Well-known Discrete Random variable

- Bernouli
- Geometric
- Binomial
- Pascal
- Discrete Uniform
- Poisson
- Hyper Geometric

Bernoulli (p) Random Variables.

↳ Models an experiment that has only two possible outcomes typically success (1) or failure (0). A Bernoulli random variable X is defined as:

$$X = \begin{cases} 1, & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

$p \in [0,1] \rightarrow$ success

$1-p \rightarrow$ failure

Bernoulli PMF:

$$P_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

CDF:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-p & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x \geq 1 \end{cases}$$

✱ $E[x] = p$, $Var[X] = p(1-p)$

For example: A biased coin:

$P[\text{Heads}] = 0.7$

$P[\text{Tails}] = 0.3$

So, $X = \begin{cases} 1, & \text{if heads} \\ 0 & \text{if tails} \end{cases}$

$$\therefore P_X(x) = \begin{cases} 0.7 & \text{if } x = 1 \\ 0.3 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example from Slide: Pick a ball randomly from a box with 8 red balls and 7 blue balls.

Sol^n: Define picking up red ball as success.

$\therefore P(\text{success}) = \frac{8}{15}$

$$\text{PMF}: p_X(x) = \begin{cases} \frac{8}{15} & \text{if } x=1 \\ \frac{7}{15} & \text{if } x=0 \\ 0, & \text{o/w} \end{cases} \qquad \text{CDF}: F_X(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{7}{15} & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

$$E[x] = p = \frac{8}{15} \qquad \text{Var}(x) = p(1-p) = \frac{8}{15} \times \frac{7}{15} = \frac{56}{225}$$

Bernouli Trials

↳ An experiment is constitud by a number of successive Bernouli experiments.

↳ A sequence of Bernouli Experiments can be considered as a Bernouli trials only if:

- Each BE is independent from the others
- Probability of success is a constant

Example From Slide:

1. Transmit a data packet from one station to another station. The sender continues to transmit the packet again and again until the transmission is successful. Assume the probability of success in each transmission is p, irrespective of the number of transmission already attempted for the packet. Therefore, this constitute a Bernoulli trials. The random variable associated with this Bernoulli trials might be the followings:

   (a) The number of transmission required to transmit the data packet successfully.

   (b) The number of unsuccessful transmissions before the packet is successfully sent.

(a) $P(\text{first success at } k) = (1-p)^{k-1} p \; ; \; k = 0, 1, 2, 3 \ldots$

(b) $P(\text{failures} = n) = (1-p)^n p$

Example From Slide:

2. Send n data packets from one station to another. Each of the packets is successfully sent with probability p, and it is dropped with probability $(1-p)$, irrespective of whatever happened for the earlier packets. Assuming an attempt of successfully sending a packet or dropping it as an Bernoulli experiment, the above example of sending n packets becomes a Bernoulli trials. The following random variables can be defined for this Bernoulli trials:

   (a) The random variable X that denotes the number of packets successfully sent.

   (b) The random variable Y that denotes the number of packets dropped.

(a) X : Number of packets successfully sent:
sum of n independent Bernouli trials,
each with success probability p

→ Binomial Distribution

(b) Also Binomial Distribution

* In a Bernouli trials, two quantities are important

→ # of repititions
→ # of successes

When one of them is a constant, the other becomes a random variable and vice versa.

# of repititions	# of successes	Random Variables	Parameters	Distributions
fixed n	variable x	$X \triangleq$ # of successes	n, p	Binomial $X \sim binom(n,p)$
Variable x	fixed 1	$X \triangleq$ # of pepitition	p	Geometric $X \sim geom(p)$
Variable n	fixed k	$X \triangleq$ # of pepitition	k, p	Pascal $X \sim pascal(k,p)$

Distributions	# of ways	# of successes	# of failures
Binomial	$\binom{n}{x}$	x p^x	$n-x$ $(1-p)^{n-x}$
Geometric	1	1 p	$x-1$ $(1-p)^{x-1}$
Pascal	$\binom{n-1}{x-1}$	x p^x	$n-x$ $(1-p)^{n-x}$

