Useful Results in Econ

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1 SSOC for utility/profit maximization

1.1 Utility maximization

Given $U = U(X_1, X_2)$, quasi-concave, utility maximization requires

$$du = u_1 dx_1 + u_2 dx_2 = 0 \tag{1}$$

$$d^2u = (u_{11}dx_1 + u_{12}dx_2)dx_1 + (u_{21}dx_1 + u_{22}dx_2)dx_2 < 0. \tag{2}$$

Therefore, we can rewrite equation (2)

$$u_{11}dx_1^2 + u_{12}dx_1dx_2 + u_{21}dx_1dx_2 + u_{22}dx_2^2 < 0 \tag{3}$$

$$u_{11}(dx_1^2 + \frac{2u_{12}}{u_{11}}dx_1dx_2) + u_{22}dx_2^2 < 0 \tag{4}$$

$$u_{11}\left(dx_1 + \frac{2u_{12}}{u_{11}}dx_1dx_2 + (\frac{u_{12}}{u_{11}}dx_2)^2 - (\frac{u_{12}}{u_{11}}dx_2)^2\right) + u_{22}dx_2^2 < 0 \tag{5}$$

$$u_{11}(dx_1 + \frac{u_{12}}{u_{11}}dx_2)^2 + dx_2^2(u_{22} - \frac{u_{12}^2}{u_{11}}) < 0 \tag{6}$$

where $u_{11} < 0$, $(dx_1 + \frac{u_{12}}{u_{11}}dx_2)^2$ and dx_2^2 are positive. Hence,

$$u_{22} - \frac{u_{12}^2}{u_{11}} < 0 \tag{7}$$

$$u_{11}u_{22} - u_{12}^2 > 0. \tag{8}$$

1.2 Profit maximization

Given $\pi = pf - w_1x_1 - w_2x_2$, where $f(\cdot)$ stands for the production function, $f = f(x_1, x_2)$.

$$d\pi = (pf_1 - w_1)dx_1 + (pf_2 - w_2)dx_2 = 0 \tag{9}$$

$$d^2\pi = p(f_{11}dx_1 + f_{12}dx_2)dx_1 + p(f_{21}dx_1 + p_{22}dx_2)dx_2 < 0. \tag{10}$$

Therefore, we can rewrite equation (10) as the following,

$$p(f_{11}dx_1 + f_{12}dx_2)dx_1 + p(f_{21}dx_1 + f_{22}dx_2)dx_2 < 0 \tag{11}$$

$$f_{11}dx_1^2 + f_{12}dx_1dx_2 + f_{12}dx_1dx_2 + f_{22}dx_2^2 < 0 \tag{12}$$

$$f_{11}\left(dx_1 + \frac{f_{12}}{f_{11}}dx_2\right)^2 + \left(f_{22} - \frac{f_{12}^2}{f_{11}}\right)dx_2^2 < 0. \tag{13}$$

Hence, we have similar results as in utility maximization problem,

$$f_{22} - \frac{f_{12}^2}{f_{11}} < 0 \tag{14}$$

$$f_{11}f_{22} - f_{12}^2 > 0. \tag{15}$$

2 Cramer's Rule and Inferior goods

2.1 Cramer's Rule

Given a comb of three equations,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \tag{16}$$

we can rewrite in matrix form, i.e., $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{17}$$

Therefore, we can solve x with the following rule,

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{det(A)}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{det(A)}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{det(A)} \tag{18}$$

2.2 Inferior good

Consider a two–good model,

$$\mathscr{L} = U(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2). \tag{19}$$

Then we can write the first order conditions (FOCs) as the following,

$$\frac{\partial \mathscr{L}}{\partial x_1} = u_1 - \lambda p_1 = 0 \tag{20}$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = u_2 - \lambda p_2 = 0 \tag{21}$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 = 0 \tag{22}$$

where u_i stands for the marginal utility of good i, i.e., $u_i = \frac{\partial U}{\partial x_i}$.

