Maths

Unit: 03

Numerical Technique

Numerical solutions of Algebraic & Transcendental equations

$a_0 + a_1 x + a_2 x^2 + \dots\dots +$

polynomial eqn

Sum of two polynomial eqn is called algebric eqn

Ex $3x^2 + 2x + 1 = 0$

$2x - 5x^0 = 0$

Transidental equation

A equation which consists of differential equation or logarithmic equation or Trignometric equation is called transidental equation.

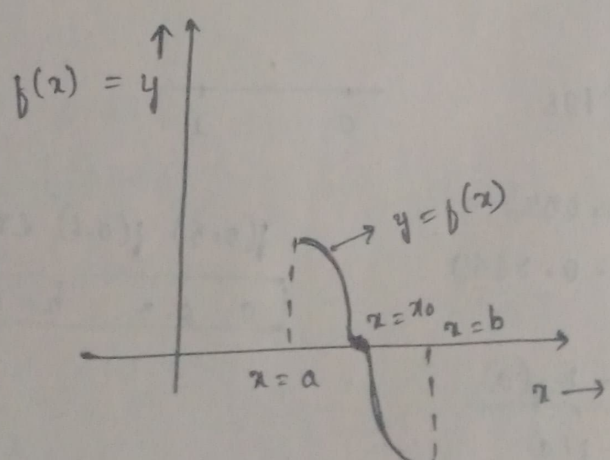
Ex :-

$x \sin x + \cos x = 0$

$x e^x + 1 = 0$

$x \log x = 2.$

Intermidiate Value theorem



Steps :- $f(x) = x^2 + x - 2$

x	$f(x)$
0	2
1	0
2	4
3	10

$i = 1$ $i = 2$ $i = 3$

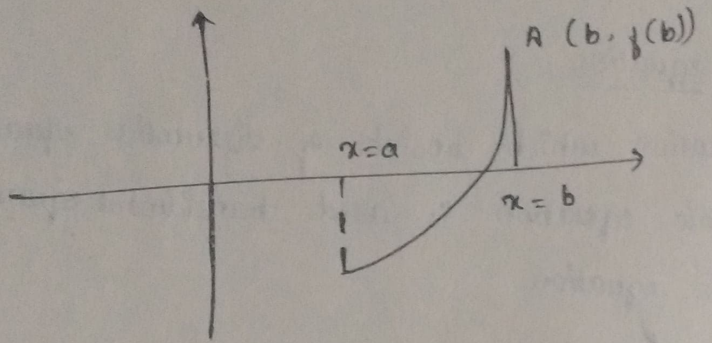
roots $(0, 1)$ $(0, 2)$ $(0, 3)$

This is because to get One +ve and -ve value [For more values more itteration]

Smaller value is good

For Intermidiate theorm One root should be -ve & Via versa.

+ Regular - False method or False position method



Formula :- $$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Problems

Q1) Find the real root of the equation $f(x) = \cos x - e^x$ using regular falsi method. Correct to 4 decimal places.

→ For $f(x) = x = 0$ Start = 1   End = 5

$f(0) = 1$.

$f(1) = -2.17798$

0   1

$f(0.5) = 0.0532$.

$f(0.6) = -0.2679$

$f(0.5)\,f(0.6) < 0$

$\boxed{a = 0.5 \quad b = 0.6}$

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x = \frac{0.5f(0.6) - 0.6f(0.5)}{f(0.6) - f(0.5)}$$

$$x_0 = \frac{x[\cos y - ye^y] - y[\cos x - xe^x]}{[\cos y - ye^y] - [\cos x - xe^x]}$$

$x_1 = 0.51657$; $f(x_1) = 3.6027 \times 10^{-3}$

$f(0.5167) = \cos(0.51657) - (0.5167)e^{0.5167}$

$x_2 = 0.5176$; $f(x_2) = 2.3893 \times 10^{-4}$

$x_3 = 0.5177$; $f(x_3) = 1.5824 \times 10^{-5}$

$x_4 = 0.5177$; $f(x_4) = 1.0479 \times 10^{-6}$

02) Find the real root of the equation $x \log_{(10)} x = 1.2$ by regular false method correct to 4 decimal points.

→ $x \log_{(10)} x = 1.2$

$f(x) = x \log_{10} x - 1.2$

start = 1	start = 1
End = 5	End = 5
step = 1	step = 0.2

$f(1) = -1.2$

$f(2) = -0.597$

$f(3) = 0.2313$

$f(2.7) \cdot f(2.8) < 0$

$f(2.7) = -0.035$

$f(2.8) = 0.052$

$a = 2.7$

$b = 2.8$

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2.7 f(2.8) - 2.8 f(2.7)}{f(2.8) - f(2.7)}$$

$$x_1 = \frac{x[y \log y - 1.2] - y[x \log x - 1.2]}{[y \log y - 1.2] - [x \log x - 1.2]}$$

$x_1 = 2.7404$; $f(x_1) = -1.904 \times 10^{-4}$

$x_2 = 2.7406$; $f(x_2) = -1.0139 \times 10^{-6}$

$x_3 = 2.7406$; $f(x_3) = -5.3995 \times 10^{-9}$

$x_4 = 2.7406$; $f(x_4) = -2.877 \times 10^{-11}$

03) Use the method of false position to find the 4th root of 32. Correct to 3 decimal place.

