MATH 226 : Test 1 (1pm)

09/30/2022

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- You have 50 minutes to complete the test.
- Before turning in your exam, YOU WILL TAKE PICTURES of every single page of the exam. Then once you leave the classroom, you will have until 3pm to turn your pictures into one single PDF and upload the PDF into Gradescope. Do not forget to TAG EACH PROBLEM.
- NO Calculators, NO books, NO notes. You are NOT allowed to discuss the test with any one else, and you are not allowed to use internet. CELL PHONES SHOULD ONLY BE USED TO TAKE PICTURES of YOUR WORK at the end.
- You may use a ONE SIDED 8.5 x 11 HANDWRITTEN formula sheet.
- Try to keep your solutions in the space provided for each question. If you use the back of a page to finish a problem, clearly indicate it.
- Show all of your work and justify every answer to receive full credit.

Problem 1 Consider the points $A = (3, 1, 0)$, $B = (1, 5, 1)$, $C = (9, 3, 4)$

1) $O = (0, 0, 0)$ is the origin. Check whether the 4 points O, A, B, C are coplanar or not.
If they are NOT coplanar, find the volume of the parallelepiped with edges OA, OB, OC.

$\vec{OA} = \langle 3,1,0 \rangle$ $\vec{OB} = \langle 1,5,1 \rangle$ $\vec{OC} = \langle 9,3,4 \rangle$

$|\vec{OC} \cdot \underbrace{(\vec{OA} \times \vec{OB})}|$

$\begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ 1 & 5 & 1 \end{vmatrix} = \langle 1, -3, 14 \rangle \cdot \overbrace{\langle 9, 3, 4 \rangle}^{\vec{OC}}$

$= 9 - 9 + 56 = 56$

$14 \times 4 = 56$

$\boxed{V = 56 \text{ units}^3}$

2) Find the equation of the plane, containing the points A, B, C. (simplify the equation to its standard form $ax + by + cz + d = 0$)

$\vec{n} = \vec{AB} \times \vec{AC}$

$= \langle -2, 4, 1 \rangle \times \langle 6, 2, 4 \rangle$

$\begin{vmatrix} i & j & k \\ -2 & 4 & 1 \\ 6 & 2 & 4 \end{vmatrix} = \langle 14, 14, -28 \rangle$

$= \langle 1, 1, -2 \rangle$

$1(x-3) + 1(y-1) - 2(z-0) = 0$

$x - 3 + y - 1 - 2z = 0$

$\boxed{x + y - 2z - 4 = 0}$

3) Find parametric equations of the line $\mathcal{L}$ that is the intersection of the plane containing A, B, C with another plane of equation $2x+y+3z+1=0$
.

$2x+\cancel{y}+3z+1=0$

$-x-\cancel{y}+2z+4=0$

$x+5z=-5$

$x=-5 \quad z=0$

$y=9$

$(-5,9,0)$

$x+y-2z-4=0$

$\vec{n}=\langle 2,1,3\rangle \times \langle 1,1,-2\rangle$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & 1 & -2 \end{vmatrix} = \langle -5, 7, 1\rangle$$

$$\boxed{\begin{aligned} x &= -5-5t \\ y &= 9+7t \\ z &= t \end{aligned}}$$

Problem 2: Consider the line (L) with parametric equations:

$$x = 6 - t \qquad y = 2 + 4t \qquad z = 1 + 3t$$

and the line (K) given by

$$x = 1 - 3t \qquad y = 12t \qquad z = -1 + 9t$$

1. Show that (L) and K are parallel but DISTINCT.

$\vec{v}_1 = \langle -1, 4, 3 \rangle$ $\qquad$ $\vec{v}_2 = \langle -3, 12, 9 \rangle$

$= \frac{1}{3} \langle -1, 4, 3 \rangle$

$\vec{v}_1$ is a scalar multiple of $\vec{v}_2$ so L is $\parallel$ to K

$t = 0$ $\qquad$ $L = (6, 2, 1)$

$K = (1, 0, -1)$ $\qquad$ $\therefore$ Because they start in different locations, they are distinct.

