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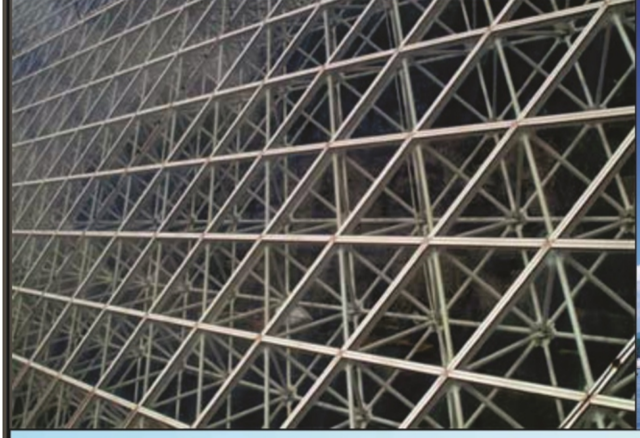
# STRENGTH OF MATERIAL




K.S.YADAV

STANDARD BOOK HOUSE

# STRENGTH OF MATERIALS

**[S.I. UNITS]**

(*A textbook for all Engineering Branches, Competitive Examination, ICS, and AMIE Examinations*)

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*Published by:*
**RAJINDER KUMAR JAIN**
Standard Book House
*Unit of:* Rajsons Publications Pvt. Ltd.
1705-A, Nai Sarak, Delhi - 110006
Post Box: 1074
Ph.: +91-(011)-23265506 Fax: +91-(011)-23250212

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Ph.: +91-(011)-43551085, +91-(011)-43551185
E-mail: sbh10@ hotmail.com
www.standardbookhouse.in

**First Edition : 2016**
**Second Edition : 2018**



*Price:* **₹ 280.00**

**ISBN: 978-81-89401-50-4**

*Typeset by:*
C.S.M.S. Computers, Delhi.

*Printed by:*
R.K. Print Media Company, New Delhi

# *Contents*

# Simple Stresses and Strains

CHAPTER 1

## 1.1 ❑ STRENGTH OF MATERIALS

It is defined as the strength of members and behaviour of materials when subjected to external forces.

## 1.2 ❑ STRESS

It is internal resistance set up per unit area due to the application of external forces. It is ratio of force ($P$) to cross-sectional area ($A$) and denoted by σ (Greek letter Sigma)

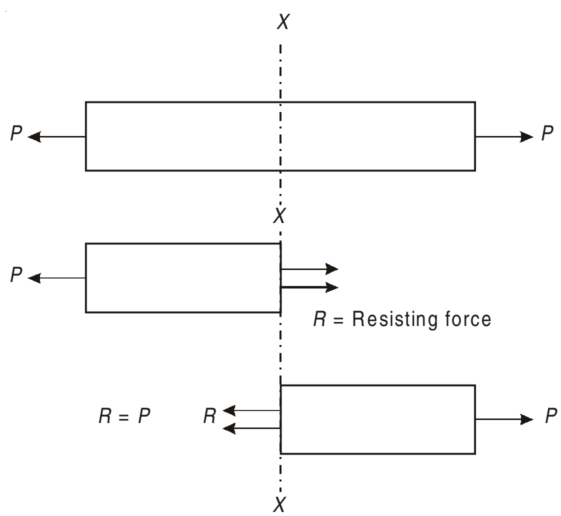



**Fig. 1.1**

Hence, $\sigma = P/A$ ...(i)

where,

$\sigma$ = Stress

$P$ = Force acting on the body, and

$A$ = Cross-sectional area of the body

In S.I. system, the unit of stress is Pascal (Pa) which is equal to 1 N/m$^2$. Bigger units of stress are kilo Pascal (kPa) Mega Pascal (MPa) and Giga Pascal (GPa).

$$1 \text{ KPa} = 10^3 \text{ Pa} = 10^3 \text{ N/m}^2 = 1 \text{ kN/m}^2$$

$$1 \text{ MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/m}^2 = \frac{10^6}{10^6} = 1\text{N/mm}^2$$

$$1 \text{ GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/m}^2 = \frac{10^9}{10^6}$$

$$= 10^3 \text{ N/mm}^2 = 1 \text{ kN/mm}^2$$

## 1.3 ❑ STRAIN

Whenever a force or a system of forces acts on a body, it undergoes some deformation. This deformation per unit length is known as *strain*. Hence, strain is defined as the ratio of change in length to original length of the member.

$$\therefore \qquad \text{Strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{\delta l}{l}$$

Mathematically, it is defined as deformation per unit length. It is denoted by Greek letter Epsilon ($\varepsilon$).

$$\therefore \qquad \varepsilon = \frac{\delta l}{l}$$

### 1.3.1 Tensile Stress

When the section is subjected to two equal and opposite pulls ($P$, $P$) and the body

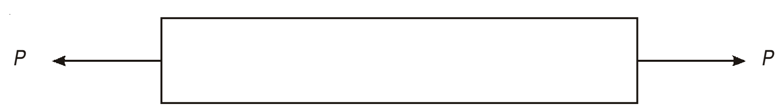



**Fig. 1.2**

tends to increase its length (Fig. 1.2), the stress induced is called *Tensile Stress* and the corresponding strain is called *Tensile Strain*. Thus, tensile stress $\sigma = \frac{P}{A}$.

### 1.3.2 Compressive Stress

When the section is subjected to two equal and opposite pushes ($P$, $P$) and the body tends to shorten its length (Fig. 1.3), the stress induced is called *compressive stress*. As a result of compressive stress the cross-sectional area of the body gets increased. Thus, compressive stress, $\sigma_c = \dfrac{P}{A}$.

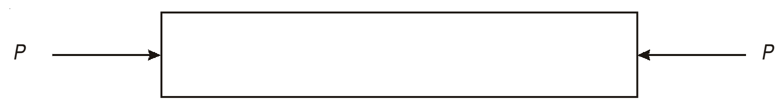



**Fig. 1.3**

Tensile and compressive stresses are known as direct stresses.

### 1.3.3 Shear Stress (Tangenital Stress)

It exists between two parts of a body in contact when each part exerts equal and opposite force on each other in a direction tangential to their surface of contact. Figure 1.3 (a) shows a section of a rivet subjected to equal and opposite forces $P$, $P$ causing sliding

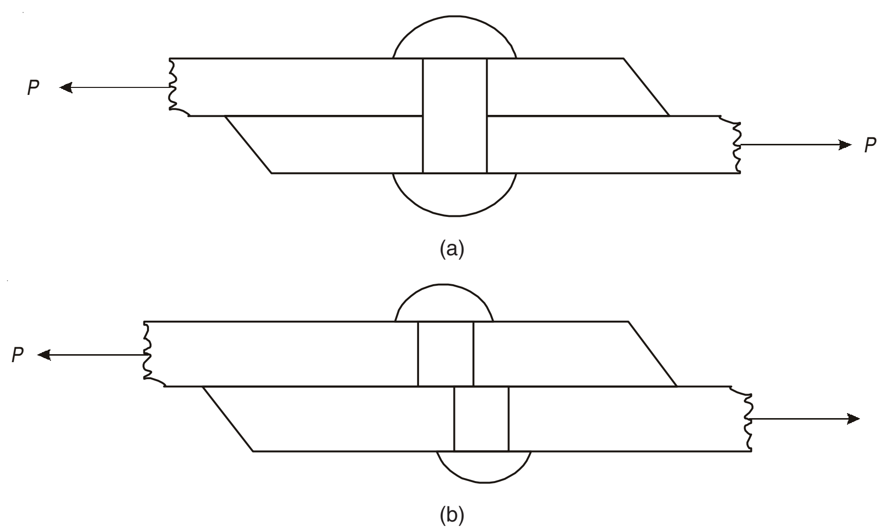



**Fig. 1.4**

of the plates by shearing off the rivet section, one over the other. From Fig. 1.4, it is clear that the resisting force of the rivet must be equal to $P$. Hence shearing stress $\tau$ is given by,

$$\tau = \frac{\text{Tangential force}}{\text{Area of cross section of rivet}} = \frac{P}{A}$$

where,

$\tau =$ Shear stress

Tensile and compressive stresses are known as *Direct stresses* and shearing stress as *Tangential stress*.

### 1.3.4 Tensile Strain

It is the ratio of increase in length to the original length of the member. It is considered as a positive strain.

$$\text{Tensile strain } (\varepsilon_t) = \frac{\text{Increase in length}}{\text{Original length}} = \frac{\delta l}{l}$$

### 1.3.5 Compressive Strain

It is the ratio of decrease in length to the original length of a member. It is considered as a *Negative strain*

$$\therefore \quad \text{Compressive strain} = \frac{\text{Decrease in length}}{\text{Original length}} = \frac{\delta l}{l}$$

### 1.3.6 Normal Strains

Tensile and compressive strains are called *Normal strains* because these are associated with normal stresses. Normal strain is the ratio of two lengths so it is a dimension less quantity i.e., it has no units.

### 1.3.7 Complementry Stress

Whenever a shear stress $\tau$ is applied on a body, an equal and opposite shear stress $\tau'$ acts in a direction perpendicular to the direction of $\tau$ so as to maintain equilibrium

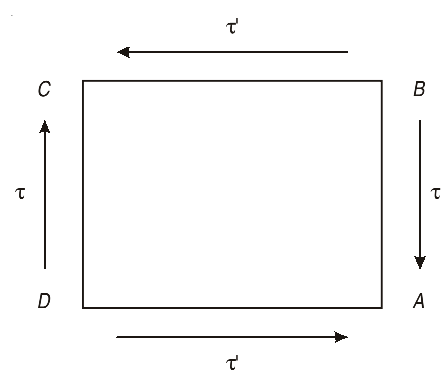



**Fig. 1.5**

of the body. These stresses will form a couple. The couple formed due to shear stress $\tau$ produces clockwise moment. For equilibrium, this applied couple is balanced by an another couple due to internal resisting stress $\tau'$ developed in the body. This resisting shear stress $\tau'$ is called as complementary shear stress. If the applied shear stress is positive in nature then the complementary shear stress will be negative. $\tau = \tau'$.

### 1.3.8 Shear Strain

Consider a cube of length $l$, fixed at the bottom face $AB$. Let force $P$ be applied at the face $CD$, tangentially to the face $AB$. As a result of force, let the cube be

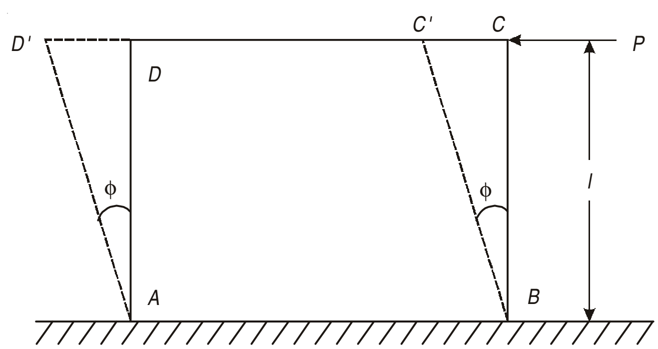



**Fig. 1.6**

distorted from $ABCD$ to $ABC'D'$ through an angle $\phi$ (Fig. 1.6). We know that, shear strain

$$(\varepsilon_s) = \frac{\text{Deformation}}{\text{Original length}}$$

$$\phi = \frac{cc'}{l}$$

and Shear stress $\tau$ $= \dfrac{P}{AB} = \dfrac{P}{l}$ (tan $\phi = \phi$ if $\phi$ = very small)

## 1.4 ❑ MODULUS OF RIGIDITY

The ratio of shear stress to the shear strain is called *Modulus of Rigidity* or *Modulus of Elasticity* in shear. It is denoted by $G$ or $C$ and is constant for a given material,

$$\therefore \qquad G = \frac{\tau}{\phi}$$

or

$$\tau = G\,\phi$$

## 1.5 ❑ ELASTIC LIMIT

It is the limiting value of force upto and within which, the deformation entirely disappears on the removal of force. The value of stress corresponding to this limiting force is called *elastic limit* of the material.

Beyond the elastic limit, the material gets into plastic stage and in this stage the deformation does not entirely disappear, on the removal of force.

## 1.6 ❑ HOOKE'S LAW

When a material is looded, within elastic limit, the stress is proportional to strain. Stress ∝ strain or $\sigma \propto \varepsilon$ or $\sigma = E.\varepsilon$ or $E = \dfrac{\sigma}{\varepsilon}$

where,

$E$ = A constant of proportionality, known as modulus of elasticity or Young's modulus

## 1.7 ❑ MODULUS OF ELASTICITY OR YOUNG'S MODULUS

Whenever a material is looded, within elastic limit, the stress is proportional to strain. Mathematically,

$$\sigma \propto \varepsilon$$

or
$$\sigma = E.\varepsilon$$

or
$$E = \frac{\sigma}{\varepsilon}$$

or
$$\frac{\text{Stress}}{\text{Strain}} = E$$

$E$ is a constant of proportionality, known as modulus of elasticity or Young's modulus.

## 1.8 ❑ GENERALIZE HOOK'S LAW

For two direction stress system:

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right)$$

$$\varepsilon_z = \frac{1}{E}\left(-\mu\sigma_y - \mu\sigma_z\right)$$

For three direction stress system:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right]$$

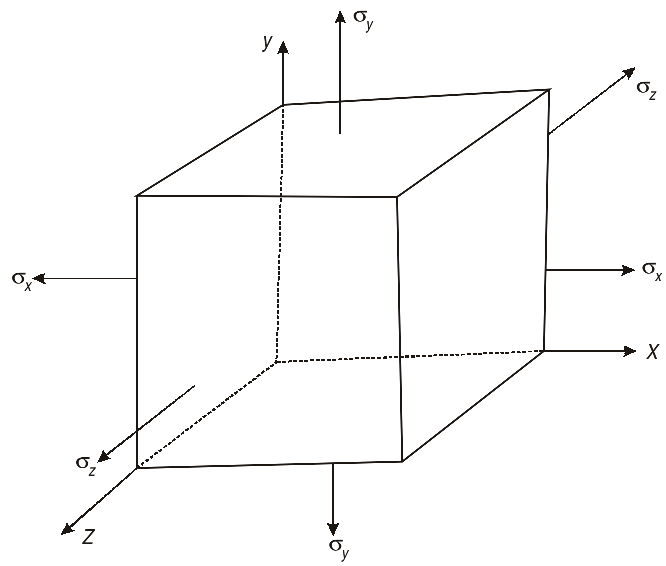



**Fig. 5 (b)**

**Example 1.1** *A metallic rectangular rod 1.5 m long, 40 mm wide and 25 mm thick is subjected to an axial tensile load of 120 kN. The elongation of rod is measured as 0.9 mm. Calculate stress, strain and modulus of elasticity.*

(*UPTU : 2004*)

**Solution**

Stress, $\sigma = \frac{\text{Load}}{\text{Area}} = \frac{\left(120 \times 10^3\right)}{40 \times 25}$

$= \mathbf{120\ N/mm^2}$

Strain, $\varepsilon = \frac{\text{Change in length}}{\text{Original length}} = \frac{\delta l}{l}$

$= \frac{0.9}{1.5 \times 10^3}$

$$= 0.0006$$

Modulus of elasticity, $E = \dfrac{\sigma}{\varepsilon}$

$$= \frac{120}{0.0006}$$

$$= \mathbf{2 \times 10^5\ N/mm^2} \qquad \textbf{Ans.}$$

## 1.9 ❑ STRESS-STRAIN CURVE FOR DUCTILE MATERIALS

A material is said to be ductile in nature, if it elongates appreciably before fracture. One such material is mild steel. The tensile test of mild steel is carried out on a specimen of uniform cross-section throughout its gauge length. The

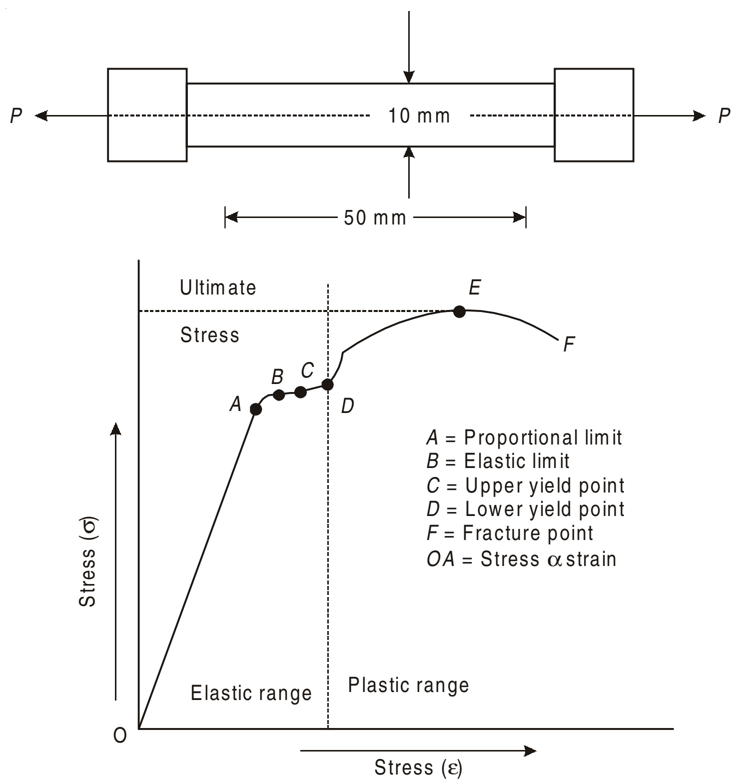



**Fig. 1.8** *Stress-strain curve*

specimen is gripped between two grips (JAWS) of the tensile testing machine, with which gradually increasing load is applied. The load and extension in the gauge length (50 mm) of the specimen are observed. A graph is drawn between load and extension i.e., stress and strain (Fig. 1.8)

**1. Proportional Limit.** From the origin $O$ to the point $A$, stress-strain diagram is a straight line i.e., stress is proportional to strain. Hook's law holds good upto this point $A$. Beyond point $A$, the stress is no longer proportional to the strain. Hence point $A$ is the proportional limit stage. Thus, the limit upto which stress is directly proportional to strain is called proportional limit.

**2. Elastic Limit**. Point $B$ indicates the elastic limit stage. Between points $A$ and $B$ although the strain increases slightly more than the stress, yet the material is elastic i.e., on the removal of load, the material will regain its original

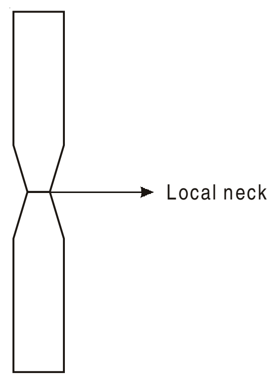



**Fig. 1.9**

shape and size. If the material is stressed beyond point $B$, the deformation will take place. Points $C$ and $D$ are upper and lower yield points respectively. Between points $B$ and $C$, the strain increases more rapidly than the stress. At this point $C$ the dial, which gives the reading of the load suddenly becomes stationary for few seconds to point $D$. Beyond point $D$, the load again starts increasing but the elongation increases at faster rate than load up to point $E$. Hence $E$ indicates maximum or ultimate stress point. The bar of specimen begins to form a local neck. Point $F$ is the breaking point. The extension remains continuous even with lesser load and fracture occurs at point $F$.

The stress corresponding to the peak load is called ultimate tensile stress or ultimate tensile strength or tensile strength. The stress corresponding to the load when the specimen ruptures is called rupture strength.

**3. Yield Point.** The yield point is the point at which considerable elongation of the test specimen occurs with no noticeable increase in the tensile load (Stress). This phenomenon is known as yielding of the material and point $C$ is called the yield point. The corresponding stress is known as the yield stress. In this region (i.e., between points $C$ and $D$) the material becomes perfectly plastic, which means that it deforms without an increase in the applied load. The elongation of specimen in the perfectly plastic region is typically 10 to 15 times the elongation that occurs up to the proportional limit.

**4. Ultimate Stress.** The ratio of maximum load to the original cross-sectional area of a bar is called ultimate stress.

$$\text{Ultimate stress} = \frac{\text{Maximum load}}{\text{Original cross-sectional area}}$$

**5. Percentage Elongation.** The ratio of percentage increase in gauge length to the original gauge length is called the percentage elongation.

$$\therefore \quad \text{Percentage elongation} = \frac{L_1 - L_0}{L_0} \times 100$$

where,

$L_0$ = original gauge length

$L_1$ = final gauge length after fracture

**6. Working Stress.** The greatest stress to which a material is ever subjected is called the working stress.

$$\therefore \quad \text{Working stress} = \frac{\text{Ultimate stress}}{\text{Factor of safety}}$$

**7. Factor of Safety.** The ratio of ultimate stress to working stress is called factor of saftey.

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Working stress}}$$

The factor of safety is also known as factor of ignorance. It is decided by experience and depends upon the number of factors such as nature of loading, degree of safety, degree of economy required, permanency of design, material used, etc. For cast iron, concrete, wood etc. the value of factor of safety may be taken as 4 to 6.

**8. Breaking Stress.** The ratio of load at the breaking or fracture point and original cross-sectional area of a material is called breaking stress or rupture stress.

$$\therefore \quad \text{Breaking stress} = \frac{\text{Load at breaking point}}{\text{Original cross-sectional area}}$$

### 1.9.1 Rupture Strength

The rupture strength is the stress at failure. The stretching of the bar beyond ultimate point is actually accompanied by a reduction in the load and fracture finally occurs at a point such as $F$.

The rupture stress at $F$ is less than the ultimate stress at $E$ (as shown by curve $EF$) is somewhat misleading. When a test specimen is stretched, lateral contraction occurs.

The resulting decrease in cross-sectional area is too small to have a noticeable effect on the calculated values of the stresses up to the yielding region, but beyond that point the reduction in area begins to alter the shape of the curve. In the vicinity of the ultimate stress, the reduction in cross-sectional area of the specimen becomes clearly visible and a pronounced necking of the bar occurs. If

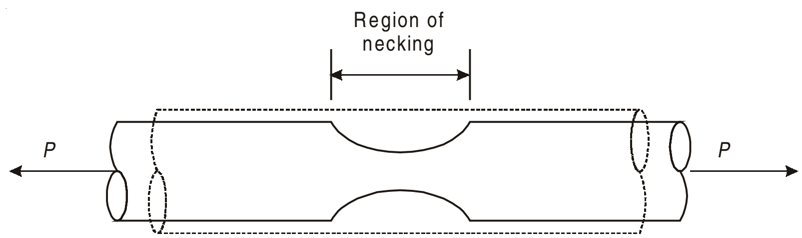



**Fig. 1.10**

the actual cross-sectional area of the narrowest part of the neck is used to calculate the stress, the true stress-strain curve may be obtained.

**9. Percentage reduction in area.** Ratio of the percentage decrease in cross-sectional area to the original cross-sectional area is called the percentage reduction in area after rupture.

$$\therefore \quad \text{Percentage reduction in area} = \frac{A_0 - A_1}{A_0} \times 100$$

where,

$A_0$ = Original cross-sectional area

$A_1$ = Final cross-sectional area

**10. Nominal Stress and True Stress.** When the initial area of the specimen (based on original diameter or nominal diameter) is used in the calculation, the stress is called the nominal stress (or conventional stress or engineering stress).

A more accurate value of the axial stress, called the true stress, can be determined by using the actual area of the bar at the cross-section where failure occurs. Since the actual area in a tension test is always less than the initial (nominal) area, the true stress is larger than the nominal stress.

**11. Nominal Strain and True Strain.** If the initial gauge length is used in the calculation, then the nominal strain is obtained. Since the distance between the gauge marks increases as the tensile load is applied, we can calculate the true strain (or natural strain) at any value of the load by using the actual distance between the gauge marks. In tension true strain is always smaller than the nominal strain.

The conventional stress strain curve *OABCDEF*, which is based upon the original cross-sectional area of the specimen and is easy to determine, provides satisfactory information for use in engineering design.

Based on the discussion on stress-strain curve, the stresses pertaining to various limits are defined as follows:

(a) Elastic stress (or normal stress)

$$= \frac{\text{Elastic load}}{\text{Initial cross-sectional area of specimen}}$$

(b) $$\text{Ultimate stress } \sigma_u = \frac{\text{Ultimate load}}{\text{Initial cross-sectional area of specimen}}$$

(c) $$\text{Fracture stress } \sigma_f = \frac{\text{Load at fracture}}{\text{Initial cross-sectional area of specimen}}$$

(d) True breaking stress (or true fracture stress)

$$= \frac{\text{Load at fracture}}{\text{Actual cross-sectional area at the point of breaking (i.e., Area of neck)}}$$

(e) $$\text{Percentage elongation} = \frac{\text{Increase in length}}{\text{Original gauge length}} \times 100$$

$$\text{i.e., \% elongation} = \left(\frac{L' - L}{L}\right) \times 100$$

$L'$ = Length of specimen at the point of breaking.

$L$ = Gauge length.

(f) $$\text{Percentage reduction in Area} = \frac{\text{Decrease in cross-sectional area}}{\text{Original area of specimen}}$$

i.e., $$\text{\% reduction in area} = \left(\frac{A - A'}{A}\right) \times 100$$

$A'$ = Minimum cross-sectional area at fracture i.e., cross-sectional area of neck

$A$ = Original area of cross-section

### 1.9.2 Stress-Strain Curve for Brittle Materials

Materials which show very little elongation before they fracture are called brittle materials. Brittle materials like concrete, aluminium and cast iron have very low proportional limit and do not show the yield point. For brittle materials the stress-strain graph is continuous curve from the beginning itself as shown in

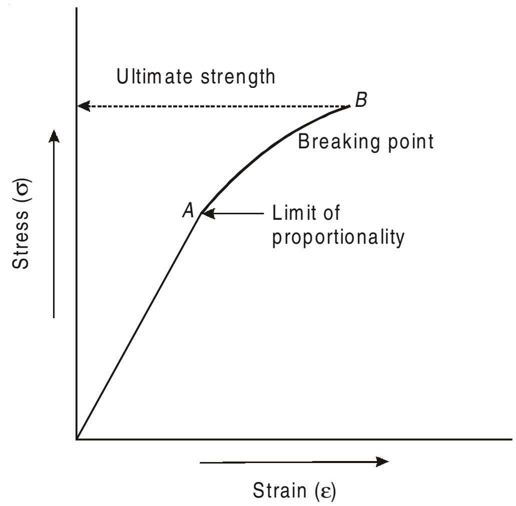



**Fig. 1.11**

Fig. 1.11. The breaking stress at point $B$ is well defined. The brittle material has relatively small tensile strain upto the breaking point.

### 1.9.3 Stress vs Strength

Every material is elastic in nature. That is why, whenever some external system of forces acts on a body, it undergoes some deformation. As the body undergoes deformation, it sets up some resistance to deformation. The resistance per unit area to the deformation is known as *stress*. Mathematically, stress may be defined as force per unit area i.e., $\sigma = \dfrac{P}{A}$,

where,

$\sigma$ = Intensity of stress (written as stress)

$P$ = Load or force acting on the body

$A$ = Cross-sectional area of the body

In S.I. units, the unit of stress are N/mm$^2$, kN/mm$^2$, MN/mm$^2$, and GN/mm$^2$ depending upon units of force.

### 1.9.4 Strength

The strength of a material is its ability to resist the application of force without rupture. In service the material may have to withstand tension, campression or shear force. The strength of a material is measured by loading it in a testing machine. The ultimate strength is the load necessary to fracture one square centimeter of the cross-section of the material. The tenacity is the ultimate strength in tension. Ultimate strength and tenacity are always expressed in stress units.

### 1.9.5 Linear (longitudinal) Strain

Whenever some external force acts on a body, it undergoes some deformation. The deformation of the bar per unit length in the direction of force i.e., $\delta l/l$ is known as linear (longitudinal) *strain*

$$\therefore \qquad \text{Linear strain} = \frac{\text{Change in linear dimension}}{\text{Original linear dimension}} = \frac{\delta l}{l}$$

$$= \frac{\sigma}{E}\left(1 - \frac{2}{m}\right)$$

### 1.9.6 Lateral Strain

Strain in the direction at right angles to the direction of applied force is known as *lateral strain.* It is the ratio of change on lateral dimension to the original lateral dimension of the member.

$$\therefore \qquad \text{Lateral strain} = \frac{\text{Change in lateral dimension}}{\text{Original lateral dimension}} = \frac{D - d}{D}$$

For a circular bar : Lateral strain, $\varepsilon_L = \dfrac{\delta d}{d}$

and For a rectangular bar : Lateral strain, $\varepsilon_L = \dfrac{\delta d}{d}$ or $\dfrac{\delta t}{t}$

where,

$d$ = original diameter

and $\delta d$ = change in diameter

$b$ = original width

$\delta d$ = change in width

and

$t$ = original thickness

$\delta t$ = change in thickness

### 1.9.7 Poisson's Ratio

The ratio of *lateral strain* to the longitudinal strain is called *Poisson's Ratio*.

$$\therefore \qquad \text{Poisson's Ratio} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= A \text{ constant} = \frac{1}{m} = \mu$$

Hence, $\quad$ Lateral strain $= \dfrac{1}{m} \cdot \varepsilon = \mu\varepsilon$

### 1.9.8 Volumetric Strain

When a body is subjected to external forces on its face, there will be a change in its volume. The ratio of change in volume to original volume is known as volumetric strain. It is denoted by $\varepsilon_\upsilon$

$$\therefore \quad \varepsilon_\upsilon = \frac{\delta\upsilon}{\upsilon}$$

$$= \varepsilon\left(1 - \frac{2}{m}\right)$$

where,

$V$ = original volume

and $\delta V$ = change in volume

$$\because \quad V = l.b.t$$

$$\therefore \quad \delta V = \delta l\,(b.t) + \delta b\,(l.t) + \delta t\,(l.b)$$

$$\therefore \quad \frac{\delta V}{V} = \frac{\delta l\,(b.t) + \delta b\,(l.t) + \delta t\,(l.b)}{l.b.t}$$

$$= \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta t}{t}$$

$$\therefore \quad \varepsilon_\upsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$\therefore$ Volumetric strain is the algebraic sum of all axial or linear strains.

where,

$e_x$ = strain in $x$-direction

$e_y$ = strain in $y$-directions

$e_z$ = strain in $z$-direction

### 1.9.9 Bulk or Volume Modulus of Elasticity

When a body is subjected to three mutually perpendicular like stresses of some intensity then the ratio of direct stress and corresponding *volumetric strain* of the

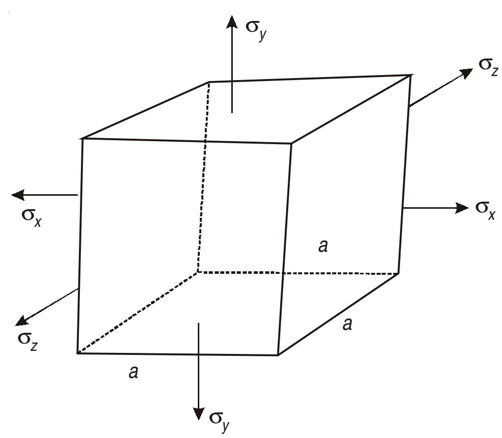



Fig. 1.12

body is constant and is known as *bulk modulus*. It is denoted by $K$ and unit N/mm$^2$ i.e.

$$K = \frac{\sigma_n}{\varepsilon_v}$$

$$K = \frac{\text{Increase in pressure}}{\text{Volumetric strain}} = \frac{dp}{\frac{dV}{V}} = \frac{\sigma_v}{\varepsilon_v}$$

where,

$V$ = volume of gas in cylinder

$dv$ = change in volume

$dp$ = change in pressure

$\sigma_v$ = volumetric stress

$\varepsilon_v$ = volumetric strain

### 1.9.10 Relation between Modulus of Elasticity (*E*) and Bulk Modulus (*K*)

As all are like stresses : $\sigma_x = \sigma_y = \sigma_z = \sigma_v$

We know that, $\frac{\delta V}{V} = \varepsilon_v = \frac{3\sigma_v}{E}(1-2\mu)$

$$E = \frac{3\sigma_v}{\varepsilon_v}(1-2\mu) = 3K(1-2\mu)$$

Bulk or volume modulus of elasticity = $\frac{\text{Normal stress}}{\text{Volumetric strain}}$

### 1.9.11 Shear Modulus or Modulus of Rigidity

Within elastic limit, *shear stress* is directly proportional to *shear strain*. The ratio of shear stress and shear strain is called *shear modulus*. It is denoted by $G$ or $C$ and units N/mm$^2$.

$\therefore$ Shear modulus = $\frac{\text{Shear stress}}{\text{Shear strain}}$

$$G = \frac{\tau}{\phi}$$

Relation between shear modulus and modulus of elasticity $E = 2G(1 + \mu)$

**Example 1.2** *The following data refers to a mild steel tensile test:*

(*i*) *Diameter of test piece = 16 mm*

(*ii*) *Gange length = 80 mm*

(*iii*) *Load at yield point = 64 kN*

(*iv*) *Maximum load = 100 kN, and*

(*v*) *Load at breaking point = 70 kN*

*Calculate,* (*i*) *yield strength,* (*ii*) *ultimate strength* (*iii*) *strength at breaking point, and* (*iv*) *why the breaking point strength is lower than the ultimate strength?*

(*UPTU : 2003 – 04*)

**Solution**

Area of test piece, $A = \frac{\pi}{4}d^2 = \frac{22}{7\times 4}(16)^2 = 201.06 \text{ mm}^2$

(i) $$\text{Yield strength} = \frac{\text{Load at yield point}}{\text{Area}} = \frac{(64\times 10^3)}{201.06} = \mathbf{318.31 \ N/mm^2}$$

(ii) $$\text{Ultimate strength} = \frac{\text{Maximum load}}{\text{Area}} = \frac{(100\times 10^3)}{201.06} = \mathbf{497.36 \ N/mm^2}$$

(iii) $$\text{Breaking strength} = \frac{\text{Breaking load}}{\text{Area}} = \frac{(70\times 10^3)}{201.06} = \mathbf{348.15 \ N/mm^2}$$

(iv) Breaking point strength is lower than the ultimate strength because original diameter of the test piece has been taken. If the diameter of the neck is considered for calculation, the breaking strength will be more than ultimate strength.

## 1.10 ❑ STRESSES IN THE BARS OF DIFFERENT SECTIONS

Sametimes a bar made up of different lengths having different cross-sectional areas (Fig. 1.13) are used. In such cases, the stresses, strains and hence changes

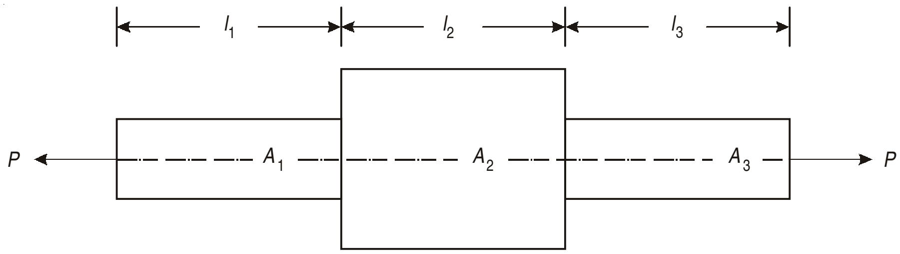



**Fig. 1.13** *Bars of different sections*

in lengths for each section are worked out separately as usual. The total changes in length is equal to the sum of changes of all the individual lengths. It may be noted that each section is subjected to the same external axial pull or push.

Hence, $$\delta l = \delta l_1 + \delta l_2 + \delta l_3 + .....$$

$$\therefore \qquad \delta l_1 = \frac{Pl_1}{A_1 E} = \frac{P}{E}\left(\frac{l_1}{A_1}\right)$$

$$\therefore \qquad \delta l = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3} + \ldots\right)$$

**Example 1.3** *Find the stresses in three parts and total extension of the bar under an axial pull of 40 kN. Take E = 2 × $10^5$ N/mm$^2$.* (*UPTU : 2005 –06*)

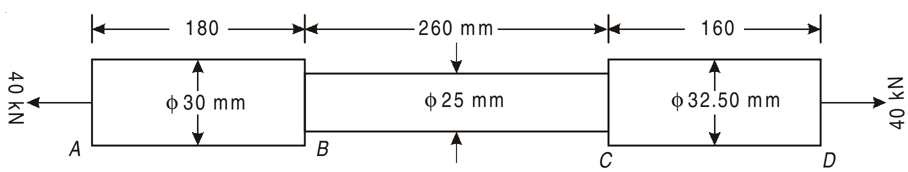



**Fig. 1.14**

**Solution**

$$\delta l = \frac{P}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right) = \frac{180}{\frac{\pi}{4}(30)^2} + \frac{260}{\frac{\pi}{4}(20)^2} + \frac{160}{\frac{\pi}{4}(32.5)^2}$$

$$\delta l = \frac{40}{200}\left(\frac{180}{706.86} + \frac{260}{314.16} + \frac{160}{829.68}\right)$$

$$\delta l = 0.20\ (0.2546 + 0.8276 + 0.1928) = 0.2\ (1.275)$$

Hence, $$\boldsymbol{\delta l = 0.255 \text{ mm}}$$

**Example 1.4** *While testing a metallic rod it is observed that the diameter of the rod is reduced by 0.025 mm under an axial pull of 20 kN. The original diameter of the rod is 15 mm. If rigidity modulus for the rod metal be 50 kN/mm$^2$, find the Young's modulus and bulk modulus.* (*UPTU : 2004–05*)

*Given :*

$$d = 15 \text{ mm}$$
$$\delta d = 0.0025 \text{ mm}$$
$$P = 20 \text{ kN} = 20 \times 10^3 \text{ N},$$
$$G = 50 \text{ kN/mm}^2 = 50 \times 10^3 \text{ N/mm}^2$$

*Area,* $$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(15)^2 = 176.7 \text{ mm}^2$$

**Solution**

Normal stress, $$\sigma = \frac{P}{A} = \frac{(20\times10^3)}{176.7} = 113.2 \text{ N/mm}^2$$

Later strain $$= \frac{\delta d}{d} = \frac{0.0025}{15} = \frac{5}{3}\times10^{-4}$$

∴ $$\varepsilon_y = 1.6667 \times 10^{-4}$$

Linear (Longitudinal) strain, $\varepsilon_x = \dfrac{\sigma}{E} = \dfrac{113.2}{E}$

$$\mu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\varepsilon_y}{\varepsilon_x}$$

or $$\mu = \frac{1.6667\times10^{-4}}{113.2}\times E = (1.4723\times10^{-6})E$$

∴ $$E = (0.6792 \times 10^6)\,\mu$$

We have, $$E = 2G(1 + \mu)$$

∴ $$(0.6792 \times 10^6)\,\mu = 2 \times 50 \times 1000\,(1 + \mu)$$

or $$(0.6792 \times 10^6)\,\mu = 10^5\,(1 + \mu)$$

or $$6.792\,\mu = 1$$

∴ $$\mu = 0.1727$$

Now, $$E = 0.6792 \times 10^6 \times \mu$$
$$= 0.6792 \times 10^6 \times 0.1727$$
$$= 0.1173 \times 10^6 \text{ N/mm}^2$$

∴ $E = 117.3 \text{ kN/mm}^2$ **= 117.3 GPa**

But, $$E = 3K(1 - 2\mu)$$

∴ $$117.3 \times 1000 = 3K(1 - 2 \times 0.1727)$$

or $$117.3 \times 1000 = (1.9638)\,K$$

∴ $$K = 59731 \text{ N/mm}^2 = 59.73 \text{ kN/mm}^2$$

**Example 1.5** *The bulk modulus of a material is 0.5 × $10^5$ N/mm². A 12 mm diameter rod of a material was subjected to an axial pull of 14 kN and the change in diameter was observed to be 3.6 × $10^{-3}$ mm. Calculate Poisson's ratio and modulus of elasticity.* *(UPTU : 2005–06)*

*Given :*

$$K = 0.5 \times 10^5 \text{ N/mm}^2$$
$$P = 14 \text{ kN} = 14 \times 10^3 \text{ N}$$
$$d = 12 \text{ mm}$$
$$\delta d = (3.6 \times 10^{-3}) \text{ mm}$$

**Solution**

Lateral strain, $\varepsilon_y = \dfrac{\delta d}{d} = \dfrac{\left(3.6 \times 10^{-3}\right)}{12} = 0.3 \times 10^{-3}$

But, $\mu = \dfrac{\text{Lateral strain}}{\text{Longitudinal strain}}$

$\therefore$ Longitudinal strain, $\varepsilon_x = \dfrac{\varepsilon_y}{\mu} = \dfrac{\left(0.3 \times 10^{-3}\right)}{\mu}$

Axial stress, $\sigma_x = \dfrac{P}{A} = \dfrac{(14 \times 1000)}{\dfrac{\pi}{4}(12)^2} = 123.8 \text{ N/mm}^2$

$\therefore$ Modulus of elasticity, $E = \dfrac{\sigma}{\varepsilon} = \dfrac{123.8}{\dfrac{0.3 \times 10^{-3}}{\mu}}$

or $E = \dfrac{123.8(\mu)}{\left(0.3 \times 10^{-3}\right)} = (412.7 \times 1000)\mu$ ...(i)

But we have,

$$E = 3K(1 - 2\mu)$$
$$= 3 \times 0.5 \times 10^5 (1 - 2\mu)$$
$$= 1.5 \times 10^5 (1 - 2\mu) \quad \text{...(ii)}$$

Equating (i) and (ii), we have

$$(412.7 \times 1000)\mu = 1.5 \times 10^5 (1 - 2\mu)$$

or $(4.127 \times 10^5)\mu = 1.5 \times 10^5 (1 - 2\mu)$

$\therefore$ $(4.127)\mu = 1.5(1 - 2\mu) = 1.5 - 3\mu$

or $4.127\mu + 3\mu = 1.5$

or $7.127\mu = 1.5$

$\therefore$ $\mu = \dfrac{1.5}{7.127} = \mathbf{0.21}$

From Eq. (i) we have,

$$E = (412.7 \times 1000)\mu$$
$$= 412.7 \times 1000 \times 0.21$$

$$= 86.67 \times 1000 \text{ N/mm}^2$$

$$\therefore \quad \mathbf{E = 86.67 \ kN/mm^2}$$

$$= 86.67 \text{ GPa}$$

**Example 1.6** *The stresses in the three principal direction are + 65 MN/m², + 20 MN/m² and –85 MN/m². Find the principal strains. Take m = 0.3 and E = 200 GN/m².* *(UPTU : 2006 – 07)*

*Given :*

$$\sigma_x = 65 \text{ MN/m}^2$$
$$\sigma_y = 20 \text{ MN/m}^2$$
$$\sigma_z = -85 \text{ MN/m}^2$$
$$E = 200 \text{ GN/m}^2 = 200 \times 10^3 \text{ GN/m}^2$$
$$\mu = 0.3$$

**Solution**

Major principal strain, $\varepsilon_x = \dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_y}{E} - \mu\dfrac{\sigma_z}{E}$

$$\therefore \quad \varepsilon_x = \frac{65}{E} - \frac{0.3(20)}{E} - \frac{0.3(-85)}{E}$$

$$= \frac{1}{E}(65 - 6 + 25.5) = \frac{84.5}{E}$$

Hence, $\varepsilon_x = \dfrac{84.5}{2\times 10^5}$

$$= \mathbf{0.4225 \ \times 10^{-3}}$$

Now, $\varepsilon_y = \dfrac{\sigma_y}{E} - \mu\dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_z}{E}$

$$= \frac{1}{E}\left[20 - 0.3(65) - 0.3(-85)\right]$$

Hence, $\varepsilon_y = \dfrac{20 - 19.50 + 25.50}{200\times 10^3}$

$$= \frac{26}{200\times 10^3}$$

$$= \mathbf{0.13 \ \times 10^{-3}}$$

Now $\varepsilon_z = \dfrac{\sigma_z}{E} - \mu\dfrac{\sigma_x}{E} - \mu\dfrac{\sigma_y}{E}$

$$= \frac{1}{E}\left[-85 - 0.3(65) - 0.3(20)\right]$$

Hence, $\varepsilon_z = \dfrac{1}{200 \times 10^3}(-85 - 19.50 - 6.00)$

$= \mathbf{-0.5525 \times 10^{-3}}$

Hence, the three principal strains are :

**(i) $\varepsilon_x$ = 0.4225 – $10^{-3}$ Tensile**

**(ii) $\varepsilon_y$ = 0.13 × $10^{-3}$ Tensile**

**(iii) $\varepsilon_z$ = 0.5525 × $10^{-3}$ Compressive**

**Example 1.7** *The principal stresses at a point in an elastic material are 60 N/mm² tensile, 20 N/mm² tensile, and 50 N/mm² compressive. Calculate the volumetric strain. Take E = 100 × 10³ N/mm², and μ = 0.3.* (UPTU : 2010–11)

*Given :*

$$\sigma_x = 60 \text{ N/mm}^2$$
$$\sigma_y = 20 \text{ N/mm}^2$$
$$\sigma_z = 50 \text{ N/mm}^2$$
$$E = 100 \times 10^3 \text{ N/mm}^2$$

**Solution**

We know that, volumetric strain,

$$\varepsilon_v = \frac{\sigma_x + \sigma_y + \sigma_z}{E}(1 - 2\mu)$$

$$= \frac{60 + 20 - 50}{100000}(1 - 2 \times 0.3)$$

$$= \mathbf{1.2 \times 10^{-4}}$$

## 1.11 ❑ RELATION BETWEEN ELASTIC CONSTANTS

*E*, *G* and *K* are known as elastic constants and their values are different for different materials.

(i) Relation between *E*, *G* and *K*

$$E = 2G\left(1 + \frac{1}{m}\right) = \frac{9KG}{3K + G} = 2G(1 + \mu)$$

Hence modulus of elasticity, $E = \dfrac{\sigma_t}{\varepsilon_t} = \dfrac{\sigma_c}{\varepsilon_c} = 3K\left(3 - \dfrac{E}{G}\right)$

Modulus of rigidity, $G = \dfrac{\sigma_s}{\varepsilon_s} = \dfrac{mE}{2(m+1)}$

Bulk or volume modulus of elasticity, $K = \dfrac{\sigma_m}{\varepsilon_v}$

$$G = \frac{E}{2(1+\mu)}, \qquad K = \frac{mE}{3(m-2)}$$

### 1.11.1 Relation between E, K and μ

Consider a cubical element subjected to volumetric stress σ which acts simultaneously along the mutually perpendicular *X*, *Y* and *Z* directions (Fig. 1.15).

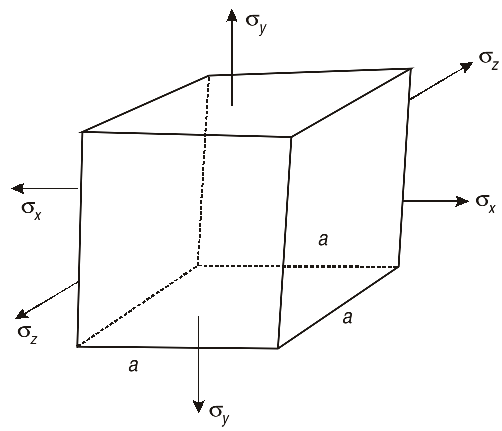



**Fig. 1.15**

The resultant strain along the three directions can be worked out by taking the effect of individual stresses.

Strain in $x$-direction.

$\varepsilon_x$ = strain in $x$-direction due to $\sigma_x$ – strain in $x$-direction due to $\sigma_y$ – strain in $x$-direction due to $\sigma_z$

$$= \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$

But $\sigma_x = \sigma_y = \sigma_z = \sigma$

$$\therefore \quad \varepsilon_x = \frac{\sigma}{E} - \mu\frac{\sigma}{E} - \mu\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\mu)$$

Likewise, $\varepsilon_y = \frac{\sigma}{E}(1-2\mu)$

and $\varepsilon_z = \frac{\sigma}{E}(1-2\mu)$

Volumetric strain, $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \dfrac{3\sigma}{E}(1-2\mu)$

Now bulk modulus, $K = \dfrac{\text{Volumetric stress}}{\text{Volumetric strain}} = \dfrac{\sigma}{\dfrac{3\sigma}{E}(1-2\mu)}$

$$= \frac{E}{3(1-2\mu)}$$

We also know that $E = 2c\,(1+\mu)$ and $c = \dfrac{E}{2(1+\mu)}$ and also

$$E = \frac{9KC}{C+3K} = 3K\,(1-2\,\mu)$$

## 1.12 ❑ CONCEPT OF STRAIN ENERGY

When a load is applied on an elastic body it may be deformed i.e., work is done by the load or force on the body (W. D. = $P$ × Displacement). This work done is stored in the body and called *strain energy* or *resilience*. When the load is removed, the body regains its original position because of stored energy. It can be easily visualised in springs. When the spring is loaded, it gets compressed and regains to its original position when the load is removed. The amount of strain energy stored in a body depends on the type of loading i.e., axial load, bending load, shear load and torsional load.

### 1.12.1 Strain Energy

The work done by the external load on a bar is stored in the material and is termed as strain energy. It is denoted by $U$.
Strain energy, $U$ = work done by the load

### 1.12.2 Resilience

It represents the ability of a material to absorb and release energy within elastic limit. The strain energy stored by a body, within elastic limit, when loaded externally, is called *resilience*.

### 1.12.3 Proof Resilience

The maximum energy which a body stores upto elastic limit is called *proof resilience*.

### 1.12.4 Modulus of Resilience

It is the energy that can be absorbed per unit volume without creating a permanent distortion.

### 1.12.5 Strain Energy dut to Gradually Applied Load

Consider a bar of length $L$ subjected to a tensile force $P$, the load is applied slowly so that it increases gradually from zero to its maximum value $P$. Such a

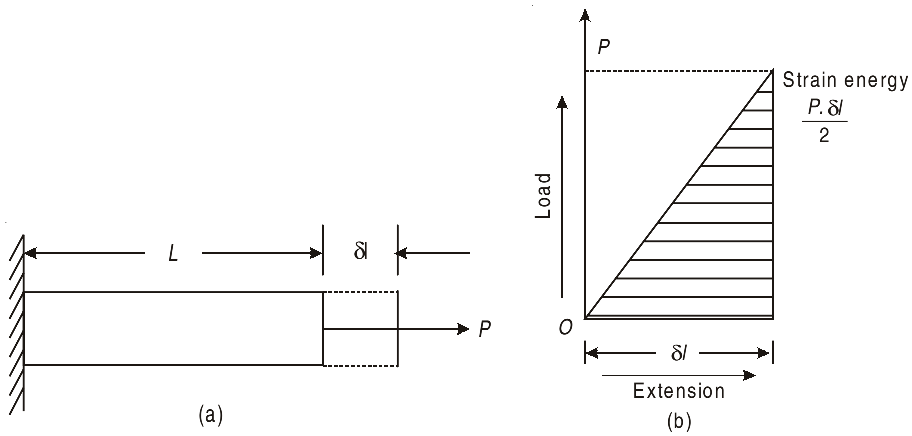



**Fig. 1.16**

load is called static load. The bar gradually elongates as the load is applied and reaches its maximum elongation $\delta l$, when load reaches its maximum value $P$ (Fig. 1.16).

During the loading process, the load moves slowly through the distance $\delta l$ and does certain amount of work which is stored in the bar material in the form of strain energy.

Strain energy ($U$) stored in the bar = work done by load

= Average load × Displacement

= Area under the curve (Shaded area)

$$= \frac{1}{2}P\delta l = \frac{1}{2}.P.\frac{PL}{AE} = \frac{1}{2}\frac{P^2L}{AE} \qquad (\because \delta l = \frac{PL}{AE})$$

$$= \frac{1}{2}.\frac{\sigma^2 AL}{E} = \frac{\sigma^2 AL}{E} = \frac{\sigma^2}{2E}.\text{Volume}$$

Hence modulus of resilience = $\dfrac{\text{Strain energy}}{\text{Volume}} = \boldsymbol{\dfrac{\sigma^2}{2E}}$

**= Energy absorbed per unit volume.**

### 1.12.6 Strain Energy due to suddenly Applied Load

The load is applied all of a sudden and remains constant throughout the process of deformation.

$$\text{Strain energy} = \text{Work done}$$
$$= \text{Area shown by shaded portion}$$

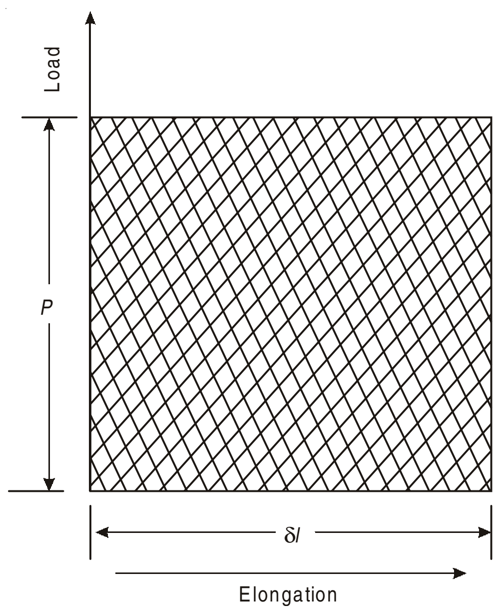



**Fig. 1.17**

or
$$U = P.\,\delta l$$

$$= P.\frac{PL}{AE} = \frac{P^2L}{AE} = \frac{P^2L}{AE}, \frac{A}{A} = \frac{P^2}{A^2}.\frac{L.A}{E}$$

$$= \frac{\sigma^2}{E} \cdot AL = \frac{\sigma^2}{E} \times \textbf{Volume}$$

The strain energy in suddenly applied load is twice than that of gradually applied load.

### 1.12.7 Strain Energy due to Impact Loading (Falling weight)

A weight $W$, drops freely through a height $h$ and strikes a stop at the lower end of the bar of length $L$. The upper end of the bar is rigidly fixed. Assume that the stress remain within elastic limit and the mass of the bar is negligible as compared to weight $W$. After striking the stop, the weight $W$, continues to move downward by stretching the bar. Let the extension and tensile stress caused by

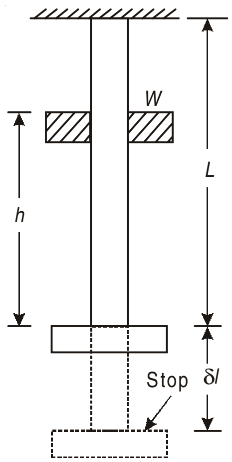



**Fig. 1.18**

impact in the bar is $\delta l$ and $\sigma$ respectively (Fig. 1.18). Therefore, strain energy stored in the bar = work done by weight $W$ in falling through the total vertical distance $(h + \delta l)$

$$W(h + \delta l) = \frac{1}{2} \cdot \frac{\sigma^2}{E} \times \text{volume} = \frac{1}{2} \cdot \frac{\sigma^2}{E} \cdot A \cdot L$$

or $$W\left(h + \frac{\sigma L}{E}\right) = \frac{\sigma^2}{2E}.AL$$

or $$\frac{\sigma^2 AL}{2E} - \frac{WL\sigma}{E} - Wh = 0$$

$$\therefore \quad \sigma = \frac{\frac{WL}{E} \pm \sqrt{\left(\frac{WL}{E}\right)^2 - 4\left(\frac{AL}{2E}\right)(-Wh)}}{2\left(\frac{AL}{2E}\right)} = \frac{W}{A} \pm \sqrt{\frac{W^2}{A^2} + \frac{2whE}{AL}}$$

Hence, $\sigma = \frac{W}{A} + \frac{W}{A}\sqrt{1 + \frac{2hAE}{WL}} = \boldsymbol{\frac{W}{A}\left[1 + \sqrt{1 + \frac{2hAE}{WL}}\right]}$

If $\sigma l$ is negligible as compared to height $h$, then $\boldsymbol{Wh = \frac{\sigma^2}{2E} \cdot AL}$ and

$$\boldsymbol{\sigma = \sqrt{\frac{2WhE}{AL}}}$$

### 1.12.8 Strain Energy due to Shear

Consider a rectangular block of length $l$, height $h$ and unit thickness perpendicular to plane of paper. The block is fixed at the bottom face. A shear force $P$ is applied on the top face moving a distance $AA'$ or $BB'$ (Fig. 1.19).

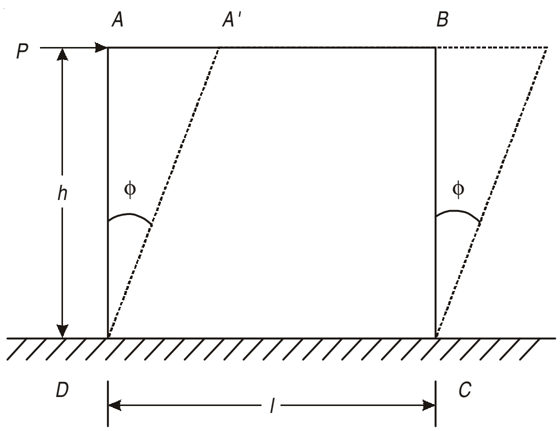



**Fig. 1.19**

Shear stress, $$\tau = \frac{P}{A} = \frac{P}{l}$$

(∵ Area of the top face of thickness of unit thickness = $l \times 1$)

or $$P = \tau.l$$

and shear strain, $$\phi = \frac{BB'}{BC} = \frac{BB'}{h}$$

or $$BB' = \phi h$$

If shear force is gradually applied, then strain energy = Work done

or Strain energy = Average Force × Displacement

$$U = \frac{P}{2} \times BB' = \frac{\tau l}{2}.\phi h = \frac{1}{2}\tau lh.\frac{\tau}{G}$$

$$U = \frac{1}{2}\cdot\frac{\tau^2}{G} \times lh = \frac{1}{2}\frac{\tau^2}{G} \times \text{volume} \qquad \left(\phi = \frac{\tau}{G}\right)$$

∴ Strain energy stored per unit volume, $$\boldsymbol{\frac{U}{V} = \frac{\tau^2}{2G}}$$

**Example 1.8** *Three bars of equal length and having cross-sectional area in ratio of 1: 2: 4, are all subjected to equal load. Compare their strain energy.*

*(UPTU : 2003 –04)*

**Solution**

Let the length of the bar = $L$, and load applied on the bar = $P$

Let $A$, $2A$ and $4A$ be the areas of the three bars respectively. Then stresses induced in the three bars are,

$$\sigma_1 = \frac{P}{A}; \qquad \sigma_2 = \frac{P}{2A}$$

and $$\sigma_3 = \frac{P}{4A}$$

∴ Strain energy, $$U = \frac{\sigma^2}{2E} \times \text{Volume} = \frac{\sigma^2}{2E} \cdot AL$$

Thus, for bar 1 :

$$U_1 = \left(\frac{P}{A}\right)^2 \cdot \frac{AL}{2E} = \frac{P^2L}{2AE}$$

For bar 2 : $$U_2 = \frac{1}{2E}\left(\frac{P}{2A}\right)^2 2AL = \frac{P^2L}{4AE}$$

For bar 3 : $$U_3 = \frac{1}{2E}\left(\frac{P}{4A}\right)^2 \times 4A.L = \frac{P^2L}{8AE}$$

∴ $$U_1 : U_2 : U_3 = \frac{P^2L}{2AE} : \frac{P^2L}{4AE} : \frac{P^2L}{8AE}$$

$$= \mathbf{\frac{1}{2} : \frac{1}{4} : \frac{1}{8} = 1 : \frac{1}{2} : \frac{1}{4}}$$

**Example 1.9** *A weight W = 5 kN attached to the end of a steel wire rope moves downward with constant velocity 1 m/s. What stresses are produced in the rope when its upper end is suddenly stripped? The free length of rope at the moment of impact is 20 m, its not cross-sectional area = 10 sq cm and E = 2 × $10^5$ N/mm$^2$.* *(UPTU : 2012 –13)*

**Solution**

As the chain gets jammed, the Kinetic Energy is transformed into strain energy

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{5 \times 10^3}{9.81}\right)(1)^2 = 254.84 \text{ N-m}$$

$$= 254.84 \times 10^3 \text{ N-mm}$$

Strain energy in rope, $$U = \frac{\sigma^2 AL}{2E} = \frac{\sigma^2 (10 \times 100)(20 \times 10^3)}{2 \times 2 \times 10^5}$$

$$= 50\,\sigma_2$$

$\therefore$ Kinetic Energy = Strain Energy

$\therefore$ $254.84 \times 10^3 = 50\,\sigma_2$

$\therefore$ $\sigma = 71.39$ N/mm$^2$

$$\delta l = \frac{\sigma L}{E} = \frac{71.39 \times 20 \times 10^3}{2 \times 10^5} = 7.14 \text{ mm}$$

**Example 1.10** *A vertical rod 2 m long, fixed at the upper end, is 13 cm$^2$ in area for 1 m and 20 cm$^2$ in area for another 1 m. A collar is attached to the free end. Through what height can a load of 100 kg fall on to the collar to cause a maximum stress of 50 N/mm$^2$?* *(UPTU : 2010 –11)*

**Solution**

Let, $W$ = Falling weight, and $W_e$ = equivalent gradually applied load which produces same maximum stress and extension as it causes due to falling load $W$

Total extension, $$\delta l = \delta l_1 + \delta l_2 = \frac{We}{E}.\frac{l_1}{A_1} + \frac{We}{E}.\frac{l_2}{A_2}$$

or $$\delta l = \frac{We}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2}\right)$$

$$= \frac{We}{2 \times 10^5}\left(\frac{1 \times 1000}{13 \times 10 \times 10} + \frac{1 \times 1000}{20 \times 10 \times 10}\right)$$

$\therefore$ $$\delta l = \frac{W_e}{2 \times 10^5}\left(\frac{1000}{1300} + \frac{1000}{2000}\right) = \frac{1.2692\, W_e}{2 \times 10^5} \qquad \text{...(i)}$$

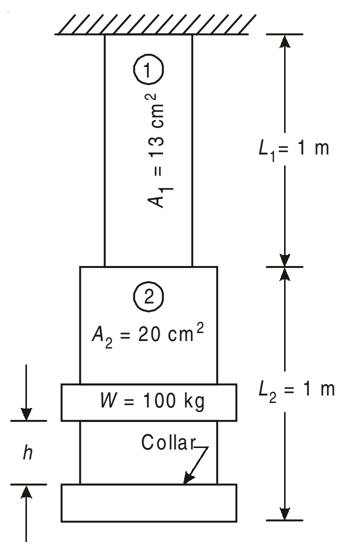



Fig. 1.18

Maximum stress occurs at smaller section

$$\therefore \quad \sigma_1 = \frac{W_e}{A_1} \text{ or } 50 = \frac{W_e}{1300}$$

$$\therefore \quad W_e = 50 \times 1300 = 65000 \text{ N}$$

Substituting this value in Eq. (i)

$$\delta l_1 = \frac{1.2692 \times 65000}{2 \times 10^5} = 0.4125 \text{ mm}$$

$$\therefore \quad W(h + \delta l) = \frac{1}{2} \times We \times \delta l$$

or $$100 \times 9.81\,(h + 0.4125) = 65000 \times 0.4125 \times \frac{1}{2}$$

or $$981\,(h + 0.4125) = \frac{65000 \times 0.4125}{2}$$

or $$h + 0.4125 = \frac{65000 \times 0.4125}{2 \times 981}$$

$$\therefore \quad \boldsymbol{h = 13.25 \text{ mm}}$$

**Example 1.11** *A one metre long rod of rectangular section 80 mm × 40 mm is subjected to an axial tensile load of 200 kN. Find the strain energy and maximum stress produced in it for the following cases:*

*(i) When load is applied gradually, and*

*(ii) Load falls through a height of 100 mm*

*Take* $E = 2 \times 105 \text{ N/mm}^2$ *(UPTU : 2004 – 2005)*

*Given:* $l = 1 \text{ m} = 1000 \text{ mm}$

*Load* = *200 kN*

$E = 2 \times 10^5 \text{ N/mm}^2$

*Area,* $A = 80 \times 40 = 3200 \text{ mm}^2$

**Solution**

(i) Stress, $$\sigma = \frac{\text{Load}}{\text{Area}} = \frac{(200 \times 10^3)}{3200}$$

$$= 62.5 \text{ N/mm}^2$$

Strain energy due to gradually applied load,

$$U = \frac{\sigma^2}{2E} \times \text{Volume}$$

$$= \frac{(62.5)^2}{2 \times 2 \times 10^5} \times 3200 \times 1000$$

$$= 31250 \text{ J} = 31.25 \text{ kJ}$$

(ii) Maximum stress

When the load falls through a height of 100 mm,

$$\sigma = \frac{W}{A}\left[1+\sqrt{1+\frac{2hAE}{WL}}\right]$$

$$= \frac{200\times10^3}{3200}\left[1+\sqrt{1+\frac{2\times100\times3200\times2\times10^5}{200\times10^3\times1000}}\right]$$

$$\therefore \quad \sigma = 62.5\left[1+\sqrt{1+640}\right]$$

$$= \mathbf{1645 \ N/mm^2}$$

Strain energy,

$$U = \frac{\sigma^2}{2E}.AL = \frac{(1645)^2\times3200\times1000}{2\times2\times10^5}$$

$$= 216648200 \text{ J}$$

$$= \mathbf{21664.82 \ kJ}$$

## 1.13 ❑ PRINCIPLE OF SUPERPOSITION

Sometimes a body is subjected to a number of forces acting on its outer edges as well as at some other sections, along the length of the body. In such a case, the forces are split up and their effects are considered on individual sections. The resulting deformation, of the body, is equal to the algebraic sum of the deformations of individual sections. Such a principle, of finding out the resultant deformation, is called principle of superposition.

The relation for the resulting deformation may be modified as:

$$\delta l = \frac{Pl}{AE} = \frac{1}{AE}(P_1l_1 + P_2l_2 + P_3l_3 + .....)$$

where,

$P_1$ = Force acting on section 1

$l_1$ = Length of section 1

$P_2l_2$ = Corresponding values of section 2

**Example 1.12** *Show that if E is assumed correct, an error of 1% in the determination of G will involve an error of about 5% in the calculation of Poisson's ratio when its correct value is 0.25* (*UPTU : 2002–2003, 2013–2014*)

*Given : Error in the value of G is 1% i.e.,* $\frac{\delta G}{G} = \frac{1}{100}$ *and* $\mu = 0.25$ *if the value* *of E is correct, in calculation no error i.e., δE = 0. To show that error in μ is 5%*

*i.e.,* $\frac{\delta\mu}{\mu} = \frac{5}{100}$ *is to be shown*

**Solution**

We have the relation, $E = 2G(1 + \mu)$

Differentiating partially, $\delta E = 2\delta G(1 + \mu) + 2G(\delta\mu)$

$$\therefore \quad O = \delta G(1 + \mu) + G(\delta\mu)$$

$$\therefore \quad -\delta G(1 + \mu) = G(\delta\mu)$$

$$\therefore \quad -\frac{\delta G}{G} = \frac{\delta\mu}{1+\mu}$$

$$\therefore \quad -\frac{1}{100} = \frac{\delta\mu}{1+0.25} \qquad (\because \mu = 0.25)$$

$$\therefore \quad \delta\mu = -\frac{1.25}{100}$$

$$\therefore \quad \frac{\delta\mu}{\mu} = \frac{-1.25}{100}\times\frac{1}{\mu} = \frac{-1.25}{100}\times\frac{1}{0.25}$$

$$\therefore \quad \frac{\delta\mu}{\mu} = -\frac{5}{100} = -5\%$$

∴ **Error in μ is 5% (–ve) proved.**

**Example 1.13** *Derive an expression for deformation of a conical bar hung to a ceiling having diameter D and height L. Weight density of bar is γ and Young's modulus is E.*

**Solution**

Consider a section $x - x$ at a distance $x$ from free end.

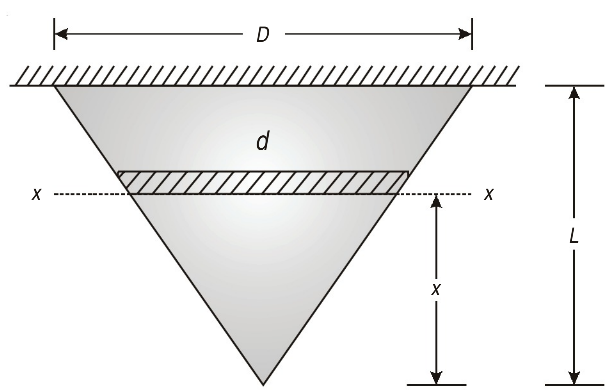



**Fig. 1.19**

By similarity of triangle $\dfrac{d}{x} = \dfrac{D}{L}$

$$d = \frac{x}{L}D$$

Weight of cone at section $x - x$, $W_x = V \times \gamma = \dfrac{\pi d^2}{12} x\gamma$

Stress at section $x - x$ $\sigma_x = \dfrac{Wx}{Ax} = \dfrac{\dfrac{\pi d^2}{12} \cdot x\gamma}{\dfrac{\pi d^2}{4}}$ $\therefore \sigma_x = \dfrac{\gamma x}{3}$

Consider a strip of length $dx$ at a distance $x - x$ from free end.

$$\text{Deformation in strip} = \frac{\sigma \times L}{E} = \frac{\gamma x\, dx}{3E}$$

$$\text{Deformation in whole bar } \delta L = \int_0^L \frac{\gamma x}{3E} dx = \left[\frac{\gamma x^2}{6E}\right]_0^L \qquad \therefore \boldsymbol{\delta L = \frac{\gamma L^2}{6E}}$$

**Example 1.14** *A bar of varying cross-section and of total length 250 mm comprises of two end parts, each of equal length and diameter 40 mm. If the total elongation of the bar is to be 0.175 mmunder a tensile load of 150 kN, and the stress in the middle part is limited to 150 N/mm², determine the diameter and length of the middle part.*

*Take* $E = 200\ GN/m^2$.

**Solution** The bar is shown in Fig. 1.20

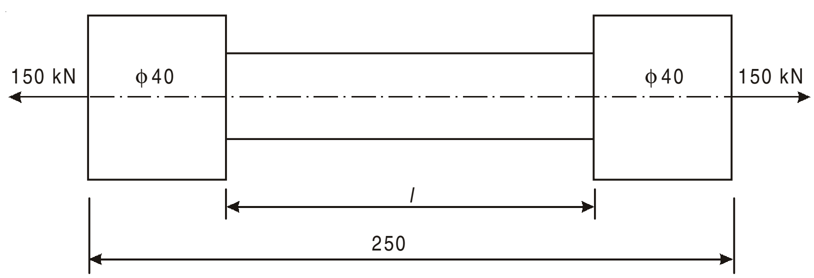



**Fig. 1.20**

**Middle Part :**

Suppose $l$ is the length, $d$ is its diameter

$$\sigma = 150 = \frac{150 \times 10^5 \times 4}{\pi d^2}$$

$$d^2 = \frac{10^3 \times 4}{\pi} \qquad \Rightarrow d = \mathbf{35.67\ mm} \quad \textbf{Ans.}$$

Elongation $= \dfrac{\sigma l}{E} = \dfrac{150l}{2\times10^5}$ mm

where $E = 200\ \text{GN/m}^2 = \dfrac{200\times10^9}{10^6}\ \text{N/mm}^2 = 2\times10^5\ \text{N/mm}^2$

**End Part :**

$$\text{Tensile stress in each} = \frac{150\times1000\times4}{\pi\times40^2} = 119.32\ \text{N/mm}^2$$

$$\text{Elongation of both the end parts} = \frac{\sigma\times\text{length}}{E} = \frac{119.32\times(250-l)}{2\times10^5}$$

**Whole Bar :**

$$\text{Total elongation} = \frac{150l}{2\times10^5} + \frac{119.32\,(250-l)}{2\times10^5}$$

or
$$0.175 = \frac{150l}{2\times10^5} + \frac{119.32\,(250-l)}{2\times10^5}$$

$$\begin{aligned} 0.175\times2\times10^5 &= 150\,l + 119.32\times250 - 119.32\,l \\ &= 30.68\,l + 29830 \\ 30.68\,l &= 5130 \\ \Rightarrow \quad l &= \mathbf{168.51\ mm} \quad \textbf{Ans.} \end{aligned}$$

**Example 1.15** *A bar of varying cross-section is shown if Fig. 1.21. If $P_1$ = 260 kN, $P_3$ = 300 kN and $P_4$ = 160 kN, determine the force $P_2$ for equilibrium of the bar.*

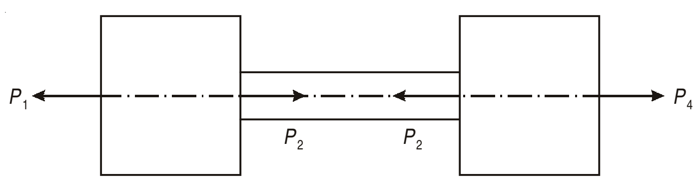



**Fig. 1.21**

**Solution** As shown, all the forces acting on the bar are horizontal, therefore, for its equilibrium $\Sigma H = 0$

or
$$\begin{aligned} P_1 + P_3 &= P_2 + P_4 \\ 260 + 360 &= P_2 + 160 \qquad \Rightarrow P_2 = \mathbf{460\ kN} \quad \textbf{Ans.} \end{aligned}$$

**Example 1.16** *A brass bar having cross-section area of $10^3$ mm$^2$ is subjected to axial forces, as shown in Fig. 1.22 (a). Find the total elongation of the bar. Modulus of elasticity of brass = 100 kN/mm$^2$.*

**Solution** Consider the equilibrium of three different portions *AB*, *BC*, and *CD*.

Portion *Ab* is in equilibrium under one tensile force of 50 kN tensile to the left and an effective tensile force 50 kN tensile i.e., to the right of it, which is the resultant of 3 forces to its right, i.e., 80 kN tensile, 20 kN compressive and 10 kN compressive.

Similarly, portion *BC* is in equilibrium under an effective compressive force of 30 kN, and portion *CD* is in equilibrium under effective 10 kN compressive force.

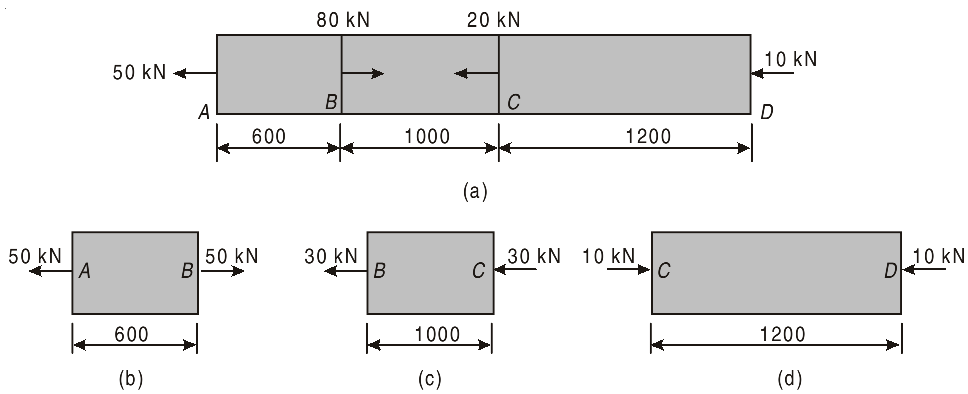



**Fig. 1.22**

The three protions can be represented separately as shown in Fig. 1.22 (b) (c) and (d).

Let $x_1$, $x_2$ and $x_3$ be the changes in length of protions *AB*, *BC* and *CD* respectively.

$\therefore \quad x_1 = \dfrac{W_1 l_1}{AE}$ ; increasae (say +ve) ; $x_2 = \dfrac{W_2 l_2}{AE}$ ; decrease (–ve); $x_3 = \dfrac{W_3 l_3}{AE}$ ; decrease (–ve)

Hence net change in length = $x_1 + x_2 + x_3$

$$= \frac{W_1 l_1}{AE} - \frac{W_2 l_2}{AE} - \frac{W_3 l_3}{AE} = \frac{1}{AE}(W_1 l_1 - W_2 l_2 - W_3 l_3)$$

$$= \frac{1}{10^3 \times 100}(50 \times 600 - 30 \times 1000 - 10 \times 1200)$$

$$= \mathbf{-\ 0.12\ mm} \quad \textbf{Ans.}$$

Negative means that the bar will be shortened by 0.12 mm.

**Example 1.17** *A bar ABCD of varying cross-section is horizontally fixed at ends A and D such that parts AB, BC and CD, each are of equal length 200 mm and of cross-sectional area of 7500, 5000 and 2500 mm² respectively. At B and C, the bar is subjected to axial loads of 50 kN and 100 kN respectively directed towards right-hand side. Neglecting any bending or buckling effect, determine the load shared by each part AB, BC and CD and displacement of section B and C.*

*Take* $E = 210 \text{ kN/mm}^2$

**Solution** The bar is shown in Fig. 1.23. The parts *AB, BC* and *CD* have been shown as 1, 2 and 3 respectively and the quantities pertaining to these shall carry *suffix* 1, 2, 3 respectively.

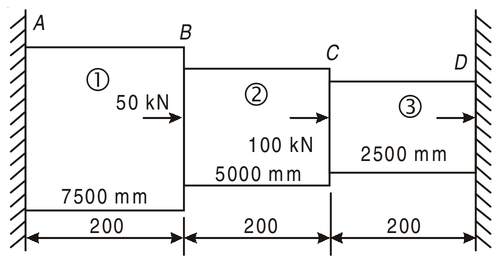



**Fig. 1.23**

As given $l_1 = l_2 = l_3 = 200$ mm

$A_1 = 7500 \text{ mm}^2$

$A_2 = 5000 \text{ mm}^2$

$A_3 = 2500 \text{ mm}^2$

Suppose load shared by *AB, BC* and *CD* are $W_1$, $W_2$ and $W_3$ respectively and that the direction of load on *AB* is an shown in Fig. 1.23 (a) and that on *BC* and *CD* as shown in Fig. 1.23 (b) and (c) respectively.

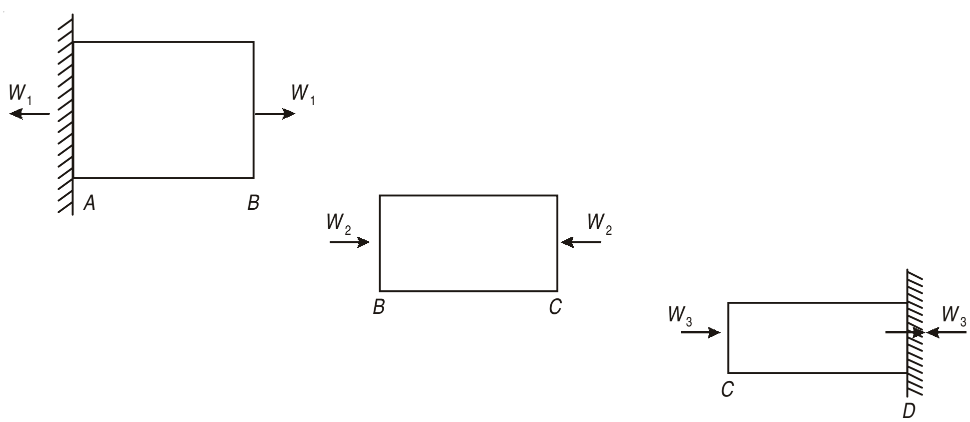



**Fig. 1.23 (a)** **Fig. 1.23 (b)** **Fig. 1.23 (c)**

$W_1$ is the reaction at $A$ and $W_3$ at $D$ as shown.

Now, since the length between $A$ and $D$ has to remain constant. Therefore, elongation of $AB$ = Compression of $BC$ + Compression of $CD$

$$\Rightarrow \quad \frac{W_1 l_1}{A_1 E} = \frac{W_2 l_2}{A_2 E} + \frac{W_3 l_3}{A_3 E}$$

$$\Rightarrow \quad \frac{W_1}{A_1} = \frac{W_2}{A_2} + \frac{W_3}{A_3} \qquad [\because l_1 = l_2 = l_3]$$

$$\frac{W_1}{75000} = \frac{W_2}{5000} + \frac{W_3}{2500}$$

$$2W_1 = 3W_2 + 6W_3$$

$$\Rightarrow \quad W_1 = 1.5\, W_2 + 3W_3 \qquad \text{... (i)}$$

## 1.14 COMPOSITE SECTIONS UNDER TENSION OR COMPRESSION

A composite section means a section made of different metals connected rigidly at both ends together. It is also called a *compound section*.

Suppose, a conposite section consists of a central copper rod, loosely surrounded byna brass tube and their both ends firmly connected together, as shown in Fig. 1.24. Let the bar be subjected to an axial tensile (or compressive) load $W$ as shown.

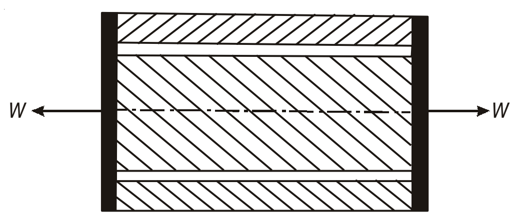



**Fig. 1.24**

Let
$A_b$ = cross-sectional area of the brass tube
$A_c$ = cross-sectional area of the copper rod
$E_b$ = Young's modulus for brass
$E_c$ = Young's modulus for copper
$l$ = their common length.

and $s_b$ and $s_c$ are the stresses induced in brass and copper respectively.

Since both the rod and the tube are held together at the ends, their elongation and *hence* the strains will be the same, but as their Young's moduli have different values, so the stress induced in each will also be different.

Since strain in both is the same,

$$\therefore \qquad \frac{\sigma_b}{E_b} = \frac{\sigma_c}{E_c} \qquad \text{... (i)}$$

But $\qquad W = W_b + W_c = \sigma_a A_b + \sigma_c A_c \qquad$ ... (ii)

where $\qquad W_c$ = load shared by copper

and $\qquad W_b$ = load shared by brass

From the above two Eqs. (i) and (ii), the stresses $\sigma_b$ and $\sigma_c$ and hence the load shared by each can be calculated.