$x = (32)^{1/4}$

$x^4 = 32.$

$f(x) = x^4 - 32.$

$f(1) = -31$

$f(2) = -16$

$f(3) = 49$

$f(2.3) = -4.015$

$f(2.4) = 1.1775.$

$f(2.3) f(2.4) < 0$

$a = 2.3$

$b = 2.4$

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2.3 f(2.4) - 2.4 f(2.3)}{f(2.4) - f(2.3)}$$

$$x_1 = \frac{x[y^4 - 32] - y[x^4 - 32]}{[y^4 - 32] - [x^4 - 32]}$$

$x_1 = 2.3773$; $f(x_1) = -0.05855$

$x_2 = 2.3783$; $f(x_2) = -7.9193 \times 10^{-4}$

$x_3 = 2.3784$; $f(x_3) = -1.0700 \times 10^{-5}$

$x_4 = 2.3784$; $f(x_4) = -1.4456 \times 10^{-7}$

Newton Raphson Method

Let x_0 be the approximation root of $f(x)=0$

let h be the small correction then x_0+h will be exact root.

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \text{------} \rightarrow (1)$$

Since h is small so we neglect $h^2, h^3 ----$

x_0+h is exact root $f(x_0+h)=0$

Substitute in eq^n (1) we get

$$0 = f(x_0) + hf'(x_0)$$

$$h = \frac{-f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 + h$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\because [x_2 = x_1 + h]$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q.1) Using NR method find the real root of $x \log_{10} x = 1.2$ correct to four decimal places

→

$$f(x) = x \log_{10} x - 1.2 = 0 \rightarrow (1)$$

$$f(2) = -0.597$$

$$f(3) = 0.2315$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = \frac{x \log x}{\log_e 10} - 1.2 \quad \text{[change to base theorem]}$$

$$= x \log_{10} e \times \log_e x - 1.2$$

$$= x(0.43429) \times \log_e x - 1.2 \quad [\because \log_{10} e = 0.4343]$$

$$f(x) = x(0.4343) \log_e x - 1.2$$

$$f'(x) = 0.4343 \left(x \times \frac{1}{x} + \log x \cdot 1 \right)$$

$$f'(x) = 0.4343 (1 + \log x) \rightarrow (2)$$

If $|f(3)| < |f(2)|$

$$x_0 = 3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x - \frac{[x \log x - 1.2]}{[0.4343(1 + \log x)]}$$

$$x_1 = 2.746$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 3 - \frac{f(3)}{f'(3)}$$

$$x_1 = 2.7461.$$

$$x_2 = 2.7461 - \frac{f(2.7461)}{f'(2.7406)}$$

$$x_3 = 2.7446.$$

$$x_4 = 2.7470.$$

Q2) Find the real root of the equation $x \sin x + \cos x = 0$ which is near $x = \pi$ using Newton Raphsons method. Correct to four decimal places.

$\rightarrow$

$$f(x) = x \sin x + \cos x = 0$$

$$f'(x) = x \cos x + \cancel{\sin x} - \cancel{\sin x} = 0$$

$$= x \cos x.$$

$$x_0 = \pi$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = \pi - \frac{f(\pi)}{f'(\pi)}$$

$$x_1 = 2.8232.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.8232 - \frac{f(2.8232)}{f'(2.8232)}$$

$$x_2 = 2.7985$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7985 - \frac{f(2.7985)}{f'(2.7985)}$$

$$x_3 = 2.7983$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.7983 - \frac{f(2.7983)}{f'(2.7983)}$$

$$x_4 = 2.7983$$

The required root is 2.7983

03) $xe^x - 2 = 0$ Correct to four decimal places.

$f(x) = xe^x - 2 = 0$ $\qquad$ $f'(x) = xe^x + e^x$.

$f(0) = -2.$

$f(1) = 0.7182.$

$|f(1)| < |f(0)|$.

$\therefore x_0 = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{f(1)}{f'(1)}$$

$$x_1 = 0.8678$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.8678 - \frac{f(0.8678)}{f'(0.8678)}$$

$$x_2 = 0.8527$$

$$x_3 = x_2 - \frac{f(x_2)}{f(x_2)}$$

$$x_3 = 0.8527 - \frac{f(0.8527)}{f'(0.8527)}$$

$$x_3 = 0.8526.$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.8526 - \frac{f(0.8526)}{f'(0.8526)}$$

$$x_4 = 0.8526$$

The required root is 0.8526.

04) $f(x) = 3x - \cos x - 1$ Correct to 3 decimal places

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

$$f(0) = -2$$

$$f(1) = 1.4596$$

$$x_0 = 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{f(1)}{f'(1)}$$

$$x_1 = 0.6200$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.6071.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.6071$$

So The required root is 0.6071 //

HW

Q1) Using regular falsi method find the real root of the eqn $xe^x = \sin x$ correct to 4 decimal places.