Geometric Random Variable

↳ A geometric random variable models the number of trials until the first success in a sequence of independent Bernoulli trials (each with success-p)

if, X = the number of trials until the first success

$$P(X=k) = (1-p)^{k-1} \cdot p \quad \text{for } k = 1, 2, 3, \ldots$$

Geometric PMF, $P_X(x) = \begin{cases} p(1-p)^{k-1} & k = 1, 2, 3 \ldots \\ 0 & \text{otherwise} \end{cases}$

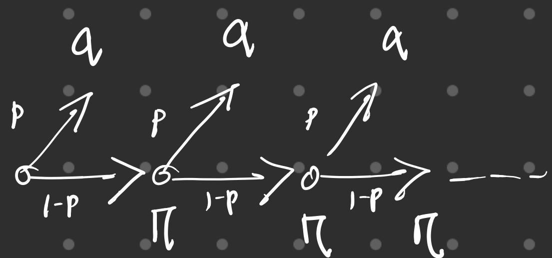
CDF, $F_X(x) = \begin{cases} 0 & , k < 1 \\ 1-(1-p)^k & k \geq 1 \end{cases}$

And, $E[X] = \dfrac{1}{p}$. $Var[X] = \dfrac{(1-p)}{p^2}$

Example: →

In a sequence of independent tests of integrated circuits, each circuit is rejected with probability p. Let Y equal the number of tests up to and including the first test that results in a reject. What is the PMF of Y ?

Soln: Here,



$P[Y=1] = p$

$P[Y=2] = p(1-p)$

$P[Y=3] = p(1-p)^2$

⋮

$P[Y=k] = p(1-p)^{k-1}$

$P_Y(y) = \begin{cases} p(1-p)^{y-1}; & y = 0, 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases}$

As we can see in the Geometric Distribution:

i) The trials are independent.

ii) Each trial results in only two outcomes, labeled as "success" and "failure".

iii) The probability of a "success" in each trial, denoted as p, remains constant for the whole sequence.

iv) The sequence of trials ends as soon as the first success occurs.

Example: From an ordinary deck of 52 cards we draw cards at random, with re-placement, and successively until an ace is drawn. What is the probability that at least 10 draws are needed?

Soln:

Here, $$P(X=n) = \begin{cases} \left(\frac{12}{13}\right)^{n-1}\left(\frac{1}{13}\right); & n=1,2,3,\ldots \\ 0, & o/w \end{cases}$$

$\therefore$ the probability that at least 10 draws are needed = first nine draws are non-aces:

$$P(X \geq 10) = (1-p)^9 = (12/13)^9 = 0.4866$$

Example: A father asks his sons to cut their backyard lawn. Since he does not specify which of the three sons is to do the job, each boy tosses a coin to determine the odd person, who must then cut the lawn. In the case that all three get heads or tails, they continue tossing until they reach a decision. Let p be the probability of heads and q = 1-p, the probability of tails.

(a) Find the PMF of the number of rounds (tosses) required.
(b) Find the expected number of rounds required.
(c) Find the probability that they reach a decision in less than n tosses.
(d) If p = 1/2, what is the minimum number of tosses required to reach a decision with probability 0.95?

Solⁿ:

$$P(\text{Head}) = p \quad ; \quad P(\text{Tail}) = q = 1-p.$$

(a) Success $\rightarrow$ Decision can be taken

$\rightarrow$ HHT, THT, TTH, THH, HTT, HTT $\rightarrow$ 6 combinations

$\rightarrow 3p^2q$ or $3q^2p \Rightarrow 3(p^2q+q^2p) \Rightarrow 3pq\underbrace{(p+q)}_{1}$

$\Rightarrow 3pq$

Failure $\rightarrow$ Decision can't be taken

$\rightarrow$ TTT, HHH $\rightarrow$ 3 combinations

$\rightarrow p^3+q^3 = 1-3pq$

$$P(X=k) = \begin{cases} (p^3+q^3)^{k-1}\cdot 3pq & \text{for } k=1,2,3 \\ 0, & o/w \end{cases}$$