**Example 1.18** *A copper rod of 30 mm diameter is surrounded tightly by a cast iron tube of 60 mm external diameter, the ends being firmly fastened together. When put to a compressive load of 12 kN what load will be shared by each ?*

*Also estimate the amount by which the compound bar shortens in a length of 1 m.*

$$E_{c.i} = 175 \text{ kN/mm}^2 \text{ and } E_c = 75 \text{ kN/mm}^2.$$

**Solution** $\qquad A_c = \dfrac{\pi}{4} \times 30^2 = 706.86 \text{ mm}^2$

$$A_{c.i} = \frac{\pi}{4}(60^2 - 30^2) = 2120.58 \text{ mm}^2$$

Strain in C.I. tube = Strain in copper rod

$$\frac{\sigma_{c.i}}{E_{c.i}} = \frac{\sigma_c}{E_c} \quad \text{or} \quad \frac{\sigma_{c.i}}{175} = \frac{\sigma_c}{75}$$

or $\qquad \dfrac{\sigma_{c.i}}{\sigma_c} = 2.33$

Total load = Load shared by C.I. tube + Load shared by copper rod

or $\qquad W = W_{c.i} + W_c$

$$= s_{c.i.} A_{c.i} + \sigma_c A_c = \sigma_{c.i} \times 2120.58 + \sigma_i \times 706.86$$

From Eqs. (i) either change $\sigma_{c.i}$ to $\sigma_c$ or *vice versa*

$$W = 2.33\,\sigma_c \times 2120.58 + \sigma_c \times 706.86 = \sigma_c \times 5647.8$$

or $\qquad 12 \times 19^3 = \sigma_c \times 5647.8 \qquad \Rightarrow \sigma_c = 2.125 \text{ N/mm}^2$

and from Eq. (i)

$$\sigma_{c.i.} = 2.125 \times 2.33 = 4.95 \text{ N/mm}^2$$

$$W_c = \sigma_c A_c = 2.125 \times 706.86 = 1502 \text{ N} \quad \text{or } 1.502 \text{ kN}$$

$$W_{c.i} = \sigma_{c.i} A_{c.i.} = 4.95 \times 2120.58 = 10498 \text{ N} \quad \text{or } 10.498 \text{ kN}$$

Strain $\qquad = \dfrac{\sigma_c}{E_c}$ or $\dfrac{\sigma_{c.i}}{E_c} = \dfrac{\text{Decrease in length}}{\text{Original length}}$

$\therefore$ Decrease in length or the amount by which the compound bar will shorten :

$$X = \frac{\sigma_c}{E_c} \times l = \frac{2.125}{7.5 \times 15^4} \times 1000 \ = \mathbf{0.00283\ mm} \quad \textbf{Ans.}$$

**Example 1.19** *Two rods of equal lengths, one of steel and the other of copper and each of 10 mm diameter, are rigidly fastened at the upper ends at a distance of 300 mm. A horizontal bar connects them at their lower ends and a load of 20 kN is placed on the bar such that it remains horizontal. Calculate the load carried by each rod and the position of the load for equal strains in the rods. Assume* $E_r/E_c = 13/7$.

**Solution** Refer Fig. 1.25. In case the load acts at the centre of the connecting bar, due to different moduli of elasticity the extension of the two rods will be unequal, as such the bar cannot remain horizontal. The bar will remain horizontal only when the load is placed in such a position that the extension or the strain in each rod is the same.

Let the load $W$ be placed at a distance $x$ mm from the centre of steel roas as shown.

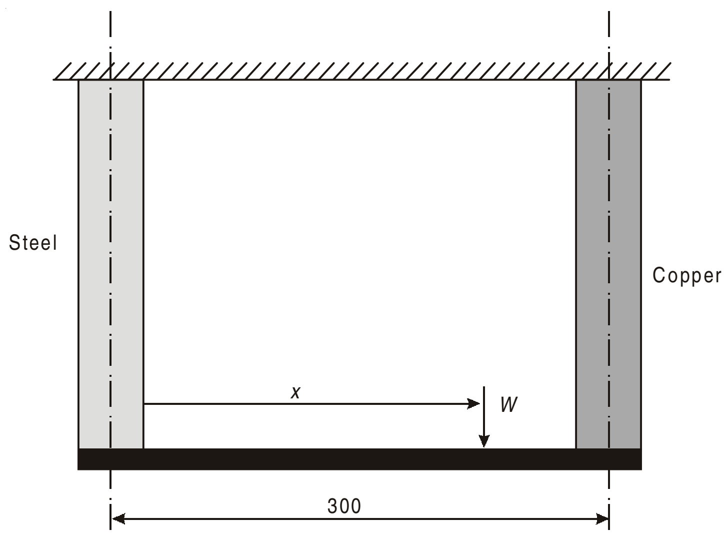



**Fig. 1.25**

Since strain is to be the same

$$\therefore \qquad \frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c} \quad \text{or} \quad \frac{\sigma_s}{\sigma_c} = \frac{E_s}{E_c} = \frac{13}{7}$$

$$W = W_s + W_c = \sigma_c A_s + \sigma_c A_c = \frac{13}{7}\ \sigma_c A_s + \sigma_c A_c$$

$$20 = \sigma_s \times \frac{\pi}{4} \times 10^2 \left(\frac{13}{7} + 1\right)$$

$$\left[\text{since } A_s = A_c = \frac{\pi}{4} \times 10^2 \text{ mm}^2\right]$$

$$\Rightarrow \qquad \sigma_c = 0.089 \text{ kN/mm}^2$$

$$W_c = \sigma_c A_c = 0.089 \times \frac{\pi}{4} 10^2 = \mathbf{7\ kN} \quad \textbf{Ans.}$$

$$W_c = W - W_c = 20 \times 7 = \mathbf{13\ kN} \quad \textbf{Ans.}$$

Now, moments of $W_s$ and $W_c$ together, about any point will be equal to the moments of their resultant or total load $W$ about the same point.

Taking moments about the axis of the steel rod,

$$W \times x = W_c \times 300 \quad \text{or } 20 \times x = 7 \times 300$$

$$x = \mathbf{105\ mm} \quad \textbf{Ans.}$$

**Example 1.20** *A steel wire 2.8 mm in diameter is covered by six bronze wires each 2.5 mm in diameter. If the working stress in the bronze is 63 N/mm², calculate (a) the strength of the combination, (b) the equivalent tensile modulus.*

$$E_s = 200 \text{ kN/mm}^2 \quad \text{and } E_b = 100 \text{ kN/mm}^2.$$

**Solution** (a) Area of steel :

$$A_s = \frac{\pi}{4} \times 2.8^2 = 6.16 \text{ mm}^2$$

Area of bronze $\qquad A_b = \frac{\pi}{4} \times 2.5^2 \times 6 = 29.45 \text{ mm}^2$

Since bronze and steel will extend eqaully under load,

Tensile strain in bronze= Tensile strain in steel

or $$\frac{\sigma_b}{E_b} = \frac{\sigma_s}{E_s} \Rightarrow \sigma_s = \sigma_b \times \frac{E_s}{E_b} = \frac{63 \times 200}{100} = 126 \text{ N/mm}^2$$

Permissible load $= \sigma_s A_s + \sigma_b A_b = 126 \times 6.16 + 63 \times 29.45$

$= 2631.5 \text{ N} = 2.632 \text{ kN}$ **Ans.**

(b) Common Strain :

$$e = \frac{\sigma_b}{E_b} = \frac{63}{10^5} = 6.3 \times 10^{-4}$$

$$\therefore \qquad \frac{\text{Total Load}}{\text{Total Section}} = \text{Equivalent modulus} \times \text{Common strain}$$

or Equivalent Modulus $$= E_{eq} = \frac{\Sigma W}{\Sigma A} \times \frac{1}{e} = \frac{2631.5}{(6.16 + 29.45)} \times \frac{1}{6.3 \times 10^{-4}}$$

$$= 117.3 \times 10^3 \text{ N/mm}^2 = 117.3 \text{ kN/mm}^2 \quad \textbf{Ans.}$$

**Example 1.21** *A solid steel bar 500 mm long and 70 mm diameter is placed inside an aluminium tube having 75 mm inside diameter and 100 mm outside diameter. The aluminium cylinder is 0.15 mm longer than steel cylinder. An axial load 600 kN is applied to the bar and cylinder through rigid cover plates as shown in Fig. 1.26. Find the stress developed in the steel bar and the aluminium tube.*

*Assume, for steel E = 220 kN/mm² and for aluminium E = 70 kN/mm².*

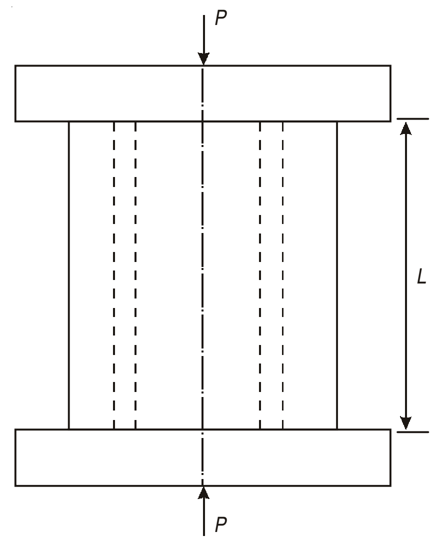



**Fig. 1.26**

**Solution** First of all, when the load is applied the aluminium cylinder, which is longer by 0.15 mm than the steel bar, will be compressed by this amount for which load required

$$= 70 \times 10^3 \times \frac{0.15}{500} \times \frac{\pi}{4} (100^2 - 75^2) = 72160$$

$$N = 72.16 \text{ kN}$$

After the aluminium cylinder and steel rod become of the same length, the rest of the load will be shared by them and the strain developed in them will be the same.

Net load now to be shared = 600 – 72.16 = 527.84 kN

Let '*e*' be the common strain and $\sigma_s$ the stress in steel rod and $\sigma_a$ the additional stress in aluminium cylinder due to sharing of 527.84 kN.

Then $$E = \frac{\sigma_s}{E_s} = \frac{\sigma_a}{E_a} \quad \text{or } \sigma_s = \sigma_s \frac{E_s}{E_a} = \sigma_a \times \frac{220}{70} = \frac{22}{7}\sigma_a$$

and $527.84 \times 10^3 = \sigma_a A_a + \sigma_s A_s$

$$= \sigma_a A_a + \sigma_a \times \frac{22}{7} \times A_s = \sigma_a \left( A_a + \frac{22}{7} A_s \right)$$

$$= \sigma_a \left[ \frac{\pi}{4}(100^2 - 75^2) + \frac{22}{7} \times \frac{\pi}{4} \times 70^2 \right] = 15531\ \sigma_a$$

$$\Rightarrow \qquad \sigma_a = \frac{527.84 \times 10^3}{15531} = 33.98 \text{ N/mm}^2$$

and $$\sigma_s = \frac{33.98 \times 22}{7} = 106.81 \text{ N/mm}^2$$

Stress in aluminium ??? due to 72.16 kN load = $\dfrac{72.16 \times 10^3}{\frac{\pi}{4}(100^2 - 75^2)}$ = 21 N/mm$^2$

Hence total stress in the aluminium tube = 33.98 + 21 = 54.98 N/mm$^2$ **Ans.**

and stress in steel rod = 106.31 N/mm$^3$ **Ans.**

**Example 1.22** *A steel rod 20 mm diameter passes centrally through a steel tube 25 mm internal diameter and 38 mm external diameter. The tube is 750 mm long and is closed by rigid washers of negligible thickness which are fastened by nuts threaded on the rod. The nuts are tightened until the compressive load on the tube is 20 kN. Calculate the stresses in the tube and the rod.*

*Find the increase in these stresses when one nut is tightened by one quarter of a turn relative to the other. Take E = 210 kN/mm$^2$ and pitch of threads = 2.5 mm.*

**Solution** Refer Fig. 1.27.

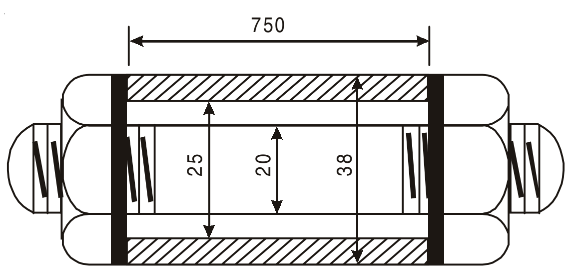



**Fig. 1.27**

Area of steel rod : $A_r = \frac{\pi}{4} \times 20^2 = 314 \text{ mm}^2$

Compressive force on tube = 20 kN = 2 × $10^4$ N.

Also, compressive force on steel tube = Tensile force in steel rod,

or $$A_t = \sigma_{t_1} = A_r \sigma_{r_1} = 2 \times 10^4$$

where $\sigma_{t1}$ and $\sigma_{r1}$ are compressive and tensile stresses in steel tube and steel rod respectively in N/mm$^2$.

$$\therefore \quad 314 \times \sigma_{r_1} = 643.24 \times \sigma_{t_1} = 2 \times 10^4$$

⇒ $\sigma_{r1}$ = 63.66 N/mm$^2$ tensile **Ans.** and $\sigma_{r_1}$ = 31.1 N/mm$^2$ compressive **Ans.**

Let $W$ be the addtional load induced in the rod and the tube such that $\sigma_{r_2}$ and $\sigma_{r_2}$ be the additional tensile and compressive stress induced in the rod and the tube respectively due to $\frac{1}{4}$ the revolution of the nut.

Axial distance moved by the nut in $\frac{1}{4}$ th of the revolution = $\frac{1}{4}$ × 2.5 = 0.625 mm

$$0.625 = \text{Constraction of the tube + Extension of rod}$$

or $$0.625 = \frac{\sigma_{t_2}}{E} \times l + \frac{\sigma_{r_2} l}{E} = \frac{l}{E}(\sigma_{t_1} + \sigma_{r_2})$$

But $$\sigma_{t_2} A_t = \sigma_{r_2} A_r$$

or $$\sigma_{t_2} = \sigma_{r_2} \frac{A_r}{A_t} = \sigma_{r_2} \frac{314}{643.24} = 0.488\sigma_{r_2}$$

$$\Rightarrow \quad 0.625 = \frac{1}{E}(0.488\sigma_{r_2} + \sigma_{r_2}) \text{ or } \frac{750}{2.1 \times 10^5}\sigma_{r_2}(0.488 + 1)$$

$$\Rightarrow \quad \sigma_{r_2} = 0.625 \times \frac{2.1 \times 10^5}{750} \times \frac{1}{1.488} = 117.6 \text{ N/mm}^2$$

**Ans.**

and $$\sigma_{t_2} = 0.488 \times 117.6 = 57.4 \text{ N/mm}^2$$ **Ans.**

**Example 1.23** *A heavy steel plate of uniform thickness, weighing 100 kN, is supported by three vertical equi-spaced coplanar bars, each of 25 mm diameter and of equal length l, as shown in Fig. 1.28. The outer bars are of copper and the inner that of brass. Assuming that the heavy plate remains horizontal, find the intensity of stress in each bar. Take E for copper = 150 kN/mm$^2$ and that for brass = 100 kN/mm$^2$.*

**Solution** Since the plate is of uniform thickness, its weight $W$ will act at its centre as shown.

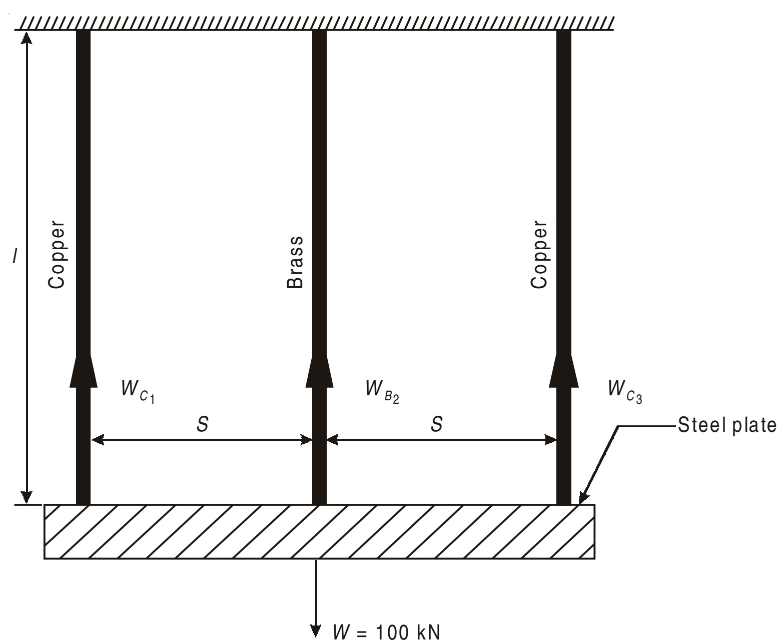



**Fig. 1.28**

Suppose $W_{C_1}$, $W_{B_2}$ and $W_{C_3}$ are loads shared by the bars as shown.

Taking moment about centre of the plate,

$$W_{C_1} \times S = W_{C_3} \times S \quad \Rightarrow W_{C_1} = W_{C_3}$$

But $$W = W_{C_1} + W_{B_2} + W_{C_3}$$

or $$100 = 2W_{C_1} + W_{B_2} \quad \text{... (i)}$$

As the plate remians horizontal, elongation of each bar will be same. Again as the length of each is the same, therefore, strain in each bar will be the same.

$\Rightarrow$ $$e_{C_1} = e_{B_2} = e_{C_3}$$

$$\frac{\sigma_{C_1} l}{E_C} = \frac{\sigma_{B_2} l}{E_B} = \frac{\sigma_{C_3} l}{E_C}$$

$$\frac{W_{C_1}}{A_{C_1} E_C} = \frac{W_{B_2}}{A_{B_2} E_B} = \frac{W_{C_3}}{A_{C_3} E_{C_3}}$$

$$\frac{W_{C_1}}{E_{C_1}} = \frac{W_{B_2}}{E_{B_2}}$$

$$W_{C_1} = \frac{E_{C_1}}{E_{B_2}} \times W_{B_2} = 1.5\, W_{B_2}$$

From Eq. (i)

$$100 = 2 \times 1.5\, W_{B_2} + W_{B_2}$$

$$\Rightarrow \quad W_{B_2} = 25 \text{ kN} ; \quad W_{C_1} = 37.5 \text{ kN}; \quad W_{C_3} = 37.5 \text{ kN}$$

$$\sigma_{C_1} = \sigma_{C_3} = \frac{37.5 \times 100}{\pi/4 \times 25^2} = \mathbf{76.37\ N/mm^2} \quad \textbf{Ans.}$$

and

$$\sigma_{B_2} = \mathbf{50.011\ N/mm^2} \quad \textbf{Ans.}$$

## 1.15 TEMPERATURE STRESS AND STRAIN

When the temperature of a body is raised or lowered, there is corresponding increase or decrease in its dimensions and if this change in dimensions due to the temperature variation is *prevented* by application of external forces, the body develops stress in it, which is called the *temperature stress* ; and the corresponding strain is called the *temperature strain*.

The extension due to rise of temperature, can be checked or suppressed by compressive forces; thereby producing compressive stresses in the body or vice versa.

Suppose, a bar of uniform section and of length $l$ is heated through temperature $T$. The length of the bar will increase, depending upon its *co-efficient of linear expansion* which is defined as the increase in length per unit rise of temperature per unit original length and is generally denoted by the Greek latter a (alpha).

Let $\alpha$ = co-efficient of linear expansion.

Extension of the bar when free to expand = $\alpha Tl$.

Now suppose, this extension due to increase of temperature is prevented by either fixing the bar at its ends or by the application of external compressive forces.

$$\text{Temperature strain so induced} = \frac{\text{Extension Prevented}}{\text{Original Length}}$$

$$= \frac{\alpha T l}{l} = \alpha T \text{ (compressive)}$$

Temperature stress = Temperature strain × $E$ = $\alpha TE$ (compressive)

Conversely, the contraction caused by lowering of the temperature can be checked by applying tensile forces to the bar resulting in temperature stress of tensile nature in it.

### 1.15.1 Temperature Stress in Composite Bars

Consider a composite bar comprising of two metals (1) and (2), as show in Fig. 1.29. Suppose $x$–$x$ be the initial level and $y$–$y$ be the final level after the rise in temperature $t$. If both bars were free to expand, the extension of material (1) would be $l\alpha_1 t$ and that of material (2) would be $l\alpha_{2t}$ on the assumption $\alpha_2 > \alpha_1$. Since both materials are rigidly connected at the ends, material (1) is forced to exrtend a distance $x'_1$ and material (2) is forced to compress a distance $x'_2$ as show in figure. Stresses in materials will be induced only due to these forced changes $x'_1$ and $x'_2$.

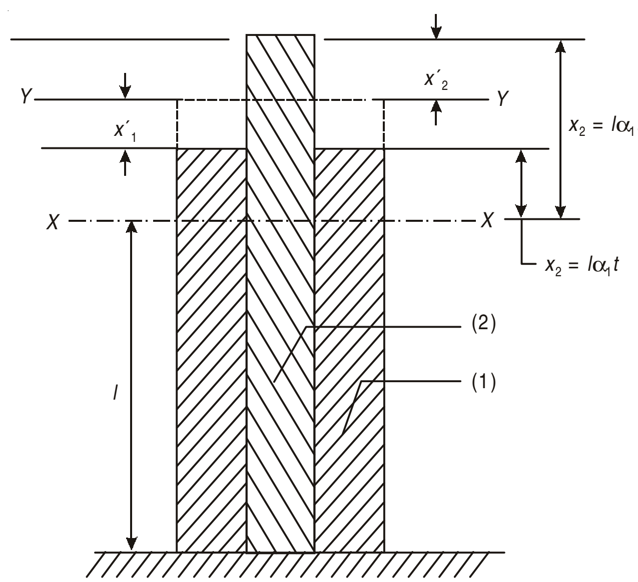



**Fig. 1.29**

From the figure it will be seen that :

$$x'_1 + x'_2 = l\alpha_2 t - l\alpha_1 t$$

or
$$\frac{\sigma_1 l}{E_1} + \frac{\sigma_2 l}{E_2} = tl\,(\alpha_2 - \alpha_1) \quad \text{... (i)}$$

Also since no external force is applied to the composite bar,

Tensile force in material (1) = Compressive force in material (2)

i.e.,
$$\alpha_1\,\alpha_1 = \alpha_2\alpha_2 \quad \text{... (ii)}$$

$\alpha_1$ and $\alpha_2$ can be evaluated from Eqs. (i) and (ii)

**In addition to rise in temperature *t*, if the bar is subjected to an external load *P*** (Fig. 1.30) material (2) will suffer further compression and

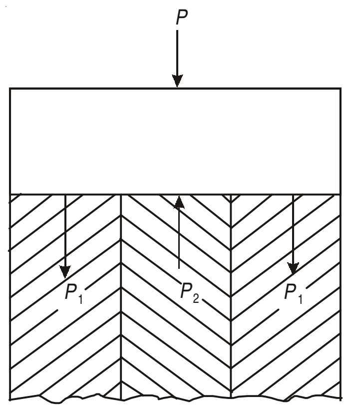



**Fig. 1.30**

exerts an additional upward reaction $P_2$, whereas material (1) will be relieved of its forced extension $x'_1$ and help $P$ in downward by force $P_1$

$\therefore$ For equilibrium$P + P_1 = P_2$

or $\alpha_1 \alpha_1 - \alpha_2\alpha_2 = P$

From expressions (i) and (iii), $\alpha_1$ and $\alpha_2$ cn be calculated.

**Example 1.24** *All undeground pipe line is laid in spring at 15°C. What stress would be induced in it when the temperature falls down to – 2°C in winter and the pipe line is unable to contract ? The co-efficient of linear contraction of the pipe is 0.000012 per °C and E = 2 × 10⁶ N/mm².*

**Solution** Fall n tempertaure = 15 – (– 2) = 17°C

Contraction when the pipe is free to contract = $\alpha 77$

Temperature strain, when the contraction is prevented $= \dfrac{\alpha Tl}{l} = \alpha T$ (tensile)

$\therefore$ Temperature stress induced $= aTE = 0.000012 \times 17 \times 2 \times 10^5$

$= 40.8$ N/mm$^2$ (tensile) **Ans.**

**Example 1.25** *A copper rod 15 mm diameter, 0.8 m long, is heated through 50°C. What is its extension when free to expand ? Suppose, the expansion is prevented by gripping it at both ends, find the stress, its nature, and the force applied by the grips when,*

*(a) the grips do not yield,*

*(b) one grip yields back by 0.5 mm.*

*Given :* $a_c = 18.5 \times 10^{-6}$ *per °C and* $E_c = 1.25 \times 10^5$ *N/mm².*

**Solution** Cross-sectional area of the rod $= \dfrac{\pi}{4} \times 15^2 = 176.7 \text{ mm}^2$

Extension when the rod is free to expand

$= \alpha 77 = 18.5 \times 10^{-6} \times 50 \times 800 = 0.74$ mm **Ans.**

(a) When the grips don't yield extension prevented = α 77

$$\text{Temperature strain} = \frac{\alpha Tl}{l} = \alpha T$$

$$\text{Temperature stress} = \alpha TE = 18.5 \times 10^{-6} \times 50 \times 1.25 \times 10^5$$

$= 115.6$ N/mm² (compressive) **Ans.**

Compressive forces applied through the grips to prevent the extension

$= 115.6 \times 176.7 = 20431$ N or 20.431 kN **Ans.**

(b) When one grip yields by 0.5 mm extension prevented

$= \alpha 77 - 0.5 = 0.74 - 0.5 = 0.24$ mm

$$\therefore \quad \text{Temperature strain} = \frac{0.24}{l} = \frac{0.24}{800}$$

$$\text{Temperature stress} = \frac{0.24}{800} \times 1.25 \times 10^5 = 37.5 \text{ N/mm}^2 \text{ (compressive)}$$

Force applied $= 37.5 \times 176.7 = 6626$ N or 6.626 kN **Ans.**

**Example 1.26** *A steel band or a ring is shrunk on a tank of 1 m diamter by raising the temperature of the ring through 60°C. Assuming the tank to be rigid, what should be the original inside diameter of the ring before heating ? Also calculate the hoop strees in the ring when it cools back to the normal temperature on the tank.* $a_s = 10^{-5}$ *per °C, and* $E_s = 2 \times 10^5$ *N/mm².*

**Solution** Let *d* mm be the original internal diameter of the band at normal tempertuare and *D* mm after being heated through 60°C. *D*, should be of course, equal to the outer diameter of the tank for slipping the ring on to it.

Circumference of the ring after heating = $\pi D$

Circumference of the ring at normal temperature = $\pi d$

The ring after having been slipped on the tank cannot contract to $\pi d$ when it cools down, resulting in tensil stress in it.

$\therefore$ Contraction prevented = $\pi (D - d)$

$$\text{Temperature strain} = \frac{\pi(D-d)}{\pi d} = \alpha_3 T \quad \Rightarrow \frac{D-d}{\alpha} = \alpha_s T$$

$$\frac{1000-d}{d} = 10^{-5} \times 60 \quad \Rightarrow d = \mathbf{999.4\ mm} \quad \textbf{Ans.}$$

Circumference temperature stress or stress due to precention of contraction of the ring.

**Example 1.27** *A 50 × 25 mm copper flat is brazed to another 50 × 50 mm steel flat as shown in Fog. 1.31 (a). The combinition is then heated through 100°*

*C. Find the stress induced in each and the shear force tending to rupture the brazing and the shear stress.*

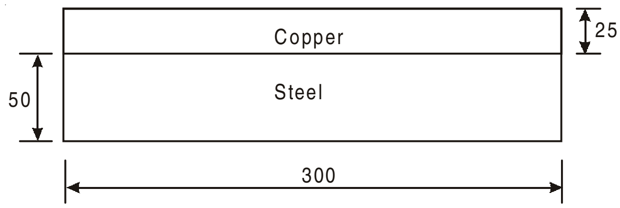



**Fig. 1.30**

$$\alpha_c = 18.5 \times 10^{-6} \text{ per } °C$$
$$\alpha_s = 12 \times 10^{-6} \text{ per } °C$$
$$E_s = 2 \times 10^5 \text{ N/mm}^2$$
$$E_c = 10^5 \text{ N/mm}^2$$
$$\text{Length of each flat} = 300 \text{ mm.}$$

**Solution** Since $\alpha_c > \alpha_s$, elongation of copper will naturally be more than that of steel for the same rise of temperature ; but since they are connected together, the copper flat will venture to pull the steel flat alongwith it, whereas the steel flat will struggle to bring the copper back. Ultimately, they will compromise and become stable at certain common position.

Extension of copper when free to expand = $\alpha_c$ 77 = $ab$ as shown in Fig. 1.30

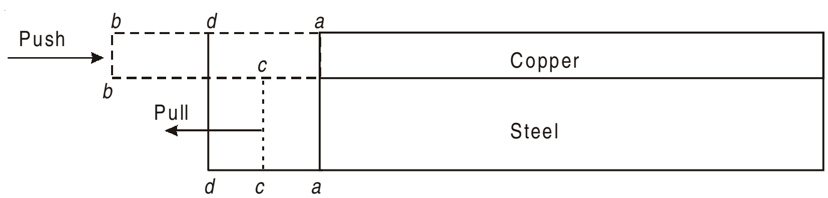



**Fig. 1.30**

Extension of steel when free to expand = $\alpha_s\, Tl = ac$.

Being connected together, suppose, they compromise at the position $d$–$d$, which means that steel will be pulled from $c$ to $d$ and copper pushed back from $b$ to $d$. In this way, steel is under tension and the copper under compression.

$\therefore$ Compressive strain in copper : $e_c = \dfrac{bd}{l} = \dfrac{ab - ad}{l}$

$$= \alpha_c\, T - \frac{ad}{l} = a_c T - e \quad \text{... (i)}$$

where $\dfrac{ad}{l}$ = common steain $e$.

Tensile strain in steel

$$e_s = \frac{cd}{l} = \frac{ad - ac}{l} = \frac{ad}{l} - \alpha_s T,$$

since $\quad ac = \alpha_s Tl = e - \alpha_s T$

Adding Eqs. (i) and (ii),

$$e_c + e_s = \alpha_c T - \alpha_s T = T(\alpha_c - \alpha_s)$$

But $\quad e_s = \dfrac{\sigma_c}{E_c}$, and $e_s = \dfrac{\sigma_s}{E_s}$

$$\Rightarrow \quad \frac{\sigma_c}{E_c} + \frac{\sigma_s}{E_s} = T(\alpha_c - \alpha_s)$$

$$\frac{\sigma_c}{10^5} + \frac{\sigma_s}{2 \times 10^5} = 100\,(18.5 \times 10^{-6} - 12 \times 10^{-6})$$

$$\Rightarrow \quad \sigma_c + 0.5\,\sigma_s = 65 \quad \text{... (iii)}$$

But the stabilised or common position $d - d$.

Push on copper = Pull on steel

$$\sigma_c A_c = \sigma_c A_s$$

$$\sigma_c \times 50 \times 25 = \sigma_s \times 50 \times 50 \quad \Rightarrow \sigma_c = 2\sigma_s$$

Substituting either $\sigma_c$ or $\sigma_s$ in Eq. (iii), we get $\sigma_c$ = 52 N/mm$^2$ and $\sigma_s$ = 26 N/mm$^2$

Shear forces tending to rupture brazing = the pull, or the push, being equal and opposite

$$= \sigma_s A_s \text{ or } \sigma_c A_c = 65000 \text{ N or } 65 \text{ kN.} \quad \textbf{Ans.}$$

$$\text{Shear stress} = \frac{\text{Shear force}}{\text{Shear area}} = \frac{65000}{300 \times 50} = \mathbf{4.33\ N/mm^2} \quad \textbf{Ans.}$$

**Example 1.28** *Two side members of a water cooler are 75 mm × 50 mm box section aluminium 5 mm thick and 600 mm long. Between them are 260 vertical copper tubes 6 mm outside diameter and 3 mm bore, of the same length. Assuming that (i) assembly was carried out at 20°C, (ii) the headers are rigid, (iii) tubes do not buckle, (iv) side members remain cold, calculate for an operating temperature of 90°C (a) the stress in the tubes, (b) the increase in the height of water coller $a_c$ = 17 × 10$^{-6}$/°C, $E_c$ = 120 G N/mm$^2$, $E_a$ = 70 G N/mm$^2$.*

**Solution** $\quad A_a = 2\,[75 \times 50 - 65 \times 40] = 2300$ mm$^2$

$$A_c = 260 \times \frac{\pi}{4}(6^2 - 3^2) = 5513.5 \text{ mm}^2$$

Temperature rise : $\quad T = 90 - 20 = 70°$ C

Extension in copper when free to expand = $\alpha_c TL$

Actual extension in copper = Extension in aluminium = $\dfrac{\sigma_a L}{E_a}$

$$\text{Compression of copper} = \alpha_c\, TL - \frac{\sigma_a L}{E_a}$$

$$\text{Strain in copper} = \frac{\alpha_c TL - \dfrac{\sigma_a L}{E_a}}{L} = \alpha_c T - \frac{\sigma_a}{E_a}$$

$$\therefore \quad \text{Stress in copper : } \quad \sigma_c = E_c \times \text{strain} = E_c \left[\alpha_c T - \frac{\sigma_a}{E_a}\right]$$

Also $\sigma_a A_a = \sigma_c A_c$ or $\sigma_a = s_c \dfrac{A_c}{A_a}$

$$\Rightarrow \quad \sigma_a = \sigma_c \frac{5513.5}{2300} = 2.397\sigma_c$$

$$\sigma_c = 120 \times 10^3 \left[\frac{17}{10^6} \times 70 - \frac{2.397\sigma_c}{70 \times 10^3}\right]$$

$$= 142.8 - 4.11\, \sigma_c$$

$\Rightarrow \quad \sigma_c = 27.95$ N/mm$^2$ **Ans.**

and $\quad \sigma_a = 27.95 \times 2.397 = 67$ N/mm$^2$

Extension in aluminium or increase in height of cooler :

$$\frac{\sigma_a L}{E_a} = \frac{67 \times 600}{80 \times 10^3} = 0.574 \text{ mm} \quad \textbf{Ans.}$$

**Example 1.29** *Three rods each of 1000 mm² cross-section are equal-spaced in a vertical plane and carry a tensile load of 150 kN. Their temperature is raised by 100°C and that load is so adjusted that they extend equally. Determine the load shared by each. The outer two rods are of brass and the middle one is of steel.*

$$\alpha_s = 0.000012 \text{ per °C}$$
$$\alpha_b = 18.5 \times 10^{-6} \text{ per °C}$$
$$E_s = 200 \text{ kN/mm}^2$$
$$E_b = 80 \text{ kN/mm}^2.$$

**Solution** The total extension $x$ of each rod is the sum of the extensions due to rise of temperature and that due to load carried by each.

Extension of each rod due to increase in temperature = α 77

Extension of each due to load = ($x$ – α 77)

$$\text{Strain due to load} = \left(\frac{x}{l} - \alpha T\right)$$

$$\text{Stress due to load} = \left(\frac{x}{l} - \alpha T\right) E$$

$$\text{Load} = \left(\frac{x}{l} - \alpha T\right) EA$$

$$\text{Total load} = \text{Sum of loads on each rod}$$

$$= \left(\frac{x}{l} - \alpha_s T\right) E_s A + 2\left(\frac{x}{l} - \alpha_b T\right) E_b A_b$$

$$150 = \left(\frac{x}{l} - 0.000012 \times 100\right) 200 \times 1000 + 2\left(\frac{x}{l} - 18.5 \times 10^{-6} \times 100\right) 80 \times 100$$

from which $\quad \dfrac{x}{l} = 1.9 \times 10^{-3}$

$$W_b = \text{Load on each brass rod} = \left(\frac{x}{l} - \alpha_b T\right) E_b \times A_b$$

**Example 1.30** *A 30 mm diameter steel rod passes vertically through a copper tube of 40 mm internal and 60 mm external diameter. The length of the tube is 1 m. On the cross-section of the tube on both ends, rigid washers are provided through which the steel rod having threads on its both ends, passes and is finally tightened against the washers at 15°C so as to exert a compressive load of 20 kN, on the tube. Calculate the net stress in each (a) at 15°C (b) when the temperature is raised to 65° C and (c) when the temperature is raised as well as one nut is tightened by 1/4 revolution.*

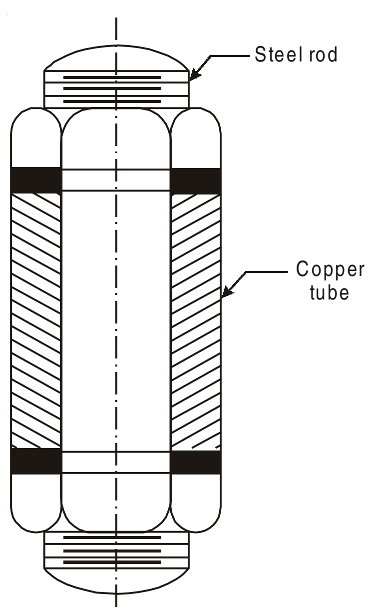



**Fig. 1.31**

$$\text{No. of threads per cm} = 3$$
$$a_s = 12 \times 10^{-6} \text{ per °C}$$
$$a_c = 18.5 \times 10^{-6} \text{ per °C}$$
$$E_s = 2 \times 10^5 \text{ N/mm}^2 \quad \text{and } E_c = 10^5 \text{ N/mm}^2$$

**Solution** The road and the tube in the position are shown in Fig. 1.31.

$$A_s = \text{Area of steel rod} = \frac{\pi}{4} \times 30^2 = 706.86 \text{ mm}^2$$

$$A_c = \text{Area of copper tube} = \frac{\pi}{4}(0^2 - 40^2)$$
$$= 1570.8 \text{ mm}^2$$

**SUMMARY**

**Stress.** It is the resistance offered by the material per unit area due to the application of external forces and is denoted by σ.

**Direct stresses.** Tensile and compressive stresses, which are caused by forces acting normal to the area on which they act are called *direct stresses*.

**Shear stress.** It is caused by the forces acting parallel i.e. tangential to the area to be sheared off.

**Strain.** The deformation per unit length is called as strain and is denoted by ε. Tensile and compressive strains are called *normal strains*.

$$\therefore \qquad \text{Strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{\delta l}{L}$$

**Hooke's law.** Within elastic limit, stress is directly proportional to strain, and change in length, $\delta l = \dfrac{PL}{AE}$.

**Modulus of elasticity or Young's modulus.** It is the ratio of direct stress and normal strain and is denoted by $E$. Hence, $E = \dfrac{\text{Direct stress}}{\text{Normal strain}}$.

**Modulus of rigidity.** The ratio of shear stress to shear strain is called *modulus of rigidity*.

*Complementary shear stresses* are always equal in magnitude but opposite in sign.

$$\text{Elastic (Normal) stress} = \frac{\text{Elastic load}}{\text{Original cross-sectional area}}$$

$$\text{Ultimate stress} = \frac{\text{Ultimate load}}{\text{Original cross-sectional area}}$$

$$\text{Fracture (Rupture) stress} = \frac{\text{Load at fracture}}{\text{Initial X-sectional area}}$$

$$\text{True breaking (Fracture) stress} = \frac{\text{Load at fracture}}{\text{X-sectional area at breaking point}}$$

$$\text{Percentage elongation} = \frac{\text{Increase in length}}{\text{Original gauge length}} \times 100$$

$$\text{Factor of safety} = \frac{\text{Actual strength}}{\text{Required strength}}$$

$$\text{Allowable stress} = \frac{\text{Yield strength}}{\text{Factor of safety}}$$

$$\text{Lateral strain} = \frac{\text{Change in diameter}}{\text{Original diameter}}$$

$$\text{Poisson's ratio} = \frac{\text{Lateral strain}}{\text{Axial strain}}$$

$$\text{Volumetric strain} = \frac{\text{Change in volume}}{\text{Original volume}}$$

$$\text{Superficial strain} = \frac{\text{Change in cross-sectional area}}{\text{Original cross-sectional area}}$$

$$\text{Bulk modulus of elasticity} = \frac{\text{Spherical or volumetric stress}}{\text{Volumetric strain}}$$

Elongation of a uniformly tapered bar

$$(\delta l) = \frac{4PL}{\pi d_1 d_2 E}$$

Extension of bar due to self weight,

$$\delta L = \frac{WL}{2AE}.$$ where, $W$ is the total weight of the bar.

Strain Energy : (a) $U = \frac{1}{2}\frac{\sigma^2}{E} \times$ Volume .....(Gradually applied load)

(b) $U = \frac{\sigma^2}{E} \times$ Volume .....(Suddenly applied load)

(c) $$U = \frac{1}{2}\frac{\tau^2}{G} \times \text{Volume} \text{ .....(Due to shear)}$$

Generalized Hook's law,

$$\text{Volumetric stress, } \varepsilon_v = \frac{1-2v}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Relation between modulus of elasticity and modulus of rigidity

$$G = \frac{E}{2(1+v)}$$

Relation between Young's modulus and Bulk modulus

$$E = \frac{9KG}{3K+G}$$

## EXERCISE

**1.1** State Hook's law. Distinguish between limit of Proportionality and Elastic limit.

**1.2.** Differentiate between (i) Force and Stress, (a) Tensile stress and Compressive stress.

**1.3.** Sketch stress-strain diagrams for the specimen of (i) Mild steel (ductile material), and (ii) cast iron (Brittle material) and define each term used in them.

**1.4.** Explain the terms (i) Modulus of rigidity, (ii) Modulus of elasticity, (iii) Shear stress, (iv) Shear strain, and (v) Bulk modulus.

**1.5.** Find the expression for strain energy stored in a body when load is applied suddenly.

**1.6.** Define in brief: (i) Factor of safefy, (ii) Poisson's ratio, (iii) Lateral strain, (iv) Longitudinal strain, and (v) Volumetric strain.

**1.7.** Distinguish between (i) Stress and strength (ii) Stress and strain (iii) Tensile and shear stress (iv) Force and stress (v) Ductility and brittleness. and (vi) Tensile and compressive stress.

**1.8.** Define (i) Resilience, (ii) Proof resilience, and (iii) Modulus of resilience.

**1.9.** A tensile axial load of 10 kN is applied to a mild steel member 4 m long having 4 $cm^2$ cross-sectional area. Find (i) Stress (ii) Strain (iii) Elongation of the member, if $E$, for mild steel is $2 \times 10^5$ N/$mm^2$.

**1.10.** A bar shown in Fig. 1.20 subjected to axial loads of 40 kN. It has 3 sections of diameters 25 mm, 20 mm and 25 mm and lengths 160 mm, 240 mm and 160 mm respectively. The applied load causes an extension of 0.3 mm. Determine the Modulus of elasticity and $E$ of the material.

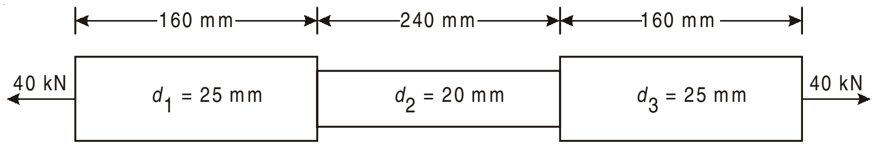



**Fig. 1.20**

**1.11.** A standard tensile test specimen gave the following results during the test on U.T.M: Load at yield point = 66 kN, Maximum load = 96 kN. Load at fracture = 68 kN. Distance between gauge points at fracture = 108 mm and minimum diameter at fracture = 9.25 mm. Gauge length = 80 mm, Initial diameter = 16 mm at the test section. Determine, (i) Upper yield stress, (ii) Ultimate tensile strength, (iii) Percentage elongation, (iv) Percentage reduction in area after fracture, and (v) Nominal and Real stresses at fracture.

**1.12.** A tensile load of 40 kN is applied on a bar of 20 mm diameter with 200 mm gauge length. Elongation due to this load is 0.13 mm and corresponding contraction in diameter is 0.0037 mm. Find Poisson's ratio and three elastic constants.

**1.13.** A steel bar of 25 mm diameter and 500 mm length is subjected to a tensile stress of 200 MPa. Determine the change in volume of the bar if $E_s$ = 210 MPa, and $\mu$ = 0.3.

**1.14.** A steel bar of uniform section 40 × 25 mm and length 1.5 m is subjected to a gradually applied load of 100 kN. Calculate proof resilience and modulus of resilience if the elastic limit of the material of the bar is 160 N/mm$^2$.

**1.15.** A bar having a collar at its lower end is fixed at the top. A 120 N load falls on the collar from a height of 50 mm. Find the instantaneous stress in the bar and also its corresponding extension if the diameter of the bar is 30 mm. What is the error in percentage if the extension of bar is neglected.

**1.16.** A mild steel bar is fixed at the upper end and carries a collar at the lower end. A weight of 3 kN falls on the collar through a height of 20 cm. If another weight of 30 kN falls through the height of 2 cm, find stresses in the rod in both situations. Take $E = 2 \times 10^5$ N mm$^2$.

**1.17.** A steel bar is stretched so that it experiences a shear stress of 60 N/mm$^2$. Find the strain energy per unit volume stored in the material due to shear stress. Take $G = 8 \times 10^4$ Nmm$^2$.

**1.18.** A bar of 12 cm$^2$ area and 3 m length carries a collar at its lower end and its upper end is fixed. A weight drops on the collar from a certain height so that the extension of the bar is 1.5 mm. Calculate the value of suddenly applied load which causes this extension. Assume $E = 2 \times 10^5$ N/mm$^2$.

**1.19.** A 600 mm long straight bar is 30 mm in diameter for the first 180 mm length, 20 mm diameter for second 260 mm length and 32.5 mm diameter for the remaining length. If the bar is subjected to an axial pull of 40 kN, find the extension of bar. Take $E = 2 \times 10^5$ kN/mm$^2$. (*UPTU : 2004 – 05*)

**[*Ans.* $\delta l$ = 0.255 mm]**

**1.20.** A rectangular block 200 mm × 80 mm × 60 mm is subjected to axial load as follows:

(i) 480 kN tensile in the direction of its length

(ii) 800 kN compressive on the 200 mm × 800 mm face

(iii) 900 kN tensile on 300 mm × 60 mm face.

Find : (i) Strain in the direction in which each stress acts, and

(ii) Change in the volume of block due to above loading.

**[*Ans.* $\varepsilon_x = 4.6875 \times 10^{-4}$, $\varepsilon_y = 3.125 \times 10^{-4}$, $\varepsilon_z = -4.6875 \times 10^{-4}$, $\delta V = 300$ mm$^3$]**

**1.21.** The following data were noted during a tensile test conducted on a mild steel bar:

Diameter of steel bar = 25 mm
Gauge length = 210 mm
Extension at a load of 80 kN = 10.95 mm
Load at elastic limit = 80 kN
Total extension = 51 mm
Diameter of the rod at failure = 21.22 mm
Maximum load = 120 kN

Calculate :

(a) The Young's modules (b) Stress at elastic limit

(c) Percentage elongation (d) Percentage decrease in area.

**[*Ans.* (a) $E$ = 3.13 kN/mm$^2$ (b) 162.98 N/mm$^2$ (c) 24.29% (d) 27.95%]**

**1.22.** The modulus of rigidity of a material is 0.8 × 10$^5$ N/mm$^2$. When a 8 mm × 8 mm rod of this material was subjected to an axial pull of 4000 N, it was found that the lateral dimensions of the rod changes to 7.9995 × 7.9995 mm. Find the Poisson's ratio and modulus of elasticity.

**[*Ans.* $v$ = 0.19, $E$ = 1.9 × 10$^5$ N/mm$^2$]**

**1.23.** A bar of variable cross-section rigidly fixed at one end is subjected to two concentrated loads $P_1$ and $P_2$, as shown in Fig. 1.21. If $P_1$ = 60 kN and $P_2$ = 45 kN, $A_1$ = 80 mm$^2$ and $A_2$ 40 mm$^2$. Find the maximum stress.

**[*Ans.* 1.31 GPa]**

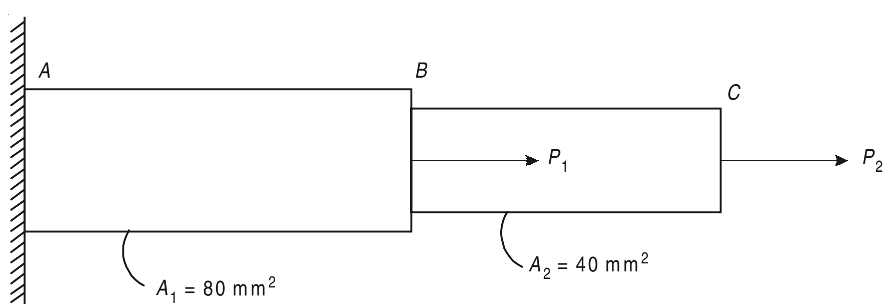



**Fig. 1.21**

**1.24.** An aluminium bar of 60 mm diameter is stressed in a tensile testing machine. At a certain instant the applied load is 100 kN, while the measured elongation is 0.22 mm in a length of 300 mm, and the diameter's dimension is decreased by 0.01215 mm. Calculate the Poisson's ratio. **[*Ans.* 0.277]**

**1.25.** A steel bar 300 mm long, 50 mm wide and 12 mm thick is subjected to a tensile pull of 100 kN. If $E = 2 \times 10^5$ N/mm$^2$ and $\frac{1}{m} = 0.32$, determine the change in volume of the bar. **[*Ans.* 54 mm$^3$]**

**1.26.** During a tensile test on a mild steel specimen, 40 mm diameter and 200 mm long, the following data was obtained:

Extension at 40 kN load = 0.030 mm

Yield load = 161 kN

Length of specimen at fracture = 249 mm.

Determine modulus of elasticity, yield stress and percentage elongation.

**[*Ans.* 209.4 GPa, 128 N/mm$^2$, 24.5%]**

**1.27.** A steel bar subjected to loads as shown in Fig. 1.22. Determine the change in length of the bar *ABCD* of 18 cm diameter. $E = 180$ kN/mm$^2$.

(*UPTU : 2005–06*)

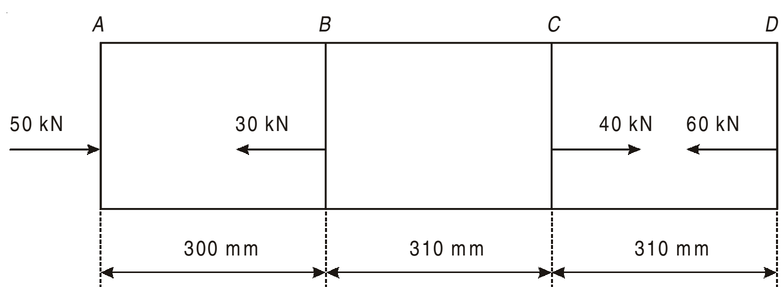



**Fig. 1.22**

**1.28.** A bar of uniform cross-sectional area *A* and length 2 hangs vertically from a rigid support. If the density of the material of the bar is *P* kg/m$^3$, drive the expression for maximum stress induced and the elongation.

**1.29.** A composite bar consists of aluminium section rigidly fastened between a bronze section and steel section as shown in figure. Axial loads are applied at positions indicated. Determine the stress in each section.

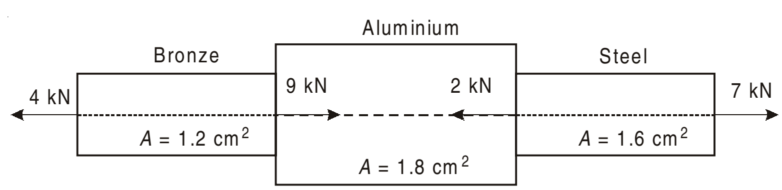



**Fig. 1.23**

**[*Ans.* 33.33 MPa, –27.78 MPa, –73.75 MPa]**

**1.30.** A steel tie rod 40 mm in diameter and 2 m long is subjected to a pull of 80 kN. To what length the bar should be bored centrally so that the total extension may increase by 20% under the same pull, the bore being 20 mm in diameter. Take $E = 2 \times 10^5$ N/mm$^2$. **[*Ans.* 1198.6 mm]**

**1.31.** Show that if $E$ is assumed correct, an error of 1% in the determination of $G$ will involve an error of about 5% in the calculation of Poisson's Ratio when its correct value is 0.25. (*UPTU : 2002–03*)

**1.32.** What do you understand by strain energy absorbed by the system, complementry strain energy and elastic strain energy? Explain these with the help of a diagram. (*UPTU : 2005 – 06*)

**[*Ans.* Section : 1.10]**

**1.33.** State the generalised Hook's law and prove for an anisotropic elastic material the maximum number of elastic constants is 21 only. Also show that for isotropic material it is 2. (*UPTU : 2008 – 09*)

**[*Ans.* Section : 1.5]**

**1.34.** A wagon whose weight is 50 kN is attached to a wire rope and is moving on a level track at a speed of 5.00 km/h, the cross-sectional area of the wire rope is 800 mm$^2$ at the time of sudden stoppage calculate the instantaneous maximum stress and elongation in the rope whose $E$ is 200 GPa. Take $g = 9.81$ m/s$^2$.

**[*Ans.* $\sigma$ = 495.78 MPa, $\delta L$ = 24.79 mm]**

**1.35.** While testing on a metallic rod it is observed that the diameter of rod is reduced by 0.0025 mm under an axial pull of 20 kN. The original diameter of the rod is 15 mm. If rigidity modulus for the rod metal be 50 kN/mm$^2$. Find the Young's modulus and Bulk modulus. (*UPTU : 2004–2005*)

**[*Ans.* Example 1.3]**

**1.36.** The bulk modulus for a material is $0.5 \times 10^5$ N/mm$^2$. A 12 mm diameter rod of the material was subjected to an axial pull of 14 kN and the change in diameter was observed to be $3.6 \times 10^{-3}$ mm. Calculate Poisson's ratio and modulus of elasticity. (*UPTU : 2005–2006*)

**[*Ans.* Example 1.3]**

**1.37.** The stresses in the three principal direction are + 65 MN/m$^2$, + 20 MN/m$^2$ and –85 MN/m$^2$, Find the principal strain. Take $m = 0.3$ and $E = 200$ GN/m$^2$.

(*UPTU : 2006–2007*)

**[*Ans.* Example 1.4]**

**1.38.** The principal stresses at a point in an elastic material are 60 N/mm$^2$ tensile, 20 N/mm$^2$ tensile and 50 N/mm$^2$ compressive. Calculate the volumetric strain. $E$ = 1,00,000 N/mm$^2$. (*UPTU : 2010–2011*)

**[*Ans.* Example 1.5]**

**1.39.** A vertical rod 2 m long, fixed at the upper end, is 13 cm$^2$ in area for 1 m and 20 cm$^2$ in area for 1 m. A collar is attached to the free end. Through what height can a load of 100 kg fall on to the collar to cause a maximum stress of 50 N/mm$^2$? $E$ = 2,00,000 N/mm$^2$. (*UPTU : 2010–2011*)

**[*Ans.* Example 1.6]**

**1.40.** A weight $W$ = 5 kN attached to the end of a steel wire rope moves downward with constant velocity 1 m/s. What stresses are produced in the rope when its upper end is suddenly stripped? The free length of rope at the moment of impact is 20 m, its net cross-sectional area is 10 sq cm. and $E = 2.00 \times 10$ N/mm$^2$. *(UPTU : 2012–13)*

**[*Ans.* Example 1.7]**

**1.41.** Derive an expression for deformation of conical bar hung to a ceiling having diameter $D$ and height $L$, weight density of bar ρ and Young's modulus is $E$. *(UPTU : 2013–14)*

**[*Ans.* Example 1.11]**

**1.42.** A steel bar of cross-section 600 mm$^2$ is acted up by forces as shown if Fig. 1.24. Determine the total elongation in the bar. $E = 2 \times 10^5$ MN/m$^2$.

**[*Ans. x* = 7.7 mm]**

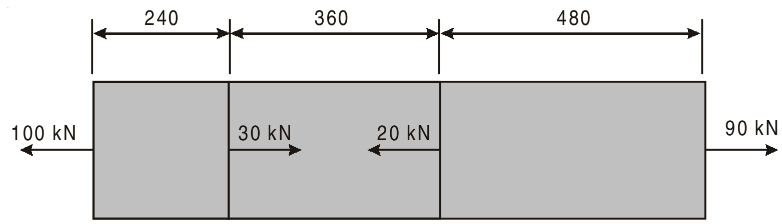



**Fig. 1.24**

**1.43.** A member *ABCD* is subjected to point loads $P_1$, $P_2$, $P_3$ as shown in Fig. 1.25. Calculate force $P_2$ necessary for equilibrium, if $P_1$ = 45 kN, $P_3$ = 450 kN and $P_4$ = 130 kN. Determine the total elongation of the member assuming modulus of elasticity to be $2.1 \times 10^2$ kN/mm$^2$. **[*Ans.* $P_2$ = *365 kN;* x = *4.914 mm*]**

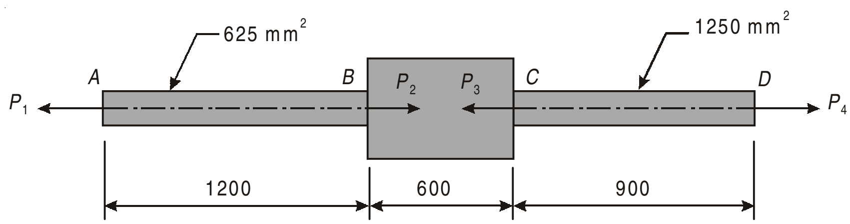



**Fig. 1.25**

**1.44.** A member is formed by connecting steel bar to an aluminium bar as shown in Fig. 1.26. Assuming thai bars arc prevented buckling sidewise. calculate the magnitude of the force $P$, that will cause the total length of the member to decrease 0.25 mm. $E_s$ = 210 kN/mm$^2$, $E_a$ = 70 kN/mm$^2$. What is the total work done by $P$ ? **[*Ans.* $P$ = 224 kN ; Work = 56 *J*]**

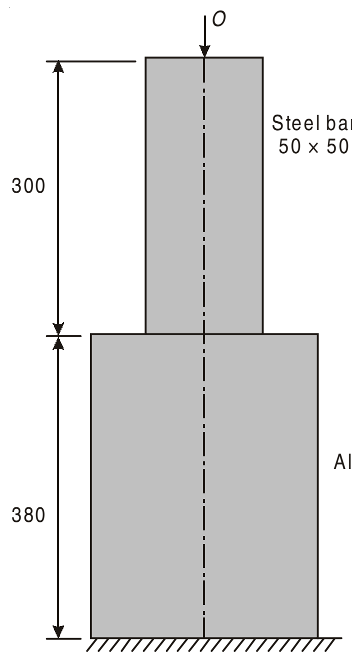



**Fig. 1.26**

**1.45.** A flat steel bar 500 mm long and 10 mm thick tapers from 100 mm at one end to 50 mm at theother. Dtermine the change in iength of the bar when a tensile force $P$ = 50 kN is acting along its axis $E$ = 200 GN/m$^2$.**[*Ans.* $x$ = 0.173 mm]**

**1.46.** A steel rod circular in section, tapers uniformly from 40 mm diameter to 20 mm diameter in a length of 0.6 m. Calculate its increase in length when subjected to an axial pull of 30 kN. $E$ = 210 kN/mm$^2$. **[*Ans.* $x$ = 0.182 mm]**

**1.47** A solid cone has base diameter $D$ and height $H$ if $E$ is the modulus of elasticity of its material and ρ its mass density, find expression tor KS extension when suspended from the base as inverted cone under its own weight.

$$\left[\textbf{\textit{Ans.}}\ x = \frac{\rho H^2}{6E}\right]$$

**1.48.** A steel rod circular in section tapers from 25 mm diameter to 12.5 mm diameter, in a length of 600 mm Find how much this length will increase under a pull of 20 kN if its Young's modulus is 200 kN/mm$^2$ ?

**[*Ans.* 0.244 mm]**

**1.49.** A steel uniform metal bar of diameter $D$ end length $L$ is hanging vertically from its upper end Obtain the total elongation of the bar due to its over weight if ρ is the mass density and $E$, the Young's modulus of material of the bar.

$$\left[\textbf{\textit{Ans.}}\ \frac{g\rho L^2}{2E}\right]$$

**1.50.** A brass tube 100 mm external diameter and 10 mm thick surrounds tightly a copper tube of an equal thickness. The composite section is subjected to a compressive load of 80 kN. Estimate (i) loads shared by brass and copper (ii) stress in brass and copper (iii) decrease in length when each is 200 mm long. $E_c$ =110 kN/mm$^2$ and $E_b$ = 90 kN/mm$^2$.

**[*Ans.* (i) $W_b$ = 41.1 kN, (ii) $a_b$ = 14.5 N/mm$^2$**
**$W_t$ = 38.99 kN, $a_c$ = 17.72 N/mm$^2$, (iii) $x$ = 0.032 mm]**

**1.51** A tube of aluminium 40 mm external diameter and 20 mm internal diameter is snugly fitted on a solid steel rod of 20 mm diameter. The composite bar is loaded in compression by an axial load *P*. Find the stress in aluminium when the load is such that the stress in steel is 70 N/mm$^2$. What is the value of *P?* $E_s$ = 200 GN/m$^2$ and $E_a$ = 70 GN/m$^2$.

**[*Ans*. $a_a$ = 24.5 N/mm$^2$, $P$ = 45.08 kN]**

**152.** Two vertical rods, one of steel and the other of bronze are suspended from a horizontal ceiling, the horizontal distance between them being 0.8 m. Each rod is 2,5 m long and 12.5 mm in diameter. A horizontal cross-piece connects the lower ends of the bars. Where should a load of 10 kN be placed on the cross-piece so that it remains horizontal after being loaded ? Estimate stress in each rod. Take $E_s$ = 200 kN/mm$^2$ and $E_b$ = 110 kN/mm$^2$. Neglect any bending in the cross-piece. **[*Ans.* $\sigma_s$ = 52.55 N/mm$^2$ ; $\sigma_b$ = 28.9 N/mm$^2$]**

**1.53.** Two vertical wires each 2.5 mm diameter and 3 m long, jointly support a load of 1.5 kN. One wire is made of sted and the other is of different metal. If each wire stretches elastically 3 mm, find the load taken by steei and the other metal. Find also Young's modulus for the metal wire if that for steel wire is 200 kN/mm$^2$. **[*Ans.* $W_s$ = 982 N, $W_m$ = 518 ; $E_m$= 105.5 kN/mm$^2$]**

**1.54.** Two vertical wires are suspended at a distance of 500 mm apart as shown in Fig. 1.27. Their upper ends are firmly secured and their lower ends support a rigid horizontal bar which carries a load *W*. The left hand wire has a diameter of 1.6 mm and is made of copper and the right hand wire has a diameter of 0.9 mm and is made of steel. Both wires are initially 4.5 m long, (a) Determine position of line of action of *W* if due to W'both wires extend by the same amount, (b) Determine the slope of the rigid bar if a load of 200 N is hung at the centre of the rigid bar. Neglect weight of the bar. $E_c$ = 1.3 × 10$^5$ N/mm$^2$, $E_s$ = 2.1× 30$^5$ N/mm$^2$. **[*Ans.* (a) 169 mm from copper wire ; (b) 0.0033 rad.]**

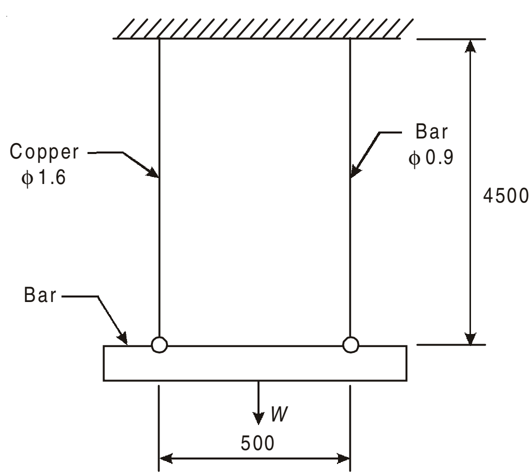



**Fig. 1.27**

**1.55.** Two copper rods and a steel rod, together support a rigid uniform beam weighing $P$ kN as shown in Fig. 1.28. The stresses in copper and steel are not to exceed 60 kN/mm$^2$ and 120 N/mrn$^2$ respectively. Find the magnitude of load $P$ that can be safely supported $\frac{E_s}{E_c} = 2$. **[*Ans.* $P$ = 372 kN]**

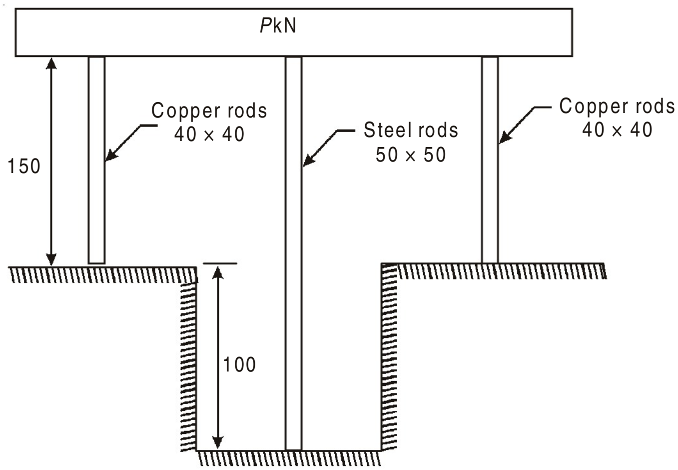



**Fig. 1.28**

**1.56.** Figure 1.29 shows a rigid har *ABC* hinged at *A* and suspended at two points *B* and *C* by two bars *BD* and *CE,* made of aluminium and steel respectively. The bar carries a load of 20 kN midway between *8* and *C* The cross-sectional area of aluminium bar *BD* is 3 mm$^2$ and that of steel bar *CE* is 2 mm$^2$, Determine the loads taken by the two bars *BD* and *CE*. $E_d$ = 70 KN/mm$^2$ and $E_s$ = 200 kN/mm$^2$. **[*Ans.* $P_a$ = 3.4807 kN ; $P_s$ = 13.26 kN]**

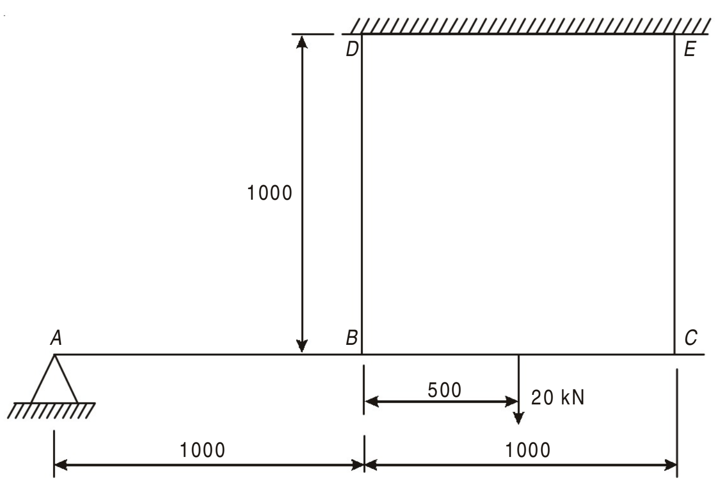



**Fig. 1.29**

**1.57.** An initially straight bar *AD* has a uniform cross-section and ii clamped to the end supports as shown in Fig. 1.30 Initially the bar is stress free. The symmetrical loads as shown are applied to the brackets (whole effect is to be neglected) and it is desired to find the resultant tensile or eompressive force acting over any normal cross-section in each of these three regions *AB, BC* and *CD,*

**[*Ans.* *AB* = 8 kN comp., *BC* = 28 kN comp., *CD* = 32 kN tensile]**

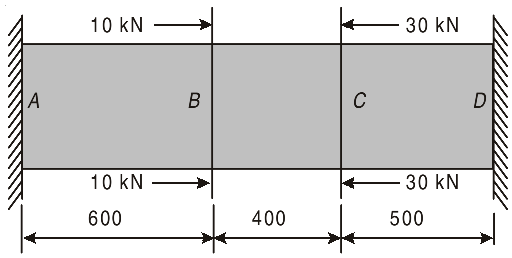



**Fig. 1.30**

**1.58.** A. sieel rod 20 mm diameter passes centals) through a hollow copper tube of external diamerer 25 mm and internal diameter 20 mm and is secured by nuts and washers of negligible thickness 1 he nuts are lightened till the tension in the rod is 75 kN, Find what tensile force should be applied to **the** rod so that the copper tube is relieved of all compressive stress '? $E_s$ = 200 GN/m ; $E_e$ = 110 GN/m$^2$. **[*Ans.* 240 kN]**

**1.59.** A rectangular base plate is fixed at each of its four corners by 20 mm diameter bolts and nuts as shown in Fig. 1.31.

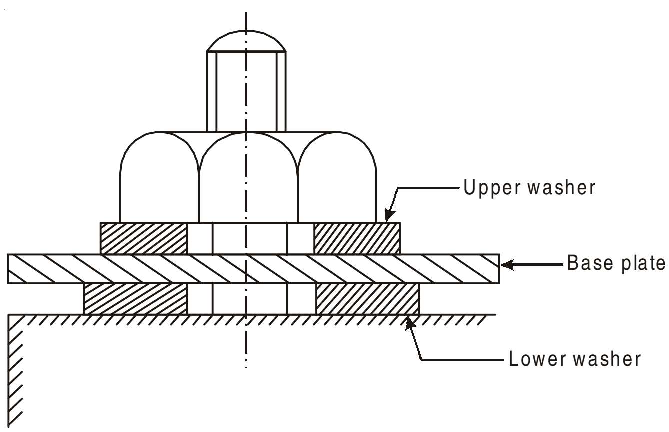



**Fig. 1.31**

The plate rests on washers of 2 mm internal diameter and 50 mm external diameter. Upper washers which are placed between nut and the plate are 22 mm internal diameter and 44 mm external diameter If the base plate carries a load of 120 kN including its self-weight which is equally distributed at four corners. Calculate the stress in the lower washers before the nuts are tightened. What could be the stress in upper and lower washers when the nuts are so tightened as to produce a tension in each bolt?

[*Ans.* 18.95 N/mm$^2$ ; 22.1 N/mm$^2$]

**1.60.** A mild steel bar, 7 m long, is of 50 mm diameter for 3 m of its length and 25 mm diameter for the remaining length, The bar is subjected to a tensile force such that the maximum stress induced in it is 120 N/mm$^2$. Determine the total elongation of the bar and the change in diameter at the smaller section. $E = 2 \times 10^5$ MN/m$^3$ and $\mu = 0.25$. **[*Ans.* $x$ = 2.85 mm ; $\delta d$ = 0.00375 mm]**

**1.61.** A steel bar 40 mm × 40 mm in section, 3 m long, is subjected to an axial pull of 128 kN, Taking $E_s = 2 \times 10^5$ N/mm$^2$ and $\mu = 0.3$, calculate the alterations in length and sides of the bar. Calculate also work done by the pull in extending the bar. **[*Ans.* $\delta l$ = 1.2 mm ; decrease in sides = 0.0048 mm , 153.6 J]**

**1.62.** An m.s. rod *ABC,* of circula? cross-section transmits an axial pull. The total length is 1.5 m. *AB* being 900 mm long and 40 mm diametei while *BC* is 30 mm diameter. Total change in length is 0.6 mm. Determine separately for portions *AB* and *BC* changes in (i) length, (ii) diameter and, (iii) volume. Take Poisson's ratio = 0.3.

**[*Ans.* *AB* (0 $\delta l$ = 0.275 mm ; (ii) $\delta d$ = 0.00366 mm ; (iii) $\delta V$ = 553 mm' ;**

***BC* (i) $\delta l$ = 0.325 mm ; (ii) $\delta d$ = 0.00487 mm ; (iii) $\delta V$ = 368 mm$^3$]**

**1.63.** A hollow cylinder of external diameter 250 mm and internal diameter 125 mm and having a length of 0.5 m is subjected to an axial load which causes an average stress of 90 N/mm$^2$. The modulus of elasticity of the material is 2.3 × 10$^5$ N/mm$^2$ and $\mu = 0.25$. Determine (a) change in linear dimension, (b) change in internal volume and (c) proportional change in density.

**[*Ans.* (a) $\delta l$ = 0.196 mm ; (b) $\delta V$ = 1199.3 mm$^3$ ; (c) $\delta \rho$ = 0.01956%]**

**1.64.** A rectangular piece of metal is subjected to tensile stresses of 100 N'mm$^2$ and 80 N/mm$^2$ on mutually perpendicular faces. Find the strain in the direction of each stress and also in a direction perpendiculat to both the stresses, $\mu = 0.3$ and $E = 10^5$ N/mm$^2$.

**[*Ans.* 76 ×10$^{-5}$ tensile ; 50 × 10$^{-5}$ tensile ; 54 ×10$^{-5}$ compressive]**

**1.65.** A cube of steel of edge 150 mm is subjected to tensile force of 250 kN on one pair of races, a tensile force of 200 kN on the second pair and a compressive force of 100 kN on the third pair Find change in volume of the cube. $E$ = 200 GN/m$^2$, $\mu$ = 0.25. **[Ans. $\delta V$= 131.25 mm$^3$ increase]**

**1.66.** A steel rod is circular in cross-section and 180 mm in length The rod is subjected to an axial compressive. stress of 150 N/rnm$^2$. Determine the intensity of radial external pressure, in N/mm$^2$, which can be applied on the surface of the piece to prevent lateral strain. What will be the change in length ? Take Poisson's ratio as 0.3 and E = 2 ×10$^5$ N/'mm$^2$.

**[*Ans.* 45 N/mm$^2$ ; 0.1107 mm]**

**1.67.** Show that in a bar subjected to longitudinal stress with the strains prevented in both the transverse directions, the longitudinal strain will be only 5/6 of what it would have been if the bar were free. μ= 0,25.

**1.68.** *A* rod is 2 m long at a temperature of 10° C. Find the expansion of the rod when the temperature is raised lo 80° C. If this expansion is prevented, find the stress in the material of the rod. Take $E = 10^{11}$ N/m$^2$ and $\alpha = 0.000012°$ C.

**[*Ans.* 1.68 mm ; 84 N/mm$^2$]**

**1.69.** A steel rod 30 mm diameter and 0.3 m long is subjected to a tensile force $W$ kN acting axiaily. The temperature of the rod is raised through 80° C and total extension measured as 0.35 mm. Calculate the value of $W$. $E_s = 2 \times 10^{-5}$ N/mm$^2$ and $\alpha_s = 12 \times 10^{-6/0}$C **[*Ans.* $W$ = 29.22 kN]**

**1.70.** The rails of a straight railway track have a cross sectional area of 6 × I0$^3$ mm$^2$. Calculate the longitudinal thrust set up for the rise of temperature of 40° C if an expansion of 7 mm is allowed in a length of 15 m. $\alpha_s = 12.5 \times 10^{-6}$ per °C. $E = 2 \times 10^5$ MN/m$^2$. **[*Ans*. 40 kN compressive]**

**1.71.** Two parallel walls 8 m apart are stayed together by a steel bar 50 mm diameter through metal plates and nuts ai each end. The nuts are just tightened up at 100° C. Find the pull exerted by the bar when it cools down to 20° C if:

(a) the ends do not yield ;

(b) the walls yield by 5 mm. $E_s = 2 \times 10^5$ N/mm$^2$ and $\alpha_s = 0.0000125$/°C.

**[*Ans,* (a) 200 N/mm$^2$ (b) 75 N/mm$^2$]**

**1.72.** A thin steel ring is heated to a temperature of 95° C. At this temperature it just fits over a steel cylinder which has a diameter of 100 mm at 20° C. If the system is allowed to cool until the temperature of both the ring and the cylinder is 20° C, what will be the stress in the ring ? Assume that the cylinder does not change its diameter.$E = 2.1 \times 10^5$ N/mm$^2$ and $\alpha_s$ =12 × I0$^6$/°C.

**[*Ans*. 189 N/mm$^2$]**

**1.73.** A brink steel rod is heated to 160° C and then suddenly clamped at both ends. It is then allowed to cool and breaks at a temperature of 85° C. Calculate the breaking stress of the steel. $E$ = 200 GN/m$^2$ and $\alpha$ = 12 I0$^6$/°C.

**[*Ans.* 180 MN/m$^2$]**

**1.74.** .......................diameter 70 mm and length 1.5 in is being turned **between** centres in the lathe. Due to heat generated in cutting, the temperature of bar rises 10°C. The natural expansion is partially restricted by the action of hvdraulically operated tail-stock which exeits a constant load of 46.2 kN on the running centre. Determine (a) increase in length of the bar (b) compressive stress in the bar. $E = 2 \times 10^5$ MN/m$^2$, $\alpha$ = 12 × I0$^6$/°C.

**[*Ans.* (a) 0.09 mm, (b) 12 MN/mm$^2$]**

**1.75.** A steel bar of cross-sectional area 6000 mm$^2$ is rigidly connected to a copper bar of 4000 mm$^2$ cross-sectional area. Both bars are 1.5 m in length. When the temperature of the compound bar is raised by 200 °C, calculate the stresses in two materials. Take $\alpha_s = 12 \times 10^{-6}$ per °C, $\alpha_c = 18 \times 10^{-5}$ per °C, $E_s$ = 200 GN/m$^2$ and $E_c$ = 120 GN/m$^2$.

**[*Ans.* $\alpha_s$ = 68,57 N/mm$^2$; $\alpha_c$ = 102.86 N/mm$^2$]**

**1.76.** A .steel tube 45 mm external diameter and 3 mm thick encloses centrally a solid copper bar of 30 mm diameter. The bar and the tube are rigidly connected together at the ends at a temperature of 30°C. Find stiess in each metal when heated to 180°C Also find the increase in length if the original length of assembly is 300 mm. $\alpha_s = 1.08 \times 10^{5/°}$C, $\alpha_s$ = 1.7 × l0$^5$/° C, $E_s = 2.1 \times 10^{11}$ N/m$^3$, $E_c = 1.1 \times 10^{11}$ N/m$^2$.

**[*Ans.* $\sigma_s$ = 94 5 N/mm$^2$; $\sigma_c$ = 52.8 N/mm$^2$; 0.62 mm]**

**1.77.** A composite bar is made up by connecting a steel membei and a copper member rigidly fixed at tlieir ends as shown in Fig. 1.31. The cross-sectional area of the copper member is $A$ mm- while that of the steel membei is 2 A mm$^2$ for half the length and $A$ mm$^2$ for the other half its length The co-etficii nt of expansions for steel and copper are $\alpha$ and 1.3 a respectively, while the elastic moduli are $E$ and $0.5E$ respectively. Estimate the stress induced in the members due to temperature rise of $t$ degree

[*Ans.* $\sigma_{s1}$, = 0.1091 at $E$ $\alpha_{s2}$ = 0.05455 $\alpha t$ $E$; $\sigma_c$ = 0.0191 $\alpha t$ $E$]

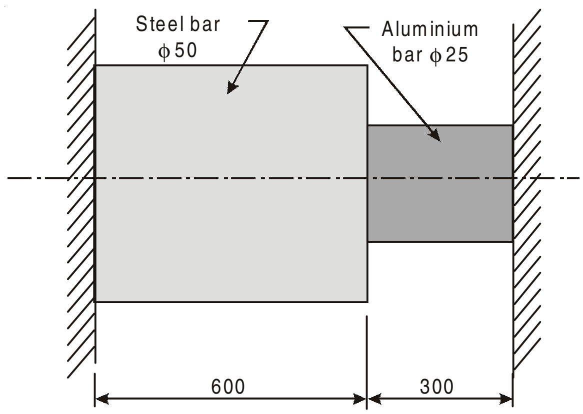



**Fig. 1.32**

**1.78.** (a) A weight of 50 kN hangs from three wires of equal length; the middle one is of steel and other two outer wires are of copper If the cross-sectional area of each wire is 320 mm$^2$, determine the load shared by each. Take $E_s = 2 \times 10^5$ N/mm$^2$ and $E_c = 1.2 \times 10^5$ N/mm$^2$.

(b) If the temperature of the wires is raised by 100°C and it is assumed that they extend an equal amount, calculate then the load shared by each.

$\alpha_s$ = 12 ×10$^{-6}$/°C, $\alpha_c$ = 18 × 10$^{6/0}$C.

**[*Ans.* (a) $W_c$ = 13.64 kN; $W_s$ = 22.72 kN; (b) $W_c$ = 3.16 kN; $W_s$ = 43.68 kN]**

**1.79.** A hollow steel cylinder surrounds a solid copper cylinder and the assembly is subjected to an axial load of 250 kN. The cross-sectional area of steel is 2000 mm$^2$ and that of copper 6200 mm$^2$. Both the cylinders are 0.5 m long before the load is applied. Determine rise in temperature of the entire system required so that copper cylinder just supports the whole of the load. $E_c = 1.1 \times 10^5$ N/mm$^2$, $E_s = 2 \times 10^5$ N/mm$^2$, $\alpha_s$= 12 × 10$^{-6/0}$C and $\alpha_c$ = 18.5 × 10$^{-6/0}$C.

**[*Ans.* 56.395°C]**

**1.80.** A steel cylinder 75 mm diameter and 0.5 m long is surrounded by an aluminium hollow cylinder 120 mm outside and 80 mm inside diameters. An axia! compressive load of 250 kN is applied to the cy'hnder through a rigid cover plate. If the aluminium cylinder is originally 0.1 mm longer than the steel before any load is applied, find normal stress in each, when temperature has dropped 20°C. $E_s = 2 \times 10^5$ MN/m$^2$, $E_{AL} = 7.5 \times 10^4$ MN/m$^2$, $\alpha_s = 12 \times 10^{-6}$ per °C, $\alpha_{AL} = 24 \times 10^6$ per °C. **[*Ans.* $s_s$ = 35.6 N/mm$^2$; $\alpha_{AL}$ = 10.3 N/mm$^2$]**

**1.81.** A composite bar made up of aluminium and steel is held between two supports as shown in Fig. 1.33. The bars aie stress free at a temperature of 38°C. What will be the stress in the two bars when temperature of 2i°C if (a) the supports are unyielding and (b) the supports come nearer to each other by 0.1 mm² it can be assumed that change in temperature is uniform all along the length of the bar $E_s$, = 210 kN/mm², $E_{AL}$ = 74 kN/mm², $\alpha_s$ = 11.7 × 10 $^{5/}$°C and $\alpha_{AL}$ = 234 × 10$^{-5}$/°C.

