Unit - 05

Joint probability distribution of markov chain

$P(X = x, Y = y) = f(x, y)$

$X = x_1, x_2, x_3 - - x_n$

$Y = y_1, y_2, y_3 - - - y_m$

i) $f(x, y) \geq 0$

ii) $\sum_i \sum_j f(x_i, x_j) = 1$

* Joint distribution table

x \ y	y_1	y_2	- - -	y_m	Sum
x_1	J_{11}	J_{12}	- - -	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	- - -	J_{2n}	$f(x_2)$
⋮	⋮				⋮
x_n	J_{n1}	J_{n2}	- - -	J_{nn}	$f(x_n)$
Sum	$g(y_1)$	$g(y_2)$	- - -	$g(y_n)$	1

* Marginal distribution of x & y.

x	x_1	x_2	- - -	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	- - - -	$f(x_n)$

y	y_1	y_2	- - - -	y_m
$g(y)$	$g(y_1)$	$g(y_2)$	- - - -	$g(y_m)$

* Mean, Variance of S.D

$m_x = E(x) = \sum x_i f(x_i)$

$m_y = E(y) = \sum y_j g(y_j)$

$E(xy) = \sum_{i,j} x_i y_j J_{ij}$

$E(x^2) = \sum_i x_i^2 f(x_i)$

$E(y^2) = \sum_j y_j^2 g(y_j)$

$E(x+y) = \sum (x_i + y_j) J_{ij}$

Variance $= \sigma_x^2 = \sum x_i^2 f(x_i) - m_x^2$

$= \sum(x^2) - [E(x)]^2$

$\sigma^2_y = \sum y_j^2 g(y_j) - m^2_y$

$= E(y^2) - [E(y)]^2$

$S.D = \sqrt{\text{variance}}$

$= \sigma_x = \sigma_y$

* Covariance $= Cov(x,y) = E(xy) - E(x)E(y)$

* Correlation $= \rho(x,y) = \dfrac{Cov(x,y)}{\sigma_x \sigma_y}$

Note:- If $Cov(x,y) = 0$ then x & y are independent.

or If $f(x_i) g(y_j) = J_{ij}$ then x & y are independent.

Q1) To joint distrubution of two random varibles for x & y as follows

$x \backslash y$	-4	2	7	$f(x)$
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	1/2
$g(y)$	3/8	3/8	1/4	1

Compute the following.

a) $E(x)$ & $E(y)$ b) $E(xy)$ c) σ_x & σ_y

d) $Cov(x,y)$ e) $\rho(x,y)$

→ Marginal distrubution of x & y

x	1	5
$f(x)$	1/2	1/2

y	-4	2	7
$g(y)$	3/8	3/8	1/4

a) $E(x) = \sum x_i f(x_i) = 1 \times \frac{1}{2} + 5 \times \frac{1}{2} = 3$

$E(y) = \sum y_j g(y_j) = -4 \times 3/8 + 2 \times 3/8 + 7 \times \frac{1}{4} = 1$

b) $E(xy) = \sum x_i y_j J_{ij}$

$= 1 \times -4 \times \frac{1}{8} + 1 \times 2 \times \frac{1}{4} + 1 \times 7 \times \frac{1}{8} + 5 \times -4 \times \frac{1}{4} +$

$5 \times 2 \times \frac{1}{8} + 5 \times 7 \times \frac{1}{8}$

$= \frac{3}{2}$

c) $\sigma_x^2 = E(x^2) - [E(x)]^2$

$= \sum x_i^2 f(x_i) - 3^2$

$= 1^2 \times \frac{1}{2} + 5^2 \times \frac{1}{2} - 9$

$= \left(\frac{1}{2} + \frac{25}{2}\right) - 9$

$= 13 - 9$

$= 4$

$\sigma_x = \sqrt{4} = 2$

$\sigma_y^2 = E(y^2) - [E(y)]^2$

$= \left((-4)^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 7^2 \times \frac{1}{4}\right) - 1^2$