$L @ t=0 \neq K @ t=0$

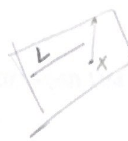
so distinct

2. Find the equation of the plane, containing the lines (L) and (K). (you can leave the equation in which ever form you want)

$\perp \; \perp \; \langle -1, 4, 3 \rangle$

$\langle x, y, z \rangle \cdot \langle -1, 4, 3 \rangle = 0$

$\boxed{-x + 4y + 3z = 0}$



$(x, y, z) = \vec{n}$

$\vec{q} \cdot \vec{n}$

$\vec{l} \cdot \vec{n} = 0$

Problem 3: Consider the 2 following curves given by

$$\vec{r_1}(t) = ((1-t)cos(t), (1-t)sin(t), t^2)$$

$$\vec{r_2}(s) = (cos(s), sin(s), s)$$

1) Find the coordinates of the COLLISION point between the two curves

↳ $t = s$

$(1-t)\cos t = \cos t$

$(1-t)\sin t = \sin t$

$t^2 = t$

Doesn't work bc

$\cancel{t=1}$ or $(1-1) = 0$ but $\cos 1 \neq 0$

$t = 0$

$t = 0$

$\boxed{(1, 0, 0)}$

2) Find the angle of Collision of the 2 curves
(namely the angle between the curves at the collision point, recall it has to be an angle between 0 and π).

$s = t = 0$

$r_1'(t) = \langle -\cos t - (1-t)\cancel{\sin t}, -\sin t + (1-t)\cos t, 2t \rangle$

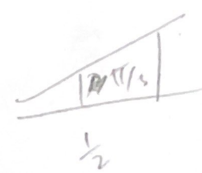
$r_2'(s) = \langle -\sin(s), \cos(s), 1 \rangle$

$r_1'(0) = \langle -1, 1, 0 \rangle$

$r_2'(0) = \langle 0, 1, 1 \rangle$

$\theta = \cos^{-1}\left(\dfrac{\langle 1,1,0\rangle \cdot \langle 0,1,1\rangle}{|\sqrt{2}||\sqrt{2}|}\right) = \left(\dfrac{0+1+0}{2}\right) = \left(\dfrac{1}{2}\right)$

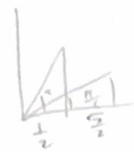
$\theta = \cos^{-1}\left(\frac{1}{2}\right)$ $\boxed{\theta = \frac{\pi}{3}}$



Problem 4 You are given the following information below about three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$:

- $||\vec{u}|| = 1, \quad ||\vec{v}|| = 3, \quad ||\vec{w}|| = 8$
- $\vec{u}$, $\vec{w}$ are orthogonal $u \cdot w = 0$
- the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$
- the angle between $\vec{w}$ and $\vec{v}$ is $\frac{\pi}{6}$

Evaluate the following:



$16 \times 3 = 48$

1. $(2\vec{u} - \vec{w}).(\vec{w} + 5\vec{u} + 2\vec{v})$

$2\overset{0}{(\cancel{\vec{u}} \cdot \vec{w})} + 10(\vec{u} \cdot \vec{u}) + 4(\vec{u} \cdot \vec{v}) - \vec{w} \cdot \vec{w} - 5\overset{0}{(\cancel{\vec{u}} \cdot \vec{w})} - 2(\vec{w} \cdot \vec{v})$

$64 - 16 = 48$

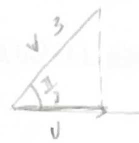
$10|(1)^2 + 4|1||3| \cos\frac{\pi}{3} - (8)^2 - 2|8||3| \cos\frac{\pi}{6}$

$10 + 12\left(\frac{1}{2}\right) - 64 - 48 \cdot \sqrt{3}/2$

$10 + 6 - 64 - 24\sqrt{3} = \boxed{-48 - 24\sqrt{3}}$

2. $|proj_{\vec{u}}\vec{v} \times proj_{\vec{w}}\vec{v}|$

$$proj_{\vec{u}}\vec{v} = \frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|^2}\vec{u}$$



$$proj_{\vec{w}}\vec{v} = \frac{|\vec{w}\cdot\vec{v}|}{|\vec{w}|^2}\vec{w}$$

$$\left(\overbrace{\frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|^2}}^{\text{scalar}}\vec{u}\right)\times\left(\overbrace{\frac{|\vec{w}\cdot\vec{v}|}{|\vec{w}|^2}}^{\text{scalar}}\vec{w}\right)$$

$$\left(\frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|^2}\right)\left(\frac{|\vec{w}\cdot\vec{v}|}{|\vec{w}|^2}\right)(\vec{u}\times\vec{w})$$