**[*Ans.* (a) $\sigma_s$= 12.5 N/mm²; $\sigma_{AL}$ = 50 N/mm², (b) $\sigma_s$ = 7.27 N/mm²; $\sigma_{AL}$ = 29.08 N/mm²]**

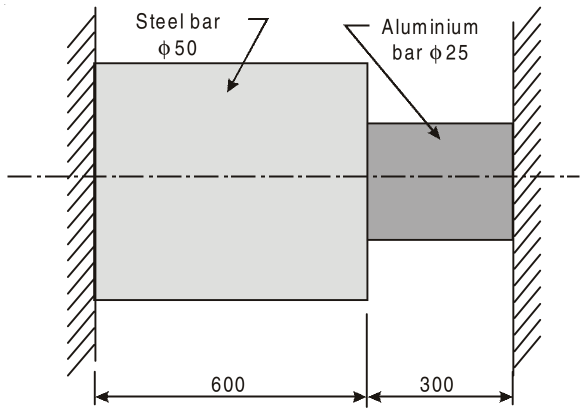



**Fig. 1.33**

# Compound Stresses and Strains

CHAPTER 2

## 2.1 ❑ INTRODUCTION

We have studied direct tensile and compressive stresses as well as simple shear. We have referred the stress in a plane, which is at right angles to the line of action of the force (in case of direct tensile or compressive stress). We have considered at a time one *type of stress*, acting in one direction only. But majority of engineering components and structures are subjected to such loading conditions that there exists a complex state of stresses, including direct tensile and compressive stress as well as shear stress in various directions. Such a combination of stresses is called compound stress.

### 2.1.1 Principal Planes

It has been observed that at any point in a strained material, there are three planes, mutually perpendicular to each other, which carry direct stresses only and no shear stress. These particular planes, which have no shear stress are known as *principal planes*.

### 2.1.2 Principal Stress

The magnitude of direct stress, across a principle plane, is known as *principal stress*. The determination of principal planes, and then principal stress is important in design of various structures and machine components.

Analytical and Graphical methods are used for the determination of stresses on an oblique section of a strained body.

## 2.2 ❑ SIGN CONVENTIONS FOR ANALYTICAL METHOD

1. All the tensile stresses and strains are taken as positive, while all compressive stresses and strains are taken as Negative.

2. The shear stress which tends to rotate the element is the clockwise direction is taken as + ve, whereas that which tends to rotate in an anticlockwise direction as Negative.

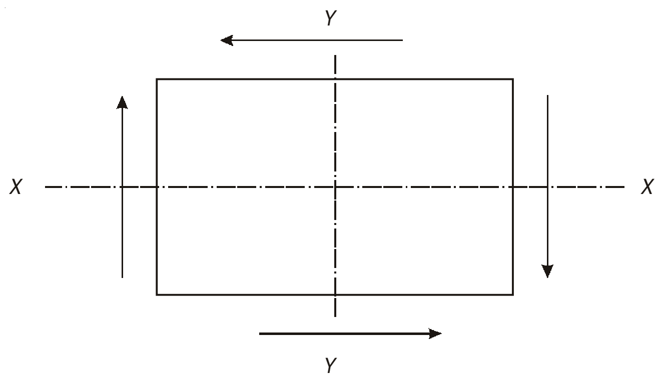



**Fig. 2.1**

In the element shown in Fig. 2.1 the shear stress on the horizontal faces (or $Y - Y$ axis) is taken as negative whereas on the vertical faces (or $X - X$ axis is taken as Positive).

## 2.3 ❑ STRESSES ON AN OBLIQUE SECTION OF BODY SUBJECTED TO A DIRECT STRESS IN ONE PLANE

Consider a rectangular body of uniform cross-sectional area of an unit thickness subjected to a direct tensile stress along $X - X$ axis (Fig. 2.2). Now consider an

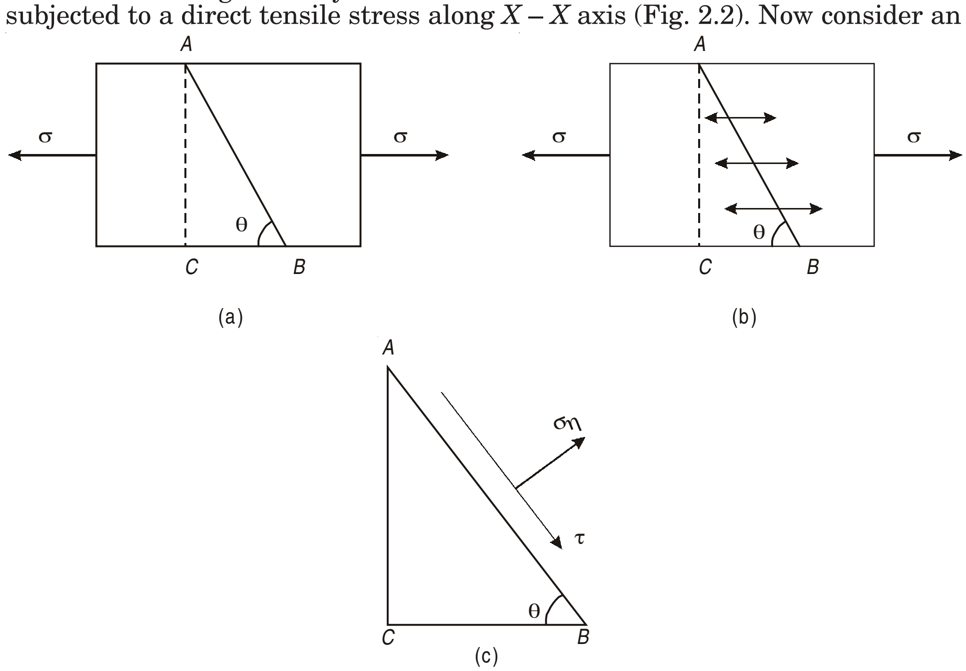



**Fig. 2.2**

oblique section $AB$ inclined with the $X - X$ axis (i.e. with the line of action of tensile stress on which we are required to find out the stresses).

Let, $\sigma$ = Tensile stress across the face $AC$, and

$\theta$ = Angle, which the oblique section $AB$ makes with $BC$ i.e. with the $X–X$ axis in the clockwise direction. First of all consider the equilibrium of wedge $ABC$ whose free-body diagram is shown in Fig. 2.2 (b) and (c) We know that horizontal force acting on the face $AC$, $P = \sigma \,.\, AC$ ($\rightarrow$)

Resolving the force perpendicular (normal) to the section $AB$

$$P_n = P \sin\theta = \sigma \,.\, AC \sin\theta \qquad ...(i)$$

Now resolving the force tangential to section $AB$

$$P_t = P \cos\theta = \sigma \,.\, AC \cos\theta \qquad ...(ii)$$

We know that normal stress across the section $AB$, $\sigma_n = \sigma \sin^2\theta$ ...(iii)

$$= \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta \qquad ...(2.1)$$

Shear stress i.e. tangential stress across the section $AB$, $\tau = \sigma \sin\theta \cos\theta$

$$= \frac{\sigma}{2}\sin 2\theta \qquad (iv)$$

As per Eq. (ii) the normal stress across the section $AB$ will be maximum, when $\sin^2\theta = 1$ or $\sin\theta = 1$ or $\theta = 90°$. Or in other words, the face $AC$ will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$. Or in other words, the shear stress will be maximum on planes inclined at 45° and 135° with maximum shear stress when $\theta$ is equal to 45°,

$$\tau_{max} = \frac{\sigma}{2}\sin 90° = \frac{\sigma}{2} \times 1 = \frac{\sigma}{2}$$

and maximum shear stress, when $\theta$ is equal to 1.35°

$$\tau_{max} = -\frac{\sigma}{2}\sin 270° = -\frac{\sigma}{2}(-1) = \frac{\sigma}{2}$$

It is thus obvious that magnitude of maximum shear stress is half of the tensile stress. Now the resultant stress may be found out from the relation

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$$

The planes of maximum and minimum normal stresses (i.e., principal planes) may be found out by equating the shear stress to *zero*. This happens as the normal stress is either maximum or minimum on a plane having *zero shear stress*. Now equating the shear stress to *zero*, $\sigma \sin\theta \cos\theta = 0$.

There are two principal planes, at right angles to each other one of them coincides with the line of action of the stress and the other at right angles to it.

**Example 2.1.** *A wooden bar is subjected to a tensile stress of 5 MPa. What will be the values of normal and shear stresses across a section, which makes an angle of 25° with the direction of the tensile stress.*

*Given : Tensile stress* ($\sigma$) = *5 MPa and angle made by the section with the direction of the tensile stress* ($\theta$) = *25°*

**Solution**

Normal stress across the section :

$$\sigma_n = \frac{\sigma}{2} - \frac{\sigma}{2}\cos 2\theta = \frac{5}{2} - \frac{5}{2}\cos(2\times 25°)\,\text{MPa}$$

$$= 2.5 - 2.5 \cos 50° = 2.5 - (2.5 \times 0.6428)\ \text{MPa}$$

$$\therefore \quad \sigma_n = 2.5 - 1.607 = \mathbf{0.89\ MPa}$$

*Shear stress across the section,*

$$\tau = \frac{\sigma}{2}\sin 2\theta \frac{\sigma}{2}\sin(2\times 25)$$

$$= 2.5 \times 0.766 = \mathbf{1.915\ MPa}$$

**Example 2.2.** *Two wooden pieces 100 mm ×100 mm in cross section are joined together along a line AB as shown in Fig. 2.3.*

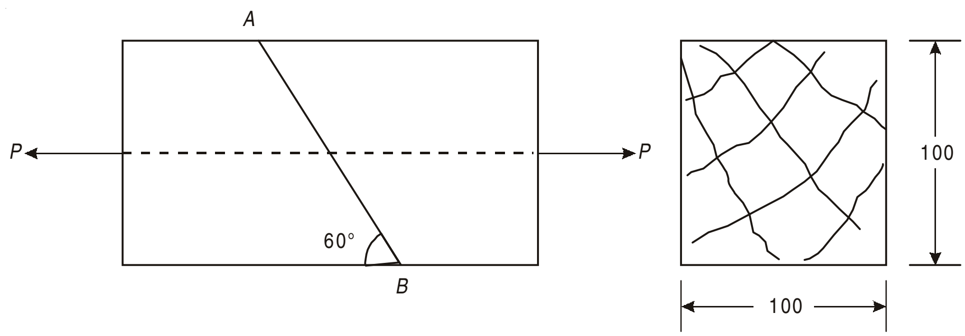



**Fig. 2.3**

*Find the maximum force (P), which can be applied if the shear stress along the joint AB is 1.3 MPa.*

**Solution** We know for shear stress

$$\therefore \quad \tau = \frac{\sigma}{2}\ \sin 2\theta$$

$$\therefore \quad 1.3 = \frac{\sigma}{2}\ \sin(2 \times 60°) = 0.433\ \sigma$$

$$\therefore \quad \sigma = \frac{1.3}{0.433} = 3.0\ \text{N/mm}^2$$

$$\therefore \quad \sigma = \frac{P}{A}$$

$\therefore$ Maximum axial force, $P = \sigma A$

$\therefore \quad P = 3.0 \times 100 \times 100$ N **= 30 kN**

**Example 2.3.** *A tension member is formed by connecting two wooden members 200 mm × 100 mm as shown in Fig. 2.4*

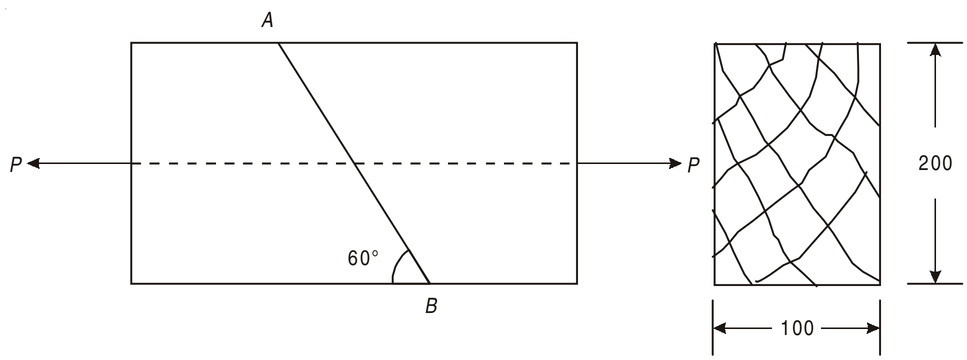



**Fig. 2.4**

*Determine the safe value of force (P), if permissible normal and shear stresses in the joint are 0.5 MPa and 1.25 MPa respectively.*

**Solution**

Let $\sigma$ = Safe stress in the joint in N/mm²

$$\because \quad \text{Normal stress, } \sigma_n = \frac{\sigma}{2} - \frac{\sigma}{2} \cos 2\theta$$

$$\therefore \quad 0.50 = \frac{\sigma}{2} - \frac{\sigma}{2}(\cos 2 \times 60)$$

$$\therefore \quad \sigma = \frac{0.5}{0.75} = 0.67 \text{ N/mm}^2$$

For shear stress ($\tau$)

$$\tau = \frac{\sigma}{2} \sin 2\theta$$

$$\therefore \quad 1.25 = \frac{\sigma}{2} \sin(2 \times 60)$$

or

$$\sigma = \frac{1.25}{0.433} = 2.89 \text{ N/mm}^2$$

From the above the values, safe stress is least of the two values i.e., 0.67 N/mm²

$\therefore$ Safe value of the force, $P = \sigma A = 0.67 \times 200 \times 100$

**= 13.4 kN**

## 2.4 ❑ STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO DIRECT STRESSES IN TWO MUTUALLY PERPENDICULAR DIRECTIONS

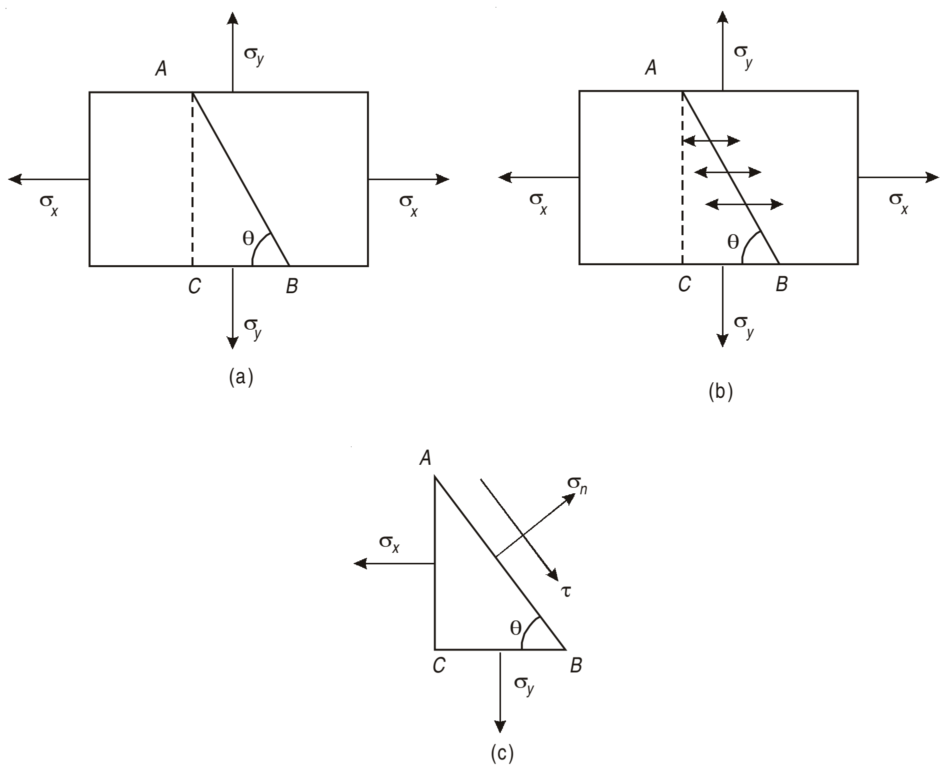



**Fig. 2.5**

$\sigma_x$ = Tensile stress along $x - x$ axis (Also termed as major tensile stress)

$\sigma_y$ = Tensile stress along $y - y$ axis (Also termed as minor tensile stress)

$\theta$ = Angle which the oblique section $AB$ makes with $x - x$ axis in clockwise directions.

Normal stress across the section $AB$,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} . \cos 2\theta \qquad \text{...(i)}$$

*Shear stress (Tangential stress) across section AB*

$$\tau = \frac{\sigma_x + \sigma_y}{2} \sin 2\theta \qquad \text{...(ii)}$$

$$\tau_{\max} = \frac{\sigma_x + \sigma_y}{2}$$

When $2\theta = 1$ or $2\theta = 90°$, or $\theta = 45°$

Resultant stress, $\sigma_R = \sqrt{\sigma_n^2 + \tau^2}$

**Example 2.4.** *A point in a strained material is subjected to two mutually perpendicular tensile stresses of 200 MPa and 100 MPa. Determine intensities of normal shear and resultant stresses on a plane inclined at 30° with the axis of minor tensile stress.*

**Solution** Normal stress on the inclined plane,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

$$= \frac{200+100}{2} - \frac{200-100}{2}\cos(2\times 30°)\,\text{MPa}$$

$$= 150 - 50 \times 0.5 = \mathbf{125\ MPa}$$

Shear stress on the inclined plane,

$$\tau = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \frac{200-100}{2}\times\sin(2\times 30°)\,\text{MPa}$$

$$= 50 \sin 60° = 50 \times 0.866 = \mathbf{43.3\ MPa}$$

Resultant stress on the inclined plane,

$$\sigma_R = \sqrt{\sigma_n^2 + \tau^2} = \sqrt{(125)^2 + (43.3)^2}$$

$$= \mathbf{132.3\ MPa}$$

## 2.5 ❑ GRAPHICAL METHOD FOR THE STRESSES ON AN OBLIQUE SECTION OF A BODY

By drawing a Mohr's circle the normal, shear and resultant stresses may be determined graphically.

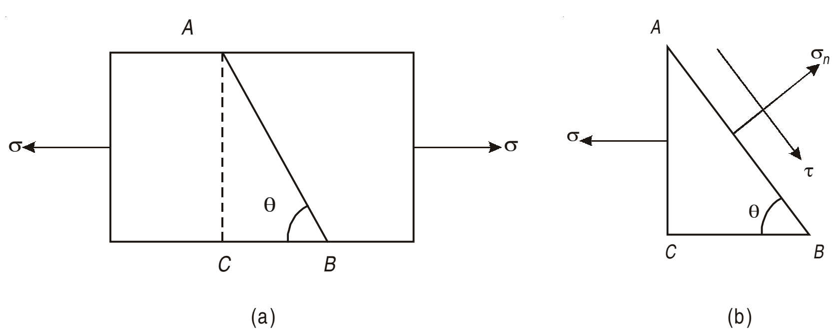



**Fig. 2.6**

Mohr's circle for stresses on an oblique section of a body subjected to a direct stress in one plane:

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to a direct tensile stress along $x - x$ axis as shown in Fig. 2.6 (a, b). Now consider an oblique section $AB$ inclined with $x - x$ axis, on which we are required to find out the stresses as shown in Fig. 2.6

Let, $\sigma$ = Tensile stress, in $x - x$ direction, and

$\theta$ = Angle which the oblique section $AB$ makes with the $x - x$ axis in clockwise direction

First of all, consider the equilibrium of the wedge $ABC$. Now draw the Mohr's circle of stresses as shown in Fig. 2.7 and as described below.

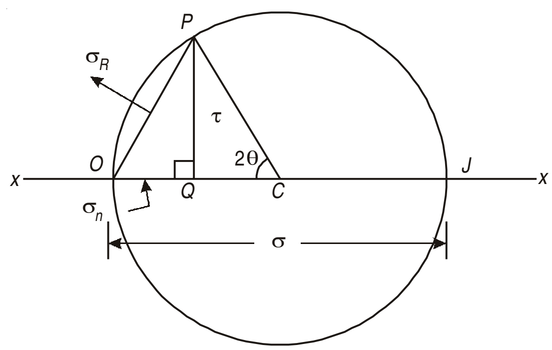



**Fig. 2.7**

1. Draw a horizontal line $X - X$ and take a suitable point $O$ on it.
2. Cut off $OJ$ equal to tensile stress ($\sigma$) to some suitable scale towards right (because $\sigma$ is tensile). Bisect $OJ$ at $C$. Now point $O$ represents the stress system on plane $BC$ and the point $J$ represents stress system on plane $AC$.
3. Now with $C$ as centre and radius equal to $CO$ or $CJ$ draw a circle. It is known as Mohr's circle for stresses.
4. Now through $C$ draw a line $CP$ making an angle of $2\theta$ with $CO$ meeting the circle at $P$. The point $P$ represents the section $AB$.
5. Through $P$, draw $PQ$ perpendicular to $OX$. Join $OP$.

Now $OQ$, $QP$ and $OP$ will give normal stress, shear stress and resultant stress and the angle $POJ$ is called the angle of obliquity ($\theta$).

## 2.6 ❑ MOHR'S CIRCLE FOR STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO DIRECT STRESSES IN TWO MUTUALLY PERPENDICULAR DIRECTIONS

Consider a rectangular body of uniform cross-sectional area and unit thickness subjected to direct tensile stresses in two mutually perpendicular disections along $X - X$ axis as shown in Fig. 2.8 (a, b). Now let us consider an oblique section $AB$ inclined with $X - X$ axis on which we are required to find out the stresses.

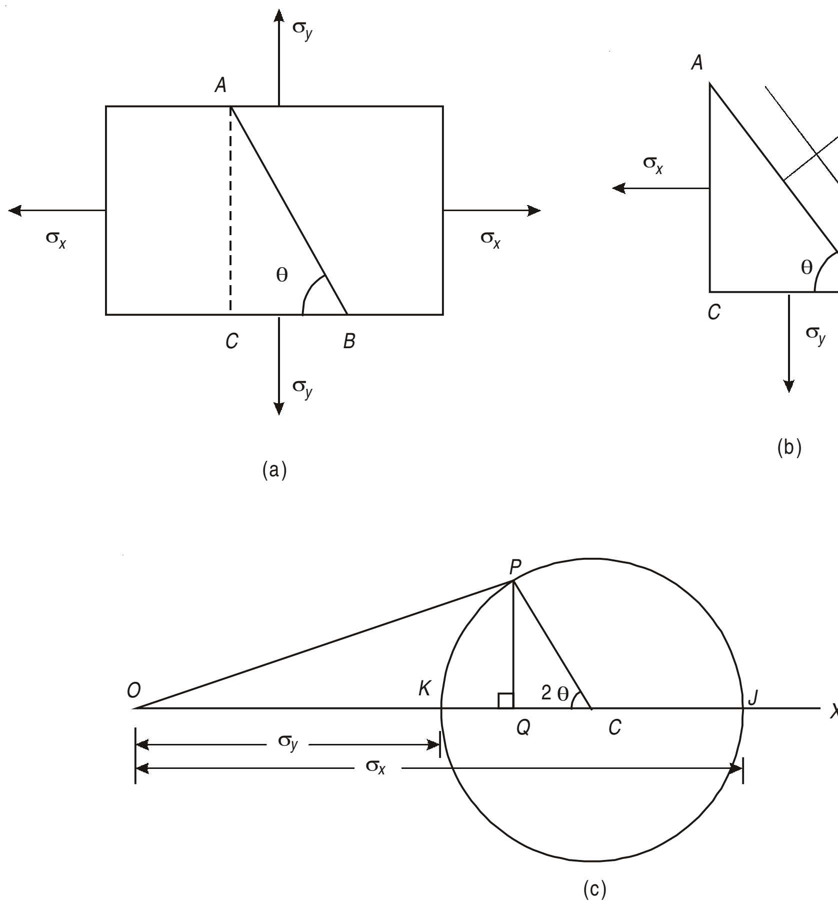



**Fig. 2.8**

Let; $\sigma_x$ = Tensile stress in $x - x$ direction (Also termed as major tensile stress)

$\sigma_y$ = Tensile stress in $y - y$ direction (Also termed as *minor tensile stress*), and

$\theta$ = Angle which the oblique section *AB* makes with $x - x$ axis is clockwise direction

First of all consider the equilibrium of the wedge *ABC*. Now draw the Mohr's circle of the stresses as shown in Fig. 2.8 and as discussed below.

1. First of all, take some suitable point *O* and draw *A* horizontal line *OX*.

2. Cut off *OJ* and *OK* equal to the tensile stresses $\sigma_x$ and $\sigma_y$ to some suitable scale towards right (because both the stresses are tensile + ve). The point *J* represents the stress system on plane *AC* and the point *K* represents the stress system on plane *BC*. Bisect *JK* at *C*.

3. Now with $C$ as centre and radius equal to $CJ$ or $CK$ draw a circle. It is known as Mohr's circle of stresses.

4. Now through $C$, draw a line $CP$ making an angle of $2\theta$ with $CK$ in clockwise direction meeting the circle at $P$. The point $P$ represents the stress systems on the section $AB$.

5. Through $P$, draw $PQ$ perpendicular to line $OX$. Join $OP$.

6. Now $OQ$, $QP$ and $OP$ will give the normal, shear and resultant stresses respectively to the scale.

The angle $POC$ is called the *angle of obliquity*.

**Example 2.5.** *The stresses at a point of a machine component are 150 MPa and 50 MPa both tensile. Find the intensities of normal, shear, and resultant stresses on a plane inclined at an angle of 55° with the axis of major tensile stress.*

*Also find the magnitude of the maximum shear stresses in the component.*

*Given :* $\sigma_x = 150$ *MPa,* $\sigma_y = 50$ *MPa,* $\theta = 55°$

**Solution**

The given stresses on the planes $AC$ and $BC$ in the machine component are shown in Fig. 2.9 (a). Now draw the Mohr's circle of stresses as shown in Fig. 2.9 (b).

1. First of all, take some suitable point $O$ and draw a horizontal line $OX$.

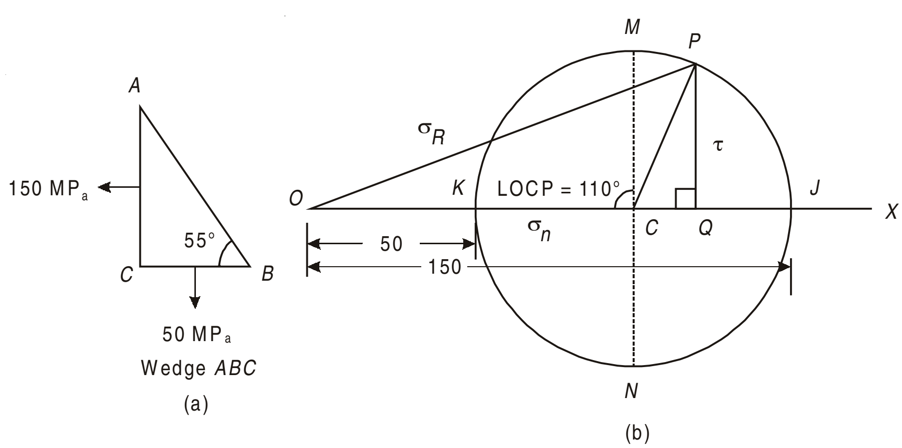



**Fig. 2.9**

2. Cut off $OJ$ and $OK$ equal to the tensile stresses $\sigma_x$ and $\sigma_y$ respectively i.e., 150 MPa and 50 MPa to some suitable scale towards right. The point $J$ represents the stress system on plane $AC$ and the point $K$ represents the stress system on plane $BC$. Bisect $KJ$ at $C$.

3. Now with $C$ as centre and radius equal to $CJ$ or $CK$ draw the Mohr's circle of stresses.

4. Now through $C$ draw two lines $CM$ and $CN$ at right angles to the line $OX$ meeting the circle at $M$ and $N$. Also through $C$ draw a line $CP$ making an angle of $2 \times 55° = 110°$ with $CK$ in clockwise direction meeting the circle at $P$. The point $P$ represents the stress systems on the plane $AB$.

5. Through $P$, draw $PQ$ perpendicular to the line $OX$. Join $PO$.

By measurement we find, normal stress $(\sigma_n) = OQ = 117.1$ MPa, shear stress $(\tau) = QP = 47.0$ MPa, resultant stress $(\sigma_R) = OP = 126.17$ MPa and maximum shear stress $(\tau_{max}) = CM = \pm 50$ MPa.

**Example 2.6.** *The stresses at a point in a component are 100 MPa (tensile) and 50 MPa (Comp.) Determine the magnitude of the normal and shear stresses on plane inclined at an angle of 25° with tensile stress. Also determine the direction of the resultant stress and the magnitude of the maximum intensity of shear stress.*

**Solution**

$$\sigma_x = 100 \text{ MPa}$$
$$\sigma_y = -50 \text{ MPa}$$
$$\theta = 25°$$

The given stresses on planes $AC$ and $BC$ are shown in Fig. 2.10 (a)

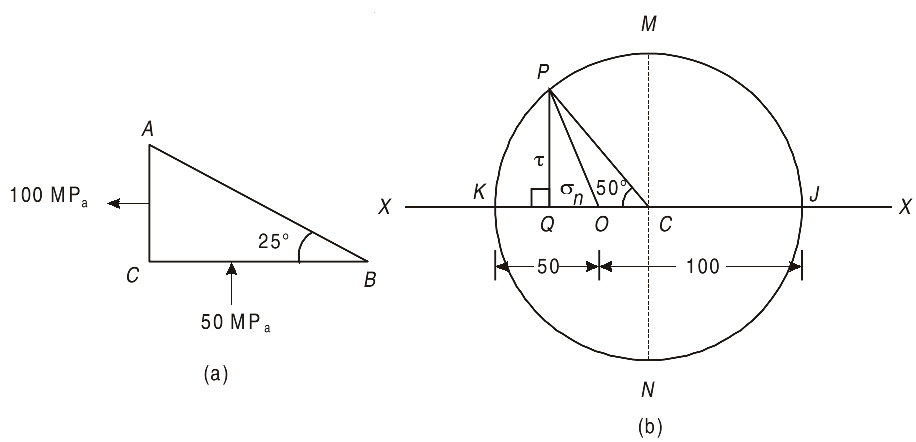



**Fig. 2.10**

Now draw Mohr's circle of stresses (Fig. 2.10 (b)) discussed below.

1. Take some suitable point $O$ and through it draw a horizontal line $XOX$.

2. Cut off $OJ$ and $OK$ equal to the stresses $\sigma_x$ and $\sigma_y$ (100 MPa and –50 MPa) to some suitable scale. Bisect $KJ$ at $C$.

3. Now with $C$ as centre and radius equal to $CJ$ or $CK$ draw Mohr's circle of stresses.

4. Now through $C$, draw two lines $CM$ and $CN$ at right angles to the line $OX$ meeting the circle at $M$ and $N$. Also through $C$, draw a line $CP$ making an angle of $2 \times 25 = 50°$ with $CK$ in clockwise direction meeting the circle at $P$. The point $P$ represents the stress system on the plane $AB$.

5. Through $P$, draw $PQ$ perpendicular to the line $OX$. Join $OP$.

By measurement, we find that the normal stress $\sigma_n = OQ = -23.21$ MPa, shear stress $(\tau) = PQ = 57.45$ MPa.

Direction of the resultant stress $\angle POQ = 68.1°$ and maximum shear stress $(\tau_{max}) = \pm CM = CN = \pm$ **75 MPa.**

## 2.7 ❑ MOHR'S CIRCLE FOR STRESSES ON AN OBLIQUE SECTION OF A BODY SUBJECTED TO DIRECT STRESSES IN ONE PLANE ACCOMPAINED BY A SIMPLE SHEAR STRESS

Consider a rectangular body of a uniform cross-sectional area and unit thickness subjected to a direct tensile stress along $x-x$ axis accompained by a positive (i.e., clockwise) shear stress along $x-x$ axis (Fig. 2.11). Now let us consider an oblique

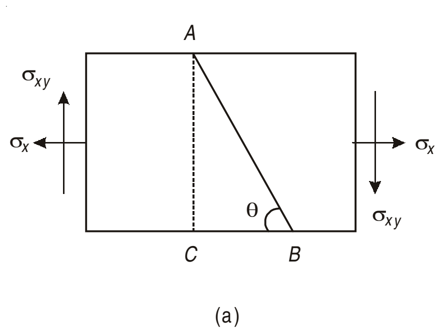



**Fig. 2.11**

section $AB$ inclined with $x-x$ axis on which we are required to find out the stresses

Let, $\sigma_x$ = Tensile stress in $x-x$ direction.

$\tau_{xy}$ = Positive (i.e. clockwise) shear stress along $x-x$ axis, and

$\theta$ = Angle which oblique section $AB$ makes with $x-x$ axis in clockwise direction.

First of all consider the equilibrium of the wedge $ABC$. We know that as per principle of simple shear the face $BC$ of the wedge will also be subjected to an anticlockwise shear stress. Now draw the Mohr's circle of stresses as shown in Fig. 2.12, and described below:

1. First of all, take some suitable point $O$ and through it draw a horizontal line $XOX$.

2. Cut off $OJ$ equal to tensile stress $\sigma_x$ to some suitable scale and towards right (As $\sigma_x$ is tensile + ve).

3. Now erect perpendicular at $J$ along the line $X-X$ (because $\tau_{xy}$ is positive along $X-X$ axis) and cut off $JD$ equal to the shear stress $\tau_{xy}$ to the scale. The point $D$ represents the stress system on plane $AC$. Similarly, erect a perpendicular below the line $X-X$ (because $\tau_{xy}$ is –ve along $Y-Y$ axis) and cut off $OE$ equal to

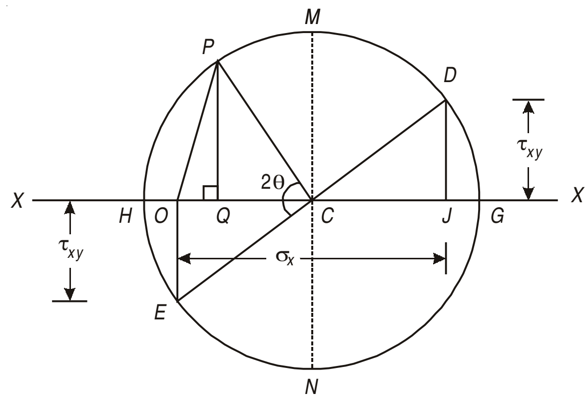



**Fig. 2.12**

shear stress $\tau_{xy}$ to the scale. The point $E$ represents the stress system on plane $BC$. Join $DE$ and bisect it at $C$.

4. Now $C$ as centre and radius equal to $CD$ or $CE$ draw a circle. It is known as *Mohr's circle of stresses*.

5. Now through $C$, draw a line $CP$ making an angle $2\theta$ with $CE$ in clockwise direction meeting, the circle at $P$. The point $P$ represents the stress system on the section $AB$.

6. Through $P$, draw $PQ$ perpendicular to the line $OX$. Join $OP$.

7. Now $OQ$, $QP$ and $OP$ will give the normal, shear and resultant stresses to the scale. And the angle $POC$ is called the *angle of obliquity*.

**Example 2.7.** *A plane element in a body is subjected to a tensile stress of 100 MPa accompained by a clockwise shear stress of 25 MPa. Find (i) The normal and shear stress on a plane inclined at an angle of 20° with the tensile stress; and (ii) the maximum shear stress on the plane.*

*Given : Tensile stress along horizontal X–X axis ($\sigma_x$) = 100 MPa, shear stress ($\tau_{xy}$) = 25 MPa and angle made by plane with tensile stress ($\theta$) = 20°.*

**Solution** The given stresses on the element and a complimentary shear stress on the $BC$ plane are shown in Fig. 2.13 (a).

Now draw the Mohr's circle of stresses as shown in Fig. 2.13 (b) and as described below:

1. First of all, take some suitable point $O$, and through it draw a horizontal line $XOX$.

2. Cut off $OJ$ equal to tensile stress on the plane $AC$ (i.e., 100 MPa) to some suitable scale towards right.

3. Now erect a perpendicular at $J$ above the line $X-X$ and cut off $JD$ equal to the positive shear stress on plane $BC$ (i.e., 25 MPa) to the scale. The point $D$ represents the stress system on the plane $AC$. Similarly erect a perpendicular at $O$ below the line $X-X$ and cut off $OE$ equal to the negative shear stress on the plane $BC$ (i.e., 25 MPa) to the scale. The point $E$ represents the stress system on the plane $BC$. Join $DE$ and bisect at $C$.

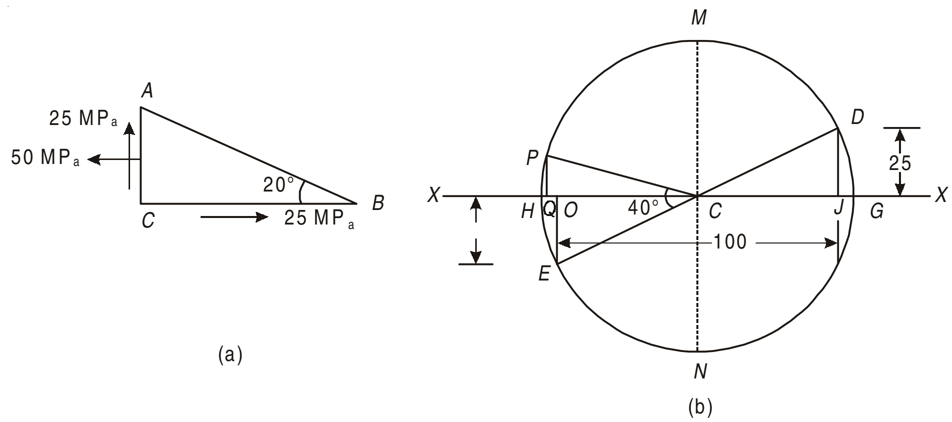



**Fig. 2.13**

4. Now with $C$ as centre and radius equal to $CD$ or $CE$ draw a Mohr's circle of stresses.

5. Now through $C$, draw two lines $CM$ and $CN$ at right angle to the line $OX$ meeting the circle at $M$ and $N$. Also through $C$, draw a line $CP$ making an angle of $2 \times 20° = 40°$ with $CE$ in clockwise direction meeting the circle at $P$. The point $P$ represents the stress system on section $AB$.

6. Through $P$, draw $PQ$ perpendicular to the line $OX$. Join $QP$.

By measurement : (i) Normal stress ($\sigma_n$) = $OQ$ = 4.4 MPa (Comp.), (ii) Shear stress $\tau$ = $QP$ = 13.0 MPa, and (iii) Maximum shear ($\tau_{max}$) = $CM$ **= 55.9 MPa**

**Example 2.8.** *An element in a strained body is subjected to a tensile stress of 150 MPa and a shear stress of 50 MPa tending to rotate the element in an anticlockwise direction. Find (i) the magnitude of the normal and shear stresses on a section inclined at 40° with the tensile stress; and (ii) the magnitude and direction of maximum shear stress that can exist on the element.*

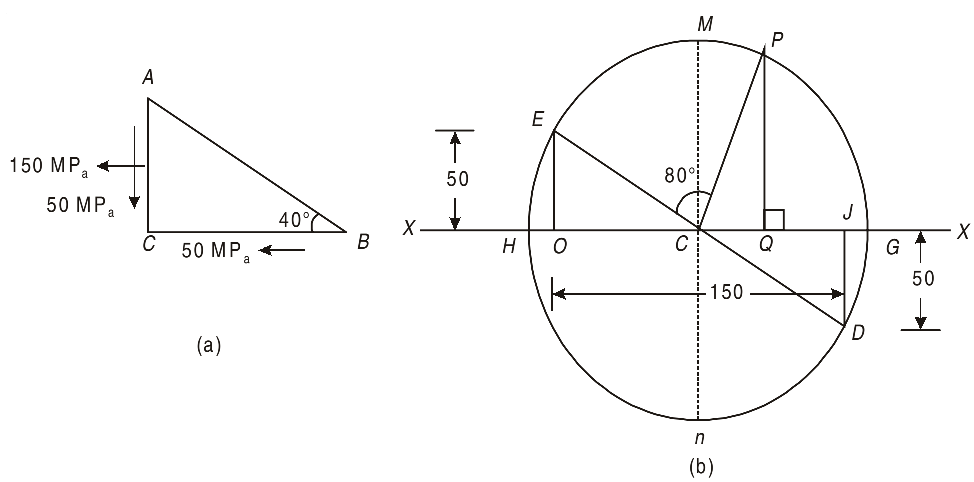



**Fig. 2.14**

*Given: Tensile stress along horizontal X – X axis* ($\sigma_x$) = *150 MPa; shear stress* ($\tau_{xy}$) = *– 50 MPa (Minus sign due to anticlockwise) and angle made by section with the tensile stress* ($\theta$) = *40°.*

**Solution** The given stresses on the plane *AB* of the element and a complimentary shear stress on the plane *BC* are shown in Fig. 2.14 (a). Now draw the Mohr's Circle of Stresses as shown in Fig. 2.14 (b) and as described below:

1. First of all take some suitable point *O*, and through it draw a horizontal line *XOX*.

2. Cut off *OJ* equal to the tensile stress on the plane *AC* (i.e. 150 MPa) to some suitable scale towards right.

3. Now erect a perpendicular at *J* below the line *X – X* and cut off *JD* equal to the negative shear stress on the plane *AC* (i.e. 50 MPa) to the scale. The point *D* represents the stress system on the plane *AC*. Similarly erect a perpendicular at *O* above the line *X – X* and cut off *OE* equal to the negative shear stress on the plane *BC* (i.e. 50 MPa) to the scale. The point *E* represents the stress systems on the plane *BC*. Join *DE* and bisect it at *C*.

4. Now with *C* as centre and radius equal to *CD* or *CE* draw the Mohr's Circle of stresses meeting the line *X – X* at *G* and *H*.

5. Through *C*, draw two lines *CM* and *CN* at right angles to the line *X – X* meeting the circle at *M* and *N*. Also through *C*, draw a line *CP* making an angle of 2 × 40° = 80° with *CE* in clockwise direction meeting the circle at *P*. The point *P* represents the stress system on the section *AB*.

6. Through *P*, draw *PQ* perpendicular to the line *OX*. Join *OP*. By measurement, we find that the Normal stress ($\sigma_n$) = *OQ* = 112.2 MPa; Shear stress ($\tau$) = *QP* = 82.5 MPa and maximum shear stress, that can exist on element ($\tau_{max}$) = ± *CM* = *CN* **= 90.14 MPa.**

**Example 2.9.** *A point in a strained material is subjected to a tensile stress of 65 N/mm² and a compressive stress of 45 N/mm², acting on two mutually perpendicular planes and a shear stress of 10 N/mm² are acting on these planes. Find the normal stress, tangential stress and resultant stress on a plane inclined to 30° with the plane of the compressive stress.* (*UPTU : 2006 – 2007*)

*Given :*

Here $\sigma_x = 65\ N/mm^2$

$\sigma_y = -45\ N/mm^2$ (*Compressive* ∴ – *ve*)

$\theta = (90 - 30) = 60°$ *with the plan of* $\sigma_x$.

$\tau = 10\ N/mm^2$

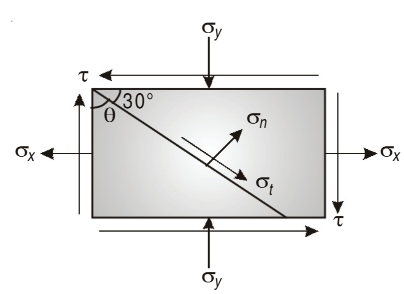



**Fig. 2.15**

**Solution** **To find :** $\sigma_n$, $\sigma_t$, $\sigma_R$.

**(i) Analytical method:**

Normal stress,

$$\sigma_n = \frac{\sigma_{x+} + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{65+(-45)}{2} + \frac{65-(-45)}{2}\cos(2\times 60) + 10\times\sin(2\times 60)$$

$$= \mathbf{-8.84\ N/mm^2}$$

Tangential stress, $\sigma_t = \dfrac{\sigma_x + \sigma_y}{2}\sin 2\theta - \tau\cos 2\theta$

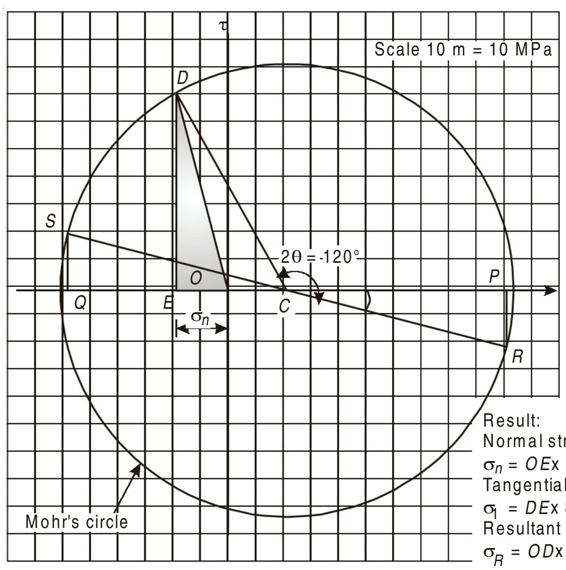
$$= \frac{65-(-45)}{2}\sin(2\times 60) - 10\cos(2\times 60)$$

$$= \mathbf{52.63\ N/mm^2}$$

Resultant stress $\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \mathbf{53.36\ N/mm^2}$

Angle of obliquity $\phi = \tan^{-1}\left(\dfrac{\sigma_t}{\sigma_n}\right) = \mathbf{80.46°}$

**(ii) Mohr's Circle Method**




Result:
Normal stress
$\sigma_n = OE$ x Scale = 0.89 x 10 = 8.9 MPa
Tangential stress
$\sigma_t = DE$ x Scale = 5.25 x 10 = 52.5 MPa
Resultant stress
$\sigma_R = OD$ x Scale = 5.3 x 10 = 53 MPa

**Fig. 2.16**

**Example 2.10.** *At a point in a body the normal and shear stresses on two perpendicular planes are given as* $\sigma_x = -100$ *MN/m*$^2$, $\sigma_y = 40$ *MN/m*$^2$, $\tau_{xy} = 50$ *MN/m*$^2$*. Using Mohr's circle determine principle stresses and their planes.*

(*UPTU : 2008–2009*)

*Given :*

$$\sigma_x = -100 \text{ MN/m}^2$$
$$\sigma_y = 40 \text{ MN/m}^2$$
$$\tau_{xy} = 50 \text{ MN/m}^2$$

**Solution To find :** $\sigma_{n_1}$, $\sigma_{n_2}$ and $\theta_p$

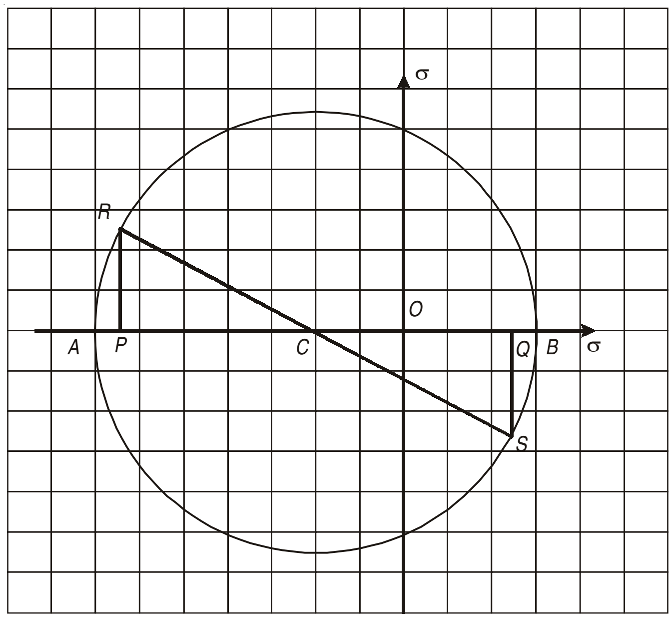



**Fig. 2.17**

**Step 1 :** Select origin $O$, the rectangular axis through $O$ represents σ and τ.

**Step 2 :** Select suitable scale, take 1 cm = 20 MN/m$^2$

**Step 3 :** Take $OP = \dfrac{\sigma_x}{20} = \dfrac{-100}{20} = -5$ cm and $OQ = \dfrac{\sigma_y}{20} = \dfrac{40}{20} = 2$ cm

**Step 4 :** Draw perpendicular from point $P$ and $Q$ such that $PR = QS = \dfrac{\tau}{20} = \dfrac{50}{20} = 2.5$ cm

**Step 5 :** Join $RS$ and mark $C$ on horizontal axis

**Step 6 :** Draw a circle of radius $CR$ from centre $C$

**Step 7 :** Mark the point $A$ and $B$ on the horizontal axis which cut by a circle.

**Step 8 :** *OA* and *OB* represent principle stress.

**Step 9 :** Angle *PCR* indicates the angle of principle plane (2θ).

From figure

$$\sigma_{n_1} = OA \times \text{Scale} = 5.8 \times 20 = \mathbf{116\ N/mm^2}\ \text{(Compressive)}$$

$$\sigma_{n_2} = OB \times \text{Scale} = 2.8 \times 20 = \mathbf{56\ N/mm^2}\ \text{(Tensile)}$$

**Example 2.11.** *Construct Mohr's circle for the case of plane stres $\sigma_x = 360$ kg/cm², $\sigma_y = 200$ kg/cm² and $\tau_{xy} = 60$ kg/cm² and determine the magnitudes of the two principal stresses $\sigma_1$ and $\sigma_2$ and the angle $\phi$ between the direction $\sigma_x$ and $\sigma_1$.* *(UPTU : 2010–2011)*

*Given :* $\sigma_x = 360$ kg/cm², $\sigma_y = 200$ kg/cm², $\tau_{xy} = 60$ kg/cm²

**Solution To Find :** $\sigma_1$, $\sigma_2$, $\phi$

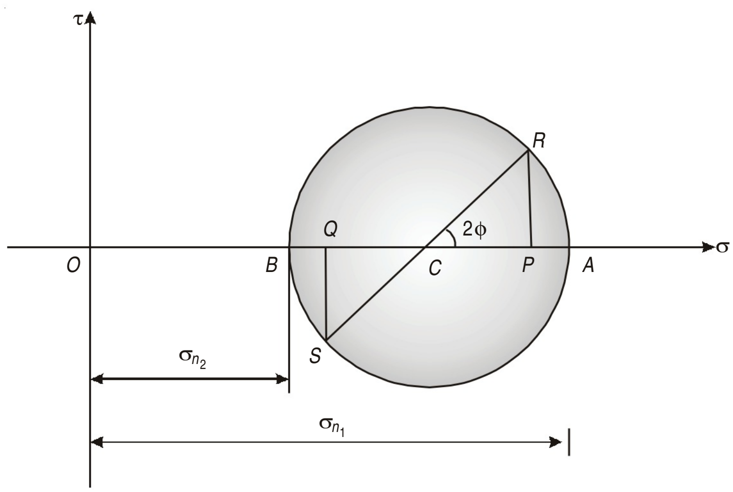



**Fig. 2.18**

$$\sigma_{n_1} = OA \times \text{scale} = 7.6 \times 50 = 380 \text{ kg/cm}^2$$

$$\sigma_{n_2} = OB \times \text{scale} = 3.5 \times 50 = 175 \text{ kg/cm}^2$$

Angle $\phi$ between the direction $\sigma_x$ and $\sigma_1$

$$\angle PCR = 2\phi = 35$$

$$\phi = \frac{35}{2} = \mathbf{17.5°}$$

## 2.8 ❑ TYPE V : PRINCIPAL PLANE AND STRESS

**Summary of important formulae :**

Major principal stress $$\sigma_{n_1} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

Minor principal stress $\sigma_{n_2} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$

Location of principal plane $\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$

Maximum shear stress $\tau_{max} = \frac{\sigma_{n_1} - \sigma_{n_2}}{2} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$

Location of plane of maximum shear stress

$$\theta = \theta_p + 45°$$

Major principal strain $e_1 = \frac{\sigma_{n_1}}{E} - \mu\frac{\sigma_{n_2}}{E}$

Minor principal strain $e_2 = \frac{\sigma_{n2}}{E} - \mu\frac{\sigma_{n1}}{E}$

**Example 2.12.** *At a point in a material there are normal stresses of 30 N/mm² and 60 N/mm² tensile, together with a shearing stress of 22.5 N/mm². Find the value of principal stresses and the inclination of the principal planes to the direction of the 60 N/mm² stress.* (*UPTU : 2012–2013*)

*Given :* $\sigma_x = 30$ N/mm², $\sigma_y = 60$ N/mm²

$\tau = 22.5$ N/mm²

**Solution To Find :** $\sigma_{n_1}, \sigma_{n_2}, \theta_{p_1}, \theta_{p_2}$

Principal stress,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{30+60}{2} \pm \sqrt{\left(\frac{30-60}{2}\right)^2 + 22.5^2}$$

$$= 40 \pm 27.04$$

Major principal stress $\sigma_{n_1} = 45 + 27.04 = 72.04$ N/mm²

Minor principal stress $\sigma_{n_2}$, $= 45 - 27.04 = 17.96$ N/mm²

Position of principal plane,

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 22.5}{30 - 60} = -1.5$$

$$\tan (180 - 2\theta) = -\tan 2\theta = 1.5$$

$$180 - 2\theta = 56.3$$

$$2\theta = 123.69$$

$$\theta_{p_1} = 61.8°$$

$$\theta_{p_2} = 90 + \theta_{p_1} = 90 + 61.8° = \mathbf{151.8}$$

**Example 2.13.** *A two dimensional state of stress is given by :* $\sigma_{xx} = 10$ *MPa,* $\sigma_{yy} = 5\ \sigma_{xy}$ *MPa and* $\sigma_{xy} = 2.5$ *MPa. Determine the following on a plane inclined at an angle of 30° from x-plane in anticlockwise direction.*

(*i*) *Normal stress* (*ii*) *Shear stress*

(*iii*) *Resultant stress and* (*iv*) *Principal stresses at the point.*

(*UPTU : 2010–2011*)

*Given :*

$\sigma_x = 10$ MPa, $\sigma_y = 5$

$\sigma_{xy} = 5 \times 2.5 = 12.5$ MPa

$\sigma_{xy} = 2.5$ MPa, $\theta = 60°$

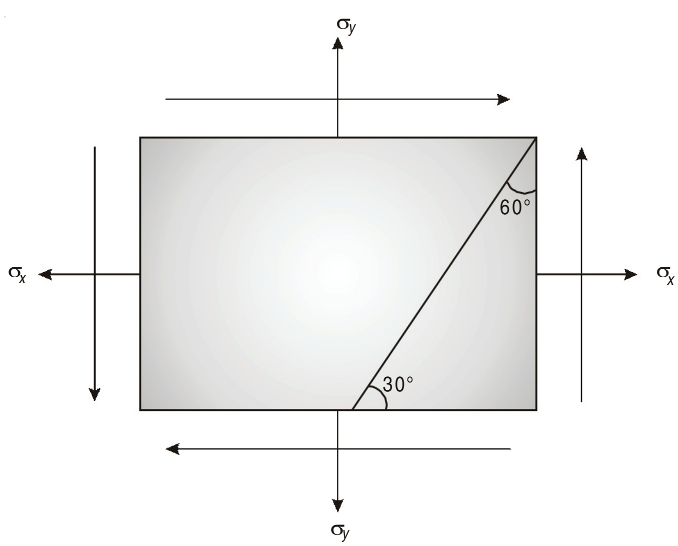



**Fig. 2.19**

**Solution To Find :** $\sigma_n$, $\sigma_t$, $\sigma_R$, and $\sigma_{n_1}$, $\sigma_{n_2}$

Normal stress,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta$$

$$= \frac{10+12.5}{2} + \frac{10+12.5}{2}\cos(2\times 60) + 2.5\sin(2\times 60)$$

$$= 11.25 + 0.625 + 2.165 = 14.04 \text{ N/mm}^2$$

Shear stress,

$$\sigma_t = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \sigma_{xy}\cos 2\theta = \frac{10-12.5}{2}\sin(2\times 60) - 2.5\cos(2\times 60)$$

$$= -2.165 + 1.25 = -0.912 \text{ N/mm}^2$$

Resultant stress,

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{(14.04)^2 + (-0.912)^2} = 14.07 \text{ N/mm}^2$$

$$\phi = \tan^{-1}\left|\frac{\sigma_t}{\sigma_n}\right| = \tan^{-1}\left|\frac{-0.912}{14.02}\right| = 3.72°$$

Principal stresses,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2}$$

$$= \frac{10+12.5}{2} \pm \sqrt{\left(\frac{10-12.5}{2}\right)^2 + 2.5^2} = 11.25 \pm 2.795$$

Major principal stress, $\sigma_{n1} = 11.25 + 2.795$

$= \mathbf{14.045 \text{ MPa}}$

Minor principal stress, $\sigma_{n2} = 11.25 - 2.795 =$ **8.455 MPa**

**Example 2.14.** *Direct stress of 120 MN/m$^2$ in tension and 90 MN/m$^2$ in compression are applied to an elastic material at a certain point on a plane at 25° with the tensile stress. If the maximum principal stress is not to exceed 150 MN/m$^2$ in tension to what shearing stress can the material be subjected ? What is then the maximum resulting shearing stress in the material and also find the magnitude of the other principal stress and its inclination to plane of 120 MN/m$^2$ stress.* *(MTU : 2012 –2013)*

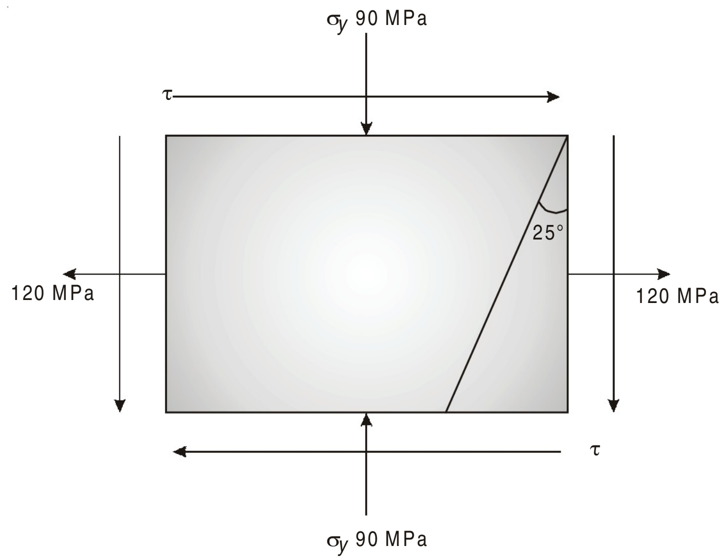



**Fig. 2.20**

*Given :* $\sigma_{n_1}$ *= 150 MPa,* $\sigma_x$ *= 120 MPa,* $\sigma$ *= –90 MPa*

**Solution To Find :** (i) $\tau$ (ii) $\tau_{max}$

(i) *Magnitude of shear stress :*

$$\text{Major principal stress } \sigma_{n_1} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$150 = \frac{120-90}{2} + \sqrt{\left(\frac{120+90}{2}\right)^2 + \tau^2}$$

$$150 = 15 + \sqrt{105^2 + \tau^2}$$

Squaring both side

$$(150-15)^2 = 105^2 + \tau^2$$

$$\boldsymbol{\tau = 84.85 \text{ N/mm}^2}$$

$$\text{Minor principal stress } \sigma_{n_2} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{120-90}{2} + \sqrt{\left(\frac{120+90}{2}\right)^2 + 84.85^2}$$

$$= -120 \text{ N/mm}^2$$

$$\tau_{max} = \frac{\sigma_{n_1} - \sigma_{n_2}}{2} = \frac{150-(-120)}{2} = \mathbf{135 \text{ N/mm}^2}$$

Position of principal plane,

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 84.85}{120 - (-90)} = 0.808$$

$$2\theta = 38.94$$

$$\theta_{p_1} = 19.47° \quad \theta_{p_2} = \theta_{p_1} + 90 = 109.47°$$

Position of maximum shear stress,

$$\theta_{s_1} = \theta_{p_1} + 45°$$
$$= 19.47° + 45° = 64.47°$$
$$\theta_{s_2} = \theta_{p_2} + 45°$$
$$= 154.47°$$

**Example 2.15.** *At a point in a strained material, there are normal stresses of 30 N/mm², tension and 20 N/mm², compression on two planes at right angles to one another, together with shearing stresses of 15 N/mm² on the same planes. If the loading on the material is increased so that the stresses reach values of K times those given, find the maximum, permissible value of K if the maximum direct stress in the material is not to exceed 80 N/mm², and maximum shear stress is not to exceed 50 N/mm².* *(UPTU : 2009–2010)*

*Given :*

$$\sigma_x = 30 \text{ N/mm}^2$$
$$\sigma_y = -20 \text{ N/mm}^2$$
$$\tau = 15 \text{ N/mm}^2$$
$$\sigma_{n1} \ngtr 80 \text{ N/mm}^2$$
$$\tau_{max} \ngtr 50 \text{ N/mm}^2$$

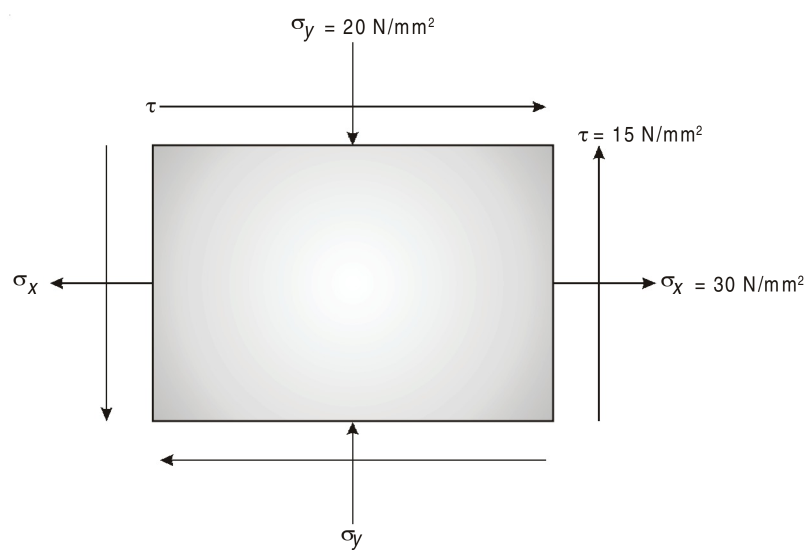



**Fig. 2.21**

**Solution To Find :** Value of $K$ (constant)

Maximum principal stress

$$\sigma_{n_1} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$$

$$= \frac{30+(-20)}{2} + \sqrt{\left(\frac{30-(-20)}{2}\right)^2 + 15^2}$$

$$= 0.05 + 29.15$$

$$\sigma_{n_1} = \mathbf{34.155\ N/mm^2}$$

Maximum shear stress

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2} = 29.15 \text{ MPa}$$

Since the loading on the material is increased so that the stresses reach vlues of $K$ times those given.

(i) *$K$ by considering maximum direct stress*

$$\sigma_{n \text{ permissible}} = K\sigma_{n_1}$$

$$80 = 34.155 \text{ K} \quad K = 2.34$$

(ii) *$K$ by considering maximum shear stress*

$$\tau_{\text{max permissible}} = K\,\tau_{\text{max (Actual)}}$$

$$50 = K \times 29.15$$

$$K = 1.715$$

∴ Permissible value of $K$ will be least value from (i) and (ii)

∴ $K = \mathbf{1.715}$

## 2.9 ❑ PRINCIPAL STRAIN

When the body subjected to principal stresses, body produce some strain, these strain is called as principal strain.

### 2.9.1 Principal Strain due to Principal Stresses

Let $\sigma_1$, $\sigma_2$, $\sigma_3$ be the three principal stresses acting on three mutually perpendicular plane, and $e_1$, $e_2$ and $e_3$ are principal strain in the direction of respective principal stresses respectively.

$$e_1 = \frac{\sigma_1}{E}\mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$e_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$e_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

where,

$\mu$ = Poisson's ratio

$E$ = Moldulus of elasticity

**Example 2.16.** *An element of material in plane strain undergoes the following strains :*

$\varepsilon_x = 340 \times 10^{-6}$, $\varepsilon_y = 110 \times 10^{-6}$, $\gamma_{xy} = 180 \times 10^{-6}$.

*Determine : (i) Strain of a line inclined at an angle of 30° from x-axis (ii) Principle strains and (iii) Maximum shear strain.*

*Given :*

$$\varepsilon_x = 340 \times 10^{-6}$$
$$\varepsilon_y = 110 \times 10^{-6}$$
$$\gamma_{xy} = 180 \times 10^{-6}$$
$$\theta = 90° - 30° = 60°$$

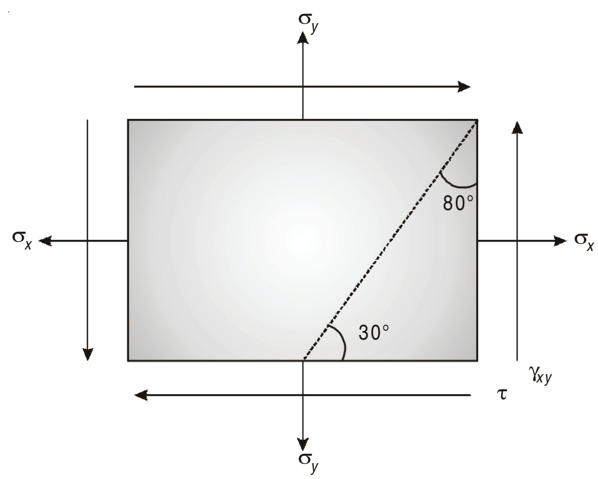



**Fig. 2.22**

**Solution To Find :** $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$ and $\gamma_{max}$

(i) *Strain of a line inclined at angle 30° from x-axis :*

$$\varepsilon = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \gamma \sin 2\theta$$

$$= \frac{(340+110)\times 10^{-6}}{2} + \frac{(340-110)\times 10^{-6}}{2}\cos(120°) + 180\times 10^{-6}\sin(120°)$$

$$= 2.25 \times 10^{-4} + (-5.75 \times 10^{-5}) + 1.56 \times 10^{-4}$$

$$= \mathbf{3.234 \times 10^{-4}}$$

(ii) *Principal strain :*

$$\varepsilon_n = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \gamma_{xy}^2}$$

$$= \frac{(340+110)\times 10^{-6}}{2} \pm \sqrt{\left[\frac{(340-110)\times 10^{-6}}{2}\right]^2 + \left(180\times 10^{-6}\right)^2}$$

$$= \mathbf{2.25 \times 10^{-4} \pm 2.136 \times 10^{-4}}$$

$$\varepsilon_{n_1} = \mathbf{4.386 \times 10^{-4}}$$

$$\varepsilon_{n_2} = \mathbf{2.25 \times 10^{-4} - 2.136 \times 10^{-4} = 0.114 \times 10^{-4}}$$

(iii) *Maximum shear strain :*

$$\gamma_{\max} = \frac{\varepsilon_{n1} - \varepsilon_{n2}}{2} = \frac{(4.38 - 0.114)\times 10^{-4}}{2}$$

$$= \mathbf{2.13 \times 10^{-4}}$$

## EXERCISE

**2.1** A two wooden pieces 10 m × 10 cm in cross section are glued together along line *AB* as shown in Fig. 2.23. What is the maximum axial force *P* can be applied if the allowable shearing stress along *AB* is 1.2 N/mm$^2$.

[*Ans. P* = 88.19 kN]

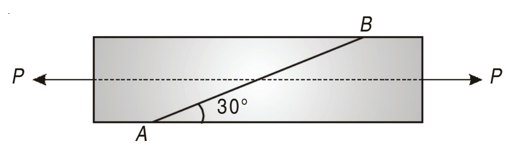



**Fig. 2.23**

**2.2.** Find the diameter of the circular bar which is subjected to an axial pull of 150 kN, if the maximum allowable shear stress on any section is 60 N/mm$^2$.

[*Ans. d* = 39.89 mm]

**2.3.** A short column of 25 mm diameter carries a compressive load of 35 kN. For a plane inclined are 60° with the direction of the load, determine the normal, shear and resultant stresses. Also find the maximum shear stress and the obliquity of the resultant stress.

[*Ans.* $\sigma_n$ = 5.35 MPa, $\sigma_t$ = 3.089 MPa, $\sigma_R$ = 6.177 MPa, $\tau_{\max}$ = 3.567 MPa, $\phi$ = 30°]

**2.4.** A bar uniform cross section 60 mm, 80 mm is subjected to an axial tensile force of 600 kN applied at each end of the bar. Determine the normal and shearing stress acting on a plane inclined at 35° to the line of action of force.

[*Ans.* $s_n$ = 41.123 MPa, $\sigma_t$ = 58.73 MPa]

**2.5.** At a point in a strained material, a shear stress of 20 MPa is acting along with a normal stress 50 MPa (tensile). On another plane right angle to this plane direct stress is zero. Find the principal stresses.

[*Ans.* $\sigma_{n1}$ = 57.01 N/mm², $\sigma_{n2}$ = –7.01 N/mm²,
$\theta_{P1}$ = 19.33° and $\theta_{P2}$ = 109.33°]

**2.6.** At a point in a strained body the normal stresses of 80 MPa (tensile) and 60 MPa (compressive) acting at a plane right angle to each other. Determine the magnitude and direction of the resultant stresses on planes making angle of 25° and 70° with the plane of + 80 MPa. Also find normal and tangential stresses on these planes.

[*Ans.* $\sigma_{n1}$ = 55 MPa, $\sigma_{t1}$ = 53.62 MPa, $\sigma_{R1}$ = 75.81 MPa,
$\sigma_{n2}$ = 43.62 MPa, $\sigma_{t2}$ = 45 MPa, $\sigma_{R2}$ = 62.67 MPa]

## REVIEW QUESTIONS

1. Define principal planes and principal stresses and explain their uses.
(*UPTU : 2012–2013*)
2. Derive an experssion for the stresses on an oblique section of a rectangular body, when it is subjected to (a) a direct stress in one plane only and (b) direct stresses in two mutually perpendicular directions.
3. Obtain an expression for the major and minor principal stresses on a plane, when the body is subject to direct stresses in two mutually perpendicular directions accompanied by a shear stress.
4. How will you find out graphically the resultant stress on an oblique section when the body is subjected to direct stresses in two mutually perpendicular directions?
5. A bar is subjected to a tensile stress of 100 MPa. Determine the normal and tangential stresses on a plane making an angle of 30° with the direction of the tensile stress. **[*Ans.* 75 MPa ; 43.3 MPa]**
6. A point in a strained material is subjected to a tensile stress of 50 MPa. Find the normal and shear stress at an angle of 50° with the direction of the stress. **[*Ans.* 29.34 MPa; 24.62 MPa]**
7. At a point in a strained material, the principal stresses are 100 MPa and 50 MPa both tensile. Find the normal and shear stresses at a section inclined at 30° with the axis of the major principal stress.
**[*Ans.* 87.5 MPa; 21.65 MPa]**
8. A point in a strained material is subjected to a tensile stress of 120 MPa and a clockwise shear stresses on a plane inclined at 45° with the normal to the tensile stress. **[*Ans.* 20 MPa; 60 MPa]**
9. The principal stresses to a point in the section of a member are 50 MPa or 20 MPa both tensile. If there is a clockwise shear stress of 30 MPs, find the

normal and shear stresses on a section inclined at an angle of 15° with the normal the major tensile stress. **[*Ans.* 32.99 MPa; 33.48 MPa]**

10. At a point in a strained material, the principal stresses are 100 MPa and 50 MPa both tensile. Find the normal and shear stresses at a section inclined at 60° with the axis of the major principal stress.

**[*Ans.* 87.5 MPa; 21.65 MPa]**

11. A point in a strained material is subjected to a tensile stress of 120 MPa and a clockwise shear stress of 40 MPa. What are the values of normal and shear stresses on a plane inclined at 25° with the normal to the tensile stress.

**[*Ans.* 20 MPa; 60 MPa]**

12. The principal stresses at a point in the section of a member are 50 MPa and 20 MPa both tensile. If there is a clockwise shear stress of 30 MPa, find graphically the normal and shear stresses on a section inclined at an angle of 15° with the normal to the major tensile stress.

**[*Ans.* 32.99 MPa; 33.48 MPa]**

13. A point is subjected to tensile stresses of 200 MPa and 150 MPa on two mutually perpendicular planes and an anticlockwise shear stress of 30 MPa. Determine by any method the value of normal and shear stresses on a plane inclined at 60° with the minor tensile stress.

**[*Ans.* 188.48 MPa: 36.65 MPa]**

14. At a point in a stressed element, the normal stress in two matually perpendicular directions are 45 MPa and 25 MPa both tesile. The complimentary shear stress in these directions is 15 MPa. By using Mohr's circle method, or otherwise, determine the maximum and minimum principal stresses. **[*Ans.* 188.48 MPa: 36.65 MPa]**

15. A point in a strained material is subjected to a tensile stress of 45 N/mm$^2$, acting on two mutually perpendicular planes. Find the normal stress, tangential stress and resultant stress on a plane inclined to 30° with the plane of the compressive stress. (*UPTU : 2006–2007*)

**[*Ans.* $\sigma_R$ = 53.36 N/mm$^2$, $\phi$ = 809.46,**
**$\sigma_n$ = – 8.84 N/mm$^2$, $\sigma_t$ = 52.63 N/mm$^2$]**

16. At a point in a body, the normal and shear stresses on two perpendicular planes are given as $\sigma_x$ = –100 MN/cm$^2$, $\sigma_y$ = 40 MN/cm$^2$, $\tau_{xy}$ = 50 MN/m$^2$. Using Mohr's circle, determine principal stresses and their planes.

(*UPTU : 2008–2009*)

17. At a point in a strained material, there are normal stresses of 30 N/mm$^2$ tension, and 20 N/mm$^2$ compression on two planes at right angles to one another, together with shearing stresses of 15 N/mm$^2$ on the same planes. If the loading on the material is increased so that the stresses reach values of $K$ times those given, find the maximum permissible value of $K$ if the maximum direct stress in the material is not to exceed 80 N/mm$^2$, and maximum shear stress is not to exceed 50 N/mm$^2$. (*UPTU : 2009–2010*)

**[*Ans.* $K$ = 1.715]**

18. Construct Mohr's circle for a case of plane stress, $\sigma_x = 360$ kg/cm$^2$, $\sigma_y = 200$ kg/cm$^2$ and $\tau_{xy} = 60$ kg/cm$^2$ and determine the magnitudes of two principal stresses $\sigma_1$ and $\sigma_2$ and the angle $\phi$ between the directions $\sigma_x$ and $\sigma_1$.
(*UPTU : 2010–2011*)
**[*Ans.* 380 kg/cm$^2$ , 175 kg/cm$^2$, $\phi$ = 175°]**

19. A two dimensional state of stress is given by : $\sigma_{xx} = 10$ MPa, $\sigma_{yy} = 5$ $\sigma_{xy}$ MPa, and $\sigma_{xy} = 2.5$ MPa. Determine the following on a plane inclined at an angle of 30° from $x$-plane in anticlockwise direction :
(a) Normal stress
(b) Shear stress
(c) Resultant stress
(d) Principal stresses at the point (*UPTU : 2010–2011*)
**[*Ans.* (a) $\sigma_x$ = 14.04 N/mm$^2$; (b) $\tau$ = – 0.912 N/mm$^2$; (c) $\sigma_R$ = 14.02 N/mm$^2$; (d) $\sigma_{xy}$ = 14.045 MPa; $\sigma_{n2}$ = 8.455 MPa]**

20. An element of material in plain strain undergoes the following strains:
$\varepsilon_x = 340 \times 10^{-6}$, $\varepsilon_y = 110 \times 10^{-6}$, $\gamma_{xy} = 180 \times 10^{-6}$
Determine (i) strain of a line inclined at an angle of 30° from $x$–axis, (ii) Principal strains, and (iii) Maximum shear strain
**[*Ans.* $\varepsilon_{n1} = 4.386 \times 10^{-4}$, $\varepsilon_{n2} = 0.114 \times 10^{-4}$, $\gamma_{max} = 2.13 \times 10^{-4}$]**

21. At a point in a material there are normal stresses of 30 N/mm$^2$ and 60 N/mm$^2$ tensile together with a shearing stress at 22.5 N/mm$^2$. Find the value of principal stresses and the inclination of the principal planes to the direction of the 60 N/mm$^2$ stress. (*UPTU* : *2012–2013*)
**[*Ans.* $\sigma_{n1}$ = 72.04 N/mm$^2$, $\sigma_{n2}$ = 17.96 N/mm$^2$, $\phi_{p1}$ = 61.8°,$\phi_{p2}$ = 151.8°]**

22. Direct stress of 120 N/mm$^2$ in tension and 90 MN/m$^2$ in compression are applied to an elastic material at a certain point on a plane at 25° with the tensile stress. If the maximum principal stress is not to exceed 150 MN/m$^2$ in tension to what shearing stress can the material be subjected? What is then the maximum resulting stress in the material and also find the magnitude of the other principal stress and its inclination to plane of 120 MN/m$^2$ stress.
(*UPTU* : *2012–2013*)
**[*Ans.* $\tau$ = 84.85 N/mm$^2$, $\tau_{max}$ = 135 N/mm$^2$, $\phi_1$ = 64.47°, $\phi_2$ = 154.47°]**

# Pure Bending and Stresses in Beams of Different Cross Sections

CHAPTER 3

## 3.1 ❑ INTRODUCTION

A beam is a structural member on which external loads act at right angles to its longitudinal axis. When a beam is subjected to a constant *bending moment,* then it is said to be in a state of *pure bending* or *simple bending*.

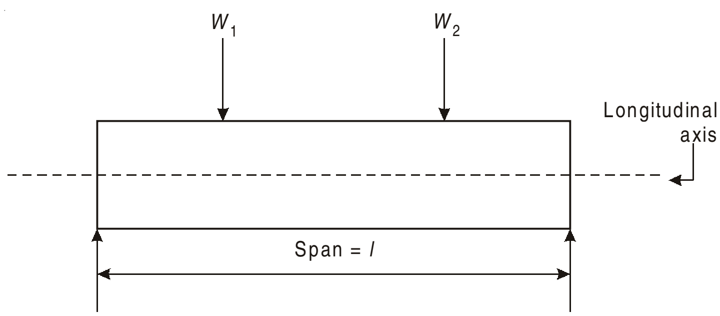



**Fig. 3.1**

The bending moment at a section tends to bend the beam and the internal stresses resist its bending. The resistance offered by the internal stresses to the bending is called *bending stress* and the relevant theory is called the *theory of simple bending.*

## 3.2 ❑ ASSUMPTIONS MADE IN THE THEORY OF SIMPLE BENDING

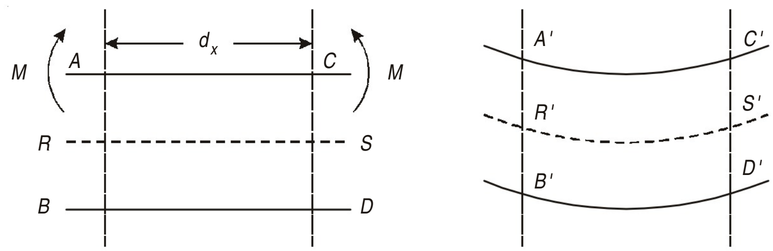
The assumption made in the theory of simple bending in explained as following.

1. The material of the beam is perfectly homogenous.
2. The beam material is stressed within its elastic limit.
3. Each layer of the beam is free to expand or contract, independently, of the layer above or below it.

4, The transverse sections which wave plane before bending remain plane after bending also.
5. The value of $E$ (Young's Modulus of Elasticity) is same in tension and compression.
6. The beam is in equilibrium i.e., there is no resultant pull or push in the beam section.

## 3.3 ❑ THEORY OF SIMPLE BENDING

Consider small length *dx* of a *simply supported beam* subjected to a bending moment *M*. Consider 2 sections *AB* and *CD* which are *normal* to the axis of beam




(a) Before bending (b) After bending

**Fig. 3.2**

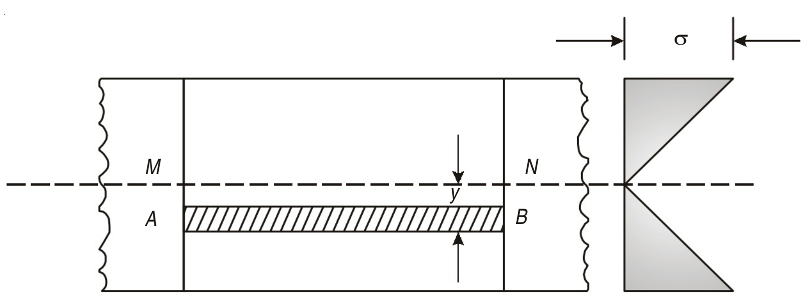
*RS*. Due to action of bending moment, the beam as a whole will bend as shown in Fig. 3.2 (b).

The top layer *AC* has suffered compression and reduced to *A´C´*. Layer *RS* has suffered no change in length, though bent to *R´S´*. Lower most layer *BD* has been stretched to *B´ D´*. Hence layers above *RS* have been compressed and those below *RS* have been stretched. This layer *RS*, which has neither compressed nor stretched is known as *neutral layer* or neutral plane.

This theory of bending is called *theory of simple bending*.

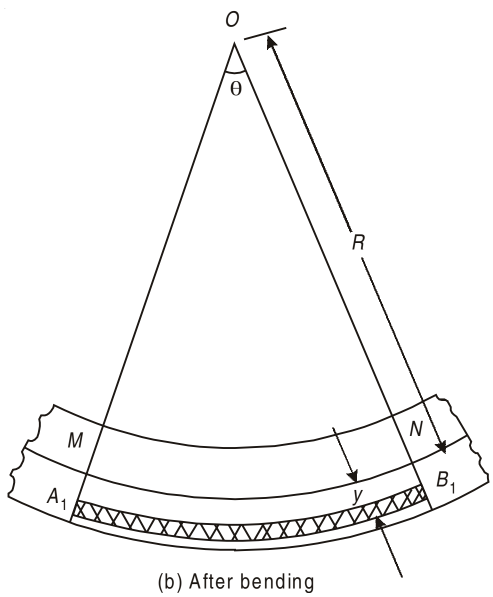
## 3.4 ❑ BENDING STRESSES

Consider a portion of pure beam subjected to *pure bending*.




(a) Before bending

**Fig. 3.3 (a)**




(b) After bending

**Fig. 3.3 (b)**

A fibre such as $AB$ at a distance $y$ from the neutral axis as shown in Fig. 3.3 (b) is chosen.

After bending it has elongated and is shown by $A_1B_1$ ( Fig.3.3 b). Let $R$ be the radius of the neutral surface after bending making an angle $\theta$ at the point $O$.