(b) $$E[X] = \frac{1}{3pq}$$

(c) Probability of Decision less than n tosses:

$$P(X<n) = 1 - P(X \geq n)$$

$$\text{CDF}: F_X(x) = \begin{cases} 0 & \text{for } n<1 \\ 1-(1-3pq)^{n-1} & \text{for } n \geq 1 \end{cases}$$

$$\therefore 1 - P(X \geq n) = 1 - (1-3pq)^{n-1}$$

$$\hookrightarrow P(X<n) = 1 - (1-3pq)^{n-1}.$$

(d) $p = 1/2$, $q = 1/2$

We want to find n so that $P(X \leq n) \geq 0.95$

$$\Rightarrow 1 - P(X>n) \geq 0.95$$

$$\Rightarrow 0.05 \geq P(X>n)$$

$$\Rightarrow 0.05 \geq P(X \geq n) - P(X=n)$$

$$\Rightarrow 0.05 \geq (1-3pq)^{n-1} - (1-3pq)^{n-1} \cdot 3pq$$

$$\Rightarrow 0.05 \geq (1-3pq)^{n}$$

$$\Rightarrow 0.05 \geq (1/4)^{n}$$

$$\Rightarrow n \geq 2.16 \Rightarrow \text{the smallest } n \text{ is } 3$$

Binomial Random Variable (n,p)

↳ Models the number of successes in a fixed number of independent trials, where each trial has only two outcomes: success (p) or failure $(1-p)$

Let: $n \rightarrow$ number of trial
$p \rightarrow$ probability of success in each trial
$X \rightarrow$ number of successes in those n trials

Then, $X \sim \text{Binomial}(n,p)$ [X distributed as Binomial (n,p)]

So, the PMF the probability of getting exactly k successes in n trials is:

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots n$$

CDF: $$F_X(x) = \begin{cases} 0, & k < 0 \\ \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}, & k = 0, 1, 2 \ldots \\ 1, & k > n \end{cases}$$

$$E[X] = np \quad \text{and} \quad Var[X] = np(1-p)$$

For Example: Flip a biased coin 5 times where the probability of heads is $p = 0.6$

Let X be the number of heads in those 5 flips.

Sol^n: $P(X=3) = \binom{5}{3} \cdot 0.6^3 \cdot 0.4^2 = 0.3456.$

Example: A restaurant serves 8 entrées of fish, 12 of beef, and 10 of poultry. If customers select from these entrées randomly, what is the probability that two of the next four customers order fish entrées?

Sol^n: $P(X=2) = \binom{4}{2} \left(\frac{4}{15}\right)^2 \left(\frac{11}{15}\right)^2 = 0.23$

Recapping this example:

Example 2.1: (Continued)

<u>Procedure</u>: Send 3 packets from a sender to a receiver.
<u>Observation</u>: Number of successes.

$S = \{FFF, FFD, FDF, FDD, DFF, DFD, DDF, DDD\}$

$E = \{E_0, E_1, E_2, E_3\}$

$X \triangleq$ Random variable that counts the number of successes

$S_X = \{0, 1, 2, 3\}$

Using Binomial Distribution: $P_X(x) = \binom{3}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{3-x}$

for $x = 0, 1, 2, 3$

$P(X=0) = \frac{1}{8}$, $P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$, $P(X=3) = \frac{1}{8}$

Example 3.12: If there is a 0.2 probability of a reject and we perform 10 tests, the PMF of the binomial (10,0.2) random variable is what?

Soln:
$$P_K(k) = \binom{10}{k}(0.2)^k(0.8)^{10-k}$$

Example 3.13: Perform independent tests of integrated circuits in which each circuit is rejected with probability p. Observe L, the number of tests performed until there are k rejects. What is the PMF of L?

Soln: Assuming, $L = l$ means the l-th test is the k-th rejection. This happens if:

- The first $l-1$ tests have exactly $k-1$ rejections.
- The l-th test is a rejection.

So, the PMF would be:
$$P(L=l) = \binom{l-1}{k-1} p^k (1-p)^{l-k} \text{ for } l \geq k$$

For Example: Rejections needed, $k=2$
Rejection rate, $p = 0.1$
Tests until the second rejection, $l=5$.

$$\therefore P(L=5) = \binom{5-1}{2-1}(0.1)^2(1-0.1)^{5-2} = 0.02916.$$

Pascal Random Variable

Pascal Random Variable (k,p)

A Pascal (k,p) random variable X represents the number of Bernoulli trials needed to achieve k successes, where each trial has success probability p.

