mathbf{x}^\mathsf{T}$ and $\mathbf{y}^\mathsf{T}$ as rows. However, a more convenient direct formula $\mathbf{l} = \mathbf{x} \times \mathbf{y}$ is also available from vector algebra. In $\mathbb{P}^3$ the analogous expression is obtained from properties of determinants and minors.

We start from the matrix $\mathtt{M} = [\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]$ which is composed of a general point $\mathbf{X}$ and the three points $\mathbf{X}_i$ which define the plane $\boldsymbol{\pi}$. The determinant $\det \mathtt{M} = 0$ when $\mathbf{X}$ lies on $\boldsymbol{\pi}$ since the point $\mathbf{X}$ is then expressible as a linear combination of the points $\mathbf{X}_i, i = 1, \ldots, 3$. Expanding the determinant about the column $\mathbf{X}$ we obtain

$$\det \mathtt{M} = \mathrm{X}_1 D_{234} - \mathrm{X}_2 D_{134} + \mathrm{X}_3 D_{124} - \mathrm{X}_4 D_{123}$$

where D_{jkl} is the determinant formed from the jkl rows of the 4×3 matrix $[\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]$. Since $\det \mathtt{M} = 0$ for points on $\boldsymbol{\pi}$ we can then read off the plane coefficients as

$$\boldsymbol{\pi} = (D_{234}, -D_{134}, D_{124}, -D_{123})^\mathsf{T}. \tag{3.4}$$

This is the solution vector (the null-space) of (3.3) above.

Example 3.1. Suppose the three points defining the plane are

$$\mathbf{X}_1 = \begin{pmatrix} \widetilde{\mathbf{X}}_1 \\ 1 \end{pmatrix} \quad \mathbf{X}_2 = \begin{pmatrix} \widetilde{\mathbf{X}}_2 \\ 1 \end{pmatrix} \quad \mathbf{X}_3 = \begin{pmatrix} \widetilde{\mathbf{X}}_3 \\ 1 \end{pmatrix}$$

where $\widetilde{\mathbf{X}} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^\mathsf{T}$. Then

$$D_{234} = \begin{vmatrix} \mathrm{Y}_1 & \mathrm{Y}_2 & \mathrm{Y}_3 \\ \mathrm{Z}_1 & \mathrm{Z}_2 & \mathrm{Z}_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \mathrm{Y}_1 - \mathrm{Y}_3 & \mathrm{Y}_2 - \mathrm{Y}_3 & \mathrm{Y}_3 \\ \mathrm{Z}_1 - \mathrm{Z}_3 & \mathrm{Z}_2 - \mathrm{Z}_3 & \mathrm{Z}_3 \\ 0 & 0 & 1 \end{vmatrix} = \left((\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_3) \times (\widetilde{\mathbf{X}}_2 - \widetilde{\mathbf{X}}_3) \right)_1$$

and similarly for the other components, giving

$$\boldsymbol{\pi} = \begin{pmatrix} (\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_3) \times (\widetilde{\mathbf{X}}_2 - \widetilde{\mathbf{X}}_3) \\ -\widetilde{\mathbf{X}}_3^\mathsf{T} (\widetilde{\mathbf{X}}_1 \times \widetilde{\mathbf{X}}_2) \end{pmatrix}.$$

This is the familiar result from Euclidean vector geometry where, for example, the plane normal is computed as $(\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_3) \times (\widetilde{\mathbf{X}}_2 - \widetilde{\mathbf{X}}_3)$. △

Three planes define a point. The development here is dual to the case of three points defining a plane. The intersection point $\mathbf{X}$ of three planes $\boldsymbol{\pi}_i$ can be computed straightforwardly as the (right) null-space of the 3×4 matrix composed of the planes as rows:

$$\begin{bmatrix} \boldsymbol{\pi}_1^\mathsf{T} \\ \boldsymbol{\pi}_2^\mathsf{T} \\ \boldsymbol{\pi}_3^\mathsf{T} \end{bmatrix} \mathbf{X} = \mathbf{0}. \tag{3.5}$$

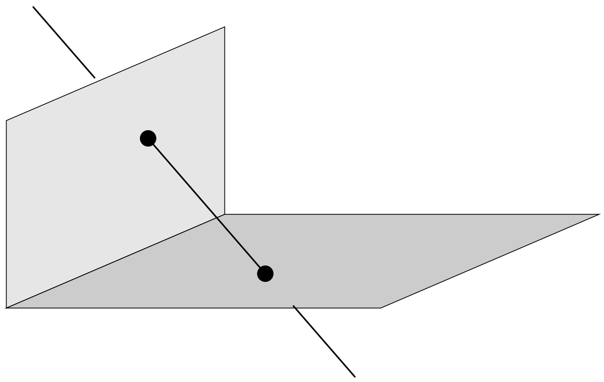


Fig. 3.1. *A line may be specified by its points of intersection with two orthogonal planes. Each intersection point has 2 degrees of freedom, which demonstrates that a line in* $\mathbb{P}^3$ *has a total of 4 degrees of freedom.*

A direct solution for $\mathbf{X}$, in terms of determinants of 3×3 submatrices, is obtained as an analogue of (3.4), though computationally a numerical solution would be obtained by algorithm A5.1(*p*589).
The two following results are direct analogues of their 2D counterparts.

Projective transformation. Under the point transformation $\mathbf{X}' = \mathtt{H}\mathbf{X}$, a plane transforms as

$$\boldsymbol{\pi}' = \mathtt{H}^{-\mathsf{T}}\boldsymbol{\pi}. \tag{3.6}$$

Parametrized points on a plane. The points $\mathbf{X}$ on the plane $\boldsymbol{\pi}$ may be written as

$$\mathbf{X} = \mathtt{M}\mathbf{x} \tag{3.7}$$

where the columns of the 4×3 matrix $\mathtt{M}$ generate the rank 3 null-space of $\boldsymbol{\pi}^{\mathsf{T}}$, i.e. $\boldsymbol{\pi}^{\mathsf{T}}\mathtt{M} = \mathbf{0}$, and the 3-vector $\mathbf{x}$ (which is a point on the projective plane $\mathbb{P}^2$) parametrizes points on the plane $\boldsymbol{\pi}$. $\mathtt{M}$ is not unique, of course. Suppose the plane is $\boldsymbol{\pi} = (a, b, c, d)^{\mathsf{T}}$ and a is non-zero, then $\mathtt{M}^{\mathsf{T}}$ can be written as $\mathtt{M}^{\mathsf{T}} = [\mathbf{p} \,|\, \mathtt{I}_{3\times3}]$, where $\mathbf{p} = (-b/a, -c/a, -d/a)^{\mathsf{T}}$.

This parametrized representation is simply the analogue in 3D of a line $\mathbf{l}$ in $\mathbb{P}^2$ defined as a linear combination of its 2D null-space as $\mathbf{x} = \mu\mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{l}^{\mathsf{T}}\mathbf{a} = \mathbf{l}^{\mathsf{T}}\mathbf{b} = 0$.

3.2.2 Lines

A line is defined by the *join* of two points or the intersection of two planes. Lines have 4 degrees of freedom in 3-space. A convincing way to count these degrees of freedom is to think of a line as defined by its intersection with two orthogonal planes, as in figure 3.1. The point of intersection on each plane is specified by two parameters, producing a total of 4 degrees of freedom for the line.

Lines are very awkward to represent in 3-space since a natural representation for an object with 4 degrees of freedom would be a homogeneous 5-vector. The problem is that a homogeneous 5 vector cannot easily be used in mathematical expressions together with the 4-vectors representing points and planes. To overcome this problem a number of line representations have been proposed, and these differ in their mathematical complexity. We survey three of these representations. In each case the representation provides mechanisms for a line to be defined by: the join of two points, a dual version where the line is defined by the intersection of two planes, and also a map between the two definitions. The representations also enable join and incidence relations to be computed, for example the point at which a line intersects a plane.

I. Null-space and span representation. This representation builds on the intuitive geometric notion that a line is a pencil (one-parameter family) of collinear points, and is defined by any two of these points. Similarly, a line is the axis of a pencil of planes, and is defined by the intersection of any two planes from the pencil. In both cases the actual points or planes are not important (in fact two points have 6 degrees of freedom and are represented by two 4-vectors – far too many parameters). This notion is captured mathematically by representing a line as the *span* of two vectors. Suppose $\mathbf{A}, \mathbf{B}$ are two (non-coincident) space points. Then the line joining these points is represented by the span of the row space of the 2×4 matrix $\mathtt{W}$ composed of $\mathbf{A}^\mathsf{T}$ and $\mathbf{B}^\mathsf{T}$ as rows:

$$\mathtt{W} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \end{bmatrix}.$$

Then:

(i) The span of $\mathtt{W}^\mathsf{T}$ is the pencil of points $\lambda\mathbf{A} + \mu\mathbf{B}$ on the line.
(ii) The span of the 2-dimensional right null-space of $\mathtt{W}$ is the pencil of planes with the line as axis.