Now, $MN = R\theta$

$A_1B_1 = (R + y)\theta$

Strain in the fibre at a distance $y$ from the neutral axis $= \dfrac{\text{Increase in length}}{\text{Original length}}$

$$= \frac{A_1B_1 - AB}{AB} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{(R+y-R)\theta}{R\theta} = \frac{y}{R}$$

But $$E = \frac{\text{Stress}}{\text{Strain}}$$

Hence $$\text{Strain} = \frac{\text{Stress}}{E} = \frac{\sigma}{E}$$

or $$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{R} = \frac{E}{R} \qquad \text{...(i)}$$

∴ *Stress intensity in any form is proportional to the distance of the fibre from the neural axis.*

## 3.5 ❑ MOMENT OF RESISTANCE

Bending stress at any point in the cross-section is proportional to its distance from *neutral axis*. The portion of the beam which is above the *neutral axis* will experience compressive stress while the portion below the *neutral axis* will be subjected to the *tensile stress*.

These compressive and tensile forces form a couple at any *cross-section* whose moment is called Moment of Resistance (M.R.). It must be equal to the external Bending Moment at the section of the beam.

$$\frac{M}{I} = \frac{E}{R}; \text{ but } \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \quad \frac{M.R.}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

It is called bending equation or Bernoulli-Euler bending equation.

## 3.6 ❑ SECTION MODULUS

From the above *bending equation* $\frac{\sigma}{y} = \frac{M}{I}$

$$\sigma = \frac{M}{I} \cdot y = \frac{M}{\left(\frac{I}{y}\right)} \qquad \therefore \quad \sigma = \frac{M}{Z}$$

The ratio $\frac{I}{y}$ is known as section modulus and is denoted by $Z$.

| *Section* | *Area* | *M.O.I. (I)* | *S. Modulus (Z)* |
|---|---|---|---|
| 1. Rectangle | $bd$ | $\frac{bd^3}{12}$ | $\frac{bd^2}{6}$ |
| 2. Square | $bb = b^2$ | $\frac{b^4}{12}$ | $\frac{b^3}{6}$ |
| 3. Circular | $\frac{\pi d^2}{4}$ | $\frac{\pi d^4}{64}$ | $\frac{\pi d^3}{32}$ |
| 4. Hollow circular | $\frac{\pi}{4}\left(d^2 - d_1^2\right)$ | $\frac{\pi}{64}\left(d^4 - d_1^4\right)$ | $\frac{\pi}{32}\left(\frac{d^4 - d_1^4}{d}\right)$ |

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad \text{bending equation, } \sigma = \frac{M}{I/y} = \frac{M}{Z}$$

$M$ = Moment of resistance
$I$ = Moment of inertia of beam section
$E$ = Young's modulus of elasticity
$R$ = Radius of curvature
$\sigma$ = Bending stress
$Z$ = Modulus section

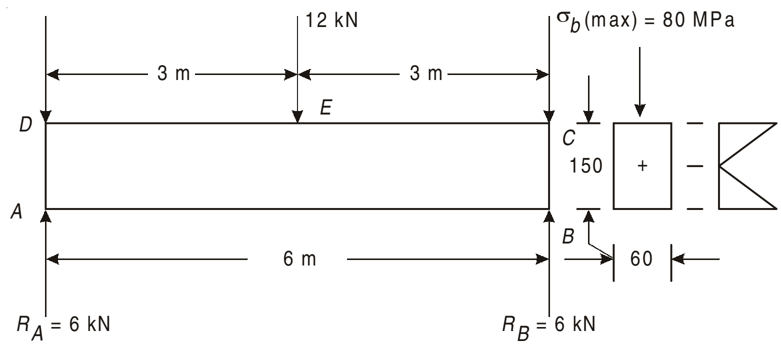



**Fig. 3.4**

**Example 3.1.** *A rectangular beam 60 mm wide and 150 mm deep is simply supported over a span of 6 metre. If the beam is subjected to central point load of 12 kN, find the maximu bending stress induced in the beam section.*

**Solution** For $\Sigma V = 0$, we have $R_A + R_B = 12$ and $R_A = R_B = \frac{12}{2} = 6$ kN

Due to central point load, the maximum bending moment at centre

i.e., $$M_e = 6\times10^3 \times 3\times10^3 = 18\times10^6 \text{ N/mm}$$

*Section modulus of the rectangular section,*

$$Z = (bd^2)/6 = (60\times150^2)/6 = 225\times10^3 \text{ mm}^3$$

Maximum Bending Stress,

$$\sigma_{max} = \frac{M}{Z} = \frac{(18\times10^6)}{225\times10^3} = 80 \text{ N/mm}^2 = \mathbf{80\ MPa}$$

**Example 3.2.** *A rectangular beam 300 mm deep is simply supported over a span of 4 m. What uniformly distributed load the beam may carry, if the bending stress is not to exceed 120 MPa. Take $I = 225 \times 10^6 mm^4$ (Fig. 3.5).*

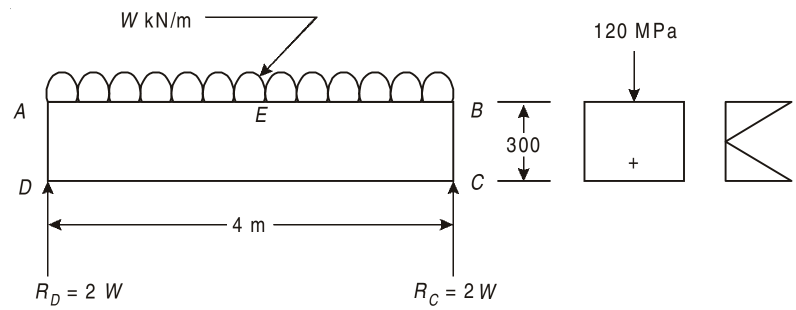



**Fig. 3.5**

**Solution** We know that distance between neutral axis of the section and external figure, $y = \frac{d}{2} = \frac{300}{2}$ = 150 mm and section modulus of rectangular section, $Z = \frac{I}{y} = \frac{(225 \times 10^6)}{150} = 1.5 \times 10^6\ \text{mm}^3$

$$\text{Moment of resistance, } M = \sigma_{max} \times Z = 120\ (1.5 \times 10^6)$$
$$= 180 \times 10^6\ \text{N/mm}$$

$$M_E = 2w \times 10^3 \times 2 \times 10^3 - w \times 10^3 \times \frac{2}{2} \times 10^3$$
$$= 4 \times 10^6\ w - 2w \times 10^6 = 2 \times 10^6\ w$$

Now $$180 \times 10^6 = 2w \times 10^6$$

$\therefore$ $$w = \frac{180}{2} = 90\ \text{N/mm}$$
$$\mathbf{= 90\ kN/m}$$

**Example 3.3.** *A wooden beam of rectangular cross section is subjected to a bending moment of 5 kN/m. If the depth of the section is to be twice the breadth and stress in wood is not to exceed 60 N/m$^2$. Find the dimension of the cross-section of the beam.*
*(UPTU : – 2006)*

*Given :* $$M = 5\ \text{kN/m},\ d = 2b,$$
$$\sigma = 60\ \text{N/cm}^2$$

**Solution** We know that, $\frac{\sigma}{y} = \frac{M}{I}$ $y = \frac{d}{2}$ $b$, and

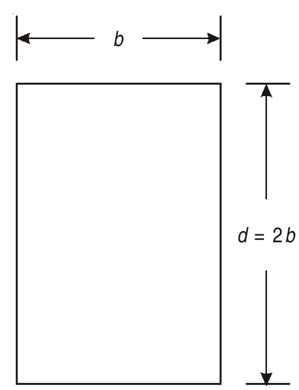



**Fig. 3.6** *Cross section of beam*

or $$M = \sigma \cdot \frac{I}{y} \quad I = \frac{bd^3}{12} = \frac{b(2b)^3}{12} = \frac{8b^4}{12} = \frac{2}{3}b^4$$

or $$5 \times 10^6 = 60 \times 10^{-2} \times \frac{\left(\frac{2}{3}\right)b^4}{b}$$

$\therefore$ $$b = 232.1 \text{ mm and } d = 2b$$
$$= \mathbf{464.2\ mm}$$

**Example 3.4.** *A rectangular beam with depth 150 mm and width 100 mm is subjected to a maximum bending moment of 300 kN/m. Calculate the maximum stress in the beam.*

**Solution** $$\frac{\sigma}{y} = \frac{M}{I}$$

$\therefore$ $$\sigma = \frac{M}{I}y \text{ and } y = \frac{d}{2} = \frac{150}{2} = 75 \text{ mm}$$

or $$I = \frac{bd^3}{12} = \frac{100(150)^3}{12} = 28125000 \text{ mm}^4$$

$\therefore$ $$\sigma = \frac{M}{I}y = \frac{300 \times 10^6 \text{Nmm}}{28125000 \text{ mm}^4} \times 75 \text{ mm}$$
$$= 800 \text{ N/mm}^2$$
$$\boldsymbol{\sigma = 800 \text{ MPa}}$$

**Example 3.5.** *A timber joist of 6 m span has to carry a load of 15 kN/m. Find the dimensions of the joist if the maximum permissible stress is limited to 8 N/mm$^2$. The depth of the joist has to be twice the width.* (*UPTU : 2009–2010*)

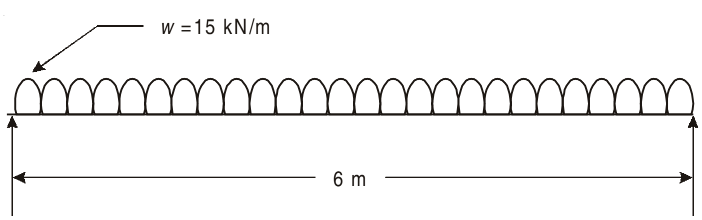



**Fig. 3.7**

*Given : Length of beam,*

$$l = 6 \text{ m, U.D.L.,} \quad w = 15 \text{ kN/m}$$

*Maximum stress,* $\sigma \ngtr 8 \text{ N/mm}^2$

*Depth,* $D = 2B$

**Solution** Let, $D$ be the depth and $B$ the width of the beam

$$M = \frac{WL^2}{8} = \frac{15 \times 6^2}{8}$$

$$= 67.5 \text{ kN/m} = 67.5 \times 10^6 \text{ N/mm}$$

Position of neutral axis, $\bar{y} = \dfrac{D}{2} = \dfrac{2B}{2} = B$

Moment of inertia, $I = \dfrac{BD^3}{12} = \dfrac{B(2B)^3}{12} = \dfrac{B^4}{12}$

Using bending formula, $\dfrac{M}{I} = \dfrac{\sigma}{y}$

$$\frac{(67.5 \times 10^6)}{(8B^4)/12} = \frac{8}{B} \text{ or } B^3 = 12.656 \times 10^6$$

$\therefore$ **B = 233 mm, D = 466 mm**

**Example 3.6.** *A beam of symmetrical I-section (Fig.3.8) is simply supported over a span of 9 m. If maximum permissible stress is 75 N/mm². What concentrated load can be carried at a distance of 3 m from one support* ($I = 31 \times 10^6 \text{ mm}^4$).

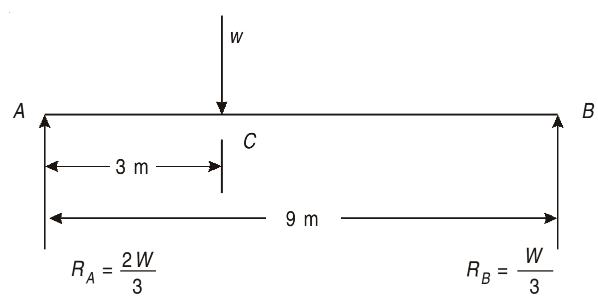



**Fig. 3.8**

**Solution** For $\Sigma M_A = 0$,

We have, $R_A(0) + 3W - 9R_B = 0$

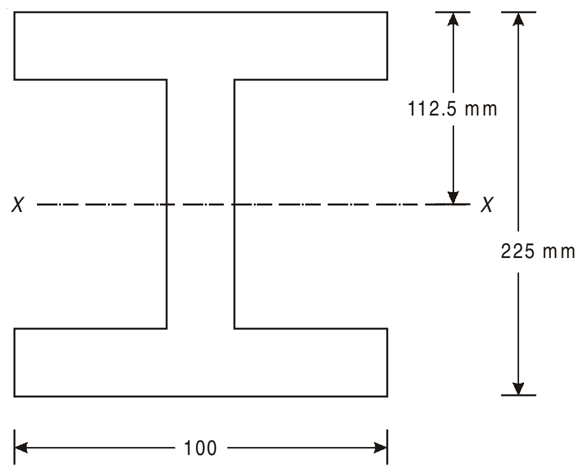



Fig. 3.9

$$\therefore \qquad R_B = \frac{3}{9}w = \frac{w}{3} \text{ and } R_A = \frac{2w}{3}$$

For $M_c = 0$; $3R_A = 6R_B$ or $3\left(\frac{2w}{3}\right) = 6\left(\frac{w}{3}\right)$

$\therefore$ $M = M_c = 2W$ N-m = 2000 $W$ N-mm

From Fig. 3.9, $y = \frac{d}{2} = \frac{225}{2} = 112.50$ mm

Using the bending equation $\frac{M}{I} = \frac{\sigma}{y}$

$$\sigma = \frac{(My)}{I} = \frac{(2000\,w \times 112.50)}{31 \times 10^6}$$

$$= 75 \text{ N/mm}^2 \qquad \text{(Given)}$$

$$\therefore \qquad W = 10333.333 \text{ N} = \mathbf{10.33\ kN}$$

**Example 3.7.** *A beam of T section (Fig. 3.10), 4 m long, with its flanges of 180 mm × 10 mm and web of 220 mm × 10 mm sizes is subjected to a sagging bending moment 15 kN-m. Determine the maximum tensile stress and maximum compression stress.*

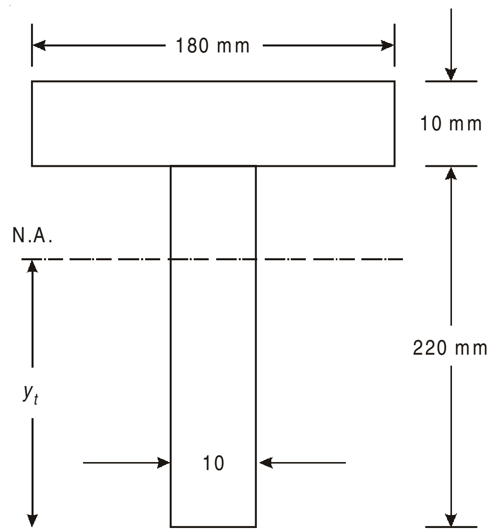



**Fig. 3.10**

*Given :* $L = 4\text{ m}, M = 15\text{ kNm}$

**Solution** Position of neutral axis, $\bar{y} = \dfrac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$ ...(i)

$A_1 = 220 \times 10 = 2200\text{ mm}^2, y_1 = \dfrac{220}{2} = 110\text{ mm}$

$A_2 = 180 \times 10 = 1800\text{ mm}^2, y_2 = 220 + \dfrac{10}{2} = 225\text{ mm}$

Now, $\bar{y} = \dfrac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \dfrac{2200 \times 110 + 1800 \times 225}{2200 + 1800}$

$= 161.75\text{ mm}$

Moment of inertia, $I_{xx} = I_{xx1} + Ixx_2$ ...(ii)

$$I_{xx_1} = \frac{bd^3}{12} + A_1(\bar{y} - y_1)^2$$

$$= \frac{10 \times 220^3}{12} + 2200(161.75 - 110)^2$$

$$= 8873333.33 + 5891738 = 14.765 \times 10^6\text{ mm}^4$$

$$I_{xx_2} = \frac{bd^3}{12} + A_2(\bar{y} - y_2)^2$$

$$= \frac{180 \times 10^3}{12} + 1800(161.75 - 225)^2$$

$$= 15000 + 7201008 = 7.216 \times 10^6 \text{ mm}^4$$

$$I_{xx} = I_{xx1} + I_{xx2}$$

$$= 14.765 \times 10^6 + 7.216 \times 10^6$$

$$= 21.98 \times 10 \text{ mm}^4$$

Applied bending moment, $M$ = 15 kN-m.

Since the beam is simply supported; $\bar{y} = y_t$ = 161.75 mm and $y_c$ = 230 – 161.75 = 68.25 mm

Let $\sigma_t$ be the tensile bending stress, then from bending equation $\frac{M}{I} = \frac{\sigma_t}{y}$

we get

$$\frac{15 \times 10^3 \times 1000}{21.98 \times 10^6} = \frac{\sigma_t}{161.75}$$

$$\therefore \qquad \sigma_t = \frac{15 \times 10^6 \times 161.75}{21.98 \times 10^6}$$

$$= \mathbf{110.98 \text{ N/mm}^2}$$

For, compressive stress

$$\frac{M}{I} = \frac{\sigma_c}{y_c}$$

or

$$\frac{15 \times 10^6}{21.98 \times 10^6} = \frac{\sigma_c}{68.25}$$

$$\therefore \qquad \sigma_c = \frac{15 \times 10^6 \times 68.25}{21.98 \times 10^6}$$

$$= \mathbf{46.576 \text{ N/mm}^2}$$

**Example 3.8.** *A cast iron water pipe of 500 mm inside diameter and 20 mm thick is supported over a span of 10 m. Detrmine the maximum stress in the pipe material when the pipe is running full. Take density of C.I. as 70.6 kN/m³ and that of water as 9.8 kN/m³.*

**Solution** Inside diameter of pipe = 500 mm

Outside diameter of pipe = 500 + 40 = 540 mm

$$\text{Area of cross-section} = \frac{\pi}{4}(D^2 - d^2)$$

$$= \frac{\pi}{4}(540^2 - 500^2)$$

$\therefore \quad A = 32673 \text{ mm}^2 = 0.03267 \text{ m}^2$

Weight of pipe = A.L. density = $\dfrac{A.\text{ density}}{m}$ = 0.03267 × 70.6

$\therefore \quad w_1 = 2.307$ kN/m

Inside area of pipe = $\dfrac{\pi}{4}(500)^2$ = 196350 mm² = 0.19635 m²

Weight of water per metre, $w_2$ = 0.19635 × 9.8 = 1.924 N/m

Uniformly distributed load,$w = w_1 + w_2$ = 2.307 + 1.924 = 4.231 kN/m

$\therefore$ Maximum B.M. = $\dfrac{wL^2}{8} = \dfrac{4.231\times10^2}{8}$ = 52.888 kN-m

$\therefore \quad M = 52.888 \times 10^6$ N-mm

Moment of inertia, $I = \dfrac{\pi}{64}\left(D^4 - d^4\right) = \dfrac{\pi}{64}\left[(540)^4 - (500)^4\right]$

$\therefore \; I = \dfrac{\pi}{64}(8.503056 - 6.250000)10^{10} = 1.106\times10^9 \text{ mm}^4$

$$y = \frac{540}{2} = 270 \text{ mm}$$

$\therefore$ Stress, $\sigma = \dfrac{My}{I} = \dfrac{52.888\times10^6\times270}{1.106\times10^9} = 12.9 \text{ N/mm}^2$

Hence, maximum stress in pipe material is **12.9 N/mm²** tensile at bottom, compressive at top.

**Example 3.9.** *A timber beam of 3 m span carries a uniformly distributed load of 5 kN/m and a point of load 1 kN at the centre of span. If the permissible bending stress be 100 N/mm², determine the section taking depth as twice the breadth.* (*UPTU : 2006–2007*)

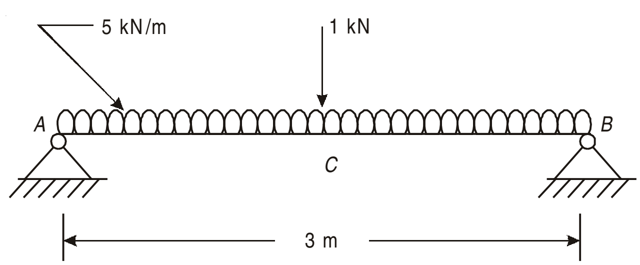



**Fig. 3.11**

**Solution** Maximum B.M. = $\dfrac{WL^2}{8} + \dfrac{WL}{4} = \dfrac{5(3)^2}{8} + \dfrac{1(3)}{4}$ = 6.375 kNm

$\therefore$ B.M. $= 6.375 \times 10^6$ Nm-m

For the rectangular section assumed. Let width $b = x$ mm

Hence, depth, $d = 2x$ mm (given)

$$I = \frac{bd^3}{12} = \frac{x(2x)^3}{12} = (0.6667)x^4 \quad \text{...(i)}$$

$$y = \frac{d}{2} = x, \quad \sigma = 100 \text{ N/mm}^2$$

Moment of resistance,

$$\text{M.R.} = \frac{\sigma.I}{y} = \frac{100 \times (0.6667)x^4}{x}$$

$$= (66.67)\,x^3$$

Equating M.R. with maximum B.M.

$$(66.67)x^3 = 6.375 \times 10^6$$

$\therefore$ $x = 45.7$ mm

$\therefore$ Width, $b = x = 45.7$ mm say 46 mm

Depth, $d = 2x = 2 \times 46 = 92$ mm

$\therefore$ **Suitable rectangular section is 92 mm × 46 mm.**

**Example 3.10.** *A cantilever 2.5 m long carries a U.D.L. of 20 kN/m run. The breadth of the section remains constant and is equal to 100 mm. Determine the depth of the section at the middle of the length of the cantilever and at fixed end if stress remains the same throughout and equal to 120 MN/m².*

*(UPTU : 2003–2004)*

*Given : Stress, $\sigma$ = 120 MN/m² = 120 N/mm², width, b = 100 mm*

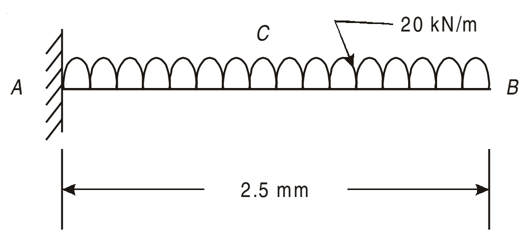



**Fig. 3.12**

**Solution** At middle of the section

$$\text{B.M.} = 20 \times 1.25 \times \frac{1.25}{2}$$

$$= 15.625 \text{ kN-m}$$

$$= 15.625 \times 10^6 \text{ N-mm}$$

$\because \quad \sigma = 120 \text{ N/mm}^2 = \dfrac{My}{I}$

$$\therefore \quad \frac{I}{y} = \frac{M}{\sigma} = \frac{15.625 \times 10^6}{120}$$

$$= 130 \times 10^3 \text{ mm}^3$$

$$I = \frac{bd^3}{12} = \frac{100d^3}{12}$$

$$= \frac{25}{3} \cdot d^3$$

But $\quad y = \dfrac{d}{2}$

$$\therefore \quad \frac{I}{y} = \frac{25}{3} \cdot d^3 \frac{2}{d} \cdot \frac{50d^3}{3}$$

$$\therefore \quad \frac{50d^3}{3} = 130 \times 10^3$$

or $\quad 50\, d^3 = 3 \times 130 \times 10^3$

$\therefore \quad d = 88.32 \text{ mm}$

At the fixed end $\quad \text{BM} = \dfrac{WL^2}{2} = \dfrac{20(2.5)^2}{2}$

$$= 62.5 \text{ kNm}$$

$$I = \frac{bd^3}{12} = \frac{100d^3}{12}$$

and $\quad y = \dfrac{d}{2} = 62.5 \times 10^6 \text{ N-mm}$

$$\therefore \quad \sigma = \frac{My}{I}$$

$$\therefore \quad 120 = \frac{My}{I}$$

or $\quad 120\, I = M \,.\, y$

or $\quad 120 \times \dfrac{100d^3}{12} = 62.5 \times 10^6 \times \dfrac{d}{2}$

$$\therefore \qquad d^2 = \frac{\left(62.5\times10^6\times12\right)}{2\times120\times100} = \frac{62500}{2}$$

$$= 31250$$

$$\therefore \qquad d = 176.78 \text{ mm}$$

**Hence depth at fixed end is 176.78 mm**

**Example 3.11.** *Three beams have the same length, same allowable stress and the same bending moment. The cross-sections of the beams are a square, a rectangular with depth twice the width and a circle (Fig. 3.13). Find the ratios of weight of the circular and the rectangular beams with respect to the square beam.*

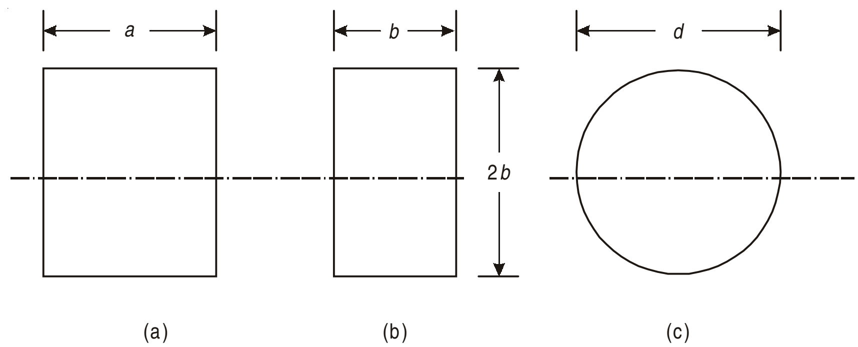



**Fig. 3.13**

**Solution** Square, rectangular and circular sections are shown in Fig. 3.13 (a, b, c)

Let,

$a$ = Side of the square beam

$b$ = Width of the rectangular beam

$2b$ = Depth of the rectangular beam, and

$d$ = Diameter of a circular section.

Since all the three beams have the same allowable stress ($\sigma$) and bending moment ($M$), therefore the Modulus of section $\left(Z = \dfrac{I}{y}\right)$ of the three beams must be equal.

Section modulus for the square beam, $Z_1 = \dfrac{bd^3}{6} = \dfrac{a.a^2}{6} = \dfrac{a^3}{6}$ ...(i)

Modulus of section for rectangular beam, $Z_2 = \dfrac{bd^2}{6} = \dfrac{b(2b)^2}{6} = \dfrac{2b^3}{3}$ ...(ii)

Modulus of section for a circular beam, $Z_3 = \dfrac{\pi}{32}d^3$ ...(iii)

Equating (i) and (ii) $\dfrac{a^3}{6} = \dfrac{2b^3}{3}$ or $a^3 = 6\times\dfrac{2b^3}{3} = 4b^3$

$\therefore$ $b = 0.63a$ ...(iv)

Now equating (i) and (iii); $\dfrac{a^3}{6} = \dfrac{\pi}{32}d^3$

or $a^3 = 6\times\dfrac{\pi}{32}d^3 = \dfrac{3\pi}{16}d^3$

$\therefore$ $d = 1.19\,a$ ...(v)

We know that weights of all the beams are proportional to the cross-sectional areas of their sections. Therefore

$$\frac{\text{Weight of square beam}}{\text{Weight of rectangular beam}} = \frac{\text{Area of square beam}}{\text{Area of rectangular beam}}$$

$$= \frac{a^2}{2b^2} = \frac{a^2}{2(0.63a)^2} = \mathbf{\frac{1}{0.79}}$$

and, $$\frac{\text{Weight of square beam}}{\text{Weight of circular beam}} = \frac{\text{Area of square beam}}{\text{Area of circular beam}}$$

$$= \frac{\mathbf{a^2}}{\frac{\pi}{4}\mathbf{d^2}} = \frac{\mathbf{a^2}}{\frac{\pi}{4}(\mathbf{1.19a})^2} = \mathbf{\frac{1}{1.12}}$$

## 3.7 ❑ BEAMS OF COMPOSITE SECTION (FLITCHED BEAM)

A composite section may be defined as a section made up of two or more different materials, joined together in such a manner that they behave like a single piece

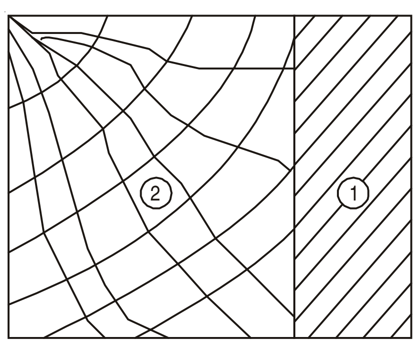



**Fig. 3.14**

and each material bends to the same radius of curvature. Such beams are used when a beam of one material, if used alone, requires quite a large cross-sectional area; which does not suit the space available. A material is then reinforced with some other materials, of higher strength, in order to reduce the cross-sectional area of the beam and to suit the space available (As is done in the case of reinforced cement concrete beams).

In such cases, the total moment of resistance will be equal to the sum of the moments of individual sections.

Consider a beam of a composite section made up of two different materials (Fig. 3.14)

Let,

$E_1$ = Modulus of elasticity of part 1

$I_1$ = Moment of inertia of part 1

$M_1$ = Moment of resistance for part 1

$\sigma_1$ = Stress in part 1, and

$Z_1$ = Modulus of section for part 1

$E_2, I_2, M_2, \sigma_2$, and $Z_2$ corresponding values for part 2 and, $R$ = Radius of the bend up beam.

We know that the moment of resistance for beam 1, $M_1 = \sigma_1 z_1$ ($\because M = \sigma z$) Similarly, $M_2 = \sigma_2\, z_2$.

Total moment of resistance of the composite section,

$$M = M_1 + M_2 = (\sigma_1 \,.\, z_1) + (\sigma_2 \,.\, z_2) \qquad \text{...(i)}$$

We also know that at any distance from the neutral axis, the strain in both the materials will be the same

$$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \quad \text{or } \sigma 1 = \frac{E_1}{E_2} \times \sigma_2 = m\sigma_2$$

where, $m = \dfrac{E_1}{E_2}$ i.e., modulus ratio.

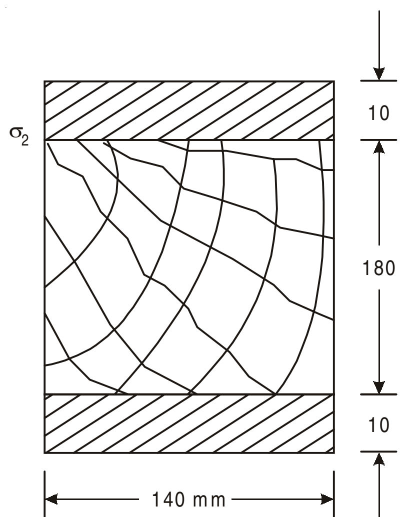



**Fig. 3.15**

**Example 3.12.** *A timber beam 140 mm wide and 180 mm deep is reinforced by 140 mm × 10 mm steel plates at top and bottom. The beam is subjected to a bending moment of 24 kNm.*

*Determine the maximum bending stress in the steel and wood. Given that the youngs modulus of steel and wood are 210 GPa and 15 GPa respectively.*

*(UPTU : 2011 –2012)*

*Given :* $M$ = 24 kN-m

= $24 \times 10^6$ N-mm

$$E_s = 210\ GPa$$
$$E_w = 15\ GPa$$

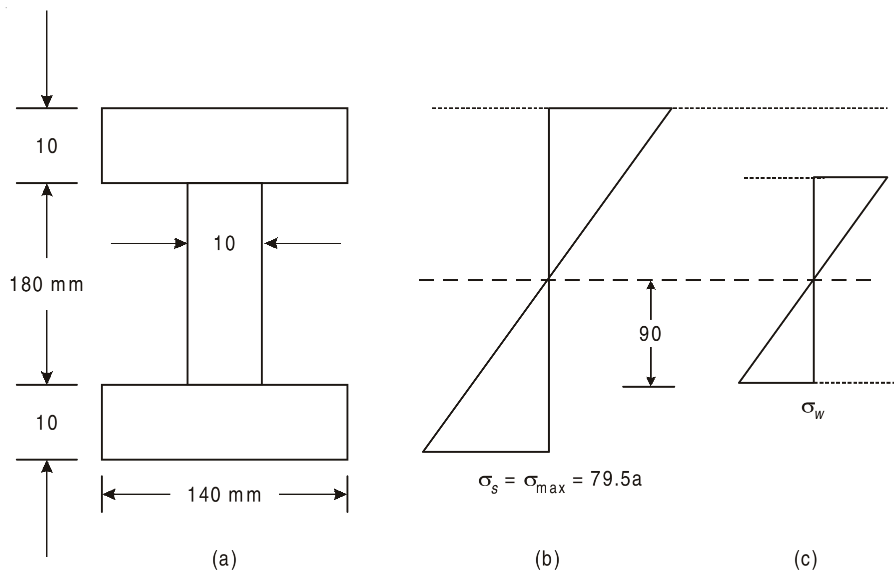



**Fig. 3.16**

**Solution To Find :**

$\sigma_s$ and $\sigma_w$

Modulus ratio, $\quad m = \dfrac{E_s}{E_w} = \dfrac{210}{15} = 14$

Equivalent width of timber in terms of steel $= \dfrac{b_w}{m} = \dfrac{140}{14} = 10$ mm

$$\bar{y} = \frac{200}{2} = 100 \text{ mm}$$

Moment of inertia, $\quad I = \dfrac{BD^3 - bd^3}{12}$

$$= \frac{140(200)^3 - 130(180)^3}{12}$$

$$= 30.153 \times 10^6 \text{ mm}^4$$

Using flexure formula, $\dfrac{M}{I} = \dfrac{\sigma_s}{y_{max}}$

$$\frac{(24 \times 10^6)}{(30.153 \times 10^6)} = \frac{\sigma_s}{100}$$

$$\therefore \qquad \boldsymbol{\sigma_s = 79.59 \text{ MPa}}$$

Stress in steel at $y = 90$ mm

$$\frac{\sigma_{s'}}{90} = \frac{79.59}{100}$$

Maximum stress in timber,

$$\therefore \qquad \sigma_{s'} = 71.63 \text{ MPa}$$

$$\sigma_w = \frac{\sigma_{s'}}{m}$$

$$= \frac{71.63}{20} = \mathbf{3.58 \text{ MPa}}$$

**Example 3.13.** *A flitched beam consists of two 50 mm × 200 mm wooden beams and 12 mm × 80 mm steel plate. The plate is placed centrally between the wooden beams and recessed into each so that, when rigidly joined, the three units form a 100 mm × 200 mm section (Fig. 3.17).*

*Determine the moment of resistance of the flitched beam when the maximum bending stress in the timber is 12 MN/m². What will then be the maximum bending stress in the steel? For steel, E = 200 GPa and for wood, E = 10 GPa.*

(*UPTU : 2005 –2006*)

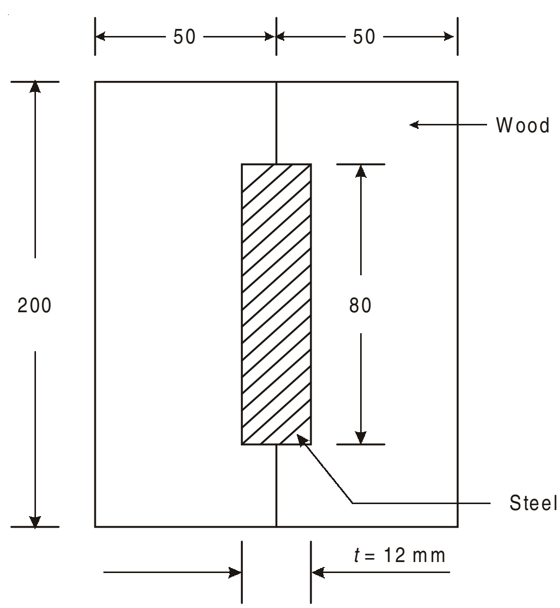



**Fig. 3.17**

*Given : Timber Joist :*

$$B = 50 \text{ mm}, D = 200 \text{ mm}$$

*Modular ratio,* $$m = \frac{E_s}{E_w} = \frac{200}{10} = 20$$

**Solution**

$$\text{M.I. of timber section} = \frac{1}{12} \times 100 \times (200)^3 - \frac{1}{2} \times 12 \times (80)^3$$

$$= 66.15 \times 10^6 \text{ mm}^4$$

$$\text{M.I. of steel plate} = \frac{1}{12} \times 12 \times (80)^3 = 0.5120 \times 10^6 \text{ mm}^4$$

M.I. of steel plate equivalent to timber

$$(m \times \text{M.I. of steel}) = 20 \times 0.5120 \times 10^6$$

$$= 10.24 \times 10^6 \text{ mm}^4$$

Total M.I. equivalent to timber $= (66.15 + 10.24)10^6$

$$= 76.39 \times 10^6 \text{ mm}^4$$

Maximum stress in timber, $\sigma = 12 \text{ MN/m}^2 = 12 \text{ N/mm}^2$

$$\text{It is developed at } y = \frac{200}{2} = 100 \text{ mm}$$

Moment of resistance, $$M = \frac{(\sigma . I)}{y} = \frac{(12 \times 76.39 \times 10^6)}{100}$$

$$\left(\frac{M}{I} = \frac{\sigma}{y}\right) = 9.1668 \times 10^6 \text{ N-mm}$$

$\therefore$ Moment of resistance **= 9.1668 kN-m**

Stress in timber at $y = 40$ is; $\sigma = \frac{12}{100} \times \frac{40}{1} = 4.8 \text{ N/mm}^2$

Maximum stress in steel = $4.8 \times 20 = 96 \text{ N/mm}^2$

Hence, Moment of resistance = **9.1668 kN-m**

Maximum stress in steel = **96 N/mm²**

## 3.8 ❑ SHEAR STRESSES

(i) In shear force and bending moment diagrams the beam is subjected to both shear force and bending moments.

(ii) In bending stresses, we consider that the beam is subjected to pure bending, shear stresses in the beam are ignored.

(iii) In actual practice, shearing stresses are also induced in the beam to resist shear force.

### 3.8.1 Shearing Stresses in the Beam

Consider a small portion *ABCD* of length $dx$ of a beam with bending moment varying from $M$ to $(M + d_m)$ due to uniformly distributed lood (U.D.L.) over its

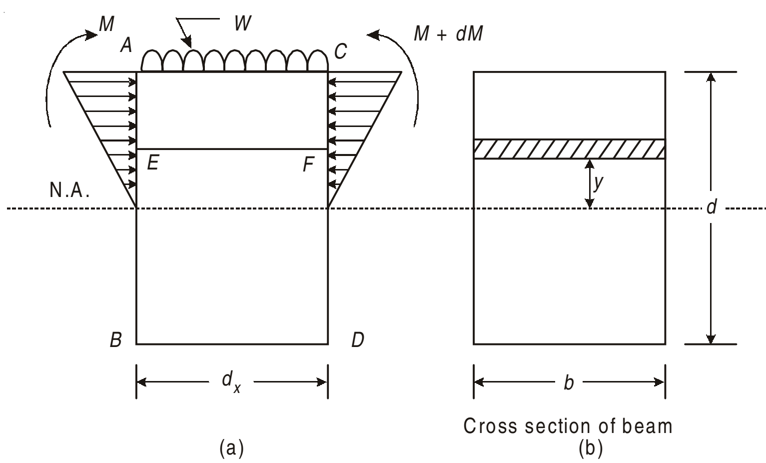



**Fig. 3.18**

length $dx$. The distribution of compressive stresses due to bending moment is shown in Fig. 3.18.

Let, $M$ = Bending moment at *AB*

$M + dM$ = B.M. at *CD*

$F$ = Shear force at *AB*

$F + dF$ = S.F. at *CD*

Consider elemental strip *EF* at a distance $y$ from neutral axis (Fig. 3.18)

Let,

$\sigma$ = Bending stress across *AB* at a distance $y$ from neutral axis

$\sigma + d\sigma$ = Bending stress across *CD*

$I$ = Moment of inertia

By using flexure formula, $\dfrac{M}{I} = \dfrac{\sigma}{y}$

On face *AB*, $\sigma = \dfrac{M}{I} \cdot y$; on face *CD*, $\sigma + d\sigma = \dfrac{M + dM}{I} \cdot y$

Force acting across *AB*; $F_{AB}$ = stress on *AB* × Area = $\sigma.a = \dfrac{M}{I} ya.$

Similarly force acting across *CD*;

$$F_{CD} = (\sigma + d\sigma)a = \frac{M + dM}{I} ya \, .$$

$\therefore$ Net unbalanced force on the strip; $dF = F_{CD} - F_{AB}$

or $$dF = \frac{M + dM}{I} \cdot y \cdot d_A - \frac{M}{I} \cdot y \cdot Da = \frac{dM}{I} \cdot y \cdot d_A$$

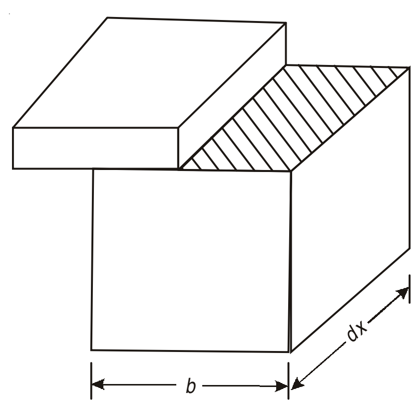
$\therefore$ The total unbalanced force $F$ above neutral axis can be found out by integrating from $O$ to $\frac{d}{2}$

$$F = \int_2^{\frac{d}{2}} d\frac{M}{I} .y.d_A = \frac{dM}{I} \int_o^{\frac{d}{2}} y.d_A$$

But $\int_o^{\frac{d}{2}} yd_A$ = First moment of area under consideration from neutral axis = A$\bar{y}$

$$\therefore \qquad F = \frac{dM}{I} \cdot A \cdot \bar{y}$$

This unbalanced force is balanced by a shearing stress $\tau$ acting along the length $d_x$ and width $b$



(c)

**Fig. 3.18**

$$\therefore \text{ Shearing stress} = \frac{\text{Shear Force}}{\text{Shear Area}}$$

$$= \frac{F}{bdx} = \frac{\left(\frac{dM}{I}.A\bar{y}\right)}{bdx}$$

$$\therefore \qquad \tau = \frac{\left(\dfrac{dm}{dx} Ay\right)}{bI}$$

But, $$\frac{dM}{dx} = S$$

$$\therefore \qquad \tau = \frac{SA\bar{y}}{bI}$$

where,

$S$ = Shear force at the section under consideration

$A$ = Area above or below the layer under consideration

$\bar{y}$ = Distance of centroid of area under consideration from neutral axis

$b$ = Width of the layer under consideration

$I$ = Moment of Inertia of the section.

**Example 3.14.** *A 100 mm × 150 mm wooden bar is to be symmetrically loaded with two equal forces, P (Fig. 3.19). Determine the position of loads and their magnitude when a bending stress of 10 MPa and shearing stress of 2.5 MPa are just reached. Neglect the weight of the beam.* (*UPTU : 2010–11*)

*Given :*

$$b = 100 \text{ mm}$$
$$d = 150 \text{ mm}$$
$$\sigma = 10 \text{ MPa}$$
$$\tau = 2.5 \text{ MPa}$$

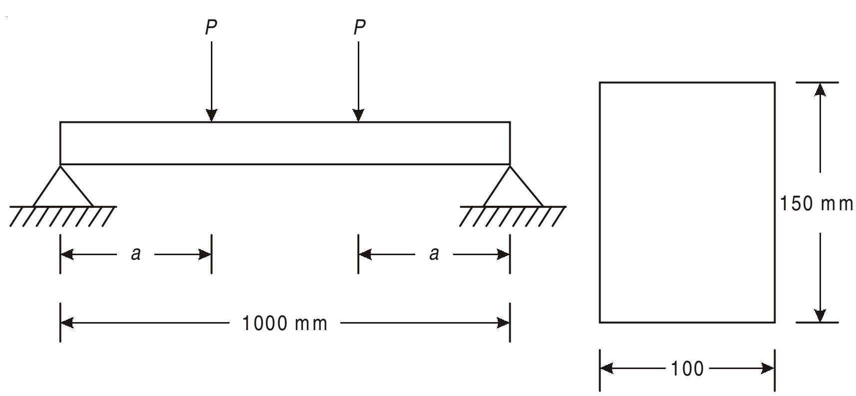



**Fig. 3.19**

**Solution** Moment of inertia,

$$I = \frac{(bd^3)}{12} = \frac{\left[100(150)^3\right]}{12}$$

$$= 28.125 \times 10^6 \text{ mm}^4$$

Maximum shear force $= S = P$

Maximum shear stress, $\tau = \frac{(SAy)}{bI}$

or $\tau = \frac{(PAy)}{bI}$

or $2.5 = \frac{P \times 100 \times 75 \times 75}{100 \times 28.125 \times 2 \times 10^6}$

$\therefore$ $P = 25000$ N

$= \mathbf{25\ kN}$

Using bending formula $\frac{M}{I} = \frac{\sigma}{y}$

$$\frac{M}{28.125 \times 10^6} = \frac{10}{75}$$

$\therefore$ $M = 3.75 \times 10^6$ N-mm

Maximum bending moment;

$M = P \times a$

or $3.75 \times 10^6 = 25 \times 10^3 \times a$

$\therefore$ $\mathbf{a = 150\ mm}$

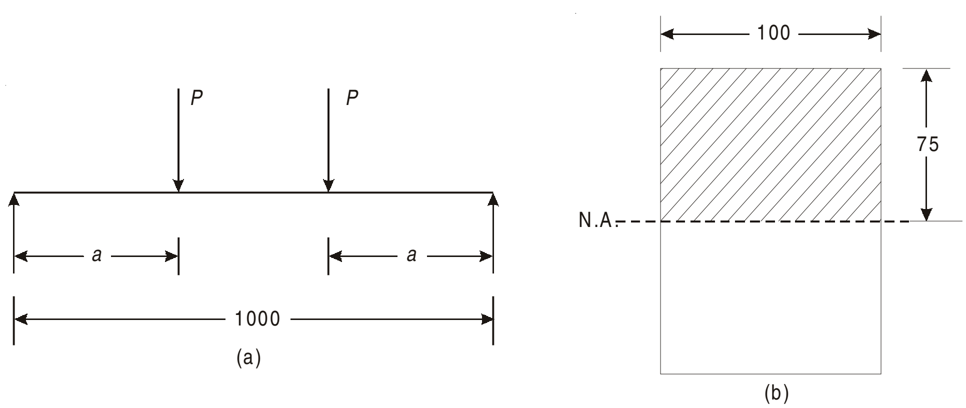



**Fig. 3.20**

**Example 3.15.** *In a thin circular tube show that the maximum shear stress is twice the average shear stress over the cross-section.* (*UPTU: 2007–2008*)

**Solution**

Shear stress $$\tau = \frac{SAy}{bI} = \frac{SQ}{BI}$$

For hollow circular section,

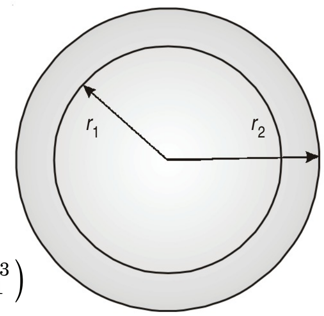



Fig. 3.21

$$I = \frac{\pi\left(r_2^4 - r_1^4\right)}{4}$$

$$Q = Ay = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

$$b = 2\,(r_2 - r_1)$$

where $r_1$ and $r_2$ are inner and outer radii of cross-section.

$$\tau_{max} = \frac{VQ}{Ib}$$

$$= \frac{V\frac{2}{3}\left(r_2^3 - r_1^3\right)}{\frac{\pi}{4}\left(r_2^4 - r_1^4\right)\times 2\left(r_2 - r_1\right)}$$

$$= \frac{4}{3}V\frac{\left(r_2 - r_1\right)\left(r_2^2 + r_2.r_1 + r_1^2\right)}{\pi\left(r_2^2 + r_1^2\right)\left(r_2^2 - r_1^2\right).\left(r_2 - r_1\right)}$$

$$= \frac{4}{3}\frac{V}{A}\frac{\left(r_2^2 + r_2.r_1 + r_1^2\right)}{\left(r_2^2 + r_1^2\right)}$$

$\because$ $$A = \pi\left(r_2^2 - r_1^2\right)$$

For a thin cylindrical tube $r_1 \approx r_2$

$$= \frac{4}{3}\frac{\left(3r_1^2\right)}{2r_1^2}.\tau_{ave}$$

$$\boldsymbol{\tau_{max} = 2\,\tau_{ave}}$$

## EXERCISE

**3.1.** What assumptions are made in theory of simple bending?

(*UPTU : 2005–06, 2007–08*)

**3.2.** Define the term, bending stress and explain the theory of simple bending.

**3.3.** Prove the relations, $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$.

**3.4.** A steel wire of 5 mm diameter is bent into a circular shape of 5 m radius. Determine the maximum stress induced in the wire. Take, $E$ = 200 GPa.

**[*Ans.* $\sigma_{max}$ = 100 MPa]**

**3.5.** A copper wire of 2 mm diameter is required to be wound around a drum. Find the minimum radius of the drum if the stress in the wire is not to exceed 80 MPa. Take modulus of elasticity for copper as 100 GPa. **[*Ans.* $R$ = 1.25 m]**

**3.6.** A metallic rod of 10 mm diameter is bent into a circular form of radius 6 m. If the maximum bending stress developed in the rod is 125 MPa, find the value of Young's modulus for the rod material. **[*Ans.* $E$ = 150 GPa]**

**3.7.** A cantilever beam is rectangular in section having 80 mm width and 120 mm depth. If the cantilever is subjected to a point load of 6 kN at the free end and the bending stress is not to exceed 40 MPa, find span of the cantilever.

**[*Ans.* $l$ = 1.28 m]**

**3.8.** A rectangular beam 60 cm wide and 150 mm deep is subjected to a uniformly distributed load of 4.5 kN/m. Find the maximum bending stress induced in the beam. **[*Ans.* $\sigma_{max}$ = 40 MPa]**

**3.9.** A rectangular beam 200 mm deep is simply supported on a span of 2 m. Find the uniformly distributed load the beam can carry if the bending stress is not to exceed 30 MPa. Take $I$ for the beam as $8 \times 10^6$ mm$^4$. **[*Ans.* $w$ = 4.8 kN/m]**

**3.10.** A beam made of C.I. having a section of 50 mm external diameter and 25 mm internal dia., is supported at two points 4 m apart. The beam carries a concentrated load of 100 N at its centre. Find the maximum bending stress induced in the beam. **[*Ans.* $\sigma_{max}$ = 8.64 N/mm$^2$]**

**3.11.** A hollow square section with outer and inner dimensions of 50 mm and 40 mm respectively is used as a cantilever of span 1m. How much concentrated load can be applied at the free end of the cantilever, if the maximum bending stress is not to exceed 35 MPa? **[*Ans.* $w$ = 430.5 N]**

**3.12.** A hollow steel tube having external and internal diameters of 100 mm and 75 mm respectively, is simply supported over a span of 5 m. The tube carries a concentrated load of $W$ at a distance of 2 m from one of the supports. What is the value of $W$, if the maximum bending stress is not to exceed 100 MPa.

**[*Ans.* $W$ = 5.6 kN]**

**3.13.** A cast iron water pipe of 500 mm inside diameter and 20 mm thick is supported over a span of 10 metres. Find the maximum stress in pipe metal, when the pipe is running full. Take density of cast iron as 70.6 kN/m$^3$ and that of water as 9.8 kN/m$^3$. **[*Ans.* $\sigma_{max}$ = 12.9 MPa]**

**3.14.** Calculate the cross-sectional dimensions of the strongest rectangular beam, that can be cut out of a cylindrical log of wood of 500 mm dia.

**[*Ans.* 288.5 mm × 408.5 mm]**

**3.15.** A rectangular beam, simply supported over a span of 4 m, is carrying a uniformly distributed load of 50 kN/m. Find the dimensions of the beam, if the depth of beam section is 2.5 times its width. Take maximum bending stress in the beam section as 60 MPa. **[*Ans.* 125 mm, 300 mm]**

**3.16.** A cast iron water pipe of 500 mm inside diameter and 20 mm thick is supported over a span of 10 metre. Determine the maximum stress in the pipe material, when the pipe is running full. Take density of C.I. as 70.6 kN/m$^3$ and that of water as 9.8 kN/m$^3$. (*UPTU : 2005–06*)

**[*Ans.* Example 3.8]**

**3.17.** A timber beam of 3m span carries a U.D.L. of 5 kN/m and a point load of 1kN at the centre of the span. If the permissible bending stress be 100 N/mm$^2$, determine the section taking depth as twice the breadth. (*UPTU : 2006–07*)

**[*Ans.* Example 3.9]**

**3.18.** A timber joist of 6 m span has to carry a load of 15 kN/metre. Find the dimensions of the joist if the maximum permissible stress is limited to 8 N/mm$^2$. Take depth of the joist as twice the width. (*UPTU : 2009–10*)

**[*Ans.* Example 3.5]**

**3.19.** A timber beam 140 mm wide and 180 mm deep is reinforced by 140 mm × 10 mm steel plates at top and bottom. The beam is subjected to a bending moment of 24 kN-m. Determine the maximum bending stress in the steel and wood, if values for $E$ for steel and wood are 210 GPa and 15 GPa respectively.

(*UPTU : 2011–2012*)

**[*Ans.* Example 3.12]**

**3.20.** A beam having T-section with its flanges of 180 mm × 10 mm and web of 220 mm × 10 mm (Fig. 3.22) is subjected to sagging bending moment 15 kN-m. Determine the maximum compressive stress. (*UPTU : 2012–13*)

**[*Ans.* Example 3.7]**

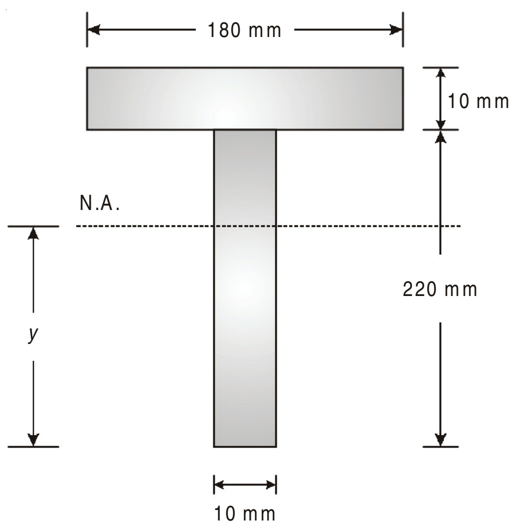



**Fig. 3.22**

**3.21.** A 100 mm × 150 mm wooden bar is to be symmetrically loaded with two equal forces $P$, $P$ (Fig. 3.23). Determine the position of loads and their magnitude when the bending of 10 MPa and shearing stress of 2.5 MPa are just reached. Neglect the weight of the beam. (*UPTU : 2010–11*)

**[*Ans.* Example 3.14]**

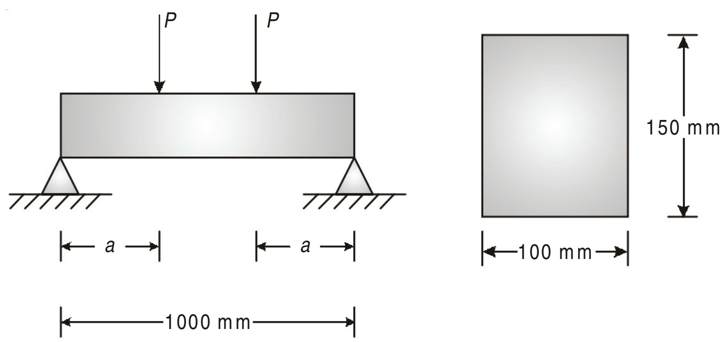



**Fig. 3.23**

**3.22.** In a thin circular tube show that the maximum shear stress is twice the average shear stress over the cross section. (*UPTU : 2007–2008*)

**[*Ans.* Example 3.15]**

CHAPTER 4

# Deflection of Beams

## 4.1 ❑ INTRODUCTION

Whenever a beam is loaded, it deflects from its original position. The amount, by which a beam deflects, depends upon its cross-section and bending moment. The beam should be strong enough to resist *bending moment* and *shear force*. The beam should also be stiff enough to resist its deflection. The beam should be stiff enough not to deflect more than permissible limit (Span/325).

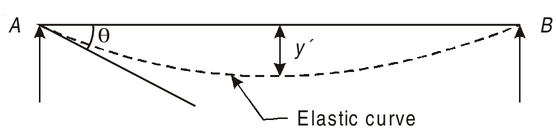



**Fig. 4.1**

The curve in which the beam bends due to loading is called *elastic curve*. The vertical distance ($y$) at any point in a beam between elastic curve and original axis of beam is called deflection at that point. Deflection is denoted by $y$, $\Delta$ or $\delta$ and is zero at the supports.

## 4.2 ❑ SLOPE OF BEAM

Slope at any point is defined as the angle in *Radian* between the tangent to the elastic curve at that point and the original axis of the beam. It is denoted by $\theta$ or $\frac{dy}{dx}$.

### 4.2.1 Boundry Conditions

(a) *Simply supported beams :*

(i) At supports deflection is always Zero

(ii) At supports slope is always maximum.
(iii) Maximum deflection occurs at the point where slope is zero (Fig. 4.1.)

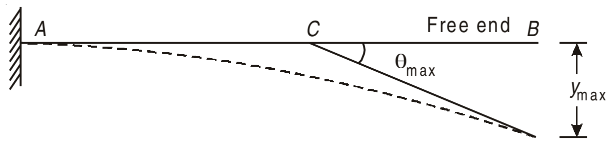



**Fig. 4.2**

(b) *Cantilever beam:*
(i) At free end both slope and deflection are maximum.
(ii) At fixed end both slope and deflection are zero

## 4.3 ❑ CURVATURE OF THE BENDING BEAM

Consider a beam *AB* subjected to a bending moment. As a result of loading let the beam deflects from *ACB* to *ADB* (Into a circular arc) Fig. 4.3.

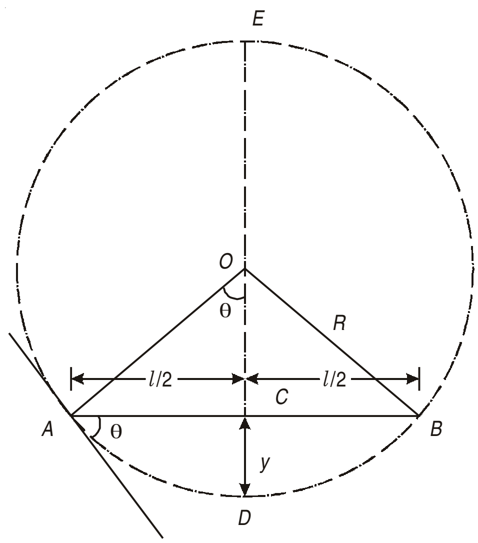



**Fig. 4.3** *Curvature of beam*

Let,

$l$ = Length of beam *AB*
$M$ = Bending moment
$R$ = Radius of curvature of the bent up beam
$y$ = Deflection of beam at centre (i.e. *CD*)
$\theta$ = Slope of the beam
$I$ = Moment of inertia of beam sections
$E$ = Modulus of elasticity of beam material

From the geometry of the circle, we know that

$$AC \times CB = EC \times CD$$

or $$\frac{l}{2} \times \frac{l}{2} = (2R - y)\, y$$

or $$\frac{l_2}{4} = 2\,Ry - y^2 = 2\,Ry$$

$$\therefore \quad y = \frac{l^2}{8R} \quad (y \text{ is very small}) \qquad \text{...(i)}$$

We have studied that for a loaded beam;

$$\frac{M}{I} = \frac{E}{R}$$

or $$R = \frac{(EI)}{M} \qquad \text{...(ii)}$$

Substituting the value of $R$ in Eq. (i)

$$y = \frac{l^2}{8 \times \frac{EI}{M}} = \frac{Ml^2}{8EI} \qquad \text{...(iii)}$$

From the geometry of the Fig. 4.3, we find that the slope of the beam at $A$ is also equal to angle $AOC$

$$\therefore \quad \sin \theta_A = \frac{AC}{AO} = \frac{\left(\frac{l}{2}\right)}{R} = \frac{l}{2R} \qquad \text{...(iv)}$$

Since the angle $\theta$ is very small, therefore $\sin \theta$ may be taken equal to $\theta$ in Radian

$$\therefore \quad \theta = \frac{l}{2R} \text{Radian} \qquad \text{...(v)}$$

Again substituting the value of $R$ in Eq. (ii)

$$\theta = \frac{l}{2R} = \frac{l}{2 \times \frac{EI}{M}} = \frac{Ml}{2EI} \text{Radian}$$

## 4.4 ❑ RELATION BETWEEN SLOPE, DEFLECTION AND RADIUS OF CURVATURE

Consider a small portion $PQ$ of a beam, bent into an arc (As shown in Fig. 4.4)

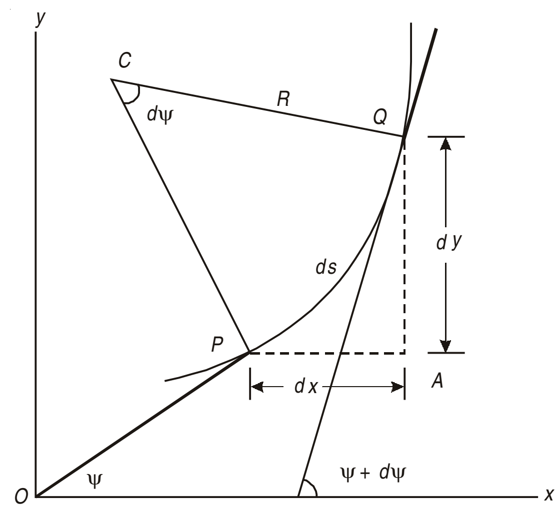



**Fig. 4.4**

Let,

$ds$ = Length of beam $PQ$

$R$ = Radius of the arc, into which the beam has been bent

Consider a small portion $PQ$ of a beam, bent into an arc (As shown in Fig. 4.4)

Let,

$C$ = Centre of the arc

$ds$ = Length of the beam $PQ$

$R$ = Radius of the arc, into which the beam has been bent.

$\psi$ = Angle, which the tangent at $P$ makes with $x - x$ axis, and

$\psi + d\psi$ = Angle which the tangent at $\theta$ makes with $x - x$ axis

From the geometry of the Fig. 4.4, we find that

$$\angle PCQ = d\psi$$

and $$ds = R\,.\,d\,\psi$$

$\therefore$ $$R = \frac{ds}{d\psi} = \frac{dx}{d\psi} \qquad \text{(Considering } ds = dx\text{)}$$

or $$\frac{1}{R} = \frac{d\psi}{dx}$$

We know that if $x$ and $y$ be the co-ordinates of point $P$, then $\tan\psi = \dfrac{dy}{dx}$

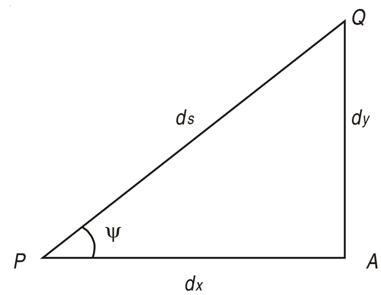



**Fig. 4.5**

Since ψ is very small angle, therefore taking tan ψ = ψ;

$$\frac{d\psi}{dx} = \frac{d^2y}{dx^2} \qquad \left(\because \frac{1}{R} = \frac{d\psi}{dx}\right)$$

We also know that

$$\frac{M}{I} = \frac{E}{R}$$

or

$$M = EI \cdot \frac{1}{R}$$

$$\therefore \qquad M = EI \cdot \frac{d^2y}{dx^2} \qquad \left(\text{Substituting value of } \frac{1}{R}\right)$$

This equation is called as *Differential equation for deflection.*

## 4.5 ❑ MACAULAY'S METHOD

The Macaulay's method is explained as following.

1. This method is similar to double integration method.
2. It is used to find the slope and deflection at any point on the beam.
3. In this method, bending moment at any section is expressed in systematic order.
4. The section $x$ is to be taken in the last portion of the beam.
5. In this method, bending moment of each force or U.D.L. is separated by a compartment of lines.
6. Integrating bending moment equation, we will get slope equation.
7. Again integrating slope equation, we will get deflection equation.

For example, consider a beam subjected to loading as shown in Fig. 4.6.

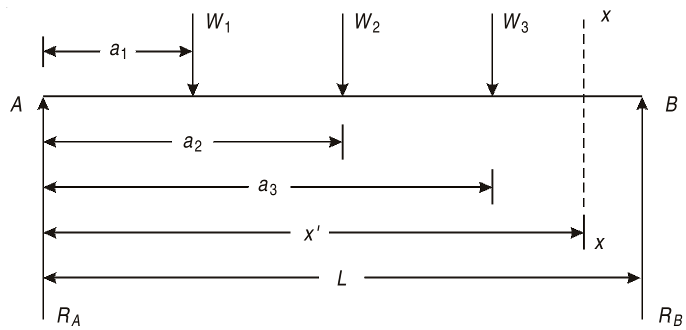



**Fig. 4.6**

**Step I.** Determine the support reactions.

**Step II.**Consider a section $x - x$ at a distance $x$ covering all the loads from left end of the beam.

**Step III.** Take moment of all forces about section $x - x$, considering sagging B.M. as positive and hogging as negative.

$$M_{xx} = R_A . x \;\vdots\; - w_1 (x - a_1) \;\vdots\; - w_2 (x - a_2) \;\vdots\; - w_3 (x - a_3)$$

**Step IV.** Use differential equation of deflection $EI \dfrac{d^2y}{dx^2} = M_{xx}$

$$EI \frac{d^2y}{dx^2} = R_A x \;\vdots\; - w_1 (x - a_1) \;\vdots\; - w_2 (x - a_2) \;\vdots\; - w_3 (x - a_3)$$

**Step V.** Integrating above equation, we get;

$$EI \frac{dy}{dx} = R_A \frac{x^2}{2} + C_1 \;\vdots\; - w_1 \frac{(x - a_1)^2}{2} \;\vdots\; - w_2 \frac{(x - a_2)^2}{2} \;\vdots\; - w_3 \frac{(x \times a_3)^2}{2}$$

**Step VI.** Again integrating above equation, we get

$$EIy = R_A \frac{x^3}{6} + c_1 x + c_2 \;\vdots\; w_1 \frac{(x - a_1)^3}{6} \;\vdots\; - w_2 \frac{(x - a_2)^3}{6} \;\vdots\; - w_3 \frac{(x - a_3)^3}{6}$$

**Step VII.** The constant of integration $c_1$ and $c_2$ are calculated by applying boundry conditions.

**Step VIII.** Substituting the value of constant, in equation obtained in step V gives general equation for slope and equation in step VI gives general equation for deflection.

## 4.6 ❑ EQUATIONS OF ELASTIC CURVES

$$EI \frac{d^4y}{dy^4} = - w = \text{Rate of loading}$$

$$EI\frac{d^3y}{dy^3} = S = \text{Shear force}$$

$$EI\frac{d^2y}{dx^2} = M = \text{Bending moment}$$

$$EI\frac{dy}{dx} = \theta = \text{Slope of the elastic curve}$$

$$EIy = \text{Deflection of beam}$$

where $EI$ is flexural rigidity

**Example 4.1.** *A simply supported beam of length 8 m carries two concentrated load of 64 kN and 48 kN magnitude in downward direction at distances of 1 m and 4 m from left end. Find the deflection below the 48 kN load. Take E = 210 GPa and I = 180 × $10^6$ mm$^4$.* (*UPTU : 2011–12*)

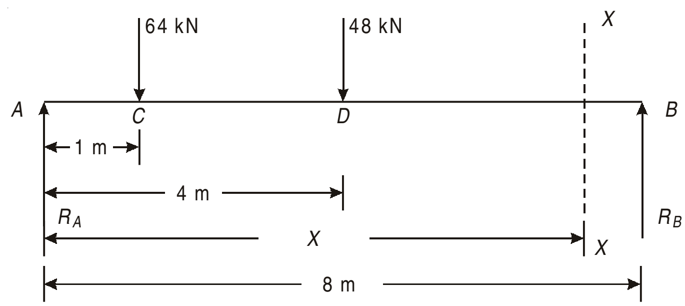



**Fig. 4.7**

**Solution**

Let support's reactions be $R_A$ and $R_B$

For $\Sigma V = 0;\quad R_A + R_B = 64 + 48 = 112$ kN

For $\Sigma M_A = 0;\quad R_A \times 0 + 64 \times 1 + 48 \times 4 - 8\,R_B = 0$

or $64 + 192 = 8\,R_B$.

$$\therefore \quad R_B = \frac{266}{8} = 32 \text{ kN} \qquad \text{...(i)}$$

$$R_A = 112 - R_B = 112 - 32 = 80 \text{ kN} \qquad \text{...(ii)}$$

*Using Macaulay's method*

Consider a section $x - x$ at a distance $x$ from support $A$

$$M_{xx} = EI\frac{d^2y}{dx^2}$$

$$= 80\,x \;\vdots\; - 64\,(x-1) \;\vdots\; - 48\,(x-4) \qquad \text{...(iii)}$$

Integrating; $EI\dfrac{dy}{dx} = \dfrac{80x^2}{2} + C_1 \Big| - \dfrac{64(x-1)^2}{6} \Big| - \dfrac{48(x-4)^2}{6}$

Again integrating; $EIy = \dfrac{80x^3}{6} + C_1x + C_2 \Big| - \dfrac{64(x-1)^3}{6} \Big| - \dfrac{48(x-4)^3}{6}$ ...(iv)

Applying boundry conditions : when $x = 0$, $y = 0$

$\therefore \quad c_2 = 0$

When, $x = B$, $y = 0$

$$0 = \frac{80 \times 8^3}{6} + 8C_1 - \frac{64 \times 7^3}{6} - \frac{48 \times 4^3}{6}$$

$$\therefore \quad C_1 = -\frac{2656}{8} = -332$$

Subtituting the value of $C_1$ and $C_2$ in Eqs. (iii) and (iv)

Deflection equation, $y = \dfrac{1}{EI}\left[\dfrac{80}{6}x^3 - 332x \Big| - \dfrac{64}{6}(x-1)^3 \Big| - \dfrac{48}{6}(x-4)^3\right]$

Deflection below 48 kN load, i.e., $x = 4$ m

$$y = \frac{1}{EI}\left[\frac{80}{6} \times 4^3 - 332 \times 4 - \frac{64}{6}(3)^3\right]$$

$$= -\frac{762.67}{EI}$$

Flexural rigidity, $EI = 210 \times 10^3 \text{ N/mm}^2 \times 180 \times 10^6 \text{ mm}^4$

$= 37.8 \times 10^9 \text{ kN-mm}^2$

$= 37800 \text{ kN-mm}$

$$y_D = -\frac{762.76}{37800}$$

$= -0.02018$ m $=$ **20.18 mm↓**

**Example 4.2.** *Derive an expression for the slope and deflection of a simply supported beam, span L, carrying a uniformly distributed load w per unit length and a point load P at the mid span. Hence, find the slope and deflection at a point L/4 from the left support.* *(UPTU : 2001–02, 2004–05)*

**Solution**

Support reactions $R_A = R_B = \dfrac{wL + P}{2}$

Consider a section $x - x$ at a distance $x$ from $A$

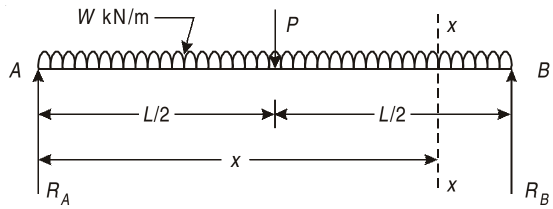



**Fig. 4.8**

Moment about section $x - x$;

$$M_{xx} = R_A \cdot x - \frac{wx^2}{2} - P\left(x - \frac{L}{2}\right)$$

$$EI\frac{d^2y}{dx^2} = \left(\frac{wL+P}{2}\right)x - \frac{wx^2}{2} - P\left(x - \frac{L}{2}\right) \qquad \text{...(i)}$$

Integrating Eq. (i) we get

$$EI\frac{dy}{dx} = \left(\frac{wL+P}{2}\right)\frac{x^2}{2} + C_1 - \frac{wx^3}{6} - \frac{P}{2}\left(x - \frac{L}{2}\right)^2 \qquad \text{...(ii)}$$

Again integrating Eq. (ii)

We get, $EIy = \left(\frac{wL+P}{2}\right)\frac{x^3}{6} + C_1x + C_2 - \frac{wx^4}{24} - \frac{P}{6}\left(x - \frac{L}{2}\right)^3$

Applying boundary condition, to find constants $C_1$ and $C_2$; when $x = 0, y = 0$,

$\therefore \qquad C_2 = 0$

When $x = L, y = 0$

$$0 = \left(\frac{wL+P}{2}\right)\frac{L^3}{6} + C_1L - \frac{wL^4}{24} - \frac{PL^3}{48}$$

$$\therefore \qquad -C_1L = \frac{wL^4}{12} + \frac{PL^3}{12} - \frac{wL^4}{24} - \frac{PL^3}{24}$$

$$C_1 = -\frac{wL^3}{24} - \frac{PL^2}{24} = -\frac{(wL+P)L^2}{24} \qquad \text{...(iii)}$$

Substituting value of $C_1$ and $C_2$ in Eqs. (ii) and (iii), we get slope equation:

$$\frac{dy}{dx} = \frac{1}{EI}\left[\left(\frac{wL+P}{2}\right)\frac{x^2}{2} - \frac{(wL+P)L^2}{24} - \frac{wx^3}{6} - \frac{P}{2}\left(x - \frac{L}{2}\right)^2\right]$$

Deflection equation

$$y = \frac{1}{EI}\left[\left(\frac{wL+P}{2}\right)\frac{x^3}{6} - \frac{(wL+P)L^2x}{24} - \frac{wx^4}{24} - \frac{P}{6}\left(x-\frac{L}{2}\right)^3\right]$$

Slope at $x = \frac{L}{4}$ from left support

$$\frac{dy}{dx} = \frac{1}{EI}\left[\left(\frac{wL+P}{4}\right)\left(\frac{L}{4}\right) - \frac{(wL+P)L^2}{24} - \frac{w}{6}\left(\frac{L}{4}\right)^3 - \frac{P}{2}\left(\frac{L}{4}-\frac{L}{2}\right)^2\right]$$

$$= \frac{1}{EI}\left[-\frac{11}{384}wL^3 - \frac{5}{192}PL^2\right]$$

$$\therefore \quad \frac{dy}{dx} = -\left[\frac{\mathbf{11wL^3}}{\mathbf{384EI}} + \frac{\mathbf{5}}{\mathbf{192}}\frac{\mathbf{PL^2}}{\mathbf{EI}}\right]$$

Deflection at $x = \frac{L}{4}$ from left support

$$y = \frac{1}{EI}\left[\left(\frac{wL+P}{4}\right)\left(\frac{L}{4}\right)^3 - \frac{(wL+P)L^3}{24}\left(\frac{L}{4}\right) - \frac{w}{24}\left(\frac{L}{4}\right)^4\right]$$

$$= -\frac{\mathbf{5}}{\mathbf{384}}\frac{\mathbf{wL^4}}{\mathbf{EI}} - \frac{\mathbf{7}}{\mathbf{768}}\frac{\mathbf{PL^3}}{\mathbf{EI}}$$

**Example 4.3.** *Cantilever of span L carries point lood w at free end. Determine the maximum slope and deflection.* (*UPTU : 2010–2011; MTU 2012–13*)

**Solution** *By Macaulay's method*

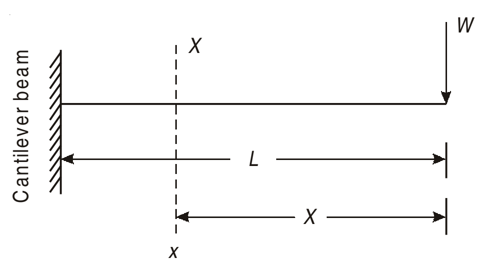



**Fig. 4.9**

Consider section $X - X$ at a distance $x$ from free end

$$M_{xx} = EI\frac{d^2y}{dx^2} = -wx \qquad ...(1)$$

Integrating Eq. (1) $EI\frac{dy}{dx} = -\frac{wx^2}{2} + C_1$ ...(2)

Again integrating Eq. (2) $EIy = -\frac{wx^3}{6} + C_1x + C_2$ ...(3)

*Applying boundary conditions*

When $x = L, \frac{dy}{dx} = 0,$ Put in Eq. (2)

$$0 = -\frac{WL^2}{2} + C_1$$

$$\therefore \quad C_1 = \frac{WL^2}{2}$$

When $x = L,\ y = O$, Put in Eq. (3)

$$O = -\frac{WL^3}{6} + \frac{WL^2}{2} \times L + C_2$$

$$\therefore \quad C_2 = -\frac{WL^3}{2} + \frac{WL^3}{6} \times L = -\frac{2WL^3}{6} = -\frac{WL^3}{3}$$

Put values of $C_1$ and $C_2$ in Eqs. (2) and (3)

Slope equation: $$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{wx^2}{2} + \frac{WL^2}{2}\right]$$

Deflection equation : $$y = \frac{1}{EI}\left[-\frac{wx^3}{6} + \frac{WL^2}{2}.x - \frac{WL^3}{3}\right]$$

Slope at free end or maximum slope; put $x = 0$ in slope equation

$$\left(\frac{dy}{dx}\right)\text{max} = \mathbf{\frac{WL^2}{2EI}}$$

For deflection at free end or maximum deflection, $x = 0$

Put in deflection equation, $y = \mathbf{\frac{WL^3}{3EI}}$

**Example 4.4.** *A beam of uniform section, 10 m long is simply supported at the ends. It carries point loads of 150 kN and 65 kN at distances of 2.5 m and 5.5 m respectively from the left end. Calculate.*

*(i) Deflection under each load, (ii) Maximum deflection*

*Take* $E = 200\ \dfrac{GN}{m^2}$ *and* $I = 118 \times 10^{-4}\ m^4$ . (*UPTU : 2006–07*)

*Given :*

$$E = \frac{200\ G\ N}{m^2}$$

$$= 200 \times 10^6\ kN/m^2$$

$$I = 118 \times 10^{-4}\ m^4$$

*Flexural rigidity,*

$$EI = 200 \times 10^6\ kN/m^2 \times 118 \times 10^{-4}\ m^4$$

$$= 2.36 \times 10^6\ kN/m^2$$

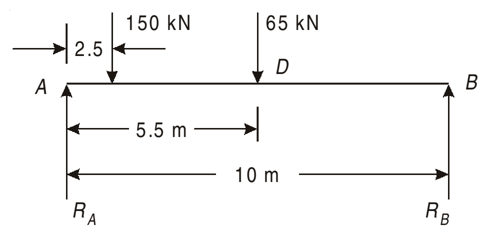



**Fig. 4.10**

**Solution** For support reactions:

$$\Sigma M_B = 0 \circlearrowright +$$

$$R_A \times 10 - 150 \times 7.5 - 65 \times 4.5 = 0$$

or $$10R_A = 1125 + 292.5 = 1417.50$$

$\therefore$ $$R_A = \frac{1417.50}{10} = 141.75 \text{ kN}$$

*By Macaulay's method:*

Consider a section $X - X$ at a distance $x$ from support $A$

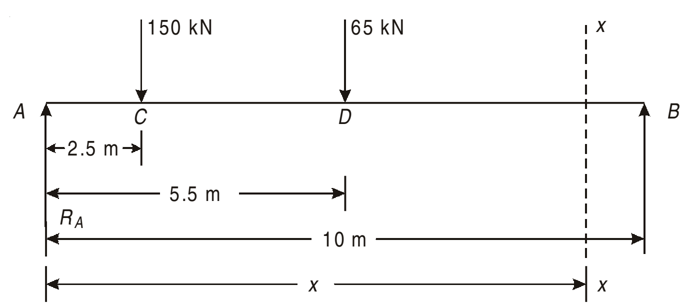



**Fig. 4.11**

Moment about section $X-X$;

$$M_{xx} = EI\frac{d^2y}{dx^2} = 141.75x \,\Big|\, -150(x-2.5) \,\Big|\, -65(x-5.5) \qquad \text{...(i)}$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = 141.75\ EI\frac{x^2}{2} + C_1 \,\Big|\, -\frac{-150(x-2.5)^3}{6} - \frac{65(x-5.5)^3}{6} \qquad \text{...(ii)}$$

Again integrating Eq. (ii), we get

$$EIy = 141.75\frac{x^3}{6} + C_1x + C_2 \,\Big|\, -\frac{-150(x-2.5)^3}{6} \,\Big|\, -\frac{65(x-5.5)^3}{6} \qquad \text{...(iii)}$$

To find constants of integration i.e. $C_1$ and $C_2$, applying boundary conditions

(i) When $x = 0$, $y = 0$, put in Eq. (ii); $C_2 = 0$

(ii) When $x = 10$ m, $y = 0$, put in Eq. (iii)

$$0 = 141.75 \times \frac{10^3}{6} + c_1 \times 10 - \frac{-150(10-2.5)^3}{6} - \frac{65(x-5.5)^3}{6}$$

$\therefore \qquad C_1 = -1209.1$

Substituting the values of $C_1$ and $C_2$ in Eq. (ii) and (iii), we get,

Slope equation,

$$\frac{dy}{dx} = \frac{1}{EI}\left[70.875x^2 - 1209.1 \,\Big|\, -75(x-2.5)^2 \,\Big|\, -32.5(x-5.5)^2\right]$$

Deflection equation,

$$y = \frac{I}{EI}\left[23.625x^3 - 1209.1x \,\Big|\, -25(x-2.5)^2 \,\Big|\, -10.833(x-5.5)^2\right]$$

Deflection at $C$ i.e., under 150 kN,

Put $x = 2.5$ m, in deflection equation

$$y_c = \frac{1}{2.36\times10^6}\left[23.625\times2.5^3 - 1209.1\times2.5\right]$$

$$= -1.1244 \times 10^{-3}\text{ m}$$

$\therefore \qquad y_c =$ **1.1244 mm** $\downarrow$

Deflection at $D$ i.e., under 65 kN, put $x = 5.5$ m in deflection equation

$$y_D = \frac{1}{2.36\times10^6}\left[23.625\times5.5^3 - 1209.1\times5.5 - 25(5.5-2.5)^2\right]$$

$$= -1.2476 \times 10^{-3}\text{ m}$$

$\therefore \qquad y_D =$ **1.2476 mm** $\downarrow$

*Location of point of maximum deflection*

Maximum deflection occurs at the point where slope is zero. Assume maximum deflection lies between $C$ and $D$. $\frac{dy}{dx} = 0$

$$70.875\,x^2 - 1209.1 - 75\,(x - 2.5)^2 = 0$$
$$70.875\,x^2 - 1209.1 - 75\,(x^2 - 5x + 6.25) = 0$$
$$70.875\,x^2 - 1209.1 - 75\,x^2 + 375x - 468.75 = 0$$
$$-4.125\,x^2 + 375x - 1677.85 = 0$$
$$\therefore \quad x = 4.69 \text{ m}$$

Maximum deflection at $x = 4.69$ m

$$y_{max} = \frac{1}{2.36\times10^6}\left[23.625\times4.69^3 - 1209.1\times4.69 - 25(4.69-2.5)^3\right]$$
$$= -1.48 \times 10^{-3} \text{ m}$$
$$= \mathbf{1.48\ mm} \downarrow$$

**Example 4.5.** *A beam, simply supported at ends A and B is located with two point loads of 60 kN and 50 kN at distance 1 metre and 3 metre respectively from end A. Determine the position and magnitude of maximum deflection. Take E = 2 × 10⁵ N/mm² and I = 8500 cm⁴.* (*UPTU : 2009 – 10*)

*Given :* $E = 2 \times 10^5\ N/mm^2,$

$I = 8500\ cm^4 = 85 \times 10^6\ mm^4$

*Flexural rigidity,* $EI = 2 \times 10^5 \times 85 \times 10^6$

$= 1.7 \times 10^{13}\ Nmm^2$

$= 17 \times 10^3\ kNm^2$

**Solution** Assume span of the beam, $L = 4$ m

To calculate support reactions. For $\Sigma M_A = 0$ ↻+ we have

$$R_A\,(0) + 60(1) + 50(3) - R_B\,(4) = 0$$
$$60 + 150 = 4R_B$$

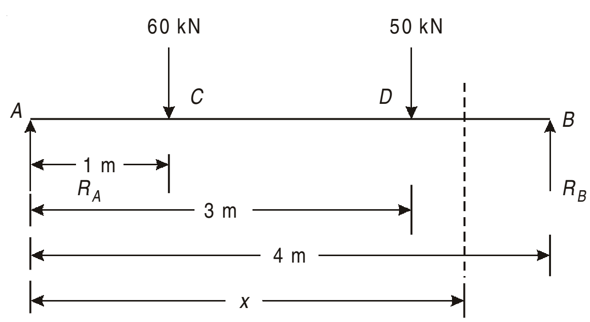



**Fig. 4.12**

$$\therefore \quad R_B = 52.50 \text{ kN}$$

For $\Sigma F_y = 0$ we have

$$R_A + R_B = 60 + 50 = 110$$

$$\therefore \quad R_A = 110 - R_B = 110 - 52.50 = 57.50 \text{ kN}$$

Using Macaulay's method for deflection

Consider a section $X - X$ at a distance $x$ from support $A$

$$M_{xx} = EI\frac{d^2y}{dx^2} = 57.50x \;\vdots\; [-60(x-1) \;\vdots\; -50(x-3) \qquad \text{...(i)}$$

Integrating the above Eq. (i):

$$EI\frac{dy}{dx} = \frac{57.50x^2}{2} + C_1 \;\vdots\; \frac{-60(x-1)^2}{2} \;\vdots\; \frac{-50(x-3)^2}{2} \qquad \text{...(ii)}$$

Integrating the above Eq. (ii)

$$EI.y = \frac{57.5x^3}{6} + C_1x + C_2 \;\vdots\; \frac{-60(x-1)^3}{6} \;\vdots\; \frac{-50(x-3)^3}{6} \qquad \text{...(iii)}$$

Applying boundry conditions to find the constants $C_1$ and $C_2$

(i) When $x = 0, y = 0$

$$\therefore \quad C_2 = 0$$

(ii) When, $x = 4, \quad y = 0$ put in Eq. (iii)

$$0 = \frac{57.5\times4^3}{6} + C_1x_4 - \frac{-60(3)^3}{6} - \frac{-50(4-3)^3}{6}$$

$$\therefore \quad C_1 = -83.75$$

Slope equation,

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{57.5x^2}{2} - 83.75 \;\vdots\; \frac{-60(x-1)^2}{2} \;\vdots\; \frac{-50(x-3)^2}{2}\right]$$

*Location of maximum deflection:*

Maximum deflection occurs at a point where slope is zero from the Fig. 4.12, it will be occurred between $CD$ portion

Consider slope equation upto $AD$ only.

$$0 = \frac{57.5x^2}{2} - 83.75 - 30(x-1)^2$$

or $$0 = 28.75x^2 - 83.75 - 30x^2 + 60x - 30$$

or $$0 = -1.25x^2 + 60x - 113.75$$

$$\therefore \quad \mathbf{x = 1.98\ m}$$

Deflection equation

$$y = \frac{I}{EI}\left[\frac{57.5x^3}{6} - 83.75x \,\Big|\, \frac{-60(x-1)^2}{6} \,\Big|\, \frac{-50(x-3)^3}{6}\right]$$

$$= \frac{1}{17000}\left[\frac{57.5\times1.98^3}{6} - 83.75\times1.98 - \frac{60}{6}(1.98-1)^3\right]$$

$$= -5.932 \times 10^{-3} \text{ m}$$

$$= -5.932 \text{ mm}$$

$\therefore \qquad y_{max} = \mathbf{5.932\ mm} \downarrow$

**Example 4.6.** *A simply supported beam has a flexural rigidity of 24 MN/m² and is loaded as shown in Fig. 4.13. Determine deflection at mid span.*

(*UPTU : 2008–09*)

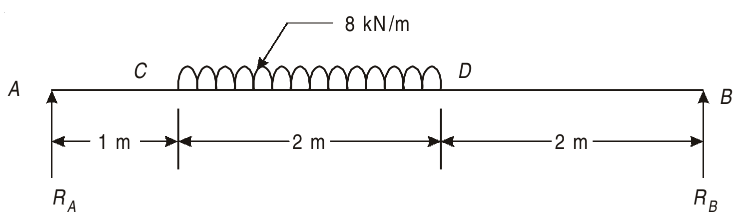



**Fig. 4.13**

**Solution** For reactions. For $\Sigma M_A = 0$

$R_A(0) + 8 \times 2\ (1 + 1) - 5\ R_B = 0$

or $\qquad 5R_B = 32$

$\therefore \qquad R_B = 6.4$ kN

For $\sum F_y = 0$; $\qquad R_A + R_B = 8 \times 2$

$\therefore \qquad R_A = 16 - R_B = 16 - 6.4 = 9.6$ kN

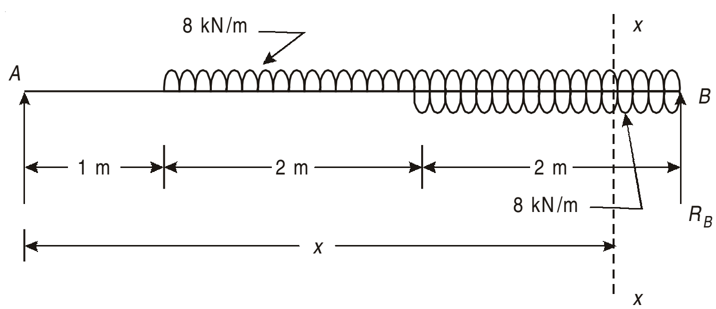



**Fig. 4.14**

**Note :** Macaulay's method has limitation, it is not applicable directly to given looding. To apply Macaulay's method, the U.D.L. is extended towards the right end $B$ where section $X - X$ is to be considered. At the same time, upward U.D.L. of same magnitude (8 kN/m should be applied for the same distance (2 m).

