Random Variables, X

↳ A function that assigns a numerical value to each outcome in a sample space of a random experiment.

↳ Types of Random Variables:

Discrete

↳ Countable number of distinct values

↳ Example: # of heads in n coin flips

Continuous

↳ Uncountable numbers within an interval

↳ Example: heights of people within six feet

1. The Random Variable *is* the Observation

- This is the **simplest** relationship.
- The outcome of a random experiment **is directly recorded as the value of the random variable.**

✅ **Example:**

You toss a coin:

- Observation: "Head"
- Define random variable X such that:
  - $X = 1$ if Head
  - $X = 0$ if Tail

So the **random variable directly represents the observed outcome.**

2. The Random Variable is a Function of the Observation

- The **observation is more complex**, and the random variable is derived by **applying a function** to that observation.

✅ **Example:**

You roll a pair of dice:

- Observation: (3, 4)
- Random variable $X = \text{sum of dice} = 3 + 4 = 7$

Here, the **random variable is a function of the observation:**
$X = f(\text{observation}) = f(3, 4) = 7$

3. The Random Variable is a Function of Another Random Variable

- You define one random variable based on another random variable.
- This is common in **transformations**, **derived distributions**, or **conditional probability models.**

✅ **Example:**

Let X = number of heads in 3 coin tosses.
Define $Y = 2X + 1$

- Here, Y is **a function of another random variable** X.
- If $X = 2$ (i.e., 2 heads), then $Y = 2(2) + 1 = 5$

📌 **Summary Table:**

Type	Description	Example
1. Random variable is the observation	Direct mapping	Toss coin → X = 1 (Head), 0 (Tail)
2. Function of observation	Derived from raw data	Dice roll (2,5) → X = sum = 7
3. Function of another random variable	Transformed variable	X = heads in tosses, Y = 2X + 1

In summary, a random variable is a numerical representation of outcomes in a random experiment.

Basic Example: Flip a coin 4 times

$\therefore S = \{ HHHH, HHHT, HHTT, \ldots\ldots TTTT \}$

if, $X = \#$ heads that occur.

$X \in \{0, 1, 2, 3, 4\}$

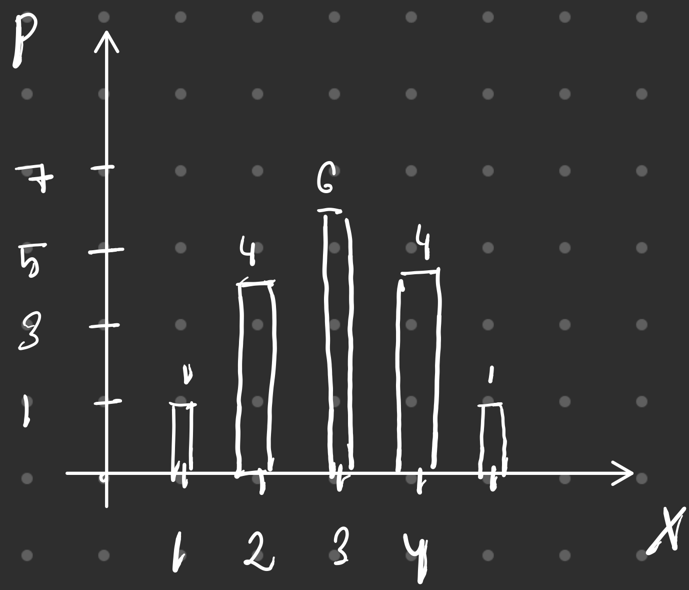
$P(X=0) = \frac{1}{16}$

$P(X=1) = \frac{4}{16}$

$P(X=2) = \frac{6}{16}$

$P(X=3) = \frac{4}{16}$

$P(X=4) = \frac{1}{16}$



So, the formal definition of a random variable $P(X)$ is a function called a distribution that contains

- values of X
- Probability P of each value

It gives a concise summary of all probabilities and helps to visualize and compute.

PMF - Probability Mass Function

- Used in Discrete Random Variables
- Gives the probability that a discrete random variable is exactly equal to a value.

* If X is a discrete random variable, then PMS is $P(X=x)$. This gives the probability that the random variable equals x.