It is evident that two other points, $\mathbf{A}'^\mathsf{T}$ and $\mathbf{B}'^\mathsf{T}$, on the line will generate a matrix $\mathtt{W}'$ with the same span as $\mathtt{W}$, so that the span, and hence the representation, is independent of the particular points used to define it.

To prove the null-space property, suppose that $\mathbf{P}$ and $\mathbf{Q}$ are a basis for the null-space. Then $\mathtt{W}\mathbf{P} = \mathbf{0}$ and consequently $\mathbf{A}^\mathsf{T}\mathbf{P} = \mathbf{B}^\mathsf{T}\mathbf{P} = 0$, so that $\mathbf{P}$ is a plane containing the points $\mathbf{A}$ and $\mathbf{B}$. Similarly, $\mathbf{Q}$ is a distinct plane also containing the points $\mathbf{A}$ and $\mathbf{B}$. Thus $\mathbf{A}$ and $\mathbf{B}$ lie on both the (linearly independent) planes $\mathbf{P}$ and $\mathbf{Q}$, so the line defined by $\mathtt{W}$ is the plane intersection. Any plane of the pencil, with the line as axis, is given by the span $\lambda'\mathbf{P} + \mu'\mathbf{Q}$.

The dual representation of a line as the intersection of two planes, $\mathbf{P}, \mathbf{Q}$, follows in a similar manner. The line is represented as the span (of the row space) of the 2×4 matrix $\mathtt{W}^*$ composed of $\mathbf{P}^\mathsf{T}$ and $\mathbf{Q}^\mathsf{T}$ as rows:

$$\mathtt{W}^* = \begin{bmatrix} \mathbf{P}^\mathsf{T} \\ \mathbf{Q}^\mathsf{T} \end{bmatrix}$$

with the properties

(i) The span of $\mathtt{W}^{*\mathsf{T}}$ is the pencil of planes $\lambda'\mathbf{P} + \mu'\mathbf{Q}$ with the line as axis.
(ii) The span of the 2-dimensional null-space of $\mathtt{W}^*$ is the pencil of points on the line.

The two representations are related by $\mathtt{W}^* \mathtt{W}^\mathsf{T} = \mathtt{W}\mathtt{W}^{*\mathsf{T}} = \mathtt{0}_{2\times 2}$, where $\mathtt{0}_{2\times 2}$ is a 2×2 null matrix.

Example 3.2. The $\mathtt{X}$-axis is represented as

$$\mathtt{W} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathtt{W}^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where the points $\mathbf{A}$ and $\mathbf{B}$ are here the origin and ideal point in the $\mathtt{X}$-direction, and the planes $\mathbf{P}$ and $\mathbf{Q}$ are the $\mathtt{XY}$- and $\mathtt{XZ}$-planes respectively. △

Join and incidence relations are also computed from null-spaces.

(i) The plane $\boldsymbol{\pi}$ defined by the join of the point $\mathbf{X}$ and line $\mathtt{W}$ is obtained from the null-space of

$$\mathtt{M} = \begin{bmatrix} \mathtt{W} \\ \mathbf{X}^\mathsf{T} \end{bmatrix}.$$

If the null-space of $\mathtt{M}$ is 2-dimensional then $\mathbf{X}$ is on $\mathtt{W}$, otherwise $\mathtt{M}\boldsymbol{\pi} = \mathbf{0}$.

(ii) The point $\mathbf{X}$ defined by the intersection of the line $\mathtt{W}$ with the plane $\boldsymbol{\pi}$ is obtained from the null-space of

$$\mathtt{M} = \begin{bmatrix} \mathtt{W}^* \\ \boldsymbol{\pi}^\mathsf{T} \end{bmatrix}.$$

If the null-space of $\mathtt{M}$ is 2-dimensional then the line $\mathtt{W}$ is on $\boldsymbol{\pi}$, otherwise $\mathtt{M}\mathbf{X} = \mathbf{0}$.

These properties can be derived almost by inspection. For example, the first is equivalent to three points defining a plane (3.3).

The span representation is very useful in practical numerical implementations where null-spaces can be computed simply by using the SVD algorithm (see section A4.4-(*p*585)) available with most matrix packages. The representation is also useful in estimation problems, where it is often not a problem that the entity being estimated is over-parametrized (see the discussion of section 4.5(*p*110)).

II. Plücker matrices. Here a line is represented by a 4×4 skew-symmetric homogeneous matrix. In particular, the line joining the two points $\mathbf{A}, \mathbf{B}$ is represented by the matrix $\mathtt{L}$ with elements

$$l_{ij} = A_i B_j - B_i A_j$$

or equivalently in vector notation as

$$\mathtt{L} = \mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T} \tag{3.8}$$

First a few properties of $\mathtt{L}$:

(i) $\mathtt{L}$ has rank 2. Its 2-dimensional null-space is spanned by the pencil of planes with the line as axis (in fact $\mathtt{L}\mathtt{W}^{*\mathsf{T}} = \mathtt{0}$, with $\mathtt{0}$ a 4×2 null-matrix).

(ii) The representation has the required 4 degrees of freedom for a line. This is accounted as follows: the skew-symmetric matrix has 6 independent non-zero elements, but only their 5 ratios are significant, and furthermore because $\det \mathtt{L} = 0$ the elements satisfy a (quadratic) constraint (see below). The net number of degrees of freedom is then 4.

(iii) The relation $\mathtt{L} = \mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T}$ is the generalization to 4-space of the vector product formula $\mathbf{l} = \mathbf{x} \times \mathbf{y}$ of $\mathbb{P}^2$ for a line $\mathbf{l}$ defined by two points $\mathbf{x}, \mathbf{y}$ all represented by 3-vectors.

(iv) The matrix $\mathtt{L}$ is independent of the points $\mathbf{A}, \mathbf{B}$ used to define it, since if a different point $\mathbf{C}$ on the line is used, with $\mathbf{C} = \mathbf{A} + \mu\mathbf{B}$, then the resulting matrix is

$$\begin{aligned}\hat{\mathtt{L}} &= \mathbf{A}\mathbf{C}^\mathsf{T} - \mathbf{C}\mathbf{A}^\mathsf{T} = \mathbf{A}(\mathbf{A}^\mathsf{T} + \mu\mathbf{B}^\mathsf{T}) - (\mathbf{A} + \mu\mathbf{B})\mathbf{A}^\mathsf{T} \\ &= \mathbf{A}\mathbf{B}^\mathsf{T} - \mathbf{B}\mathbf{A}^\mathsf{T} = \mathtt{L}.\end{aligned}$$

(v) Under the point transformation $\mathbf{X}' = \mathtt{H}\mathbf{X}$, the matrix transforms as $\mathtt{L}' = \mathtt{H}\mathtt{L}\mathtt{H}^\mathsf{T}$, i.e. it is a valency-2 tensor (see appendix 1(*p*562)).

Example 3.3. From (3.8) the $\mathtt{X}$-axis is represented as

$$\mathtt{L} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where the points $\mathbf{A}$ and $\mathbf{B}$ are (as in the previous example) the origin and ideal point in the $\mathtt{X}$-direction respectively. △

A dual Plücker representation $\mathtt{L}^*$ is obtained for a line formed by the intersection of two planes $\mathbf{P}, \mathbf{Q}$,

$$\mathtt{L}^* = \mathbf{P}\mathbf{Q}^\mathsf{T} - \mathbf{Q}\mathbf{P}^\mathsf{T} \tag{3.9}$$

and has similar properties to $\mathtt{L}$. Under the point transformation $\mathbf{X}' = \mathtt{H}\mathbf{X}$, the matrix $\mathtt{L}^*$ transforms as $\mathtt{L}^{*\prime} = \mathtt{H}^{-\mathsf{T}}\mathtt{L}\mathtt{H}^{-1}$. The matrix $\mathtt{L}^*$ can be obtained directly from $\mathtt{L}$ by a simple rewrite rule:

$$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}. \tag{3.10}$$

The correspondence rule is very simple: the indices of the dual and original component always include all the numbers $\{1, 2, 3, 4\}$, so if the original is ij then the dual is those numbers of $\{1, 2, 3, 4\}$ which are not ij. For example $12 \mapsto 34$.

Join and incidence properties are very nicely represented in this notation:

(i) The plane defined by the join of the point $\mathbf{X}$ and line $\mathtt{L}$ is

$$\boldsymbol{\pi} = \mathtt{L}^*\mathbf{X}$$

and $\mathtt{L}^*\mathbf{X} = \mathbf{0}$ if, and only if, $\mathbf{X}$ is on $\mathtt{L}$.

