Appendix 1

Tensor Notation

Since tensor notation is not commonly used in computer vision, it seems appropriate to give a brief introduction to its use. For more details, the reader is referred to [Triggs-95]. For simplicity, these concepts will be explained here in the context of low-dimensional projective spaces, rather than in their full generality. However, the ideas apply in arbitrary dimensional vector spaces.

Consider a set of basis vectors $\mathbf{e}_i,\ i = 1,\ldots,3$ for a 2-dimensional projective space $\mathbb{P}^2$. For reasons to become clear, we will write the indices as subscripts. With respect to this basis, a point in $\mathbb{P}^2$ is represented by a set of coordinates x^i, which represents the point $\sum_{i=1}^{3} x^i\,\mathbf{e}_i$. We write the coordinates with an upper index, as shown. Let $\mathbf{x}$ represent the triple of coordinates, $\mathbf{x} = (x^1, x^2, x^3)^\mathsf{T}$.

Now, consider a change of coordinate axes in which the basis vectors $\mathbf{e}_i$ are replaced by a new basis set $\hat{\mathbf{e}}_j$, where $\hat{\mathbf{e}}_j = \sum_i H^i_j\,\mathbf{e}_i$, and $\mathtt{H}$ is the basis transformation matrix with entries H^i_j. If $\hat{\mathbf{x}} = (\hat{x}^1, \hat{x}^2, \hat{x}^3)^\mathsf{T}$ are the coordinates of the vector with respect to the new basis, then we may verify that $\hat{\mathbf{x}} = \mathtt{H}^{-1}\mathbf{x}$. Thus, if the basis vectors transform according to $\mathtt{H}$ the coordinates of points transform according to the inverse transformation $\mathtt{H}^{-1}$.

Next, consider a line in $\mathbb{P}^2$ represented by coordinates $\mathbf{l}$ with respect to the original basis. With respect to the new basis, it may be verified that the line is represented by a new set of coordinates $\hat{\mathbf{l}} = \mathtt{H}^\mathsf{T}\mathbf{l}$. Thus coordinates of the line transform according to $\mathtt{H}^\mathsf{T}$.

As a further example, let $\mathtt{P}$ be a matrix representing a mapping between projective (or vector) spaces. If $\mathtt{G}$ and $\mathtt{H}$ represent basis transformations in the domain and range spaces, then with respect to the new bases, the mapping is represented by a new matrx $\hat{\mathtt{P}} = \mathtt{H}^{-1}\mathtt{P}\mathtt{G}$. Note in these examples that sometimes the matrix $\mathtt{H}$ or $\mathtt{H}^\mathsf{T}$ is used in the transformation, and sometimes $\mathtt{H}^{-1}$.

These three examples of coordinate transformations may be written explicitly as follows.

$$\hat{x}^i = (H^{-1})^i_j\,x^j \qquad \hat{l}_i = H^j_i\,l_j \qquad \hat{P}^i_j = (H^{-1})^i_k\,G^l_j\,P^k_l$$

where we use the tensor summation convention that an index repeated in upper and lower positions in a product represents summation over the range of the index. Note that those indices that are written as superscripts transform according to $\mathtt{H}^{-1}$, whereas

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those that are written as subscripts transform as $\mathtt{H}$ (or $\mathtt{G}$). Note that there is no distinction in tensor notation between indices that are transformed by $\mathtt{H}$, and those that are transformed by $\mathtt{H}^\mathsf{T}$. In general, tensor indices will transform by either $\mathtt{H}$ or $\mathtt{H}^{-1}$ – in fact this is the characteristic of a tensor. Those indices that transform according to $\mathtt{H}$ are known as *covariant* indices and are written as subscripts. Those indices that transform according to $\mathtt{H}^{-1}$ are known as *contravariant* indices, and are written as superscripts. The number of indices is the *valency* of the tensor. The sum over an index, e.g. $H_i^j\, l_j$, is referred to as a *contraction*, in this case the tensor H_i^j is contracted with the line l_j.

A1.1 The tensor ϵ_{rst}

The tensor ϵ_{rst} is defined for $r, s, t = 1, \ldots, 3$ as follows:

$$\epsilon_{rst} = \begin{cases} \;\;0 \text{ unless } r, s \text{ and } t \text{ are distinct} \\ +1 \text{ if } rst \text{ is an even permutation of } 123 \\ -1 \text{ if } rst \text{ is an odd permutation of } 123 \end{cases}$$

The tensor ϵ_{ijk} (or its contravariant counterpart, ϵ^{ijk}) is connected with the cross product of two vectors. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, and $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ is their cross product, then the following formula may easily be verified.

$$c_i = (\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a^j b^k .$$

Related to this is the expression (A4.5–*p*581) for the skew-symmetric matrix $[\mathbf{a}]_\times$. Using tensor notation one writes this as

$$([\mathbf{a}]_\times)_{ik} = \epsilon_{ijk} a^j .$$

Thus, one sees that if $\mathbf{a}$ is a contravariant vector, then $[\mathbf{a}]_\times$ is a matrix with two covariant indices. A similar formula holds for $[\mathbf{v}]_\times$ where $\mathbf{v}$ is covariant, namely $([\mathbf{v}]_\times)^{ik} = \epsilon^{ijk} v_j$.

