Representing a Circle or a Sphere with NURBS

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Created: January 13, 2003
Last Modified: April 24, 2015

Contents

This is just a brief note on representing circles and spheres with NURBS. For more information about NURBS, a good engineering-style approach to NURBS is [1]. An excellent book covering the mathematics and modeling with splines is [2]. A more mathematically advanced presentation is [3].

1 Introduction

A NURBS curve in 2D is generated by a homogeneous B-spline curve in 3D,

$$(\bar{x}(u), \bar{y}(u), w(u)) = \sum_{i=0}^{n} N_{i,d}(u) w_i (\mathbf{C}_i, 1) \tag{1}$$

where the 3D curve has $n+1$ control points $(\mathbf{C}_i, 1)$ [the $\mathbf{C}_i$ are 2-tuples], $n+1$ weights w_i, and degree d with $1 \leq d \leq n$. The functions $N_{i,d}(u)$ are the B-spline basis functions, which are defined recursively and require selection of a sequence of nondecreasing scalars u_i for $0 \leq i \leq n+d+1$. Each u_i is called a knot and the total sequence is called a knot vector. The basis function that starts the recursive definition is

$$N_{i,0}(u) = \begin{cases} 1, & u_i \leq u < u_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

for $0 \leq i \leq n+d$. The recursion is

$$N_{i,j}(u) = \frac{u - u_i}{u_{i+j} - u_i} N_{i,j-1}(u) + \frac{u_{i+j+1} - u}{u_{i+j+1} - u_{i+1}} N_{i+1,j-1}(u) \tag{3}$$

for $1 \leq j \leq d$ and $0 \leq i \leq n+d-j$. Equation (1) uses only the final evaluations in the recursion, $N_{i,d}(u)$ for which $j = d$ and $0 \leq i \leq n$. The 2D curve is obtained by the perspective division,

$$(x(u), y(u)) = (\bar{x}(u), \bar{y}(u))/w(u) \tag{4}$$

In the section on circle representations, I will discuss NURBS curves that produce circular arcs; each curve will be listed with its n, d, u_i, $\mathbf{C}_i$, and w_i.

A tensor-product NURBS surface in 3D is generated by a homogeneous B-spline surface in 4D,

$$(\bar{x}(u,v), \bar{y}(u,v), \bar{z}(u,v), w(u,v)) = \sum_{i_0=0}^{n_0} \sum_{i_1=0}^{n_1} N_{i_0,d_0}(u) N_{i_1,d_1}(v) w_{i_0 i_1} (\mathbf{C}_{i_0 i_1}, 1) \tag{5}$$

where the 4D surface has an $(n_0+1) \times (n_1+1)$ array of control points $(\mathbf{C}_{i_0 i_1}, 1)$ [the $\mathbf{C}_{i_0 i_1}$ are 3-tuples], an $(n_0+1) \times (n_1+1)$ array of weights $w_{i_0 i_1}$, and degrees d_0 and d_1 with $1 \leq d_0 \leq n_0$ and $1 \leq d_1 \leq n_1$. The functions $N_{i_0,d_0}(u)$ and $N_{i_1,d_1}(v)$ each have a knot vector and are B-spline basis functions generated by equations (2) and (3). The 3D surface is obtained by the perspective division,

$$(x(u,v), y(u,v), z(u,v)) = (\bar{x}(u,v), \bar{y}(u,v), \bar{z}(u,v))/w(u,v) \tag{6}$$

In the section on sphere representations, I will discuss NURBS surfaces that produce spherical regions; each surface will be listed with its n_0, n_1, d_0, d_1, u_i, v_i, $\mathbf{C}_{i_0 i_1}$, and $w_{i_0 i_1}$.

2 Representing a Circle

Several representations are provided for a quarter of a circle in the first quadrant. A representation is also provided for half a circle. All these representation use the same knot pattern involving only the repeated zeros and ones at the beginning and end of the knot array. Finally, a representation for a full circle is created from that of the half circle by introducing a repeated knot of one-half that has the effect of splicing together two regular NURBS surfaces.

2.1 Quarter of a Circle

Consider the quarter circle in the first quadrant, $x^2 + y^2 = 1$ with $x \geq 0$ and $y \geq 0$.