(ii) The point defined by the intersection of the line $\mathtt{L}$ with the plane $\boldsymbol{\pi}$ is

$$\mathbf{X} = \mathtt{L}\boldsymbol{\pi}$$

and $\mathtt{L}\boldsymbol{\pi} = \mathbf{0}$ if, and only if, $\mathtt{L}$ is on $\boldsymbol{\pi}$.

The properties of two (or more) lines $\mathtt{L}_1, \mathtt{L}_2, \ldots$ can be obtained from the null-space of the matrix $\mathtt{M} = [\mathtt{L}_1, \mathtt{L}_2, \ldots]$. For example if the lines are coplanar then $\mathtt{M}^\mathsf{T}$ has a 1-dimensional null-space corresponding to the plane $\boldsymbol{\pi}$ of the lines.

Example 3.4. The intersection of the X-axis with the plane $\mathrm{X} = 1$ is given by $\mathbf{X} = \mathtt{L}\boldsymbol{\pi}$ as

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which is the inhomogeneous point $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})^\mathsf{T} = (1, 0, 0)^\mathsf{T}$. △

III. Plücker line coordinates. The Plücker line coordinates are the six non-zero elements of the 4×4 skew-symmetric Plücker matrix (3.8) $\mathtt{L}$, namely[1]

$$\mathcal{L} = \{l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34}\}. \tag{3.11}$$

This is a homogeneous 6-vector, and thus is an element of $\mathbb{P}^5$. It follows from evaluating $\det \mathtt{L} = 0$ that the coordinates satisfy the equation

$$l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0. \tag{3.12}$$

A 6-vector $\mathcal{L}$ only corresponds to a line in 3-space if it satisfies (3.12). The geometric interpretation of this constraint is that the lines of $\mathbb{P}^3$ define a (co-dimension 1) surface in $\mathbb{P}^5$ which is known as the *Klein quadric*, a quadric because the terms of (3.12) are quadratic in the Plücker line coordinates.

Suppose two lines $\mathcal{L}$, $\hat{\mathcal{L}}$ are the joins of the points $\mathbf{A}, \mathbf{B}$ and $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}$ respectively. The lines intersect if and only if the four points are coplanar. A necessary and sufficient condition for this is that $\det[\mathbf{A}, \mathbf{B}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}}] = 0$. It can be shown that the determinant expands as

$$\begin{aligned} \det[\mathbf{A}, \mathbf{B}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}}] &= l_{12}\hat{l}_{34} + \hat{l}_{12}l_{34} + l_{13}\hat{l}_{42} + \hat{l}_{13}l_{42} + l_{14}\hat{l}_{23} + \hat{l}_{14}l_{23} \\ &= (\mathcal{L}|\hat{\mathcal{L}}). \end{aligned} \tag{3.13}$$

Since the Plücker coordinates are independent of the particular points used to define them, the bilinear product $(\mathcal{L}|\hat{\mathcal{L}})$ is independent of the points used in the derivation and only depends on the lines $\mathcal{L}$ and $\hat{\mathcal{L}}$. Then we have

Result 3.5. *Two lines $\mathcal{L}$ and $\hat{\mathcal{L}}$ are coplanar (and thus intersect) if and only if $(\mathcal{L}|\hat{\mathcal{L}}) = 0$.*

This product appears in a number of useful formulae:

[1] The element l_{42} is conventionally used instead of l_{24} as it eliminates negatives in many of the subsequent formulae.

(i) A 6-vector $\mathcal{L}$ only represents a line in $\mathbb{P}^3$ if $(\mathcal{L}|\mathcal{L}) = 0$. This is simply repeating the Klein quadric constraint (3.12) above.

(ii) Suppose two lines $\mathcal{L}$, $\hat{\mathcal{L}}$ are the intersections of the planes $\mathbf{P}$, $\mathbf{Q}$ and $\widehat{\mathbf{P}}$, $\widehat{\mathbf{Q}}$ respectively. Then

$$(\mathcal{L}|\hat{\mathcal{L}}) = \det[\mathbf{P}, \mathbf{Q}, \widehat{\mathbf{P}}, \widehat{\mathbf{Q}}]$$

and again the lines intersect if and only if $(\mathcal{L}|\hat{\mathcal{L}}) = 0$.

(iii) If $\mathcal{L}$ is the intersection of two planes $\mathbf{P}$ and $\mathbf{Q}$ and $\hat{\mathcal{L}}$ is the join of two points $\mathbf{A}$ and $\mathbf{B}$, then

$$(\mathcal{L}|\hat{\mathcal{L}}) = (\mathbf{P}^\mathsf{T}\mathbf{A})(\mathbf{Q}^\mathsf{T}\mathbf{B}) - (\mathbf{Q}^\mathsf{T}\mathbf{A})(\mathbf{P}^\mathsf{T}\mathbf{B}). \tag{3.14}$$

Plücker coordinates are useful in algebraic derivations. They will be used in defining the map from a line in 3-space to its image in chapter 8.

3.2.3 Quadrics and dual quadrics

A quadric is a surface in $\mathbb{P}^3$ defined by the equation

$$\mathbf{X}^\mathsf{T}\mathtt{Q}\mathbf{X} = 0 \tag{3.15}$$

where $\mathtt{Q}$ is a symmetric 4×4 matrix. Often the matrix $\mathtt{Q}$ and the quadric surface it defines are not distinguished, and we will simply refer to the quadric $\mathtt{Q}$.

Many of the properties of quadrics follow directly from those of conics in section 2.2.3(*p*30). To highlight a few:

(i) A quadric has 9 degrees of freedom. These correspond to the ten independent elements of a 4×4 symmetric matrix less one for scale.

(ii) Nine points in general position define a quadric.

(iii) If the matrix $\mathtt{Q}$ is singular, then the quadric is *degenerate*, and may be defined by fewer points.

(iv) A quadric defines a polarity between a point and a plane, in a similar manner to the polarity defined by a conic between a point and a line (section 2.8.1). The plane $\boldsymbol{\pi} = \mathtt{Q}\mathbf{X}$ is the polar plane of $\mathbf{X}$ with respect to $\mathtt{Q}$. In the case that $\mathtt{Q}$ is non-singular and $\mathbf{X}$ is outside the quadric, the polar plane is defined by the points of contact with $\mathtt{Q}$ of the cone of rays through $\mathbf{X}$ tangent to $\mathtt{Q}$. If $\mathbf{X}$ lies on $\mathtt{Q}$, then $\mathtt{Q}\mathbf{X}$ is the tangent plane to $\mathtt{Q}$ at $\mathbf{X}$.

(v) The intersection of a plane $\boldsymbol{\pi}$ with a quadric $\mathtt{Q}$ is a conic $\mathtt{C}$. Computing the conic can be tricky because it requires a coordinate system for the plane. Recall from (3.7) that a coordinate system for the plane can be defined by the complement space to $\boldsymbol{\pi}$ as $\mathbf{X} = \mathtt{M}\mathbf{x}$. Points on $\boldsymbol{\pi}$ are on $\mathtt{Q}$ if $\mathbf{X}^\mathsf{T}\mathtt{Q}\mathbf{X} = \mathbf{x}^\mathsf{T}\mathtt{M}^\mathsf{T}\mathtt{Q}\mathtt{M}\mathbf{x} = 0$. These points lie on a conic $\mathtt{C}$, since $\mathbf{x}^\mathsf{T}\mathtt{C}\mathbf{x} = 0$, with $\mathtt{C} = \mathtt{M}^\mathsf{T}\mathtt{Q}\mathtt{M}$.

(vi) Under the point transformation $\mathbf{X}' = \mathtt{H}\mathbf{X}$, a (point) quadric transforms as

$$\mathtt{Q}' = \mathtt{H}^{-\mathsf{T}}\mathtt{Q}\mathtt{H}^{-1}. \tag{3.16}$$

The dual of a quadric is also a quadric. Dual quadrics are equations on planes: the tangent planes $\boldsymbol{\pi}$ to the point quadric $\mathtt{Q}$ satisfy $\boldsymbol{\pi}^\mathsf{T}\mathtt{Q}^*\boldsymbol{\pi} = 0$, where $\mathtt{Q}^* = \text{adjoint } \mathtt{Q}$, or $\mathtt{Q}^{-1}$ if $\mathtt{Q}$ is invertible. Under the point transformation $\mathbf{X}' = \mathtt{H}\mathbf{X}$, a dual quadric transforms as

$$\mathtt{Q}^{*\prime} = \mathtt{H}\mathtt{Q}^{*}\mathtt{H}^{\mathsf{T}}. \tag{3.17}$$

The algebra of imaging a quadric is far simpler for a dual quadric than a point quadric. This is detailed in chapter 8.