Now, we can obtain boarder Hessian matrix by differentiating equation (20) – (22) with respect to income, M.

$$\frac{\partial}{\partial M}\frac{\partial \mathscr{L}}{\partial x_1} = u_{11}\frac{\partial x_1}{\partial M} + u_{12}\frac{\partial x_2}{\partial M} - p_1\frac{\partial \lambda}{\partial M} = 0 \tag{23}$$

$$\frac{\partial}{\partial M}\frac{\partial \mathscr{L}}{\partial x_2} = u_{21}\frac{\partial x_1}{\partial M} + u_{22}\frac{\partial x_2}{\partial M} - p_2\frac{\partial \lambda}{\partial M} = 0 \tag{24}$$

$$\frac{\partial}{\partial M}\frac{\partial \mathscr{L}}{\partial \lambda} = -p_1\frac{\partial x_1}{\partial M} - p_2\frac{\partial x_2}{\partial M} + 0 = -1 \tag{25}$$

$$\begin{bmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x_1}{\partial M} \\ \frac{\partial x_2}{\partial M} \\ \frac{\partial \lambda}{\partial M} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{26}$$

Suppose we want to verify if x_1 is inferior,

$$\frac{\partial x_1}{\partial M} = \frac{\begin{vmatrix} 0 & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ -1 & -p_2 & 0 \end{vmatrix}}{det(H)}, \text{ where } H = \begin{bmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}. \tag{27}$$

In two-good case, $det(H) > 0$, so we only need to check the sign of the numerator.

$$\begin{vmatrix} 0 & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ -1 & -p_2 & 0 \end{vmatrix} = (-1)^{3+1} \cdot (-1) \cdot \begin{vmatrix} u_{12} & -p_1 \\ u_{22} & -p_2 \end{vmatrix} = u_{12}p_2 - u_{22}p_1. \tag{28}$$

Hence,

$$\frac{\partial x_1}{\partial M} = \frac{u_{12}p_2 - u_{22}p_1}{det(H)}, \tag{29}$$

where prices are non-negative, u_{22} is negative. We do not know the sign of u_{12}.

If u_{12} is known, we know the sign of $\frac{\partial x_1}{\partial M}$. Good x_1 is inferior if the partial derivative w.r.t M is negative.

3 Hessian and optimization

We are going to test if Hessian matrix shows the same result as the convexity of the indifference curve (IDC) in two–good optimization problem, i.e., the proof of $det(H) > 0$ is also the proof of convexity of IDC.

MRS stands for the slope of the IDC,

$$MRS_{12} = \frac{dx_2}{dx_1} = -\frac{u_1}{u_2} \tag{30}$$

If IDC is convex, $\frac{dMRS}{dx_1} > 0$. To show this,

$$\begin{aligned}
dMRS &= d(-\frac{u_1}{u_2}) && (31)\\
&= -\frac{1}{u_2^2}(u_2(u_{11}dx_1 + u_{12}dx_2) - u_1(u_{21}dx_1 + u_{22}dx_2)) && (32)\\
\frac{dMRS}{dx_1} &= -\frac{1}{u_2^2}\left(u_2(u_{11} + u_{12}\frac{dx_2}{dx_1}) - u_1(u_{21} + u_{22}\frac{dx_2}{dx_1})\right) && (33)\\
&= -\frac{1}{u_2^3}\left(u_2^2u_{11} + u_2^2u_{12}\frac{dx_2}{dx_1} - u_1u_2u_{12} - u_1u_2u_{22}\frac{dx_2}{dx_1}\right). && (34)
\end{aligned}$$

Recall, $\frac{dx_2}{dx_1} = -\frac{u_1}{u_2}$. We can substitute it into equation (34),

$$\begin{aligned}
\frac{dMRS}{dx_1} &= -\frac{1}{u_2^3}\left(u_2^2u_{11} + u_2^2u_{12}(-\frac{u_1}{u_2}) - u_1u_2u_{12} - u_1u_2u_{22}(-\frac{u_1}{u_2})\right) && (35)\\
&= -\frac{1}{u_2^3}(u_2^2u_{11} - 2u_1u_2u_{12} + u_1^2u_{22}) > 0. && (36)
\end{aligned}$$

Now, let's see if Hessian matrix can tell us the same thing.