→ $f(x) = \sin x - xe^x$

$f(0) = 0$ $\qquad$ $f(-3) = 8.1163$

$f(1) = -1.876.$ $\qquad$ $f(-2.9) = -0.079$

$f(0) \cdot f(1) < 0$

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{(-3) \cdot f(-2.9) - (2.9)f(-3)}{f(-2.9) - f(-3)} = \frac{(-3)f(-2.9) - f(-3)(-2.9)}{f(-2.9) - f(-3)}$$

$$x_1 = \frac{x(\sin y - ye^y) - y(\sin x - xe^x)}{(\sin y - ye^y) - (\sin x - xe^x)}$$

$x_1 = -2.9906$

$f(x_1) = -9.633 \times 10^{-5}$

$x_2 = -2.9907$

$f(x_2) = 1.1438 \times 10^{-6}$

$x_3 = -2.9907$

$f(x_3) = -1.3579 \times 10^{-8}$

$x_4 = -2.9907$

$f(x_4) = 1.61209 \times 10^{-10}$

The required root is -2.9907

02) By using NR method $xe^x = \sin x$ Correct to 4 decimal places.

→ $f(x) = \sin x - xe^x$ ⇒ $f'(x) = \cos x - xe^x - e^x$

$f(-3) = 8.2163$

$f(-2.9) = -0.079$

$|f(-3)| > |f(-2.9)|$

$x_0 = -3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_1 = -2.9907$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$x_2 = -2.9907$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$x_3 = -2.9907$

The required root is -2.9907

hence the root lies in between $(-3, -2.9)$ //

Fixed Point itteration

* Value should be in such a way that - $|\phi(x)|$ should less then 1.

let $f(x) = 0$ be the given eqn

let us write this equation in the form

$x = \phi(x) \longrightarrow (1)$

let x_0 be the initial approximation value to the actual root α & substituting

$x = x_0$ in RHS of (1) we get

$x_1 = \phi(x_0) \longrightarrow (2)$

Again put $x = x_1$ in (2) we get

$x_2 = \phi(x_1)$

$x_n = \phi(x_{n-1})$

The sequence of approximate roots $x_1, x_{n-1} \cdots x_n$

If it Converges to α is taken as the root of the eqn $f(x) = 0$

Note:-

01) Smaller the $\phi'(x)$, more rapid will be the Convergence

02) The sufficient condition for the Convergence is $|\phi'(x)| < 1$.

$\forall x$ in the interval I (containing root $x = \alpha$)

[$f(x) = 0$ Can be algebraically expressed as $x = \phi(x)$

A point says α is fixed point if it satisfies $x = \phi(x)$]

Q1) Find the root of the equation $x^2+x-1=0$.
using the fixed point iteration

→ $f(x) = x^2+x-1$

$f(0) = -1$

$f(1) = 1$

$f(x) = x^2+x-1$

$\cancel{f'(x)} = \cancel{2x}+1$

$\cancel{f'(1) = 2+1 = 3}$

$x^2+x-1=0$

$x(x+1)-1=0$

$\phi(x) = x = \left(\frac{1}{x+1}\right)$

$\phi(x) = \frac{1}{x+(1)}$

$\phi'(x) = \frac{-1}{(x+1)^2}$

put $x=1$

$\frac{-\cancel{1}}{(2)^2} < 1$

Condition satisfied

$-0.25 < 0$

$\phi(x) = x^2 + x - x \cdot \frac{1}{x+1}$

$f(0.6) = -0.04$

$f(0.7) = 0.15$

$\phi'(x) = \frac{-1}{(x+1)^2}$

$= \frac{-1}{(0.7+1)^2}$

$= -0.346 < 1$

So $x_0 = 0.7$

$x_1 = \phi(x) = \phi(0.7) = 0.5882$

$x_2 = \phi(0.5882) = 0.6296$

$x_3 = \phi(0.6296) = 0.6136$

$x_4 = \phi(0.6136) = 0.6197$

$x_5 = \phi(0.6197) = 0.6173$

$x_6 = \phi(0.6173) = 0.6182$

$x_7 = \phi(0.6182) = 0.6179$

$x_8 = \phi(0.6179) = 0.6180$

$x_9 = \phi(0.6180) = 0.6180$

The required root is 0.6180

02) Find the root of $x = \frac{1}{2} + \sin x$ by fixed point itteration method.

$$f(x) = \frac{1}{2} + \sin x - x$$

$$x = \frac{1}{2} + \sin x.$$

$$\phi(x) = \frac{1}{2} + \sin x.$$

$$f(x) = \frac{1}{2} + \sin x - x.$$

roots are

$f(1.4) = 0.0854$

$f(1.5) = -2.163.$

$\phi'(x) = \cos x.$

So $x = 1.4$

$\cos(1.4) = 0.9997$

$0.9997 < 1$

So Condition satisfied.

