RSA, ECC, & Pairing-based Cryptography

A basic introduction from a math perspective

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For filling the void in the cc

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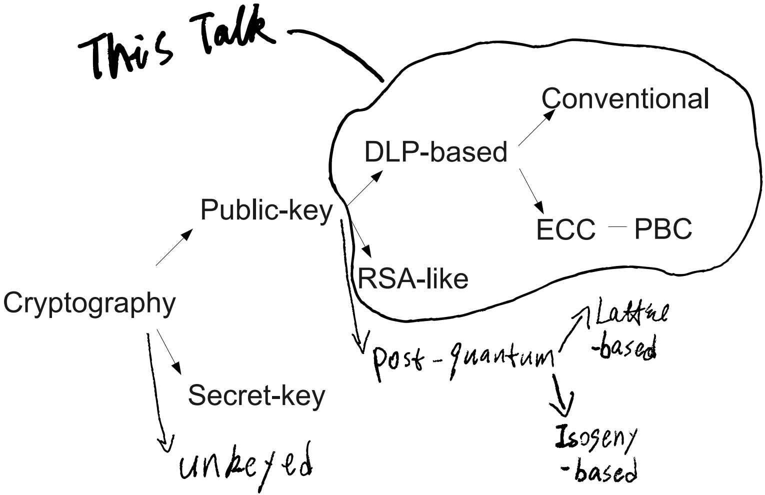
Chapter One: Overview of Cryptography

Cryptography: Overview

What is cryptography?

Study of mathematics techniques related to aspects of information security, e.g., how to hide information, data integrity, identification and authentication.

Cryptography: Overview (cont'd)



Chapter Two:
RSA (Factoring-based Crypto)

Rivest-Shamir-Adleman

Setup/KeyGen:

- Choose two large primes p and q.
- Let $n = p \cdot q$.
- Randomly choose e such that $1 < e < n - 1$ and $\gcd(e, \phi(n)) = 1$.
  Here $\phi(n) = (p-1)(q-1)$ is the Euler's totient function.
- Compute d such that $e \cdot d \equiv 1 \pmod{\phi(n)}$.
- The values p and q are never revealed.

RSA public key: (n, e).
RSA private key: d.

Remark: for RSA to work, factorization of the modulus n should be hard.

RSA Encryption (Basic Scheme)

Plaintext message: $M \in \{1, 2, \ldots, n-1\}$.
Encryption: $C := E(M) = M^e \pmod{n}$.
Decryption: $M := D(C) = C^d \pmod{n}$.

RSA Signatures (Basic Scheme)

Message to sign: $M \in \{1, 2, \ldots, n-1\}$.
Sign: signature $\sigma = D(M) = M^d \pmod{n}$.
Verify: check $M = E(\sigma) = \sigma^e \pmod{n}$.

Remark: The scheme shown above is not secure. In practice, one should sign the hash of the message instead of the message itself.

Chapter Three:
Elliptic Curve Crypto

Elliptic Curves in Cryptography

Application of elliptic curves in cryptography

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- Hash function construction [Charles et al., 2007]
- Zero-knowledge proofs
- And so on

A Mathematician Quote

"It is possible to write endlessly on elliptic curves. (This is not a threat.)" — Serge Lang

We Follow The Wise

We are not going to talk about everything ...

Focus of this talk: ECC

Elliptic curve cryptography: use elliptic curves as an approach to public-key cryptography

Elliptic Curve

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q,\ q \text{ odd } 3 \nmid q$$

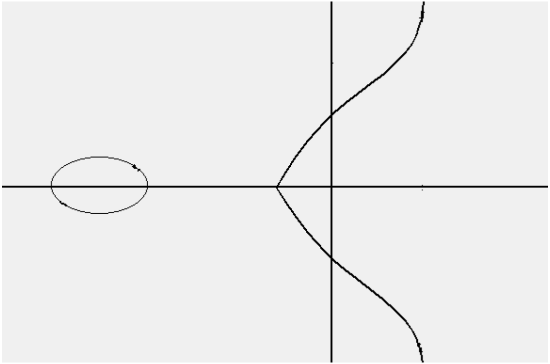

Figure: An example of elliptic curve over $\mathbb{R}$

Review: Group

A group is a set G together with a binary operation $\circ$ that satisfies

1. closure: $a, b \in G \Rightarrow a \circ b \in G$
2. associativity: for any $a, b, c \in G$, have $(a \circ b) \circ c = a \circ (b \circ c)$
3. identity: there exists an identity element i such that $a \circ i = i \circ a = a$, for any $a \in G$
4. inverse: for any element $a \in G$, there is an element $b \in G$ such that $a \circ b = b \circ a = i$.