Given boarder Hessian H, where

$$H = \begin{bmatrix} u_{11} & u_{12} & u_1 \\ u_{21} & u_{22} & u_2 \\ u_1 & u_2 & 0 \end{bmatrix}. \tag{37}$$

$$det(H) = (-1)^{1+1} \cdot u_{11} \cdot \begin{vmatrix} u_{22} & u_2 \\ u_2 & 0 \end{vmatrix} + (-1)^{1+2} \cdot u_{12} \cdot \begin{vmatrix} u_{12} & u_2 \\ u_1 & 0 \end{vmatrix} + (-1)^{1+3} \cdot u_1 \cdot \begin{vmatrix} u_{12} & u_{22} \\ u_1 & u_2 \end{vmatrix} \quad (38)$$

$$= -u_2^2 u_{11} + u_1 u_2 u_{12} + u_1(u_2 u_{12} - u_1 u_{22}) \quad (39)$$

$$= -(u_2^2 u_{11} - 2u_1 u_2 u_{12} + u_1^2 u_{22}) > 0 \quad (40)$$

Clearly, equation (36) and (40) are exactly the same thing.

4 Conditional factor demand and Shephard's Lemma

4.1 An example for Shephard's Lemma

Consider Cobb-Douglas (CD) production function, $y = x_1^\alpha x_2^\beta$. We can write the cost minimization problem as the following,

$$\mathscr{L} = w_1 x_1 + w_2 x_2 + \lambda(y_0 - x_1^\alpha x_2^\beta) \quad (41)$$

where w_i stands for the price of factor i, and we have a given output level y_0. Write down the FOCs,

$$\frac{\partial \mathscr{L}}{\partial x_1} = w_1 - \lambda \alpha x_1^{\alpha-1} x_2^\beta = 0 \quad (42)$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = w_2 - \lambda \beta x_1^\alpha x_2^{\beta-1} = 0 \quad (43)$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = y_0 - x_1^\alpha x_2^\beta = 0 \quad (44)$$

Divide (42) by (43), we have $x_2 = \frac{w_1}{w_2}\frac{\beta}{\alpha}x_1$. Substitute back to (44)

$$y_0 = x_1^{\alpha}(\frac{w_1}{w_2})^{\beta}(\frac{\beta}{\alpha})^{\beta}x_1^{\beta} \tag{45}$$

$$x_1^* = y_0^{\frac{1}{\alpha+\beta}}(\frac{w_2}{w_1})^{\frac{\beta}{\alpha+\beta}}(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} \tag{46}$$

$$x_2^* = y_0^{\frac{1}{\alpha+\beta}}(\frac{w_1}{w_2})^{\frac{\alpha}{\alpha+\beta}}(\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}} \tag{47}$$

Now we can substitute the optimal factor demand back to the cost function, then we have $C^* = C^*(x_1^*, x_2^*)$,

$$C^* = w_1x_1^* + w_2x_2^* \tag{48}$$

$$= \frac{\alpha}{\beta}w_2x_2 + w_2x_2 \tag{49}$$

$$= \frac{\alpha+\beta}{\beta}w_2x_2 \tag{50}$$

$$= \frac{\alpha+\beta}{\beta}w_2y_0^{\frac{1}{\alpha+\beta}}(\frac{w_1}{w_2})^{\frac{\alpha}{\alpha+\beta}}(\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}} \tag{51}$$

$$\frac{\partial C^*}{\partial w_1} = y_0^{\frac{1}{\alpha+\beta}}(\frac{w_2}{w_1})^{\frac{\beta}{\alpha+\beta}}(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} \tag{52}$$

$$= X_1^*. \tag{53}$$

4.2 Shephard's lemma

Consider a cost minimization problem in two–factor model,

$$\mathscr{L} = \sum_{i=1}^{n} w_ix_i + \lambda(\overline{y} - f(x_1, \cdots, x_n)), \tag{54}$$

where $f(\cdot)$ stands for the production function.