↳ You are repeating Bernoulli trials until you get k successes.

↳ x is the total number of trials needed to get these k successes.

$x = k, k+1, k+2, \ldots$

$p \in (0,1)$

$k \in \mathbb{Z}^+$ (positive integers)

So, the PMF would be:

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

$$\text{CDF}: F_X(x) = P(X \le n) = \sum_{x=k}^{n} \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

$$E[X] = \frac{k}{p} \; ; \; \text{Var}(X) = \frac{k(1-p)}{p^2}$$

✱ If k=1, the Pascal distribution becomes a Geometric distribution (which models the number of trials until the first success)

So, Pascal (1, p) = Geometric (p)

For Example: Flip a biased coin where the probability of getting heads $p = 0.6$. What's the probability to get the 3^{rd} Head on the 5^{th} flip?

Sol^n:

$$P(X=5) = \binom{5-1}{3-1}(0.6)^3(0.4)^{5-3} = 0.20736$$

Example: Two players, A and B, compete in a series of games where the first player to win 5 games wins the series. Each game is independent, and player A wins any given game with probability 0.58.

a) What is the probability that the series lasts exactly seven games?
b) What is the probability that the series ends in fewer than seven games?
c) If the series goes to seven games, what is the probability that player A wins the series?

Sol^n: (a) Case 1: A wins the 7^{th} game:

$$P_1 = \binom{7-1}{5-1}(0.58)^5(0.42)^{7-5} = 0.1736$$

Case 2: B wins the 7^{th} game:

$$P_2 = \binom{7-1}{5-1}(0.42)^5(0.58)^{7-5} = 0.066$$

$\therefore$ Total Probability, $P = P_1 + P_2 = 0.2446$

(b) $P(< 7 \text{ games}) = 1 - P = 1 - 0.2446 = 0.7554$

(c) $P(A \text{ wins} \mid 7 \text{ games}) = \dfrac{P_1}{P} = \dfrac{0.1736}{0.2446} = 0.709$

Example 3.14 If there is a 0.2 probability of a reject and we seek four defective circuits, the random variable L is the number of tests necessary to find the four circuits. The PMF of the Pascal(4,0.2) random variable is what?

Solⁿ: $k=4, \quad p=0.2.$

$$\therefore \quad P_L(l) = \binom{l-1}{3}(0.2)^3(0.8)^{l-4}$$

Discrete Uniform (k,l) Random Variable:

A discrete Uniform Random variable X is one where each value between k and l (inclusive) is equally likely. The PMF:

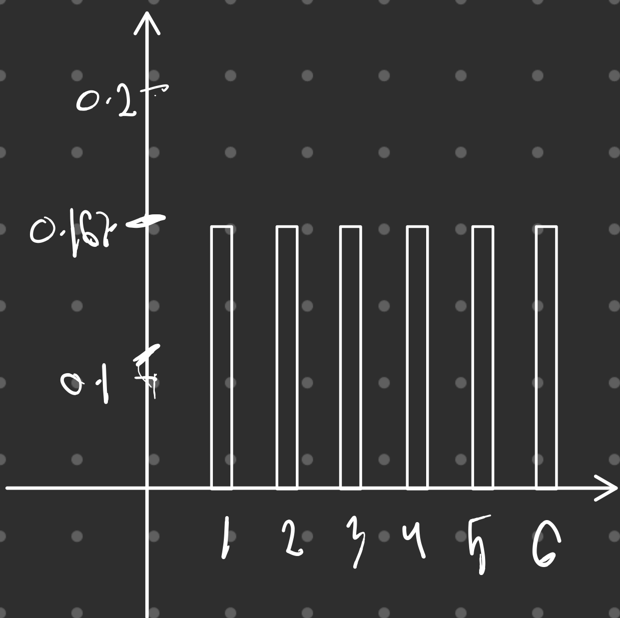
$$P_N(n) = \begin{cases} \dfrac{1}{l-k+1} & \text{if, } n = k, k+1, \ldots, l \\ 0 & \text{otherwise} \end{cases}$$

☞ All values in the range have equal probability. $k < l$, and both must be integers.