$x_1 = \phi(1.4) = 0.5244$

$x_2 = \phi(0.5244) = 1.4854$

$x_3 = \phi(1.4854) = 1.4963$

$x_4 = \phi(1.4963) = 1.4972$

$x_5 = \phi(1.4972) = 1.4972.$

So the required root is 1.4972

03) Find the root of the equation $x^3 - 9x + 1$ upto 3 decimal

→ $f(x) = x^3 - 9x + 1$

$f(0.1) = 0.101$

$f(0.2) = -0.792$

$x^3 - 9x + 1 = 0$

$9x = x^3 + 1$

$$x = \frac{x^3 + 1}{9}$$

$$\phi(x) = \frac{x^3+1}{9}$$

$$\phi'(x) = \frac{\cancel{3}x^2}{\cancel{9}\,3} = \frac{x^2}{3}$$

put $x = 0.1$

$$\phi'(0.1) = \frac{(0.1)^2}{3}$$

$= 0.0033 < 1$

Condition satisfied.

$x_1 = \phi(0.1) = 0.1112$

$x_2 = \phi(0.1112) = 0.1126$

$x_3 = \phi(0.1126) = 0.11126$

$x_4 = \phi(0.11126) = 0.11126$

Hence the required root is 0.1112.

* ## Taylars Serie Method

Consider the initial value problem

$$\frac{dy}{dx} = F(x,y) \text{ \& } y(x_0) = y_0$$

The solution $y(x)$ is approximation to a power series in $(x-x_0)$ using Taylars series. then we can find the value of y for various values of x in nighbourhood of x. we have Taylar's series expansion of $y(x)$ about the point x_0 in the form.

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) +$$

$$\frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots\cdots$$

Hence $y'(x_0), y''(x_0) \cdots\cdots$ denote the values of the derivatives dy/dx, $d^2y/dx^2 \cdots\cdots$ at x_0 which can be found by making use of data.

* Working Rule:-

i) Given equation $\frac{dy}{dx} = y'$

$y(x_0) = y_0$, initial approximation

ii) Substitute in taylar series decring on degree take $y', y'', y''' \cdots$

iii) Find y', y'', y''', y^{iv} using Taylar Series.

we get the result.

05) Use Taylor's series method then find y at $x = 0.1, 0.2, 0.3$ Considering terms upto 3rd degree given that $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$

→ Taylor Series expansion of $y(x)$ is given by $y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \dots$

data; $x_0 = 0$, $y_0 = 1$ & $y' = x^2 + y^2$

$$y(x) = y(0) + x\, y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0).$$

Consider $y' = x^2 + y^2$

$$y'(0) = 0^2 + [y(0)]^2$$

$$= 0 + 1 = 1$$

$$y'' = 2x + 2yy'$$

$$y''(0) = 2(0) + 2\, y(0)\, y'(0)$$

$$= 0 + 2(1)(1)$$

$$= 2$$

$$y''' = 2 + 2\,[yy'' + (y')^2]$$

$$y'''(0) = 2 + 2\,[y(0)\, y''(0) + (y'(0))^2]$$

$$= 2 + 2\,[1(2) + 1^2]$$

$$= 2 + 2\,[3]$$

$$= 8$$

Substituting in Taylor Series.

$$y(x) = 1 + x + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(8)$$

$$y(x) = 1 + x + x^2 + 4x^3/3$$

$$y(0.1) = 1.1113$$

$$y(0.2) = 1.2506$$

$$y(0.3) = 1.426$$

Q2) Employ Taylors method to obtain the approximate value of y at $x = 0.2$ for $\frac{dy}{dx} = 2y + 3e^x$, $y(0) = 1$.

Sol:- $y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0)$ ....

by data; $x_0 = 0$; $y(0) = 1$.

$y' = 2y + 3e^x$.

$y(x) = y(0) + x y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \ldots$

Consider $y' = 2y + 3e^x$

$y'(0) = 2(y_0) + 3e^0$

$= 2 + 3 = 5$

$y'' = 2y' + 3e^x$

$y''(0) = 2(y'(0)) + 3e^0$

$= 2(5) + 3 = 13.$

$y''' = 2y'' + 3e^x$

$y'''(0) = 2y''(0) + 3e^0$

$= 2(13) + 3$

$= 29$

Substitute in Taylors Series

$y(x) = 1 + x(5) + \frac{x^2}{2!}(13) + \frac{x^3}{3!}(29)$

$= 1 + 5x + \frac{13x^2}{2} + \frac{29x^3}{6}$

$y(0.2) = 1 + (0.2)5 + \frac{13}{2}(0.2)^2 + \frac{29}{6}(0.2)^3$

$y(0.2) = 2.2986.$

03) Find an approximate value of y when $x = 0.1$ if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$ using Taylors Series

→ let $y' = x - y^2$, $y_0 = 1$, $x_0 = 0$

$$y(0.1) = y(0) + (0.1)\,y'(0) + \frac{(0.1)^2}{2!}y''(0) + \frac{(0.1)^3}{3!}y'''(0)$$

$$y' = x - y^2$$

$$y'(x) = x - y^2$$

$$y'(0) = 0 - 1^2 = -1$$

$$y''(x) = 1 - 2yy'$$

$$y''(0) = 1 - 2(1)(-1)$$

$$= 1 + 2$$

$$= 3.$$

$$y'''(x) = -2\left[yy'' + (y')^2\right]$$

$$= -2\left[(1)(3) + (-1)^2\right]$$

$$= -2\left[3 + 1\right]$$

$$= -8$$

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8)$$

$$y(0.1) = 1 - 0.1 + \frac{3(0.1)^2}{2} - \frac{(8)^4 \times (0.1)^3}{6_3}$$

$$y(0.1) = 1 - 0.1 + 0.015 - 0.001$$

$$y(0.1) = 1.015 - 0.101$$

$$y(0.1) = 0.914$$

04) Find y at $x = 1.02$ Correct to 5 decimal places given $dy = (xy-1)dx$ and $y=2$ at $x=1$. applying Taylors Series.