2.1.1 Degree 2

A quadrant of a circle can be represented as a NURBS curve of degree 2. The curve is $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$. The general parameterization is

$$(x(u), y(u)) = \frac{w_0(1-u)^2(1,0) + w_1 2u(1-u)(1,1) + w_2 u^2(0,1)}{w_0(1-u)^2 + w_1 2u(1-u) + w_2 u^2} \tag{7}$$

for $u \in [0,1]$. The requirement that $x^2 + y^2 = 1$ leads to the weights constraint $2w_1^2 = w_0 w_2$. The choice of weights $w_0 = 1$, $w_1 = 1$, and $w_2 = 2$ leads to a commonly mentioned parameterization

$$(x(u), y(u)) = \frac{(1-u^2, 2u)}{1+u^2} \tag{8}$$

If you were to tessellate the curve with an odd number of uniform samples of u, say $u_i = i/(2n)$ for $0 \leq i \leq 2n$, then the resulting polyline is not symmetric about the midpoint $u = 1/2$. To obtain a symmetric tessellation you need to choose $w_0 = w_2$. The weight constraint then implies $w_0 = w_1\sqrt{2}$. The parameterization is then

$$(x(u), y(u)) = \frac{(\sqrt{2}(1-u)^2 + 2u(1-u), 2u(1-u) + \sqrt{2}u^2)}{\sqrt{2}(1-u)^2 + 2u(1-u) + \sqrt{2}u^2} \tag{9}$$

In either case we have a ratio of quadratic polynomials.

The NURBS parameters are the following. There are 3 control points, so $n = 2$. The degree is $d = 2$. The knot vector is $\{u_0, u_1, u_2, u_3, u_4, u_5\} = \{0, 0, 0, 1, 1, 1\}$. The control values are $\mathbf{C}_0 = (1, 0)$, $\mathbf{C}_1 = (1, 1)$, and $\mathbf{C}_2 = (0, 1)$. The weights are w_0, w_1, and w_2 subject to the constraint $2w_1^2 = w_0 w_2$.

2.1.2 Degree 4

An algebraic construction of the same type produces a ratio of quartic polynomials. The control points and control weights are required to be symmetric to obtain a tessellation that is symmetric about its midpoint. By the homogeneity in the weights, we have one degree of freedom. We can choose $w_0 = 1$, effectively dividing the numerator and denominator by w_0 if it were not 1.

$$(x(u), y(u)) = \frac{(1-u)^4(1,0) + 4(1-u)^3 u w_1(x_1, y_1) + 6(1-u)^2 u^2 w_2(x_2, x_2) + 4(1-u)u^3 w_1(y_1, x_1) + u^4(0,1)}{(1-u)^4 + 4(1-u)^3 u w_1 + 6(1-u)^2 u^2 w_2 + 4(1-u)u^3 w_1 + u^4} \tag{10}$$

Let the denominator of this fraction be named $w(u)$. To be a circle, we need $x(u)^2 + y(u)^2 = 1$. Using Mathematica 10.1 [4], this equation forces $x_1 = 1$ and generates 3 functionally independent equations in the 4 unknowns w_1, w_2, x_2, and y_1:

$$\begin{aligned} &4w_1^2 y_1^2 + 3w_2(x_2 - 1) = 0 \\ &1 - y_1 + 6w_2(1 - x_2 - x_2 y_1) = 0 \\ &32w_1^2 y_1 + 36w_2^2 x_2^2 - 16w_1^2 - 18w_2^2 - 1 = 0 \end{aligned} \tag{11}$$

Thus, we have at least one more degree of freedom and there are infinitely many solutions. For example, if we choose $w_1 = 1$, then there are 2 possibilities for the remaining parameters: $(y_1, x_2, w_2) = (1/(2\sqrt{2}), 1 - \sqrt{2}/8, 2\sqrt{2}/3)$ and $(y_1, x_2, w_2) = (\sqrt{2}-1, (44\sqrt{2}-1)/79, (16-7\sqrt{2})/6)$. With these choices, Mathematica 10.1 verified that $x(u)^2 + y(u)^2 = 1$ and parametric plots of $(x(u), y(u))$ were indeed quarter circles. Mathematica 10.1 also symbolically generated solutions to the constraints when asked to solve for y_1, x_2, and w_2 in terms of w_1.