$= \frac{\cancel{16}^{\,2} \times 3}{\cancel{8}_{\,1}} + 4 \times \frac{3}{\cancel{8}_{\,2}} + \frac{49}{4} - 1$

$\sigma_y^2 = \frac{75}{4}$

$\sigma_y = \sqrt{\frac{75}{4}} = \frac{\sqrt{75}}{2}$

d) $Cov(x, y) = E(xy) - E(x)E(y)$

$= \frac{3}{2} - 3 \times 1$

$= \frac{-3}{2}$

e) $\rho(x, y) = \frac{Cov(x, y)}{\sigma_x \sigma_y}$

$= \frac{-3/2}{\cancel{2} \times \sqrt{75}/\cancel{2}} = \frac{-3}{2\sqrt{75}}$

Q) The joint probability distrubution of two discrete random variables x & y is given by.

$f(x,y) = k(2x+y)$ where x & y integers that $0 \le x \le 2$, $0 \le y \le 3$.

i) Find the value of the constant k.

ii) Find the marginal probability distrubution of x & y

iii) Show that the random variables x & y are dependent.

→ $f(x,y) = k(2x+y)$

$0 \le x \le 2 \qquad 0 \le y \le 3$

$x = \{0, 1, 2\}$

$y = \{0, 1, 2, 3\}$

$x \backslash y$	0	1	2	3	$f(x)$
0	0	k	$2k$	$3k$	$6k$
1	$2k$	$3k$	$4k$	$5k$	$14k$
2	$4k$	$5k$	$6k$	$7k$	$22k$
$g(y)$	$6k$	$9k$	$12k$	$15k$	1

i) $6k + 14k + 22k = 1$

$$\boxed{k = \frac{1}{42}}$$

or $6k + 9k + 12k + 15k = 1$

$\boxed{k = 1/}$

ii) Marginal distrubution of x & y

x	0	1	2
$f(x)$	$6/42 = 1/7$	$14/42 = 1/3$	$\frac{22}{42} = 11/21$

y	0	1	2	3
$g(y)$	$\frac{6}{42} = 1/7$	$\frac{9}{42} = 3/14$	$\frac{12}{42} = 2/7$	$\frac{15}{42} = \frac{5}{14}$

iii) x & y are not independent.

$$f(x_1)\, g(y_1) = \frac{1}{7} \times \frac{1}{7} = \frac{1}{49} \neq 0 = f_{11}$$

$\therefore$ x & y are independent.

03) Given the following joint distrubution of the random variables x & y. find the correspondi[ng] distrubution also find the co-variance and the corelation of the random variables.

x/y	1	3	9	$f(x)$
2	$1/8$	$1/24$	$1/12$	$1/4$
4	$1/4$	$1/4$	0	$1/2$
6	$1/8$	$1/24$	$1/12$	$1/4$
$g(y)$	$1/2$	$1/3$	$1/6$	1

Marginal distrubution of x & y.

x	2	4	6
$f(x)$	$1/4$	$1/2$	$1/4$

y	1	3	9
$g(y)$	$1/2$	$1/3$	$1/6$

Q4) A pair coin is tossed thrise the random variables x & y are defined as. x denotes 0 or 1 according to as head or tail occours on 1st toss; y denotes the number of heads

find. i) Marginal distrubution of x & y.

ii) The joint distrubution of x & y.

iii) expections of x, & y & xy.

iv) SD of xy is and v) Covarience & Corelation of x & y.