Example: Roll a fair six-sided die. Let N be the number of spots on the face top. Find PMF.

Solⁿ:

$$P_N(n) = \begin{cases} \dfrac{1}{6}, & \text{for } n = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$



Poisson Random variable (α)

A poisson random variable X describes the number of occurrences (events) in a fixed interval of time or space, assuming these events happen independently and at a constant average rate. So, the PMF:

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Let us derive it:

Firstly, the poisson distribution models:

- → Number of rare events in a fixed time or space
- → When these events occur independently and at a constant average rate.

Step 01: Start with Binomial PMF:

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Step 02: Make the Binomial → Poisson connection

We approximate rare events by:

1. $n \to \infty$; Assuming the number of trials are very large.
2. $p \to 0$; Each single trial has a tiny chance of success.
3. $np = \alpha$; Keeping the expected number of events fixed. $\Rightarrow p = \frac{\alpha}{n}$

Step 03: Now, $P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$= \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \left(\frac{\alpha}{n}\right)^x \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots 2\cdot 1}{x!\,(n-x)(n-x-1)\cdots 2\cdot 1} \left(\frac{\alpha}{n}\right)^x \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \frac{\alpha^x}{n^x} \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x} \frac{\alpha^x}{x!} \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \underbrace{\lim_{n\to\infty} \left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]}_{1} \frac{\alpha^x}{x!} \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= 1 \times \lim_{n\to\infty} \frac{\alpha^x}{x!} \left(1-\frac{\alpha}{n}\right)^{n} \left(1-\frac{\alpha}{n}\right)^{-x}$$

$$= \frac{\alpha^x}{x!} \underbrace{\lim_{n\to\infty} \left(1-\frac{\alpha}{n}\right)^{n}}_{e^{-\alpha}} \left(1-\frac{\alpha}{n}\right)^{-x}$$

$$= \frac{\alpha^x}{x!} e^{-\alpha} (1-0)^{-x} = \frac{\alpha^x e^{-\alpha}}{x!}$$

$$\lim_{n\to\infty} \left(1-\frac{\alpha}{n}\right)^n = e^{-\alpha}$$

(L'Hopital Rule)

[You don't need the derivation, I just wanted to study it]

$\therefore\ P(X=x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & ; x = 0, 1, 2, \ldots \\ 0 & ; \text{otherwise} \end{cases}$

CDF: $F_X(x) = P(X \leq x) = \sum_{i=0}^{x} \dfrac{\alpha^i e^{-\alpha}}{i!}$

$(\alpha = \lambda T)$

$E[X] = \alpha = \lambda T$; $V[X] = \alpha = \lambda T$;

↳ In poisson distribution, events are independent, one occurrence doesn't affect another.

↳ Average rate α remains constant over the interval.

Example: Average of 3.2 alpha particles emitted per second. What is the probability of exactly 3 particles to be emitted?

Soln:

$$P(X=3) = \frac{(3.2)^3 e^{-3.2}}{3!}$$

$$\approx 0.195$$

Example: Suppose email arrive at a rate of 2 per minute. What's the probability of receiving 3 emails in 2 mintutes?

Solⁿ:

$$P(X=3) = \frac{4^3 e^{-4}}{3!}$$

$\alpha = n \times p = 2 \times 2 = 4$

$x = 3$

$$= 0.19536$$

Example: A call center receives an average of 10 calls per hour. What's the probability they receive exactly 15 calls in an hour?

Solⁿ:

$$P(X=15) = \frac{10^{15} e^{-10}}{15!}$$

$\alpha = np = 10 \times 1$

$x = 15$

$$= 0.0347$$

Example: A server experiences 1 crash every 2 months. What's the probability that there are no crashes in the next month?

Solⁿ:

$$P(X=0) = \frac{\left(\frac{1}{2}\right)^0 \times e^{-\frac{1}{2}}}{0!} = 0.6065$$

$\alpha = np = 1 \times \frac{1}{2}$

$x = 0$

Example:

Example 3.17

The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second. What is the probability that there are no hits in an interval of 0.25 seconds? What is the probability that there are no more than two hits in an interval of one second?