→ $x_0 = 1$, $y_0 = 2$, $y' = xy - 1$

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \frac{(x-x_0)^3}{3!}\,y'''(x_0)$$

$$y(1.02) = y(1) + (1.02-1)\,y'(1) + \frac{(1.02-1)^2}{2!}\,y''(1) + \frac{(1.02-1)^3}{3!}\,y'''(1)$$

Consider; $y' = xy - 1$.

$$y'(1) = (1)(2) - 1$$
$$= 1.$$

$$y''(1) = xy' + y(1)$$
$$= (1)(1) + 2(1)$$
$$= 3.$$

$$y'''(1) = xy'' + y'(1) + y'(1)$$
$$= (1)(3) + (1)(1) + (1)(1)$$
$$= 3 + 1 + 1$$
$$= 5.$$

$$y(1.02) = 2 + (0.02)(1) + \frac{(0.02)^2}{2!} \times (3) + \frac{(0.02)^3}{3 \times 2} \times (5)$$

$$= 2 + 0.02 + 0.0006 + 0.00000666.$$

$$= 2 + 0.02 + 0.0006 + 0.000006.$$

$$= 2.0206$$

$$y(1.02) = 2.0206$$

05) $\frac{dy}{dx} = x - y$, $y(0) = 1$, find $y(0.1)$ upto 4th degree.

→ $x_0 = 0$; $y_0 = 1$, $y' = x - y$

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) \rightarrow (1)$$

Consider; $y' = x - y$

$y'(0) = 0 - 1 = -1$

$y'' = 1 - y'$

$y''(0) = 1 - (-1)$

$= 2.$

$y''' = -y''$

$y'''(0) = -2.$

$y'''(0) = -2.$

Substitue the values in eqn (1).

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0)$$

$$= 1 + x(-1) + \frac{x^2}{\cancel{2}}(\cancel{2}) + \frac{x^3}{3 \times \cancel{2}}(-\cancel{2})$$

$$= 1 - x + x^2 - \frac{x^3}{3}$$

$$y(0.1) = 1 - 0.1 + (0.1)^2 - \frac{(0.1)^3}{3}$$

$= 0.9096$

$y(0.1) = 0.9096$ //

06) $y' = x^2y - 1$, $y(0) = 1$ find $y(0.03)$ upto 2nd degree.

→ Taylors Series:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) - - -$$

Given data;

$x_0 = 0$

$y_0 = 1$.

Consider; $y' = x^2y - 1$

$y'(0) = (0)1 - 1$

$= -1$.

$y'' = 2xy + x^2y'$

$= 2(0)(1) + 0^2(-1)$

$= 0$

Substitute the values we get.

$$y(x) = y(0) + x \times y'(0) + \frac{x^2}{2!} \times y''(0)$$

$$y(x) = 1 + x(-1) + \frac{x^2}{2!}(0)$$

$= 1 - x + 0$

$= 1 - x$

$y(0.03) = 1 - 0.03$

$= 0.97$

$y(0.03) = 0.97$ //

* Euler's Method

Consider the initial value problem

$$\frac{dy}{dx} = f(x,y);$$

$$y(x_0) = y_0$$

We need to find y at $x_1 = x + h$ we first obtain $y(x_1) = y_1$ by applying Euler's formula and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$ also called as the predicted value of y_1. Euler's formula is given by

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

Problems :-

Q1) Using Euler's method find an approximate value of 'y' corresponding to $x = 1$ given that $\frac{dy}{dx} = x+y$ and $y = 1$ when $x = 0$ take $h = 0.1$?

→ Let us take $n = 10$.