- For example: Toss a fair coin once:
  Let X = number of heads. | $P(X=1) = 0.5$; $P(X=0) = 0.5$

PDF - Probability Density Function (will be discussed in next chapter)

- Used in Continuous Random Variables.
- Gives the density of the probability at a point. It doesn't give the actual probability of a value. The probability, is computed over an interval using integration

* If X is a continuous random variable:

$$P(a \le X \le b) = \int_a^b f(x)\,dx \quad ; \quad f(x) \to PDF$$

- For example: let $X \sim U(0,1)$ (uniform distribution from 0 to 1) then:

$f(x) = 1$ for $0 \le x \le 1$ || $P(0.2 \le x \le 0.6) = \int_{0.2}^{0.6} 1\,dx = 0.4$

✱ For a discrete random variable X with PMF $P_X(x)$ and range S_X.

ⓐ For any x, $P_X(x) \geq 0$ [Non-negativity]

ⓑ $\sum_{x \in S_X} P(x) = 1$ [Normalization]

ⓒ For any event $B \subset S_X$, the probability that X is in the set B is $P[B] = \sum_{x \in B} P_X(x)$ [Additivity for events]

Example:

The random variable N has PMF

$$P_N(n) = \begin{cases} c/n & n = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) The value of the constant c

(b) $P[N = 1]$

(c) $P[N \geq 2]$

(d) $P[N > 3]$

Sol^n:

ⓐ $P_N(n)$ is a PMF.

Here, $\sum_{n \in S_N} P(n) = 1$

$$\hookrightarrow \sum_{n=1}^{3} P_N(n) = 1 \Rightarrow \frac{c}{1} + \frac{c}{2} + \frac{c}{3} = 1 \Rightarrow C = \frac{6}{11}$$

ⓑ If, $P[N=1] = \frac{c}{1} = \frac{6}{11}$

$\therefore P[N=1] = \frac{6}{11}$.

$N \rightarrow$ random variable

$n \rightarrow$ specific value N takes.

(c) $P[N \geq 2] = P[N=2] + P[N=3] = \frac{c}{2} + \frac{c}{3}$

$$= \frac{6}{11}\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{5}{11}.$$

(d) $P[N > 3] = 0$; PMF is defined only for $n = 1, 2, 3, 0$

Example from Slide:

Example 2.1: (Continued)

Procedure: Send 3 packets from a sender to a receiver.

Observation: Number of successes.

$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$

$E = \{E_0, E_1, E_2, E_3\}$

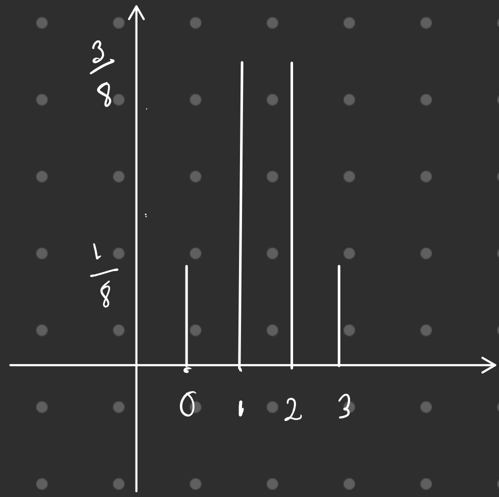
X ≜ Random variable that counts the number of successes

$S_X = \{0, 1, 2, 3\}$

x (successes)	E_x	s (sequences)	$P[X = x]$	$P[X \leq x]$
0	E_0	DDD	1/8	1/8
1	E_1	FDD, DFD, DDF	3/8	1/8 + 3/8
2	E_2	FFD, FDF, DFF	3/8	1/8 + 3/8 + 3/8
3	E_3	FFF	1/8	1/8 + 3/8 + 3/8 + 1/8

So, the PMF would be:

$$P_X(x) = \begin{cases} 1/8 & \text{if } x = 0, 3 \\ 3/8 & \text{if } x = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$



(*) Falls onto Binomial Distribution Technique.
This will be shown later on.