The NURBS parameters are the following. There are 5 control points, so $n = 4$. The degree is $d = 4$. The knot vector has 10 components with $u_i = 0$ for $0 \le i \le 4$ and $u_i = 1$ for $5 \le i \le 9$. Assuming the constraints of equation (11) are satisfied, the control values are $\mathbf{C}_0 = (1, 0)$, $\mathbf{C}_1 = (x_1, y_1)$, $\mathbf{C}_2 = (x_2, x_2)$, $\mathbf{C}_3 = (y_1, x_1)$, and $\mathbf{C}_4 = (0, 1)$ and the weights are $w_0 = w_4 = 1$, w_2, and $w_1 = w_3$.

2.2 Half a Circle

I discuss a symmetric parameterization of the half circle $x^2 + y^2 = 1$ for $x \ge 0$. The NURBS curve has degree 3. The control values are chosen to be $(0, 1)$, $(\alpha, 1)$, $(\alpha, -1)$ and $(0, -1)$ and the weights are chosen to be $w_0 = w_3 = 1$ and $w_1 = w_2 = \omega$, where α and ω must be chosen so that we indeed obtain a half circle. The curve is

$$(x(u), y(u)) = \frac{(1-u)^3(0,1) + 3u(1-u)^2\omega(\alpha,1) + 3u^2(1-u)\omega(\alpha,-1) + u^3(0,-1)}{(1-u)^3 + 3u(1-u)^2\omega + 3u^2(1-u)\omega + u^3} \tag{12}$$

Symbolic manipulation of $x(u)^2 + y(u)^2 = 1$ leads to the constraints $\omega = 1/3$ and $\alpha^2 = 4$. Because we want the half circle where $x \ge 0$, we choose $\alpha = 2$.

The NURBS parameters are the following. There are 4 control points, so $n = 3$. The degree is $d = 3$. The knot vector has 8 components with $u_i = 0$ for $0 \le i \le 3$ and $u_i = 1$ for $4 \le i \le 7$. The control values are $\mathbf{C}_0 = (1, 0)$, $\mathbf{C}_1 = (2, 1)$, $\mathbf{C}_2 = (2, -1)$, and $\mathbf{C}_3 = (0, -1)$. The weights are $w_0 = w_3 = 1$ and $w_1 = w_2 = 1/3$.

2.3 Full Circle

The full-circle curve can be obtained by splicing together two half-circle curves as constructed in the previous section ($d = 3$). The 7 control values ($n = 6$) are in the clockwise order $(1, 0)$, $(2, 1)$, $(2, -1)$, $(0, -1)$, $(-2, -1)$, $(-2, 1)$, and $(1, 0)$. Observe that the first and last controls are duplicated in order to close the curve. The splicing occurs at the control value $(0, -1)$. Rather than duplicate the control, the splicing is accomplished by choosing an interior knot of $1/2$ with multiplicity 3; the knots are $u_i = 0$ for $0 \le i \le 3$, $u_i = 1/2$ for $4 \le i \le 6$, and $u_i = 1$ for $7 \le i \le 10$. Note that the number of knots is as required: $n + d + 2 = 11$.

Equation (2) provides the initial functions for starting the recursion. These are $N_{i,0}(u) = 0$ for $i \in \{0, 1, 2, 4, 5, 7, 8, 9\}$ and

$$\begin{aligned}
N_{3,0}(u) &= \{1, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{6,0}(u) &= \{0, \quad u \in [0, 1/2); \quad 1, \quad u \in [1/2, 1)\}
\end{aligned} \tag{13}$$

For $j = 1$, equation (3) produces $N_{i,1}(u) = 0$ for $i \in \{0, 1, 4, 7, 8\}$ and

$$\begin{aligned}
N_{2,1}(u) &= \{1 - 2u, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{3,1}(u) &= \{2u, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{5,1}(u) &= \{0, \quad u \in [0, 1/2); \quad 2 - 2u, \quad u \in [1/2, 1)\} \\
N_{6,1}(u) &= \{0, \quad u \in [0, 1/2); \quad 2u - 1, \quad u \in [1/2, 1)\}
\end{aligned} \tag{14}$$