3.2.4 Classification of quadrics

Since the matrix, $\mathtt{Q}$, representing a quadric is symmetric, it may be decomposed as $\mathtt{Q} = \mathtt{U}^{\mathsf{T}}\mathtt{D}\mathtt{U}$ where $\mathtt{U}$ is a real orthogonal matrix and $\mathtt{D}$ is a real diagonal matrix. Further, by appropriate scaling of the rows of $\mathtt{U}$, one may write $\mathtt{Q} = \mathtt{H}^{\mathsf{T}}\mathtt{D}\mathtt{H}$ where $\mathtt{D}$ is diagonal with entries equal to $0, 1$, or -1. We may further ensure that the zero entries of $\mathtt{D}$ appear last along the diagonal, and that the $+1$ entries appear first. Now, replacement of $\mathtt{Q} = \mathtt{H}^{\mathsf{T}}\mathtt{D}\mathtt{H}$ by $\mathtt{D}$ is equivalent to a projective transformation effected by the matrix $\mathtt{H}$ (see (3.16)). Thus, up to projective equivalence, we may assume that the quadric is represented by a matrix $\mathtt{D}$ of the given simple form.

The *signature* of a diagonal matrix $\mathtt{D}$, denoted $\sigma(\mathtt{D})$, is defined to be the number of $+1$ entries minus the number of -1 entries. This definition is extended to arbitrary real symmetric matrices $\mathtt{Q}$ by defining $\sigma(\mathtt{Q}) = \sigma(\mathtt{D})$ such that $\mathtt{Q} = \mathtt{H}^{\mathsf{T}}\mathtt{D}\mathtt{H}$, where $\mathtt{H}$ is a real matrix. It may be proved that the signature is well defined, being independent of the particular choice of $\mathtt{H}$. Since the matrix representing a quadric is defined only up to sign, we may assume that its signature is non-negative. Then, the projective type of a quadric is uniquely determined by its rank and signature. This will allow us to enumerate the different projective equivalence classes of quadrics.

A quadric represented by a diagonal matrix $\mathrm{diag}(d_1, d_2, d_3, d_4)$ corresponds to a set of points satisfying an equation $d_1\mathrm{X}^2 + d_2\mathrm{Y}^2 + d_3\mathrm{Z}^2 + d_4\mathrm{T}^2 = 0$. One may set $\mathrm{T} = 1$ to get an equation for the non-infinite points on the quadric. See table 3.1. Examples of quadric surfaces are shown in figure 3.2 – figure 3.4.

Rank	σ	Diagonal	Equation	Realization
4	4	$(1,1,1,1)$	$\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2+1=0$	No real points
	2	$(1,1,1,-1)$	$\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2=1$	Sphere
	0	$(1,1,-1,-1)$	$\mathrm{X}^2+\mathrm{Y}^2=\mathrm{Z}^2+1$	Hyperboloid of one sheet
3	3	$(1,1,1,0)$	$\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2=0$	One point $(0,0,0,1)^{\mathsf{T}}$
	1	$(1,1,-1,0)$	$\mathrm{X}^2+\mathrm{Y}^2=\mathrm{Z}^2$	Cone at the origin
2	2	$(1,1,0,0)$	$\mathrm{X}^2+\mathrm{Y}^2=0$	Single line (Z-axis)
	0	$(1,-1,0,0)$	$\mathrm{X}^2=\mathrm{Y}^2$	Two planes $\mathrm{X}=\pm\mathrm{Y}$
1	1	$(1,0,0,0)$	$\mathrm{X}^2=0$	The plane $\mathrm{X}=0$

Table 3.1. *Categorization of point quadrics.*

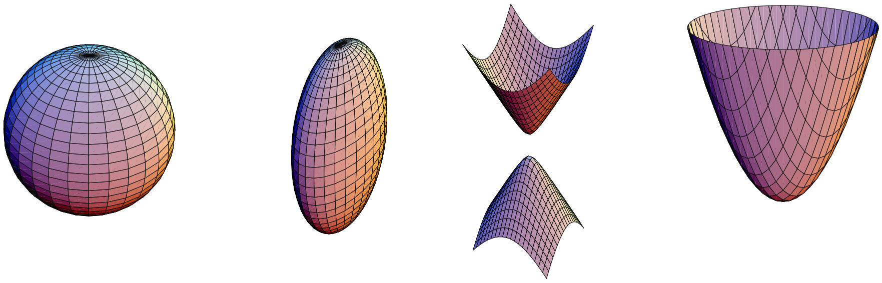


Fig. 3.2. **Non-ruled quadrics.** *This shows plots of a sphere, ellipsoid, hyperboloid of two sheets and paraboloid. They are all projectively equivalent.*

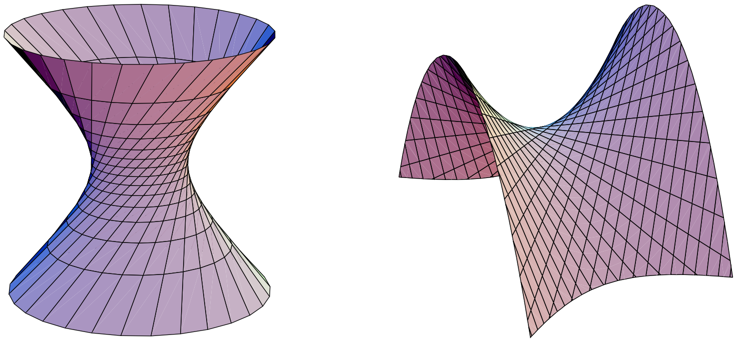


Fig. 3.3. **Ruled quadrics.** *Two examples of a hyperboloid of one sheet are given. These surfaces are given by equations* $\mathrm{X}^2 + \mathrm{Y}^2 = \mathrm{Z}^2 + 1$ *and* $\mathrm{XY} = \mathrm{Z}$ *respectively, and are projectively equivalent. Note that these two surfaces are made up of two sets of disjoint straight lines, and that each line from one set meets each line from the other set. The two quadrics shown here are projectively (though not affinely) equivalent.*

Ruled quadrics. Quadrics fall into two classes – ruled and unruled quadrics. A ruled quadric is one that contains a straight line. More particularly, as shown in figure 3.3, the non-degenerate ruled quadric (hyperboloid of one sheet) contains two families of straight lines called *generators*. For more properties of ruled quadrics, refer to [Semple-79].

The most interesting of the quadrics are the two quadrics of rank 4. Note that these two quadrics differ even in their topological type. The quadric of signature 2 (the sphere) is (obviously enough) topologically a sphere. On the other hand, the hyperboloid of 1 sheet is *not* topologically equivalent (homeomorphic) to a sphere. In fact, it is topologically a torus (topologically equivalent to $S^1 \times S^1$). This gives the clearest indication that they are not projectively equivalent.

3.3 Twisted cubics

The twisted cubic may be considered to be a 3-dimensional analogue of a 2D conic (although in other ways it is a quadric surface which is the 3-dimensional analogue of a 2D conic.)

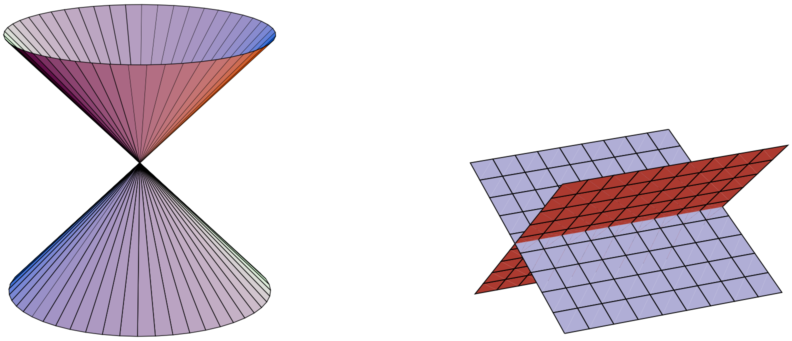


Fig. 3.4. **Degenerate quadrics.** *The two most important degenerate quadrics are shown, the cone and two planes. Both these quadrics are ruled. The matrix representing the cone has rank 3, and the null-vector represents the nodal point of the cone. The matrix representing the two (non-coincident) planes has rank 2, and the two generators of the rank 2 null-space are two points on the intersection line of the planes.*

A conic in the 2-dimensional projective plane may be described as a parametrized curve given by the equation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathtt{A} \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\theta + a_{13}\theta^2 \\ a_{21} + a_{22}\theta + a_{23}\theta^2 \\ a_{31} + a_{32}\theta + a_{33}\theta^2 \end{pmatrix} \tag{3.18}$$

where $\mathtt{A}$ is a non-singular 3×3 matrix.