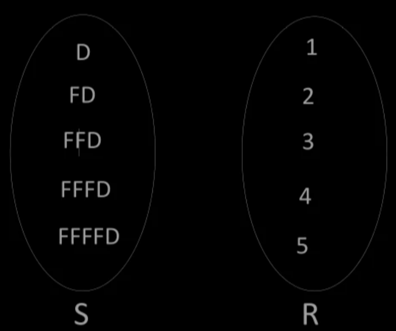
Example from Slides

Example 2.2 (Example 1.13 Continued)

Procedure: Keep sending packets from a sender to a receiver until 1 packet is delivered

Observation: Number of attempts

$S = \{D, FD, FFD, FFFD, \ldots\}$ $\quad s_X = \{1, 2, 3, \ldots\}$



X = number of attempts until success.

Possible values: $S_X = \{1, 2, 3, \ldots\}$

$p(\text{success}) = p = 0.5$
$p(\text{failure}) = 1 - p = 0.5$

	E_1	E_2	E_3	E_4	E_5	E_6
S	D	FD	FFD	FFFD	FFFFD	FFFFFD
x	1	2	3	4	5	6
$P[X=x]$	0.5	0.5^2	0.5^3	0.5^4	0.5^5	0.5^6
$P[X\leq x]$	0.5	$0.5+0.5^2$	$0.5+0.5^2+0.5^3$	$0.5+0.5^2+0.5^3+0.5^4$	$0.5+0.5^2+0.5^3+0.5^4+0.5^5$	$0.5+0.5^2+0.5^3+0.5^4+0.5^5+0.5^6$

$$\therefore \text{PMF}: P(X=x) = \begin{cases} (0.5)^x & \text{for } x = 1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Falls onto Geometric Distribution Technique.
This will be shown later on.

Example 2.3: (Example 1.14 Continued)

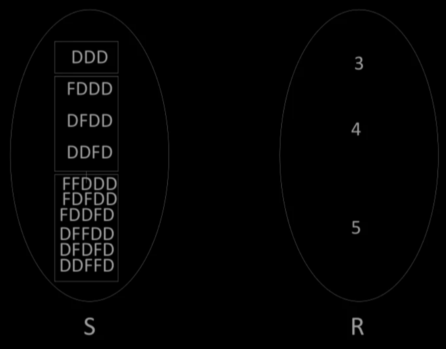
Example 1.14 (**Experiment 1.5**)

Procedure: Keep sending packets from a sender to a receiver until 3 packets are delivered

Observation: Number of attempts

$S = \{DDD, FDDD, DFDD, DDFD, FFDDD, \ldots\}$

$S_X = \{3, 4, 5, \ldots\}$



Example from Slide:

X = number of packets until three successes

$P(\text{success}) = p = 0.5$

$r = 3$ (number of success we want)

	E_3	E_4	E_5
S	DDD	FDDD, DFDD, DDFD	FFDDD, FDFDD, FDDFD, DFFDD, DFDFD, DDFFD.
x	3	4	5
$P[X=x]$	p^3	$3 \cdot p^3(1-p)$	$6 \cdot p^3(1-p)^2$
$P[X \le x]$	p^3	$3p^3(1-p) + p^3$	$p^3 + 3p^3(1-p) + 6p^3(1-p)^2$

$\therefore$ PMF : $P(X=x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r}, & x = 3, 4, 5 \ldots \\ 0 & \text{otherwise} \end{cases}$

Falls onto Pascal Distribution Technique

This will be shown later on.

CDF - Cumulative Distribution Function

↳ denoted by $F_X(x)$, gives the probability that X takes a value less than or equal to x,

$$F_X(x) = P(X \leq x)$$

This accumulates the probabilities of all outcomes up to and including x. So, for any real number x, the CDF is the probability that the random variable X is no larger than x.

If X is discrete, the CDF is the sum of the PMF values up to x:

$$F_X(x) = \sum_{k=-\infty}^{\lfloor x \rfloor} P_X(k)$$

(*) All random variables have cumulative distribution functions but only discrete random variables have probability mass functions.

Properties of CDF:

(i) Non-decreasing: CDF always increases or stays constant.
For all $x' \geq x$, $F_X(x') \geq F_X(x)$.