Review: Group (cont'd)

Abelian Group

A group G is called **abelian** if $a \circ b = b \circ a$, for any $a, b \in G$.

Examples of abelian groups

- $(\mathbb{Z}, +)$
- $(\mathbb{F}_p^\times, \cdot)$
- $(E(\mathbb{F}_q), +)$: elliptic curve over finite field

Finite abelian groups are convenient for cryptography, because the group law behaves well!

No need to worry about things like $a + b \neq b + a$

Discrete Logarithm Problem

Discrete logarithm problem (DLP)

$G = \langle \alpha \rangle$: finite cyclic group of order N with generator α; written multiplicatively. $\beta = \alpha^n$ for some $n \in \{0, 1, 2, \ldots, N-1\}$.
DLP: given α, β, find n.

Discrete Logarithm Problem (cont'd)

Example

$G = (\mathbb{Z}/(1152921504606847363))^{\times} = \langle\alpha\rangle, \alpha = \overline{12345678}$.
Given $n = 640511947003800044$, it is easy to compute
$\beta = \alpha^n = \overline{243069074995667794}$: $O(\log N)$.
If only $\alpha = \overline{12345678}$ and $\beta = \overline{243069074995667794}$ are given, how to recover n?

Discrete Logarithm Problem (cont'd)

Bad news: this is in general a hard problem.
Best generic methods take time $O(\sqrt{N})$ – Shank's BSGS, Pollard's ρ and λ methods.

Discrete Logarithm Problem (cont'd)

Good news: we can use it to do cryptography!

Discrete Logarithm Based Cryptography

Example: Diffie-Hellman(-Merkle) key exchange protocol

Enables two parties to share a common secret (e.g. an symmetric encryption key) over an insecure communications channel.
How it works:

1. Alice and Bob agree on a finite cyclic group G and a generator α.
2. Alice chooses a secret integer m, computes $\beta = \alpha^m$, and sends β to Bob.
3. Bob chooses a secret integer n, computes $\gamma = \alpha^n$, and sends γ to Alice.
4. Alice computes $\gamma^m = \alpha^{nm}$; Bob computes $\beta^n = \alpha^{mn}$; this information is used as their shared secret.

Groups Suitable for DLP Based Cryptography

Requirements

- Intractability of DLP:
  - group order is desired to be "almost prime": to resist the Pohlig-Hellman attack.
  - No "easy" transformation into another group where DLP can be solvable in less time: to resist attacks like MOV, GHS, ...
- Compact representation of group elements.
- Efficient group operations.

Group Suitable for DLP Based Cryptography (cont'd)

Candidates:

- Multiplicative groups of finite fields $\mathbb{F}_q^\times$.
- Class groups of orders in number fields.
- Abelian varieties over finite fields, e.g., Jacobians of algebraic curves, **ELLIPTIC CURVES**.
- Others.

Elliptic Curve Group

Elliptic curve $E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q$, q odd, $3 \nmid q$

$\mathbb{F}_q$-rational points of E,

$$E(\mathbb{F}_q) := \{(x, y) \in (\mathbb{F}_q)^2 : y^2 = x^3 + ax + b\} \cup \{\infty\}$$

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Important fact about $E(\mathbb{F}_q)$

$E(\mathbb{F}_q)$ is a finite abelian group
NON-TRIVIAL! Do Not Try This at Home!

Group operation in $E(\mathbb{F}_q)$ is often written as addition $(+)$
Scalar multiplication: $[m]P := \underbrace{P + P + \ldots + P}_{m \text{ times}}$

Elliptic Curve Diffie-Hellman(-Merkle) key exchange

Rewrite Diffie-Hellman key exchange in language of elliptic curves.