In an analogous manner, a twisted cubic is defined to be a curve in $\mathbb{P}^3$ given in parametric form as

$$\begin{pmatrix} \mathrm{X}_1 \\ \mathrm{X}_2 \\ \mathrm{X}_3 \\ \mathrm{X}_4 \end{pmatrix} = \mathtt{A} \begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\theta + a_{13}\theta^2 + a_{14}\theta^3 \\ a_{21} + a_{22}\theta + a_{23}\theta^2 + a_{24}\theta^3 \\ a_{31} + a_{32}\theta + a_{33}\theta^2 + a_{34}\theta^3 \\ a_{41} + a_{42}\theta + a_{43}\theta^2 + a_{44}\theta^3 \end{pmatrix} \tag{3.19}$$

where $\mathtt{A}$ is a non-singular 4×4 matrix.

Since a twisted cubic is perhaps an unfamiliar object, various views of the curve are shown in figure 3.5. In fact, a twisted cubic is a quite benign space curve.

Properties of a twisted cubic. Let $\mathbf{c}$ be a non-singular twisted cubic. Then $\mathbf{c}$ is not contained within any plane of $\mathbb{P}^3$; it intersects a general plane at three distinct points. A twisted cubic has 12 degrees of freedom (counted as 15 for the matrix $\mathtt{A}$, less 3 for a 1D projectivity on the parametrization θ, which leaves the curve unaltered). Requiring the curve to pass through a point $\mathbf{X}$ places two constraints on $\mathbf{c}$, since $\mathbf{X} = \mathtt{A}(1, \theta, \theta^2, \theta^3)^\mathsf{T}$ is three independent ratios, but only two constraints once θ is eliminated. Thus, there is a unique $\mathbf{c}$ through six points in general position. Finally, all non-degenerate twisted cubics are projectively equivalent. This is clear from the definition (3.19): a projective transformation $\mathtt{A}^{-1}$ maps $\mathbf{c}$ to the standard form $\mathbf{c}(\theta') = (1, \theta', \theta'^2, \theta'^3)^\mathsf{T}$, and since

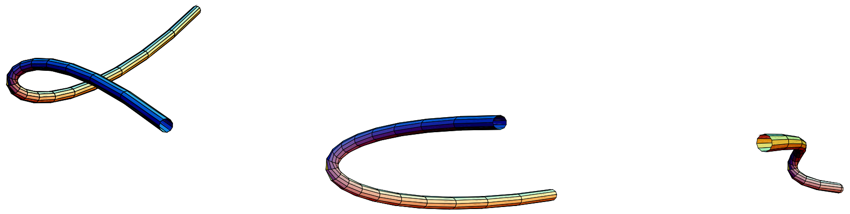


Fig. 3.5. *Various views of the twisted cubic* $(t^3, t^2, t,)^\mathsf{T}$. *The curve is thickened to a tube to aid in visualization.*

all twisted cubics can be mapped to this curve, it follows that all twisted cubics are projectively equivalent.

A classification of the various special cases of a twisted cubic, such as a conic and coincident line, are given in [Semple-79]. The twisted cubic makes an appearance as the horopter for two-view geometry (chapter 9), and plays the central role in defining the degenerate set for camera resectioning (chapter 22).

3.4 The hierarchy of transformations

There are a number of specializations of a projective transformation of 3-space which will appear frequently throughout this book. The specializations are analogous to the strata of section 2.4(*p*37) for planar transformations. Each specialization is a subgroup, and is identified by its matrix form, or equivalently by its invariants. These are summarized in table 3.2. This table lists only the *additional* properties of the 3-space transformations over their 2-space counterparts – the transformations of 3-space also have the invariants listed in table 2.1(*p*44) for the corresponding 2-space transformations.

The 15 degrees of freedom of a projective transformation are accounted for as seven for a similarity (three for rotation, three for translation, one for isotropic scaling), five for affine scalings, and three for the projective part of the transformation.

Two of the most important characterizations of these transformations are parallelism and angles. For example, after an affine transformation lines which were originally parallel remain parallel, but angles are skewed; and after a projective transformation parallelism is lost.

In the following we briefly describe a decomposition of a Euclidean transformation that will be useful when discussing special motions later in this book.

3.4.1 The screw decomposition

A Euclidean transformation on the plane may be considered as a specialization of a Euclidean transformation of 3-space with the restrictions that the translation vector $\mathbf{t}$ lies in the plane, and the rotation axis is perpendicular to the plane. However, Euclidean actions on 3-space are more general than this because the rotation axis and translation are not perpendicular in general. The screw decomposition enables any Euclidean

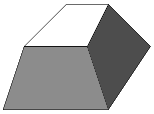
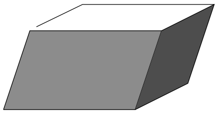
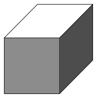
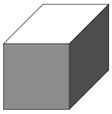
Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, $\boldsymbol{\pi}_\infty$, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$		Volume.

Table 3.2. **Geometric properties invariant to commonly occurring transformations of 3-space.** *The matrix* $\mathtt{A}$ *is an invertible* 3×3 *matrix,* $\mathtt{R}$ *is a 3D rotation matrix,* $\mathbf{t} = (t_X, t_Y, t_Z)^{\mathsf{T}}$ *a 3D translation,* $\mathbf{v}$ *a general 3-vector,* v *a scalar, and* $\mathbf{0} = (0, 0, 0)^{\mathsf{T}}$ *a null 3-vector. The distortion column shows typical effects of the transformations on a cube. Transformations higher in the table can produce all the actions of the ones below. These range from Euclidean, where only translations and rotations occur, to projective where five points can be transformed to any other five points (provided no three points are collinear, or four coplanar).*

action (a rotation composed with a translation) to be reduced to a situation almost as simple as the 2D case. The screw decomposition is that

Result 3.6. *Any particular translation and rotation is equivalent to a rotation about a* screw axis *together with a translation along the screw axis. The screw axis is parallel to the rotation axis.*

In the case of a translation and an *orthogonal* rotation axis (termed *planar motion*), the motion is equivalent to a rotation *alone* about the screw axis.

Proof. We will sketch a constructive geometric proof that can easily be visualized. Consider first the 2D case – a Euclidean transformation on the plane. It is evident from figure 3.6 that a screw axis exists for such 2D transformations. For the 3D case, decompose the translation $\mathbf{t}$ into two components $\mathbf{t} = \mathbf{t}_{\|} + \mathbf{t}_{\perp}$, parallel and orthogonal respectively to the rotation axis direction ($\mathbf{t}_{\|} = (\mathbf{t}.\mathbf{a})\mathbf{a}$, $\mathbf{t}_{\perp} = \mathbf{t} - (\mathbf{t}.\mathbf{a})\mathbf{a}$).
Then the Euclidean motion is partitioned into two parts: first a rotation about the screw

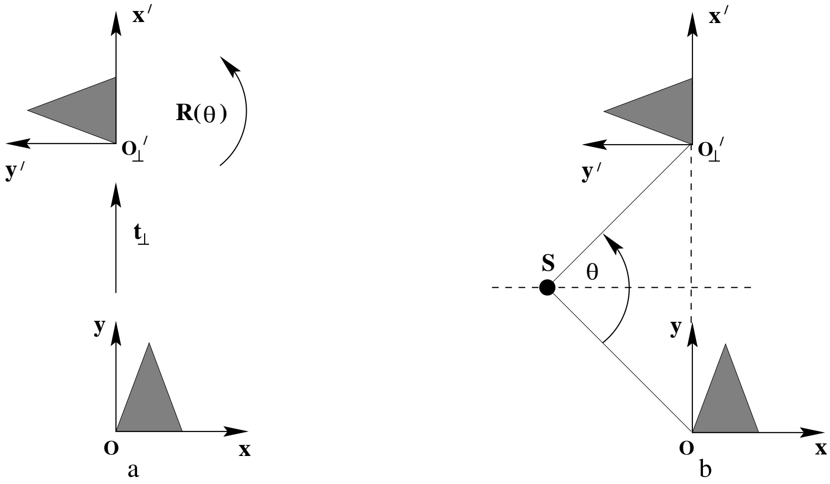


Fig. 3.6. **2D Euclidean motion and a "screw" axis.** *(a) The frame $\{x, y\}$ undergoes a translation $\mathbf{t}_\perp$ and a rotation by θ to reach the frame $\{x', y'\}$. The motion is in the plane orthogonal to the rotation axis. (b) This motion is equivalent to a single rotation about the screw axis $\mathbf{S}$. The screw axis lies on the perpendicular bisector of the line joining corresponding points, such that the angle between the lines joining $\mathbf{S}$ to the corresponding points is θ. In the figure the corresponding points are the two frame origins and θ has the value $90°$.*