$|\vec{u}||\vec{v}|\cos\theta$

$|\vec{w}||\vec{v}|\cos\theta$

$|\vec{u}||\vec{w}|\sin\theta$

$$\left(\frac{3\cdot\cos\frac{\pi}{3}}{1}\right)\left(\frac{2\cancel{4}^{3}\cdot\cos\frac{\pi}{6}}{\cancel{64}\,\cancel{8}}\right)(1)(\cancel{8})\sin\cancel{\frac{\pi}{2}}^{1}$$

$$9\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{9\sqrt{3}}{4}}$$

Problem 5 Consider the curve $\vec{r(t)} = (t - e^t, 2 - t, e^t)$.

1) Find the equation of the NORMAL plane to the curve at the point $(-1, 2, 1)$.

$t_0 = 0$

$r'(t) = \langle 1 - e^t, -1, e^t \rangle$

$r'(0) = \langle 0, -1, 1 \rangle$ $\quad |r'(0)| = \sqrt{2}$

$\vec{T}(0) = \dfrac{\langle 0, -1, 1 \rangle}{\sqrt{2}}$

$$\boxed{\frac{-1}{\sqrt{2}}(y-2) + \frac{1}{\sqrt{2}}(z-1) = 0}$$

2) Find the equation of the OSCULATING plane to the curve at the point $(-1, 2, 1)$.

$r''(t) = \langle -e^t, 0, e^t \rangle$

$r''(0) = \langle -1, 0, 1 \rangle$ $\quad |r'(0) \times r''(0)| = \sqrt{3}$

$r'(0) \times r''(0) = \langle 0, -1, 1 \rangle \times \langle -1, 0, 1 \rangle$

$$\begin{vmatrix} i & j & k \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1, -1, -1 \rangle$$

$\vec{B}(0) = \dfrac{\langle -1, -1, -1 \rangle}{\sqrt{3}}$

$$\boxed{-\frac{1}{\sqrt{3}}(x+1) - \frac{1}{\sqrt{3}}(y-2) - \frac{1}{\sqrt{3}}(z-1) = 0}$$

3) Compute the curvature at the point $(-1, 2, 1)$.
At that point, is the curve bending towards its tangent vector or towards its normal vector? (justify)

$$\kappa(t) = |r'(t) \times r''(t)| \Big/ |r'(t)|^3$$

$$\kappa(0) = \frac{|\langle -1, -1, 1 \rangle|}{|\sqrt{2}|^3} = \frac{\sqrt{3}}{(\sqrt{2})^3} = \frac{(3)^{1/2}}{(2)^{3/2}}$$

It is pointing towards the tangent vector because

$\kappa < 1$.

4) Compute the TORSION $\tau(t)$ for ALL t.

$r'''(t) = \langle e^{-t}, 0, e^{t} \rangle$

$$\tau(t) = \frac{(r'(t) \times r''(t)) \cdot r'''(t)}{|r'(t) \times r''(t)|^2}$$

$\langle 1-e^{t}, -1, e^{t} \rangle \times \langle -e^{t}, 0, e^{t} \rangle$

$$\begin{vmatrix} i & j & k \\ 1-e^{t} & -1 & e^{t} \\ -e^{t} & 0 & e^{t} \end{vmatrix} = \langle -e^{t}, -e^{t}, -e^{t} \rangle$$

$\langle -e^{t}, -e^{t}, -e^{t} \rangle \cdot \langle e^{t}, 0, e^{t} \rangle =$

$-e^{2t} - e^{2t} = -2e^{2t}$

$|r'(t) \times r''(t)|^2 = (\sqrt{3})^2 = 3$

$$\tau(t) = \boxed{\frac{-2e^{2t}}{3}}$$

5) Is the curve planar or not? Justify.
If it is planar, give the equation of the plane containing the curve.

No it is not because $\tau(t)$ is not 0 for all t

~~yes b/c $\tau(t)$ is~~ 0 for all t.

~~Plane: osculating plane: $\frac{1}{\sqrt{3}}(x+1) - \frac{1}{\sqrt{3}}(y-2) - \frac{1}{\sqrt{3}}(z-1) = 0$~~