For $j = 2$, equation (3) produces $N_{2,2}(u) = 0$, $N_{7,2}(u) = 0$, and

$$\begin{aligned}
N_{1,2}(u) &= \{(1 - 2u)^2, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{2,2}(u) &= \{2(2u)(1 - 2u), \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{3,2}(u) &= \{(2u)^2, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{4,2}(u) &= \{0, \quad u \in [0, 1/2); \quad (2 - 2u)^2, \quad u \in [1/2, 1)\} \\
N_{5,2}(u) &= \{0, \quad u \in [0, 1/2); \quad 2(2u - 1)(2 - 2u), \quad u \in [1/2, 1)\} \\
N_{6,2}(u) &= \{0, \quad u \in [0, 1/2); \quad (2u - 1)^2, \quad u \in [1/2, 1)\}
\end{aligned} \tag{15}$$

For $j = 3$, equation (3) produces

$$\begin{aligned}
N_{0,3}(u) &= \{(1 - 2u)^3, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{1,3}(u) &= \{3(2u)(1 - 2u)^2, \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{2,3}(u) &= \{3(2u)^2(1 - 2u), \quad u \in [0, 1/2); \quad 0, \quad u \in [1/2, 1)\} \\
N_{3,3}(u) &= \{(2u)^3, \quad u \in [0, 1/2); \quad (2 - 2u)^3, \quad u \in [1/2, 1)\} \\
N_{4,3}(u) &= \{0, \quad u \in [0, 1/2); \quad 3(2u - 1)(2 - 2u)^2, \quad u \in [1/2, 1)\} \\
N_{5,3}(u) &= \{0, \quad u \in [0, 1/2); \quad 3(2u - 1)^2(2 - 2u), \quad u \in [1/2, 1)\} \\
N_{6,3}(u) &= \{0, \quad u \in [0, 1/2); \quad (2u - 1)^3, \quad u \in [1/2, 1)\}
\end{aligned} \tag{16}$$

The last set of functions, $N_{i,3}(u)$, are what are used in the NURBS curve definition. Observe that the only function that has nonzero expressions for both $u \in [0, 1/2)$ and $u \in [1/2, 1)$ is $N_{3,3}(u)$. This is where the splicing occurs. The weights are the same as in the half-circle case. Table 1 summarizes the control points and weights.

Table 1. The control points, weights, and intervals of the domain for the circle.

controls	$(0,1)$	$(2,1)$	$(2,-1)$	$(0,-1)$	$(-2,-1)$	$(-2,1)$	$(0,-1)$
weights	1	1/3	1/3	1	1/3	1/3	1
$u \in [0,1/2)$	$(1-2u)^3$	$3(2u)(1-2u)^2$	$3(2u)^2(1-2u)$	$(2u)^3$	0	0	0
$u \in [1/2,1)$	0	0	0	$(2-2u)^3$	$3(2u-1)(2-2u)^2$	$3(2u-1)^2(2-2u)$	$(2u-1)^3$

For $u \in [0,1/2)$, the NURBS curve has nonzero weights only for the first four control points. For $u \in [1/2,1)$, the NURBS curve has nonzero weights only for the last four control points.

3 Representing a Sphere

3.1 One Octant of a Sphere

3.1.1 Degree 2

An octant of a sphere can be represented as a triangular NURBS surface patch of degree 4. A simple parameterization of $x^2+y^2+z^2=1$ can be made by setting $r^2=x^2+y^2$. The sphere is then $r^2+z^2=1$. Now apply the parameterization for a circle,

$$(r,z)=\frac{(1-u^2,2u)}{1+u^2}$$

But $(x/r)^2+(y/r)^2=1$, so another application of the parameterization for a circle is

$$\frac{(x,y)}{r}=\frac{(1-v^2,2v)}{1+v^2}$$

Combining these produces

$$(x(u,v),y(u,v),z(u,v))=\frac{((1-u^2)(1-v^2),(1-u^2)2v,2u(1+v^2))}{(1+u^2)(1+v^2)}$$

The components are ratios of quartic polynomials. The domain is $u \geq 0$, $v \geq 0$, and $u+v \leq 1$. In barycentric coordinates, set $w=1-u-v$ so that $u+v+w=1$ with u, v, and w nonnegative. In this setting, you can think of the octant of the sphere as a mapping from the uvw-triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Although a valid parameterization, a symmetric subdivision of the the uvw-triangle does not lead to a symmetric tessellation of the sphere.