(ii) $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

(iii) Can be used to compute probability ranges:

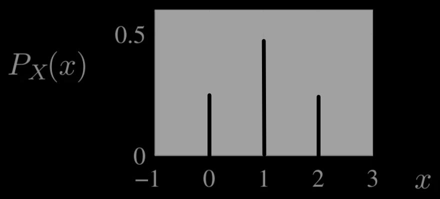
$$P(a < X \leq b) = F_X(b) - F_X(a)$$

(iv) $F_X(x) = F_X(x_i)$ for all x such that $x_i \leq x < x_{i+1}$

Example:

Example 3.21

In Example 3.5, random variable X has PMF



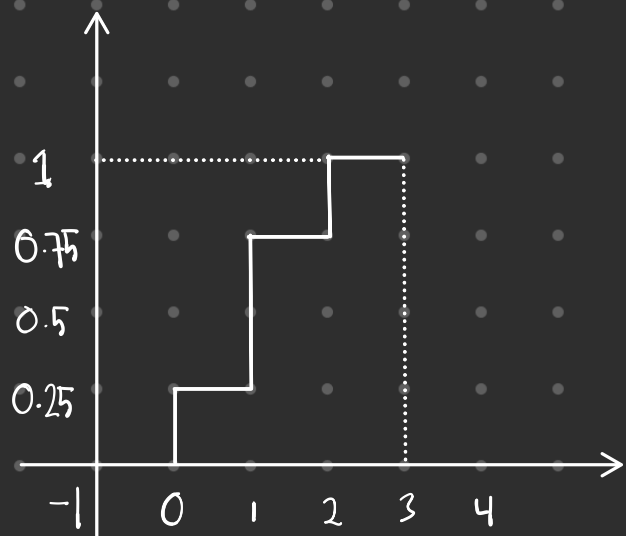
$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.28}$$

Find and sketch the CDF of random variable X.

Sol^n:

$$F_X(x) = P[X \leq x]$$

$$= \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq x < 1 \\ 3/4 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



Example from the slide:

Example 2.4: Let a discrete random variable X assumes values $-1, 1, 2$, and 3, with probabilities 0.6, 0.3, 0.08, and 0.02 , respectively.

PMF: $$P_X(x) = \begin{cases} 0.6, & x = -1 \\ 0.3, & x = 1 \\ 0.008, & x = 2 \\ 0.002, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

CDF: $$F_X(x) = \begin{cases} 0, & x \leq -1 \\ 0.6, & -1 \leq x < 1 \\ 0.9, & 1 \leq x < 2 \\ 0.98, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Average and Expected Value

An average is a number that describes a set of experimental observations. In probability, this is called the expected value or mean, denoted as: $E[X]$.

The expected value of X is

$$E[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

For Example: A die is rolled. The random variable X is the number on the face.

Here, $P_X(x) = 1/6$ for $x = 1, 2, 3, 4, 5, 6$

$$E[X] = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

$$= 3.5$$

Even though 3.5 is not a possible outcome it's the average of many rolls.

Example: Suppose you play a game, where in each play you lose $1 with probability 0.60 and you win $1 $2 and $3 with probabilities 0.30 0.08 and 0.02, respectively. Find your net gain in each play of the game.

-If you play only a few times, your gain might depend on your luck

- If you play a large number of times - your luck no longer determines the total gain - rather it will be multiple of average gain

Say, you play the game n times, approximately in 0.6 x n games you will loose $1 per game and will win $1, $2 and $3 in 0.3 x n, 0.08 x n and 0.02 x n games, respectively.

x	-1	1	2	3	Total
$P_X(x)$	0.60	0.30	0.08	0.02	1.00
$x \cdot P_X(x)$	-0.60	0.30	0.16	0.06	-0.08

$$P_X(x) = \begin{cases} 0.60 & x=-1 \\ 0.30 & x=1 \\ 0.08 & x=2 \\ 0.02 & x=3 \\ 0. & o/w \end{cases}$$

Here, Total gain $= .60n(-1) + .30n(1) + .08n(2) + .02n(3) = -0.08n$

$\therefore$ Expected Value $= \dfrac{-0.08n}{n} = -0.08$

Example: Three packets are sent from one computer to another. Each packet is successfully sent with a probability of 0.8 irrespective of the others. Each successfully sent packet results in a reward of 1 unit whereas 1 unit penalty (negative reward) is imposed if none of the packets is sent successfully. Find the expected net reward.