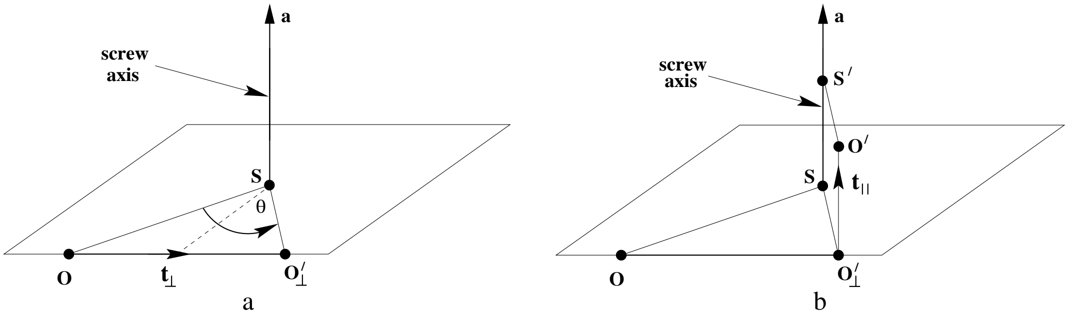


Fig. 3.7. **3D Euclidean motion and the screw decomposition.** *Any Euclidean rotation $\mathtt{R}$ and translation $\mathbf{t}$ may be achieved by (a) a rotation about the screw axis, followed by (b) a translation along the screw axis by $\mathbf{t}_{\|}$. Here $\mathbf{a}$ is the (unit) direction of the rotation axis (so that $\mathtt{R}\mathbf{a} = \mathbf{a}$), and $\mathbf{t}$ is decomposed as $\mathbf{t} = \mathbf{t}_{\|} + \mathbf{t}_\perp$, which are vector components parallel and orthogonal respectively to the rotation axis direction The point $\mathbf{S}$ is closest to $\mathbf{O}$ on the screw axis (so that the line from $\mathbf{S}$ to $\mathbf{O}$ is perpendicular to the direction of $\mathbf{a}$). Similarly $\mathbf{S}'$ is the point on the screw axis closest to $\mathbf{O}'$.*

axis, which covers the rotation and $\mathbf{t}_\perp$; second a translation by $\mathbf{t}_{\|}$ along the screw axis. The complete motion is illustrated in figure 3.7. □

The screw decomposition can be determined from the fixed points of the 4×4 matrix representing the Euclidean transformation. This idea is examined in the exercises at the end of the chapter.

3.5 The plane at infinity

In planar projective geometry identifying the line at infinity, $\mathbf{l}_\infty$, allowed affine properties of the plane to be measured. Identifying the circular points on $\mathbf{l}_\infty$ then allowed the measurement of metric properties. In the projective geometry of 3-space the corresponding geometric entities are the plane at infinity, $\boldsymbol{\pi}_\infty$, and the absolute conic, Ω_∞.

The plane at infinity has the canonical position $\boldsymbol{\pi}_\infty = (0, 0, 0, 1)^\mathsf{T}$ in affine 3-space. It contains the directions $\mathbf{D} = (\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, 0)^\mathsf{T}$, and enables the identification of affine properties such as parallelism. In particular:

- Two planes are parallel if, and only if, their line of intersection is on $\boldsymbol{\pi}_\infty$.
- A line is parallel to another line, or to a plane, if the point of intersection is on $\boldsymbol{\pi}_\infty$.

We then have in $\mathbb{P}^3$ that any pair of planes intersect in a line, with parallel planes intersecting in a line on the plane at infinity.

The plane $\boldsymbol{\pi}_\infty$ is a geometric representation of the 3 degrees of freedom required to specify affine properties in a projective coordinate frame. In loose terms, the plane at infinity is a fixed plane under any affine transformation, but "sees" (is moved by) a projective transformation. The 3 degrees of freedom of $\boldsymbol{\pi}_\infty$ thus measure the projective component of a general homography – they account for the 15 degrees of freedom of this general transformation compared to an affinity (12 dof). More formally:

Result 3.7. *The plane at infinity, $\boldsymbol{\pi}_\infty$, is a fixed plane under the projective transformation* $\mathtt{H}$ *if, and only if,* $\mathtt{H}$ *is an affinity.*

The proof is the analogue of the derivation of result 2.17(*p*48). It is worth clarifying two points:

(i) The plane $\boldsymbol{\pi}_\infty$ is, in general, only fixed as a set under an affinity; it is not fixed pointwise.
(ii) Under a particular affinity (for example a Euclidean motion) there may be planes in addition to $\boldsymbol{\pi}_\infty$ which are fixed. However, only $\boldsymbol{\pi}_\infty$ is fixed under any affinity.

These points are illustrated in more detail by the following example.

Example 3.8. Consider the Euclidean transformation represented by the matrix

$$\mathtt{H}_\mathrm{E} = \begin{bmatrix} \mathtt{R} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.20}$$

This is a rotation by θ about the Z-axis with a zero translation (it is a planar screw motion, see section 3.4.1). Geometrically it is evident that the family of XY-planes orthogonal to the rotation axis are simply rotated about the Z-axis by this transformation. This means that there is a pencil of fixed planes orthogonal to the Z-axis. The planes are fixed as sets, but not pointwise as any (finite) point (not on the axis) is rotated in horizontal circles by this Euclidean action. Algebraically, the fixed planes of $\mathtt{H}$ are the eigenvectors of $\mathtt{H}^\mathsf{T}$ (refer to section 2.9). In this case the eigenvalues are $\{e^{i\theta}, e^{-i\theta}, 1, 1\}$

and the corresponding eigenvectors of $\mathtt{H}_{\mathrm{E}}^{\mathsf{T}}$ are

$$\mathbf{E}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{E}_2 = \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{E}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{E}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvectors $\mathbf{E}_1$ and $\mathbf{E}_2$ do not correspond to real planes, and will not be discussed further here. The eigenvectors $\mathbf{E}_3$ and $\mathbf{E}_4$ are degenerate. Thus there is a pencil of fixed planes which is spanned by these eigenvectors. The axis of this pencil is the line of intersection of the the planes (perpendicular to the Z-axis) with $\boldsymbol{\pi}_\infty$, and the pencil includes $\boldsymbol{\pi}_\infty$. △

The example also illustrates the connection between the geometry of the projective plane, $\mathbb{P}^2$, and projective 3-space, $\mathbb{P}^3$. A plane $\boldsymbol{\pi}$ intersects $\boldsymbol{\pi}_\infty$ in a line which is the line at infinity, $\mathbf{l}_\infty$, of the plane $\boldsymbol{\pi}$. A projective transformation of $\mathbb{P}^3$ induces a *subordinate* plane projective transformation on $\boldsymbol{\pi}$.

Affine properties of a reconstruction. In later chapters on reconstruction, for example chapter 10, it will be seen that the projective coordinates of the (Euclidean) scene can be reconstructed from multiple views. Once $\boldsymbol{\pi}_\infty$ is identified in projective 3-space, i.e. its projective coordinates are known, it is then possible to determine affine properties of the reconstruction such as whether geometric entities are parallel – they are parallel if they intersect on $\boldsymbol{\pi}_\infty$.

A more algorithmic approach is to transform $\mathbb{P}^3$ so that the identified $\boldsymbol{\pi}_\infty$ is moved to its canonical position at $\boldsymbol{\pi}_\infty = (0,0,0,1)^{\mathsf{T}}$. After this mapping we then have the situation that the Euclidean scene, where $\boldsymbol{\pi}_\infty$ has the coordinates $(0,0,0,1)^{\mathsf{T}}$, and our reconstruction are related by a projective transformation that fixes $\boldsymbol{\pi}_\infty$ at $(0,0,0,1)^{\mathsf{T}}$. It follows from result 3.7 that the scene and reconstruction are related by an affine transformation. Thus affine properties can now be measured directly from the coordinates of the entities.

3.6 The absolute conic

The absolute conic, Ω_∞, is a (point) conic on $\boldsymbol{\pi}_\infty$. In a metric frame $\boldsymbol{\pi}_\infty = (0,0,0,1)^{\mathsf{T}}$, and points on Ω_∞ satisfy

$$\left.\begin{array}{r} \mathrm{X}_1^2 + \mathrm{X}_2^2 + \mathrm{X}_3^2 \\ \mathrm{X}_4 \end{array}\right\} = 0. \tag{3.21}$$

Note that two equations are required to define Ω_∞.