Hence consider a U.D.L. of 8 kN/m upward and downward in the portion $DB$ (Fig. 4.14). Consider a section $X - X$ at distance $x$ from the end $A$.

$$Mx = EI\frac{d^2y}{dx^2} = 9.6x \;\vdots\; -\frac{8}{2}(x-1)^2 \;\vdots\; \frac{8}{2}(x-2)^2 \qquad \text{...(i)}$$

Integrating above equation

$$EI\frac{dy}{dx} = \frac{9.6x^2}{2} + C_1 \;\vdots\; -\frac{8}{6}(x-1)^3 \;\vdots\; \frac{8}{6}(x-2)^3 \qquad \text{...(ii)}$$

Integrating equation (ii)

$$EIy = \frac{9.6x^3}{6} + C_1x + C_2 \;\vdots\; -\frac{8}{24}(x-1)^4 \;\vdots\; \frac{8}{24}(x-3)^4 \qquad \text{...(iii)}$$

Applying boundary conditions;

At $x = 0, y = 0,$

$\therefore$ $C_2 = 0$

At $x = 5$, $y = 0$, put in (iii)

$$0 = \frac{9.6(5)^3}{6} + C_1(5) - \frac{8}{24}(5-1)^4 + \frac{8}{24}(5-3)^4$$

$$0 = 200 + 5C_1 - 85.33 + 5.33$$

$$\therefore \quad C_1 = -\frac{120}{5} = -24$$

Substituting values of $C_1$ and $C_2$ in Eqs. (ii) and (iii)

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{9.6}{2}x^2 - 24 \;\vdots\; -\frac{8}{6}(x-1)^3 \;\vdots\; \frac{8}{6}(x-3)^3\right]$$

$$y = \frac{1}{EI}\left[\frac{9.6}{6}x^3 - 24x \;\vdots\; -\frac{8}{24}(x-1)^4 \;\vdots\; \frac{8}{24}(x-3)^4\right]$$

Deflection at mid point, i.e.

$$x = 2.5 \text{ m}$$

$$y = \frac{1}{EI}\left[\frac{9.6}{6}(2.5)^3 - 24(2.5) \;\vdots\; -\frac{8}{24}(2.5-1)^4\right]$$

$$= \mathbf{\frac{-36.687}{EI}}$$

**Example 4.7.** *Determine the deflection at point B and C of the beam (Fig. 4.15) using Macaulay's method.*

*Take E = 200 GPa, I = 19802.8 cm*$^4$. (*UPTU : 2005–06*)

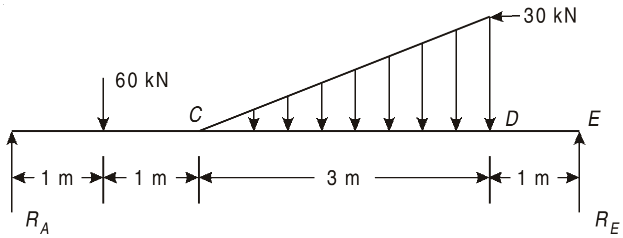



**Fig. 4.15**

**Solution** For $\Sigma M_A = 0$ ↻+ we have

$$R_A(0)+\frac{1}{2}(3)(30)\left(2+\frac{2}{3}\times 3\right)+60(1)-6R_E = 0$$

$$0 + 60 + 45\,(2 + 2) - 6R_E = 0$$

or $$R_E = \frac{240}{6} = 40 \text{ kN} \uparrow$$

For $\Sigma F_y = 0$ we have $R_A + R_E = 60 + \frac{1}{2}(3 \times 30) = 105$ kN

$\therefore$ $$R_A = 105 - R_E = 105 - 40 = 65 \text{ kN} \uparrow$$

Macaulay's method :

$$M = EI\frac{d^2y}{dx^2} = 65x \;\Big|\; (-60)(x-1) \;\Big|\; -\frac{1}{2}(x-2)10(x-2)\times\frac{1}{3}(x-2) \;\Big|$$

$$+\,30(x-5)\left(\frac{x-5}{2}\right)+\frac{1}{2}(x-5)10(x-5)\frac{1}{3}(x-5)$$

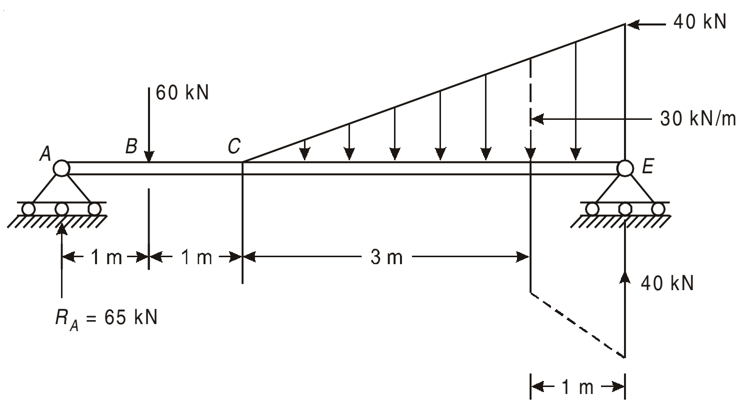



**Fig. 4.16**

$$\therefore\ EI\frac{d^2y}{dx^2} = 65x \;\vdots\; -60(x-1) \;\vdots\; \frac{-5}{3}(x-2)^3 \;\vdots\; +15(x-5)^2 + \frac{5}{3}(x-5)^3$$

$$EI\frac{dy}{dx} = 32.5x^2 + C_1 \;\vdots\; -30(x-1)^2 \;\vdots\; \frac{-5}{12}(x-2)^4 \;\vdots\; +5(x-5)^3 + \frac{5}{12}(x-5)^4$$

$$\therefore\ EI\frac{dy}{dx} = 32.5x^2 + C \;\vdots\; -30(x-1)^2 \;\vdots\; \frac{-5}{12}(x-2)^4 \;\vdots\; +5(x-5)^3 + \frac{5}{12}(x-5)^4$$

$$\therefore\ EIy = \frac{32.5}{3}x^3 + C_1x + C_2 \;\vdots\; -10(x-1)^3 \;\vdots\; \frac{-5}{12\times 5}(x-2)^5 \;\vdots\; +\frac{5}{4}(x-5)^4 + \frac{1}{12}(x-5)^5$$

At $A$: $x = 0, y = 0$

$\therefore$ $C_2 = 0$ and at $B : x = 6, y = 0$

$$\therefore\quad 0 = \frac{32.5}{3}(6)^3 + 6C_1 - 10\times 5^3 - \frac{1}{12}(4)^3 + \frac{5}{4}(1) + \frac{1}{12}(1)$$

$\therefore$ $C_1 = -181$

$$\therefore\ EIy = \frac{32.5}{3}x^3 - 181x \;\vdots\; -10(x-1)^3 \;\vdots\; -\frac{1}{12}(x-2)^5 \;\vdots\; +\frac{5}{4}(x-5)^4 + \frac{1}{12}(x-5)^5$$

At $B$ : $x = 1, y = \delta_B$

$$\therefore\quad EI\,\delta_B = \frac{32.5}{3} - 181$$

$$\therefore\quad \delta_B = \frac{-170.2}{EI}(\text{kNm units are used})$$

$E = 200$ GPa

$= 200 \times 10$ N/m$^2$

$= (200 \times 10^6)$ kN/m$^2$

$I = 19802.8$ cm$^4$

$= (19802.8 \times 10^{-8})$ m$^4$

Putting these values, $$\delta_B = \frac{(-170.2)}{(200\times 10^6)(19802.8)\times 10^{-8}}$$

$= -(4.3 \times 10^{-3})$ m

$\therefore$ $\boldsymbol{\delta_B}$ **= 4.3 mm ↓**

At $C$ : $x = 2$ m, $y = \delta_C$

$$\therefore\quad EI\,\delta_C = \frac{32.5}{3}\times 8 - 181\times 2 = -10(1)^3$$

$$\therefore \qquad \delta_C = \frac{(-285)}{EI} = \frac{(-285)}{\left(200\times10^6\right)}\times19802.8\times10^{-8}$$

$$= (-7.2 \times 10^{-3})\text{ m} = -7.2\text{ mm}$$

Hence, deflection at **C = 7.2 mm ↓**

**Example 4.8.** *A cantilever subjected to U.V.L. in such a way that zero at free end and maximum load w kN/m at fixed end. Determine maximum slope and deflection.* (*UPTU : 2005–06*)

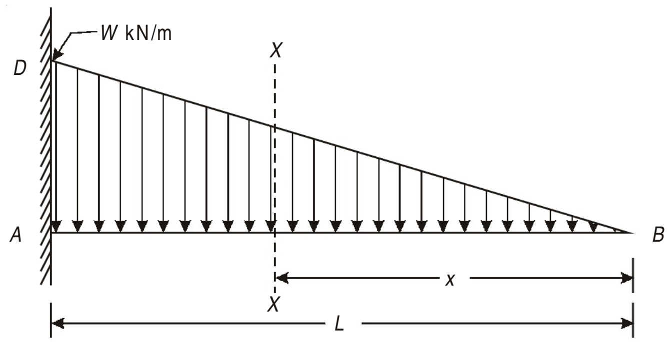



**Fig. 4.17**

**Solution** Consider a section $X - X$ at a distance $x$ from the end $B$. Load at section $X - X$, $\quad wx = \dfrac{wx}{L}$

By Macaulay's method;

$$M_{xx} = EI\frac{d^2y}{dx^2} = -\left(\frac{1}{2}\cdot x\frac{wx}{L}\right)\frac{x}{3}$$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6L} \qquad \text{...(i)}$$

Integrating $$EI\frac{dy}{dx} = -\frac{wx^4}{24L}+C_1 \qquad \text{...(ii)}$$

Again integrating, $$EIy = \frac{-wx^5}{120L}+C_1x+C_2$$

Apply boundary condition, to find contents of integration

(i) At $$x = L,\ \frac{dy}{dx} = 0,\quad 0 = \frac{-wL^4}{24L}+C_1$$

$$\therefore \qquad C_1 = \frac{wL^3}{24}$$

(ii) At $x = L, \quad y = 0,$

$$\therefore \qquad 0 = \frac{-wx^5}{120L} + \frac{wL^3}{24} \times L + C_2$$

$$\therefore \qquad C_2 = \frac{wL^4}{120} - \frac{wL^4}{24}$$

$$= \frac{-4wL^4}{120} = \frac{-wL^4}{30}$$

Put values of $C_1$ and $C_2$ in Eqs. (i), (ii) and (iii)

Slope equation, $$EI\frac{dy}{dx} = \frac{-wx^4}{24L} + \frac{wL^3}{24}$$

Deflection equation, $$EIy = \frac{-wx^5}{120L} + \frac{-WL^3x}{24} - \frac{wL^4}{30}$$

Maximum deflection occurs at free end. Deflection at $B$ i.e.,
At $x = 0$

$$y_{\max} = \frac{1}{EI}\left[-\frac{w}{120L} \times 0 + \frac{wL^3}{24} \times 0 - \frac{wL}{30}\right]$$

$$= -\frac{wL^4}{30EI} = -\frac{\mathbf{wL^4}}{\mathbf{30EI}} \downarrow$$

Maximum slope also occurs at free end
Hence slope at $B$ i.e., at $x = 0$

$$\left(\frac{dy}{dx}\right)_{\max} = \frac{1}{EI}\left[\frac{-w}{24} \times 0 + \frac{wL^3}{24}\right] = \frac{\mathbf{wL^3}}{\mathbf{24EI}}$$

**Example 4.9.** *A beam of uniform section 9 m long is carried on three supports at the same level, one at each end and one at 6 m from the left end. A uniformly distributed load of 16 kN/m is carried across the whole span, and a point load of 20 kN at 4.5 m from the end. Draw the S.F. and B.M. diagrams.*

(*UPTU : 2010–2011*)

**Solution**

Consider beam 1 $$R_A = R_C = \frac{wL}{2} + \frac{W}{2} = \frac{16 \times 9}{2} + \frac{20}{2} = 82 \text{ kN}$$

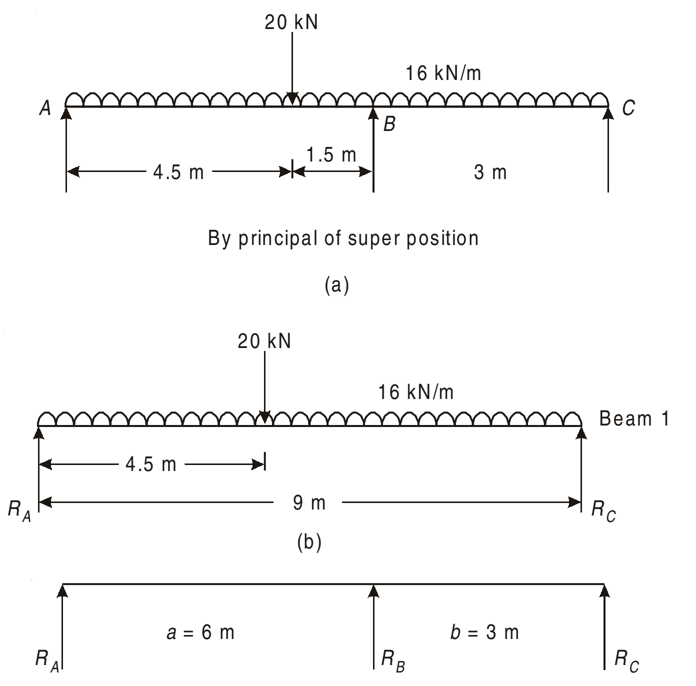
Consider a section $x - x$ at a distance $x$ from $A$

$$M_{xx} = EI\frac{d^2y}{dx^2} = 82x \Big| \frac{-16x^2}{2} \Big| - 20(x - 4.5)$$

Integrate,

$$EI\frac{dy}{dx} = \frac{82x^2}{2} + C_1 \Big| \frac{-8x^3}{3} \Big| - 10(x - 4.5)^2$$

$$EI\,y = \frac{82x^3}{6} + C_1x + C_2 \Big| \frac{-8x^4}{12} \Big| - \frac{10(x-4.5)^3}{3}$$




(c)

**Fig. 4.18**

Applying boundary condition

At $\quad x = 0, y = 0, C_2 = 0$

At $\quad x = 9 \text{ m}, y = 0$

$$0 = \frac{82 \times 9^3}{6} + C_1(9) - \frac{8 \times 9^4}{12} - \frac{10(4.5)^3}{3}$$

$$C_1 = -587.25$$

Deflection at $x$ = 6 m

$$EI\,y = \frac{82(6)^3}{6} - 587.25(6) - \frac{8\times 6^4}{12} \frac{-10(6\times 4.5)^3}{3}$$

$$y_B = -\frac{1446.75}{EI}$$

Consider beam 2

Deflection under load (Reaction) by standard formula

$$y_B = \frac{Wa^2b^2}{3EIL} = \frac{R\times 6^2\times 3^2}{3EI\times 9} = \frac{12R}{EI}\uparrow$$

as the point $B$ is supported by support, deflection at point $B$ must be zero

$$y_B + y_{B2} = 0$$

$$\frac{-1446.75}{EI} + \frac{12R}{EI} = 0$$

$$R = \frac{1446.75}{12} = 120.6 \text{ kN}$$

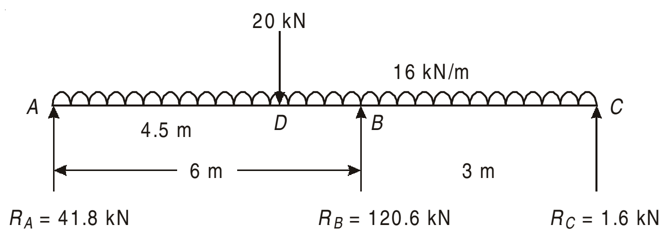



**Fig. 4.19**

$$\circlearrowright{+}\Sigma m_A = 0$$

$$R_A \times 9 - 20 \times 4.5 - 16 \times \frac{9^2}{2} + 120.6 \times 3 = 0$$

$$R_A = 41.8 \text{ kN}$$

$$+\uparrow\Sigma Fy = 0$$

$$R_A + R_c + 120.6 - 20 - 16 \times 9 = 0$$

$$R_C = 1.6 \text{ kN}$$

Shear force calculation

$$SF_A = 41.8 \text{ kN}$$

$$SF_{DL} = 41.8 - 16 \times 4.5 = -30.2 \text{ kN}$$

$$SF_{DR} = -30.3 - 20 = -50.3 \text{ kN}$$

$$SF_{BL} = -50.3 - 16 \times 1.5 = -74.3 \text{ kN}$$

$$SF_{BR} = -74.3 + 120.6 = 46.3$$

$$SF_{CL} = 46.3 - 16 \times 3 = -1.6$$
$$SF_{CR} = -1.6 + 1.6 = 0$$

B.M. calculation

$$BM_A = 0$$

$$BM_D = 41.8 \times 4.5 - 16 \times \frac{4.5^2}{2} = 26.1 \text{ kN/m}$$

$$BM_B = 41.8 \times 6 - 16 \times \frac{6^2}{2} - 20 \times 1.5 = -67.2 \text{ kNm}$$

$$BM_C = 0$$

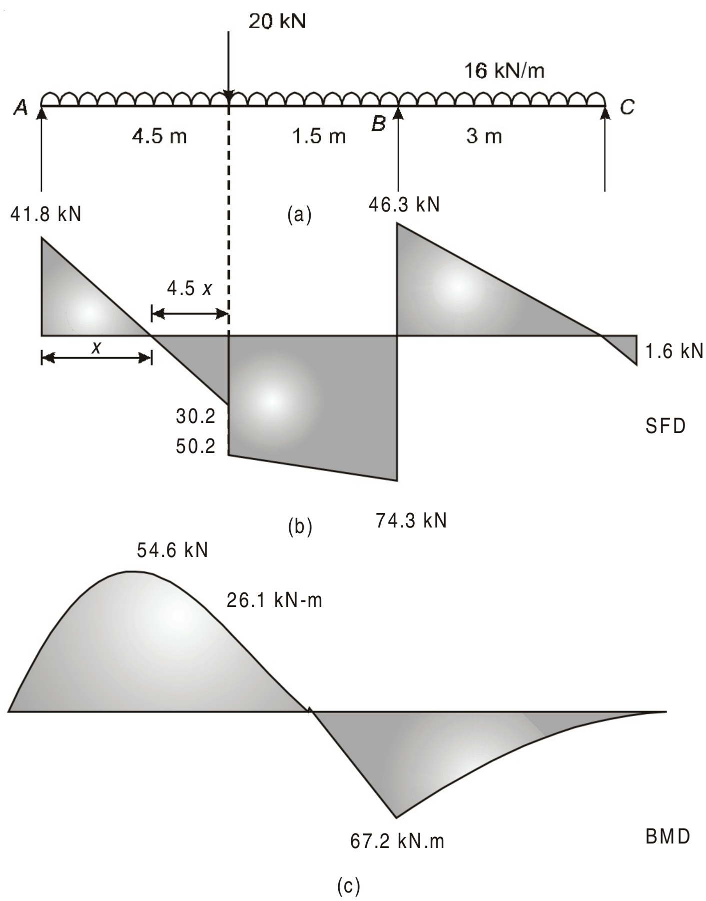



**Fig. 4.20**

By similarity of triangles

$$\frac{30.2}{4.5 - x} = \frac{41.8}{x}$$

$$0.722\,x = 4.5\,x$$
$$x = 2.6 \text{ m}$$

$$\text{BM}_{x\,=\,2.6} = 41.8 \times 2.6 - 16 \times \frac{2.6^2}{2} = 54.6 \text{ kNm}$$

**Example 4.10.** *A simply supported beam with point load W at a distance a from support A. Determine slope at supports, deflection under load and also find maximum deflection.* (*UPTU : 2011–2012*)

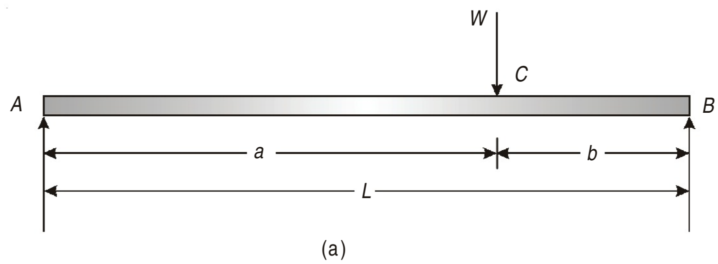



**Fig. 4.21**

**Solution**

$\therefore$ Support reaction
$$\Sigma R_A = 0$$
$$-R_B \times L + Wa = 0$$
$$R_B = \frac{Wa}{L}$$
$$\text{Similarly } R_A = \frac{Wb}{L}$$

Consider section $x - x$ at a distance $x$ from support $A$ as shown in Fig. 21 (b).

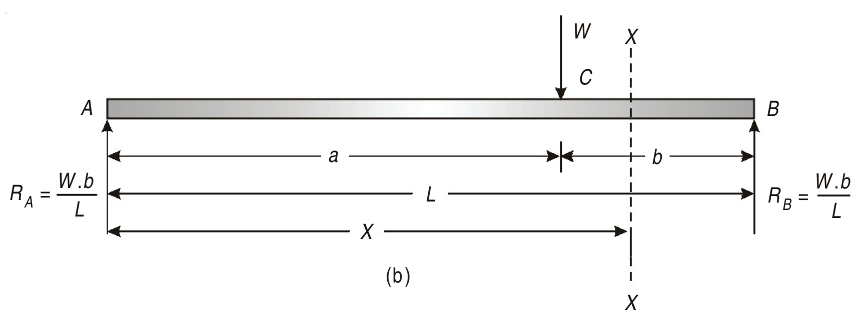



**Fig. 4.21**

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{Wb}{L}x \;\Big|\; -W(x-a) \qquad \text{...(1)}$$

Integrate $$EI\frac{dy}{dx} = \frac{Wb}{L}\left(\frac{x^2}{2}\right)+C_1 \Bigg| -\frac{W(x-a)^2}{2} \qquad ...(2)$$

Again integrate $$Ely = \frac{Wbx^3}{6L}+C_1x+C_2 \Bigg| -\frac{W(x-a)^3}{6} \qquad ...(3)$$

Apply boundary condition

When $x = 0, y = 0$

$\therefore$ $C_2 = 0$

When $x = L$, $y = 0$ put in Eq. (3)

$$0 = \frac{WbL^3}{6L}+C_1L-\frac{W(L-a)^3}{6}$$

$$C_1 L = \frac{Wb^3}{6}-\frac{WbL^3}{6L}$$

$\because$ $L-a = b$

$$= \frac{Wb}{6}\left(b^2-L^2\right) = \frac{Wb}{6}\ (b^2-[a^2+2\,ab+b^2)]$$

$$C_1 L = \frac{Wb}{6}\left(b^2-a^2-2\,ab-b^2\right)$$

$$\therefore \qquad C_1 = \frac{Wab}{C_1}(a+2b)$$

Put in Eqs. (2) and (3)

Slope equation $$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{Wb}{2L}x^2-\frac{Wab}{6L}(a+2b)\Bigg| -\frac{W(x-a)^2}{2}\right]$$

Deflection equation $$y = \frac{1}{EI}\left[\frac{Wbx^3}{6L}-\frac{Wab}{6L}(a+2b)x\Bigg| -\frac{W(x-a)^3}{6}\right]$$

$\therefore$ Slope at supports

$x = 0$ put in slope equation

$$\therefore \qquad \left(\frac{dy}{dx}\right)_A = \mathbf{-\frac{Wab}{6LEI}(a+2b)}$$

Similarly put $x = L$

$$\left(\frac{dy}{dx}\right)_B = \mathbf{\frac{Wab}{6LEI}(2a+b)}$$

Deflection under load

Put $x = a$
$$y = \frac{1}{EI}\left[\frac{Wa^3b}{6L} - \frac{Wa^2b}{6L}(a+2b)\right]$$

$$= \frac{1}{EI}\frac{Wa^2b}{6L}[a-a-2b]$$

$$= -\frac{2Wa^2b^2}{6LEI}$$

$$\mathbf{y = -\frac{Wa^2b^2}{3LEI}}$$

Maximum deflection occur where slope is zero, $\frac{dy}{dx} = 0$ in slope equation

Assume maximum deflection between $A$ and $C$.

$$0 = \frac{Wb}{2L}x^2 - \frac{Wab}{6L}(a+2b)$$

$$\frac{Wb}{2L}x^2 = \frac{W \cdot ab}{36L}(a+2b) = \frac{a^2+2ab}{3}$$

$$x = \sqrt{\frac{L^2-b^2}{3}}$$

This is position where maximum deflection occurs.

Put $x = \sqrt{\frac{L^2-b^2}{3}}$ in deflection equation

$$y = \frac{1}{EI}\left[\frac{Wb}{6L}\left(\sqrt{\frac{L^2-b^2}{3}}\right)^3 - \frac{Wab}{6L}(a+2b)\left(\sqrt{\frac{L^2-b^2}{3}}\right)\right]$$

$$y = \frac{Wb}{6LEI}\left[\left(\frac{L^2-b^2}{3}\right)^{3/2} - a(a+2b)\left(\frac{L^2-b^2}{3}\right)^{1/2}\right]$$

$$= \frac{Wb}{6LEI}\left[\left(\frac{L^2-b^2}{3}\right)^{3/2} - \left(a^2+2ab+b^2-b^2\right)\left(\frac{L^2-b^2}{3}\right)^{1/2}\right]$$

$$= \frac{Wb}{6LEI}\left[\left(\frac{L^2-b^2}{3}\right)^{3/2} - 3\left(\frac{L^2-b^2}{3}\right)^{1}\left(\frac{L^2-b^2}{3}\right)^{1/2}\right]$$

$$= \frac{Wb}{6LEI}\left[\left(\frac{L^2-b^2}{3}\right)^{3/2} - 3\left(\frac{L^2-b^2}{3}\right)^{3/2}\right] = \frac{2Wb}{6EI}\left(\frac{L^2-b^2}{3}\right)^{3/2}$$

$$= \frac{-Wb\left(L^2-b^2\right)^{3/2}}{3\times 3^{3/2}} \qquad 3^{3/2} = \sqrt{27} = 3\sqrt{3}$$

$$\mathbf{y_{max}} = \frac{\mathbf{-Wb\left(L^2-b^2\right)^{3/2}}}{\mathbf{9\sqrt{3}}}$$

**Example 4.11.** *A simply supported beam of L carrying two equal point load W at L/4 from each support. Find slope at support and maximum deflection (at centre).* *(UPTU : 2004–2005)*

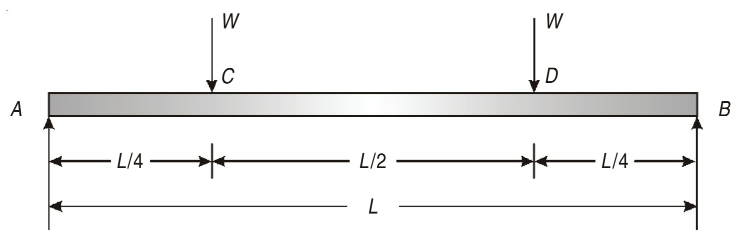



**Fig. 4.22 (a)**

**Solution**

Support reaction $R_A = R_B = W$

Consider section $x - x$ at distance $x$ from left support as shown in Fig. 4.22 (b)

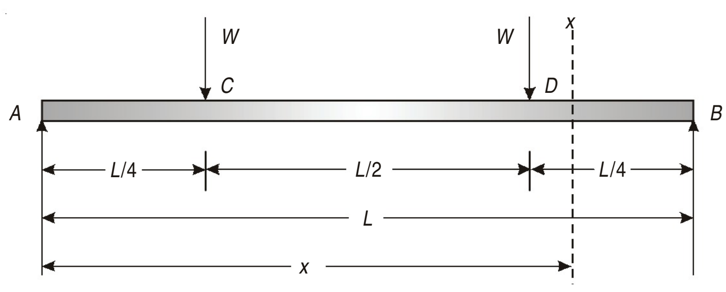



**Fig. 4.22 (b)**

$$M_{xx} = EI\frac{d^2y}{dx^2} = Wx \Big| -w\left(x-\frac{L}{4}\right) \Big| -W\left(x-\frac{3L}{4}\right) \qquad ...(1)$$

Integrating

$$EI\frac{dy}{dx} = \frac{Wx^2}{2} + C_1 \Big| -\frac{W(x-L/4)^2}{2} \Big| -\frac{W(x-3L/4)^2}{2} \qquad ...(2)$$

Again integrating

$$EIy = \frac{Wx^3}{6} + C_1x + C_2 \Big| -\frac{W(x-L/4)^3}{6} \Big| -\frac{W(x-3L/4)^3}{6} \qquad ...(3)$$

When $x = 0, y = 0$

$\therefore$ $C_2 = 0$

When $x = L, y = 0$

Put in Eq. (3) $0 = \frac{WL^3}{6} + C_1L - \frac{W(3L/4)^3}{6} - \frac{W(L/4)^3}{6}$

$$C_1 = \frac{WL^2}{354}[-64+27+1]$$

$$C_1 = \frac{-3\,WL^2}{32}$$

Slope equation

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{Wx^2}{2} - \frac{3WL^2}{32} - \frac{W(x-L/4)^2}{2} - \frac{W(x-3L/4)^2}{2}\right]$$

Deflection equation

$$y = \frac{1}{EI}\left[\frac{Wx^3}{6} - \frac{3WL^2}{32}k - \frac{W(x-L/4)^3}{6} - \frac{W(x-32/4)^3}{6}\right]$$

$\therefore$ Slope at $A$, at $x = 0$

$\therefore$ $$\theta_A = -\frac{\mathbf{3WL^2}}{\mathbf{32EI}}$$

Similarly $\therefore$ $$\theta_B = +\frac{\mathbf{3WL^3}}{\mathbf{32EI}}$$

Deflection at centre i.e. at $x = \frac{1}{2}$

$$\text{Put } y_{\max} = \frac{I}{EI}\left[\frac{WL^3}{6\times 8} - \frac{3WL^3}{32}\times\frac{1}{2} - \frac{W\left(\frac{L}{2}\right)^3}{6}\right] = \frac{WL^3}{EI}\left[\frac{1}{48} - \frac{3}{64}\times\frac{1}{48}\right]$$

$$\therefore \qquad \mathbf{y_{max}} = \frac{\mathbf{3WL^3}}{\mathbf{64EI}}$$

## 4.7 ❑ MOMENT AREA METHOD

It is graphical method for the slope and deflection of beams and cantilevers, which is based on Mohr's theorems.

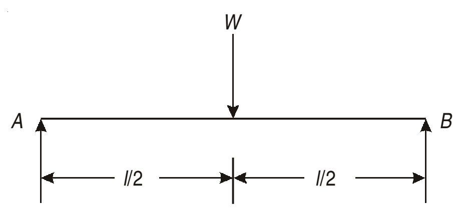



**Fig. 4.23**

1. *Simply supported beam carries central point load W.*
Slope at $A$ i.e.,

$$\theta_A = \frac{Wl^2}{16EI}$$

$$\theta_B = \frac{Wl^2}{16EI}$$

Deflection at $C$ i.e. $\quad y_c = \dfrac{Wl^3}{48EI}$

2. *Simply supported beam carrying U.D.L.*

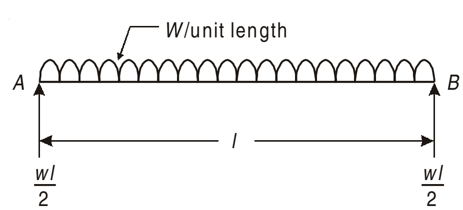



**Fig. 4.24**

Slope at $A$ i.e., $\theta_A = \frac{Wl^3}{24EI}$ ↻

$\theta_B = \frac{Wl^3}{24EI}$ ↻

Deflection at $C$, $y_C = \frac{5wl^4}{384EI}$ ↓

**Example 4.12.** *A simply supported beam of 2 m span carries a point load of 20 kN at its mid point. Determine the maximum slope and deflection of the beam. Take flexural rigidity of the beam as 500 × 10⁹ N-mm².*

**Solution**

(i) Slope $\theta_B = \frac{Wl^2}{16EI} = \frac{20\times10^3\left(2\times10^3\right)^2}{16\left(500\times10^9\right)}$ = **0.01 Rad**

Deflection, $y_c = \frac{wl^3}{48EI} = \frac{\left(20\times10^3\right)\left(2\times10^3\right)^3}{48\left(500\times10^9\right)}$ = **6.67 mm**

**Example 4.13.** *A simply supported beam of 2.4 m span is subjected to a uniformly distributed load of 6 kN/m over the entire span. Calculate maximum slope and deflection of the beam if its flexural rigidity is 8 × 10¹² N-mm².*

**Sloution**

Slope of the beam, $\theta_A = \frac{Wl^3}{24EI}$

$\theta_A = \frac{Wl^3}{24EI} = \frac{6\left(2.4\times10^3\right)^3}{24\left(8\times10^{12}\right)}$ = **0.00043 rad**

Deflection of the beam, $y_c = \frac{5Wl^4}{384EI} = \frac{5\times6\left(24\times10^3\right)^4}{384\left(8\times10^{12}\right)}$

= **0.324 mm**

### 4.7.1 Cantilever with a Point Load at Free End

Slope, $\theta_B = \frac{Wl^2}{2EI}$ Radians

Deflection, $y_B = \dfrac{Wl^3}{3EI}$

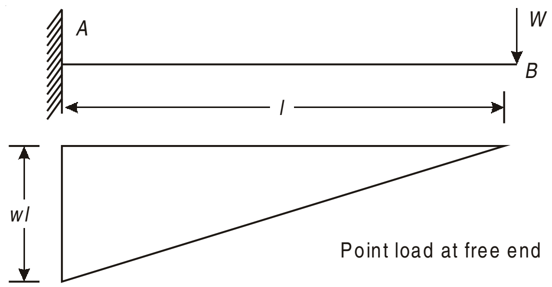



Fig. 4.25

### 4.7.2 Cantilever with a U.D.L.

Slope, $\theta_B = \dfrac{Wl^3}{6EI}$ Radians

Deflection, $y_B = \dfrac{Wl^4}{8EI}$

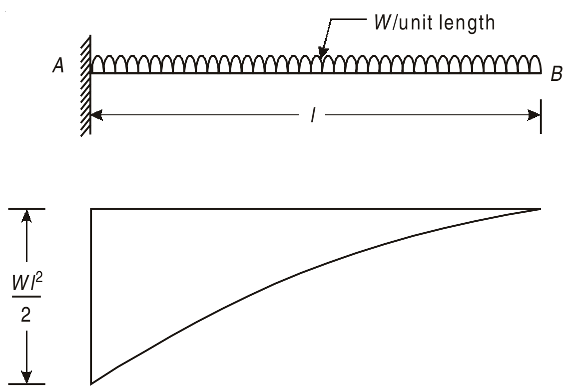



Fig. 4.26

**Example 4.14.** *A cantilever beam at 2m span is subject to a point load of 30 kN at its free end. Find the slope and deflection of the free end. Take EI of the beam as $8 \times 10^{12}$ N-mm$^2$.*

*Given :* Span, $l = 2$ m

$= 2 \times 10^3$ mm, point load

$= 30$ kN

$= 30 \times 10^3$ N

**Sloution** Flexural rigidity,

$$EI = 8 \times 10^{12} \text{ N-mm}^2$$

Slope at the free end, $\theta_B = \dfrac{wl^2}{2EI}$

$$= \frac{(30\times10^3)(2\times10^3)^2}{2(8\times10^{12})} = \textbf{0.0075 Radian}$$

$$= \textbf{0.0075 radians}$$

Deflection at free end, $y_B = \dfrac{wl^3}{3EI}$

$$= \frac{(30\times10^3)(2\times10^3)^3}{3(8\times10^{12})} = \textbf{10 mm}$$

**Example 4.15.** *A cantilever beam 120 mm wide and 150 mm deep carries a uniformly distributed load of 10 kN/m over its entire length of 2.4 metres. Find the slope and deflection of the beam at its free end. Table E = 180 GPa.*

**Solution**

$$I = \frac{(bd^3)}{12}$$

$$= \frac{[120(150)^3]}{12} = 33.75\times10^6 \text{mm}^4$$

Slope at free end, $\theta_B = \dfrac{wl^3}{6EI} = \dfrac{10(2.4\times10^3)^3}{6(180\times10^3)(33.75\times10^6)}$

$$= \textbf{0.0038 Radians}$$

Deflection at free end, $y_B = \dfrac{wl^4}{8EI} = \dfrac{10(2.4\times10^3)^4}{8(180\times10^3)(33.75\times10^6)}$

$$= \textbf{6.83 mm}$$

## EXERCISE

**4.1.** Determine by Macaulay's method slope and deflection at point $C$ and $D$ in the beam shown in Fig. 4.27. Take $E$ = 200 GPa and $I = 20 \times 10^{-6}$ m$^4$.

**[*Ans.* $y_C$= 9.306 mm (↓), $y_D$= 5.486 mm (↓)]**

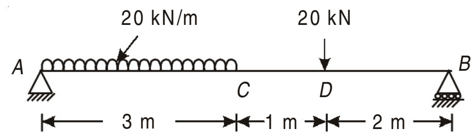



**Fig. 4.27**

**4.2.** A simply supported beam $ABCD$ 6 m long carries U.D.L 32 kN/m over $BC$. $AB$ = 2 m, $BC$ = 3 m, $CD$ = 1m. The cross-section of beam is hollow rectangular section as shown in Fig. 4.28. Find maximum deflection for the beam. Take $E$ = 200 GPa. **[*Ans.* $y_{max}$ = 0.303 mm ↓]**

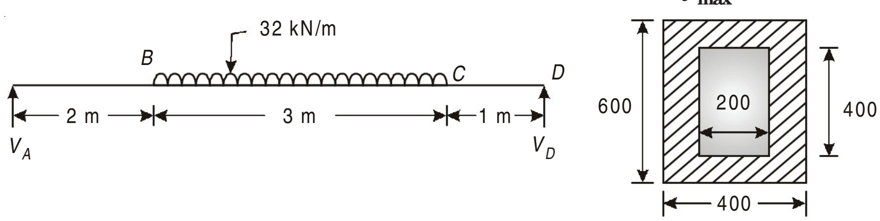



**Fig. 4.28**

**4.3.** Determine the deflection at the end of the overhang for the beam $ABC$ as shown in Fig. 4.29.

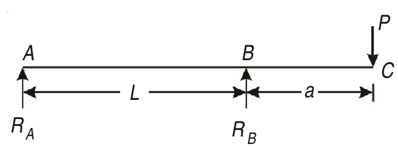



**Fig. 4.29**

**[*Ans.* Deflection at point** $y_c = \dfrac{-Pa^2(L+a)}{3EI}$ **]**

**4.4.** A cantilever beam subjected to the point load $P$. It also carries U.D.L of $W$ kN/m on its entire span. Determine maximum slope and deflection.

**[*Ans.*** $y = \left(\dfrac{PL^2}{3EI} + \dfrac{WL^4}{8EI}\right)$ **]**

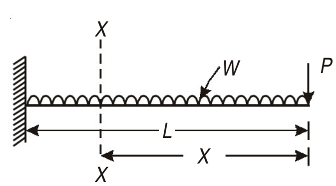



**Fig. 4.30**

**4.5.** Figure 4.31 shows the simply supported beam of span 14 m loaded with the two concentrated loads of 120 kN and 80 kN.

Calculate the deflection of the girder at points under the two loads. Take $I = 16 \times 10^8$ mm$^4$ and $E = 2.1 \times 10^5$ N/mm$^2$.

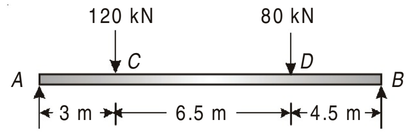



**Fig. 4.31**

**[*Ans.* $y_c$ = 15.64 mm, $y_D$ = 19.93 mm]**

**4.6.** A beam *AB* of 4 m span is simply supported at the ends and is loaded as shown in Fig. 4.32. Determine; (i) Deflection at *C*. (ii) Maximum deflection and (iii) Slope at the end *A*.

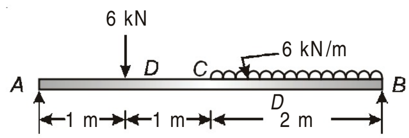



**Fig. 4.32**

Take $E = 2 \times 10^5$ N/mm$^2$. and $I = 1100 \times 10^4$ mm$^4$.

**[*Ans.* $y_c$ = 9.545 mm; $y_{max}$ = 9.55 mm at $x$ = 1.958 m;**
**$\theta_A$ = 0° 27' 20" (Anticlockwise)]**

**4.7.** A simply supported beam *AB* of span 7.5 m is loaded as shown in Fig. 4.33. Determine Deflections at points *C* and *D*.

Take flexural rigidity $EI = (30 \times 10^{12})$ N/mm$^2$.

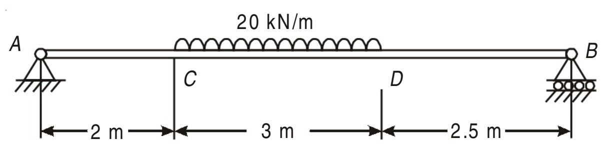



**Fig. 4.33**

**[*Ans.* $\delta_C$ = 12.089 mm, $\delta_D$ = 13.806 mm]**

**4.8.** Cantilever *AB* is fixed at *A*. It is 6 m long. It carries loads as shown in Fig. 4.34 Calculate slope and deflection at the free end *B* in terms of flexural rigidity *EI*.

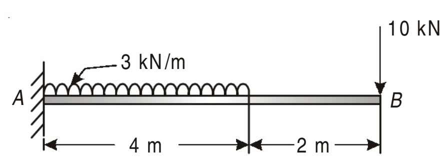



**Fig. 4.34**

**[*Ans.* $\theta = \dfrac{212}{EI}, \delta = \dfrac{880}{EI}$]**

**4.9.** In Fig. 4.35, simply supported beam *AB* of 9 m span is loaded as shown in Fig. 4.35. Calculate slope at *C* and deflection at *D* in terms of *EI*.

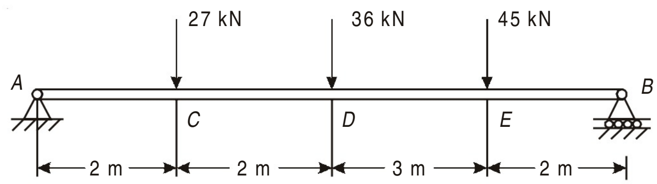



**Fig. 4.35**

**[*Ans.* $\theta = \dfrac{417}{EI}, \delta_D = \dfrac{1200}{EI} \downarrow$ ]**

**4.10.** Determine the maximum deflection, in terms of *EI* for a beam as shown in Fig. 4.36.

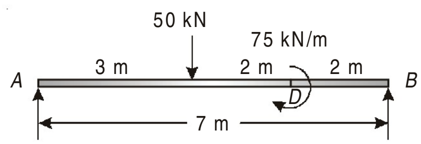



**Fig. 4.36**

**[*Ans.* $y_{max}$ = 194.032/*EI* at $x$ = 3.214 m from *A*]**

**4.11.** A cantilever 2 m long is loaded as shown in Fig. 4.37. Cross-section of the beam is rectangular 100 mm × 200 mm deep. Take $E$ = 10.5 GPa. Calculate the deflection at the free end.

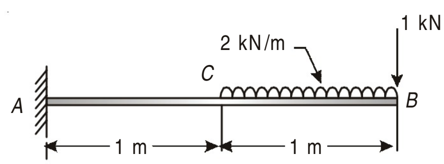



**Fig. 4.37**

**4.12.** A fixed beam *AB* of span 6 m carries a U.D.L. of 30 kN/m over the entire span. Find the fixed end moments by using first principle and draw S.F. and B.M. diagram. Also locate the point of contraflexure.

**[*Ans.* $M_A = M_B$ = 90 kN.m (Hogging), $R_A = R_B$ = 90 kN, point of centre flexure lies at 1.266 m from both the ends]**

**4.13.** A beam *AB* of span 6 m is fixed at *A* and *B*. It is loaded with a U.D.L of 3 kN/m over the entire span in addition to a concentrated load of 4 kN at 4 m from *A*.

Calculate the fixed end moment.

**[*Ans.* $M_A$ = – 10.78 kN.m, $M_B$ = –12.55 kN.m]**

**4.14.** An encastre beam of span 4 m carries a U.D.L of 7.5 kN/m, and two point loads of 40 kN and 60 kN at 1 m and 2 m from the left hand support. Find the fixed end moments, support reaction and draw S.F.D and B.M.D.

**[Ans. $M_A$ = – 62.5 kN.m, $M_B$ = – 47.5 kN.m, $R_A$ = 78.75 kN, $R_B$ = 51.25 kN]**

**4.15.** A fixed beam *AB* of span 10 m carries U.D.L of 80 kN/m over $\frac{1}{3^{rd}}$ span from *A*. Find fixed end moment. **[Ans. $M_A$ = – 270.9 kN.m, $M_B$ = 74.18 kN.m]**

**4.16.** Analysis the fixed beam as shown in Fig. 4.39.

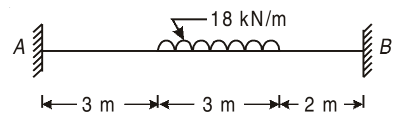



**Fig. 4.39**

**[Ans. $M_A$ = – 44.935 kN.m, $M_B$ = –56.32 kN.m]**

**4.17.** Analysis the fixed beam as shown in Fig. 4.40.

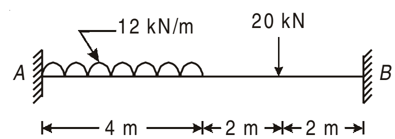



**Fig. 4.40**

**4.18.** What is Macaulay's method? Where is it used? Find an expression for a simply supported beam with an eccentric point load, using Macaulay's method. (*UPTU : 2011–2012*)

**[Ans. Section : 2.5]**

**4.19.** Explain in brief Macaulay's method. (*UPTU : 2012–13*)

**[Ans. Section : 2.5]**

**4.20.** Cantilever of span *L* carries point load *W* at free end. Determine the maximum slop and deflection. (*UPTU : 2011–2012*)

**[Ans. Example 4.3]**

**4.21.** Cantilever of span *L* carries point load *W* at free end. Determine the maximum slop and deflection. (*UPTU : 2012–13*)

**[Ans. Example 4.3]**

**4.22.** A beam of uniform section 9 m long is carried on three supports at the same level, one at each end and one at 6 m from the left end. A uniformly distributed load of 16 kN/m is carried across the whole span, and a point load of 20 kN at 4.5 m from the end. Draw the S.F and B.M diagrams.

(*UPTU : 2010–2011*)

**[Ans. Example 4.11]**

**4.23.** A simply supported beam carries a point load *W* at a distance *a* from support *A*. Determine slope at supports, deflection under load and also find maximum deflection. (*UPTU : 2011–2012*)

**[Ans. Example 4.12]**

**4.24.** A simply supported beam of length 8 m carries two concentrated forces of magnitude 64 kN and 48 kN in downward direction at distances of 1 m and 4 m from left end. Find the deflection below the 48 kN load.

Take $E$ = 210 GPa and $I = 180 \times 10^6$ mm$^4$. (*UPTU : 2011–2012*)

**[*Ans.* Example 4.1]**

**4.25.** Derive an expression for the slope and deflecton of a simpleys supported beam, span $L$, carrying a uniformly distributed load $w$ per unit length and a point load $P$ at the mid span. Hence, find the slope and deflection at a point $L/4$ from the left support. (*UPTU : 2004–2005*)

**[*Ans.* Exampe 4.2]**

**4.26.** A cantilever subject to UVL in such a way that zero at free end and maximum load intensity $w$ kN/m at fixed end. Determine maximum slope and deflection. (*UPTU : 2005–2006*)

**[*Ans.* Example 4.8]**

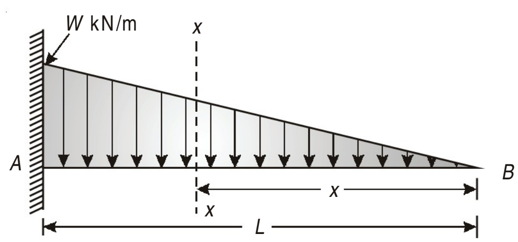



**Fig. 4.42**

**4.27.** Determine the deflections at point $B$ and $C$ of the beam shown in Fig. 4.43 using Macaulay's method. Take $E$ = 200 GPa, $I$ = 19802.8 cm$^4$.

(*UPTU : 2005–06*)

**[*Ans.* Example 4.7]**

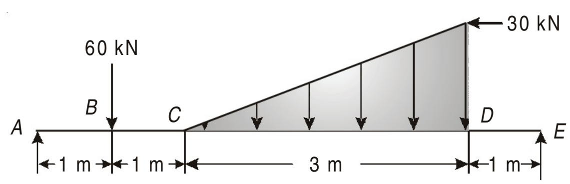



**Fig. 4.43**

**4.28.** A beam of uniform section, 10 m long is simply supported at the ends. It carries point loads of 150 kN and 65 kN at distances of 2.5 m and 5.5 m respectively from the left end. Calculate :

(i) Deflection under each load,

(ii) Maximum deflection.

Take $E$ = 200 GN/m$^2$ and $I = 118 \times 10^{-4}$ m$^4$.

(*UPTU : 2006–07*)

**[*Ans.* Example 4.4]**

**4.29.** A simply supported beam has a flexural rigidity of 24 $MN/m^2$ and is loaded as shown in Fig. 4.44. Determine the deflections at mid-span.

(*UPTU : 2008–09*)

**[*Ans.* Example 4.6]**

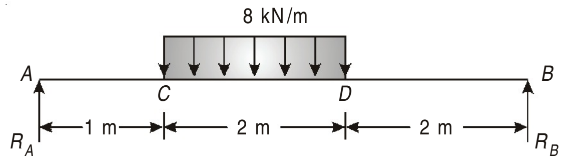



**Fig. 4.44**

**4.30.** A beam of length $L$ and flexural rigidity $EI$ is fixed at both ends at the same level and carries U.D.L. of intensity per unit length over whole span. Obtain expressions for maximum deflection of the beam. (*UPTU : 2008–09*)

**[*Ans.* Section 2.4]**

**4.31.** A beam, simply supported at ends $A$ and $B$ is loaded with two point loads of 60 kN and 50 kN at distance 1 metre and 3 metre respectively from end $A$. Determine the position and magnitude of maximum deflection. Take $E = 2 \times$ 105 $N/mm^2$ and $I = 8500\ cm^4$. (*UPTU : 2009–2010*)

**[*Ans.* Example 4.5]**

**4.32.** A simply supported beam of length $L$ carrying two equal point loads $W$ at $L/4$ from each support. Find slope at support and maximum deflection (at centre). (*UPTU : 2004–2005*)

**[*Ans.* Example 4.13]**

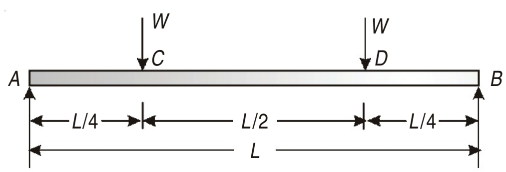



**Fig. 4.45**

# Shear Force and Bending Moment Diagrams

CHAPTER 5

## 5.1 ❑ SHEAR FORCE

It is unbalanced net vertical force on either side of a section.

## 5.2 ❑ BENDING MOMENT

The bending moment (B.M.) at a cross-section of a beam may be defined as the algebraic sum of the moments of the forces, to the right or left of the section.

## 5.3 ❑ BEAM

A beam is a horizontal *structural* member subjected ganerally to *vertical loads*.

## 5.4 ❑ CANTILEVER BEAM

The beam which is fixed at one end and free at the other is called *cantilever beam*. At the *fixed support* the beam can neither translate nor rotate, consequently both force and moment reactions may exist at the fixed support.

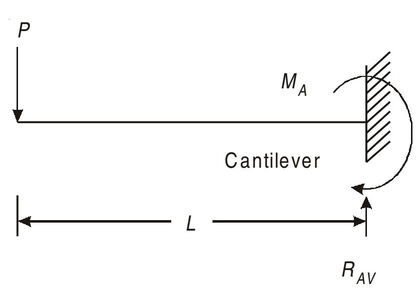



**Fig. 5.1**

## 5.5 ❑ SIGN CONVENTION

Shear force is taken as positive if it tends to move the left portion upward with respect to *right portion* and *vice-versa*. A better rule of sign, which is


**Fig. 5.2**

easier to remember is to give a *positive sign* to those forces acting upward on the left portion.

*Bending moment* is taken *positive* if it tends to *sag* (*concave form*) the beam, and if it is *negative* it tends to hog (*convex shape*)

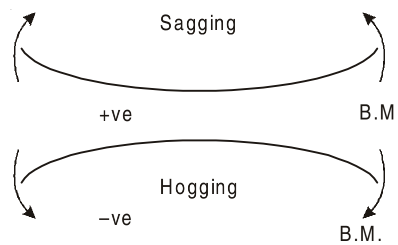



**Fig. 5.3**

The *shear force* and *bending moment* vary, along the length of the beam and the variation is represented graphically. The plots are known as *shear force* and *bending moment* diagrams. These plots help to determine the maximum value of each of these quantities. In these diagrams, the abscissa indicates the position of the section along the beam and the ordinate represents the value of S.F. and B.M. respectively.

### 5.3.1 Fixed Beam

A beam whose both ends are rigidly fixed is called *fixed* (*builtin*) *beam.*

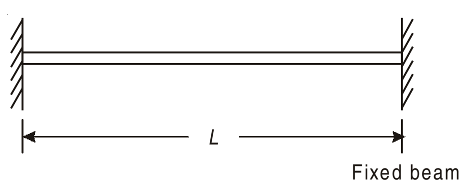



**Fig. 5.4**

### 5.3.2 Simply Supported Beam

A beam supported at both the ends is known as *simply supported beam*. Generally the beam is supported with a hinge or *pin support* at one end and a roller support at the other. A pin support prevents translation at the end of the beam. Thus end $A$ of the beam can not move horizontally or vertically.

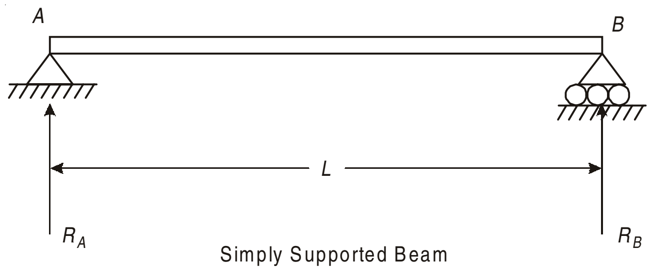



**Fig. 5.5**

The roller support prevents translation in vertical directions but not in the horizontal direction, hence this support can resist a *Vertical* force $R_B$ but not a horizontal force.

### 5.3.3 Overhanging Beam

An overhanging beam is supported with either one or *both ends* extending beyond the supports. The extended portion of the beam acts like a *cantilever*.

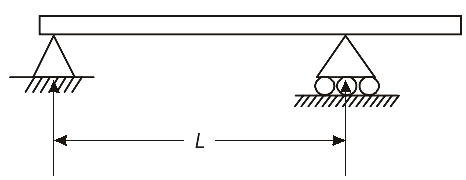



**Fig. 5.6** *One end overhanging*

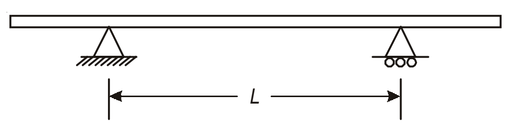



**Fig. 5.7** *Both ends overhanging*

### 5.3.4 Continuous Beam

A beam supported at more than *two points* is called *continuous beam.*

### 5.3.5 Statically Determinate Beams

Beams supported in such a way that their external support reaction can be determined directly from the equation of *static equilibrium*, are statically determinate beams. Examples, simply supported beam, overhanging beam and cantilever beams.

Continuous and fixed beams are called *statically indeterminate beams*. The methods of *statics* are not sufficient to determine support reactions.

## 5.4 ❑ TYPES OF LOADING

### 5.4.1 Concentrated or Point Load

A load which is acted at a point is called as point load.

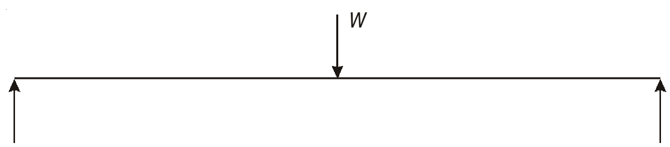



**Fig. 5.8**

### 5.4.2 Uniformly Distributed Load

A load which is uniformly spread over the beam is called as uniformly distributed load. (U.D.L).

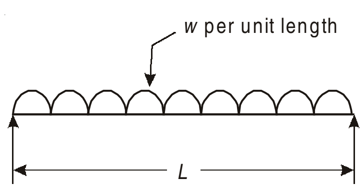



**Fig. 5.9**

Equivalent point load = Intensity of load × Length = $w \times L$

Act at C.G. of loading i.e. $\frac{L}{2}$.

### 5.4.3 Uniformly Varying Load (U.V.L)

The intensity of loading increases or decreases at a constant rate is called as uniformly varying load (U.V.L) or triangular loading.

Equivalent point load = Intensity of load × Length

$$= \left(\frac{0+w}{2}\right) \times L = \frac{wL}{2}$$

Act at C.G. of loading i.e., at $\frac{2L}{3}$ or $\frac{L}{3}$

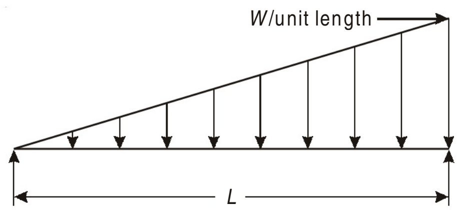



Fig. 5.10

## 5.5 ❑ TYPES OF SUPPORTS

Types of supports

(1) Roller support
(2) Hinged support
(3) Fixed support

### 5.5.1 Roller Support

Roller support has only one reaction which is always normal to the contact surface.

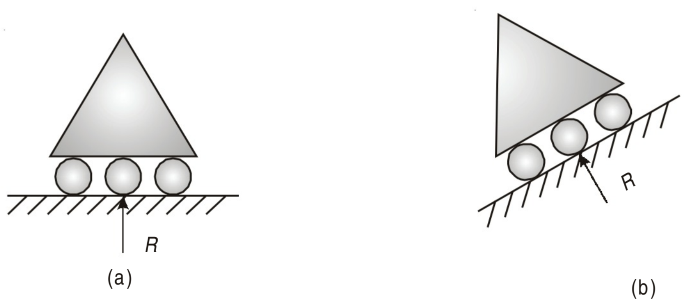



Fig. 5.11

### 5.5.2 Hinged Support

Hinged support has restriction to movement in $x$ and $y$ direction, therefore, it has two reactions $R_x$ and $R_y$.

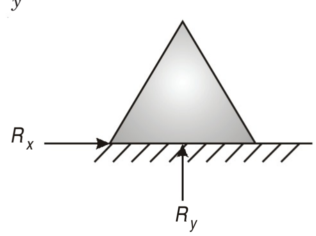



Fig. 5.12

### 5.5.3 Fixed Support

Fixed support has restriction to move in $x$ and $y$ direction as well as rotation about that point. It has three reaction i.e., $V_A$, $H_A$, $M_A$.

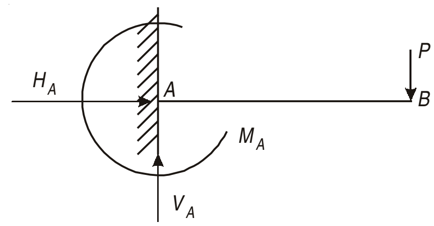



**Fig. 5.13**

**Example 5.1.** *Construct shear force and bending moment diagrams for the cantilever as loaded in Fig. 5.14.*

**Solution**

**S.F. Diagram :** We know that

$$
\begin{aligned}
F_A &= -5 \text{ kN} \\
F_B &= -(5+4) \\
&= -9 \text{ kN} \\
F_C &= -(9+3) \\
&= -12 \text{ kN} \\
F_D &= -12 \text{ kN}
\end{aligned}
$$

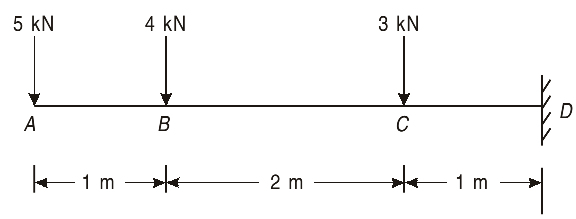
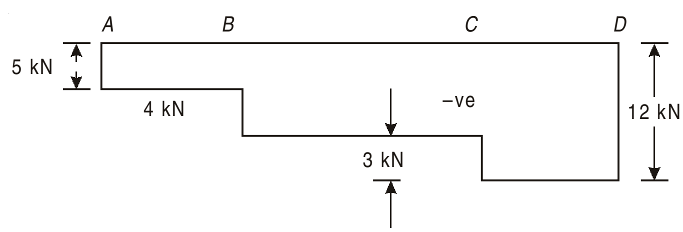
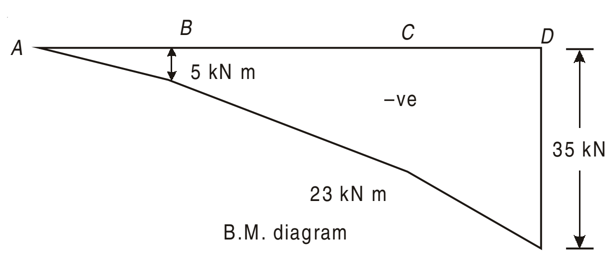



**Fig. 5.14**




S.F. diagram



For B.M. Diagram :

$$M_A = 0$$
$$M_b = -(5 \times 1)$$
$$= -5 \text{ kN m}$$
$$M_c = -(5 \times 3) + (-4 \times 2)$$
$$= -23 \text{ kN m}$$
$$M_d = -(5 \times 4) + (-4 \times 3) + (-3 \times 1)$$
$$= -35 \text{ kN m}$$

**Example 5.2.** *Draw S.F. and B.M. diagram for the cantilever beam shown in Fig. 5.15.*

**Solution**

For S.F. diagram

$$F_C = 0, \quad F_B = -(60 \times 1.8) = -108 \text{ N}$$
$$F_A = -108 \text{ N}$$

For B.M. diagram

$$M_C = 0$$
$$M_B = -(60 \times 1.8 \times 0.9) = -97.2 \text{ N-m}$$
$$M_A = -(60 \times 1.8 \times 1.6) = -172.8 \text{ N-m}$$

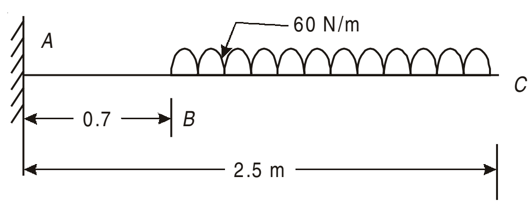
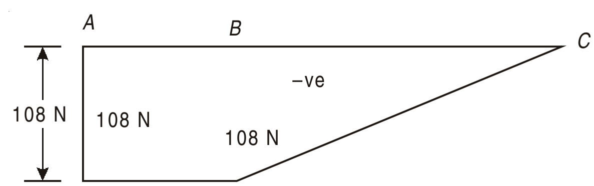
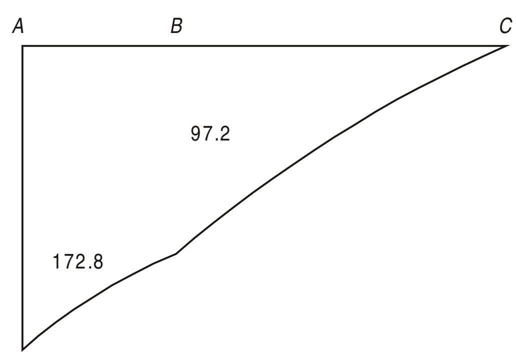



**Fig. 5.15**



**Example 5.3.** *Construct the S.F. and B.M. diagrams for the cantilever beam loaded as shown in Fig. 5.16.*

**Solution**

For S.F. Diagram :

$$F_A = -10 \text{ kN}$$
$$F_B = -(10 + 10 \times 1 + 20) = -40 \text{ kN}$$
$$F_c = -(10 + 20 + 10 \times 3) = -60 \text{ kN}$$

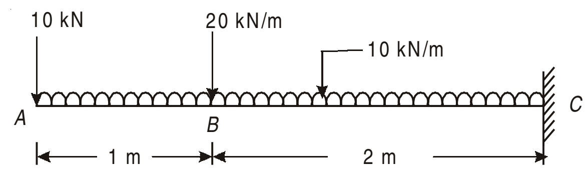
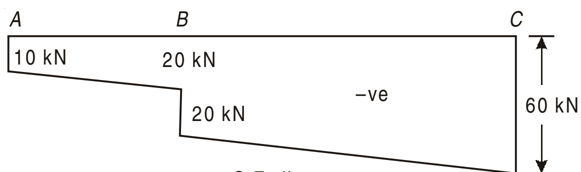
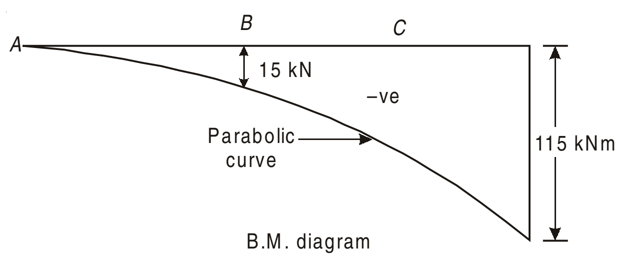



**Fig. 5.16**




The S.F. diagram indicates the values of S.F. at salient points




For B.M. Diagram :

$$M_A = 0$$
$$M_B = -(10 \times 1 + 10 \times 1 \times 0.5) = -15 \text{ kN m}$$
$$M_C = -(10 \times 3 + 20 \times 2 + 10 \times 3 \times 1.5)$$
$$= -(30 + 40 + 45) = -115 \text{ kN m}$$
$$M_A = 0$$
$$M_x = -w(x) \cdot \frac{x}{2} = -w\frac{x^2}{2}$$
$$M_L = -w(l) \cdot \frac{l}{2} = -w\frac{l^2}{2}$$

B.M. at $A = 0$ (where $x = 0$ and increases in the form of a Parabolic Curve to $-\dfrac{wl^2}{2}$ at $C$ (where $x = l$) (Refer Fig.5.17).