λ (or α) $= 2 \therefore \lambda' = \lambda T = 2 \times 0.25 = 0.5$

$$P(X=0) = \frac{e^{-0.5} \times 0.5^0}{0!} = e^{-0.5} \approx 0.6065$$

$\therefore P(0 \text{ hits in } 0.25 \text{ sec}) \approx 0.6065$

Now, $\lambda' = \lambda T = 2 \times 1 = 2$

$$P(X \le 2) = P(0) + P(1) + P(2) = \frac{e^{-2} 2^0}{0!} + \frac{e^{-2} 2^1}{1!} + \frac{e^{-2} 2^2}{2!} \approx 0.6767$$

$\therefore P(\le 2 \text{ hits in } 1 \text{ sec}) \approx 0.6767.$

Example

Example 3.18

The number of database queries processed by a computer in any 10-second interval is a Poisson random variable, K, with $\alpha = 5$ queries. What is the probability that there will be no queries processed in a 10-second interval? What is the probability that at least two queries will be processed in a 2-second interval?

Sol^n: Given, $\alpha = 5$ (while $T = 10$ second)

$$\therefore P(X=0) = \frac{5^0 e^{-5}}{0!} \approx 0.0067$$

Now, $\alpha = 5$ (for 10 s query)

$\Rightarrow \alpha' = \frac{5 \times 2}{10}$ (for 2s query) $= 1$ s

$$\therefore P(X \geq 2) = 1 - P(X=1) - P(X=0) = 1 - \frac{1^1 e^{-1}}{1!} - \frac{1^0 e^{-1}}{0!}$$

$$\approx 0.264$$

Example:

An office receives wrong number calls that follow a Poisson distribution with an average rate of λ = 2 calls per day. What is the probability that there will be at least 2 wrong number calls by tomorrow?

Sol^n:

At least two wrong calls by tomorrow

$$= P(X \geq 2)$$

$$= 1 - P(X < 2)$$

$$= 1 - P(X=0) - P(X=1)$$

$$= 1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!}$$

$$= 1 - 0.1353 - 0.2707$$

$$= 0.594$$

Hypergeometric Random Variable (N, K, n)

↳ Models the number of successes in n draws without replacement from a finite population of size N containing exactly K success states.

↳ For $X \sim \text{Hypergeometric}(N, k, n)$:

PMF: $$P(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$k \leq K$ (successes in population)

$n-k \leq N-K$ (failures in population)

CDF: $$F_X(m) = P(X \leq m) = \sum_{k=\max(0,n+K-N)}^{\min(m,K)} \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$E[X] = n \cdot \frac{K}{N} \ ; \ Var[X] = n \cdot \frac{K}{N}\left(1 - \frac{K}{N}\right)\frac{N-n}{N-1}$$

Example: A deck of $N=52$ cards has $K=13$ hearts. Draw 7 cards.

(a) What is the probability of exactly 3 hearts?

(b) What is the probability of at most 2 hearts?

Solⁿ:

Sol^n:

ⓐ $P(X=3) = \dfrac{\binom{13}{3}\binom{52-13}{7-3}}{\binom{52}{7}} \approx 0.1758$

ⓑ $P(X \geq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= \frac{\binom{13}{0}\binom{52-13}{7-0} + \binom{13}{1}\binom{52-13}{7-1} + \binom{13}{2}\binom{52-13}{7-2}}{\binom{52}{7}}$$

$= 0.1149 + 0.317 + 0.335$

$= 0.7669$

Example:

In a box, there are r red and g green balls. Pick n balls randomly
(a) with replacement and
(b) without replacement.
Find the probability that exactly X red balls are picked.
($x \leq r$ and $n \leq r + g$ for without replacement.)

Sol^n:

ⓐ Total Balls $= r+g$

Here, balls are replaced after each draw, the probability of drawing a red ball remains constant. So:

$$P(X=x) = \binom{n}{x}\left(\frac{r}{r+g}\right)^{x}\left(\frac{g}{r+g}\right)^{n-x} \; ; \; x=0,1,2,\dots n$$

⇒ follows the binomial distribution

(b) When the balls are drawn without replacement, the probability changes with each draw. The number of red balls X follows a hypergeometric distribution:

$$P(X=x) = \frac{\binom{r}{x}\binom{g}{n-x}}{\binom{r+g}{n}}$$