x	y	$y' = x + y$	$y_{n+1} = y_n + hf(x_n, y_n)$
x_0 0	y_0 1	$y'_0 = 0 + 1 = 1$	$y_{0+1} = y_0 + (0.1) f(x_0, y_0)$ $y_1 = 1 + (0.1)(0+1) = 1.1$
x_1 0.1	y_1 1.22	$y'_1 = x_1 + y_1$ $= 1.2$	$y_2 = y_1 + (0.1)(x_1 + y_1)$ $= 1.1 + 0.1\,(1.2)$ $= 1.22$
x_2 0.2	y_2 1.22	$y'_2 = x_2 + y_2$ $= 1.42$	$y_3 = y_2 + (0.1)(x_2 + y_2)$ $= 1.36$.
x_3 0.3	y_3 1.36	$y'_3 = x_3 + y_3$ $= 1.66$	$y_4 = 1.53$
x_4 0.4	y_4 1.53	$y'_4 = x_4 + y_4$ $= 1.93$	$y_5 = 1.72$.
x_5 0.5	y_5 1.72	$y'_5 = \cancel{x_4} + y_5$ $= 2.22$	$y_6 = 1.94$
x_6 0.6	y_6 1.94	$y'_6 = 2.54$	$y_7 = 2.19$
x_7 0.7	y_7 2.19	$y'_7 = 2.89$	$y_8 = 2.48$
x_8 0.8	y_8 2.48	$y'_8 = 3.89$	$y_9 = 2.81$
x_9 0.9	y_9 2.81	$y'_9 = 3.79$	$y_{10} = 3.18$
x_{10} 1.	y_{10} 3.18	$y'_{10} = 3.71$	

$$\boxed{y(1) = 3.18}$$

Modified Eulers Method.

Consider the initial value problem

$\frac{dy}{dx} = F(x,y)$, $y(x_0) = y_0$ we need to

and y at $x_1 = x_0 h$

We first obtain $y(x_1) = y_1$ by applying Eulers formula

i.e. $y_1^{(0)} = y_0 + hf(x_0, y_0)$

To the designed degree of accuracy we use fall. modified Eulers formula where the successive approximations are denoted by.

$y_1^{(0)}, y_2^{(1)}, y_3^{(3)}$ ------ etc

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

Q1) Using Eulers modified method obtain a solution of equation $\frac{dy}{dx} = \log(x+y)$ $y(0) = 2$. Find y at $x = 0.2$ with $h = 0.2$.

→ let $y_0 = 2$, $x_0 = 0$, $x_1 = 0.2$, $h = 0.2$.

by Eulers formula

$y_1^{(0)} = y_0 + hf(x_0\ y_0) = 2 + (0.2)(\log(x+y))$

$= 2 + (0.2)\log 2.$

$= 2.060$

By Modified formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= y_0 + \frac{h}{2}\left[\log(x_0 + y_0) + \log(x_0 + y_1^{(0)})\right]$$

$$= 2 + \frac{0.2}{2}\left[\log(0+2) + \log(0.2 + 2.0602)\right]$$

$$y_1^{(1)} = 2.0655$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= y_0 + \frac{h}{2}\left[\log(0+2) + \log(0.2 + 2.0655)\right]$$

$$= 2 + \frac{0.2}{2}\left[\log(0+2) + \log(2.2655)\right]$$

$$y_1^{(2)} = 2.0656$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= y_0 + \frac{h}{2}\left(f(x_0 + y_0) + f(x_1 + y_1^{(2)})\right]$$

$$= y_0 + \frac{h}{2}\left[\log(x_0 + y_0) + \log(x_1 + y_1^{(2)})\right]$$

$$= 2 + \frac{0.2}{2}\left[\log(0+2) + \log(0.2 + 2.0656)\right]$$

$$y_1^{(3)} = 2.06562$$

y at $x = 0.4$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[\log(x_0+y_0) + \log(x_0 + y_1^{(0)})\right]$$

$$= 2 + \frac{0.2}{2}\left[\log 2 + \log(0.4 + 2.0602)\right]$$

$$y_1^{(1)} = 2.0692.$$

$$y_1^{(2)} = 2.0693.$$

$$y_1^{(3)} = 2.06936.$$

y at $x = 0.6$

$$y_1^{(1)} = 2.0726.$$

$$y_1^{(2)} = 2.0728$$

$$y_1^{(3)} = 2.0729.$$

Q2) Using Euler modified method. obtain the solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with initial conditions.

$y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2.

$0 \le x \le 0.6$

x at 0

x at 0.2.

x at 0.4

x at 0.6.