→ $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$x = \{0, 0, 0, 1, 1, 1, 0, 1\} = \{0, 1\}$

$y = \{3, 2, 2, 2, 1, 1, 1, 0\} = \{0, 1, 2, 3\}$.

a)

x	0	1
$f(x)$	$4/8 = 1/2$	$4/8 = 1/2$

y	0	1	2	3
$g(y)$	$1/8$	$3/8$	$3/8$	$1/8$

$x \backslash y$	0	1	2	3	$f(x)$
0	0	$1/8$	$2/8 = 1/4$	$1/8$	
1	$1/8$	$2/8 = 1/4$	$1/8$	0	
$g(y)$					

a) $E(x) = \sum x_i f(x_i) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

$E(y) = \sum y_i g(y_i) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{6}{8} + 3 \times \frac{1}{8} = \frac{3}{2}$.

b) $E(xy) = \sum x_i y_j J_{ij} =$

$= 0 \times 0 \times 0 + 0 \times 1 \times 1/8 + 0 \times 2 \times 2/8 + 0 \times 3 \times 1/8 +$

$1 \times 0 \times 1/8 + 1 \times 1 \times 2/8 + 1 \times 2 \times 1/8 + 1 \times 3 \times 0$

$= 1/2$.

c) $\sigma_x^2 = E(x^2) - [E(x)]^2 = \sum x_i^2 f(x_i) - (1/2)^2$

$= \left[0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2}\right] - \frac{1}{4}$

$\sigma_x^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow \sigma_x = \frac{1}{2}$

$\sigma_y^2 = E(y^2) - [E(y)]^2 = \sum y_i^2 g(y_i) - (3/2)^2$

$= \left[0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8}\right] - \frac{9}{4}$

$= \frac{3}{8} + \frac{12}{8} + \frac{9}{8} - \frac{9}{4} = \frac{3}{4}$

$\sigma_y = \frac{\sqrt{3}}{2}$.

d) $Cov(xy) = E(xy) - E(x)E(y)$

$= \frac{1}{2} - \frac{1}{2} \times \frac{3}{2} = \frac{-1}{4}$

e) $\rho(xy) = \frac{Cov(xy)}{\sigma_x \times \sigma_y} = \frac{-1/4}{\frac{1}{2} \times \frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}$.

Markov chain :-

$S \to$ Sample space $\quad R \to$ Real no

$f : S \to R$

$X = f(s)$ where $s \in S$

Stochastic function

$T \subset R$ parameter t such as time.

$\{x(t), t \in T\}$ on S with parameter t

$x_0 = x(0)$

$\{x_1(t), x_2(t) \text{ - - - - - - - - }\} \to$ finite $\to$ chain

$\downarrow$ States

* Probability Vector.

$V = (V_1, V_2 \text{ - - - - - - } V_n)$ are non-negative

& $\sum v_i = 1$.

Eg. $V = (1/2, 1/2)$; $v = [0, 1]$ $\quad \sum v_i = 1$.

* Stochastic matrix.

A Square matrix $P = [P_{ij}]$ is said to be stochastic matrix if every row is a probability vector.

Eg

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{array}{l} \sum v_i = 1 \\ \to 1 \\ \to 1 \\ \to 1 \end{array}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{array}{l} \sum v_i = 1 \\ \to 1 \\ \to 1 \end{array}$$

* Regular stochastic matrix.

A stochastic matrix P is said to be regular stochastic matrix if all the entries of some power P^n are positive.

Ex

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/6 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 1/3 & 1/6 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$ It is regular stochastic matrix.

Q1) Find the uniqe probability vector of the regular stochastic matrix. $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$.

$\Rightarrow$ $V = [x \quad y]$ st $x + y = 1$.

$VA = V$.

$$[x \quad y] \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x \quad y]$$

$$\begin{bmatrix} \frac{3}{4}x + \frac{y}{2} & \frac{x}{4} + \frac{y}{2} \end{bmatrix} = [x \quad y]$$

$$\frac{3x}{4} + \frac{y}{2}^{\times 2} = x \qquad \frac{x}{4} + \frac{y}{2} = y.$$

$$3x + 2y = 4x \rightarrow (1) \qquad x + 2y = 4y \rightarrow (2)$$

WKT $x+y=$

$y = 1-x$.

sub in eqn (2)

$x + 2(1-x) = 4(1-x)$

$x + 2 - 2x = 4 - 4x$.

$3x = 2$.