Finally, the tensor ϵ_{ijk} is related to determinants: for three contravariant tensors a^i, b^j and c^k, one verifies that $a^i b^j c^k \epsilon_{ijk}$ is the determinant of the 3×3 matrix with rows a^i, b^j and c^k.

A1.2 The trifocal tensor

The trifocal tensor $\mathcal{T}_i^{jk}$ has one covariant and two contravariant indices. For vectors and matrices, such as x^i, l_i and P_j^i, it is possible to write the transformation rules using standard linear algebra notation, e.g. $\mathbf{x}' = \mathtt{H}\mathbf{x}$. However, for tensors with three or more indices, this cannot conveniently be done. There is really no choice but to use tensor notation when dealing with the trifocal tensor.

Transformation rule. The arrangement of indices for the trifocal tensor implies a transformation rule

$$\hat{\mathcal{T}}_i^{jk} = F_i^r \, (G^{-1})_s^j \, (H^{-1})_t^k \mathcal{T}_r^{st} \tag{A1.1}$$

with respect to changes of basis in the three images. It is worthwhile pointing out one possible source of confusion here. The transformation rule (A1.1) shows how the

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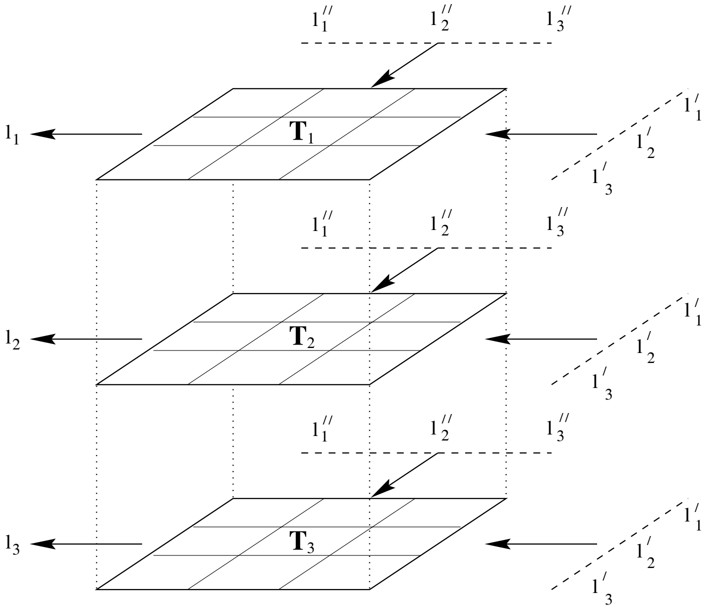

Fig. A1.1. **A 3-dimensional representation of the trifocal tensor** – *figure after Faugeras and Papadopoulo [Faugeras-97]. The picture represents* $l_i = l'_j l''_k \mathcal{T}_i^{jk}$, *which is the contraction of the tensor with the lines* $\mathbf{l}'$ *and* $\mathbf{l}''$ *to produce a line* $\mathbf{l}$. *In pseudo-matrix notation this can be written as* $l_i = \mathbf{l}'^{\mathsf{T}} \mathtt{T}_i \mathbf{l}''^{\mathsf{T}}$, *where* $(\mathtt{T}_i)_{jk} = \mathcal{T}_i^{jk}$.

tensor is transformed in terms of *basis* transformations in the three images. Often, we are concerned instead with point coordinate transformations. Thus, if $\mathtt{F}'$, $\mathtt{G}'$ and $\mathtt{H}'$ represent *coordinate* transformations in the images, in the sense that $\hat{x}^j = F'^j_i\, x^i$, and $\mathtt{G}'$ and $\mathtt{H}'$ are similarly defined for the other images, then the transformation rule may be written as

$$\hat{\mathcal{T}}_i^{jk} = (F'^{-1})_i^r\, G'^j_s\, H'^k_t \mathcal{T}_r^{st}.$$

Picture of tensors. A vector $\mathbf{x}$ may be thought of as a set of numbers arranged in a column or row, and a matrix $\mathtt{H}$ as a 2D array of numbers. Similarly, a tensor with three indices may be thought of as a 3D array of numbers. In particular the trifocal tensor is a $3 \times 3 \times 3$ cube of cells as illustrated in figure A1.1.

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