For directions on $\boldsymbol{\pi}_\infty$ (i.e. points with $\mathrm{X}_4 = 0$) the defining equation can be written

$$(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)\mathtt{I}(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)^{\mathsf{T}} = 0$$

so that Ω_∞ corresponds to a conic $\mathtt{C}$ with matrix $\mathtt{C} = \mathtt{I}$. It is thus a conic of purely imaginary points on $\boldsymbol{\pi}_\infty$.

The conic Ω_∞ is a geometric representation of the 5 additional degrees of freedom that are required to specify metric properties in an affine coordinate frame. A key property of Ω_∞ is that it is a fixed conic under any similarity transformation. More formally:

Result 3.9. *The absolute conic, Ω_∞, is a fixed conic under the projective transformation* $\mathtt{H}$ *if, and only if,* $\mathtt{H}$ *is a similarity transformation.*

Proof. Since the absolute conic lies in the plane at infinity, a transformation fixing it must fix the plane at infinity, and hence must be affine. Such a transformation is of the form

$$\mathtt{H}_{\mathtt{A}} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}.$$

Restricting to the plane at infinity, the absolute conic is represented by the matrix $\mathtt{I}_{3\times 3}$, and since it is fixed by $\mathtt{H}_{\mathtt{A}}$, one has $\mathtt{A}^{-\mathsf{T}}\mathtt{I}\mathtt{A}^{-1} = \mathtt{I}$ (up to scale), and taking inverses gives $\mathtt{A}\mathtt{A}^{\mathsf{T}} = \mathtt{I}$. This means that $\mathtt{A}$ is orthogonal, hence a scaled rotation, or scaled rotation with reflection. This completes the proof. □

Even though Ω_∞ does not have any real points, it shares the properties of any conic – such as that a line intersects a conic in two points; the pole–polar relationship etc. Here are a few particular properties of Ω_∞:

(i) Ω_∞ is only fixed as a set by a general similarity; it is not fixed pointwise. This means that under a similarity a point on Ω_∞ may travel to another point on Ω_∞, but it is not mapped to a point off the conic.

(ii) All circles intersect Ω_∞ in two points. Suppose the support plane of the circle is $\boldsymbol{\pi}$. Then $\boldsymbol{\pi}$ intersects $\boldsymbol{\pi}_\infty$ in a line, and this line intersects Ω_∞ in two points. These two points are the circular points of $\boldsymbol{\pi}$.

(iii) All spheres intersect $\boldsymbol{\pi}_\infty$ in Ω_∞.

Metric properties. Once Ω_∞ (and its support plane $\boldsymbol{\pi}_\infty$) have been identified in projective 3-space then metric properties, such as angles and relative lengths, can be measured.

Consider two lines with directions (3-vectors) $\mathbf{d}_1$ and $\mathbf{d}_2$. The angle between these directions in a Euclidean world frame is given by

$$\cos\theta = \frac{(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_1)(\mathbf{d}_2^{\mathsf{T}}\mathbf{d}_2)}}. \tag{3.22}$$

This may be written as

$$\cos\theta = \frac{(\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_1)(\mathbf{d}_2^{\mathsf{T}}\Omega_\infty\mathbf{d}_2)}} \tag{3.23}$$

where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the points of intersection of the lines with the plane $\boldsymbol{\pi}_\infty$ containing the conic Ω_∞, and Ω_∞ is the matrix representation of the absolute conic in that plane.

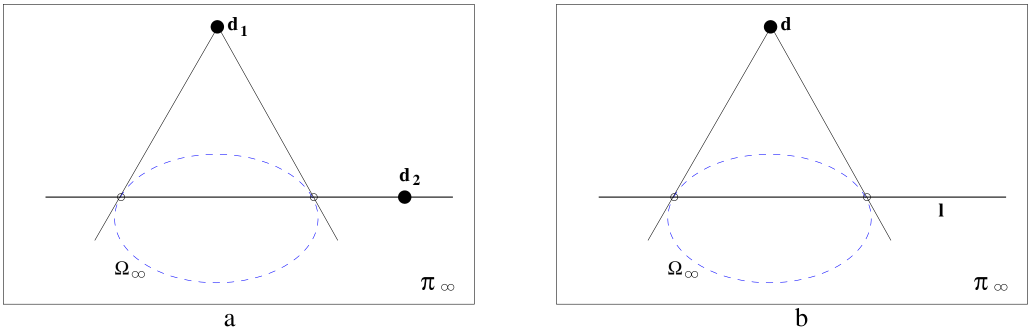


Fig. 3.8. **Orthogonality and Ω_∞.** *(a) On $\boldsymbol{\pi}_\infty$ orthogonal directions $\mathbf{d}_1$, $\mathbf{d}_2$ are conjugate with respect to Ω_∞. (b) A plane normal direction $\mathbf{d}$ and the intersection line $\mathbf{l}$ of the plane with $\boldsymbol{\pi}_\infty$ are in pole–polar relation with respect to Ω_∞.*

The expression (3.23) reduces to (3.22) in a Euclidean world frame where $\Omega_\infty = \mathtt{I}$. However, the expression is valid in any projective coordinate frame as may be verified from the transformation properties of points and conics (see (iv)(b) on page 63).

There is no simple formula for the angle between two planes computed from the directions of their surface normals.

Orthogonality and polarity. We now give a geometric representation of orthogonality in a projective space based on the absolute conic. The main device will be the pole–polar relationship between a point and line induced by a conic.

An immediate consequence of (3.23) is that two directions $\mathbf{d}_1$ and $\mathbf{d}_2$ are orthogonal if $\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_2 = 0$. Thus orthogonality is encoded by *conjugacy* with respect to Ω_∞. The great advantage of this is that conjugacy is a projective relation, so that in a projective frame (obtained by a projective transformation of Euclidean 3-space) directions can be identified as orthogonal if they are conjugate with respect to Ω_∞ in that frame (in general the matrix of Ω_∞ is not $\mathtt{I}$ in a projective frame). The geometric representation of orthogonality is shown in figure 3.8.

This representation is helpful when considering orthogonality between rays in a camera, for example in determining the normal to a plane through the camera centre (see section 8.6(*p*213)). If image points are conjugate with respect to the *image* of Ω_∞ then the corresponding rays are orthogonal.

Again, a more algorithmic approach is to projectively transform the coordinates so that Ω_∞ is mapped to its canonical position (3.21), and then metric properties can be determined directly from coordinates.

3.7 The absolute dual quadric

Recall that Ω_∞ is defined by *two* equations – it is a conic on the plane at infinity. The dual of the absolute conic Ω_∞ is a degenerate dual *quadric* in 3-space called the *absolute dual quadric*, and denoted $\mathtt{Q}^*_\infty$. Geometrically $\mathtt{Q}^*_\infty$ consists of the planes tangent to Ω_∞, so that Ω_∞ is the "rim" of $\mathtt{Q}^*_\infty$. This is called a *rim quadric*. Think of the set of planes tangent to an ellipsoid, and then squash the ellipsoid to a pancake.

Algebraically $\mathtt{Q}^*_\infty$ is represented by a 4×4 homogeneous matrix of rank 3, which in metric 3-space has the canonical form

$$\mathtt{Q}^*_\infty = \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}. \tag{3.24}$$

We will show that any plane in the dual absolute quadric envelope is indeed tangent to Ω_∞, so the $\mathtt{Q}^*_\infty$ is truly a dual of Ω_∞. Consider a plane represented by $\boldsymbol{\pi} = (\mathbf{v}^\mathsf{T}, k)^\mathsf{T}$. This plane is in the envelope defined by $\mathtt{Q}^*_\infty$ if and only if $\boldsymbol{\pi}^\mathsf{T}\mathtt{Q}^*_\infty\boldsymbol{\pi} = 0$, which given the form (3.24) is equivalent to $\mathbf{v}^\mathsf{T}\mathbf{v} = 0$. Now, (see section 8.6(*p*213)) $\mathbf{v}$ represents the line in which the plane $(\mathbf{v}^\mathsf{T}, k)^\mathsf{T}$ meets the plane at infinity. This line is tangent to the absolute conic if and only if $\mathbf{v}^\mathsf{T}\mathtt{I}\mathbf{v} = 0$. Thus, the envelope of $\mathtt{Q}^*_\infty$ is made up of just those planes tangent to the absolute conic.

Since this is an important fact, we consider it from another angle. Consider the absolute conic as the limit of a series of squashed ellipsoids, namely quadrics represented by the matrix $\mathtt{Q} = \mathrm{diag}(1, 1, 1, k)$. As $k \to \infty$, these quadrics become increasingly close to the plane at infinity, and in the limit the only points they contain are the points $(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, 0)^\mathsf{T}$ with $\mathrm{X}_1^2 + \mathrm{X}_2^2 + \mathrm{X}_3^2 = 0$, that is points on the absolute conic. However, the dual of $\mathtt{Q}$ is the quadric $\mathtt{Q}^* = \mathtt{Q}^{-1} = \mathrm{diag}(1, 1, 1, k^{-1})$, which in the limit becomes the absolute dual quadric $\mathtt{Q}^*_\infty = \mathrm{diag}(1, 1, 1, 0)$.