1. Alice and Bob agree on an elliptic curve $E(\mathbb{F}_q)$ and a point $P \in E(\mathbb{E}_q)$ as base point.
2. Alice chooses a secret integer m, computes $Q = [m]P$, and sends Q to Bob.
3. Bob chooses a secret integer n, computes $R = [n]P$, and sends R to Alice.
4. Alice computes $[m]R = [nm]P$; Bob computes $[n]Q = [mn]P$; this information is used as their shared secret.

Group Operation in $E(\mathbb{F}_q)$

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q,\ q \text{ odd},\ 3 \nmid q$$

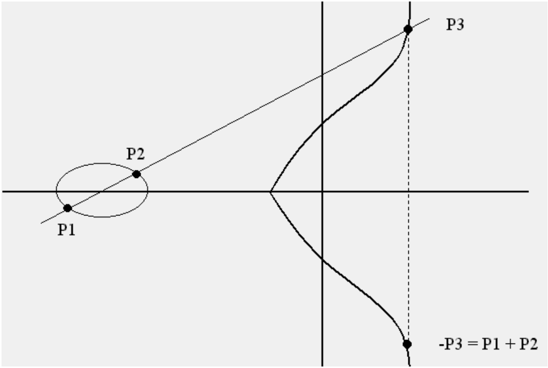

Figure: Illustration of points addition (courtesy of garykessler.net)

Group Law for $E(\mathbb{F}_q)$

Input: $P_1(x_1, y_1), \quad P_2(x_2, y_2)$
Output $Q(x, y) = P_1 + P_2$

Case $P_1 \neq P_2$: I, 3M/S

$$\lambda = (y_2 - y_1)/(x_2 - x_1)$$
$$x = \lambda^2 - x_1 - x_2$$
$$y = (x_1 - x)\lambda - x - y_1$$

Case $P_1 = P_2$: I, 4M/S

$$\lambda = (3x_1^2 + a)/(2y_1)$$
$$x = \lambda^2 - 2x_1$$
$$y = (x_1 - x)\lambda - x - y_1$$

Elliptic curve $E(\mathbb{F}_q)$ vs. $\mathbb{F}_p^\times$

Reason to replace "conventional" choice of the multiplicative group $\mathbb{F}_p^\times$?

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EFFICIENCY

At same level of security, crypto schemes implemented with elliptic curves run faster than their $\mathbb{F}_q^\times$ counterparts, with much shorter key lengths.

Elliptic curve $E(\mathbb{F}_q)$ vs. $\mathbb{F}_p^\times$ (cont'd)

n: bit length of q
N: bit length of p

Complexity of best known attacks to DLP

- Elliptic curve $E(\mathbb{F}_q)$:

$$C_{\mathrm{ec}} = 2^{n/2}$$

- Multiplicative group $\mathbb{F}_q^\times$:

$$C_{\mathrm{conv}} = \exp(1.92 N^{1/3} (\log(N \log 2))^{2/3})$$

Elliptic curve $E(\mathbb{F}_q)$ vs. $\mathbb{F}_p^\times$ (cont'd)

Crude Estimate

$$C_{\mathrm{ec}} = C_{\mathrm{conv}} \Longrightarrow n = 4.91 N^{1/3} (\log(N \log 2))^{2/3}$$

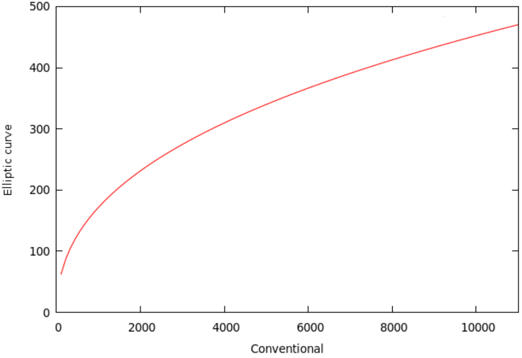

Figure: Elliptic curve vs. $\mathbb{F}_p^\times$ key sizes (in bits) for similar security level

Elliptic Curve $E(\mathbb{F}_q)$ vs. $\mathbb{F}_p^\times$ (cont'd)

Estimate above tells us

bit-strength	size q (bits)	size p (bits)
87	173	1024
117	233	2048
157	313	4096
209	417	8192

The estimate is at best crude, but it at least gives us some ideas about the low-costness of ECC over conventional public-key cryptosystems.