$x = \frac{2}{3}$

$y = 1 - x = 1 - 2/3 = 1/3$

Or from (1) $-x + 2y = 0$

from (3). $x + y = 1$

· Solving (1) & (2)

$x = \frac{2}{3}$. $y = \frac{1}{3}$

The unique fixed probability vector. $V = \left[\frac{2}{3} \ \frac{1}{3}\right]$

02) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$

let $V = [x \ y \ z]$

$VA = V$.

$[x \ y \ z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x \ y \ z]$

$\left[\frac{y}{6}, \ x + \frac{y}{2} + \frac{2z}{3} \ \ \frac{y}{3} + \frac{z}{3}\right] = [x \ y \ z]$

$\frac{y}{6} = x$; $\quad x + \frac{y}{2} + \frac{2z}{3} = y \quad \frac{y}{3} + \frac{z}{3} = z$

Scanned with CamScanner

$y = 6x$

$6x - y = 0 \longrightarrow (1)$

$6x + 3y + 4z = 6y$

$6x - 3y + 4z = 0 \longrightarrow (2)$

$y + z = 3z$

$y = 2z$

$y - 2z = 0 \longrightarrow (3)$

put eq no (1) in (3)

$\overset{3}{\cancel{6}}x = 2z$

$z = 3x \longrightarrow (4)$

Solve (3) & (4)

$y - 6x = 0 \quad = 2z - 6x = 0$

$y = 6x. \quad = z = 3x$

$6x - 3y + 4z = 0$

$6x - 18x + 12x = 0$

$x + y + z = 1.$

$x + z = 1 - y$

$x + z = 1 - 6x.$

$7x + z = 1 \longrightarrow (4) \Rightarrow 7x + 3x = 1 \Rightarrow x = 1/10$

solving eq (3) & (4) we get

$x = \frac{1}{10}$

$z = \frac{3}{10}$

$y = \frac{6}{10}$

Q3) $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$VA = V$

$$[x \ y \ z]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \ y \ z]$$

$$\left[\frac{z}{2} \quad x+\frac{z}{2} \quad y\right] = [x \ y, z]$$

$\frac{z}{2} = x$ $\qquad$ $\frac{x}{1}+\frac{z}{2} = y$ $\qquad$ $y = z. \rightarrow (3)$

$z = 2x. \rightarrow (1)$ $\qquad$ $2x + z = 2y \rightarrow (2)$

$x + y + z = 1 \longrightarrow (4)$

$x + z + 2x = 1$

$3x + z = 1$

$3x + 2x = 1$

$5x = 1.$

$x = 1/5$

$z = 2/5$

$y = 2/5$

V is a regular stochastic probability vector

$$V = \left[\frac{1}{5} \ \frac{2}{5} \ \frac{2}{5}\right]$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

It is regular stochastic matrix.

★ Markov chains

$S \to$ Sample space

$R \to$ Real no.

$f : S \to R$

$\{x_1, x_2, x_3 - - \}$

$\{x_1(t), x_2(t) - - - - \}$

$x_1, x_2, x_3 - -$. Sequence of trials

i) $\{a_1, a_2 - - a_m\}$

ii) $P = [P_{ij}] = \begin{bmatrix} P_{11} & P_{12} & - - - & P_{1m} \\ P_{21} & P_{22} & - - - - & P_{2m} \\ & & & \\ P_{m1} & P_{m2} & - - - - - & P_{mm} \end{bmatrix}$

$P_{ij} \; (a_i, a_j)$

$a_i \to a_j$

$a_i \quad P_{11} \quad P_{12} \; \cdots \; P_{1m}$

$i) \; 0 \le P_{ij} \le 1 \qquad ii) \; \sum P_{ij} = 1$

n steps $\to a_j$

$P_{ij}^{(n)} \qquad a_i \to a_j \qquad a_{i1} \to a_{i2} \to \cdots a_{r_{n-1}}$

$$P^{(1)} = P^{(0)} \cdot P \quad P^{(2)} = P^{(1)} \cdot P = P^{(0)} P^2 \cdots P^{(n)} = P^{(0)} P^n$$

* irreducible $\to$ if is regular stochastic matrix

Q1) Proove that the Markov chain whose transition probability matrix.