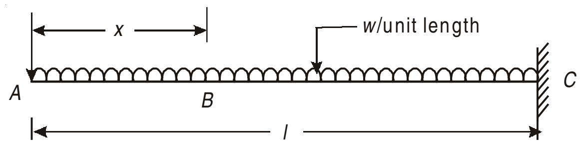



**Fig. 5.17**

**Example 5.4.** *A cantilever beam of 1.5 m span is loaded as shown in Fig. 5.18. Draw S.F. and B.M. Diagrams.*

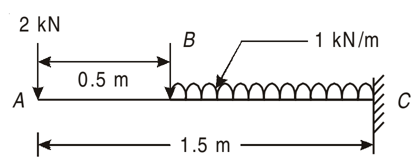
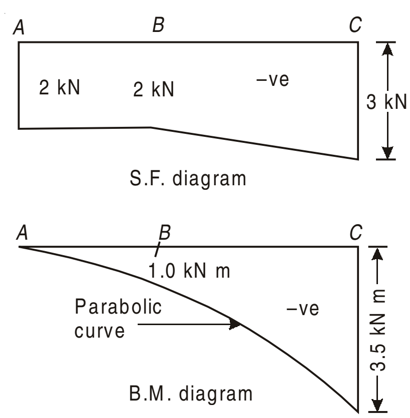



**Fig. 5.18**




**Solution**

For S.F. Diagram :

$$F_A = -2\text{ kN}$$
$$F_B = -2\text{ kN}$$
$$F_C = -(2 + 1 \times 1)$$
$$= -3\text{ kN}$$

For B.M. diagram :

$$M_A = 0$$
$$M_B = -(2 \times 0.5)$$

$$= -1 \text{ kN m}$$

$$M_C = -\left[(2 \times 1.5) + (1 \times 1)\frac{1}{2}\right]$$

$$= -3.5 \text{ kN m}$$

**Example 5.5.** *A cantilever beam of 2 m span is subjected to a gradually varying load from 2 kN/m to 5 kN/m as shown in Fig. 5.21. Draw S.F. and B.M. diagrams for the beam.*

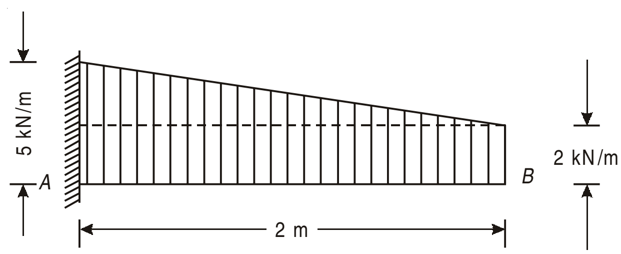
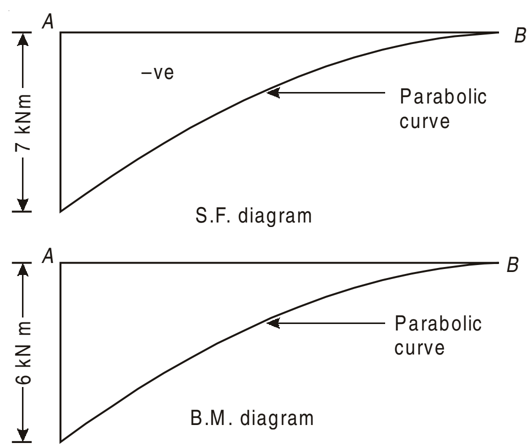



**Fig. 5.19**




**Solution**

For S.F. diagram :

$$F_B = 0$$

$$F_A = -\left[(2\times2)+\left(\frac{3\times2}{2}\right)\right]$$

$$= -7 \text{ kNm}$$

For B.M. diagram :

$$M_B = 0$$

$$M_A = -\left[\left(2\times2\times\frac{2}{2}\right)+\frac{3\times2\times2}{2\times3}\right]$$

$$= -6 \text{ kN m}$$

**Example 5.6.** *A simply supported beam AB of span 2.5 m is carrying two point loads as shown in Fig. 5.20. Draw S.F. and B.M. diagrams for beam.*

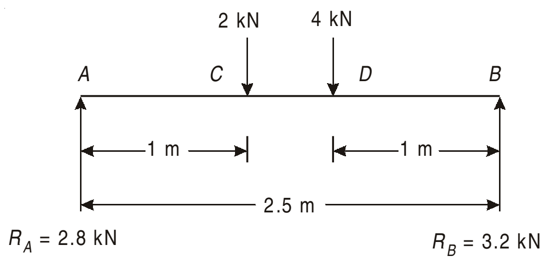
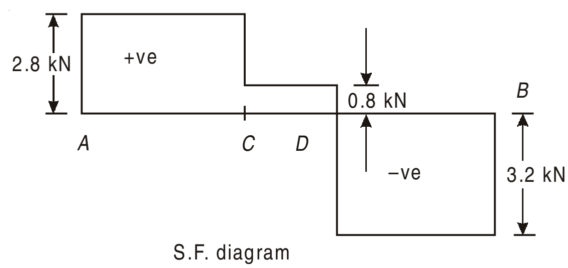
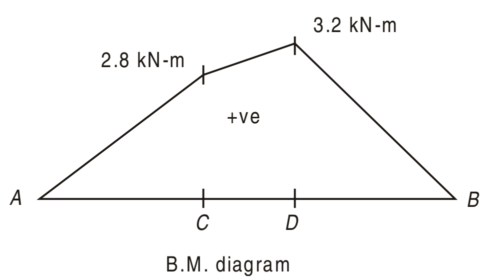



**Fig. 5.20**




**Solution**

For reactions $R_A$ and $R_B$

For $\Sigma M_A = 0$

$R_A$ (0) + 2 ×1 + 4 × 1.5 – 2.5 $R_B$ = 0

$$R_B = \frac{8}{2.5} = 3.2 \text{ kN}$$

But $R_A + R_B = 6$

$$R_A = 6 - R_B = 6 - 3.2$$
$$= 2.8 \text{ kN}$$
$$R_A = 2.8 \text{ kN}$$

For S.F. diagram

$$F_A = R_A = 2.8 \text{ kN}$$
$$F_C = 2.8 - 2$$
$$= 0.8 \text{ kN}$$
$$F_D = 0.8 - 4$$
$$= -3.2 \text{ kN}$$
$$F_B = -3.2 \text{ kN}$$

For B.M. Diagram :

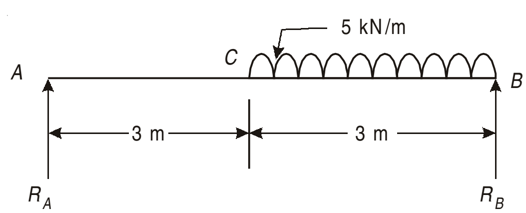
$$M_A = M_B = 0$$
$$M_C = 2.8 \times 1 = 2.8 \text{ kN-m}$$
$$M_D = 3.2 \times 1 = 3.2 \text{ kN-m}$$

**Example 5.7.** *A simply supported beam 6 m long is carrying a uniformly distributed load of 5 kN/m over a length of 3 m from the right end. Draw S.F. and B.M. diagrams for the beam and also calculate maximum B.M. on the section.*

**Solution**

For $\Sigma M_A = 0$

$$R_A(0) + 5(3)\left(3 + \frac{3}{2}\right) - 6R_B = 0$$




(a)

$$R_B = 0$$
$$R_B = 11.25 \text{ kN}$$
$$R_A = 3.75 \text{ kN}$$

For $\Sigma V = 0,$

We have $R_A + R_B = 5 \times 3 = 15 \text{ kN}$

$\therefore$ $R_A = 15 - 11.25 = 3.75 \text{ kN}$

For S.F. Diagram :

$$F_A = R_A$$
$$= 3.75 \text{ kN}$$

$$F_C = 3.75 \text{ kN}$$
$$F_B = 3.75 - (5 \times 3)$$
$$= -11.25 \text{ kN}$$

For B.M. diagram :

$$M_A = M_B = 0$$
$$M_C = 3.75 \times 3$$
$$= 11.25 \text{ kN-m}$$

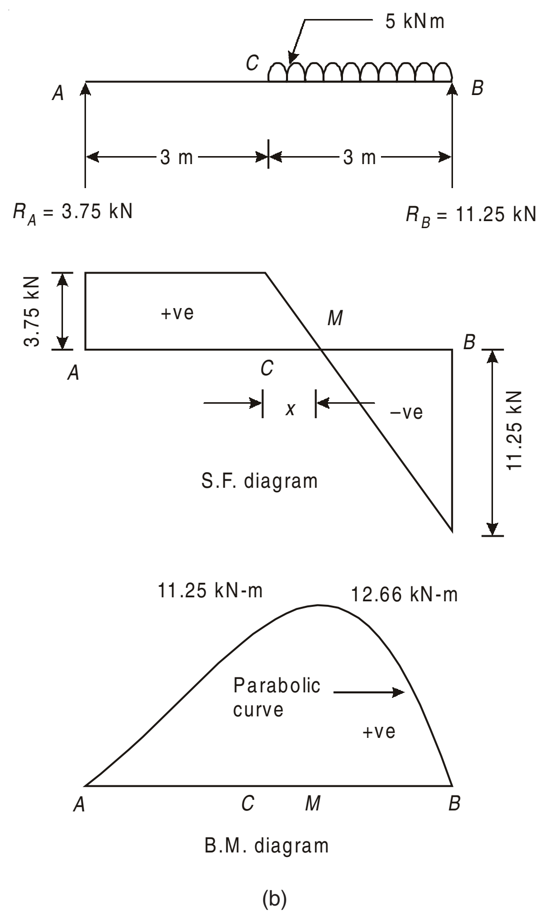



**Fig. 5. 21**

We know that the maximum bending moment will occur at *M,* where the S.F. changes sign. Let *x* be the distance between *C* and *M*. Now from the geometry of the figure bewteen *C* and *B,* we find that,

$$\frac{x}{3.75} = \frac{3-x}{11.25}$$

or
$$11.25\,x = (3-x)\,3.75$$

or $$x = 0.75 \text{ m}$$

$$M_{max} = 3.75\,(3 + 0.75) - 5 \times 0.75 \times 0.75\,(0.5)$$

$$= \mathbf{12.66\ kN\text{-}m}$$

**Example 5.8.** *Draw S.F. and B.M diagram for the simply supported beam shown in Fig. 5.22.* *(UPTU : 2004–2005)*

**Solution**

For symmetric U.D.L., we have

$$R_A = R_B = \frac{20 \times 18}{2} = 180 \text{ kN}$$

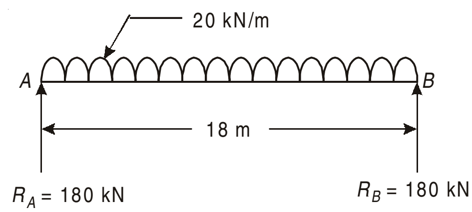
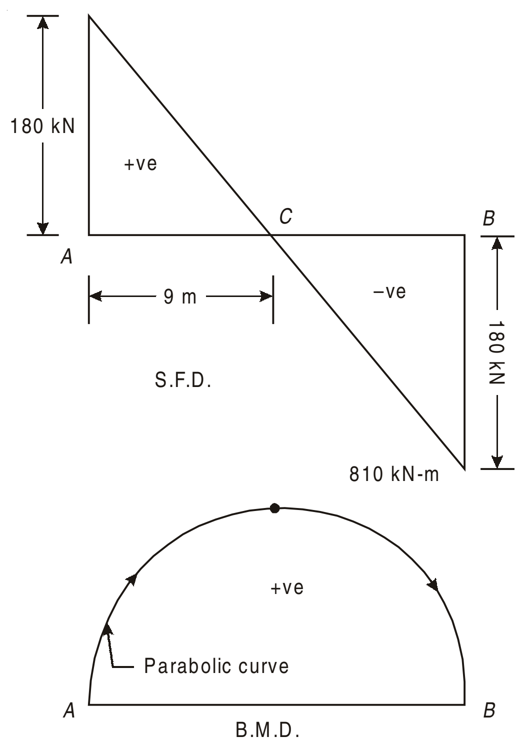



**Fig. 5.22 (a)**




(b)

For S.F. diagram :

$$F_A = R_A = 180 \text{ kN}$$

$$F_B = 180 - 20 \times 18 = -180 \text{ kN}$$

S.F. is zero (Changes sign) at 9 m from *A*

For B.M. diagram :

$$M_A = M_B = 0$$

$$M_C = 180 \times 9 - 20 \times 9 \times 4.5 = 810 \text{ kN-m}$$

**Example 5.9.** *A simply supported beam of 4 m span is carrying loads as shown in Fig. 5.26. Draw S.F. and B.M. diagrams for the beam.*

**Solution**

For $\Sigma M_A = 0$, we have

$$R_A(0) + 4(1.5) + 2 \times 1\left(1.5 + \frac{1}{2}\right) - 4R_B = 0$$

$$\therefore \qquad R_B = \frac{10}{4} = 2.5 \text{ kN}$$

$$R_A = 3.5 \text{ kN}$$

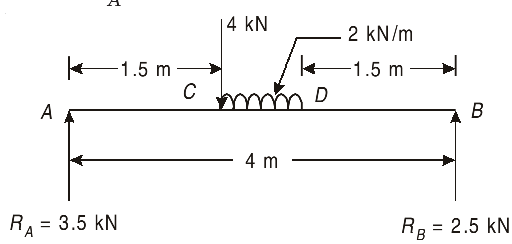
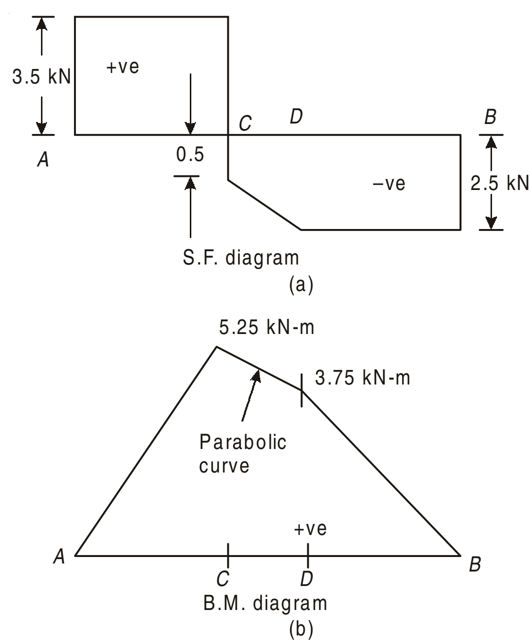



**Fig. 5. 23**



For S.F. diagram :

$$F_A = R_A = 3.5 \text{ kN}$$
$$F_C = 3.5 - 4 = -0.5 \text{ kN}$$
$$F_D = -0.5 - (2 \times 1) = -2.5 \text{ kN}$$
$$F_B = -2.5 \text{ kN} = -2.5 + 2.5 = 0 \text{ kN}$$

For B.M. diagram :

$$M_A = M_B = 0$$
$$M_C = 3.5 \times 1.5 = 5.25 \text{ kN-m}$$
$$M_D = 2.5 \times 1.5 = 3.75 \text{ kN-m}$$
$$M_D = 3.5 \times 2.5 - 4 \times 1 - 2 \times 1 \times 0.5$$
$$= 8.75 - 4 - 1$$
$$= 3.75 \text{ kN-m}$$

We know that the maximum bending moment will occur at $C$, where the shear force diagram changes sign, i.e., at $C$ (Fig. 5.23)

**Example 5.10.** *Draw shear force and bending moment diagrams for the beam shown in Fig. 5.27.*

(a) *For S.F. diagram*:

$$F_A = R_A = 10 \text{ kN}$$
$$F_C = 10 - 1 \times 5 - 8 = 3 \text{ kN}$$
$$F_D = -3 - 4 = -7 \text{ kN}$$
$$F_B = -7 + 11 = 4 \text{ kN}$$
$$F_E = 4 - (1.6 \times 2.5)$$
$$= 4 - 4 = 0$$

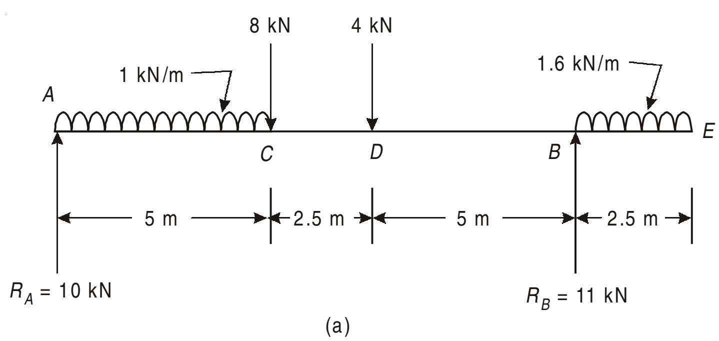
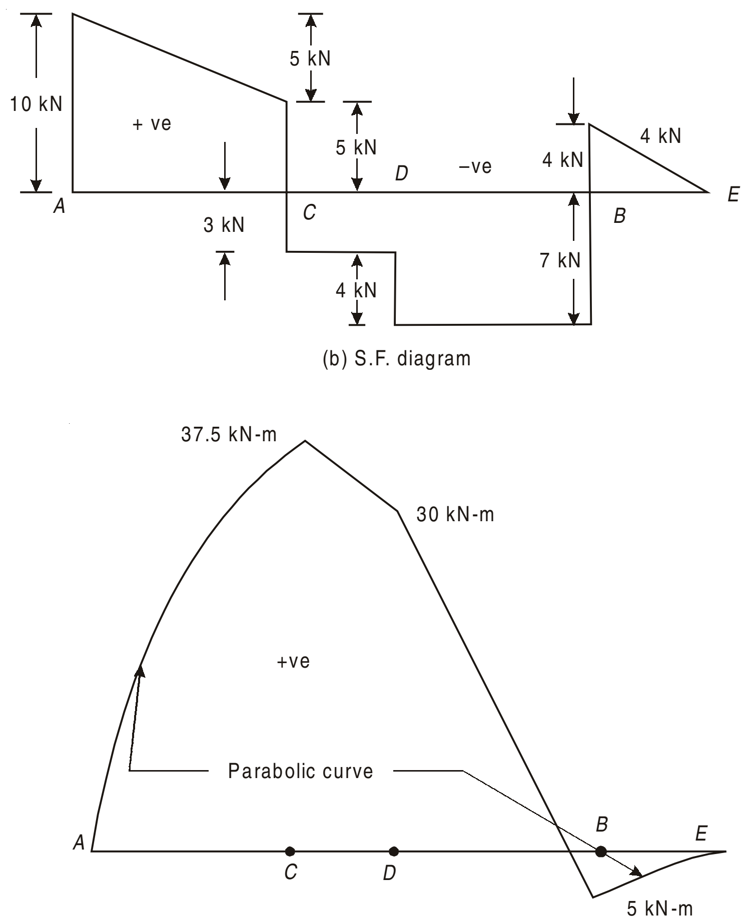



**Fig. 5.24**

(b) *For B.M. diagram* :

$$M_A = M_E = 0$$
$$M_C = 10 \times 5 - 1 \times 5 \times 2.5 = 37.50 \text{ kN-m}$$
$$M_D = 10 \times 7.5 - 1 \times 5\ (2.5 + 2.5)$$
$$= -8\ (2.5) + 75 - 45 = 30 \text{ kN-m}$$

$$M_B = -1.6 \times 2.5 \times 1.25 = -5 \text{ kN-m}$$




(b) S.F. diagram

(c) B.M. diagram

**Fig. 5.24**

**Solution**

For $\sum M_A = 0$, we have

$$R_A (0) + 1 \times 2.5 + 8 (5) + 4 (7.5) - 12.5 R_B + 1.6 \times 2.5 \left(12.5 + \frac{2.5}{7}\right) = 0$$

$$0 + 12.50 + 40 + 30 - 12.5 R_B + 55 = 0$$

$$\therefore \quad 137.5 - 12.5 R_B = 0$$

$$R_B = \frac{137.5}{12.5} = 11 \text{ kN}$$

$$R_A = 21 - 11 = 10 \text{ kN}$$

For $\Sigma V = 0$, we have

$$R_A + R_B = 1\times 5 + 8 + 4 + 1.6 \times 2.5$$
$$= 17 + 4 = 21 \text{ kN} \qquad ...(1)$$

**Maximum Bending Moment** : The maximum bending moment will occur at $C$ (+ve) and $B$ (–ve) because the S.F. changes sign at both these points.

## 5.5 ❑ POINT OF CONTRAFLEXURE

In an overhanging beam, there will be a point, where the bending moment will change sign from negative to positive or vice versa. Such a point where the bending moment changes sign, is known as a point of *contraflexure*.

**Example 5.11.** *An ourhanging beam ABC is shown in Fig. 5.25. Draw the shear force and bending moment diagrams and find the point of contraflexure.*

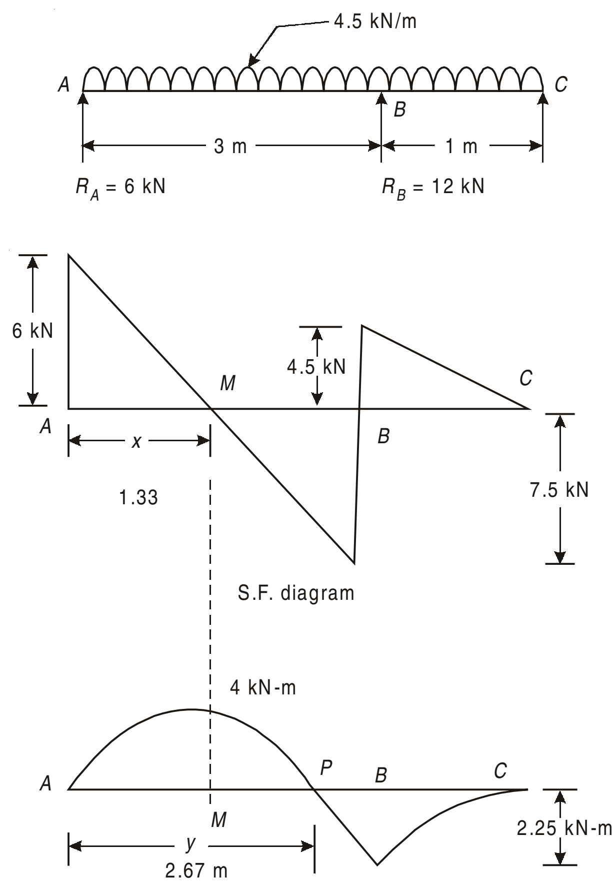



Fig. 5.25

**Solution**

For $\sum V = 0$

$$R_A + R_B = 4.5 \times 4 = 18 \text{ kN}$$

For $M_A = 0$, we have

$$R_A \times 0 + 4.5 \times 4 \times 2 - 3\,R_B = 0$$

$$\therefore \quad R_B = \frac{36}{3} = 12 \text{ kN}$$

$$\therefore \quad R_A = 18 - R_B = 18 - 12 = 6 \text{ kN}$$

For S.F. diagram:

$$F_A = R_A = 6 \text{ kN}$$

$$F_B = 6 - (13.50) + 12 = 4.5 \text{ kN},$$

$$F_C = 4.5 - (4.5 \times 1) = 0$$

For B.M. Diagram, we have

$$M_A = M_C = 0$$

$$M_B = 6 \times 3 - 4.5 \times 3 \times 1.5 = -2.25 \text{ kN-m}$$

We know that maximum B.M. will coccur at $M$, where S.F. changes sign.

Let $x$ be the distance between $A$ and $M$. From the geometry of the Fig. 5.25 between $A$ and $B$.

$$\frac{x}{6} = \frac{(3 - x)}{7.5}$$

or $$7.5\,x = 18 - 6x$$

or $$13.5x = 18$$

$$\therefore \quad x = 1.33 \text{ m}$$

$$\therefore \quad M_{\max} = (6 \times 1.39) - 4.5 \times 1.33 \times \frac{1.33}{2} = 4 \text{ kN-m}$$

**Point of Contraflexure.** Let $P$ be the point of contraflexure at a distance $y$ from the support $A$. As B.M. at $P$ should be zero.

$$M_P = 6 \times y - 4.5 \times y \times \frac{y}{2} = 0$$

or $$6\,y = 2.25\,y^2$$

or $$2.25\,y = 6$$

$$\therefore \quad y = \frac{6}{2.25}$$

$$= \mathbf{2.67 \ m}$$

**Example 5.12.** *The shear force diagram (S.F.D.) of a simply supported beam is given in Fig. 5.20. Calculate the support reactions of the beam and also draw the bending moment diagram of the beam.* (*UPTU : 2002*)

*Given : Simply supported beam, let supported at the ends A and B and reactions be $R_A$ and $R_B$. The shear force curve in the region AC and EB is an inclined straight line, it means the beam is loaded with U.D.L. in the region AC and EB. Let the intensity of UDL be $w_1$ and $w_2$ kN per m respectively.*

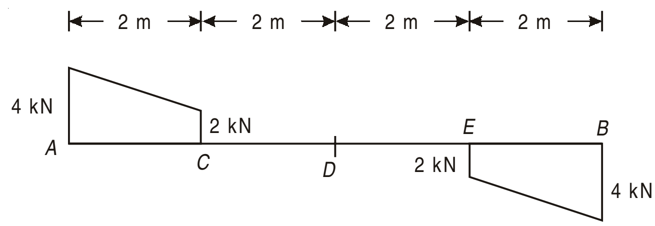



**Fig. 5.26**

**Solution**

For S.F.D.

$$F_A = R_A = 4 \text{ kN}$$
$$F_B = R_B = 4 \text{ kN}$$
$$F_C = 4 - (w_1 \times 2 + 2) = 4 - 2 - 2w_1$$
$$w_1 = \frac{2}{2} = 1 \text{ kN/m}$$
$$F_D = 0$$

Hence no load at $D$.

$$F_E = -2 \text{ kN} = \text{Vertical line}$$

**Example 5.13.** *Draw B.M. and S.F. diagrams for the beam shown in Fig. 5.27 (a), indicating the values at all salient points.*

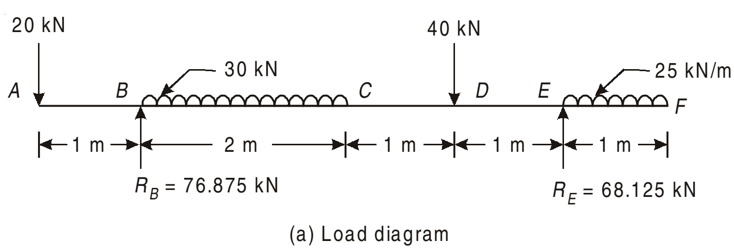
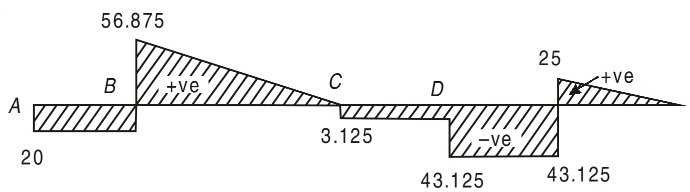
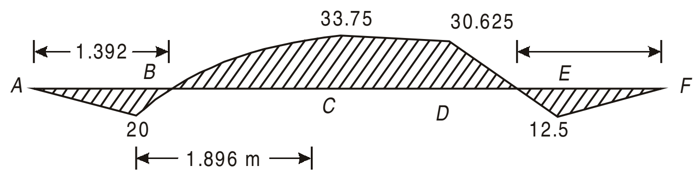



**Fig. 5. 27**



(b) SFD



(c) BMD

**Solution**

For $\sum M_B = 0$

$$-4\,R_E - 20 \times 1 + 30 \times 2 \times \frac{2}{2} - 40 \times 3 + 25 \times 1 \times 4.5 = 0$$

or $$-20 + 120 + 60 - 4\,R_E + 112.50 = 0$$

$$292.50 - 20 - 4\,R_E = 0$$

$$272.50 - 4\,R_{E,} \quad R_E = \frac{272.50}{4}$$

$\therefore$ $\quad R_E = 68.125$ kN

For $\sum V = 0$

$R_B = 20 + 30 \times 2 + 40 + 25 \times 1 - 68.125$

$R_B = 76.875$ kN

For S.F. $\quad F_A = -20$ kN

$F_B = -20 + 76.875$

$= 56.875$ kN

$F_C = 56.875 - 30 \times 2 = -3.125$ kN

$F_D = -3.125 - 40 = -43.125$ kN

$F_E = -43.125 + 68.125 = 25$ kN

Portion $AB$ : $\quad F_F = 25 - 25 = 0$

Measuring $x$ from $A$, $\quad F = -20$ kN, constant

$M = -20x$, linear variation

At $\quad x = 0, M_A = 0$

At $\quad x = 1$ m, $M_B = -20$ kN-m

Portion $BC$ :

Measuring $x$ from $B$, $\quad F = -20 + 76.875 - 30\,x$ , linear variation

At $\quad x = 0, F = 56.875$ kN

At $x = 2$ m, $F = -3.125$ kN

The point of zero shear force is given by

$$0 = -20 + 76.85 - 30x$$

or $x = 1.896$ m from $B$.

At distance $x$ from $B$ the moment is given by

$$M = -20(x+1) + 76.875x - 30x\frac{x}{2}$$

$$M_A = M_F = 0, M_B = -20 \times 1 = -20 \text{ kN-m}$$

$$M_C = -20 \times 3 + 76.875 \times 2 - 30 \times 2 \times 1$$
$$= 153.75 - 120 = 33.75 \text{ kN-m}$$

$$M_D = (25 \times 1 \times 1.5) + 68.125 \times 1$$
$$= -37.5 + 68.125 = 30.625 \text{ kN-m}$$

$$\boldsymbol{M_E = -25 \times 1 \times \frac{1}{2} = -12.50 \text{ kN-m}}$$

$$M = -20 + 56.875x - 15x^2, \text{ parabolic variation}$$

At $x = 0$, $M_B = -20$ kN-m

At $x = 2$ m, $M_C = -20 + 56.875 \times 2 - 15 \times 4$

$$M_C = 33.75 \text{ kN-m}$$

Maximum moment occurs where SF = 0, i.e., at $x = 1.896$ m

$$M_{max} = -20 + 56.875 \times 1.876 - 15 \times 18.96^2$$
$$= 33.913 \text{ kN-m}$$

The bending moment is changing its sign in this portion. Hence the point of contraflexure exists in this portion. It is given by

$$0 = -20 + 56.875x - 15x^2$$
$$x = 0.392 \text{ m},$$

i.e., the point of contraflexure is at 1.392 m from the free end $A$.

Portion *CD* :

Measuring $x$ from $F$,

$$\text{Shear force} = 25 \times 1 - 68.125 - 40$$
$$= 3.125 \text{ kN, constant}$$

$$M = -25 \times 1(x - 05) + 68.125(x-1) - 40(x-2),$$

linear variation

At $x = 3$ m, $M_C = 33.75$ kN-m

At $x = 2$ m, $M_D = -25 \times 1.5 + 68.125$

$\therefore$ $M_D = 30.625$ kN-m

Portion *DE*:

Measuring $x$ from free end $F$,

$$\text{Shear force} = 25 - 68.125$$
$$= -43.125 \text{ kN, constant}$$

$$M = -25(x - 0.5) + 68.125(x-1), \text{ linear variation}$$

At $x = 2$ m, $M_D = 30.625$ kN-m

At $x = 1$m, $M_E = -12.5$ kN-m

Portion *EF* :

Measuring $x$ from free end $F$,

Shear force $F = 25\,x$, linear variation

At $x = 0$, $F_F = 0$

At $x = 1$m, $F_E = 25$ kN

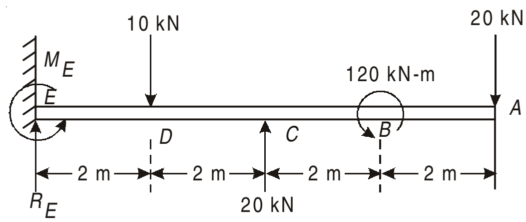
$$M = -25x, \frac{x}{2} \text{ parabolic variation}$$

At $x = 0$, $M_F = 0$

At $x = 1$ m, $\boldsymbol{M_E = -12.5}$ **kN-m**.

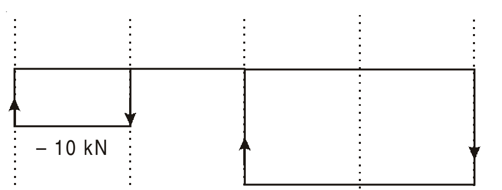
S.F. and B.M. diagrams are as shown in Fig. 5.27 (b) and (c) respectively.

**Example 5.14.** *Draw shear force and bending moment diagrams for the beam shown in Fig. 5.28* (*a*).

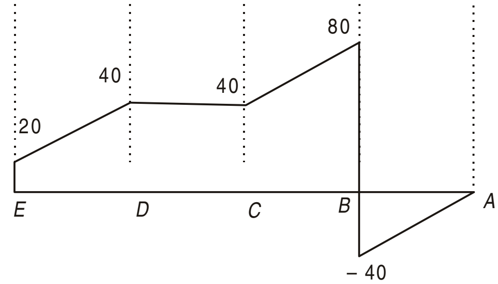



(a) Load diagram

**Fig. 5.28**




(b) Shear force diagram




(c) Bending moment diagram

**Solution**

(Solve the problem by considering portion of beam right to end $E$)

(1) **Find Reaction as**

$$\Sigma y = 0 ; ( + ve)$$
$$R_E - 10 - 20 + 20 = 0$$
$$R_E = 10 \text{ kN}$$
$$M_E = 0; ( + ve)$$
$$20 \times 8 + 10 \times 2 - 20 \times 4 - 120 = 0$$
$$ME = 20 \text{ kN-m, Anticlockwise}$$

(2) **Shear Force Portion AC**

S.F. = – 20 kN, constant

**Portion, CD** S.F. = – 20 + 20 = 0 ;

**Portion DE** S.F. = –20 +20 – 10 = – 10 kN (Constant)

(3) **Bending Moment**

**Portion *AB*: (0 < *x* < 2)**

BM = – 20 $x$—varies linearly with $x$

at $x$ = 0, $BM_A = 0$.

at $x$ = 2 m, $BM_B = -40$ kN-m

**Portion *BC* : (0 < *x* < 2)**

Take section between $B$ and $C$, at a distance $x$ from $B$.

$$\boxed{BM = -20(2 + x) + 120}$$

at $x$ = 0, $BM_B = +80$ kNm

at $x$ = 2, $BM_C = -20(4) + 120$

$BM_C$ = + 40 kN-m

**Portion *CD*:** Take section between $C$ and $D$, at a distance $x$ from $C$.

$$\boxed{BM = -20\,(4 + x) + 120 + 20\,(x)}$$

at $x$ = 0, $BM_C = -80 + 120 = +40$ kNm

at $x$ = 2 m, $BM_D = -120 + 120 + 40 = +40$ kNm

**Portion *DE*:** (0 < $x$ < 2)

Take section between $D$ and $E$, at a distance $x$ from $D$.

$$\boxed{BM = -20\,(6 + x) + 120 + 20\,(2 + x) - 10\,(x)}$$

at $x$ = 0, $BM_D = -120 + 120 + 40 - 0 = +40$ kNm

at $x$ = 2 m, $BM_E = -160 + 120 + 80 - 20 = +20$ kNm

**Example 5.15.** *A simply supported beam is subjected to various loadings as shown in Fig. 5.29 (a). Sketch the shear force and bending moment diagrams showing their values at significant locations.*

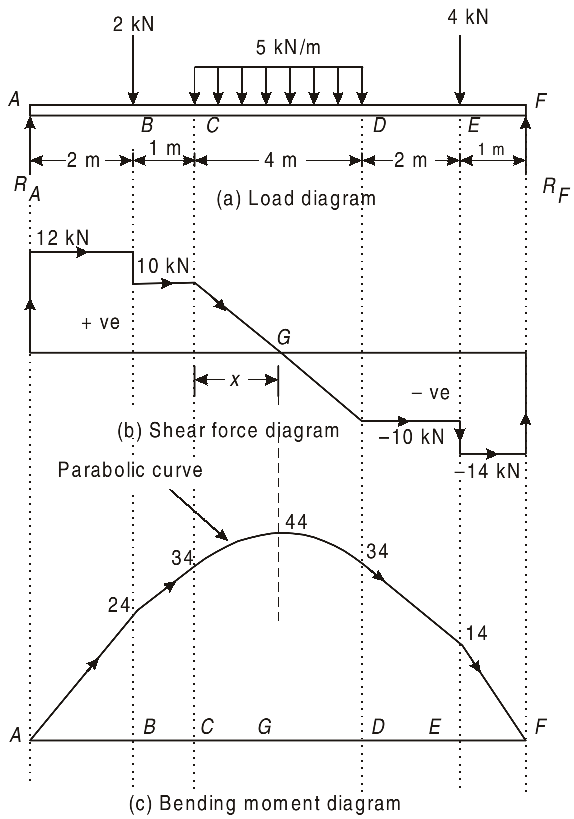



**Fig. 5.33**

**Reaction:**

For $\quad M_A = 0, \quad (\curvearrowright +\text{ve})$

$$2 \times 2 + (5 \times 4 \times 5) + 4 \times 9 - R_F \times 10 = 0,$$

$$R_F = 14 \text{ kN}$$

$$\Sigma y = 0, (\uparrow + \text{ve}), \quad R_A = 12 \text{ kN}$$

**Shear force:**

$$SF_A = R_A = 12 \text{ kN}$$

$$SF_B = 12 - 2 = 10 \text{ kN}$$

$$SF_C = 10 \text{ kN}$$

$$SF_D = 10 - 5 \times 4 = -10 \text{ kN}$$

The location of zero SF, from Fig. 5.29.

(b) $\dfrac{10}{x} = \dfrac{10}{(4-x)}$, $x = 2$ m from $C$. Let this point is $G$.

$$\begin{aligned} SF_E &= 10 - 4 = -14 \text{ kN} \\ SF_F &= -14 + 14 = 0 \text{ kN} \end{aligned}$$

Bending Moment:

$$\begin{aligned} BM_A &= 0,\ BM_B = 12 \times 2 = 24 \text{ kN-m} \\ BM_C &= 12 \times 3 - 2 \times 1 = 34 \text{ kN-m} \\ BM_D &= 12 \times 7 - 2 \times 5\ (5 \times 4)\ (2) = 34 \text{ kN-m} \\ \text{Max. BM, i.e.,}\quad BM_G &= 12 \times 5 - 2 \times 3 - 5 \times 2 \times 1 = 44 \text{ kN-m} \\ BM_E &= 12 \times 9 - 2 \times 7 - (5 \times 4) \times 4 = 14 \text{ kNm} \\ BM_F &= 0 \end{aligned}$$

**Example 5.16.** *Find out reactions at A and B and draw SF and BM diagram for simply supported beam shown in Fig. 5.30 (a).* (*UPTU : 2003*)

**Solution**

$$\begin{aligned} \Sigma M_A &= 0; \\ 30 \times 3 - R_B \times 6 &= 0; \\ R_B &= 15 \text{ kN},\ \text{and } R_A = 15 \text{ kN} \end{aligned}$$

**Shear Force:**

$$\begin{aligned} SF_A &= 15 \text{ kN},\ SF_C = 15 \text{ kN} \\ SF_D &= 15 - \frac{1}{2} \times 3 \times 20 = -15 \text{ kN} \end{aligned}$$

Location of zero shear,

i.e., $$15 - \frac{1}{2}x \cdot \frac{20}{3}x = 0$$

Between $C$ and $D$, Shear Curve is parabola with increasing nagative slope.

Bending Moment

For
$$\begin{aligned} BM_A &= 0 \\ BM_C &= 15 \times 1 = 15 \text{ kN-m} \end{aligned}$$

BM between $C$ and $D$,

$$BM_C = 15\,(1 + x) - \frac{1}{2}x \cdot \frac{20}{3}x \cdot \frac{x}{3}$$

Cubic equation

at $x = 0$, $BM_C = 15$ kNm

at $x = 0$, $BM_D = 30$ kNm

BM at $x = 2.12$ m, i.e. maximum

$$BM_E = 36.21 \text{ kNm}$$

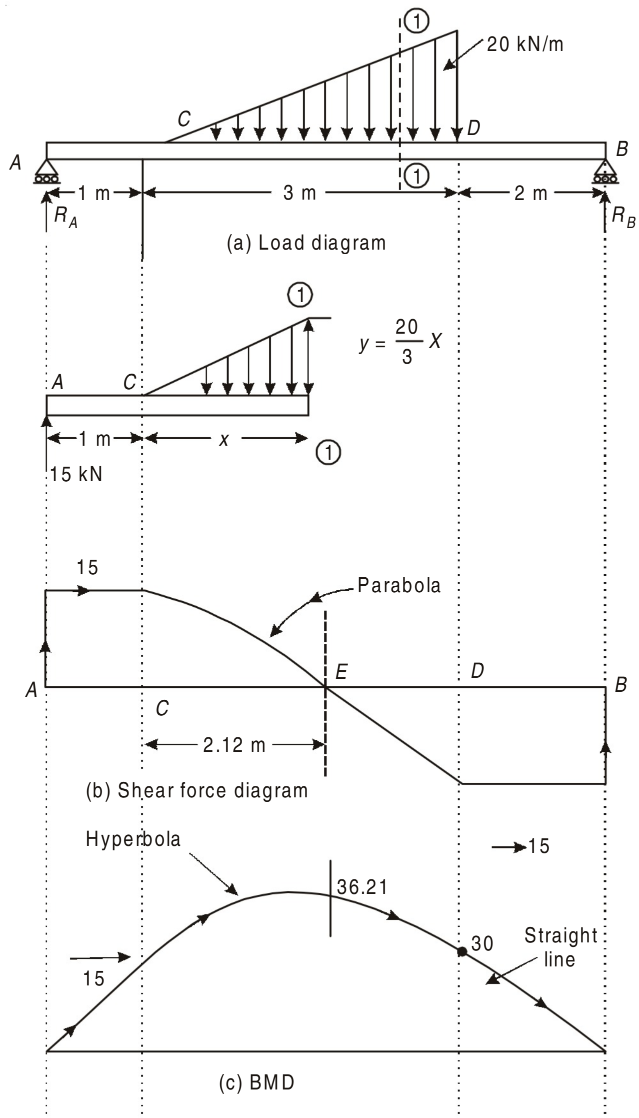



**Fig. 5.30**

**Example 5.17.** *Shear force diagram of a simply supported beam is given in Fig. 5.31* (*a*). *Calculate the support reactions and draw the bending moment diagram of the beam.* (*UPTU : 2004–2005*)

**Solution**

Abrupt change in SFD at *A*, *B* and *C* indicates the presence of point load or reactions at the points *A*, *B* and *C*.

SFD is horizontal between $AB$ and $BC$, it means there is no load between these points.

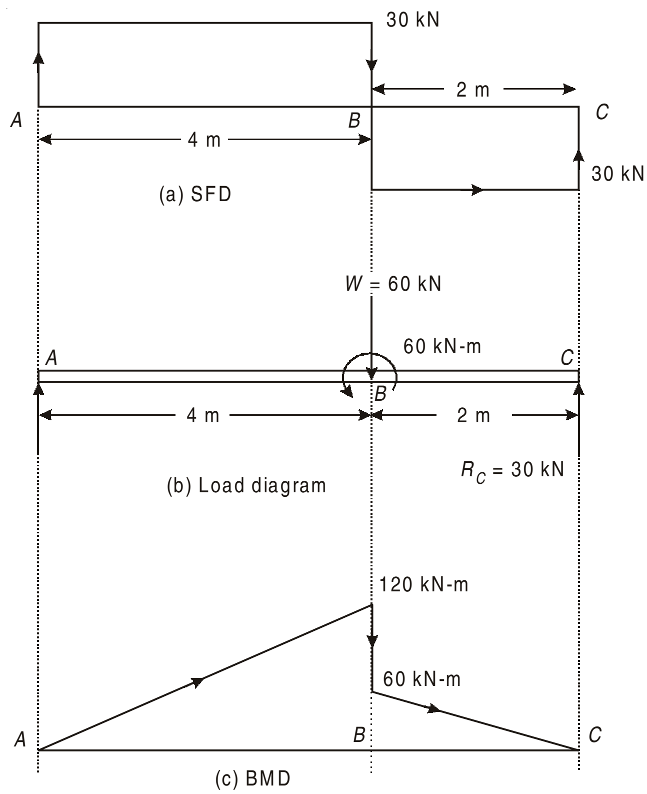



**Fig. 5. 31**

Let $R_A$ and $R_C$ be the reaction

$$SF_A = R_A = 30 \text{ kN}$$
$$SF_B = R_A - W = -30 \text{ kN}$$
$$\therefore \quad W = 60 \text{ kN}$$
$$SF_C = R_A - W + R_C = 0$$
$$R_C = 30 \text{ kN}$$

**Bending Moment:**

$$BM_A = 0;$$
$$BM_B = R_A \times 4 = 120$$

at $B$, anticlock-wise moment 60 kN-m will reduce the value of BM from 120 kN-m to 60 kN-m.

$$BM_C = R_A \times 6 - 60 \times 2 - 60 = 0$$

**Example 5.18.** *Draw the SF and BM diagram for the beam loaded and supported as shown in Fig. 5.32 (a).*

**Solution**

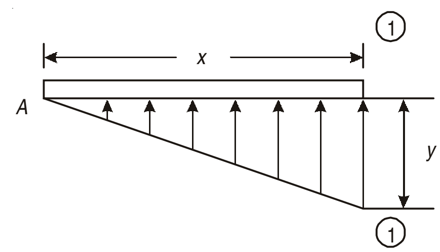
(i) $\sum M_A = 0$; ( +ve)

$$3W \times 4 + 8W \times 10 + 3W \times 16 - 500 \times 6 - 3200 \times 10 - 500 \times 14 = 0;$$

$$W = 300 \text{ N}$$

(ii) **SFD and BMD**

Take a section 1–1 anywhere between $A$ and $B$ at a distance $x$ from $A$. Draw $FD_B$ of left to section 1–1 $(0 < x < 6)$




(a)

$$y = \frac{300}{6}x = 50x$$

$$\boxed{SF_{1-1} = \frac{1}{2}x \cdot y}.$$

$$SF_{1-1} = 25\,x^2 \qquad \text{... (i)}$$

$$SF_A = 0;$$

$$SF_B = 900 \text{ N}.$$

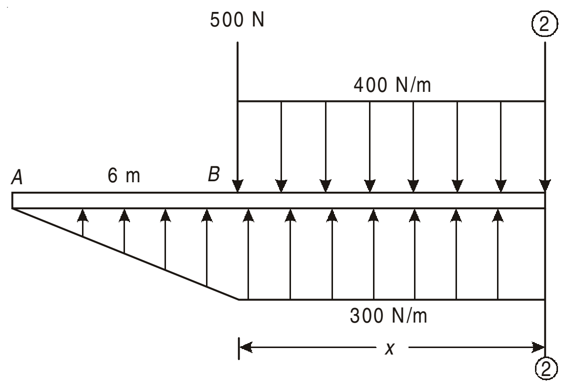
Equation (i) is a equation of parabola and its slope increases with increase in value of $x$.

$$BM_{1-1} = 25x^2\left(\frac{x}{3}\right) = \frac{25}{3}x^3 \qquad \text{...(ii)}$$

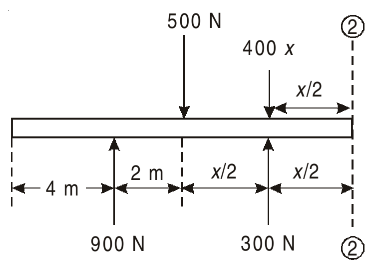
For $\quad BM_A = 0, BM_B = 1800$ Nm

Equation (ii) is a equation of cubic curve whose slope increases with increase in value of $x$.

(3) Take a section 2–2, anywhere between $B$ and $C$, at a distance $x$ from $B$. Draw FBD of left portion to section 2–2.




(b)




(c)

**Fig. 5.32**

$$SF_{2-2} = 900 - 500 - 400\,x + 300\,x$$

$$\boxed{SF_{2-2} = 400 - 100\,x} \quad \text{...(c)}$$

at $x = 0$, $SF_B = 400$ N,

at $x = 8$ m, $SF_C = -400$ N.

Location of zero shear.

$$400 - 100x = 0; \quad x = 4 \text{ m from } B.$$

$$BM_{2-2} = 900\,(2 + x) - 500x$$

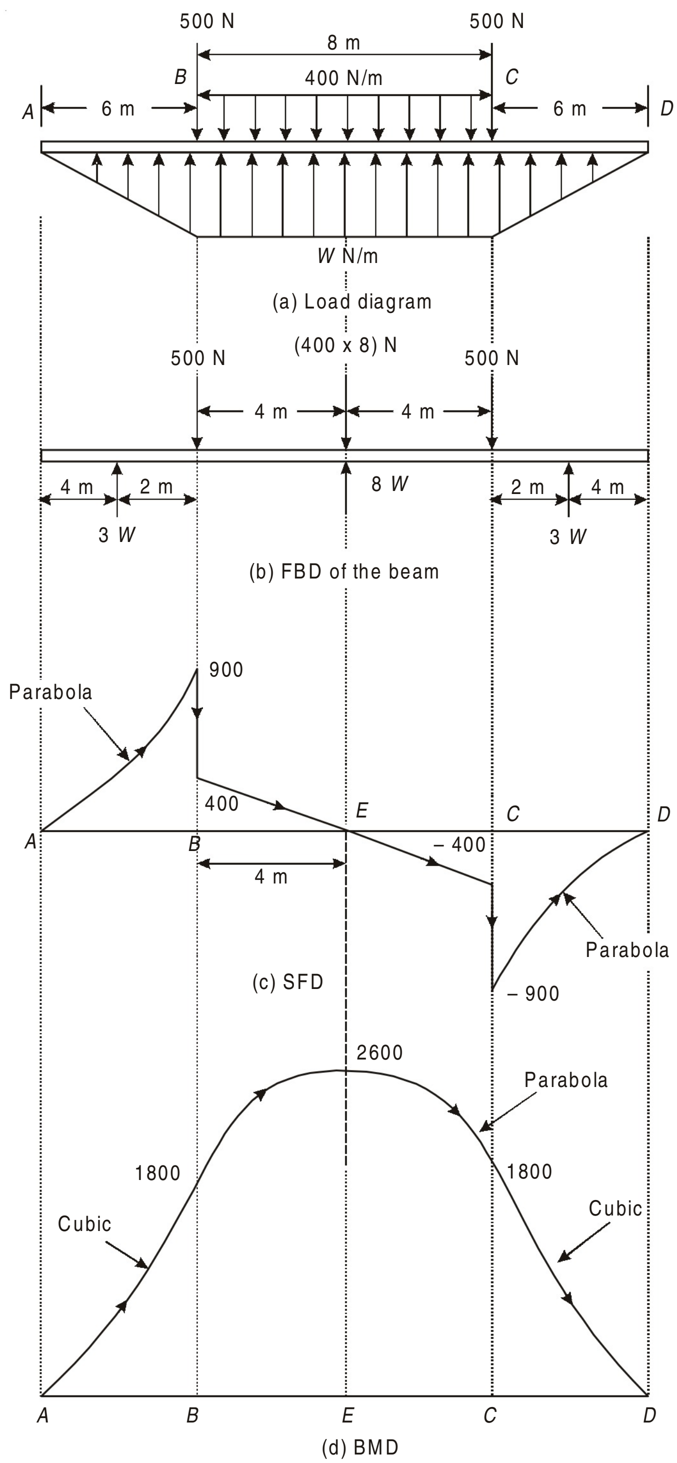
$$\boxed{BM_{2-2} = 1800 + 400\,x - 50\,x^2} \quad \text{...(d)}$$

at $x = 0$, $BM_B = 1800$ N-m

at $x = 8$ m, $BMc = 1800$ N-m Pl. Check

at $x = 4v$ m, $BM_{max} = 2600$ N-m

(4) Similarly the third section can be taken between $C$ and $D$, and SF and BM diagrams can be completed as shown in Fig. 5.33 (c) and (d)



500 N
500 N
8 m
B
C
400 N/m
A
6 m
6 m
D
W N/m
(a) Load diagram
(400 x 8) N
500 N
500 N
4 m
4 m
4 m
2 m
8 W
2 m
4 m
3 W
3 W
(b) FBD of the beam
900
Parabola
400
E
C
D
A
B
– 400
4 m
Parabola
(c) SFD
– 900
2600
Parabola
1800
1800
Cubic
Cubic
A
B
E
C
D
(d) BMD


**Fig. 5. 33**

## SUMMARY


Beam is a structural member, subjected to a system of loads at right angle to its axis.

Depending upon the type of supports, beams may be classified as - (i) cantilever, (ii) simply supported beam, (iii) continuous beam, (iv) fixed beam, (v) overhanging beam.

The shear force at the cross section of a beam is the unbalanced vertical force to the right or left of the section.

i.e. SF = $(\sum y)\, L$ or SF = $(\sum y) R$

The bending moment at the cross section of a beam is the algebraic sum of the moments of the forces, to the right or left of the section

i.e. BM = $(\sum M)\, L = (\sum M) R$

Upward acting forces or loads cause positive shear and moment and vice versa.

The SF and BM should be computed in terms of the forces to the left of the section being considered.

Relations among load, shear and moment are given by,

Load, $W = \dfrac{d}{dx}(\text{SF})$ and shear force, $\dfrac{d}{dx}(\text{BM})$

Change in shear force = $(\sum\text{SF})$ = (Area) Load

Change in bending moment = $(\sum\text{BM})$ = (Area) Shear

Intensity of load = Corresponding slope of shear diagram and intensity of shear = Corresponding slope of moment diagram.

The positive values of SF and BM are plotted above the base line and the negative below it.

A hinge in the compound beam can transmit a shear force but not a bending moment.

The maximum positive and negative bending moments in a beam may occur at the following places:

(i) A cross section where a concentrated load is applied and the shear force changes sign.

(ii) A cross section where the shear force equals zero,

(iii) A point of support where a vertical reaction is present

(iv) A cross section where a couple is applied.

The point of inflexion or point of contraflexure is a point, where BM changes its sign (or the BM is zero).

If a beam is subjected to a couple, the SF remains unaffected but the BM suddenly changes in magnitude equal to that of the couple.

If the beam is subjected to inclined loads, resolve them into vertical and horizontal components. The SF and BM are caused by vertical components only. The horizontal components cause only axial thrust.

## EXERCISE

**5.1.** Define a beam. What is a cantilever, a fixed beam and an overhanging beam ?

**5.2.** List the various types of loads to which a beam can be subjected. Differentiate between UDL and UVL.

**5.3.** Define shear force and bending moment.

**5.4.** What are sagging and hogging moments.

**5.5.** Discuss the utility, of shear force and bending moment diagrams.

**5.6.** What do you understand by the term, point of contraflexure ? Explain with reasons if it exists in a

(i) Cantilever

(ii) Simply supported beam

(iii) Overhanging beam.

**5.7.** Describe the effect of a couple on the SF and BM diagram of a beam.

**5.8.** Draw the SF and BM diagrams for the cantilever beam shown in Fig. 5.38.

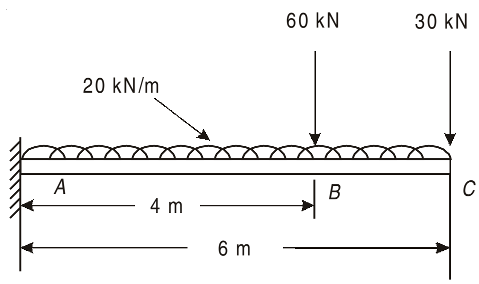



**Fig. 5.34**

**5.9.** A cantilever 3 m long carries loads 2 kN, 1.5 kN and 2.5 kN at free end, 1 m and 2 m from free end respectively. Draw SFD and BMD.

**[*Ans.* Max. S.F. = 6 kN, and Max. B.M. = –11.5 kN-m]**

**5.10.** A horizontal cantilever 6 m long carries loads of 20 kN and 30 kN at 2 m and 5m respectively from fixed end. A uniformly distributed load (U.D.L) of 100 kN is also spread over its entire length. Draw shear force diagran (SFD) and bending moment diagrams (BMD) indicating maximum values.

**[*Ans.* Max. S.F. = 150 kN, and Max. BM. = – 490 kN-m]**

**5.11.** A cantilever *ABCD* of 7 m length is fixed at *A* such that *AB* = *BC* = 2 m and *CD* = 3 m. It carries loads of 50 kN, 30 kN, and 20 kN at *B*, *C* and *D* respetively. In addition to it a U.D.L. of 10 kN/ m run between *AB* and 20 kN/m run between *CD* exist. Draw SFD. and BMD.

**[*Ans.* Max. S.F. = 180 kN, Max. B.M. = –710 kN-m]**

**5.12.** A simply supported horizontal beam, 5 m long carries concentrated loads of 70 kN, 90 kN, 50 kN and 80 kN at distances 1 m, 3 m, 4 m and 4,5 m respectively from the left hand support. Find support reactions and draw SFD. and BMD.

**[*Ans.* $R_A$ = 180 kN, $R_B$ = 110 kN, and Max. B.M. = 190 kN under 90 kN load]**

**5.13.** For all the problems given below draw SFD and BMD indicating values at salient point.

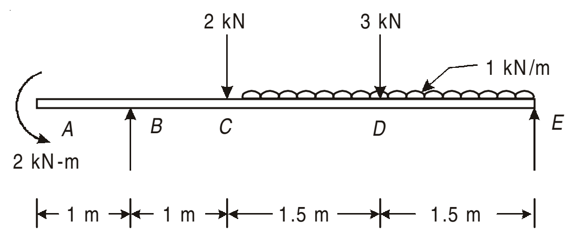



**Fig. 5. 35**

**[*Ans.* $F_B$ = 4.25 kN, $F_B$ = 3.75 kN, $M_{max}$ = 4.5 kN-m at *D* point of contraflexure at 1.57 m from *A*]**

**5.14.**

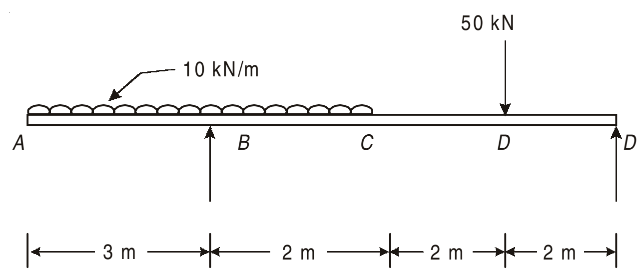



**Fig. 5.36**

**[*Ans.* $F_B$ = 70.83 kN, $F_E$ = 29.17 kN, $M_{max}$ = 45 kN-m at *B* Point of contraflexure at 4.313 m from *A*]**

**5.15.**

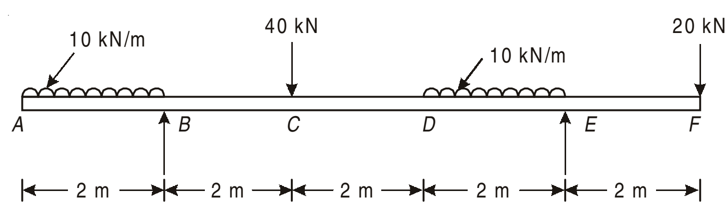



**Fig. 5.37**

**[*Ans.* $R_B$ = *m* 46.67 kN, $R_E$ = 53.33 kN, $M_B$ = – 20 kN-m, $M_E$ = – 40 kN-m, $M_C$ = 33.33 kN-m, Point of contraflexure: 2.75 m from *A* and 3.57 m from *F*]**

**5.16.**

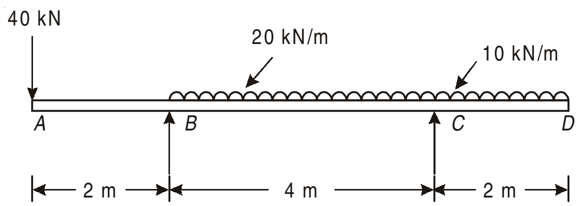



Fig. 5.38

**[*Ans.* $R_B$ = 95 kN, $R_C$ = 45 kN, $M_{max}$ = – 4.375 at 4.75 m from *A* No +ve moment anywhere]**

**5.17.**

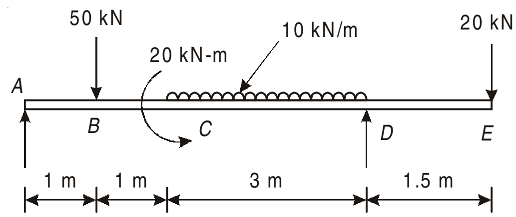



Fig. 5.39

**5.18.** Determine load *P* such that reactions at supports *A* and *B* are equal in the beam shown in Fig. 5.40. Draw SFand BM diagram marking the values at salient points.

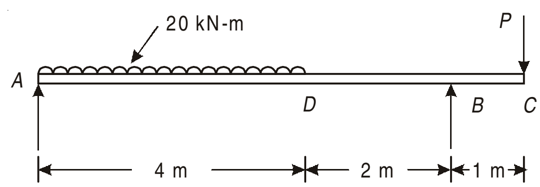



Fig. 5.40

**[*Ans.* *P* = 20 kN, $R_A$ = $R_B$ = 50 kN, $M_{max}$ = 62.5 kN-m at 2.5 m from *A*, Point of contraflexure 1.667 m from *C*]**

**5.19.** Draw the SF and BM diagrams for the beam shown in Fig. 5.41.

**[*Ans.* ($SF_A$= 108 N, $SF_B$ = –108, $SF_C$ = 0 $BM_A$= –172.8 Nm, $BM_B$ – 97.2 Nm $BM_C$ = 0]**

**5.20.** Draw SFD and BMD for the beam shown in Fig. 5.41.

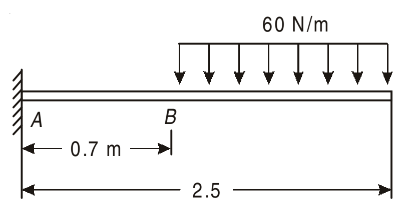



Fig. 5.41

**[*Ans.* ($SF_A$ = – 100 N, $SF_B$ = – 150 N, $SF_C$ = – 310 N, $BM_A$ = 0, $BM_B$ = – 100 Nm, $BM_C$ = – 560 Nm, $BM_D$ = – 870 Nm]**

**5.21.** Find the reaction at the fixed end of the cantilever loaded as shown in Fig. 5.42. Draw SFD and BMD.

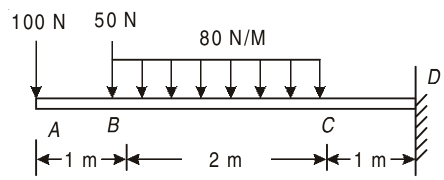



**Fig. 5.42**

**[*Ans.* ($SF_A$ = – 20N, $SF_C$ = – 50 N**
**$BM_B$ = – 40 + (–30) Nm, $BM_E$ = –290 Nm]**

**5.22.** Draw SFD and BMD of the beam shown in Fig. 5.43.

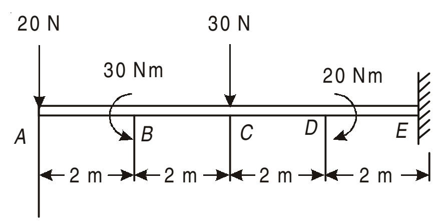



**Fig. 5.43**

**[*Ans.* ($SF_B$ = –4 kN: Parabolic curve)**
**$BM_B$ = – 5.33 kNm (Hyperbolic curve)]**

**5.23.** Draw shear force and bending moment diagrams and label all critical ordinates, including the maximum and minimum values for the beams given below:

In the following problems, draw load diagrams and bending moment diagrams corresponding to the given shear force diagrams. Specify values at all change of load positions and at all points of zero shear.

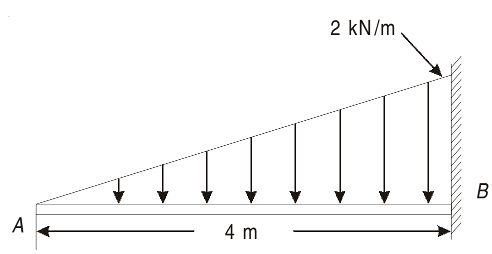



**Fig. 5.44**

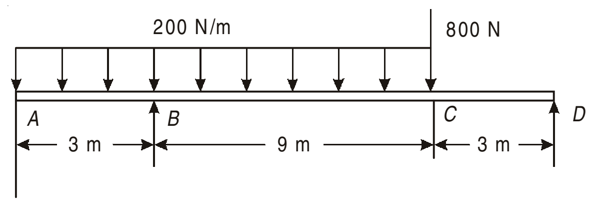



Fig. 5.45

**[*Ans.* $SF_B$ = – 600 N, and 1400 N ; $SF_C$ = – 400 N, and 1200 N; $BM_B$ = – 900 Nm; $BM_C$ = 3600 Nm; $BM_{max}$ = 4000 Nm]**

**5.24.**

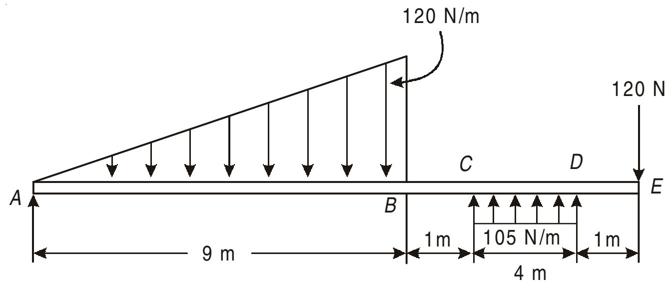



Fig. 5.46

**[*Ans.* $SF_A$ = 240 N; $SF_B$ = – 300 N, $SF_D$ = 120 N; $BM_B$ = 540 Nm; $BM_C$ = 240 Nm; $BM_D$ = – 120 Nm; $BM_{max}$ = 960 Nm; $BM_{min}$ = – 188.4 Nm]**

**5.25.**

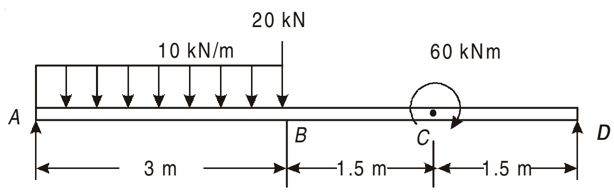



Fig. 5.47

**[*Ans.* $SF_A$ = 22.5 kN; $SF_B$ = –7.5kN; $SF_D$ = – 27.5 kN; Max BM = 25.31 kNm; $BM_B$ = 22.5 kNm; $BM_C$ = – 18.75 kNm; and + 41.25 kNm]**

**5.26.**

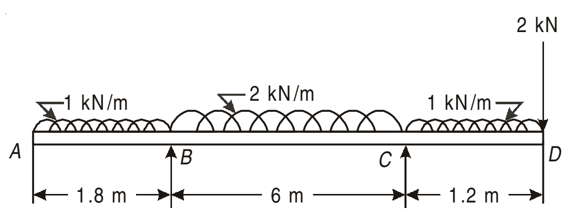



Fig. 5.48

**[*Ans.* $SF_B$ = – 1.8 and + 5.75 kN; $SF_C$ = – 6.25 and +3.2 kN; $BM_B$ = – 1.62 kNm; $BM_C$ = – 3.12 kNm; Max BM = 6.655 kNm; Point of cantraflexure at 2.1 m and 7.25 m from left end *A*]**

**5.27.**

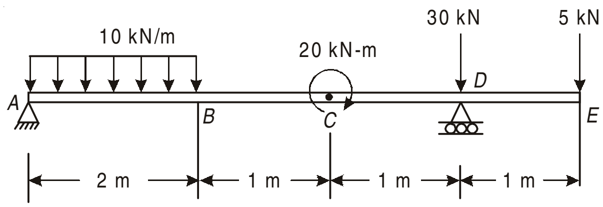



**Fig. 5.49**

**[*Ans.* $R_A$ = 8.75 kN; $R_D$ = 46.25 kN; $BM_D$ = – 5 kNm; $BM_B$ = – 2.5 kNm; Max BM = 3.86 kNm]**

**5.28.**

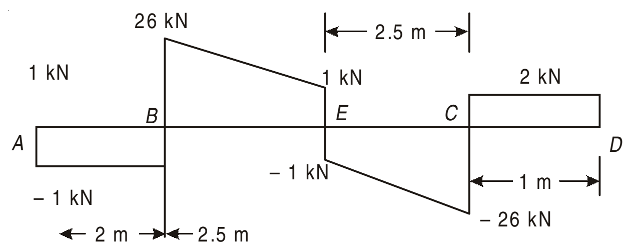



**Fig. 5.50**

**[*Ans.* $M_B$ = – 2 kNm; $M_C$ = – 2 kNm; $M_E$ = 31.75 kNm]**

**5.29.**

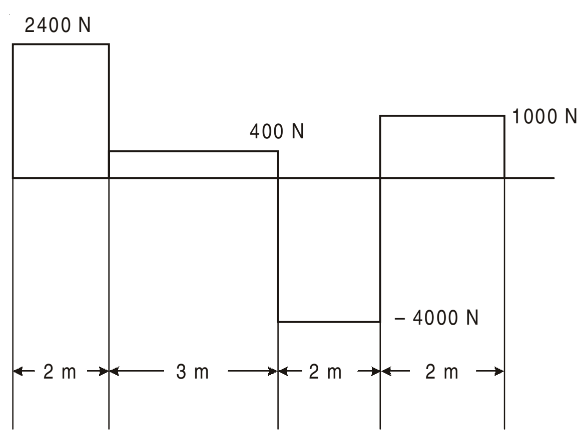



**Fig. 5.51**

**[*Ans.* Max. BM = 6000 Nm]**

**5.30.**

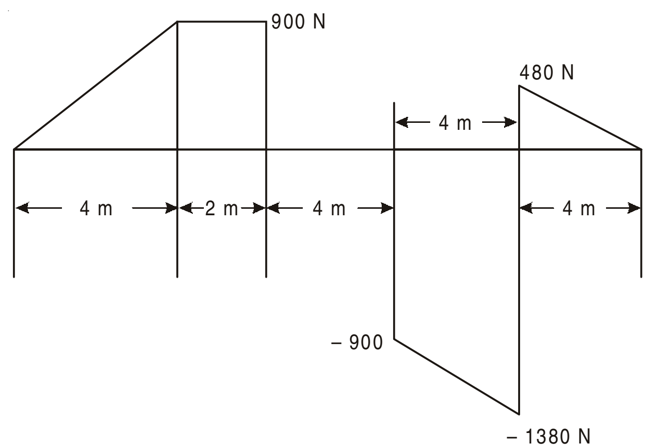



**Fig. 5.52**

**[*Ans.* Max. BM = 3600 Nm]**

# Torsion of Circular Shafts

CHAPTER 6

## 6.1 ❑ TORSION

A shaft is said to be in *torsion*, when equal and opposite forces are applied at the two ends of the shaft. The product of force applied (tangential to the ends of the shaft) and radius of the shaft is called *torque*, *turning moment* or *twisting moment* and the shaft is said to be subjected to *torsion*. Due to this torque the cross-section of the shaft is subjected to some *shear stresses* and *strains* in the material of the shaft.

## 6.2 ❑ SHEAR STRESSES PRODUCED IN A CIRCULAR SHAFT SUBJECTED TO TORSION

When the circular shaft is subjected to *torsion*, *shear stresses* are developed in the material of the shaft. To determine the magnitude of shear stress at any point on the shaft, consider a shaft fixed at one end *AA* and free at the end *BB* (Fig. 6.1).

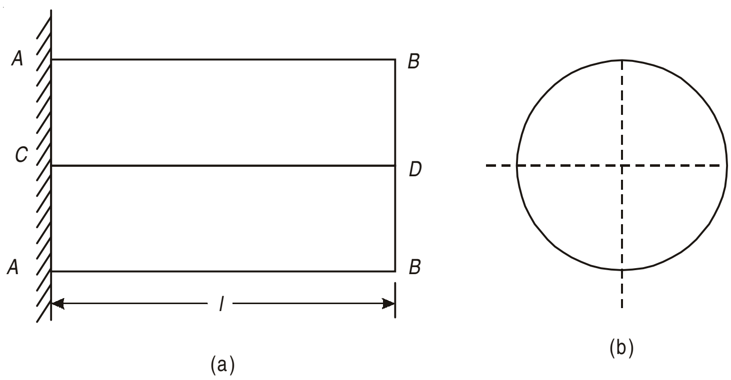



**Fig. 6.1** *Shaft fixed at one end before torque T is applied*

Let $CD$ is any line on the outer surface of the shaft. Now let the shaft is subjected to a *torque* $T$ at the end $BB$ (Fig. 6.2). As a result of this torque $T$, the shaft at

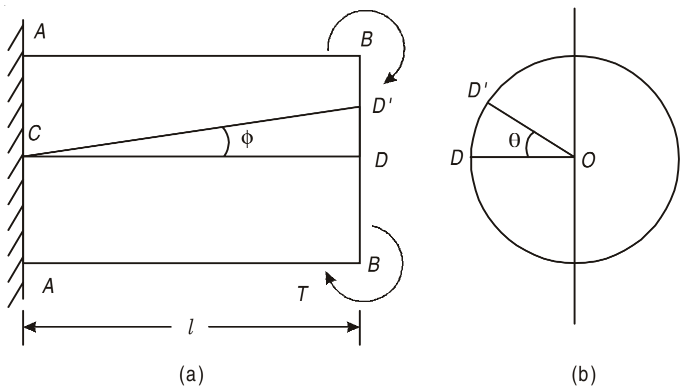



**Fig. 6.2** *Shaft fixed at AA and subjected to torque at BB*

the end $BB$ will rotate clockwise and every cross-section of the shaft will be subjected to *shear stresses*. The point $D$ will shift to $D'$ and hence line $CD$ will be deflected to $CD'$ (Fig. 6.2a). The line $OD$ will be shifted to $OD'$ (Fig. 6.2b)

Let,

$R$ = Radius of shaft
$l$ = Length of shaft
$T$ = Torque applied at the end $BB$
$\tau$ = Shear stress induced at the surface of the shaft due to torque
$G$ = Modulus of rigidity of shaft material
$\phi$ = $\angle DCD'$ also equal to shear strain
$\theta$ = $DOD'$ and is also called angle of twist

Now distortion at the outer surface due to torque, $T = DD'$ shear strain at outer surface = Distortion per unit length

$$= \frac{\text{Distortion at outer surface}}{\text{Shaft length}}$$

$$= \frac{DD'}{l}$$

$$= \frac{DD'}{CD} = \tan\phi = \phi$$

(If $\phi$ is very small then $\tan\phi = \phi$)

$$\therefore \text{ Shear strain at outer surface, } \phi = \frac{DD'}{l} \qquad \text{...(i)}$$

From Fig. 6.2(b) : Arc $DD' = OD.\ (\theta) = R\theta$ ($\because$ $OD = R$ = Radius of the shaft)
Substituting the value of $DD'$ in Eq. (i),
We get shear strain at outer surface,

$$\phi = \frac{R\theta}{l} \quad \text{...(ii)}$$

Now modulus of rigidity ($G$) of the material of the shaft is given as,

$$G = \frac{\text{Shear stress induced}}{\text{Shear strain produced}}$$

$$G = \frac{\text{Shear stress at outer surface}}{\text{Shear strain at outer surface}}$$

$$G = \tau/\frac{R\theta}{l} = \frac{\tau l}{R\theta}$$

$$\therefore \quad \frac{G\theta}{l} = \frac{\tau}{R} \quad \text{...(1)}$$

$$\therefore \quad \tau = \frac{G\theta R}{l}$$

Now for a given shaft subjected to a given torque ($T$), the values of $G$, $\theta$ and $l$ are constant. Hence shear stress produced is proportional to the radius $R$.

$$\therefore \quad \tau \propto R \text{ or } \frac{\tau}{R} = \text{constant} \quad \text{...(iii)}$$

If $q$ is the shear stress induced at a radius $r$ from the centre of the shaft then,

$$\frac{\tau}{R} = \frac{q}{r} \quad \text{...(2)}$$

But,

$$\frac{\tau}{R} = \frac{G\theta}{l}$$

$$\therefore \quad \boxed{\frac{\tau}{R} = \frac{G\theta}{l}}$$

$$= \frac{q}{r} \quad \text{...(3)}$$

From Eq. (iii), it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft. Hence, it is maximum at outer surface and zero at the axis of the shaft.

$$\frac{T}{J} = \frac{G\theta}{l} \text{ is known as stiffness equation}$$

$$\frac{T}{J} = \frac{\tau}{r} \text{ is known as strength equation}$$

$$\theta = \frac{Tl}{GJ} \quad GJ \text{ is known as torsional rigidity}$$

∴ Torsion equation :

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$$

Let us now consider transverse section of the shaft of radius *Ro*. For any concentric tube of thickness *dR* at radius *R* from the centre, let us consider shear stress τ (Fig. 6.3).

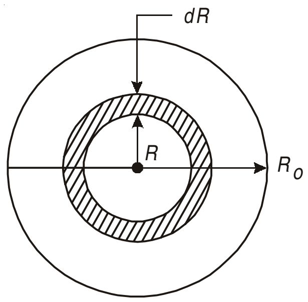



**Fig. 6.3**

Total twisting moment,

$$T = \int_o^{Ro} \tau(2\pi R \cdot dR)R$$

$$= \int_o^{Ro} \frac{G\theta R}{l} \times 2\pi R^2 dR$$

$$\text{From equation } \tau = \frac{G\theta R}{l}$$

$$T = \frac{2\pi G\theta}{l} \times \frac{R_0^4}{4}$$

$$= \frac{G\theta}{l} \times \frac{\pi R_0^4}{2}$$

$$= \frac{G\theta}{l} \times J$$

where, $J$ = polar moment of inertia of the shaft section

$$\therefore \qquad \frac{T}{J} = \frac{G\theta}{l}$$

From Eq. (3) i.e., $\frac{\tau}{R} = \frac{G\theta}{l}$, we get

$$\frac{T}{J} = \frac{G\theta}{l} = \frac{\tau}{R}$$

This derivation is based on the following assumptions :

1. The shaft is straight and of uniform cross-section along its length.
2. Torque $T$ is constant throughout the shaft length.
3. Material of the shaft is homogenous and follows Hook's law.
4. Cross-sections of the shaft which are plane before twist remain plane after twist.
5. All radii which are straight before twist will remain straight after twist.

## 6.3 ❑ COMBINED BENDING AND TORSION

We have studied effect of *pure bending*, but it is observed that the shafts transmitting power are also subjected to bending moment due to :

(i) Self weight,
(ii) Weight of the pulleys, and
(iii) Belt tension, etc.

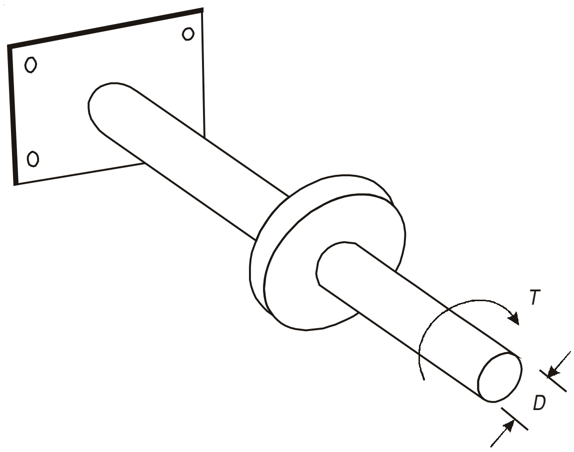



**Fig. 6.4**

Due to *bending moment*, *bending stresses* are set up in the shaft. We have seen that a shaft transmitting power torque or power is subjected to *shear stresses* also. Hence a shaft subjected to bending and torsion produces bending stress and shear stress respectively.

Consider any point on the cross-section of the shaft.

Let,

$T$ = Torque at the section,

$D$ = Diameter of the shaft

$M$ = Bending moment at the section.

The torque ($T$) will produce shear stress at the point and bending moment ($M$) will produce bending stress.

Let,

$\tau$ = Shear stress at the point produced by torque ($T$) and, $\sigma_b$ = bending stress at the point produced by B.M. ($M$)

Now, the shear stress at the point due to torque $T$.

$$\tau = \frac{T}{J} \cdot r = \frac{16T}{\pi D^3} \quad \text{...(i)}$$

$$\left[\because \frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l} \text{ and } J = \frac{\pi}{32} D^4\right]$$

Similarly the bending stress at a point due to bending moment ($M$) is given by

$$\sigma_b = \frac{M}{I} \cdot y = \frac{M}{I} \cdot \frac{D}{2} \quad \left(\because y = \frac{D}{2}\right)$$

$$= \frac{M}{\frac{\pi}{64} D^4} \cdot \frac{D}{2} = \frac{32M}{\pi D^3} \quad \text{...(2)}$$

The principal stresses are :

$$\sigma_{1,2} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_{max}^2} \quad \text{...(3)}$$

## 6.4 ❑ EFFECT OF END THRUST ON SHAFT

Besides the above two stresses i.e., shear stress ($\tau$) and bending stress ($\sigma_b$), sometimes the shafts are also subjected to end thrust. For example, in a propeller shaft, end thrust is developed, which causes a compressive stress in the shaft. Under such circumstances, total direct stress in the shaft.

= Stress due to B.M. ($M$) + Stress due to end thrust ($P$).

End thrust being compressive in nature will have negative (–) sign for evaluating total stress

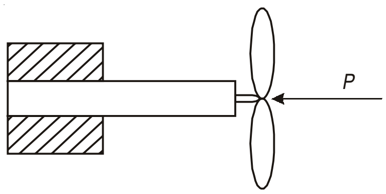



**Fig. 6.5**

Thus at the ends of vertical diameter, total direct stress,

$$\sigma_{\text{Direct}} = \pm \sigma_b - \frac{P}{A}$$

## 6.5 ❑ SOME IMPORTANT TERMS

### 6.5.1 Polar Moment of Inertia

The moment of inertia of a plane area with respect to an axis perpendicular to the plane of the figure is called as polar moment of inertia. It is denoted by $J$ having unit mm$^4$.

For solid shaft, $$J = I_{xx} + I_{yy} = 2I = \frac{2\pi d^4}{64} = \frac{\pi}{32} d^4$$

For hollow shaft, $$J = I_{xx} + I_{yy} = 2I = \frac{2\pi\left(D^4 - d^4\right)}{64} = \frac{\pi\left(D^4 - d^4\right)}{32}$$

### 6.5.2 Polar Sectional Modulus

It is defined as the quantity which is obtained by the polar moment of inertia of the cross section of the member by the distance of extreme fibre from centroidal axis. It is denoted by $Z_P$ having S.I. unit mm$^3$. Hence, $z_p = \dfrac{J}{R}$

For sold section $$Z_p = \frac{J}{R} = \frac{\pi}{32}\frac{d^4}{d/2} = \frac{\pi d^3}{16}$$

For hollow shaft $$Z_p = \frac{J}{R} = \frac{\frac{\pi}{32}\left(D^4 - d^4\right)}{D/2} = \frac{\pi\left(D^4 - d^4\right)}{16D}$$

### 6.5.3 Strength of a Shaft

(i) Solid $= \tau z_p = \dfrac{\tau\pi d^3}{16}$

(ii) Hollow = $\tau z_p = \dfrac{\tau\pi\left(D^4 - d^4\right)}{16D}$

where,

$D$ = External diameter

$d$ = Internal diameter

### 6.5.4 Torsional Stiffness

It is the torque required to produce unit angle of twist. Torsional stiffness = $\dfrac{G\theta}{L}$

having S.I. units N-mm/radian $\left(1° = \dfrac{\pi}{180}\text{Radian}\right)$

### 6.5.5 Torsional Flexibility

It is angle of twist to produce by unit torque applied.

$$\therefore \text{ Torsional flexibility} = \frac{L}{GJ} = \frac{1}{\text{Torsional stiffness}}$$

### 6.5.6 Torsional Rigidity

It is the product of modulus of rigidity($G$) and polar moment of inertia ($J$), denoted by $k$, having S.I. Units N.m$^2$

$$\therefore \quad K = GJ$$

## 6.6 ❑ POWER TRANSMITTED BY SHAFT

It is the product of average torque and corresponding angle turned per unit duration of time.

∴ Power = Average torque × Angle of rotation per sec.

$$P = T \times \frac{2\pi N}{60} = \frac{2\pi NT}{60} \text{watt,}$$

$$1 \text{ H.P.} = 746 \text{ watt}$$

where,

$T$ = Average (mean) torque

$N$ = No. of revolutions per minute (R.P.M.)

## 6.7 ❑ ELASTIC CONSTANTS

Modulus of elasticity : $E = \dfrac{\sigma}{\varepsilon} = \dfrac{P}{A}.\dfrac{L}{\delta L} = \dfrac{4PL}{\pi d^2 \delta L} = \dfrac{2G}{1+\mu}$

Bulk modulus : $K = \frac{E}{3}(1-2\mu)$

Shear modulus : $G = \frac{\tau}{\phi}$

Torsional formula : $\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{R}$

## 6.8 ❑ DESIGN OF SHAFT

If power transmitted by shaft is given, average torque can be calculated by,

$$P = \frac{2\pi N T_{ave}}{60} \text{ watt}$$

If the relation between maximum torque and mean torque is given, i.e.

$$T_{max} = K.T_{mean}$$

where $K$ = Total percentage of torque increament

### 6.8.1 To Find Suitable Diameter

1. Diameter based on shear stress (strength) : $\frac{T}{J} = \frac{\tau}{R}$

2. Diameter based on angle of twist (strength) : $\frac{T}{J} = \frac{G\theta}{L}$

Select greater value of shaft diameter for design purpose.