→ let $y_0 = 1$

$x_0 = 0.$

$h = 0.2.$

$$y_1^{(0)} = y_0 + hf(x_0\ y_0)$$

$$= 1 + 0.2f[x_0 + y_0]$$

$$= 1 + 0.2f\left[0 + |\sqrt{1}|\right]$$

$$= 1 + 0.2\,[1]$$

x	$y' = x + \lvert\sqrt{y}\rvert$	Mean slope	~~old~~ $y_0 + ($ … $)$
			$= 1 + (1)(0.2) = 1.2$
0	$= 0 + 1 = 1$	1	
0.2	$= 0.2 + \lvert\sqrt{1.2}\rvert$ $= 1.295$	$= \frac{1}{2}(1 + 1.2954)$ $= 1.1477$	$= 1 + (1.1477)(0.2)$ $= 1.2295$
0.2	$= 0.2 + \lvert\sqrt{1.2295}\rvert$ $= 1.3088$	$= \frac{1}{2}(1 + 1.3088)$ $= 1.1544$	$= 1 + (1.1544)(0.2)$ $= 1.2309$
0.2	$= 0.2 + \lvert\sqrt{1.2309}\rvert$ $= 1.3094$	$= \frac{1}{2}(1 + 1.3094)$ $= 1.1547$	$= 1 + (1.1547)(0.2)$ $= 1.2309$
0.2	$= 1.3094$	$= 1.3094$	$= 1.2309 + 0.2(1.309$ $= 1.4927$
0.4	$= 0.4 + \lvert\sqrt{1.4927}\rvert$ $= 1.6218$	$= \frac{1}{2}(1.3094 + 1.6218)$ $= 1.4654$	$= 1.2309 + 0.2(1.46$ $= 1.5240$
0.4	$= 0.4 + \lvert\sqrt{1.5240}\rvert$ $= 1.6345$	$= \frac{1}{2}(1.6345 + 1.3094)$ $= 1.4718$	$= 1.2309 + 0.2(1.47$ $= 1.5253$
0.4	$= 0.4 + \lvert\sqrt{1.5253}\rvert$ $= 1.6350$	$= \frac{1}{2}[1.3094 + 1.6345]$ $= 1.4721$	$= 1.2309 + 0.2(1.472$ $= 1.5253$
0.4	$= 1.6350$	$= 1.6350$	$= 1.5253 + 0.2(1.63$ $= 1.8523$
0.6	$= 0.6 + \lvert\sqrt{1.8523}\rvert$ $= 1.9610$	$= \frac{1}{2}(1.6360 + 1.9610)$ $= 1.798$	$= 1.5253 + 0.2(1.798)$ $= 1.8849$
0.6	$= 0.6 + \lvert\sqrt{1.8849}\rvert$ $= 1.9729$	$= \frac{1}{2}(1.6360 + 1.9729)$ $= 1.8040$	$= 1.5253 + 0.2(1.804)$ $= 1.8861$
0.6	$= 0.6 + \sqrt{1.8861}$ $= 1.9734$	$= \frac{1}{2}(1.6360 + 1.9734)$ $= 1.8042$	$= 1.5253 + 0.2(1.8042$ $= 1.8861$

$\therefore\ y(0.2) = 1.2309$

$y(0.4) = 1.5253$

$y(0.6) = 1.8861$ //

02) Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler method.

→ Let $n = 5$ & $h = 0.02$

x	y	$\frac{dy}{dx} = \frac{y-x}{y+x}$	New y = old $y + 0.02\left(\frac{dy}{dx}\right)$
0	1	$= \frac{1-0}{1+0} = 1$	$= 1 + 0.02(1) = 1.02$
0.02	1.02	$= \frac{1.02-0.02}{1.02+0.02} = 0.9615$	$= 1.02 + 0.02(0.9615) = 1.0392$
0.04	1.0392	$= \frac{1.0392-0.04}{1.0392+0.04} = 0.926$	$= 1.0392 + 0.04(0.962) = 1.0577$
0.06	1.0577	$= \frac{1.0577-0.06}{1.0577+0.06} = 0.893$	$= 1.0577 + 0.06(0.893) = 1.0796$
0.08	1.0796	$= \frac{1.0796-0.08}{1.0796+0.08} = 0.862$	$= 1.0796 + 0.08(0.862) = 1.0928$
0.1	1.0928		

$\therefore y(0.1) = 1.0928$

Runge - Kutta Method of fourth Order

[RK Method]

* Consider an initial value problem

$\frac{dy}{dx} = f(x,y)$. $y(x_0) = y_0$ we find $y(x_0+h)$

where h is step length

We Compute :- $K_1 = hf(x_0\ y_0)$

$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2)$

$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2)$

$K_4 = hf(x_0 + h,\ y_0 + K_3)$

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

Q1) Apply RK method to find an approximate value of y where $x = 2$ find $y(0.2)$ given that $\frac{dy}{dx} = x + y^2$ and $y = 1$ at $x = 0$; $h = 0.1$

→ $\frac{dy}{dx} = x + y^2$; $x_0 = 0$; $y_0 = 1$; $h = 0.1$

* $K_1 = hf(x_0, y_0)$

$= h[x_0 + y_0]$

$= 0.1[0 + 1]$

$= 0.1.$

* $K_2 = hf[x_0 + h/2,\ y_0 + K_1/2]$

$= h[(x_0 + h/2) + (y_0 + K_1/2)]$

$= 0.1[(0 + 0.1/2) + (1 + 0.1/2)^2]$

$= 0.1[0.05 + (1.05)^2]$

$= 0.1[0.05 + (1.05)^2]$

$= 0.11525$

$K_3 = hf(x_0 + h/2, y_0 + h/2)$

$= 0.1 \left[(x_0 + h/2) + (y_0 + K_2/2)^2 \right]$

$= 0.1 \left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1152}{2}\right)^2 \right]$

$= 0.1 \left[0.05 + 1.1185 \right]$

$K_3 = 0.1168$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$K_4 = 0.1 \left[(0 + 0.1) + (1 + 0.1168)^2 \right]$

$= 0.1 \left[0.1 + 1.247 \right]$

$K_4 = 0.1347$

$y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y(0 + 0.1) = 1 + \frac{1}{6} \left[0.1 + 2(0.1152) + 2(0.1168) + 0.1347 \right]$