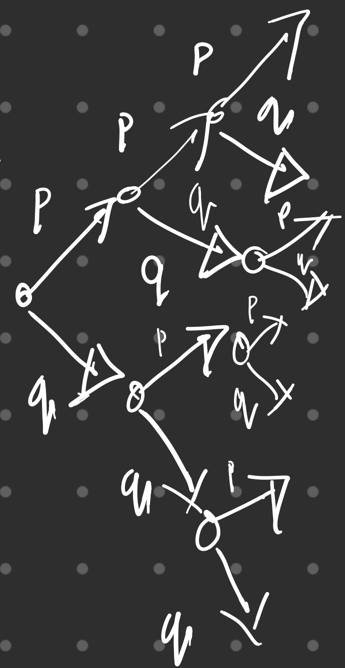
Solⁿ: Possible Values $S_X = \{-1, 1, 2, 3\}$

Three Steps:

1. Find Random Variable
2. Find PMF.
3. Finally find $E[X]$

$p = 0.8$

$q = 0.2$



$$P(-1) = q^3 = 0.008$$

$$P(1) = 3pq^2 = 0.096$$

$$P(2) = 3p^2q = 0.384$$

$$P(3) = p^3 = 0.512$$

$$P_X(x) = \begin{cases} 0.08 & x = -1 \\ 0.096 & x = 1 \\ 0.384 & x = 2 \\ 0.512 & x = 3 \\ 0 & o/w \end{cases}$$

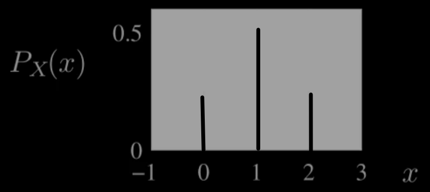
$$\therefore E[X] = (-1)\times 0.08 + 1\times 0.096 + 2\times 0.384 + 0.512\times 3$$

$$= 2.32$$

Example:

Example 3.24

Random variable X in Example 3.5 has PMF



$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases} \quad (3.36)$$

What is $E[X]$?

Soln:

Here, $E[X] = \left(\frac{1}{4} \times 0\right) + \left(1/2 \times 1\right) + \left(\frac{1}{4} \times 2\right)$

$= 1.$

Distribution	Expected Values, $E[X]$
Bernoulli (p)	p
Binomial (n, p)	np
Geometric (p)	$1/p$
Discrete Uniform (k, l)	$(k+l)/2$
Pascal (k, p)	k/p
Poisson (α)	α

(*) proof can be done using PMF.

Example:

Quiz 3.5

In a pay-as-you go cellphone plan, the cost of sending an SMS text message is 10 cents and the cost of receiving a text is 5 cents. For a certain subscriber, the probability of sending a text is 1/3 and the probability of receiving a text is 2/3. Let C equal the cost (in cents) of one text message and find

(a) The PMF $P_C(c)$

(b) The expected value $E[C]$

(c) The probability that the subscriber receives four texts before sending a text.

(d) The expected number of texts received by the subscriber before the subscriber sends a text.

Soln:

(a) the PMF is: $P_C(c) = \begin{cases} 1/3, & \text{if } c = 10 \\ 2/3, & \text{if } c = 5 \\ 0, & \text{otherwise} \end{cases}$

(b) $$E[C] = \sum_{c} c \cdot P_C(c)$$

$$= 10 \cdot \frac{1}{3} + 5 \cdot \frac{2}{3} + 0 = \frac{20}{3} = 6.67 \text{ cents}$$

(c) $P(4 \text{ receives, then } 1 \text{ send}) = \left(2/3\right)^4 \cdot \left(1/3\right)$

$$= \frac{16}{243} \approx 0.0658$$

Example: A college mathematics department sends 8 to 12 professors to the annual meeting of the American Mathematical Society, which lasts five days. The hotel at which the conference is held offers a bargain rate of a dollars per day per person if reservations are made 45 or more days in advance, but charges a cancellation fee of 2a dollars per person. The department is not certain how many professors will go. However, from past experience it is known that the probability of the attendance of i professors is 1/5 for i = 8, 9, 10, 11, and 12. If the regular rate of the hotel is 2a dollars per day per person, should the department make any reservations? If so, how many?

Sol^n:

$$S_X = \{8, 9, 10, 11, 12\}$$

i = professors
r = reservations

Three possible Cases:

Case 1: if all reservations are used as planned
$(i = r)$ Cost $= 5ar$

Case 2: if extra rooms $(i-r)$ are needed.
Cost $= 5\left(ra + (i-r)2a\right)$

Case 3: if only i rooms are used, the rest $(r-i)$ rooms are cancelled.
Cost $= 5ia + (r-i)2a$.