### 6.8.2 To Find Maximum Torque

1. Torque based on strength (Shear stress) : $\frac{T}{J} = \frac{\tau}{R}$

2. Torque based on angle of twist (Stiffness) : $\frac{T}{J} = \frac{G\theta}{L}$

Select smaller value as a safe torque

**Example 6.1.** *Find the internal and external diameters required for a hollow shaft, which is to transmit 40 kW of power at 240 rev./minute. The shear stress is to be limited to 100 MN/m*$^2$. *Take outside diameter to be twice the inside diameter.*

**Solution**

$\therefore$ Power $$P = \frac{2\pi NT}{60}$$

$$\therefore \quad 40 \times 10^3 = \frac{(2\pi \times 240 \times T)}{60}$$

$$\therefore \qquad T = 1591.55 \text{ N-mm}$$
$$= 1.592 \times 10^6 \text{ N-mm}$$

Using torsion formula, $\dfrac{T}{J} = \dfrac{\tau}{R}; \quad \dfrac{J}{R} = \dfrac{T}{\tau}$

$$\therefore \qquad Z_p = \frac{T}{\tau} = \frac{1.592 \times 10^6}{100}$$
$$= 1.592 \times 10^4$$

But
$$Z_p = \frac{\pi\left(D^4 - d^4\right)}{16D}$$
$$= 1.592 \times 10^4$$

or
$$\frac{\pi\left[(2d)^4 - d^4\right]}{16(2d)} = 1.592 \times 10^4$$

or
$$\frac{\pi\left[\left(16d^4 - d^4\right)\right]}{32d} = 1.592 \times 10^4$$

or
$$\frac{\pi\left[15d^3\right]}{32} = 1.592 \times 10^4$$

or $\qquad d^3 = 10807.6$

$\therefore \qquad d = 22.109 = 22.11$ mm

$\therefore \qquad D = 2d = 2 \times 22.11 = 44.22$ mm

**Hence internal diameter = 22.11 mm and external diameter = 44.22 mm**

**Example 6.2.** *A propeller shaft of a ship has 180 mm diameter, transmits 1000 kW at 120 r.p.m. The shaft is subjected to a bending moment of $10^4$ N-m and an end thrust of 100 kN. Calculate principal stresses.*

**Solution**

Let $T$ Newton metre be the torque transmitted

$$\because \qquad P = \frac{2\pi NT}{60}$$

$$\therefore \qquad T = \frac{P \times 60}{2\pi N} = \frac{1000 \times 60 \times 10^3 \times 7}{2 \times 22 \times 120}$$
$$= 79545.45 \text{ N-m}$$

Direct stress due to end thrust $= \dfrac{P}{A} = \dfrac{-10^5}{\frac{\pi}{4}(180)^2}$

$$= -3.928 \text{ N/mm}^2 = -3.93 \text{ N/mm}^2$$

Bending stress,
$$\sigma_b = \pm\frac{M}{I}\cdot y = \pm\frac{M}{I_y/y} = \pm\frac{M}{Z}$$

$$Z = \frac{I}{y} = \frac{\left(\frac{\pi}{64}d^4\right)}{\frac{d}{2}}$$

$$= \frac{\pi d^4}{64}\cdot\frac{2}{d} = \frac{\pi d^3}{32}$$

∴
$$\sigma_b = \pm\frac{M}{Z} = \frac{10^4\times 10^3}{\frac{\pi}{32}d^3}$$

$$= \pm\frac{32\times 10^4\times 10^3\times 7}{22(180)^3} = \pm\frac{224\times 10^4}{128304}$$

$$= \pm 17.46 \text{ N/mm}^2.$$

∵
$$\frac{T}{J} = \frac{\tau}{r}$$

∴ Shear stress,
$$\tau = \frac{T\cdot r}{J} = \frac{T}{\frac{J}{r}}$$

Now
$$\frac{J}{r} = \frac{\left(\frac{\pi d^4}{32}\right)}{r} = \frac{\frac{\pi d^4}{32}}{\frac{d}{2}}$$

$$= \frac{\pi d^4}{32}\cdot\frac{2}{d} = \frac{\pi}{16}d^3$$

∴
$$\tau = \frac{T}{\frac{J}{r}} = \frac{79545.45\times 10^3}{\frac{\pi}{16}d^3}$$

$$= \frac{79545.45\times 10^3\times 7\times 16}{22\times(180)^3}$$

$$= 69.43 \text{ N/mm}^2$$

$$\text{Total direct net stress}(\sigma_d) = \left(\pm\sigma_b - \frac{P}{A}\right)$$

$$= \pm 17.46 - 3.93$$

$$= 13.53 \text{ N/mm}^2$$

or $$= -21.39 \text{ N/mm}^2$$

*Principal stresses on tension side*

$$\sigma_{1,2} = \frac{\sigma d}{2} \pm \sqrt{\left(\frac{\sigma d}{2}\right)^2 + \tau_{max}^2}$$

$$= \frac{13.53}{2} \pm \sqrt{\left(\frac{13.53}{2}\right)^2 + (69.49)^2}$$

$$= \mathbf{6.76 \pm 69.82 = 76.58\ MPa\ or\ -63.06\ MPa}$$

*Principal stresses on compression sides*

$$\sigma_{1,2} = -\frac{21.39}{2} \pm \sqrt{-\left(\frac{21.39}{2}\right)^2 + (69.49)^2}$$

$$= -10.7 \pm 70.3$$

$$= 59.6 \text{ MPa or } = \mathbf{-81\ MPa}$$

**Example 6.3.** *A hollow circular shaft has the external diameter 100 mm and internal diameter 60 mm. The allowable shear stress in the shaft material is 55 N/mm². Determine the angle of twist in a length of 20 times the external diameter of the shaft. Take G = 8.5 × 10⁴ N/mm².* (*UPTU : 2006–07*)

*Given :*

$$D = 100 \text{ mm}$$

$$d = 60 \text{ mm}$$

$$\tau = 55 \text{ N/mm}^2$$

$$l = 20\,D = 20 \times 100 = 2000 \text{ mm}$$

$$G = 8.5 \times 10^4 \text{ N/mm}^2$$

$$\theta = ?$$

**Solution** We know that,

$$\frac{\tau}{r} = \frac{(G\theta)}{l}$$

Hence, $$\theta = \frac{\tau l}{RG} = \frac{55 \times 2000}{50 \times 8.5 \times 10^4}$$

$$= 0.02588 \text{ Radian}$$

$$= \frac{0.2588 \times 180}{\pi} = \mathbf{1.433°}$$

**Example 6.4.** *A solid shaft rotating at 500 r.p.m. transmits 300 kW. The maximum torque is 20% more than mean torque, material of shaft has the allowable shear stress of 65 MPa and modulus of rigidity of 81 GPa. The angle of twist in the shaft should not exceed 1° in 1 metre length. Determine the diameter of the shaft* (*UPTU : 2012–13*)

*Given :*

$$\text{Power, } P = 300 \text{ kW} = 300 \times 10^3 \text{ W}$$

$$\text{Speed, } N = 500 \text{ R.P.M}$$

$$T_{max} = 1.2\, T_{mean}$$

$$\text{Length } L, = 1\text{m} = 1000 \text{ mm}$$

$$\text{Shear stress, } \tau_{max} \ngtr 65 \text{ MPa}$$

$$\text{Angle of twist} \ngtr 1°$$

$$\text{Shear modulus, } G = 81 \text{ GPa} = 81 \times 10^3 \text{ MPa}$$

**Solution** We know that, power,

$$P = \frac{2\pi N T_{\text{mean}}}{60}$$

$$\therefore \quad 300 \times 10^3 = \frac{(2\pi \times 500 \times T_{\text{mean}})}{60}$$

$$\therefore \quad T_{\text{mean}} = \frac{(60 \times 300 \times 10^3)}{100\pi} = \frac{(18 \times 10^6)}{3141.592} = 5729.579 \text{ N-m} = 5.73 \times 10^6 \text{ N-mm}$$

$$\therefore \quad T_{\text{max}} = 1.2\, T_{\text{mean}} = 1.2\,(5.73 \times 10^6) = 6.88 \times 10^6 \text{ N-mm}$$

(i) Diameter based on shear stress : $\dfrac{T}{J} = \dfrac{\tau}{R}$

$$\therefore \quad \frac{6.88 \times 10^6}{\dfrac{(\pi D^4)}{32}} = \frac{65}{\dfrac{D}{2}}$$

or

$$6.88 \times 10^6 \times D = 65 \times 2 \times \pi D^4$$

$$\therefore \quad D^3 = \frac{(6.88 \times 10^6 \times 32)}{130\pi}$$

$$\frac{\left(220 \times 10^6\right)}{408.40} = 539.07 \times 10^3$$

$$\therefore \qquad D = 81.39 \text{ mm}$$

(ii) Diameter based on angle of twist : $\dfrac{T}{J} = \dfrac{G\theta}{L}$

$$\therefore \qquad \frac{6.88 \times 10^6}{J} = \frac{81 \times 10^3 \times \left(\frac{1 \times \pi}{180}\right)}{1000}$$

$$= \frac{81 \times 10^3 \times (0.0175)}{1000}$$

$$\therefore \qquad J = 4.85 \times 10^6$$

But, $\qquad J = \dfrac{\pi D^4}{32}$

$$\therefore \qquad \frac{\pi D^4}{32} = 4.85 \times 10^6$$

or $\qquad \pi D^4 = 32 \times 4.85 \times 10^6$

$$\therefore \qquad D = 83.84 \text{ mm}$$

$\therefore$ Safe diameter, $\qquad \mathbf{D = 83.84 \text{ mm}}$

**Example 6.5.** *Determine the dimensions of hollow shaft with a diameter ratio of 3 : 4, which is to transmit 60 kN at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m² and the angle of twist to 3.8° in the length of 4 m. For the shaft material, G = 80 GPa.* (UPTU : 2007–08)

*Given : Let external diameter = D and internal dia. = d*

$$\therefore \qquad \frac{d}{D} = \frac{3}{4}$$

$$\therefore \qquad d = 0.75\, D$$

*Power,* $\qquad P = 60 \text{ kW} = 60 \times 10^3 \text{ watt}$

$$N = 200 \text{ rev/min}$$

$$\tau \ngtr 70 \text{ MN/m}^2$$

$$L = 4 \text{ m} = 400 \text{ mm}$$

*Angle of twist,* $\qquad \theta \ngtr 3.8° = 3.8 \times \dfrac{\pi}{180}$ Radian

$$= 0.0663 \text{ Radian}$$

$$G = 80 \text{ GPa}$$

$$= 80 \times 10^3 \text{ MPa}$$

**Solution**

Let the diameters (External and internal) of the shaft be $D$ and $d$

Power transmitted, $P = \dfrac{2\pi NT}{60}$ ;

$$60 \times 10^3 = \frac{2\pi \times 200 \times J}{60}$$

$\therefore$ $T = 2.865 \times 10^3$ N-m

(i) Diameter based on shear stress : $\dfrac{T}{J} = \dfrac{\tau}{R}$

$$\frac{2.865 \times 10^6}{\frac{\pi}{32}\left(D^4 - d^4\right)} = \frac{70}{\frac{D}{2}}$$

or $2.865 \times 10^6 \times D \times 32 = \pi\,(D^4 - d^4)\,(2)\,(70)$

or $$\frac{\left(2.865 \times 10^6 \times 32\right)}{170\pi} = \frac{D^4 - d^4}{D}$$

$\therefore$ $$\frac{D^4 - d^4}{D} = 208.447 \times 10^3$$

$$\frac{\left[D^4 - (0.75D)^4\right]}{D} = 208.447 \times 10^3$$

or $0.6836\, D^3 = 208.447 \times 10^3$

$\therefore$ $D = 67.3$ mm

and $d = 0.75\,(67.3)$

$= 50.48$ mm

(ii) Diameter based on angle of twist : $\dfrac{T}{J} = \dfrac{G\theta}{L}$

$$\frac{2.865 \times 10^6}{\frac{\pi}{32}\left(D^4 - d^4\right)} = \frac{80 \times 10^3 \times 0.0663}{4000}$$

or $D^4 - (0.75D)^4 = 22 \times 10^6$

$\therefore$ $D^4 = 32.1837 \times 10^6$

and $D = 75.32$ mm

$d = 0.75(75.32) = 56.49$ mm

Select greater diameter to satisfy both conditions

$\therefore$ External diameter = **75.32 mm**

Internal dia = **56.49 mm**

**Example 6.6.** *A 20 mm diameter shaft of length 500 mm is fixed at one end. A torque, T is applied at its free end. The linear strain at surface of shaft at an angle of 45° from the axis is 4.0 × $10^{-3}$. Determine : (i) Torque, (ii) Angle of twist, and (iii) Shear stress in the shaft. Take, E = 2.0 × $10^{11}$ and G = 8.0 × $10^{10}$.*

*(UPTU : 2010–11)*

*Given :* 

$$\text{Diameter } d = 20 \text{ mm}$$
$$\text{Length } L = 500 \text{ mm}$$
$$\text{Linear strain } \phi = 45°$$
$$= \frac{\pi}{180} \times 45 \text{ Radian}$$
$$= 0.785 \text{ radian,}$$
$$E = 2 \times 10^{10}$$

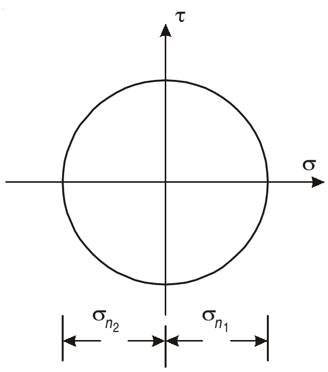



**Fig. 6.6**

**Solution** When the member is subjected to pure shear, principal stresses are

$$\sigma_1 = \tau$$

and $$\sigma_2 = -\tau$$

Using the relation, $$E = 2G(1 + \mu)$$

$$2 \times 10^{11} = 2 \times 8.0 \times 10^{10} (1 + \mu)$$

∴ $$\mu = 0.25$$

Linear strain, $$\varepsilon = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E}$$

$$= \frac{\tau}{E} - \mu\left(\frac{-\tau}{E}\right)$$

$$= \frac{\tau}{E}(1 + \mu)$$

$$4 \times 10^{-3} = \frac{\tau}{2 \times 10^5}(1 + 0.25)$$

$\therefore \qquad \tau = 640 \text{ N/mm}^2$

*Using torsion formula :*

$$\frac{T}{J} = \frac{\tau}{R}$$

or
$$T = \frac{J}{R}\tau = Zp\tau$$

$$\tau = \frac{\pi d^3}{16}.\tau$$

$$= \frac{\pi}{16} \times 20^3 \times 640$$

$$= 1.005 \times 10^6 \text{ N-mm}$$

Angle of twist;

$$\frac{G\theta}{L} = \frac{\tau}{R}$$

or
$$\frac{80 \times 10^3 (\theta)}{500} = \frac{640}{10}$$

$\therefore$
$$\theta = 0.4 \text{ rad} = 0.4 \times \frac{180}{\pi}$$

$$= 22.92°$$

Hence, $\qquad T = 1.005 \times 10^6$ N-mm

$\qquad \theta = 22.92°$,

and $\qquad \boldsymbol{\tau = 640 \textbf{ N/mm}^2}$

**Example 6.7.** *A solid shaft of 80 mm diameter is to be replaced by a hollow shaft of external diameter 100 mm. Determine the internal diameter of hollow shaft if the same power is to be transmitted by both the shafts at same angular velocity and shear stress.* (*UPTU : 2005–2006*).

*Given :*

*Solid shaft diameter,* $\quad ds = 80$ *mm*

*Hollow shaft, external*

*Diameter,* $\quad D = 100$ *mm*

*Power, angular velocity and shear stress are same.*

**Solution** As same power is to be transmitted at same speed and torque,

Hence, $\qquad T_1 = T_2$

Also shear stress is same,

Hence,
$$\frac{T_1 R_1}{J_1} = \frac{T_2 R_2}{J_2}$$

or
$$R_1 J_2 = R_2 J_1 \qquad (\because T_1 = T_2)$$

$R_1$ is for solid shaft

Let $d$ mm be the internal diameter of hollow shaft

Hence from the equation,

$$R_1 J_2 = R_2 J_1$$

$$\therefore \quad \frac{80}{2} \times \frac{\pi}{32}(80)^4 = \frac{100}{2} \times \frac{\pi}{32}\left[(100)^4 - (d)^4\right]$$

or $$40 \times \frac{\pi}{32}(80)^4 = 50 \times \frac{\pi}{32}\left[(100)^4 - (d)^4\right]$$

or $$0.8 \times (80)^4 = (100)^4 - d^4$$

$$\therefore \quad d = 90.60 \text{ mm}$$

**Internal diameter of the shaft should be 90.6 mm**

## 6.9 ❑ SHAFT SUBJECTED TO COMBINED BENDING MOMENT AND TWISTING MOMENT

When the shaft is subjected to resisting bending moment $M$ along with a twisting moment, $T$ the stresses induced in the shaft are due to combined effect of shear stress $\tau$ and bending stress $\sigma$.

Torque applied on the shaft due to the resultant effect of bending moment and twisting moment is called as *equivalent torque* ($Te$). Hence, $Te = \sqrt{M^2 + T^2}$ .

### 6.9.1 Derivation for Shaft Subjected to Combined Bending and Torsion

Consider a shaft subjected to the bending moment $M$ and twisting moment $T$

Bending moment by using bending formula :

$$\frac{M}{I} = \frac{\sigma}{y}$$

or $$M = \frac{\sigma I}{y} = \frac{\sigma I}{R} \qquad (\because y = R)$$

$$\therefore \quad M = \frac{\sigma I}{R} \text{ and,}$$

Bending stress, $$\sigma = \frac{MR}{I} \qquad \text{...(i)}$$

Twisting moment by torsional :

$$\frac{T}{J} = \frac{\tau}{R}$$

or $$T = \frac{\tau J}{R} = \frac{2\tau I}{R} \qquad (\because J = 2I)$$

Hence shear stress, $$\tau = \frac{TR}{2I} \quad \text{...(ii)}$$

If σ is the normal stress and τ is the shear stress the principal stresses are given by,

$$\sigma = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Substituting the values from Eqs. (i) and (ii)

$$\sigma = \frac{MR}{2I} \pm \sqrt{\left(\frac{MR}{2I}\right)^2 + \left(\frac{TR}{2I}\right)^2}$$

$$= \frac{MR}{2I} \pm \frac{R}{2I}\sqrt{M^2 + T^2}$$

$$\sigma = \frac{R}{2I}\left[M \pm \sqrt{M^2 + T^2}\right]$$

But $$2I = J$$

$$\therefore \quad \sigma = \frac{R}{J}\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$\therefore \quad \sigma = \frac{1}{Z_p}\left[M \pm \sqrt{M^2 + T^2}\right]$$

where

$$Z_p = \frac{J}{R} = \text{Polar sectional modulus}$$

$$\therefore \quad \sigma_{max} = \frac{1}{Z_p}\left[M + \sqrt{M^2 + T^2}\right]$$

$$\sigma_{min} = \frac{1}{Z_p}\left[M - \sqrt{M^2 + T^2}\right]$$

Equivalent bending moment :

$$\frac{M_e}{I} = \frac{\sigma_{max}}{R}$$

$$M_e = \frac{1}{R} \times \frac{R}{2I}\left[M + \sqrt{M^2 + T^2}\right]$$

$$= \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$$

Maximum shear stress

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sqrt{M^2 + T^2}}{Z_p}$$

Equivalent torque $\quad \dfrac{T_e}{J} = \dfrac{\tau_{max}}{R}$

$$\therefore \quad T_e = \frac{\sqrt{M^2+T^2}}{Z_p} \times \frac{J}{R} = \sqrt{M^2+T^2}$$

Location of principal plane:

$$\tan 2\theta = \frac{2\tau}{\sigma} = \frac{T}{M}$$

**Example 6.8.** *A flywheel weighing 500 kg is mounted on a shaft 75 mm diameter and midway between bearings 0.6 m apart. If the shaft is transmitting 30 kW at 360 R.P.M., calculate the principal stresses and maximum shear stress at the ends of a vertical and horizontal diameter in a plane close to the flywheel.* (*UPTU : 2010–11*)

*Given :*

$$W = 500 \text{ kg} = 500 \times 9.81 = 4905 \text{ N}$$
$$P = 30 \text{ kW} = 30 \times 10^3 \text{ W}$$
$$D = 75 \text{ mm}$$
$$N = 360 \text{ R.P.M.}$$

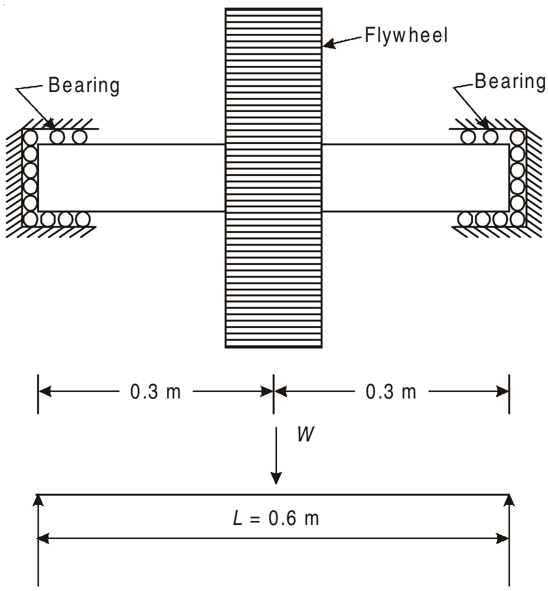



**Fig. 6.7**

**Solution** Maximum Bending moment,

$$M = \frac{WL}{4} = \frac{4905 \times 600}{4}$$

$$= 7.3575 \times 10^6 \text{ N-mm}$$

$\because$ Torque, $$T = \frac{2\pi NT}{60}$$

$\therefore$ $$30 \times 10^3 = \frac{2\pi \times 360 \times T}{60}$$

$$T = 795.77 \text{ N-m}$$

$$= 0.796 \times 10^6 \text{ N-mm}$$

Polar sectional modulus,

$$Z_p = \frac{\pi D^3}{16} = \frac{\pi(75)^3}{16}$$

$$= 82834.96 \text{ mm}^3$$

Principal stress, $$\sigma = \frac{M \pm \sqrt{M^2 + T^2}}{Z_p}$$

$$= \frac{1}{82834.96}\left[7.3575 \times 10^6 \pm \sqrt{\left(7.3575 \times 10^6\right)^2 + \left(0.796 \times 10^6\right)^2}\right]$$

$\therefore$ Principal stress, $\sigma = 88.29 \pm 89.34$

Maximum principal stress,

$$\sigma_{n1} = 88.29 + 89.34 = \mathbf{177.62\ N/mm^2}$$

Minimum principal stress,

$$\sigma_{n2} = 88.29 - 89.34 = \mathbf{-1.04\ N/mm^2}$$

Shear stress, $$\tau = \frac{\sqrt{M^2 + T^2}}{Z_p}$$

$$= \frac{\sqrt{\left(7.3575 \times 10^6\right)^2 + \left(0.796 \times 10^6\right)^2}}{82834.96}$$

$$= \mathbf{89.34\ N/mm^2}$$

**Example 6.9.** *A 800 mm long shaft of diameter 80 mm is supported on ball bearing at its ends. It carries a flywheel of weight 4 kN at its middle and transmitting a power of 24 kW at 240 R.P.M. Determine the principal stress at the ends of a vertical diameter at a section which is just before the middle of the beam.* *(UPTU : 2011 – 2012)*

*Given :*

| | | |
|---|---|---|
| *Diameter of shaft* | $D$ | $= 80\ mm$ |
| *Power* | $P$ | $= 24\ kW = 24 \times 10^3\ watt$ |
| *Speed* | $N$ | $= 240\ RPM$ |
| *Length* | $L$ | $= 800\ mm$ |
| *Point load* | $W$ | $= 4\ kN = 4000\ N$ |

**Solution** Maximum bending moment at centre,

$$M = \frac{WL}{4} = \frac{4000 \times 800}{4}$$

$$= 800 \times 10^3 \text{ N-mm}$$

Torque acting on shaft :

$$\because \quad P = \frac{2\pi NT}{60}$$

$$\therefore \quad 24 \times 10^3 = \frac{2\pi \times 240 \times T}{60}$$

or

$$T = \frac{24 \times 10^3 \times 60}{2\pi \times 240}$$

$$\therefore \quad T = 954.93 \text{ N-m}$$

$$= 945.93 \times 10^3 \text{ N-mm}$$

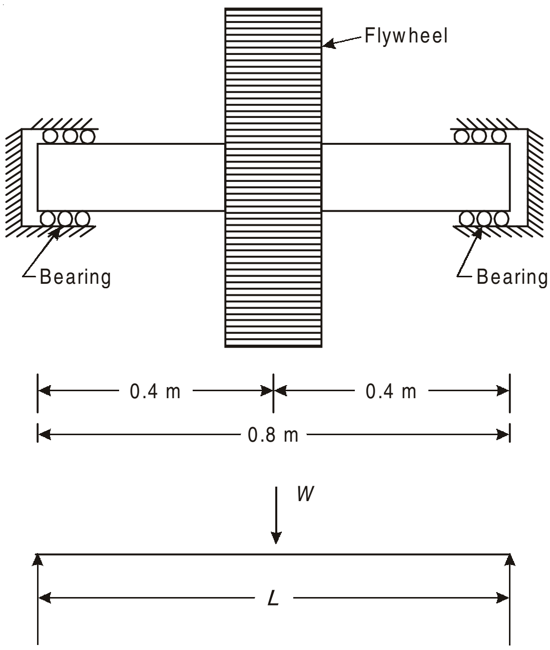



Fig. 6.8

Polar sectional modulus;

$$Z_p = \frac{\pi D^3}{16} = \frac{\pi(80)^3}{16}$$

$$= 100.53 \times 10^3 \text{ mm}^4$$

Maximum stress at centre

$$\sigma_{max} = \frac{M + \sqrt{M^2 + T^2}}{Z_p}$$

$$= \frac{800(10)^3 + \sqrt{800^2 + 945.93^2}}{100.53 \times 10^3}$$

$$\mathbf{= 20.28 \ N/mm^2 \ (Tensile)}$$

Minimum principal stress,

$$\sigma_{min} = \frac{M - \sqrt{M^2 + T^2}}{Z_p}$$

$$= \frac{800(10)^3 + \sqrt{800^2 + 945.93^2}}{100.53 \times 10^3}$$

$$\mathbf{= - \ 4.386 \ N/mm^2 \ (Compressive)}$$

**Example 6.10.** *A solid shaft is subjected to a bending moment of 2.3 kN-m and twisting moment of 3.45 kN-m. Find the diameter of the shaft if the allowable tensile and shear stresses for the shaft material are limited to 703 MN/m² and 421.8 MN/m² respectively.* *(UPTU : 2008–2009)*

*Given :*

$$M = 2.3 \text{ kN-m} = 2.3 \times 10^6 \text{ N-mm}$$

$$T = 3.45 \text{ kN-m} = 3.45 \times 10^6 \text{ N-mm}$$

$$\sigma = 703 \text{ MN/m}^2 = 703 \text{ N/mm}^2$$

$$\tau = 421.8 \text{ MN/m}^2 = 421.8 \text{ N/mm}^2$$

**Solution**

(i) Maximum bending stress :

$$\because \quad \sigma = \frac{1}{Z_p}\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$\therefore \quad 703 = \frac{1}{Z_p}\left[2.3 \times 10^6 + \sqrt{\left(2.3 \times 10^6\right)^2 + \left(3.45 \times 10^6\right)^2}\right]$$

$$\therefore \quad Z_p = 3275.1$$

Now, $$\frac{\pi d^3}{16} = 3275.1,$$

Hence, $d = 25.55$ mm

(ii) For maximum shear stress :

$$\because \quad \tau = \frac{\sqrt{M^2 + T^2}}{Z_p}$$

$$\therefore \quad 421.8 = \frac{\sqrt{\left(2.3\times10^6\right)^2 + \left(3.45\times10^6\right)^2}}{Z_p}$$

$$\therefore \quad \frac{\pi d^3}{16} = 9830.21$$

Hence, $d = 36.86$ mm

Provide the maximum diameter,

$$\mathbf{d = 36.86 \text{ mm}}$$

**Example 6.11.** *A hollow steel shaft 10 cm external diameter, 5 cm internal diameter, transmits 600 kN at 500 r.p.m. and is subjected to an end thrust of 60 kN. Find what bending moment may be safely applied to the shaft if the greater principal stress is not to exceed 100 N/mm².* (*UPTU : 2012–2013*)

*Given :*

*Speed of shaft* $N = 500$ *R.P.M.*

$\sigma_n = 100$ *N/mm²*

*External diameter* $D = 10$ *cm*

$= 100$ *mm*

*Internal dia.* $d = 50$ *mm*

*Power* $P = 600$ *kW* $= 600 \times 10^3$

*End thrust* $F = 60$ *kN* $= 60 \times 10^3$ *N*

**Solution**

Area, $$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(100^2 - 50^2\right)$$

$$= 5890.5 \text{ mm}^2$$

Moment of inertia, $$I = \frac{\pi}{64}\left(D^4 - d^4\right) = \frac{\pi}{64}\left(100^4 - 50^4\right)$$

$$= 4.602 \times 10^6 \text{ mm}^4$$

Polar moment of inertia, $J = 2I = 9.204 \times 10^6 \text{ mm}^4$

Direct stress, $$\sigma_d = \frac{F}{A} = \frac{60\times10^3}{5890.5} = 10.19 \text{ N/mm}^2$$

Power transmitted, $$P = \frac{(2\pi NT)}{60}$$

$$\therefore \quad 600 \times 10^3 = \frac{2\pi\times500\times T}{60}$$

$$\therefore \quad T = 11.46 \text{ kN-m} = 11.46 \times 10^6 \text{ N-mm}$$

Maximum shear stress due to torsion :

$$\frac{\tau}{R} = \frac{T}{J}$$

$$\therefore \quad \tau = \frac{(TR)}{J} = \frac{\left[(11.46)(10^6)(50)\right]}{9.204\times10^6}$$

$$= 62.25 \text{ N/mm}^2$$

Principal stress due to compression :

$$\because \quad \sigma_n = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau}$$

$$\therefore \quad 100 = \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + (62.25)^2}$$

or $$100 - \frac{\sigma}{2} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + (62.25)^2}$$

$$\left(100 - \frac{\sigma}{2}\right)^2 = \left(\frac{\sigma}{2}\right)^2 + (62.25)^2$$

or $$100^2 - 100\,\sigma + \frac{\sigma^2}{4} = \left(\frac{\sigma^2}{4}\right) + (62.25)^2$$

or $$100\,\sigma = (100)^2 - (62.25)^2$$

$$= 6124.8$$

$\therefore \qquad \sigma = 61.25 \text{ MPa}$

But $\qquad \sigma = \sigma_d + \sigma_b$

$\therefore \qquad 61.25 = 10.19 + \sigma_b$

$\therefore \qquad \sigma_b = 51.06 \text{ N/mm}^2$

But $\qquad \sigma_b = \dfrac{M}{I} \times y$

and
$$M = \frac{\sigma_b \times I}{y} = \frac{51.06\left(4.602 \times 10^6\right)}{50}$$
$$= 4.699 \times 10^6$$

$\therefore \qquad \mathbf{M = 4.7 \ kN\text{-}m}$

**Example 6.12.** *A circular steel rod AB of diameter d, length a, modulus of elasticity E and modulus of rigidity G, is loaded as shown in Fig. 6.9. Rigid bar BC of length b is rigidly fixed to AB at B such that BC is perpendicular to AB and lies in the horizontal plane. Find the deflection at point C due to*

(*i*) *bending of AB*

(*ii*) *torsion of AB*.

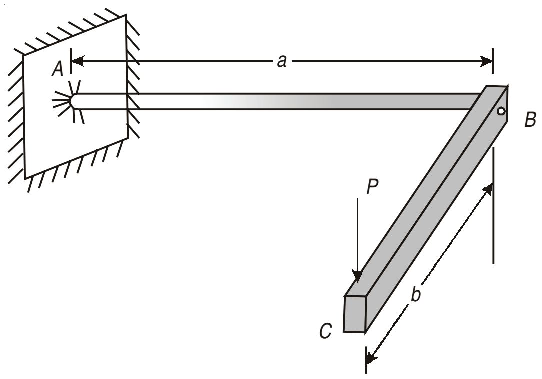



**Fig. 6.9**

**Solution** (i) Adding equal and opposite vertical loads $P$ and $P$ at $B$, Load acting at $B$ is $P$ downward.

Torsional moment acting on $AB$ is

$$T = P \times a$$

$$\text{Moment of Inertia } I = \frac{\pi}{64} \times d^4$$

$\therefore$ Deflection due to bending at $B$ is,

$$\delta_1 = \frac{Pa^3}{3EI} = \frac{Pa^3}{3E} \times \frac{64}{\pi d^4} = \frac{64}{3\pi E} \times \left(\frac{Pa^3}{d^4}\right)$$

As $BC$ is rigid, deflection at $C$ is also $\delta_1$.

$\therefore$ Deflection at $C$ due to bending of $AB$ is

$$\delta_1 = \frac{64}{3\pi E} \times \frac{Pa^3}{d^4} \text{ downward.}$$

(ii) Due to torsion of $AB$ :

$$\text{Torque } T = (P \times a)$$

$$\text{Polar M.I } J = \frac{\pi}{32}\left(d^4\right)$$

$$\therefore \quad \text{Angle of twist } \theta = \frac{TL}{GJ} = \frac{(P \times a \times b)}{G} \times \frac{32}{\pi d^4} = \frac{32}{\pi G} \times \frac{P_a b}{d^4}$$

$$\therefore \quad \text{Deflection of } C = b \times \theta = \frac{32}{\pi G} \times \left(\frac{P_a b^2}{d^4}\right)$$

$\therefore$ Deflection of $C$ due to torsion of $AB$ is,

$$\delta_2 = \frac{32}{\pi G} \times \frac{Pab^2}{d^4} \downarrow$$

**Example 6.13.** *A shaft is subjected to a torque T and bending moment M (both equal) and another similar shaft is subjected to a single torque* $T\sqrt{2}$ . *Show that ratio of maximum principal stresses in these two cases will be* $\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)$ *or 1.707.* *(UPTU : 2001–2002)*

*Given : Torque T and bending moment M both are equal*

**Note :** Maximum principal stress is produced by equivalent B.M. Formula for equivalent B.M. is:

**Solution**

$$M_E = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right)$$

In the first case : $M = T$ (given)

$$\therefore \quad (M_E)_1 = \frac{1}{2}\left(T + \sqrt{2T^2}\right)$$

$$\therefore \quad (M_E)_1 = \frac{1}{2} T\left(1 + \sqrt{2}\right) \quad \text{...(i)}$$

In the second case : $M = 0$, Torque $= \sqrt{(2)}T$

$$\therefore \quad (M_E)_2 = \frac{1}{2}\left(0+\sqrt{0+2T^2}\right)$$

$$(M_E)_2 = \frac{1}{2}\times\left(\sqrt{2}\right)T \quad \text{...(ii)}$$

As the shafts are identical, ratio of maximum principal stress will be same as ratio of equivalent moments.

$$\therefore \quad \text{Ratio} = \frac{(M_E)_1}{(M_E)_2} = \left(\frac{1+\sqrt{2}}{\sqrt{2}}\right) = \mathbf{1.707}$$

## EXERCISE

**6.1.** Write briefly on : 'Equivalent torque' (*UPTU : 2001–2002*)

**[*Ans.* Section : 6.9]**

**6.2.** Derive the relation to obtain polar modulus for hollow circular section. Also derive the equation for the power transmitted by shaft. (*UPTU : 2005–2006*)

**[*Ans.* Section : 6.5, 6.6]**

**6.3.** Determine equivalent bending moment for the shafts subjected to combined bending and torsion. (*UPTU : 2007 – 2008*)

**[*Ans.* Section : 6.9.1]**

**6.4.** What do you mean by strength of a shaft? (*UPTU : 2011 – 12*)

**[*Ans.* Section : 6.5]**

**6.5.** Write down the expression for power transmitted by shaft. (*UPTU : 2012 – 2013, 2005 – 2006*)

**[*Ans.* Section : 6.6]**

**6.6.** Compare the weights of hollow and solid shafts of equal lengths to transmit a given torque of the same maximum shear stress, if the inside diameter is 2/3 of the outside. (*UPTU : 2002 – 2003*)

**[*Ans. WH* = 0.6432 WT. solid]**

**6.7.** A circular bar *ABC*, 3 m long, is rigidly fixed at its ends *A* and *C*. The portion *AB* is 1.8 m long and 50 mm diameter and *BC* is 1.2 m long and 25 mm diameter. If a twisting moment of 680 Nm is applied at *B*, determine the values of the resisting moments at *A* and *C* and the maximum stress in each section of shaft. What will be the angle of twist of each portion ? For the material of shaft, $G$ = 80 GPa. (*UPTU : 2005 – 06*)

**[*Ans.* $\theta$ = 1.31°]**

**6.8.** A shaft is subjected to a torque $T$ and bending moment $M$ (both equal) and another similar shaft is subjected to a single torque $T\sqrt{2}$. Show that ratio of maximum principal stresses in these two cases will be 1.707. (*UPTU* : 2001 – 2002)

**6.9.** A flywheel weighing 500 kg is mounted on a shaft 75 mm diameter and midway between bearings 0.6 m apart. If the shaft is transmitting 30 kW at 360 R.P.M., calculate the principal stresses and the maximum shear stress at the ends of a vertical diameter in a plane close to the flywheel.

(*UPTU* : 2003 – 2004, 2010 – 11)

**[*Ans.* Example 6.8, $\sigma_1$ = 15.02 N/mm², $\sigma_2$ = –6.15 N/mm², $t_{max}$ = 10.585 N/mm²]**

**6.10.** Define : (i) Polar sectional modulus, (ii) Torsional stiffness, and (iii) Torsional flexibility. (*UPTU* : 2013 – 14)

**[*Ans.* Section : 6.5]**

**6.11.** A circular shaft of 80 mm diameter is required to transmit torque in a factory. Find the torque, which the shaft can transmit, if the allowable shear stress is 50 MPa. **[*Ans.* 5.03 kN-m]**

**6.12.** A solid shaft is required to transmit a torque of 6.5 kN-m. What should be the minimum diameter of the shaft, if the maximum shear stress is 40 MPa?

**6.13.** A solid shaft of 40 mm diameter is subjected to a torque of 0.8 kN-m. Find the maximum shear stress induced in the shaft. **[*Ans.* 63.7 MPa]**

**6.14.** A circular shaft of 80 mm diameter is required to transmit power at 120 r.p.m. If the shear stress is not to exceed 40 MPa, find the power transmitted by the shaft. **[*Ans.* 50.5 kW]**

**6.15.** A hollow shaft of external and internal diameters of 60 mm and 40 mm is transmitting torque. Find the torque it can transmit, if the shear stress is not to exceed 40 MPa. **[*Ans.* 1.36 kN-m]**

**6.16.** A solid circular shaft of 100 mm diameter is transmitting 120 kW at 150 r.p.m. Find the intensity of shear stress in the shaft.

**[*Ans.* $\tau$ = 39/mm² = 39 MPa]**

**6.17.** A hollow shaft is to transmit 200 kW at 80 r.p.m. If the shear stress is not to exceed 60 MPa and internal diameter is 0.6 of the external diameter, find the diameters of the shaft. **[*Ans.* $D$ = 132 mm, and $d$ = 79.2 mm]**

**6.18.** A hollow shaft of external and internal diameters as 80 mm and 50 mm respectively is transmitting power at 150 r.p.m. Determine the power, which the shaft can transmit, if the shearing stress is not to exceed 40 MPa.

**[*Ans.* 53.6 kW]**

**6.19.** A hollow shaft has to transmit 53 kW at 160 r.p.m. If maximum shear stress is 50 MPa and internal diameter is half of the external diameter, find the diameters of the shaft. **[*Ans.* $D$ = 70 mm, $d$ = 35 mm]**

**6.20.** Find the torque a solid shaft of 100 mm diameter can transmit. If the maximum angle of twist is 1.5° in a length of 2 m. Take $G$ = 70 GPa. **[*Ans.* 9.0 kN-m]**

**6.21.** A shaft is transmitting 100 kW at 180 r.p.m. If the allowable shear stress in the shaft material is 60 MPa, determine the suitable diameter for the shaft. The shaft is not to twist more than 1° in a length of 3 metres. Take $G$ = 80 GPa. **[*Ans.* $D$ = 103.8 mm]**

**6.22.** A hollow shaft of external and internal diameters as 80 mm and 40 mm is required to transmit torque from one pulley to another. What is the value of torque transmitted, if angle of twist is not to exceed 1° in a length of a 2m. Take $G$ = 80 GPa. **[*Ans.* $T$ = 2.63 kN-m ]**

**6.23.** A solid shaft of 150 mm diameter is to be replaced by a hollow shaft of the same material with internal diameter equal to 60% of the external diameter. Find the saving in material, if maximum allowable shear stress is the same for both the shafts. **[*Ans.* 30.9% ]**

**6.24.** A solid shaft and a hollow circular shaft, whose inside diameter is 3/4 of the outside diameter are of equal lengths and are required to transmit a given torque. Compare the weights of these two shafts, if maximum shear stress developed in both the shafts is also equal. **[*Ans.* 1.76]**

**6.25.** A solid shaft of 200 mm diameter has the same cross-sectional area as a hollow shaft of the same material with inside diameter of 150 mm. Find the ratio of :

(i) Power transmitted by both the shafts at the same angular velocity, and

(ii) Angles of twist in equal lengths of these shafts, when stressed to the same intensity. **[*Ans.* 1.7 and 0.8]**

**6.26.** A solid steel shaft of 60 mm diameter is to be replaced by a hollow steel shaft of the same material with internal diameter equal to half of the external diameter. Find the diameters of the hollow shaft and saving in material, if the maximum allowable shear stress is same for both shafts.

**[*Ans.* $D_1$ = 61.30 mm, $d$ = 30.65 mm and 21.7%]**

**6.27.** A solid shaft of 80 mm diameter is to be replaced by a hollow shaft of external diameter 100 mm. Determine the internal diameter of the hollow shaft if the same power is to be transmitted by both the shafts at the same angular velocity and shear stress. **[*Ans.* $d$ = 83.6 mm]**

**6.28.** A solid aluminium shaft 1 m long and 50 mm diameter is to be replaced by a hollow shaft of the same length and same outside diameter, so that the hollow shaft could carry the same torque and has the same angle of twist. What must be the inner diameter of the hollow shaft? Take modulus of rigidity for aluminium as 28 GPa and that for steel as 85 GPa. **[*Ans.* $d$ = 45, 25 mm]**

**6.29.** A hollow steel shaft of 300 mm external diameter and 200 mm internal diameter has to be replaced by a solid alloy shaft. Assuming the same values of polar modulus for both, calculate the diameter of the latter and workout the ratio of their torsional rigidities. Take $G$ for steel as 2.4 $G$ for alloy.

**[*Ans.* $D_1$ = 278.8 and Ratio = 2.58]**

**6.30.** A hollow shaft of 3 m length is subjected to torque such that the shaft experiences a maximum shear stress of 75 MN/m$^2$. Find the angle of twist if $G$ = 75 GN/m$^2$ and external and internal diameters of shaft are 150 mm and 100 mm respectively. Also find the shear stress at the inner surface of the shaft and show the distribution of the shear stress on the section.

**[*Ans.* $\theta$ = 2.29°, and $\tau$ = 50 MN/m$^2$]**

**6.31.** A hollow shaft transmits power at 60. r.p.m. Its maximum permissible shear stress is 70 MN/ m$^2$ and external and internal diameters are 250 mm and 160 mm respectively. If maximum torque is 40 percent more than the mean or average torque, Calculate

(i) The power transmitted by the shaft, and

(ii) Maximum angle of twist if length of the shaft is 5 m. Assume $G = 80$ GN/m$^2$.

**[*Ans.* 802 kW]**

**6.32.** A solid shaft of 80 mm diameter is to be replaced by a hollow shaft of external diameter 100 mm. Determine the internal diameter of hollow shaft if the same power is to be transmitted by both the shaft at same angular velocity and shear stress. (*UPTU : 2005–06*)

**[*Ans.* Example 6.7]**

**6.33.** A shaft of hollow circular section, has external diameter 100 mm and internal diameter is 60 mm. The allowable shear stress in the shaft material is 55 N / mm$^2$. Determine the angle of twist in a length of twenty times the external diameter of the shaft.

Take : $G = 8.5 \times 10^4$ N/mm$^2$. (*UPTU : 2006–2007*)

**[*Ans.* Example 6.3]**

**6.34.** Determine the dimensions of hollow shaft with a diameter ratio of 3 : 4, which is to transmit 60 kN at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m$^2$ and the angle of twist to 3.8° in a length of 4 m. For the shaft material. $G = 8$ GPa. (*UPTU : 2009–2008*)

[*Ans.* Example 6.5]

**6.35.** Find the internal and external diameters required for a hollow shaft, which is to transmit 40 kW of power at 240 rev/minute. The shear stress is to be limited to 100 MN/m$^2$. Take outside diameter to be twice the inside diameter. (*UPTU : 2009–2010*)

**[*Ans.* Example 6.1]**

**6.36.** A 20 mm diameter shaft of length 500 mm is fixed at one end. A torque, $T$ is applied at its free end. The linear strain at surface of shaft at angle of 45° from the axis is $4.0 \times 10^{-3}$. Determine (i) Torque (ii) Angle of twist and, (iii) Shear stress in the shaft.

Take $E = 20 \times 10^{11}$ and $G = 80$ GPa (*UPTU : 2010–2011*)

**[*Ans.* Example 6.6]**

**6.37.** A 800 mm long shaft of diameter 80 mm is supported on ball bearing at its ends. It carries a flywheel of weight 4 kN at its middle and transmits a power of 24 kW at a ends of 240 rpm. Determine the principal stress at the ends of a vertical diameter at a section which is just before the middle of the beam.

(*UPTU : 2011–2012*)

**[*Ans.* Example 6.9]**

**6.38.** A solid shaft rotating at 500 rpm transmits 300 kW. The maximum torque is 2% more than mean torque. Material of shaft has the allowable shear stress of 65 MPa and modulus of rigidity of 81 GPa, the angle of twist in the shaft should not exceed 1° in 1 metre length. Determine the diameter of the shaft. (*UPTU : 2012–13*)

**[*Ans.* Example 6.4]**

**6.39.** A hollow steel shaft 10 cm external diameter, 5 cm internal diameter, transmits 600 kW at 500 rpm and is subjected to an end thrust of 60 kN. Find what bending moment may be safely applied to the shaft if the greater principal stress is not to exceed 100 N/mm$^2$. (*UPTU : 2012–2013*)

**[*Ans.* Example 6.11]**

**6.40.** A solid shaft is subjected to a bending moment of 2.3 kN/m and a twisting moment of 3.45 kN/m. Find the diameter of the shaft if the allowable tensile and shear stresses for shaft material are limited to 703 MN/m$^2$ and 421.8 MN/m$^2$ respectively. (*UPTU : 2012–2013*)

**[*Ans.* Example 6.10]**

**6.41.** A circular steel rod *AB* of diameter *d*, length *a*, modulus of elasticity *E* and modulus of rigidity *G*, is loaded as shown in Fig. 6.10. Rigid bar *BC* of length

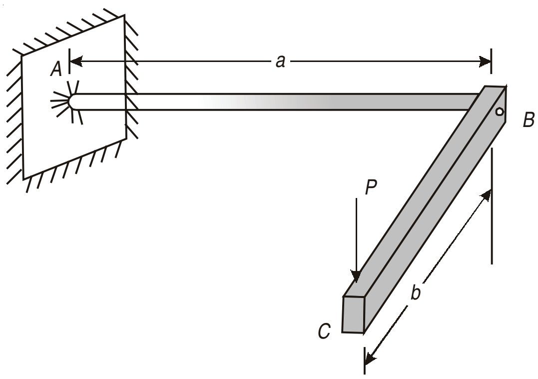



**Fig. 6.10**

*b* is rigidly fixed to *AB* at *B* such that *BC* is perpendicular to *AB* and lies in the horizontal plane. Find the deflection at point *C* due to

(i) bending of *AB*

(ii) torsion cf *AE*. (*UPTU : 2001–2002*)

**[*Ans.* Example 6.12]**

**6.42.** A shaft is subjected to a torque *T* and bending moment *M* (both equal) and another similar shaft is subjected to a single torque *T* . Show that the ratio of maximum principal stresses in these two cases will be 1.707

(*UPTU : 2001–02*)

**[*Ans.* Exampel 6.13]**

CHAPTER 7

# Columns and Struts

## 7.1 ❑ AXIALLY LOADED BEAM

The column having a load which coincides with the centroidal axis of the column is called as axially loaded column. (Fig. 7.1)

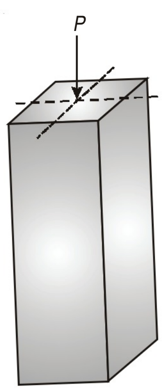



**Fig. 7.1**

## 7.2 ❑ ECCENTRIC LOADING

(i) The load whose lines of action does not coincide with the axis of a column is called as an eccentric load (Fig. 7.2a)

(ii) Due to eccentric load, both direct stress and bending stress induced.

Combined maximum stress = Direct stress ± Bending stress

$$\sigma = \sigma_0 \pm \sigma_b = \frac{P}{A} \pm \frac{M}{Z}$$

Maximum stress $\quad \sigma_{max} = \frac{P}{A} \pm \frac{M}{Z}$ (Compressive)

Minimum stress $\sigma_{min} = \dfrac{P}{A} - \dfrac{M}{Z}$ (Tensile)

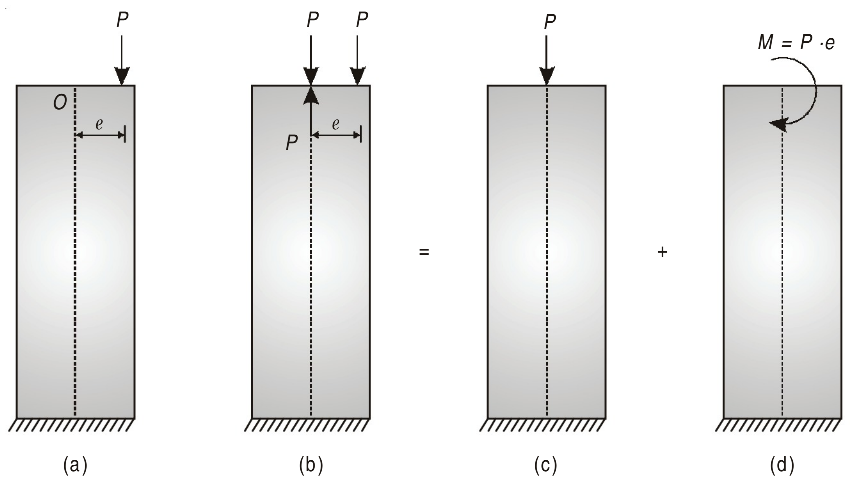
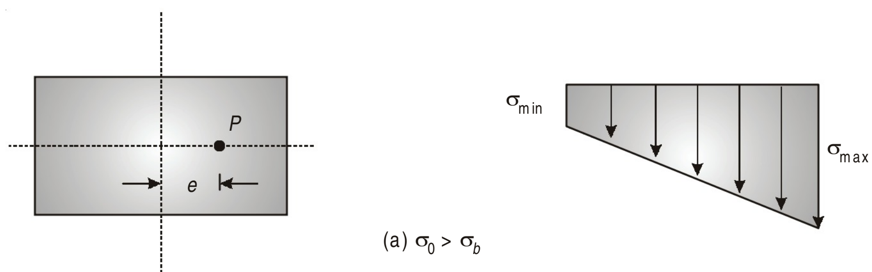



Fig. 7.2

Condition for stress distribution at the base :

(i) If $\sigma_0 > \sigma_b$, the stress throughout the section will be of same nature i.e. compressive. Refer Fig. 7.3(a).




(ii) If $\sigma_0 = \sigma_b$, the stress throughout the section will be of same nature i.e., compressive. In this case, $\sigma_{max} = 2\,\sigma_0$ and $\sigma_{min} = 0$. Refer Fig. 7.3 (b).

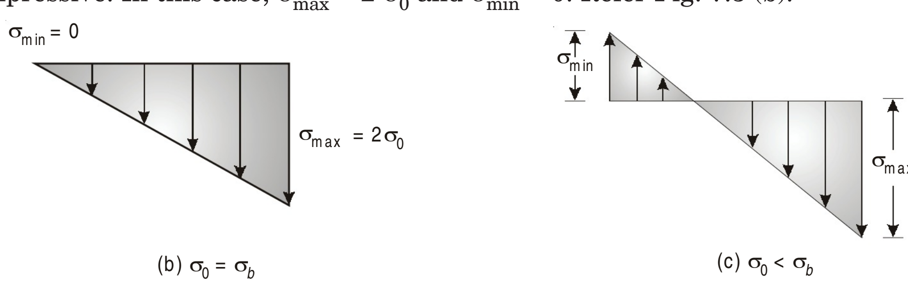



Fig. 7.3

(iii) If $\sigma_0 < \sigma_b$, the stress will be partly tensile and partly compression. Refer Fig. 7.3 (c).

## 7.3 ❑ CONDITION FOR NO TENSION

1. When the eccentric load acts on a column, it produces both direct stress and bending stress.

2. If the $\sigma_0 > \sigma_b$, the resultant stress is compressive.

3. If the $\sigma_0 = \sigma_b$, the resultant stress will be compressive such that $\sigma_{max} = 2\sigma_0$ and $\sigma_{min} = 0$.

4. If the $\sigma_0 < \sigma_b$, the resultant stress in the section is partly compressive and partly tensile.

5. Tensile stress at the base of column is harmful as it produces tensile crack.

6. To avoid this tensile crack or for no tension condition, direct stress should be greater than or equal to bending stress.

For no tension condition,

$$\sigma_0 \geq \sigma_b \qquad \frac{P}{A} \geq \frac{M}{Z}$$

$$\frac{P}{A} \geq \frac{P.e}{Z} \qquad \frac{1}{A} \geq \frac{e}{Z}$$

$$e \geq \frac{Z}{A}$$

Hence, for no tension condition, eccentricity should be less than or equal to $\frac{Z}{A}$.

## 7.4 ❑ MIDDLE THIRD RULE

**Rectangular Section :**

For no tension

$$e = \frac{Z}{A}$$

Area $\qquad A = bd$

Sectional Modulus

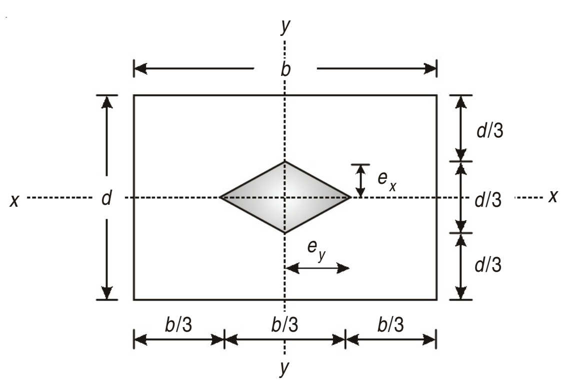



**Fig. 7.4**

$$Z_{XX} = \frac{Ixx}{\bar{y}} = \frac{\frac{bd^3}{12}}{\frac{d}{2}} = \frac{bd^2}{6}$$

$$Z_{YY} = \frac{Iyy}{\bar{x}} = \frac{\frac{db^3}{12}}{\frac{d}{2}} = \frac{db^2}{6}$$

$$e_X \leq \frac{Zxx}{A} \quad \text{and} \quad e_Y = \frac{Zyy}{A}$$

$$e \leq \frac{\frac{bd^2}{6}}{bd} \qquad e \leq \frac{\frac{db^2}{6}}{bd}$$

$$e \leq \frac{d}{6} \qquad \leq \frac{b}{6}$$

$$\therefore \quad \mathbf{2\,e \leq \frac{d}{3}} \qquad \mathbf{2\,e \leq \frac{b}{3}}$$

Therefore no tension condition, the load must lies within the middle third shaded area of eccentricity 2*e* as shown in Fig. 7.4. The central shaded portion is known as core or kernel of section.

### 7.4.1 Middle Quarter Rule

**Circular Section :**

Sectional modulus

$$Z = \frac{Ixx}{\bar{y}} \qquad Z = \frac{\frac{\pi}{64}D^4}{\frac{D}{2}} = \frac{\pi D^3}{32}$$

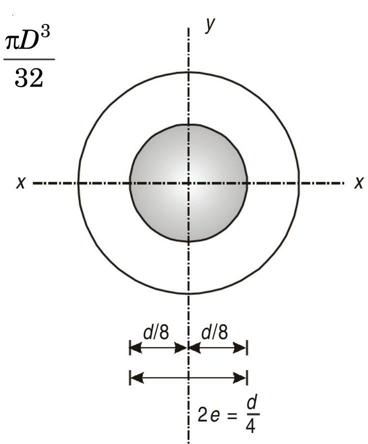


Fig. 7.5

Area $$e = \frac{\pi}{4}D^2$$

$\therefore$ For no tension condition,

$$e = \frac{Z}{A} = \frac{\frac{\pi D^3}{32}}{\frac{\pi D^2}{4}}$$

$$e \leq \frac{D}{8} \qquad \therefore 2e \leq \frac{D}{4}$$

∴ For no tension developed in the section, the load should be applied within one-fourth diameter of diameter of circle as shown by shaded area.

## SOLVED EXAMPLE

**Type I :** Column subjected to the eccentric load about one axis only.

$$\sigma = \sigma_0 \pm \sigma_b$$

**Example 7.1.** *A rectangular column of 200 mm × 150 mm is subjected to the compressive load 225 kN at an eccentricity of 28 mm in a plane bisecting the thickness. Find the maximum and minimum intensities of stress in the section.*

*Given : b = 200 mm, d = 150 mm, P = 225 kN, e = 28 mm*

**Solution To find :** $\sigma_{max}$ and $\sigma_{min}$

Area $A = bd = 200 \times 150 = 30{,}000 \text{ mm}^2$

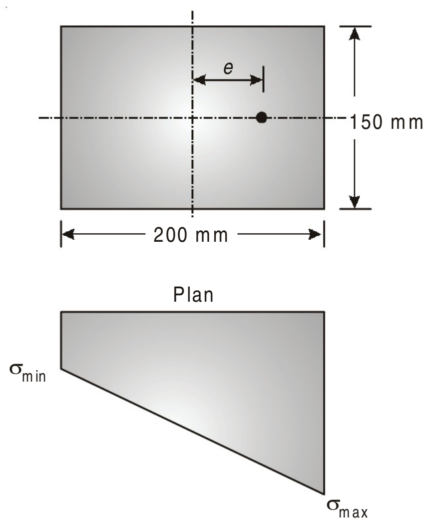



**Fig. 7.6** *Stress distribution diagram*

$$\text{Direct stress } \sigma_0 = \frac{P}{A} = \frac{225 \times 10^3}{30000}$$

$$= 7.5 \text{ N/mm}^2$$

$$\text{Bending moment } M = P.e = 225 \times 10^3 \times 28$$

$$= 6.3 \times 10^6 \text{ N.mm}$$

$$\text{Sectional modulus } Z = \frac{db^2}{6} = \frac{150 \times 200^2}{6}$$

$$= 1 \times 10^6 \text{ mm}^3$$

## 7.5 ❑ STRUT

A member of structure in any position other than vertical and carrying an axial compressive load is called strut, e.g. connecting rod of I.C. engines.

## 7.6 ❑ COLUMN

A vertical strut is usually called column, carrying axial compressive load. Examples of columns are vertical pillar of building and structural column in the form of I beam or box column or column made up of pipe etc.

### 7.6.1 Slenderness Ratio $\left(\frac{l}{k}\right)$

$$\text{Slenderness ratio of a column} = \frac{\text{Length of column}}{\text{Least radius of gyration}}$$

$$= \frac{l}{k} \;;$$

where,

$k$ = least radius of gyration

classification :

$\frac{l}{k} < 30$ : Short column

$\frac{l}{k} > 120$ : Long column

$\frac{l}{k}$ Lies between 30 and 120 : Intermediate column

### 7.6.2 Buckling Load, Cripling Load or Critical Load

It is the maximum axial load at which the column tends to have lateral displacement. Buckling always takes place about the axis having least moment of inertia ($I_{xx}$ or $I_{yy}$). Buckling should be prevented by applying safe load.

$$\text{Safe load} = \frac{\text{Critical load}}{\text{Factor of safety}}$$

## 7.7 ❑ EULER'S COLUMN THEORY

Mr. Euler has derived an equation, for buckling load of long columns based on bending stress. Hence this equation (Formula) can not be used for short columns.

### 7.7.1 Assumptions in the Euler's Column Theory

1. The cross-section of the column is uniform throughout its length.
2. The column material is homogeneous and obeys Hook's law.
3. Initially the column is perfectly straight and load applied is truly axial.
4. The length of column is very large as compared to its cross-sectional dimensions.
5. The shortening of column, due to direct compression (being very small) is neglected.
6. The failure of column occurs due to buckling alone.

### 7.7.2 Sign Conventions

1. A moment, which tends to bend the column with convexity towards initial central line (Fig. 7.7a) is taken as positive.

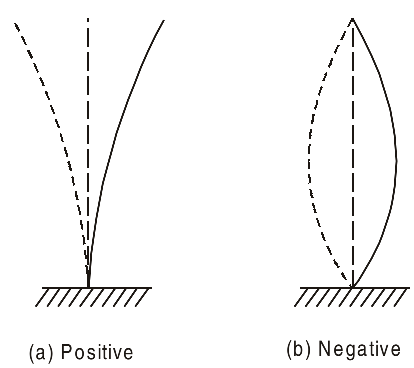



**Fig. 7.7**

2. A moment, which tends to bend the column with its concavity towards its initial central line (Fig. 7.7 b) is taken as negative.

## 7.8 ❑ TYPES OF END CONDITIONS OF COLUMNS

1. Columns with both ends hinged :

$$P = \frac{\pi^2 EI}{l^2} \quad \text{... (i)}$$

2. Columns with one end fixed and the other free :

$$P = \frac{\pi^2 EI}{4l^2} \quad \text{... (ii)}$$

3. Columns with both ends fixed :

$$P = \frac{\left(4\pi^2 EI\right)}{l^2} \quad \text{... (iii)}$$

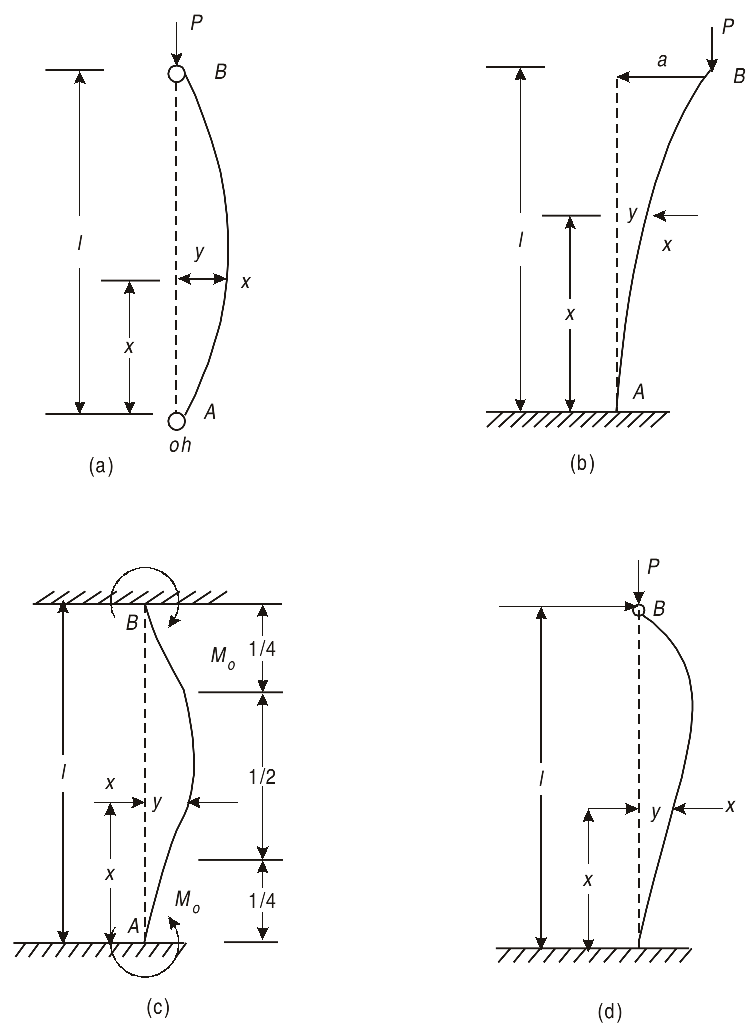



**Fig. 7.8**

4. Columns with one end fixed and other hinged :

$$P = \frac{\left(2\pi^2 EI\right)}{l^2}$$

$P$ = Critical load on column
$y$ = Deflection of column at $x$

### 7.8.1 Equivalent Length of a Column

Sometimes all of 4 cases are represented by a general equation, $P = \dfrac{(\pi^2 EI)}{Cl^2}$

where $C$ is a constant, representing end conditions of the column. Value for this constant :

(i) $C$ is 1 for a column with both ends hinged

(ii) $C$ is 4 for a column with 1 end fixed and other free

(iii) $C$ is $\frac{1}{4}$ for a column with both ends fixed

(iv) $C$ is $\frac{1}{2}$ for a column with 1 end fixed and other hinged.

There is another way of representing the equation for cripling load by an equivalent length or effective length of the column.

| *S.No.* | *End conditions* | *Crippling load* | | *Equivalent length, le* |
|---|---|---|---|---|
| 1 | Both ends hinged | $\dfrac{\pi^2 EI}{l^2}$ | $\dfrac{\pi^2 EI}{l^2 e}$ | $le = l$ |
| 2 | 1 end fixed and other free | $\dfrac{\pi^2 EI}{4l^2}$ | $\dfrac{\pi^2 EI}{l^2 e}$ | $le = 2l$ |
| 3 | Both ends fixed | $\dfrac{4\pi^2 EI}{l^2}$ | $\dfrac{\pi^2 EI}{l^2 e}$ | $le = \dfrac{l}{2}$ |
| 4 | 1 end fixed and other hinged | $\dfrac{2\pi^2 EI}{l^2}$ | $\dfrac{\pi^2 EI}{l^2 e}$ | $le = \dfrac{l}{\sqrt{2}}$ |

**Example 7.2.** *A steel rod 5 m long and 40 mm diameter is used as a column, with one end fixed and other free. Determine crippling load by Euler's formula. Take E as 200 GPa.*

*Given : Length,* $l = 5 \text{ m} = 5 \times 10^3 \text{ mm}$

*Diameter of column,* $d = 40 \text{ mm}$

$E = 200 \text{ GPa} = 200 \times 10^3 \text{ N/mm}^2$

**Solution** We know that moment of inertia of column section,

$$I = \frac{\pi}{64} d^4$$

$$= \frac{\pi}{64} (40)^4$$

$$= 40{,}000\,\pi \text{ mm}^4$$

Since the column is fixed at one end and free at the other, therefore

Equivalent length of the column,

$$le = 2l$$

$$= 2\,(5 \times 10^3)$$

$$= 10 \times 10^3 \text{ mm}$$

Crippling load,

$$P = \frac{\pi^2 EI}{le^2}$$

$$= \frac{\pi^2 \left(200 \times 10^3\right)\left(40000\pi\right)}{\left(10 \times 10^3\right)^2}$$

$$= 2480 \text{ N} = \mathbf{2.48\ kN}$$

*Or*

$$P = \frac{\pi^2 EI}{4l^2}$$

$$= \frac{\pi^2 \left(200 \times 10^3\right) 40{,}000\pi}{4\left(5 \times 10^3\right)^2}$$

$$P = 2480 \text{ N} = \mathbf{2.48\ kN}$$

## 7.9 ❑ RANKINE FORMULA

Prof. Rankine devised an emperical formula for determining critical load, which is applicable to all columns.

Crippling load by Rankine formula,

$$P_{CR} = \frac{(\sigma_C . A)}{1 + a.\left(\frac{le}{k}\right)^2}$$

$\sigma_c$ = Crushing stress of the column

$A$ = cross-sectional area of the column

$a$ = Rankine constant = $\dfrac{\sigma_c}{\pi^2 E}$

$k$ = least radius of gyration

$le$ = equivalent length of column

**Example 7.3.** *What are the limitations of Euler's theory of columns?*

(*UPTU : 2011–2012, 2012–2013, MTU : 2012–2013*)

**Solution** Crippling stress $\sigma_{CR} = \dfrac{\pi^2 E}{\left(\dfrac{L_e}{k}\right)^2}$

1. The critical stress is directly proportional to the modulus of elasticity of material, and it is inversely proportional to square of slenderness ratio of the column.

2. The graph plotted between critical stress $\sigma_{CR}$ and slenderness ratio is shown in Fig. 7.9.

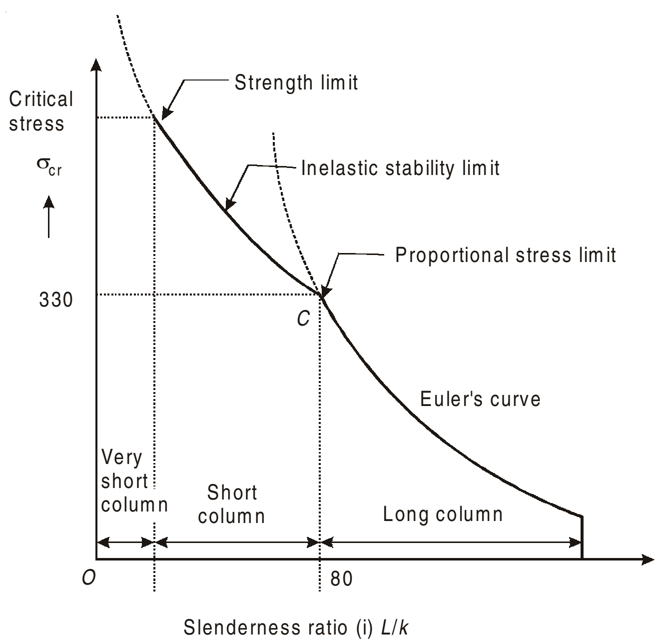



**Fig. 7.9**

3. When the slenderness ratio decreases, the critical stress increases.

4. The value of $\dfrac{L_e}{k}$ at the point $C$ is called as critical value of slenderness ratio.

5. For any slenderness ratio above this value, column fails by buckling and for any value of slenderness ratio less than this value, the column fails by crushing and not by buckling.
6. Euler's formula not applicable for short column, this is the limitation of Euler's formula.
7. Mathematically, Euler's formula is applicable,

If crushing stress $\geq$ Buckling stress

$$\sigma_{cr} \geq \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}, \qquad \left(\frac{L_e}{k}\right)^2 \geq \frac{\pi^2 E}{\sigma_c} \qquad \frac{L_e}{k} \geq \sqrt{\frac{\pi^2 E}{\sigma_c}}$$

When slenderness ratio is greater than $\sqrt{\frac{\pi^2 E}{\sigma_c}}$, the Euler's formula is applicable. This is Euler's limitation.

**For example :**

Maximum value of slenderness ratio for mild steel column.

Crushing stress for mild steel

$$\sigma_c = 330 \text{ MPa}$$

Young's modulus for mild steel $E = 2.1 \times 10^5$ MPa

$$\therefore \qquad \frac{L_e}{k} \geq \sqrt{\frac{\pi^2 E}{\sigma_c}} \geq \sqrt{\frac{\pi^2 \times 2.1 \times 10^5}{330}}$$

$$\geq \mathbf{79.25 \approx 80}$$

$\therefore$ When slenderness ratio for mild steel is less than 80, the Euler's formula is not applicable.

**Example 7.3** Derive an expression to obtain buckling load for the column which is neither too short nor too long.

## 7.10 ❑ RANKINE–GORDON FORMULA

(i) It is commonly known as Rankine formula.

(ii) Rankine devised an empirical formula applicable for both short and long column.

(iii) Short columns fails by crushing load $P_c = \sigma_c A$ where $\sigma_c$ = Crushing stress.

(iv) Long columns fails by buckling load $P_E = \dfrac{\pi^2 EI}{L_e^2}$

(v) In practice, the column or struts *d* fails due to combined effect of crushing stress and bending stress.

**Rankine's formula :**

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E}$$

$$\frac{1}{P} = \frac{P_E + P_C}{P_C \, P_E} \qquad \therefore \quad P = \frac{P_E P_C}{P_C + P_E} = \frac{P_C}{1 + \dfrac{P_C}{P_E}}$$

$$P = \frac{\sigma_C A}{1 + \dfrac{\sigma_C A}{\dfrac{\pi^2 EI}{L_e^2}}} = \frac{\sigma_C A}{1 + \dfrac{\sigma_C A}{\dfrac{\pi^2 E\, Ak^2}{L_e^2}}} = \frac{\sigma_C A}{1 + \dfrac{\sigma_C}{\pi^2 E}\left(\dfrac{L_e}{k}\right)^2}$$

But $$I = Ak^2$$

$$\mathbf{P} = \frac{\boldsymbol{\sigma_C A}}{\mathbf{1 + a\left(\dfrac{L_e}{k}\right)^2}}$$

where $a$ = Rankine's constant = $\dfrac{\sigma_C}{\pi^2 E}$

Rankine's constant is a property of the material of the column. Value of Rankine's constant '$a$' and crushing value $\sigma_c$ for some materials are given below.

**Table 7.1.** *Rankine's constant 'a' for various materials*

| *S.No.* | *Material* | *Crushing stress* $\sigma_C$ | *Rankine's constant* |
|---|---|---|---|
| 1 | Wrought iron | 250 MPa | $\frac{1}{9000}$ |
| 2 | Cast iron | 550 MPa | $\frac{1}{1600}$ |
| 3 | Mild steel | 320 MPa | $\frac{1}{7500}$ |
| 4 | Timber | 50 MPa | $\frac{1}{750}$ |

**Example 7.4.** *Derive Euler's expression for buckling load for column with both ends hinged.* (*UPTU : 2011–2012*)

**Solution**

**Columns with Both Ends Hinged**

Consider a column $AB$ of length $l$ hinged at both of its ends $A$ and $B$ and carrying a critical load at $B$. As a result of loading, let the column deflect into a curved form $AXB$ as shown in Fig. 7.10.

Now consider any section $X$, at a distance $x$ from $A$.

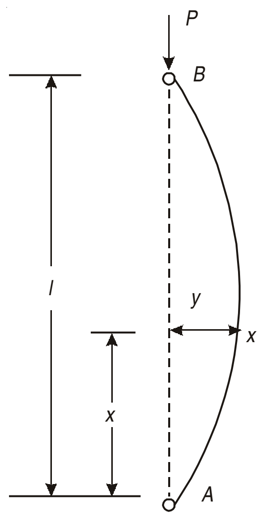



**Fig. 7.10**

Let $P$ = Critical load on the column

$y$ = Deflection of the column at $X$.

$\therefore$ Moment due to the critical load $P$,

$$M = -P \cdot y$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -P \cdot y$$

... (Minus sign due to concavity towards initial centre line)

$$\therefore \quad EI\frac{d^2y}{dx^2} + P \cdot y = 0$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The general solution of the above differential equation is

$$y = A \cdot \cos x\frac{\sqrt{P}}{EI} + b\sin x\frac{\sqrt{P}}{EI}$$

where $A$ and $B$ are the constants of integration. We know that when $x = 0$, $y = 0$. Therefore, $A = 0$. Similarly, when $x = l$, then $y = 0$. Therefore

$$0 = B\sin l\frac{\sqrt{P}}{EI}$$

A little consideration will show that either $B$ is equal to zero or $\sin l \frac{\sqrt{P}}{EI}$ is equal to zero. Now if we consider $B$ to be equal to zero, then it indicates that the column has not bent at all. But if

$$\sin l \frac{\sqrt{P}}{EI} = 0$$

$$\therefore \qquad l \frac{\sqrt{P}}{EI} = 0 = \pi = 2\pi = 3\pi = .....$$

Now taking the least significant value,

$$l \frac{\sqrt{P}}{EI} = \pi$$

or

$$P = \frac{\pi^2 EI}{l^2}$$

**Example 7.5.** *A slender column of length l is built in at its lower end, and free at upper end. Find the critical value of the compressive load P.*

(*UPTU : 2003–2004*)

**Solution**

**Columns with One End Fixed and the Other Free**

Consider a column $AB$ of length $l$ fixed at $A$ and free at $B$ and carrying a critical load at $B$. As a result of loading, let the beam deflect into a curved form $AXB$ such that the free end $B$ deflects through $a$ and occupies a new position $B_1$ as shown in Fig. 7.11.

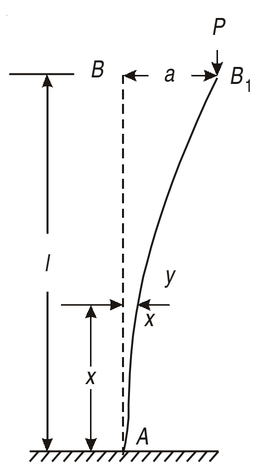



**Fig. 7.11**

Now consider any section $X$ at a distance $x$ from $A$.

Let $P$ = Critical load on the column and

$y$ = Deflection of the column at $X$.

$\therefore$ Moment due to the critical load $P$,

$$M = +P(a-y)$$

...(Plus sign due to convexity towards initial centre line)

$$= P.a - P.y$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = P.a - P.y$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{P.a}{EI}$$

The general solution of the above differential equation is,

$$y = A\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right) + a \qquad \text{...(i)}$$

where $A$ and $B$ are the constants of integration. We know that when $x = 0$, then $y = 0$, therefore $A = -a$. Now differentiating the above equation,

$$\frac{dy}{dx} = -A\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + B\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right)$$

We also know that when $x = 0$, then $\frac{dy}{dx} = 0$. Therefore

$$0 = B\sqrt{\frac{P}{EI}}$$

A little consideration will show that either $B$ is equal to zero or $\sqrt{\frac{P}{EI}}$ is equal to zero. Since the load $P$ is not equal to zero, it is thus obvious that $B$ is equal to zero. Now substituting the values $A = -a$ and $B = 0$ in Eq. (i),

$$y = -a\cos\left(x\sqrt{\frac{P}{EI}}\right) + a = a\left[1 - \cos x\sqrt{\frac{P}{EI}}\right]$$

We also know that when $x = l$, then $y = a$. Therefore

$$a = a\left[1 - \cos l\sqrt{\frac{P}{EI}}\right]$$

$$\therefore \qquad \cos l\sqrt{\frac{P}{EI}} = 0$$

or
$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2} = \frac{3\pi}{2} = \frac{5\pi}{2}$$

Now taking the least significant value,

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$$

$$P = \frac{\pi^2 EI}{4l^2}$$

**Example 7.6.** *Derive the equation to obtain buckling load for the columns with both ends fixed.*

**Solution**

**Columns with Both Ends Fixed**

Consider a column $AB$ of length $l$ fixed at both of its ends $A$ and $B$ and carrying a critical load at $B$. As a result of loading, let the column deflects as shown in Fig. 7.12.

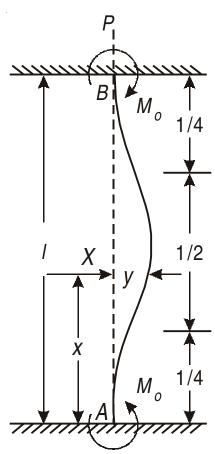



Fig. 7.12

Now consider any section $X$ at a distance $x$ from $A$.

Let $\quad P$ = Critical load on the column and

$\quad y$ = Deflection of the column at $X$.

A little consideration will show that since both the ends of the beam $AB$ are fixed and it is carrying a load, therefore there will be some fixed end moments at $A$ and $B$.

Let $\quad M_0$ = Fixed end moments at $A$ and $B$.

$\therefore$ Moment due to the critical load $P$,

$$M = -P \cdot y$$

... (Minus sign due to concavity initial centre line)

$$El \frac{d^2 y}{dx^2} = M_0 - P \cdot y$$

$$\therefore \quad \frac{d^2 y}{dx^2} + \frac{P}{EI} \cdot y = \frac{M_0}{EI}$$

The general solution of the above differential equation is :

$$y = A \cos\left(x \frac{\sqrt{P}}{EI}\right) + B \sin\left(x \frac{\sqrt{P}}{EI}\right) + \frac{M_0}{P} \quad \text{...(i)}$$

where $A$ and $B$ are the constants of integration. We know that when $x = 0$, then $y = 0$. Therefore $A = -\frac{M_0}{P}$. Now differentiating the above equation,

$$\frac{dy}{dx} = -A \frac{\sqrt{P}}{EI} \sin\left(x \frac{\sqrt{P}}{EI}\right) + B \frac{\sqrt{P}}{EI} \cos\left(x \frac{\sqrt{P}}{EI}\right)$$

We also know that when $x = 0$, then $\frac{dy}{dx} = 0$. Therefore

$$0 = B \frac{\sqrt{P}}{EI}$$

A little consideration will show, that either $B$ is equal to zero, or $\frac{\sqrt{P}}{EI}$ is equal to zero. Since the load $P$ is not equal to zero, it is thus obvious that $B$ is equal to zero. Substituting the values $A = \frac{M_0}{P}$ and $B = 0$ in Eq. (i),

$$y = -\frac{M_0}{P} \cos\left(x \frac{\sqrt{P}}{EI}\right) + \frac{M_0}{P} = \frac{M_0}{P}\left[1 - \cos\left(l \frac{\sqrt{P}}{EI}\right)\right]$$

We also know that when $x = l$, then $y = 0$. Therefore

$$0 = \frac{M_0}{P}\left[1 - \cos\left(l \frac{\sqrt{P}}{EI}\right)\right]$$

$$\therefore \quad \cos\left[l \frac{\sqrt{P}}{EI}\right] = 1$$

or $$l\frac{\sqrt{P}}{EI} = 0 = 2\pi = 4\pi = 6\pi = .....$$

Now taking the least significant value,

$$l\frac{\sqrt{P}}{EI} = 2\pi$$

$$\therefore \quad P = \frac{4\pi^2 EI}{l^2}$$

**Alternative methods**

1. The fixed beam *AB* may be considered as equivalent to a column of length $\frac{l}{2}$ with ends hinged (i.e., middle portion of the column.

$$\therefore \text{ Critical load,} \quad P = \frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2} = \frac{4\pi^2 EI}{l^2}$$

2. The fixed beam *AB* may also be considered as equivalent to a column of length $\frac{l}{4}$ with one end fixed and the other free (i.e., lower one-fourth portion of the beam as shown in Fig. 7.12.

$$\therefore \text{ Critical load,} \quad P = \frac{\pi^2 EI}{4\left(\frac{l}{4}\right)^2}. = \frac{4\pi^2 EI}{l^2}$$

**Example 7.7.** *Derive the equation to obtain buckling load of the column having one end fixed and other hinged.* (*UPTU : 2006*)

**Solution**

**Columns with One End Fixed and the Other Hinged**

Consider a column *AB* of length *l* fixed at *A* and hinged at *B* and carrying a critical load at *B*. As a result of loading, let the column deflect as shown in Fig. 7.13

Now consider any section *X* at a distance *x* from *A*.

Let $P$ = Critical load on the column, and

$y$ = Deflection of the beam at *X*.

A little consideration will show, that since the beam *AB* is fixed at *A* and it is carrying a load, therefore, there will be some fixed end moment at *A*. In order to balance the fixing moment at *A*, there will be a horizontal reaction at *B*.

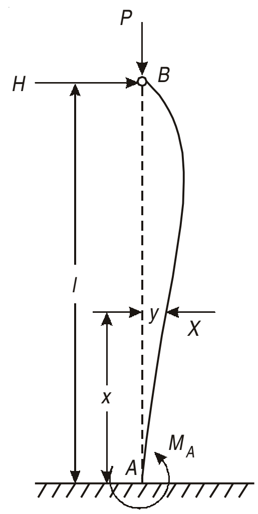



**Fig. 7.13**

Let $M_A$ = Fixed end moment at $A$ and

$H$ = Horizontal reaction at $B$.

∴ Moment due to critical load $P$,

$$M = -P \cdot y$$

... (Minus sign due to concavity towards initial centre line)

or
$$EI\frac{d^2y}{dx^2} = H(l-x) - P \cdot y$$

∴
$$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = \frac{H(l-x)}{EI}$$

The general solution of the above differential equation is

$$A = y\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H(l-x)}{P} \quad \text{...(i)}$$

where $A$ and $B$ are the constants of integration. We know that when $x = 0$, then $y = 0$. Therefore $A = -\frac{Hl}{P}$.

Now differentiating the above equation,

$$\frac{dy}{dx} = -A\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + B\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) - \frac{H}{P}$$

We know that when $x = 0$, $\frac{dy}{dx} = 0$. Therefore

$$0 = B\frac{\sqrt{P}}{EI} - \frac{H}{P}$$

$$\therefore \quad B = \frac{H}{P} \times \frac{EI}{\sqrt{P}}$$

We also know that when $x = l$, then $y = 0$. Therefore substituting these values of $x$, $A$ and $B$ in Eq. (i),

$$0 = -\frac{Hl}{P}\cos\left(l\frac{\sqrt{P}}{EI}\right) + \frac{H}{P}\frac{EI}{\sqrt{P}}\sin\left(l\frac{\sqrt{P}}{EI}\right)$$

$$\therefore \quad \frac{H}{P}\frac{EI}{\sqrt{P}}\sin\left(l\frac{\sqrt{P}}{EI}\right) = \frac{HI}{P}\cos\left(l\frac{\sqrt{P}}{EI}\right)$$

or

$$\tan\left(l\frac{\sqrt{P}}{EI}\right) = \left(l\frac{\sqrt{P}}{EI}\right)$$

A little consideration will show that the value of $\left(l\frac{\sqrt{P}}{EI}\right)$ in radians, has to be such that its tangent is equal to itself. We know that the only angle, the value of whose tangent is equal to itself, is about 4.5 radians.

$$\therefore \quad l\frac{\sqrt{P}}{EI} = 4.5 \qquad \text{or} \qquad l^2 \times \frac{P}{EI} = 20.25$$

$$\therefore \quad P = \frac{20.25EI}{l^2} = \frac{2\pi^2 EI}{l^2}$$

**Note :** A little consideration will show that 20.25 is not exactly euqal to $2\pi^2$, but approximately equal to $2\pi^2$. This has been done to rationalise the value of $P$, i.e., crippling load in various cases.

### 7.10.1 Slenderness Ratio

We have already discussed that the general equation for the crippling load,

$$P = \frac{\pi^2 EI}{l^2} \qquad \text{...(i)}$$

We know that the buckling of a column under the crippling load will take place about the axis of least resistance. Now substituting $I = Ak^2$ (where $A$ is the area and $k$ is the least radius of gyration of the section) in the above equation,

$$P = \frac{\pi^2 E A k^2}{l^2} = \frac{\pi^2 E A}{\left(\frac{l}{k}\right)^2} \quad \text{...(ii)}$$

where $\frac{l}{k}$ is known as slenderness ratio.

**Note.** It may be noted that the formula for crippling load, in the previous articles, have been derived on the assumption that the slenderness ratio $\frac{l}{k}$ is so large, that the failure of the column occurs only due to bending, the effect of direct stress (i.e., $\frac{P}{A}$ ) being negligible.