$y(0.1) = 1.1335$

$y_1 = 1.1335$; $x_1 = 0.1$; $h = 0.1$; $f(x, y) = x + y^2$

$K_1 = hf(x_1, y_1)$

$= h \left[x_1 + y_1^2 \right]$

$= 0.1 \left[0.1 + (1.1335)^2 \right]$

$K_1 = 0.1384$

$K_2 = hf \left[x_1 + h/2, y_1 + K_1/2 \right]$

$= 0.1 \left[(0.1 + 0.1/2) + (1.1335 + 0.1384/2)^2 \right]$

$K_2 = 0.1596$

$k_3 = hf[x_0 + h/2,\ y_0 + k_2/2]$

$= 0.1\,[(0.1 + 0.1/2) + (1.1335 + 0.1596/2)^2]$

$k_3 = 0.1622$

$k_4 = hf[x_0 + h,\ y_0 + k_3]$

$= 0.1\,[(0.1 + 0.1) + (1.1335 + 0.1622)^2]$

$k_4 = 0.1878$

$y(x_1 + h) = y_1 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.1 + 0.1) = 1.1335 + \frac{1}{6}\,[0.1384 + 2(0.1596) + 2(0.1622) + 0.1878]$

$y(0.2) = 1.2951$

02) Apply RK method to find an approximate value of y when $x = 0.2$. given that $\frac{dy}{dx} = x + y$ & $y = 1$ when $x = 0$.

→ $f(x, y) = x + y;\ y_0 = 1;\ x_0 = 0;\ h = 0.2.$

$k_1 = hf(x_0\, y_0) = 0.2\,[0 + 1] = 0.2$

$k_2 = hf[(x_0 + h/2), (y_0 + k_1/2)] = 0.2 f(0.1 + 1.1) = 0.24$

$k_3 = hf[(x_0 + h/2), (y_0 + k_2/2)] = 0.2 f(0.1 + 1.2) = 0.244$

$k_4 = hf[(x_0 + h), (y_0 + k_3)] = 0.2 f(0.2 + 1.244) = 0.2888$

$y(x_0 + h) = y_0 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.2) = 1 + \frac{1}{6}\,[0.2 + 2(0.24) + 2(0.244) + 0.2888]$

$y(0.2) = 1.2428$

03) Using RK method solve $\frac{dy}{dx} = 2x - y$ at $x = 1.1$

given that $y = 3$ at $x = 1$

$\rightarrow$ $h = x_1 - x_0$

$h = 1.1 - 1 = 0.1$.

We have $\frac{dy}{dx} = 2x - y$, $h = 0.1$; $x_0 = 1$

Compute k_1, k_2, k_3, k_4 we get

$k_1 = hf(x_0, y_0) = 0.1 [2(x_0) + y_0]$

$= 0.1 [2(1) + (-3)] = -0.1$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$= 0.1 f [2(1 + 0.1/2) - (3 - 0.1/2)]$

$= -0.085$

$k_3 = hf [x_0 + h/2, y_0 + k_2/2]$

$= 0.1 f [2(1 + 0.1/2) - (3 - 0.085/2)]$

$= -0.08575$

$k_4 = hf [x_0 + h, y_0 + k_3]$

$= 0.1 f [2(1 + 0.1) - (3 - 0.08575)]$

$= -0.071425$

$y(x_0 + h) = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.1 + 1) = 3 + \frac{1}{6} [(-0.1) + 2(-0.085) + 2(-0.08575) + (-0.0714)]$

$y(1.1) = 2.91451$ //

04) Solve using RK method $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^3}$ for $x = 0.2, 0.4$ given that $y = 1$ at $x = 0$

→ We have $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0$; $y_0 = 1$ $h = 0.2$

$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$; Compute K_1, K_2, K_3, K_4

$K_1 = h f(x_0, y_0)$ $= 0.2 \left[\frac{1 - 0}{1 + 0}\right] = 0.2$

$= 0.2 f(0 + 1) = 0.2$

$K_2 = hf((x_0 + h/2), (y_0 + K_1/2))$

$= \cancel{0.2} f \left[\cancel{(0 + 0.2/2)} + \cancel{(1 + 0.2/2)}\right]$

$$K_2 = 0.2 \left[\frac{(y_0 + K_1/2)^2 - (x_0 + h/2)^2}{(y_0 + K_1/2)^2 + (x_0 + h/2)^2}\right]$$

$$= 0.2 \left[\frac{(1 + 0.2/2)^2 - (0 + 0.2/2)^2}{(1 + 0.2/2)^2 + (0 + 0.2/2)^2}\right]$$

$= 0.196$

$K_3 = hf(x_0 + h/2, y_0 + K_2/h)$

$= 0.1967$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1891$

$y(0.2) = 1 + \frac{1}{6} [0.2 + 2(0.196) + 2(0.1967) + 0.1891]$

$y(0.2) = 1.196$

Step II $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0.2$; $h = 0.2$; $y_0 = 1.196$

$K_1 = 0.1891$; $K_2 = 0.1795$; $K_3 = 0.1793$; $K_4 = 0.1688$

$y(0.4) = 1.3753$ //.