3.1.2 Degree 4

Another parameterization is provided in [3]. This one chooses symmetric control points and symmetric weights,

$$(x(u,v),y(u,v),z(u,v))=\frac{\sum_{i=0}^{4}\sum_{j=0}^{4-i}w_{i,j,4-i-j}\mathbf{P}_{i,j,4-i-j}B_{i,j}(u,v)}{\sum_{i=0}^{4}\sum_{j=0}^{4-i}w_{i,j,4-i-j}B_{i,j}(u,v)}$$

where

$$B_{i,j}(u,v) = \frac{4!}{i!j!(4-i-j)!}\,u^i v^j (1-u-v)^{4-i-j}, \;\; u \ge 0, \;\; v \ge 0, \;\; u+v \le 1$$

are the Bernstein polynomials. The control points $\mathbf{P}_{i,j,k}$ are defined in terms of three constants $a_0 = (\sqrt{3}-1)/\sqrt{3}$, $a_1 = (\sqrt{3}+1)/(2\sqrt{3})$, and $a_2 = 1-(5-\sqrt{2})(7-\sqrt{3})/46$,

$$\begin{array}{lllll} \mathbf{P}_{040} & & & & \\ \mathbf{P}_{031} & \mathbf{P}_{130} & & & \\ \mathbf{P}_{022} & \mathbf{P}_{121} & \mathbf{P}_{220} & & \\ \mathbf{P}_{013} & \mathbf{P}_{112} & \mathbf{P}_{211} & \mathbf{P}_{310} & \\ \mathbf{P}_{004} & \mathbf{P}_{103} & \mathbf{P}_{202} & \mathbf{P}_{301} & \mathbf{P}_{400} \end{array} = \begin{array}{lllll} (0,1,0) & & & & \\ (0,1,a_0) & (a_0,1,0) & & & \\ (0,a_1,a_1) & (a_2,1,a_2) & (a_1,a_1,0) & & \\ (0,a_0,1) & (a_2,a_2,1) & (1,a_2,a_2) & (1,a_0,0) & \\ (0,0,1) & (a_0,0,1) & (a_1,0,a_1) & (1,0,a_0) & (1,0,0) \end{array}$$

The control weights $w_{i,j,k}$ are defined in terms of four constants, $b_0 = 4\sqrt{3}(\sqrt{3}-1)$, $b_1 = 3\sqrt{2}$, $b_2 = 4$, and $b_3 = \sqrt{2}(3+2\sqrt{2}-\sqrt{3})/\sqrt{3}$,

$$\begin{array}{lllll} w_{040} & & & & \\ w_{031} & w_{130} & & & \\ w_{022} & w_{121} & w_{220} & & \\ w_{013} & w_{112} & w_{211} & w_{310} & \\ w_{004} & w_{103} & w_{202} & w_{301} & w_{400} \end{array} = \begin{array}{lllll} b_0 & & & & \\ b_1 & b_1 & & & \\ b_2 & b_3 & b_2 & & \\ b_1 & b_3 & b_3 & b_1 & \\ b_0 & b_1 & b_2 & b_1 & b_0 \end{array}$$

Both the numerator and denominator polynomial are quartic polynomials. Notice that each boundary curve of the triangle patch is a quartic polynomial of one variable that is exactly what was shown earlier for a quadrant of a circle.

3.2 A Hemisphere

The construction of a hemisphere as a NURBS surface of degree 3 in each of u and v is similar to that for the half circle. For the half circle, we had control points at the circular poles $(0,1)$ and $(0,-1)$, each with associated weight 1. We postulated two other control points, $(\alpha,\pm 1)$ and determined that $\alpha = 2$ and that the weight $w = 1/3$.

The idea extends to 3D. We will select two control points at the circular poles $(0,0,1)$ and $(0,0,-1)$. We can add control points of the form $(2,0,\pm 1)$ to form a hemicircle; each point has an associated weight $1/3$. Now we can add more control points of the form $(2,\beta,\pm 1)$ to extrude the hemicircle to a hemisphere; each point has an associated weight w. To obtain a tensor product surface, we need a rectangular array of control points. The poles account for 2 and the other points account for 8. To pinch off the surface at the poles, we can require each pole to occur 4 times. We then have 16 control points to work with. Symbolic manipulation to force $x(u)^2 + y(u)^2 + z(u)^2 = 1$ leads to $\beta^2 = 16$ and $w = 1/9$. Choose the hemisphere where $y \ge 0$, so $\beta = 4$.