Using the formulas, find expected values:

$$E(X_8) = \frac{1}{5}(40a + 50a + 60a + 70a + 80a) = 60a$$

$$E(X_9) = \frac{1}{5}(42a + 45a + 55a + 65a + 75a) = 56.4a$$

$$E(X_{10}) = \frac{1}{5}(44a + 47a + 50a + 60a + 70a) = 54.2a$$

$$E(X_{11}) = \frac{1}{5}(46a + 49a + 52a + 55a + 65a) = 53.4a$$

$$E(X_{12}) = \frac{1}{5}(48a + 51a + 54a + 57a + 60a) = 54a$$

We can see that X_{11} has the smallest expected value. Thus making 11 reservations is the most reasonable policy.

Derived Random Variable

↳ A derived random variable is formed by applying a function to another random variable.

Let X be a random variable, and let $Y = g(X)$, where g is some function. Then Y is called a derived or transformed random variable.

For Example:

$$P_X(x) = \begin{cases} 0.60 & x=-1 \\ 0.30 & x=1 \\ 0.08 & x=2 \\ 0.02 & x=3 \\ 0. & o/w \end{cases}$$ and if $Y = X^2$

What will be the PMF of Y?

Soln:

Here, $S_X = \{-1, 1, 2, 3\}$

So if, $Y = X^2 \to S_Y = \{1, 4, 9\}$

$$\therefore P_Y(y) = \begin{cases} .90 \; ; & y=1 \\ .08 \; ; & y=4 \\ .02 \; ; & y=9 \end{cases}$$

$P[Y=1] = P[X=-1] + P[X=1]$
$= .90$

$P[Y=4] = P[X=2]$
$= .08$

$P[Y=9] = P[X=3]$
$= .02$

For a discrete random variable Y, if there exists another random variable X, where $Y = g(X)$, then the PMF of X is

$$P_Y(y) = \sum_{x:\, g(x)=y} P_X(x)$$

Expected Value of a Derived Random Variable

$$\hookrightarrow E[Y] = E[g(X)] = \sum_x g(x) \cdot P_X(x)$$

For Example:

$$P_X(x) = \begin{cases} 0.60 & x=-1 \\ 0.30 & x=1 \\ 0.08 & x=2 \\ 0.02 & x=3 \\ 0. & o/w \end{cases} \quad \text{and} \quad Y = X^2$$

$$\therefore E[Y] = (-1)^2 0.60 + (1)^2 0.30 + (2)^2 0.08 + (3)^2 0.02 + 0$$

$$= 1.4$$

(*) $E[X - \mu_X] = 0$

(*) $E[X+Y] = E[X] + E[Y]$

(*) $E[aX+b] = aE[X] + b$

N^{th} Moment of a Random Variable

$$\hookrightarrow E[X^n] = \sum_x x^n \cdot P_X(x)$$

So, if, $n=0 \rightarrow E[X^0] = 1$

$n=1 \rightarrow E[X] \Rightarrow$ expected Value

$n=2 \rightarrow E[X^2] \Rightarrow$ second moment

$$= \sum_x x^2 P_X(x)$$

Variance

↳ A measure of how much a random variable deviates from its mean, on average – it quantifies spread or variability.

The variance is defined as the 2nd central moment:

$$\mathrm{Var}(X) = E[(X-\mu)^2] \quad \text{where, } \mu = E[X]$$

or, $\mathrm{Var}(X) = E[X^2] - (E[X])^2$

Standard Deviation

↳ the square root of the variance – it measures the average distance of the values from the mean, in original units of the data. It directly tells you how spread out the values are from the mean.

↳ $SD(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{E(X-\mu)^2}$

Nth Central Moment → $E[(X-\mu)^n]$

if, $n=0$ → $E[(X-\mu)^0] = 1$

if, $n=1$ → $E[(X-\mu)^1] = 0$

if $n=2$ → $E[(X-\mu)^2] = V[X]$ → Variance

if $n=3$ → $E[(X-\mu)^3]$ → Skewness

if $n=4$ → $E[(X-\mu)^4]$ → Kurtosis