The dual quadric $\mathtt{Q}^*_\infty$ is a degenerate quadric and has 8 degrees of freedom (a symmetric matrix has 10 independent elements, but the irrelevant scale and zero determinant condition each reduce the degrees of freedom by 1). It is a geometric representation of the 8 degrees of freedom that are required to specify metric properties in a projective coordinate frame. $\mathtt{Q}^*_\infty$ has a significant advantage over Ω_∞ in algebraic manipulations because both $\boldsymbol{\pi}_\infty$ and Ω_∞ are contained in a single geometric object (unlike Ω_∞ which requires two equations (3.21) in order to specify it). In the following we give its three most important properties.

Result 3.10. *The absolute dual quadric, $\mathtt{Q}^*_\infty$, is fixed under the projective transformation $\mathtt{H}$ if, and only if, $\mathtt{H}$ is a similarity.*

Proof. This follows directly from the invariance of the absolute conic under a similarity transform, since the planar tangency relationship between $\mathtt{Q}^*_\infty$ and Ω_∞ is transformation invariant. Nevertheless, we give an independent direct proof.
Since $\mathtt{Q}^*_\infty$ is a dual quadric, it transforms according to (3.17–*p*74), so it is fixed under $\mathtt{H}$ if and only if $\mathtt{Q}^*_\infty = \mathtt{H}\mathtt{Q}^*_\infty\mathtt{H}^\mathsf{T}$. Applying this with an arbitrary transform

$$\mathtt{H} = \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & k \end{bmatrix}$$

we find

$$\begin{aligned} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} &= \begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & k \end{bmatrix} \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathtt{A}^\mathsf{T} & \mathbf{v} \\ \mathbf{t}^\mathsf{T} & k \end{bmatrix} \\ &= \begin{bmatrix} \mathtt{A}\mathtt{A}^\mathsf{T} & \mathtt{A}\mathbf{v} \\ \mathbf{v}^\mathsf{T}\mathtt{A}^\mathsf{T} & \mathbf{v}^\mathsf{T}\mathbf{v} \end{bmatrix} \end{aligned}$$

which must be true up to scale. By inspection, this equation holds if and only if $\mathbf{v} = \mathbf{0}$ and $\mathtt{A}$ is a scaled orthogonal matrix (scaling, rotation and possible reflection). In other words, $\mathtt{H}$ is a similarity transform. □

Result 3.11. *The plane at infinity $\boldsymbol{\pi}_\infty$ is the null-vector of $\mathtt{Q}^*_\infty$.*

This is easily verified when $\mathtt{Q}^*_\infty$ has its canonical form (3.24) in a metric frame since then, with $\boldsymbol{\pi}_\infty = (0, 0, 0, 1)^\mathsf{T}$, $\mathtt{Q}^*_\infty \boldsymbol{\pi}_\infty = \mathbf{0}$. This property holds in any frame as may be readily seen algebraically from the transformation properties of planes and dual quadrics: if $\mathbf{X}' = \mathtt{H}\mathbf{X}$, then $\mathtt{Q}^{*\,\prime}_\infty = \mathtt{H}\,\mathtt{Q}^*_\infty\,\mathtt{H}^\mathsf{T}$, $\boldsymbol{\pi}'_\infty = \mathtt{H}^{-\mathsf{T}}\boldsymbol{\pi}_\infty$, and

$$\mathtt{Q}^{*\,\prime}_\infty \boldsymbol{\pi}'_\infty = (\mathtt{H}\,\mathtt{Q}^*_\infty\,\mathtt{H}^\mathsf{T})\mathtt{H}^{-\mathsf{T}}\boldsymbol{\pi}_\infty = \mathtt{H}\mathtt{Q}^*_\infty \boldsymbol{\pi}_\infty = \mathbf{0}.$$

Result 3.12. *The angle between two planes $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ is given by*

$$\cos\theta = \frac{\boldsymbol{\pi}_1^\mathsf{T}\mathtt{Q}^*_\infty\boldsymbol{\pi}_2}{\sqrt{(\boldsymbol{\pi}_1^\mathsf{T}\mathtt{Q}^*_\infty\boldsymbol{\pi}_1)\,(\boldsymbol{\pi}_2^\mathsf{T}\mathtt{Q}^*_\infty\boldsymbol{\pi}_2)}}. \tag{3.25}$$

Proof. Consider two planes with Euclidean coordinates $\boldsymbol{\pi}_1 = (\mathbf{n}_1^\mathsf{T}, d_1)^\mathsf{T}$, $\boldsymbol{\pi}_2 = (\mathbf{n}_2^\mathsf{T}, d_2)^\mathsf{T}$. In a Euclidean frame, $\mathtt{Q}^*_\infty$ has the form (3.24), and (3.25) reduces to

$$\cos\theta = \frac{\mathbf{n}_1^\mathsf{T}\mathbf{n}_2}{\sqrt{(\mathbf{n}_1^\mathsf{T}\mathbf{n}_1)\,(\mathbf{n}_2^\mathsf{T}\mathbf{n}_2)}}$$

which is the angle between the planes expressed in terms of a scalar product of their normals.

If the planes and $\mathtt{Q}^*_\infty$ are projectively transformed, (3.25) will still determine the angle between planes due to the (covariant) transformation properties of planes and dual quadrics. □

The details of the last part of the proof are left as an exercise, but are a direct 3D analogue of the derivation of result 2.23(*p*54) on the angle between two lines in $\mathbb{P}^2$ computed using the dual of the circular points. Planes in $\mathbb{P}^3$ are the analogue of lines in $\mathbb{P}^2$, and the absolute dual quadric is the analogue of the dual of the circular points.

3.8 Closure

3.8.1 The literature

The textbooks cited in chapter 2 are also relevant here. See also [Boehm-94] for a general background from the perspective of descriptive geometry, and Hilbert and Cohn-Vossen [Hilbert-56] for many clearly explained properties of curves and surfaces.

An important representation for points, lines and planes in $\mathbb{P}^3$, which is omitted in this chapter, is the Grassmann–Cayley algebra. In this representation geometric operations such as incidence and joins are represented by a "bracket algebra" based on matrix determinants. A good introduction to this area is given by [Carlsson-94], and its application to multiple view tensors is illustrated in [Triggs-95].

Faugeras and Maybank [Faugeras-90] introduced Ω_∞ into the computer vision literature (in order to determine the multiplicity of solutions for relative orientation), and Triggs introduced $\mathtt{Q}^*_\infty$ in [Triggs-97] for use in auto-calibration.

3.8.2 Notes and exercises

(i) **Plücker coordinates.**

(a) Using Plücker line coordinates, $\mathcal{L}$, write an expression for the point of intersection of a line with a plane, and the plane defined by a point and a line.

(b) Now derive the condition for a point to be on a line, and a line to be on a plane.

(c) Show that parallel planes intersect in a line on $\boldsymbol{\pi}_\infty$. Hint, start from (3.9–*p*71) to determine the line of intersection of two parallel planes $\mathtt{L}^*$.

(d) Show that parallel lines intersect on $\boldsymbol{\pi}_\infty$.

(ii) **Projective transformations.** Show that a (real) projective transformation of 3-space can map an ellipsoid to a paraboloid or hyperboloid of two sheets, but cannot map an ellipsoid to a hyperboloid of one sheet (i.e. a surface with real rulings).

(iii) **Screw decomposition.** Show that the 4×4 matrix representing the Euclidean transformation $\{\mathtt{R}, \mathbf{t}\}$ (with $\mathbf{a}$ the direction of the rotation axis, i.e. $\mathtt{R}\mathbf{a} = \mathbf{a}$) has two complex conjugate eigenvalues, and two equal real eigenvalues, and the following eigenvector structure:

(a) if $\mathbf{a}$ is perpendicular to $\mathbf{t}$, then the eigenvectors corresponding to the real eigenvalues are distinct;

(b) otherwise, the eigenvectors corresponding to the real eigenvalues are coincident, and on $\boldsymbol{\pi}_\infty$.

(E.g. choose simple cases such as (3.20), another case is given on page 495). In the first case the two real points corresponding to the real eigenvalues define a line of fixed points. This is the screw axis for planar motion. In the second case, the direction of the screw axis is defined, but it is not a line of fixed points. What do the eigenvectors corresponding to the complex eigenvalues represent?