We can write down the FOC,

$$\frac{\partial \mathscr{L}}{\partial x_i} = w_i - \lambda f_i = 0, \tag{55}$$

where f_i stands for the marginal product of factor i, i.e., $f_i = \frac{\partial f}{\partial x_i}$. Then, solving optimal factor demand x_i^* is quite straightforward , $x_i^* = x_i^*(\boldsymbol{w}, \overline{y})$.

Plug x_i^* back to cost function, we have $C^* = C^*(\boldsymbol{w}, \overline{y})$.

Then, take partial derivative w.r.t. w_i,

$$\frac{\partial C^*}{\partial w_j} = x_j^* + \sum_{i=1}^{n} w_i \frac{\partial x_i^*}{\partial w_j} \tag{56}$$

From FOC, we know $w_i = \lambda f_i$, plug into equation (56),

$$\frac{\partial C^*}{\partial w_j} = x_j^* + \sum_{i=1}^{n} \lambda f_i \frac{\partial x_i^*}{\partial w_j} \tag{57}$$

Clearly,

$$\frac{\partial f(x_1^*, \cdots, x_n^*)}{\partial w_j} = \sum_{i=1}^{n} f_i \frac{\partial x_i^*}{\partial w_j} = \frac{\partial \overline{y}}{\partial w_j} = 0 \tag{58}$$

Note, $\overline{y}$ is constant, so equation (58) = 0. Hence, the second term in equation (57) is zero, and we have

$$\frac{\partial C^*}{\partial w_j} = x_j^*. \tag{59}$$

Shephard's lemma says C^* is differentiable in $\boldsymbol{w}$ at $(\boldsymbol{w}^0, \overline{y}^0)$ with $\boldsymbol{w}^0 >> 0$, and

$$\frac{\partial C^*}{\partial w_j} = x_j^* = x_j^H(\boldsymbol{w}^0, \overline{y}^0), \quad j = 1, \cdots, n. \tag{60}$$

We are going to prove this by invoking Envelope theorem in next section.

4.3 Envelope theorem and Shephard's lemma

Consider a cost minimization problem in two–factor model, we obtain optimal factor demand x_j^* and Lagrangian multiplier λ^* by solving Lagrangian equation, where $x_j^* = x_j^*(\boldsymbol{w}, \overline{y})$ and $\lambda^* = \lambda^*(\boldsymbol{w}, \overline{y})$. Plug the optimal solution back to Lagrangian,

$$\mathscr{L}^*(\boldsymbol{w}, \overline{y}) = \sum_{i=1}^{n} w_i x_i^* + \lambda^*(\overline{y} - f(\boldsymbol{x}^*)) \tag{61}$$

where $\boldsymbol{w}$ and $\boldsymbol{x}$ are vectors, $\boldsymbol{w} = [w_1, w_2, \cdots, w_n]$, $\boldsymbol{x} = [x_1, \cdots, x_n]$.

Differentiate Lagrangian w.r.t w_1,

$$\frac{\partial \mathscr{L}^*}{\partial w_1} = x_1^* + \sum_{i=1}^{n} w_i \frac{\partial x_i^*}{\partial w_1} + \lambda^* \left(- \sum_{i=1}^{n} f_i \frac{\partial x_i^*}{\partial w_1} \right) + \frac{\partial \lambda^*}{\partial w_1} (\overline{y} - f(x_1^*, \cdots, x_n^*)) \tag{62}$$

$$= x_1^* + \left(\sum_{i=1}^{n} (w_i - \lambda^* f_i^*) \frac{\partial x_i^*}{\partial w_1} \right) + \frac{\partial \lambda^*}{\partial w_1} (\overline{y} - f(x_1^*, \cdots, x_n^*)). \tag{63}$$

From FOC, we know $w_i - \lambda f_i = 0$, $\overline{y} - f(\boldsymbol{x}^*) = 0$, so the last two terms in equation (63) equals to zero. Hence,

$$\frac{\partial \mathscr{L}^*}{\partial w_1} = \frac{\partial C^*}{\partial w_1} = x_1^*. \tag{64}$$

In general,

$$\frac{\partial \mathscr{L}^*}{\partial w_i} = \frac{\partial C^*}{\partial w_i} = x_i^* = x_i^H \tag{65}$$