The knot vectors are the standard uniform ones, $u_i = v_i = 0$ for $0 \le i \le 3$ and $u_i = v_i = 1$ for $4 \le i \le 7$. Table 2 lists the control points, weights, and basis functions.

Table 2. The control points, weights, and basis functions for the half sphere. Each cell has the control point $\mathbf{C}_{ij}$ and the weight w_{ij}. The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	$(1-u)^3$	$3(1-u)^2u$	$3(1-u)u^2$	u^3
$(1-v)^3$	$(0,0,1),\quad 1$	$(0,0,1),\quad 1/3$	$(0,0,1),\quad 1/3$	$(0,0,1),\quad 1$
$3(1-v)^2v$	$(2,0,1),\quad 1/3$	$(2,4,1),\quad 1/9$	$(-2,4,1),\quad 1/9$	$(-2,0,1),\quad 1/3$
$3(1-v)v^2$	$(2,0,-1),\quad 1/3$	$(2,4,-1),\quad 1/9$	$(-2,4,-1),\quad 1/9$	$(-2,0,-1),\quad 1/3$
v^3	$(0,0,-1),\quad 1$	$(0,0,-1),\quad 1/3$	$(0,0,-1),\quad 1/3$	$(0,0,-1),\quad 1$

Mathematica 10.1 was used to verify symbolically that $x^2+y^2+z^2=1$.

3.3 Full Sphere

A full sphere can be formed from two hemispheres by splicing in a manner similar to that for generating a circle from two half circles. The u-knot vector is the standard uniform one with $u_i=0$ for $0\le i\le 3$ and $u_i=1$ for $4\le 7$. However, the v-knot vector has 11 elements with $v_i=0$ for $0\le i\le 3$, $v_i=1/2$ for $4\le i\le 6$, and $v_i=1$ for $7\le i\le 10$. Tables 3 and 4 show the control points, weights, and B-spline functions.

Table 3. The control points, weights, and basis functions for the sphere when $v\in[0,1/2)$. The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	$(1-2v)^3$	$3(2v)(1-2v)^2$	$3(2v)^2(1-2v)$	$(2v)^3$	0	0	0
$(1-u)^3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$
$3(1-u)^2u$	$(2,0,1),1/3,$	$(2,4,1),1/9$	$(-2,4,1),1/9$	$(-2,0,1),1/3$	$(-2,-4,1),1/9$	$(2,-4,1),1/9$	$(2,0,1),1/3$
$3(1-u)u^2$	$(2,0,-1),1/3,$	$(2,4,-1),1/9$	$(-2,4,-1),1/9$	$(-2,0,-1),1/3$	$(-2,-4,-1),1/9$	$(2,-4,-1),1/9$	$(2,0,-1),1/3$
u^3	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$

Table 4. The control points, weights, and basis functions for the sphere when $v\in[1/2,1)$. The basis function corresponding to that term is formed from the polynomials that tag the row and column.

	0	0	0	$(2-2v)^3$	$3(2v-1)(2-2v)^2$	$3(2v-1)^2(2-2v)$	$(2v-1)^3$
$(1-u)^3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$	$(0,0,1),1/3$	$(0,0,1),1/3$	$(0,0,1),1$
$3(1-u)^2u$	$(2,0,1),1/3,$	$(2,4,1),1/9$	$(-2,4,1),1/9$	$(-2,0,1),1/3$	$(-2,-4,1),1/9$	$(2,-4,1),1/9$	$(2,0,1),1/3$
$3(1-u)u^2$	$(2,0,-1),1/3,$	$(2,4,-1),1/9$	$(-2,4,-1),1/9$	$(-2,0,-1),1/3$	$(-2,-4,-1),1/9$	$(2,-4,-1),1/9$	$(2,0,-1),1/3$
u^3	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$	$(0,0,-1),1/3$	$(0,0,-1),1/3$	$(0,0,-1),1$

The control points $(0, 0, 1)$ and $(0, 0, -1)$ each occur 7 times to pinch off the surface at the poles. The control points $(2, 0, 1)$ and $(2, 0, -1)$ each occur 2 times to wrap the surface around the up-axis.

References

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