Elliptic Curve vs. RSA

Similar arguments apply to cryptosystems based on RSA.

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In practice, the performance comparison relies on implementation.

Commercial Time

In the commercial cryptographic literature,
1024-bit RSA $\approx$ 160-bit ECC [Lenstra and Verheul, 2001].

DSA vs. ECDSA

q: order of prime field $\mathbb{F}_q$
r: order of subgroup of $\mathbb{F}_r^\times$
n: order of subgroup of $E(\mathbb{F}_p)$
h: cofactor such that $|E(\mathbb{F}_p)| = n \cdot h$

The Digital Signature Standard (FIPS PUB 186-3) recommends

DSA

size q	1024	2048	2048	3072
size r	160	224	256	256

ECDSA

size n	161-223	224-255	256-383	384-511	≥ 512
max h	2^{10}	2^{14}	2^{16}	2^{24}	2^{32}

In Real World

Elliptic curve cryptography

- Elliptic curve based public-key cryptography is part of NSA Suite B.
- ECC has been written into various industrial standards eg. Digital Signature Standard (FIPS PUB 186-3).
- ECC has been integrated in many software and hardware (eg. latest version of OpenSSL, NSS v3.8, Firefox).
- A number of companies dedicated to ECC (eg. Certicom)
- Numerous white papers, presentations and publications (of course)

Chapter Four: Pairing-Based Crypto

Beyond Efficiency: Bilinear Maps

Many useful cryptographic protocols and applications, eg. IBE, short signatures, aggregate signatures, require use of a bilinear map

$$e : G_1 \times G_2 \to G_T$$

Bilineality

For all $P_1 \in G_1$, $P_2 \in G_2$, $m, n \in \mathbb{Z}$, $e([m]P_1, [n]P_2) = e(P_1, P_2)^{mn}$

Non-degeneracy

For $\mathcal{O} \neq P_1 \in G_1$, there exists $P_2 \in G_2$ such that $e(P_1, P_2) \neq 1$

Beyond Efficiency: Bilinear Maps (cont'd)

For certain kinds of elliptic curves, there exist efficient implementation of bilinear maps – the elliptic curve pairings (eg. Weil pairings, Tate pairings, Ate pairings, ...)

$$e : G_1 \times G_2 \to G_T$$

G_1 and G_2 are subgroups of $E(\mathbb{F}_q)$, G_T is a subgroup of $\mathbb{F}_{q^k}^{\times}$ for some small integer k. $|G_1| = |G_2| = |G_T| = r$.

Elliptic curve pairings are so far the only known efficient implementation of bilinear maps suitable for cryptography.

Miller's algorithm for Tate pairings

Algorithm 1 Basic Miller's algorithm

Input: $P \in G_1 \subseteq E(\mathbb{F}_{q^k})$, $Q \in G_2 \subseteq E(\mathbb{F}_{q^k})$, where r is the order of P

Output: $e(P, Q)$

$T \leftarrow P, f \leftarrow 1$
for $i = \lfloor \lg(r) \rfloor - 1$ to 0 **do**
    $f = f^2 \cdot l_{T,T}(Q)/v_{2T}(Q)$
    $T = 2T$
    if the i-th bit (from right) of r is 1 **then**
        $f = f \cdot l_{T,P}(Q)/v_{T+P}(Q)$
        $T = T + P$
    end if
end for
$f \leftarrow f^{(p^k-1)/r}$
return f

Here $l_{A,B}(Q)$ and $v_{A+B}(Q)$ are the "line" and "vertical" functions, resp.

Elliptic Curves Suitable for PBC

"Pairing-friendly curves"

- Present additional mathematical structures.
- Provide additional crypto hardness assumptions.
- Must satisfy all requirements for (regular) ECC.
- Have low density in all elliptic curves suitable for ECC

Applications of PBC

You cannot do (or do better) the following things without PBC.

- Tripartite key exchange [Joux, 2000]
- Identity-based encryption (IBE) [Boneh and Franklin, 2003], attributed-based encryption (ABE)
- Short signature scheme [Boneh et al., 2001]
- Aggregate signature scheme with different signing keys [Boneh et al., 2003]
- Efficient non-interactive zero-knowledge proofs [Groth and Sahai, 2008]
- Broadcast encryption scheme [Boneh and Waters, 2006]


The
END

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