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$ is irreducible.

$\to$

$$P^2 = P \times P = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

P is irreducible since all entries of P^2 are positive

Q2) A students study habbits are as follows if he studies 1 night he is 70%. sure, not to study till next night on the other hand. if he does not study 1 night he is 60%. sure not to study the next night in the long run how often does he study.

Q2) 3 boys A B C are throwing ball to each other. A always throws the ball to B & B always throws the ball to C. C is just as lightly to throw the ball to B as to A. If C was the 1st person to through the ball find the probabilities that after 3 throws

i) A has ball.

ii) B has ball.

iii) C has ball.

→

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array} \Rightarrow P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2$$

$$P^{(0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$P^{(3)} = P^{(0)} P^3 \longrightarrow (1)$$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

1) reduces to

$$P^{(3)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 1/4 & 1/4 & 1/2 \end{bmatrix} \end{array}$$

∴ The probability after three throws.

i) A has ball is 1/4 ii) B has the ball is 1/4 iii) C has the ball is 1/2

Scanned with CamScanner

Q4) A ~~gambler~~ the luck as follows a pattern if he win a game the probability of winning the next game is 0.6. however if he looses a game. The probability of loosing the next game is 0.7 there is an even chance of gambler winning the first game if so i) what is the probability of he winning the second game

ii) what is the probability of he winning the 3rd game

iii) On the long run how often he will win.

→ $w \rightarrow$ win

$l \rightarrow$ loses

$$P = \begin{array}{c|cc} & W & L \\ \hline W & 0.6 & 0.4 \\ L & 0.3 & 0.7 \end{array} = \begin{bmatrix} 6/10 & 4/10 \\ 3/10 & 7/10 \end{bmatrix}$$

$$p^{(0)} = \begin{array}{cc} W & L \\ \end{array} \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$$

i) $p^{(1)} = p^{(0)} P$

$$= \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 6/10 & 4/10 \\ 3/10 & 7/10 \end{bmatrix}$$

$$= \begin{array}{cc} W & L \end{array} \begin{bmatrix} 9/20 & 10/20 \end{bmatrix}$$

prob of winning the 2nd game is 9/20.

ii) $p^{(2)} = p^{(1)} P$

$$= \begin{bmatrix} 9/20 & 11/20 \end{bmatrix} \begin{bmatrix} 6/10 & 4/10 \\ 3/10 & 7/10 \end{bmatrix} = \begin{array}{cc} W & L \end{array} \begin{bmatrix} \frac{87}{200} & \frac{113}{200} \end{bmatrix}$$

Prob of he winning the 3rd game is $\frac{87}{200}$.

$p^{(n)} \Rightarrow a_i \rightarrow a_j$

Scanned with CamScanner

iii) $v = [x \quad y]$ st $x + y = 1$

$vP = v$.

$$[x \quad y] \begin{bmatrix} 6/10 & 4/10 \\ 3/10 & 7/10 \end{bmatrix} = [x \quad y]$$

$$\begin{bmatrix} \frac{6x}{10} + \frac{3y}{10} & \frac{4x}{10} + \frac{7y}{10} \end{bmatrix} = [x \quad y]$$

$$\frac{6x}{10} + \frac{3y}{10} = x \qquad \frac{4x}{10} + \frac{7y}{10} = y$$

$$6x + 3y = 10x \qquad 4x + 7y = 10y.$$

$$-4x + 3y = 0 \rightarrow ①. \qquad 4x - 3y = 0 \rightarrow ②$$

WKT $x + y = 1 \rightarrow ③$

Solving ① & ②

$$x = \frac{3}{7} \qquad y = \frac{4}{7}.$$

$$v = \begin{bmatrix} 3/7 & 4/7 \end{bmatrix}$$

$\therefore$ In the long run he winning is $\frac{3}{7}$