### 7.10.2 Limitation of Euler's Formula

We have discussed earlier that the general equation for the crippling load,

$$P = \frac{\pi^2 E A}{\left(\frac{l}{k}\right)^2}$$

$\therefore$ Crippling stress,

$$\sigma_c = \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler's formula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for the mild steel is 320 MPa or 320 N/m$^2$ and Young's modulus for the mild steel is 200 GPa or $200 \times 10^3$ N/mm$^2$.

Now equating the crippling stress to the crushing stress,

$$320 = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} = \frac{\pi^2 \times \left(200 \times 10^3\right)}{\left(\frac{l}{k}\right)^2}$$

$$\therefore \qquad \left(\frac{l}{k}\right)^2 = \frac{\pi^2 \times 200 \times 10^3}{320}$$

$$\text{or} \qquad \frac{l}{k} = 78.5 \text{ say } 80$$

Thus, if the slenderness ratio is less than 80 the Euler's formula for a mild steel column is not valid.

Sometimes, the columns, whose slendemess ratio is *more than 80* are known as *long columns* and those whose slenderness ratio is less than 80 are known as short columns. It is thus obvious that the Euler's formula holds good only for long columns.

**Note.** In the Euler's formula, for crippling load, we have not taken into account the direct stresses induced in the material due to the load, (which increases gradually from zero to its crippling value). As a matter of fact, the combined stress, due to direct load and slight bending reaches its allowable value at a load, lower than that required for buckling; and therefore this will be the limiting value of the safe load.

### 7.10.3 Difference between Column and Strut

1. Column is a vertical structural member carrying an axial compressive load. It is the main member of the structure. It is usually long as compared to its cross-section. There is possibility of buckling of the column.

2. Struct is a structural member carrying an axial compressive load. It is component member of truss or mechanism. It is usually short in length. There is no possibility of buckling of struts. Load carried by strut is small as compared to that by a column. It may be horizontal, inclined or even vertical.

**Example 7.8.** *A hollow alloy tube having internal and external diameters of 36 mm and 52 mm respectively is 6 m long. It extends by 3 mm when an axial force of 50 kN is applied. Determine crippling load for the tube when used as column with both ends pinned.* (*UPTU : 2011–2012*)

*Given :*

$$D = 52 \text{ mm}$$

$$d = 36 \text{ mm}$$

$$L = 6 \text{ m} = 6000 \text{ mm}$$

$$\delta l = 3 \text{ mm}$$

$$P = 50 \text{ kN}$$

*Condition : Both end pinned*

**Solution**

Area,

$$A = \frac{\pi}{4}\left(D^2 - d^2\right)$$

$$= \frac{\pi}{4}\left(52^2 - 36^2\right) = 1105.8 \text{ mm}^2$$

Moment of inertia, $I = \frac{\pi}{64}\left(D^4 - d^4\right)$

$$= \frac{\pi\left(52^4 - 36^4\right)}{64} = 276.46 \times 10^3 \text{ mm}^4$$

Young's modulus using deformation formula

$$E = \frac{PL}{A\delta l} = \frac{50 \times 10^3 \times 6000}{1105.8 \times 3}$$

$$= 90.43 \times 10^3 \text{ N/mm}^2$$

Effective length for both ends hinged column,

$$le = l = 6000 \text{ mm}$$

Crippling load, $P_C = \frac{\pi^2 EI}{l^2 e}$

$$= \frac{\pi^2 \times 90.43 \times 10^3 \times 276.46 \times 10^3}{(6000)^2}$$

$$= 6854.14 \text{ N}$$

$\therefore$ $\mathbf{P_c = 6854.14\ N = 6.854\ kN}$

**Example 7.9.** *In a column section, the length of the column is 40 times the length of each side of the square section. If both ends of the column are pinned and E = 2 × $10^4$ kN/m$^2$, determine the critical stress set up in the column.*

(*UPTU : 2004–2005*)

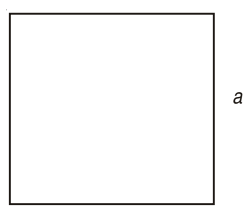



**Fig. 7.14**

**Solution**

$$E = 2 \times 10^4 \text{ kN/cm}^4$$

$$= \frac{\left(2 \times 10^4 \times 10^3\right)}{10^2}$$

$$= 2 \times 10^5 \text{ N/mm}^2$$

Let the side of the column be $a$

$\therefore$ Length of the column, $l = 40\ a$

Effective length of the column, $le = l = 40\ a$ ($\because$ both ends pinned)

Moment of inertia, $$I = \frac{a^4}{12}$$

By using Euler's formula, crippling load

$$P_c = \frac{(\pi^2 EI)}{l_2^2}$$

$$= \frac{\left[\pi^2 \times 2 \times 10^5 \left(\frac{a^4}{12}\right)\right]}{(40a)^2} = 102.81\ a^2\ \text{N}$$

Crippling stress set up in the column,

$$\sigma_C = \frac{P_C}{A} = \frac{102.81\ a^2}{a^2}$$

$$= \mathbf{102.81\ \ N/mm^2}$$

**Example 7.10.** *A rectangular masonary column has a cross-section 500 mm × 400 mm and is subjected to a vertical compressive load of 100 kN applied at point P as shown in Fig. 7.15.*

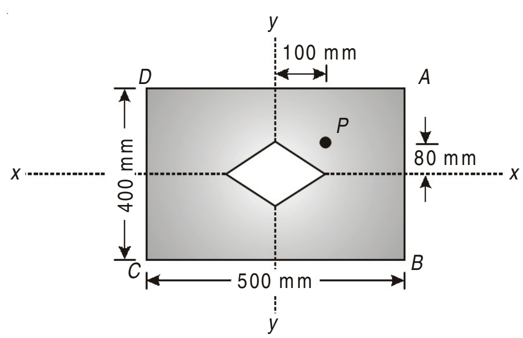



**Fig. 7.15**

*Determine the value of the maximum stress produced in the section. Is the section at any point subjected to tensile stresses?* (*UPTU : 2005–2006*)

*Given :*

$b$ = *500 mm*

$d$ = *400 mm*

$e_x$ = *80 mm*

$e_y$ = *100 mm*

$P$ = *100 kN*

**Solution**

Area, $A = bd = 500 \times 400 = 200000 \text{ mm}^2$

Direct stress, $$\sigma_0 = \frac{P}{A} = \frac{(100 \times 10^3)}{20 \times 10^4} = 0.5 \text{ N/mm}^2$$

Sectional modulus, $$Z_{xx} = \frac{db^2}{6} = \frac{(500 \times 400^2)}{6} = 13.33 \times 10^6 \text{ mm}^3$$

$$Z_{yy} = \frac{bd^2}{6} = \frac{(400 \times 500)^2}{6} = 16.67 \times 10^6 \text{mm}^3$$

Bending moment,

$$M_x = P\,.\,ex = 100 \times 10^3 \times 80 = 8 \times 10^6 \text{ N-mm}$$
$$M_y = P\,.\,ey = 100 \times 10^3 \times 100 = 10 \times 10^6 \text{ N-mm}$$

Bending stress

$$\sigma_{bx} = \frac{M_x}{Z_x} = \frac{(8 \times 10^6)}{13.33 \times 10^6} = 6.6 \text{ N/mm}^2$$

$$\sigma_{by} = \frac{M_y}{Z_y} = \frac{(10 \times 10^6)}{16.67 \times 10^6} = 0.6 \text{ N/mm}^2$$

Resultant bending stress at the base :

$$\sigma_R = \sigma_0 + \sigma_{bx} + \sigma_{by}$$

Since the load $P$ is acting in the first quadrant i.e., near corner $A$, the maximum compressive stress developed at $A$ and tensile stress may be developed near corner $C$.

$$\therefore \quad \sigma_A = \sigma_0 + \sigma_{bx} + \sigma_{by}$$
$$= 0.5 + 0.6 + 0.6 = \mathbf{1.7 \text{ N/mm}^2}$$

Maximum compressive stress :

$$\sigma_c = \sigma_0 - \sigma_{bx} - \sigma_{by}$$
$$= 0.5 - 0.6 - 0.6 = \mathbf{-0.7 \text{ N/mm}^2}$$

Hence maximum :

(i) **Compressive stress = 1.7 N/mm²**

(ii) **Tensile stress = –0.7 N/mm²**

## 7.11 ❑ TYPE III : NUMERICALS ON RANKINE'S FORMULA

**Step 1 :** To find effective length depends on its end condition.

**Step 2 :** To find least moment of inertia.

**Step 3 :** To find cross-sectional area.

**Step 4 :** To find least radius of gyration

$$k = \sqrt{\frac{I_{\text{Least}}}{\text{Area}}}$$

**Step 5 :** Rankine's crippling load.

$$P_c = \frac{\sigma_c A}{1 + a\left(\frac{L}{k}\right)^2}$$

**Step 6 :** Safe load,

$$P_c = \frac{\text{Crippling load}}{\text{Factor of safety}}$$

**Example 7.11.** *A mild steel hollow column has 100 mm external diameter and 60 mm internal diameter and 4 m length is used as a column. Determine the crippling load by Rankine's formula when both ends are hinged.*

*Take* $\sigma_c = 320\ N/mm^2$*, Rankine's constant* $a = \frac{1}{7500}$.

*Given : For the column* $L = 4\ m = 4000\ mm$

*Diameter* $d_1 = 100\ mm$

$d_2 = 60\ mm$

$$\sigma_c = 320\ MPa,\ a = \frac{1}{7500}$$

**Solution To Find :** Crippling load $P_c$

$\therefore$ Moment of inertia, $I = \frac{\pi}{64}\left(d_1^4 - d_2^4\right) = \frac{\pi}{64} \times \left(100^4 - 60^4\right)$

$= 4.2726 \times 10^6\ \text{mm}^4$

Area $A = \frac{\pi}{4}\left(d_1^2 - d_2^2\right) = 5027\ \text{mm}^2$

$\therefore$ Radius of gyration, $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{4.2726 \times 10^6}{5027}} = 29.15\ \text{mm}$

$\therefore$ Rankine's load $P_c = \frac{\sigma_c \times A}{1 + \left[a \times \left(\frac{L}{k}\right)^2\right]} = \frac{320 \times 5027}{1 + \frac{1}{7500}\left(\frac{4000}{29.15}\right)^2}$

$\therefore \qquad P_c = 458.3 \times 10^3$ N

$\therefore$ **Rankine's load = 458.3 kN.**

The crippling load by Rankine's formula is 458.3 kN

**Example 7.12.** *A hollow C.I. column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both ends. Calculate the safe load by Rankine formula using a factor of safety of 4.*

*Take* $\sigma_c = 550$ *MPa,* $a = \dfrac{1}{1600}$. *(UPTU : 2003–2004)*

*Given :*

$$\sigma_c = 550 \text{ MN/m}^2 = 550 \text{ N/mm}^2$$

$$a = \frac{1}{1600} \text{ (Rankine's constant)}$$

$$d_1 = 200 \text{ mm}$$

$$d_2 = (200 - 2 \times 20) = 160 \text{ mm}$$

$$L = 4.5 \text{ m} = 4500 \text{ mm}$$

$$FOS = 4$$

**Solution To Find :** Safe load $P_s$ and $\dfrac{P_E}{P_R}, \dfrac{L_e}{k}$

(i) Since the column is fixed at both ends,

Effective length, $$L_e = \frac{L}{2} = \frac{4.5}{2} = 2.25 \text{ m} = 2250 \text{ mm}$$

Area, $$A = \frac{\pi}{4}\left(d_1^2 - d_2^2\right) = \frac{\pi}{4}\left(200^2 - 160^2\right)$$

$\therefore$ $$A = 11309.7 \text{ mm}^2$$

$\therefore$ Moment of inertia, $$I = \frac{\pi}{64}\left(d_1^4 - d_2^4\right) = \frac{\pi}{64} \times \left(200^4 - 160^4\right)$$

$$= 46.37 \times 10^6 \text{ mm}^4$$

$\therefore$ Radius of gyration, $$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{46.37 \times 10^6}{11310}} = 64.03 \text{ mm}$$

$\therefore$ Rankine's load $$P_R = \frac{\sigma_c \times A}{1 + a \times \left(\dfrac{L}{k}\right)^2} = \frac{550 \times 11309.7}{1 + \dfrac{1}{1600}\left(\dfrac{2250}{64.03}\right)^2}$$

$$P_R = 3.510 \times 10^6 \text{ N} = 3510 \text{ kN}$$

$\therefore$ Safe load $\qquad P_s = \dfrac{3510}{\text{Factor of safety}} = \dfrac{3510}{4}$ **= 877.5 kN**

$\therefore$ The safe load by Rankine's formula is 877.5 kN

**Example 7.13.** *A 5 m long hollow column with fixed ends supports an axial load of 800 kN. The external diameter of the column is 240 mm. Determine the thickness of the column using Rankine's formula. Given that a =* $\dfrac{1}{6400}$ *and working stress of 80 MPa.* *(UPTU : 2010–2011)*

*Given :*

$$L = 5 \text{ m} = 5000 \text{ mm}$$
$$P = 800 \text{ kN}$$
$$D = 240 \text{ mm}$$
$$\alpha = \frac{1}{6400}$$
$$\sigma = 80 \text{ MPa}$$

**Solution To Find :** Thickness $t$

For both end fixed column,

Effective length $\qquad L_e = \dfrac{L}{2} = \dfrac{5000}{2} = 2500$ mm

Area $\qquad A = \dfrac{\pi}{4}\left(D^2 - d^2\right) = \dfrac{\pi}{4}\left(240^2 - d^2\right)$

Radius of gyration for hollow column

$$K^2 = \frac{I}{A} = \frac{D^2 + d^2}{16}$$

Using Rankine's formula,

$$P = \frac{\sigma_c A}{1 + \alpha\left(\dfrac{L_e}{K}\right)^2}$$

$$800 \times 10^3 = \frac{80 \times \dfrac{\pi}{4}\left(240^2 - d^2\right)}{1 + \dfrac{1}{6400}\left(\dfrac{2500^2 \times 16}{240^2 + d^2}\right)}$$

$$12.732 \times 10^3 = \frac{240^2 - d^2}{1 + \dfrac{15625}{240^2 + d^2}} = \frac{\left(240^2 - d^2\right)}{\dfrac{240^2 + d^2 + 15625}{240^2 + d^2}}$$

$$932.3 \times 10^6 + 12.732 \times 10^3 d^2 = (240^2 - d^2)(240^2 + d^2) = 240^4 - d^4$$

$$d^4 + 12.732 \times 10^3 d^2 - 2.385 \times 10^9 = 0$$

$$d^2 = 42883.6$$

$$d = 207 \text{ mm}$$

∴ Thickness of column $t = \frac{D-d}{2} = \frac{240-207}{2}$ **= 16.5 mm**

**Example 7.14.** *A 1.5 m long Cast Iron (C.I.) column has a circular cross section of 5 cm diameter. The one end of the column is fixed and other end is free. By taking a factor of safty as 3, find the safe load on column by using;*

*(i) Rankine – Gordon formula ; using yield stress σ 560 MN/mm² and a = $\frac{1}{1600}$ for pinned end, and a = $\frac{1}{1600}$ for pinned end.*

*(ii) Euler's formula; assume Young's modulus for C.I. as 120 GN/m².*

*(MTU : 2012–2013)*

*Given :*

$$D = 5 \text{ cm} = 50 \text{ mm}$$

$$L = 1.5 \text{ m} = 1500 \text{ mm}$$

$$\alpha = \frac{1}{1600}$$

$$\sigma_c = 560 \text{ N/mm}^2$$

$$E = 120 \times 10^3 \text{ N/mm}^2$$

**Solution To Find :** $P_E$, $P_R$

Internal diameter $d = D = 50$ mm

Area $A = \frac{\pi}{4}(D)^2 = \frac{\pi}{4}\left(50^2\right) = 1963.5 \text{ mm}^2$

Moment of inertia $I = \frac{\pi}{64}\left(D^4\right) = \frac{\pi}{64}\left(50^4\right) = 306.8 \times 10^3 \text{ mm}^4$

Radius of gyration $k = \sqrt{\frac{I}{A}} = \sqrt{\frac{306.8 \times 10^3}{1963.5}}$

$= 12.5$ mm

(i) Euler's crippling load

$$P_E = \frac{\pi^2 EI}{L^2}$$

$L_e = 2L = 2 \times 1.5 = 3$ m one end is fixed and other free

$$= \frac{\pi^2 \times 120 \times 10^3 \times 306.8 \times 10^3}{3000^2} = 40.37 \text{ kN}$$

$$= \mathbf{40.37 \times 10^3\ kN}$$

(ii) Rankine's critical load

$$P_R = \frac{\sigma_c A}{1+\alpha\left(\frac{L}{K}\right)^2} = \frac{560\times 1963.5}{1+\frac{1}{1600}\left(\frac{3000}{12.5}\right)^2}$$

$$= 29.72 \times 10^3 \text{ N}$$

$$\mathbf{P_R = 29.72\ kN}$$

**Example 7.15.** *Determine the section of a cast iron hollow cylindrical column 5 metre long with ends firmly built-in if it carries an axial load of 300 kN. The ratio of internal to external diameter is $\frac{3}{4}$. Use factor of safety of 8. Take $\sigma_c$ = 567 N/mm$^2$ and Rankine's constant $a = \frac{1}{1600}$.* *(UPTU : 2009 – 2010)*

*Given :*

$$L = 5 \text{ m}$$

$$\sigma_c = 567 \text{ N/mm}^2$$

$$P = 300 \text{ kN}$$

$$a = \frac{1}{1600}$$

$$FOS = 8$$

$$\frac{d}{D} = \frac{3}{4} = 0.75$$

**Solution** **To Find :** $D$ and $d$

Crippling load = Axial load × FOS = 300 × 8 = 2400 kN

Area

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(D^2 - (0.75D)^2\right)$$

$$= 0.3436\, D^2$$

Radius of gyration

$$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{\frac{\pi}{64}\left(D^4 - d^4\right)}{\frac{\pi}{4}\left(D^2 - d^2\right)}}$$

$$= \sqrt{\frac{D^2 + d^2}{16}} = \sqrt{\frac{D^2 - (0.75D)^2}{16}}$$

$$= 0.3125\, D$$

Effective length

Since the column ends are firmly built in i.e., both ends are fixed.

$$L_e = \frac{L}{2} = \frac{5}{2} = 2.5 \text{ m} = 2500 \text{ mm}$$

Using Rankine Loads

$$P_c = \frac{\sigma_c A}{1+\alpha\left(\frac{L_e}{K}\right)^2}$$

$$2400 \times 10^3 = \frac{567 \times 0.3436 D^2}{1+\frac{1}{1600}\left(\frac{2500}{0.3125 D}\right)^2}$$

$$2400 \times 10^3 = \frac{194.82 D^4}{D^2 + 40000}$$

$$194.84\, D^4 - 2400 \times 10^3\, D^2 - 9.6 \times 10^{10} = 0$$

$$D^2 = 29194.6$$

$$D = \mathbf{170.86 \text{ mm}}$$

$$d = 0.75 \times 170.86 = \mathbf{128.15 \text{ mm}}$$

## EXERCISE

**7.1.** What do you understand by the term column and strut? Distinguish clearly between long column and short columns.

**7.2.** Explain the failure of long columns and short columns.

**7.3.** Describe the assumptions in the Euler's column theory.

**7.4.** Derive the relations for Euler's crippling load for a column when it has (i) Both ends fixed, and (ii) Both ends hinged.

**7.5.** Define the term 'Equivalent length'. Discuss its uses.

**7.6.** Explain the term 'slenderness ratio' and describe with mathematical expression, how it limits the use of Euler's formula for crippling load.

**7.7.** Obtain a relation for the Rankine's crippling load for columns.

**7.8.** Explain the effect of eccentric loading on a column. Derive the relation for the maximum stress in an eccentrically loaded columns.

**7.9.** A 1.5 m long column has circular cross-section of 50 mm diameter. One end of the column is fixed and the other is free. Determine the safe load on the column.

**7.10.** A hollow circular column of 200 mm external diameter and 160 mm internal diameter is 4 m long with both ends fixed. If the column carries a load of 150 kN at an eccentricity of 25 cm, find the extreme stress in the column.

**[*Ans.* 21.5 MPa]**

**7.11.** An alloy tube 60 mm diameter and 2.8 m length is used as a strut with both ends hinged. If the tube is subjected to eccentric load equal to 60% of the Euler's crippling load, find the value of eccentricity. Take yield strength as 320 MPa and modulus of elasticity a 210 GPa. **[*Ans.* 12.1 mm]**

**7.12.** A solid circular bar 5 m long and 4 cm in diameter was found to extend 4.5 mm under a tensile load of 48 kN. The bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 3.0. **[*Ans.* 2105.5 N and 701.8 N]**

**7.13.** A pinned-end strut of aluminium ($E$ = 73 GPa) with length $L$ = 2 m is constructed of circular tubing with outside diameter $d$ = 50 mm. The strut must resist an axial load $P$ = 14 kN with a factor of safety $n$ = 2 with respect to buckling. Determine the required thickness $t$ of the tube.

**[*Ans.* $t$ = 4.05 mm]**

**7.14.** What is the minimum actual length of column for which Euler's formula will hold good, if the cross-section of the uniform bar is 120 × 120 mm, the column having both its ends fixed ? The yield stress may be taken as 250 MPa and the modulus of elasticity as 205 kN/mm$^2$. **[*Ans.* 6.233 metres]**

**7.15.** A column of hollow circular cross-section has external diameter of 120 mm and internal diameter 80 mm. The column is 4 m long and hinged at both ends. Find the slenderness ratio of the column. **[*Ans.* 110.94]**

**7.16.** Calculate the safe compressive load on a hollow cast iron column (one end rigidly fixed and other hinged) of 10 cm external diameter, 7 cm internal diameter and 8 m in length. Use Euler's formula with a factor of safety of 4 and $E$ = 95 kN/mm$^2$. **[*Ans.* 27.3 kN]**

**7.17.** Show that for mild steel columns, Euler's formula is applicable only when the slenderness ratio is greater than 80, if both ends are hinged. Take crippling stress and modulus of elasticity as 330 MPa and 214 GPa respectively. What will be its value if, instead, both ends are fixed? **[*Ans.* 160]**

**7.18.** A hollow cylindrical cast iron column is 6 m long with both ends fixed. Determine the maximum diameter of the column if it has to carry a safe load 300 kN with a factor of safety of 4. Take the internal diameter as 0.7 times the external diameter. Take $f_c$ = 550 N/mm$^2$ and $a = \dfrac{1}{1600}$ in Rankine's formula. **[*Ans.* $D$ = 9.53 cm, $d$ = 6.67 cm]**

**7.19.** A 2.0 m long column has a circular cross-section of 6 cm diameter. One of the ends of the column is fixed in direction and position and other end is free. Taking factor of safety as 3, calculate the safe load using

(i) Rankine's formula take yield stress $f_c$ = 550 N/mm$^2$ and $a = \dfrac{1}{1600}$ for pinned ends.

(ii) Euler's formula, Young's modulus for C.I. = 1.3 × 10$^5$ N/mm$^2$.

**[*Ans.* 11.4 kN and (ii) 170 kN]**

**7.20.** A straight cylindrical bar is 15 mm in diameter and 1.2 m long. Where it is freely supported horizontally at its two ends and loaded at center by a load of

100 N, its central deflection is found to be 3 mm. If this bar is now placed in a vertical position and loaded along its axis with one end hinged and the other ends fixed, under what load will it buckle? What is the ratio of maximum stresses in the two cases. **[*Ans.* 16449.27 N, $\sigma_c/\sigma_d$ = 1.028]**

$E$ = 200 GN/m$^2$. For what length of strut of this cross section does the Euler's formula cease to apply ? **[*Ans.* 17.24 kN, 17.2 kN, 0.98 m]**

**7.21.** A rolled T-section 150 mm × 75 mm deep, thickness of flange 9 mm, thickness of web 8.4 mm, is used as a strut of 4250 mm length. The strut has one end hinged and the other end is fixed. Calculate the safe axial load using Rankine's formula with a factor of safety 2.5. Rankine's constants are $f_c$ = 315 MPa and

$a = \dfrac{1}{7500}$. **[*Ans.* 59.91 kN]**

**7.22.** A mild steel stanchions built up of one R.S.J. 160 mm × 100 mm with one 120 mm plate riverted to each flange. If it is 4000 mm long with one end fixed and the other end is hinged, calculate the safe axial load using Rankine's formula with a factor of safety. For the joist, $A$ = 2167 mm$^2$, $I_{xx}$ = 8.39 × 10$^6$ mm$^4$, and $I_{yy}$ = 94.8 × 10$^4$ mm.

Rankine's constants as $f_c$ = 315 MPa and $a = \dfrac{1}{7500}$. **[*Ans.* 238.45 kN]**

**7.23.** A compression member 3500 mm long in a roof truss consists of two unequal angles ISA 90 × 60 × 8 with their longer legs connected back to back with a separating distance of 10 mm between them as shown in Fig. 7.16. It is fixed at one end and hinged at the other. Find the safe axial load for the compression member using a factor of safety 3.

$f_c$ = 320, $\quad a = \dfrac{1}{7500}$

For ISA 90 × 60 ×8, $\quad I_{xx}$ = 91.5 ×10$^4$ mm$^4$

$I_{yy}$ = 32.4 ×10$^4$ mm$^4$ $\quad A$ = 1137 mm$^2$

$C_{xx}$ = 29.6 mm, $\quad C_{yy}$ = 14.8 mm

**[*Ans.* 109.94 kN]**

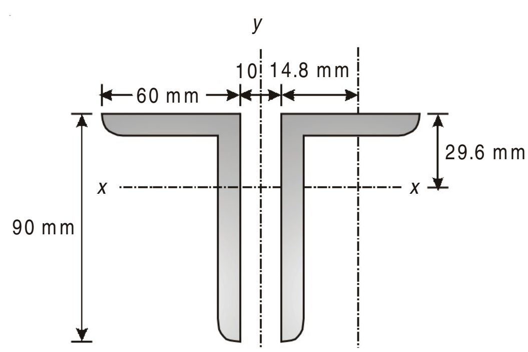



**Fig. 7.16**

## UNIVERSITY QUESTIONS

1. From the first principles derive the expression for the critical buckling load for a column having one end fixed and one end hinged. (*UPTU : 2001–2002*)

   **[*Ans.* Example 7.7]**

2. A straight steel bar, 1.5 m long and section 20 mm × 5 mm is compressed longitudinally until if buckles. Using Euler's formula, estimate the maximum central deflection before the steel passes the yield point at 330 N/mm$^2$. Take $E$ = 200,000 N/mm$^2$. (*UPTU : 2001–2002*)

   **[*Ans.* 150.46 mm]**

3. A slender column of length $l$ is built in at its lower end $A$ and laterally supported at its upper end. Find the first critical value of the compression load $P$. (*UPTU : 2002–2003*)

   **[*Ans.* Example 7.5]**

4. Calculate the critical load of a strut which is made of a bar circular in section and 5 m long and which is pin jointed at both ends. The same bar when freely supported gives mid-span deflection of 10 mm with a load of 80 N at the centre. (*UPTU : 2002–2003*)

   **[*Ans.* $P_c$ = 8.225 kN)]**

5. A slender column of length $l$ is built in at its lower end and free at upper end. Find the first critical value of the compressive load $P$. (*UPTU : 2003 – 2004*)

   **[*Ans.* Example 7.5]**

6. A hollow C.I. column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both ends. Calculate the safe load of Rankine formula using a factor of safety of 4.

   Take $\sigma_c$ = 550 MPa, $a = \dfrac{1}{1600}$. (*UPTU : 2003 – 2004*)

   **[*Ans.* Example 7.12]**

7. In a column section, the length of the column is 40 times the length of each side of the square section. If both ends of the column are pinned and $E = 2 \times 10^4$ kN/cm$^2$, determine the critical stress set up in the column. (*UPTU : 2004–2005*)

   **[*Ans.* $\sigma_c$ = 102.81 N/mm$^2$)]**

8. How will you justify that Rankine's formula is applicable for all lengths of columns, ranging from short to long columns. (*UPTU : 2004–2005*)

   **[*Ans.* Section 7.10]**

9. A rectangular masonary column has a cross-section 500 mm × 400 mm and is subjected to a vertical compressive load of 100 kN applied at point $P$ shown in Fig. 7.17. Determine the value of the maximum stress produced in the section. Is the section at any point subjected to tensile stresses? (*UPTU : 2005–2006*)

   **[*Ans.* Example 7.10]**

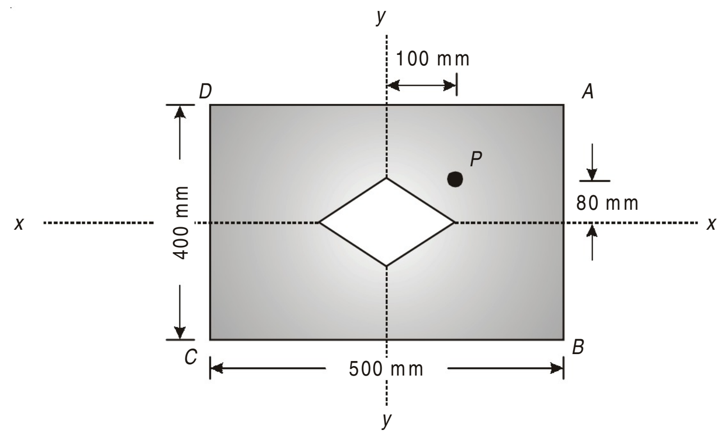



**Fig. 7.17**

10. Derive the equation to obtain buckling load for the column having one end hinged and other fixed. (*UPTU : 2005–2006*)

**[*Ans.* Example 7.7]**

11. What are the limitation of Euler's formula for the column? Explain with any one example. (*UPTU : 2005–2006*)

**[*Ans.* Section 7.7.1]**

12. A mild steel hollow column has 100 mm external diameter and 60 mm internal diameter and 4 m length is used as a column. Determine the crippling load by Rankine's formula when both ends are hinged.

    Take $\sigma_c$ = 320 N/mm$^2$, Rankine's constant $a = \dfrac{1}{7500}$. (*UPTU : 2006–2007*)

**[*Ans.* Example 7.11]**

13. Differentate in columns and struts with a short description of each classification. (*UPTU : 2006–2007*)

**[*Ans.* Section 7.7.2]**

14. Write the limitation of Euler's formula for critical load. (*UPTU : 2006*)

**[*Ans.* Section 7.7.1]**

15. Explain middle quanta and middle third rules. What are the importance of these rules for concrete sections? (*UPTU : 2007–2008*)

**[*Ans.* Section 7.4, 7.4.1]**

16. State the assumptions made during the Euler"s formula for a strut with pin jointed ends. Derive the Euler's crippling load for such a strut—the general equations of bending and also the solution of the differential equation may be assumed. (*UPTU : 2007–2008*)

**[*Ans.* Section 7.7.1]**

17. Explain middle third rule for rectangular sections. (*UPTU : 2008–2009*)

**[*Ans.* Section 7.4]**

18. Derive an expression to obtain buckling load for the column which is neither too short nor too long. (*MTU : 2008–09*)

**[*Ans.* Section 7.10]**

19. Determine the section of a cast iron hollow cylindrical column 5 metre long with ends firmly built-in if it carries an axial load of 300 kN. The ratio of internal to external diameter is $\frac{3}{4}$. Use factor of safety of 8. Take $\sigma_C = 567$ N/mm$^2$ and Rankine's constant $a = \frac{1}{1600}$. (*UPTU : 2009 – 2010*)

**[*Ans.* Example 7.15]**

20. From the first principles derive the expression for the critical buckling for a column having one end fixed and one end hinged. (*UPTU : 2009–2010*)

**[*Ans.* Example 7.7]**

21. A 5 m long hollow column with fixed ends supports an axial load of 800 kN. The external diameter of the column is 240 mm. Determine the thickness of the column using Rankine's formula. Given that $a = \frac{1}{1600}$ and working stress of 80 MPa. (*UPTU : 2010–2011*)

**[*Ans.* Example 7.13]**

22. Explain the term core of section with reference to short columns. Derive an expression for finding out the core of a rectangular section. (*UPTU : 2010–2011*)

**[*Ans.* Section 7.4]**

23. A slender column of length $l$ is built-in at its lower end and laterally supported (pin jointed) at its upper end. Find the first critical value of the compression load $P$. (*UPTU : 2011–2011*)

**[*Ans.* Example 7.7]**

24. What are the limitations of Euler's formula? (*UPTU : 2011–12*)

**[*Ans.* Section 7.7.1]**

25. Find the expression for cripping load for a long column when one end of the column is fixed and other end is hinged. (*UPTU : 2011–2012*)

**[*Ans.* Example 7.7)]**

26. A hollow alloy tube having internal and external diameters of 36 mm and 52 mm respectively is 6 m long. It extends by 3 mm when an axial force of 50 kN is applied. Determine crippling load for the tube when used as column with both ends pinned. (*UPTU : 2011–2012*)
27. A slender column of length $l$ is built-in at its lower end and laterally supported (pin jointed) at its upper end. Find the first critical value of the compression load $P$. (*UPTU : 2011–2012*)

**[*Ans.* Example 7.7]**

28. What are the limitations of Euler's theory of columns? (*UPTU : 2012–13*)

**[*Ans.* Example 7.3]**

29. Define slenderness ratio and derive Euler's expression for buckling load for column with both ends hinged. *(UPTU : 2012–13)*

**[*Ans.* Section 7.6.1, Example 7.4]**

30. A 1.5 m long Cast Iron (C.I.) column has a circular cross section of 5 cm diameter. The one end of the column is fixed and other end is free. By taking a factor of safety as 3, find the safe load on column by using ;

    (i) Rankine-Gordon formula; using yield stress as 560 $MN/mm^2$ and $a = \frac{1}{1600}$ for pinned end.

    (ii) Euler's formula, assume Young's modulus for C.I. as 120 $GN/m^2$.

*(UPTU : 2012–13)*

**[*Ans.* Example 7.14]**

31. What are the limitations of Euler Theory for buckling? Discuss Rankine-Gordon formula. *(UPTU : 2012–13)*

**[*Ans.* Example 3,7.10]**

# Helical and Leaf Springs

CHAPTER 8

## 8.1 ❑ INTRODUCTION

A spring is a device, in which the material is arranged in such a way that it can undergo a considerable change, without getting permanently distorted. A spring is used to absorb energy due to resilience*, which may be restored as and when required. The quality of a spring is judged from the energy it can absorb e.g., in a watch the spring is wound to absorb strain energy. This energy is released to run the watch, when the spring regains its original shape. A carnage spring is used to absorb shocks. It is thus obvious that a spring, which is capable of absorbing the greatest amount of energy for the given stress is known to be the best one.

## 8.2 ❑ STIFFNESS OF A SPRING

The load required to produce a unit deflection in a spring is called spring stiffness or *stiffness of a spring.*

## 8.3 ❑ TYPES OF SPRINGS

We have already discussed that a spring is used for absorbing energy due to resilience. Thus in general, the springs are of the following two types depending upon the type of resilience.

1. Bending spring and
2. Torsion spring.

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Work done by External Load on the Bar is stored in the material and is termed as strain energy (*U*). Hence, S.E. = W.D. by load. Strain energy stored by the body, within elastic limit, when loaded *externally* is called *resilience*.

## 8.4 ❑ BENDING SPRINGS

A spring which is subjected to bending only and the resilience is also due to it, is known as *bending spring*. Laminated springs or leaf springs are also called bending springs.

## 8.5 ❑ TORSION SPRINGS

A spring which is subjected to torsion or twisting moment only and the resilience is also due to it, is known as a *torsion spring*. Helical springs are also called torsion springs. Some springs are subjected to bending as well as torsion.

## 8.6 ❑ FORMS OF SPRINGS

Though there are many forms of springs, which are made by the manufacturers, yet the following types of springs are commonly used in Engineering practice.

1. Carriage springs or leaf springs
2. Helical springs.

### 8.6.1 Functions of Springs

1. Carriage springs are used to absorb shocks or impact loading
2. Clock springs are used to store energy
3. Spring balances are used to measure forces
4. Brakes and clutches are used to apply forces and control motions.

## 8.7 ❑ CARRIAGE SPRINGS OR SEMI-ELLIPTICAL TYPE LEAF SPRINGS

These are also called laminated springs and are of two types: (1) semi-elliptical types (i.e., simply supported at its ends subjected to central load) and (2) quarter-elliptical (i.e., cantilever) types.

The carriage springs are widely used in railway wagons, coaches and road vehicles these days. These are used to absorb shocks, which give an unpleasant feeling to the passengers. The energy absorbed by a laminated spring, during a shock, is released immediately without doing any useful work.

A laminated spring, in its simplest form, consists of a number of parallel strips of a meta] having different lengths but same width and placed one over the other in laminations as shown in Fig. 8.1. All the plates are initially bent to the same radius and are free to slide one over the other. When the spring is loaded to the designed load, all the plates become flat and the central deflection disappears. The purpose of this type of arrangement of plates is to make the spring of uniform strength throughout. This is achieved by tapering the ends of the laminations. The semi-elliptical type spring rests on the axis of the vehicle and its top plate is pinned at the ends to the chassis of the vehicle.

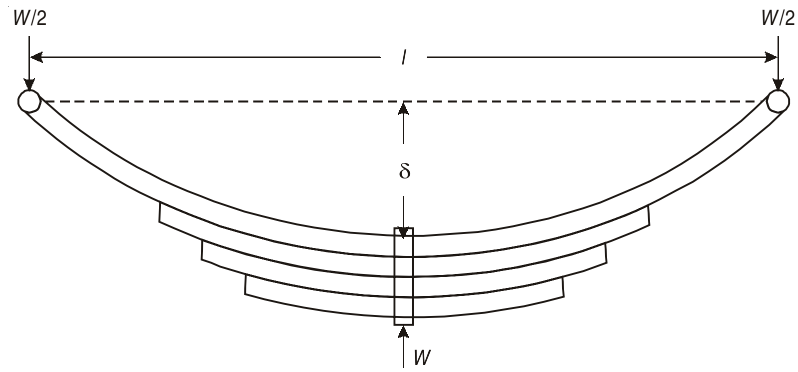



**Fig. 8.1** *Carriage spring*

Now consider a carriage spring pinned at its both ends, and carrying an upward load at its centre as shown in Fig. 8.1.

Let $l$ = Span of the spring

$t$ = Thickness of plates

$b$ = Width of the plates

$n$ = Number of plates

$W$ = Load acting on the spring

$\sigma$ = Maximum bending stress developed in the plates

$\delta$ = Original deflection of the top spring and

$R$ = Radius of the spring

A little consideration will show, that the load will be acting on the spring on the lowermost plate and it will be shared equally on the two ends of the top plate as shown in Fig. 8.1. We know that the bending moment, at the centre of the span due to this load,

$$M = \frac{Wl}{4} \quad \text{...(i)}$$

and moment resisted by one plate

$$= \frac{\sigma \cdot I}{y} \quad \text{...}\left(\because \frac{M}{I} = \frac{\sigma}{y}\right)$$

$$= \frac{\sigma \times \frac{bt^3}{12}}{\frac{t}{2}} = \frac{\sigma \cdot bt^2}{6} \quad \text{...}\left(\because I = \frac{bt^3}{12} \text{ and } y = \frac{t}{2}\right)$$

$\therefore$ Total moment resisted by $n$ plates,

$$M = \frac{n\sigma bt^2}{6} \qquad \text{...(ii)}$$

Since the maximum bending moment due to load is equal to the total resisting moment, therefore equating (i) and (ii),

$$\frac{Wl}{4} = \frac{n\,\sigma\, bt^2}{6}$$

or

$$\sigma = \frac{3Wl}{2nbt^2}$$

From the geometry of the spring figure, we know that the central deflection,

$$\delta = \frac{l^2}{8R} \qquad \text{...(iii)}$$

We also know that in the case of a bending beam,

$$\frac{\sigma}{y} = \frac{E}{R}$$

or

$$R = \frac{E \cdot y}{\sigma} = \frac{Et}{2\sigma} \qquad \text{...}\left(\because y = \frac{t}{2}\right)$$

Substituting this value of $R$ in Eq. (iii)

$$\delta = \frac{l^2}{8 \times \dfrac{E \cdot t}{2\sigma}} = \frac{\sigma l^2}{4Et}$$

Now substituting the value of $\sigma$ in the above equation,

$$\delta = \frac{3Wl}{2nbt^2} \times \frac{l^2}{4Et} = \frac{3Wl^3}{8Enbt^3}$$

**Example 8.1.** *What are Leaf springs?*

(*i*) *Leaf spring or carriage spring consists of number of overlapping layers (leaves).*

(*ii*) *Each layer having same width and thickness different length as shown in Fig. 8.2.*

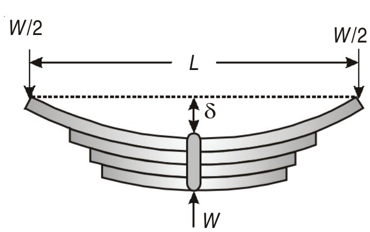



**Fig. 8.2**

*(iii) All plates are initially bent to the same radius and are free to slide one over the other.*

*(iv) The centre deflection δ will disappear, when the spring is loaded.*

*(v) The plates are arranged in this manner to achieve uniform strength in each plate.*

*(vi) There are two types of carriage spring :*

*1. Semi-elliptical type spring*

*2. Quarter-elliptical type spring.*

*(i) It is commonly used in railway wagons, coaches and road vehicles.*

*(ii) It is used to observe shocks and provide comfortable travelling to the passengers.*

**Example 8.2.** *In elliptical type of leaf spring is 800 mm long. Static deflection of spring under a load of 3 kN is 100 mm. Determine the number of leaves required and maximum stress if the leaves are 75 mm wide and 8 mm thick E = 204 GPa.*

(*UPTU : 2011–2012*)

*Given :*

$$L = 800 \text{ mm}$$
$$\delta = 100 \text{ mm}$$
$$W = 3 \text{ kN} = 3000 \text{ N}$$
$$b = 75 \text{ mm}$$
$$t = 8 \text{ mm}$$
$$E = 204 \text{ GPa} = 204 \times 10^3 \text{ MPa}$$

**Solution**

$$\delta = \frac{(3WL^3)}{8\, nbt^3}$$

$$E = \frac{(3 \times 3000 \times 800^2)}{8N \times 75 \times 204 \times 10^3}$$

or
$$100 = \frac{(73.529)}{n}$$

or
$$n = \frac{73.529}{100}$$
$$= 0.735, \text{ Say } 1$$

Bending stress,
$$\sigma = \frac{(3Wt)}{2nbt^2}$$
$$= \frac{(3 \times 3000 \times 8)}{2 \times 1 \times 75 \times 8^2}$$

∴
$$\sigma = \frac{72000}{9600}$$
$$= \mathbf{7.5 \ N/mm^2}$$

## Numerical on Semi-elliptical Leaf Spring

Maximum central deflection

$$\sigma = \frac{3WL}{2\,nbt^2}$$

Central deflection $\qquad \delta = \dfrac{3WL^3}{8\,nbt^2E}$

**Example 8.3.** *A leaf spring is made of plates 50 mm wide and 8 mm thick. The spring has a span of 700 mm. Determine the number of plates required to carry a central load of 45 kN. The maximum allowable stress in the plates is 200 MPa. What is the maximum deflection under this load ?* (*UPTU : 2007–2008*)

*Given :*

$$L = 700 \text{ mm}$$
$$W = 45 \text{ kN} = 45{,}000 \text{ N}$$
$$T = 8 \text{ mm}, \quad \sigma \ngtr 200 \text{ N/mm}^2$$
$$b = 50 \text{ mm}$$
$$E = 200 \times 10 \text{ N/mm}^2$$

**Solution To find :** $n$, $\delta$ and $R$

Number of plates $\qquad \sigma = \dfrac{3\,WL}{2\,nbt^2}$

$$200 = \frac{3}{2} \times \frac{45000 \times 700}{n \times 50 \times 8^2}$$

$$n = \mathbf{73.83 = 74 \text{ plates}}$$

Initial radius of curvature

$$\frac{\sigma}{y} = \frac{E}{R}$$

$\therefore \qquad R = \dfrac{E}{\sigma} \cdot y \qquad \qquad \therefore y = \dfrac{t}{2}$

$$= \frac{200 \times 10^3}{200} \times \frac{8}{2}$$

$$= 4000 \text{ mm} = \mathbf{4 \text{ m}}$$

Central deflection

$$\delta = \frac{3WL^3}{8nbt^3E} = \frac{3}{8} \times \frac{45000 \times 700^3}{74 \times 60 \times 8^3 \times 200 \times 10^3}$$

$$= \mathbf{12.73 \text{ mm}}$$

**Example 8.4.** *A leaf spring has 12 plates each 50 mm wide and 5 mm thick, the longest plate being 600 mm long. The greatest bending stress is not to exceed 180 N/mm² and the central deflection is 15 mm. Estimate the magnitude of the greatest central load that can be applied to the spring. E = 0.206 × 10⁶ N/mm².*

(*UPTU : 2009–2010*)

*Given : Number of plate*

$$n = 12, \quad b = 50 \text{ mm}, \quad t = 5 \text{ mm}, \quad L = 600 \text{ mm}$$
$$\sigma \not> 180 \text{ N/mm}^2$$
$$\delta \not> 15 \text{ mm}$$
$$E = 0.206 \times 10^6 \text{ N/mm}^2$$

**Solution To Find :** Central load $W$.

(i) Load based on maximum stress

$$\sigma = \frac{3\,WL}{2\,nbt^2}$$

$$180 = \frac{3}{2} \times \frac{W \times 600}{12 \times 50 \times 5^2}$$

$$W = 3000 \text{ N}$$

(ii) Load based on deflection

$$\delta = \frac{3\,WL^3}{8\,nbt^3E}$$

$$15 = \frac{3 \times W \times 600^3}{8 \times 12 \times 50 \times 5^3 \times 0.206 \times 10^6}$$

$$\mathbf{W = 2861.11 \text{ N}}$$

Greatest central load that can be applied will be the least value from above two conditions.

$\therefore$ $W =$ **2861.11 N**

**Example 8.5.** *A contilever graduated leaf spring is to absorb 600 J of energy without exceeding a deflection of 15 cm and the permissible stress of 900 MN/m². Length of spring is 70 cm. Calculate the suitable width and thickness of 5 leaves of the spring. Take E = 210 GPa*

*Given :*

$$U = 600$$
$$J = 600 \times 10^3 \text{ Nmm}$$
$$L = 70 \text{ cm} = 700 \text{ mm}$$
$$\delta = \not> 15 \text{ cm} = 150 \text{ mm}$$
$$n = 5$$
$$\delta_{max} = \not> 900 \text{ MN/m}^2$$
$$E = 210 \text{ GPa}$$
$$= 210 \times 10^3 \text{ MPa}$$

**Solution To Find :** Width $b$ and thickness $t$

$$U = \frac{1}{2} W\delta$$

$$\therefore \quad W = \frac{2U}{\delta} = \frac{2 \times 600 \times 10^3}{150} = 8000 \text{ N}$$

Thickness of plate $$t = \frac{\sigma L^2}{\delta E} = \frac{900 \times 700^2}{150 \times 210 \times 10^3} = \mathbf{14\ mm}$$

Width of plate $$\sigma = \frac{6WL}{nbt^2}$$

$$900 = \frac{6 \times 8000 \times 700}{5 \times b \times 14^2}$$

$$\mathbf{b = 38\ mm}$$

**Example 8.5.** *A cantilever leaf spring of length 0.43 m has four leaves of thickness 9 mm. If an end load of 2.5 kN causes a deflection of 36 mm find the width of the leaves.*

*E = 200,000 N/mm²*

*Given :* $L = 0.43 \text{ m} = 430 \text{ mm}$ $\quad W = 2.5 \text{ kN} = 2500 \text{ N}$

$N = 4 \text{ leaves}$ $\quad 8 = 36 \text{ mm}$

$T = 9 \text{ mm}$ $\quad E = 200.000 \text{ N/mm}^2$

**Solution To find :** Width of leaves

$$\delta = \frac{6\ WL^3}{nEbt^3}$$

$$36 = \frac{6 \times 2500 \times 430^3}{4 \times 200000 \times b \times 9^3}$$

$$b = \frac{1.1926 \times 10^{12}}{2.09952 \times 10^{10}}$$

$$= \mathbf{56.80\ mm}$$

## 8.8 ❑ QUARTER-ELLIPTICAL TYPE LEAF SPRINGS

The quarter-elliptical type leaf springs are rarely used, except as certain parts in some machines. Like a carriage spring, a quarter-elliptical type leaf spring consists of a number of parallel strips of a metal having different lengths but same width and placed one over the other in laminations as shown in Fig. 8.3. All the plates are initially bent to the same radius and are free to slide one over the other.

Now consider a quarter-elliptical type leaf spring subjected to a load at its free end as shown in Fig. 8.3

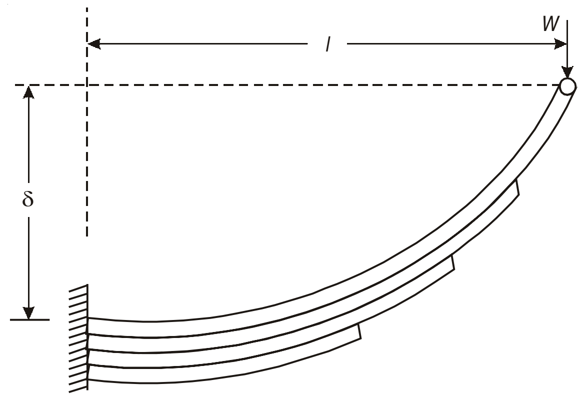



**Fig. 8.3** *Semi-elliptical leaf spring*

Let $l$ = Length of the spring

$t$ = Thickness of the plates

$b$ = Width of the plates

$n$ = Number of plates

$W$ = Load acting at the free end of the spring, and

$\delta$ = Original deflection of the spring.

We know that the bending moment at the fixed end of the leaf.

$$M = W\,l \qquad \text{...(i)}$$

and moment resisted by one plate

$$= \frac{\sigma \cdot I}{y} \qquad \text{...}\left(\because \frac{M}{I} = \frac{\sigma}{y}\right)$$

$$= \frac{\sigma \cdot \frac{bt^3}{12}}{\frac{t}{2}} = \frac{\sigma bt^2}{6} \qquad \text{...}\left(\because I = \frac{bt^3}{12} \text{ and } y = \frac{t}{2}\right)$$

∴ Total moment resisted by $n$ plates,

$$M = \frac{n\sigma bt^2}{6} \qquad \text{...(ii)}$$

Since the maximum bending moment due to load is equal to the total resisting moment, therefore equating (i) and (ii),

$$Wl = \frac{n\sigma bt^2}{6}$$

or $$\sigma = \frac{6\ Wl}{nbt^2}$$

From the geometry of the spring figure, we know that

$$\delta\,(2R - \delta) = l\,.\,l = l^2$$

$$\therefore \quad \delta = \frac{l^2}{2R} \quad \text{...(Neglecting } \delta^2\text{)} \qquad \text{...(iii)}$$

We know that in the case of a bending cantilever,

$$\frac{\sigma}{y} = \frac{E}{R}$$

or $$R = \frac{E \cdot y}{\sigma} = \frac{Et}{2\sigma} \qquad \ldots\left(\because\ y = \frac{t}{2}\right)$$

Substituting this value of $R$ in Eq. (iii)

$$\delta = \frac{l^2}{2 \times \dfrac{Et}{2\sigma}} = \frac{\sigma l^2}{Et}$$

Now substituting the value of σ in the above equation,

$$\delta = \frac{6\ Wl}{nbt^2} \times \frac{l^2}{Et} = \frac{6\ Wl^3}{Enbt^3}$$

**Example 8.6.** *A cantilever type laminated spring (quarter elleptic) has a span of 0.5 metre. If each leaf be 8 mm thick and 72 mm wide, find the number of leaves so that the spring deflects 60 mm under an end load of 3 kN. Determine maximum bending stress at this load. Also determine the height from which this load may be allowed to fall so that maximum bending stress induced is 700 N/mm².*

*Given :* $L = 0.5\ m = 500\ mm$    $t = 8\ mm$

$b = 72\ mm$    $\delta = 60\ mm$

$W = 3\ kN = 3000\ N$    $E = 2 \times 10^5\ N/mm^2$

**Solution**

$$\delta = \frac{6WL^3}{nEbt^3}$$

$$60 = \frac{6 \times 3000 \times 500^3}{N \times 2 \times 10^5 \times 72 \times 8^3}$$

$$n = 5 \text{ Nos.}$$

Maximum bending stress

$$\sigma_{max} = \frac{6WL}{nbt^2} = \frac{6\times3000\times500}{5\times72\times8^2} = 390.625 \text{ N/mm}^2$$

Strain energy stored in the spring = P.E. by spring

$$\frac{W_e\,\delta_1^2}{2} = W\,(h + \delta_1)$$

Equivalent load $\sigma_{max} = \dfrac{6W_e\,L}{nbt^2}$

$$700 = \frac{6\times W_e\times500}{5\times72\times8^2}$$

$$W_e = 5376 \text{ N}$$

$$\therefore \quad \delta_1 = \frac{6\times5376\times500^3}{5\times2\times10^5\times72\times8^3} = 109.375 \text{ mm}$$

$$= \frac{5376\times109.375^2}{2} = 2\times10^5\,(h + 109.335)$$

$$\mathbf{h = 51.4 \text{ mm}}$$

## 8.9 ❑ HELICAL SPRINGS

It is a torsion spring and made up of a wire coiled into a helix. Though there are many types of helical springs, yet the following are important from the subject point of view:

1. Closely-coiled helical springs and
2. Open-coiled helical springs.

## 8.10 ❑ CLOSELY-COILED HELICAL SPRINGS

In a closely coiled helical spring, the spring wire is coiled so close that the each turn is practically a plane at right angles to the axis of the helix and the wire is subjected to torsion. The bending stress is negligible as compared to the torsional stress. A closely-coiled helical spring may be subjected to (1) axial loading and (2) axial twist. In this chapter, we shall discuss both the cases one by one.

## 8.11 ❑ CLOSELY-COILED HELICAL SPRINGS SUBJECTED TO AN AXIAL LOAD

Consider a closely-coiled helical spring subjected to an axial load as shown in Fig. 8.4.

Let $d$ = Diameter of the spring wire

$R$ = Mean radius of the spring coil

$n$ = No. of turns of coils
$C$ = Modulus of rigidity for the spring material
$W$ = Axial load on the spring
$\tau$ = Maximum shear stress induced in the wire due to twisting
$\theta$ = Angle of twist in the spring wire and
$\delta$ = Deflection of the spring, as a result of axial load.

A little consideration will show that the load $W$ will cause a twisting moment,

$$T = W \,.\, R \qquad \text{...(i)}$$

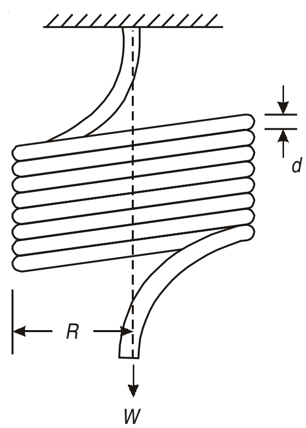



**Fig. 8.4**

We know that the twisting moment

$$T = \frac{\pi}{16} \times \tau \times d^3 \qquad \text{...(ii)}$$

$$\therefore \qquad W \,.\, R = \frac{\pi}{16} \times \tau \times d^3$$

We also know that the length of the wire

$$l = \text{Length of one coil} \times \text{No. of coils}$$
$$= 2\,\pi R \,.\, n$$

We have already discussed in Sec. 8.5 that

$$\frac{T}{J} = \frac{C \cdot \theta}{l}$$

or

$$\theta = \frac{T \cdot l}{J \cdot C} = \frac{WR \cdot 2\pi\, Rn}{\frac{\pi}{32} \times d^4 C} \qquad \text{...}(\because T = WR)$$

$$= \frac{64WR^2n}{Cd^4}$$

$\therefore$ Deflection of the spring,

$$\delta = R \,.\, \theta = R \times \frac{64WR^2n}{Cd^4} = \frac{64WR^3n}{Cd^4}$$

We know that the energy stored in the spring,

$$U = \frac{1}{2} W \cdot \delta$$

and stiffness of the spring, $$\delta = \frac{W}{\delta} = \frac{Cd^4}{64R^3n}$$

## 8.12 ❑ CLOSELY-COILED HELICAL SPRINGS SUBJECTED TO AN AXIAL TWIST

Consider a closely-coiled helical spring subjected to an axial twist as shown in Fig. 8.5.

Let $d$ = Diameter of the spring wire
$R$ = Mean radius of the spring coil
$n$ = No. of turns of coils
$C$ = Modulus of rigidity for the spring material and
$M$ = Moment or axial twist applied on the spring.

A little consideration will show that the number of spring coils will tend to increase or decrease depending upon the sense of the moment. Moreover, if the number of turns tend to increase then the mean radius of the spring coil will

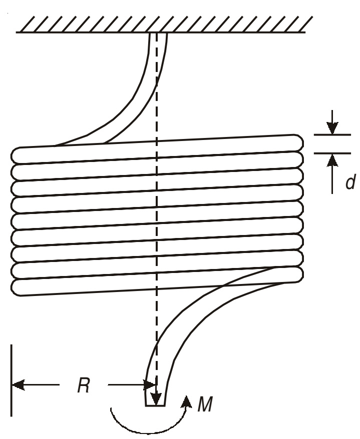



**Fig. 8.5** *Closely coiled helical spring*

decrease. Now let us consider that the number of turns increase from $n$ to $n'$ and the mean radius decreases from $R$ to $R'$.

Now length of the spring,

$$l = 2\pi Rn = 2\pi R' n' \quad \text{...(i)}$$

$$\therefore \quad \frac{1}{R} = \frac{2\pi n}{l} \text{ and } \frac{1}{R'} = \frac{2\pi n'}{l}$$

We know that $$\frac{M}{I} = E \times \text{Change of curvature}$$

$$= E\left(\frac{1}{R'} - \frac{1}{R}\right) = E\left(\frac{2\pi n'}{l} - \frac{2\pi n}{l}\right) = \frac{2\pi E}{l}(n' - n)$$

or $$2\pi(n' - n) = \frac{Ml}{EI} \quad \text{...(ii)}$$

We also know that the total angle of bend,

$$\phi = 2\pi(n' - n)$$

Substituting the value of $2\pi(n' - n)$ from Eq. (ii),

$$\phi = \frac{Ml}{EI}$$

Differentiating the above equation with respect to $l$,

$$\frac{d\phi}{dl} = \frac{Ml}{EI}$$

It is thus obvious that the change in curvature or angle of bend per unit length, is constant throughout the spring.

We know that the energy stored in the spring,

$$U = \frac{1}{2}M \cdot \phi$$

**Example 8.7.** *A close coiled helical spring made of 10 mm diameter steel bar has 8 coils of 150 mm mean diameier. Calculate the elongation, maximum shear stress and strain energy per unit voiume when the spring is subjected to an axial load of 130 N. Take G = 80 GPa.*

*Given :*

$D$ = *150 mm* $\quad$ $W$ = *130 N* $\quad$ $n = 8$

$d$ = *10 mm* $\quad$ $G = 80 \times 10^3$ *N/mm*$^2$

**Solution**

Mean radius $$R = \frac{D}{2} = \frac{150}{2} = 75 \text{ mm}$$

(i) Shearing stress $\tau = \dfrac{16\,WR}{\pi d^3} = \dfrac{16 \times 130 \times 75}{\pi (10)^3}$ **= 49.66 N/mm²**

(ii) Deflection ot spring $\delta = \dfrac{64WR^3 n}{Gd^4} = \dfrac{64 \times 130 \times 75^3 \times 8}{80 \times 10^3 \times 10^4}$

$\delta$ = 35.1 mm

(iii) Stiffness of the spring $k = \dfrac{W}{\delta} = \dfrac{130}{35.1}$ = **3.704 N/mm**

(iv) Strain energy per unit volume $= \dfrac{u}{v} = \left( \dfrac{\dfrac{32W^2R^3n}{cd^4}}{\dfrac{\pi}{4}d^2 \times 2\pi R n} \right) = \dfrac{64W^2R^2}{\pi^2 d^6 G}$

$= \dfrac{64 \times 130^2 \times 75^2}{\pi^2 \times 10^6 \times 80 \times 10^3} = 7.705 \times 10^{-3}$ N/mm²

**Example 8.8.** *A closed coiled helical spring made of 10 mm diameter steel bar 8 coils of 150 mm mean diameter. Calculate the elongation, torsional stress and strain energy/unit volume when the spring is subjected to an axial load of 130 N. Take G = 80 GPa. If instead of axial twist bending stress and strain energy/volume. Take E = 200 GPa.* *(UPTU : 2010–2011)*

*Given :*

$d$ = *10 mm* $n = 8$

*Mean diameter* $D$ = *150 mm*

*Modulus of rigidity* $G$ = *80 GPa = 80 × 10³ MPa*

*Axial load* $W$ = *130 N*

**Solution**

**Case I : Spring subjected to an axial load :**

Mean radius $R = \dfrac{D}{2} = \dfrac{150}{2}$ = 75 mm

Deflection $\delta = \dfrac{64WR^3n}{Gd^4} = \dfrac{64 \times 130 \times 75^3 \times 8}{80 \times 10^3 \times 10^4}$ = **35.1 mm**

Shear stress $\tau = \dfrac{64WR}{\pi d^4} = \dfrac{64 \times 130 \times 75}{\pi \times 10^3}$ = **49.66 N/mm²**

Strain energy per unit volume $= \dfrac{\tau^2}{4G} = \dfrac{49.66^2}{4 \times 80 \times 10^3}$

$$= \mathbf{7.705 \times 10^{-3}\ N/m^2}$$

**Case II : Spring subjected to axial Torque :**

$$T = 9 \text{ N.m}, \quad E = 200 \text{ GPa} = 200 \times 10^3 \text{ MPa}$$

Axial twist
$$\phi = \frac{128\, TRn}{Ed^4} = \frac{128 \times 9 \times 10^3 \times 75 \times 8}{200 \times 10^3 \times 10^4} = \mathbf{0.3456\ rad}$$

Bending stress
$$\sigma = \frac{32T}{\pi d^3} = \frac{32 \times 9 \times 10^3}{\pi \times 10^3} = \mathbf{91.67\ MPa}$$

## 8.13 ❑ OPEN-COILED HELICAL SPRINGS

In an open helical spring, the spring wire is coiled in such a way, that there is large gap between the two consecutive turns. As a result of this the spring can take compressive load also.

An open helical spring, like a closed helical spring, may be subjected to (i) axial loading or (ii) axial twist. In this chapter, we shall discuss only the first case.

Now consider an open coiled helical spring subjected to an axial load as shown in Fig. 8.6.

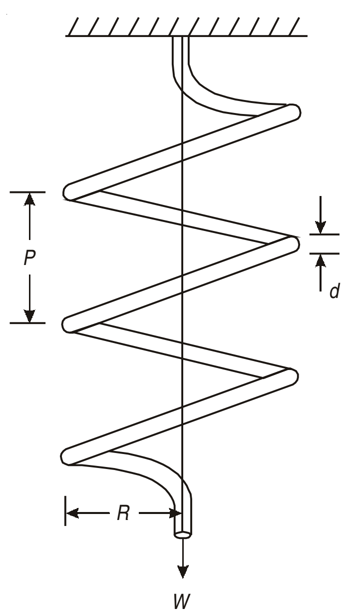



**Fig. 8.6** *Open coiled helical spring*

Let $d$ = Diameter of the spring wire

$R$ = Mean radius of the spring coil

$p$ = Pitch of the spring coils
$n$ = No. of turns of coils
$C$ = Modulus of rigidity for the spring materials
$W$ = Axial load on the spring
$\tau$ = Maximum shear stress induced in the spring wire due to loading
$\sigma_b$ = Bending stress induced in the spring wire due to bending
$\delta$ = Deflection of the spring as a result of axial load and
$\alpha$ = Angle of helix.

A little consideration will show that the load $W$ will cause a moment $WR$. This moment may be resolved into the following two components,

$$T = WR \cos \alpha \qquad \text{... (It causes twisting of coils)}$$

$$M = WR \sin \alpha \qquad \text{... (It causes bending of coils)}$$

Let $\delta$ = Angle of twist, as a result of twisting moment, and

$\phi$ = Angle of bend, as a result of bending moment.

We know that the length of the spring wire,

$$l = 2\pi nR \sec \alpha \qquad \text{...(i)}$$

and twisting moment,

$$W \,.\, R \cos \alpha = \frac{\pi}{16} \times \tau \times d^3 \qquad \text{...(ii)}$$

We also know that bending stress,

$$\sigma_b = \frac{M \cdot y}{I} = \frac{WR \sin \alpha \cdot \dfrac{d}{2}}{\dfrac{\pi}{64} \times d^4} \qquad \text{...}\left(\because \frac{M}{I} = \frac{\sigma_b}{y}\right)$$

$$= \frac{32\, WR \sin \alpha}{\pi d^3} \qquad \text{...(iii)}$$

and angle of twist $(\theta)$

$$\theta = \frac{Tl}{JC} = \frac{WR \sin \alpha \cdot l}{JC} \qquad \left(\because \frac{T}{J} = \frac{C\theta}{l}\right)$$

We have also seen in the previous article, that angle of bend due to bending moment,

$$\phi = \frac{Ml}{El} = \frac{WR \sin \alpha \cdot l}{El}$$

We know that the work done by the load in deflecting the spring, is equal to the stress energy of the spring.

$$\therefore \qquad \frac{1}{2}W \cdot \delta = \frac{1}{2}T + \frac{1}{2}M \cdot \phi$$

or

$$W \,.\, \delta = T \,.\, \theta + M \,.\, \phi$$

$$= \left[WR\cos\alpha \times \frac{WR\cos\alpha \cdot l}{JC}\right] + \left[WR\sin\alpha \times \frac{WR\sin\alpha \cdot l}{EI}\right]$$

or

$$\delta = WR^2 l \left[\frac{\cos^2\alpha}{JC} + \frac{\sin^2\alpha}{EI}\right] \qquad \text{...(iv)}$$

Now substituting the values of $l = 2\pi nR \sec\alpha$, $J = \frac{\pi}{32}(d)^4$ and $I = \frac{\pi}{64}(d)^4$ in the above equation,

$$\delta = WR^2 \times 2\pi nR \sec\alpha \left[\frac{\cos^2\alpha}{\frac{\pi}{32}d^4 C} + \frac{\sin^2\alpha}{E \times \frac{\pi}{64}d^4}\right]$$

$$= \frac{64\ WR^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{C} + \frac{2\sin^2\alpha}{E}\right] \qquad \text{... (v)}$$

**Note.** If we substitute $\alpha = 0$ in the above equation, it gives deflection of a closed coiled spring i.e.,

$$\delta = \frac{64\ WR^3 n}{Cd^4}$$

**Angular rotation**

Let $\psi$ is the resultant of the rotations $\theta$ and $\alpha$.

$$\psi = \theta \sin\alpha - \phi\cos\alpha = \frac{TL}{GJ}\sin\alpha - \frac{ML}{EI}\cos\alpha$$

$$= \frac{WR\cos\alpha \times 2\pi R\sec\alpha \times n}{GJ}\sin\alpha - \frac{WR\sin\alpha \times 2\pi R\sec\alpha \times n}{EI}$$

$$\psi = 2WR^2 n\pi \sin\alpha \left[\frac{1}{GJ} - \frac{1}{EI}\right] \qquad \text{...(v)}$$

$$= 2WR^2 n\pi \sin\alpha \left[\frac{1}{G \times \frac{\pi}{32}d^4} - \frac{1}{E \times \frac{\pi}{64}d^4}\right]$$

$$\boldsymbol{\psi = \frac{64\ WR^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right]}$$

**Numerical on Open Coiled Helical Spring**

$$\text{Bending stress } \sigma = \frac{32\, WR \sin\alpha}{\pi\, d^3}$$

$$\text{Shear stress } \tau = \frac{16\, WR \cos\alpha}{\pi\, d^3}$$

$$\text{Deflection } \delta = \frac{64\, WR^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right]$$

$$\text{Rotation } \psi = \frac{64\, WR^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right]$$

**Example 8.9.** *Find the mean radius of an open coiled spring of helix angle 30°, to give a vertical a displacement of 23 mm and angular rotation of the load end of 0.02 radtans under an axial load of 35 N. The material available is steel rod 6 mm diameter.*

*$E = 2 \times 10^5 N/mm^2$, $G= 8.0 \times 10^4\ N/mm^2$* (*UPTU : 2010–2013*)

*Given :*

$W$ = *35 N*

$D$ = *6 mm*

$\alpha$ = *30°*

$\psi$ = *0.02 radian*

$\delta$ = *23 mm*

$G$ = *$80 \times 10^3 MPa$*

$E$ = *$200 \times 10^3 MPa$*

**Solution**

**Case I : Axial deflection**

$$\delta = \frac{64\, WR^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right]$$

$$23 = \frac{64 \times 35 \times R^3 \times n \sec 30^\circ}{6^4}\left[\frac{\cos^2 30^\circ}{80 \times 10^3} + \frac{2\sin^2 30^\circ}{200 \times 10^3}\right]$$

$$\mathbf{R^3\, n = 970.47 \times 10^3} \quad \text{...(i)}$$

**Case II: Angular rotation**

$$\psi = \frac{64\, WR^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right]$$

$$0.02 = \frac{64 \times 35 \times R^2 \times n \sin 30^\circ}{6^4}\left[\frac{1}{80 \times 10^3} + \frac{2}{200 \times 10^3}\right]$$

$$R^2 n = 1.0286 \times 10^3 \qquad ...(ii)$$

Divide (i) to (ii), we get $\dfrac{R^3 n}{R^2 n} = \dfrac{970.47 \times 10^3}{1.0286 \times 10^3}$

$\therefore$ $$\mathbf{R = 943.5 \ mm}$$

**Example 8.10.** *Determine the maximum angle of helix for which, the error in calculating the extension of a helical spring under axial load by the "close-coiled" formula is less than 1 %.* (*UPTU : 2010-2011*)

**Solution**

Length of open coiled spring $= L_1 = 2\pi Rn \sec \alpha$

Length of closed coiled spring $= L_2 = 2\pi Rn$

Extension in open coiled spring is $\delta_1$ and

Extension in closed coiled spring is $\delta_2$

$$\text{Error in extension} = 1\% = 0.01$$

$$\frac{\delta_1 - \delta_2}{\delta_1} = 0.01$$

$$1 - \frac{\delta_2}{\delta_1} = 0.01$$

$$\delta_2 = 0.99\,\delta_1 \qquad ...(i)$$

Assume Poisson's ratio $\mu = 0.3$

$$E = 2N(1 + \mu)$$

$$\frac{2N}{E} = \frac{1}{1+\mu} = \frac{1}{1+0.3} = 0.7692 \qquad ...(ii)$$

$$\delta_2 = 0.99\,\delta_1$$

$$\frac{64WR^3 n}{Nd^4} = 0.99\left[\frac{64WR^3 n}{d^4 \cos\alpha}\left(\frac{\cos^2\alpha}{N} + \frac{2\sin^2\alpha}{E}\right)\right]$$

$$1 = 0.99\left[\frac{1}{\cos\alpha}\left(\cos^2\alpha + \frac{2N}{E}\sin^2\alpha\right)\right]$$

$$1.0101\cos\alpha = \cos^2\alpha + 0.7692\,(1 - \cos\alpha)$$

$$1.0101\cos\alpha = -0.2308\cos^2\alpha + 0.7692$$

$$0.2308\cos^2\alpha + 1.0101\cos\alpha - 0.7692 = 0$$

$\therefore$ $$\cos\alpha = 0.6615$$

$$\boldsymbol{\alpha = \cos^{-1}(0.6615) = 48.59^\circ}$$

**Example 8.11.** *A helical spring having 12 coils of mean coil diameter of 20 cm is made of 10 mm diameter steel rod. The helix angle is 25°.*

*Find the angular twist and the axial deflection of one end of the spring relative* *to the other if it is subjected to an axial couple of 14 Nm. Calculate the maximum bending stress and maximum torsional stress in the wire.*

*Take $E = 200\ GN/m^2$ and $G = 80\ GN/m^2$* (*UPTU : 2006–2007*)

*Given :*

$$n = 12$$
$$D = 20\ cm = 200\ mm$$
$$d = 10\ mm,\ \alpha = 25°$$
$$E = 200\ GPa$$
$$G = 80\ GPa$$
$$M = 14\ N$$
$$m = 14000\ N\text{-}mm$$

**Solution To find:** Angular twist $\theta$. $\sigma_b$ and $\tau$,

Moment of inertia $$I = \frac{\pi d^4}{64} = \frac{\pi(10)^4}{64} = 490.87\ \text{mm}^4$$

Polar moment of inertia $$J = 2I = 981.75\ \text{mm}^4$$

Helix angle $$\alpha = 25 \times \frac{\pi}{180} = 0.4363°$$

Angular twist

$$\theta = 2\pi n\, R \sin\alpha\, M \left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

$$= 2\pi \times 12 \times 100 \sin 25 \times 14000 \left[\frac{1}{80\times10^3 \times 981.75} - \frac{1}{200\times10^3\times490.87}\right]$$

$$= 0.1136 \text{ radian}$$

$$\theta = 0.1136 \times \frac{180}{\pi}$$

$$\boldsymbol{\theta = 6.508°}$$

Maximum stress in spring wire

$$\sigma_b = \frac{32M\cos a}{\pi d^3} = \frac{32\times14000\cos 25}{\pi(10)^3}$$

$$= \mathbf{129.24\ \ N/mm^2}$$

Maximum shear stress in spring

$$\tau = \frac{16M\cos a}{\pi d^3} = \frac{32\times14000\sin 25}{\pi(10)^3}$$

$$= \mathbf{30.132\ \ N/mm^2}$$

## EXERCISE

**8.1.** A laminated spring 1 m long is built in 100 mm × 10 mm plates. If the spring is to carry a load of 10 kN at its centre determine the number of plates required for the spring. Take allowable bending stress as 150 MPa. **[*Ans.* $n$ = 10]**

**8.2.** A leaf spring 1 m long is made up with steel plates with width equal to 6 times its thickness. Design the spring for a load of 15 kN when the maximum stress is 100 MPa and deflection is not to exceed 16 mm.

**[*Ans.* $t$ = 12.5 mm; $b$ = 75 mm; $n$ = 12]**

**8.3.** A carriage spring 800 mm long is made of 12 plates of 40 mm width. Determine the thickness of the plates if bending stress is not to exceed 200 MPa and spring is to carry a load of 6 kN at its centre. Also determine the central deflection of the spring. Take $E$ as 200 GPa. **[*Ans.* 9 mm; 16.5 mm]**

**8.4.** A laminated spring of the quarter elliptic type 600 mm long is to provide a deflection of 75 mm under an end load of 1960 N. If the leaf material is 60 mm wide and 6 mm thick, find the number of leaves required and the maximum stress. **[*Ans.* 13; 252 N/mm$^2$]**

**8.5.** A closely coiled helical spring of mean diameter 140 mm is made up of 12 mm diameter steel wire. Calculate the direct axial load, the spring can carry if the maximum stress is not to exceed 100 MPa. **[*Ans.* 484 N]**

**8.6.** A closely coiled helical spring is made of 6 mm wire. The maximum shear stress and deflection under a 200 N load is not to exceed 80 MPa and 11 mm respectively. Determine the number of coils and their mean diameter. Take $C$ = 84 MPa. **[*Ans.* 20; 34 mm]**

**8.7.** An open coil helical spring made of 10 mm diameter wire has 15 coils of 50 mm radius with a 20° angle of helix. Determine the deflection of the spring, when subjected to an axial load of 300 N. Take $E$ = 200 GPa and $C$ = 80 GPa.

**[*Ans.* 47.4 mm]**

**8.8.** A semi-elliptic carriage spring 100 cm long is to absorb 150 J of energy under a proof load 4500 N (Proof load is a load to straighten the spring). Assuming a working stress of 350 MN/m$^2$, find the thickness of leaves. What is the initial curvature of the spring ? $E$ = 20 GN/m$^2$. **[*Ans.* 6.3 mm, 189 cm]**

**8.9.** A single leaf spring of uniform rectangular section is to be fixed one end and loaded at the other end. Length of spring is 75 mm and the load is 90 N. If the maximum deflection and the maximum bending stress are 6.25 mm and 425.MN/m$^2$ respectively, find the thickness and breadth of spring.

**[*Ans.* $t$ = 1.2 mm and $b$ = 65 mm]**

**8.10.** A semi-elliptic laminated spring 105 cm long has 9 leaves and is held at the centre by a 6.25 cm wide band. The spring is to carry a maximum toad of 5400 N. Taking allowable stress for the spring material as 490 MN/m$^2$, calculate the width and thickness of leaves if deflection is not to exceed 7.5 cm when (a) leaves are stressed initially to get equal stress at maximum load and (b) leaves are not stressed initially. Assume that spring has two full length leaves. $E$ = 210 GN/m$^2$.

**[*Ans.* (a) $t$ = 6.8 mm, $b$ = 39 mm, (b) $t$ = 5 mm, $b$ = 98 mm]**

**8.11.** A quarter elliptic spring 90 cm long is composed of 8 graduated leaves and one extra full length leaf (i.e. total leaves = 9). The leaves are 40 mm wide. The spring is to carry a load of 2200 N at the free end with a maximum deflection of 75 mm. Determine the thickness of leaves and the maximum bending stress assuming (i) the leaves are prestressed to provide same maximum stress in leaves and (ii) the leaves are not prestressed. $E$ = 210 GN/m$^2$. **[*Ans.* (i) 1.17 cm, 241 MN/m$^2$; (ii)1.17 cm; $\sigma_t$ = 342 MN/m$^2$]**

**8.12.** A carriage spring made of laminations each 100 mm wide and 8 mm thick, is 0.8 m long. Determine the number of plates required when the spring is designed for a maximum central load of 6 kN and maximum allowable stress in the material is not to exceed 200 MN/m$^2$. What would be the deflection under the load when $E$ = 200 GN/m$^2$ ? **[*Ans*. 5.625 say 6; 18.75 mm]**

**8.13.** A leaf of semi-elliptical type is built-up with 9 leaves of 0.5 cm × 5 cm cross-section. The length of spring is 0.75 m and the leaves are of steel having proof stress of 400 MN/m$^2$. Determine to what radius should the leaves be initially bent? Also calculate the value of proof load. From what maximum height should a load of 200 N fall without exceeding the proof stress in the leaves of the spring? $E$ = 200 GN/m$^2$. **[*Ans.* 1.25 m, 4 kN, 50.63 cm]**

**8.14.** A semi-elliptic carriage spring of 120 cm length is to carry a maximum load of 55 kN with a maximum deflection of 9 cm. Taking safe bending stress of 650 MN/m$^2$, determine the suitable thickness of leaves. If number of leaves is 10, find the width. **[*Ans.* 13 mm, 90 mm]**

**8.15.** A graduated cantilever leaf spring is to absorb 620 J of energy without exceeding a deflection of 15 cm and a permissible stress of 875 MN/m$^2$. If length of spring is 60 cm and $E$ = 210 GN/m$^2$ find suitable thickness of leaves. If width is 5 cm, find the number of leaves. **[*Ans.* $t$ = 10 mm, $n$ = 7]**

**8.16.** A cantilever spring 60 cm long is made of 12 graduated leaves each 6 cm wide. Maximum load at the free end is 10 kN. What is thickness of each leaf for a max. stress of 1000 MN/m$^2$? Find the end deflection. Take $E$ = 210 GN/m$^2$.

**8.17.** A quarter elliptic spring 60 cm long is composed of 12 leaves each 6 cm wide. For a load of 10 kN at the free end and a maximum bending stress of 500 MN/m$^2$, find the thickness of each leaf and the maximum deflection. What are theoretical lengths of various leaves ? $E$ = 200 GN/m$^2$.

**[*Ans.* 1 cm, 9 cm and lengths are 5, 10, 15, 20, 25, 30, 35,40, 45, 50, 55 and 60 cm]**

**8.18.** An open-coiled helical spring made from wire if circular cross-section is required to carry a load of 100 N. The wire diameter is 8 mm and the mean coil radius is 40 mm, if the helix angle of the spring is 30° and number of turns is 12, calculate :

(i) Axial deflection;

(ii) Angular rotation of free end with respect to the fixed end of the spring.

**[*Ans.* (i) 16.45 mm; (ii) 0.0376 radian]**

**8.19.** An open-coiled helical spring of wire diameter 10 mm, mean coil radius 70 mm, helix angle 20° carries as axial load of 400 N. Determine the shear stress and direct stress developed at the inner radius of the coil.

**[*Ans.* $\tau$ = 139.09 MN/m$^2$, $\sigma_b$ = 97.54 MN/m$^2$]**

**8.20.** An open-coiled spring consists of 10 coils, each of mean diameter 50 mm. The wire forming the coil being 6 mm in diameter. Each coil makes an angle of 30° with plane perpendicular to the axis of the spring.

(i) Determine the load required to elongate the spring by 20 mm and the bending and shear stresses caused by that load.

(ii) Calculate the axial twist that would cause a bending stress of 60 MN/m$^2$. Take : $E$ = 200 GN/m$^2$, and $C$ = 82 GN/m$^2$.

**[*Ans.* (i) W = 192.7 N, 113.6 MN/m$^2$, 98.37 MN/m$^2$ (ii) 1.469 NM]**

## UNIVERSITY QUESTIONS

1. Deduce an expression for the extension of an open-coiled helical spring carrying an axial load $W$. Take $\alpha$ as the inclination of the coils, $d$ as the diameter of the wire and $R$ as the mean radius of the coils. Find by what percentage the axial extension is underestimated if the inclination of the coil is neglected for a spring in which $\alpha$ = 25°. Assume $n$ and $R$ remain constant.

   (*UPTU : 2005–2006*)

   **[*Ans.* Section 8.13]**

2. A helical spring having 12 coils of mean coil diameter of 20 cm is made of 10 mm diameter steel rod. The helix angle is 25°.

   Find the angular twist and the axial deflection of one end of the spring relative to the other if it is subjected to an axial couple of 14 Nm. Calculate the maximun bending stress and maximum torsional stress in the wire.

   Take $E$ = 200 GN/m$^2$ and $G$ = 80 GN/m$^2$ (*UPTU : 2006–2007*)

   **[*Ans.* Example 8.10]**

3. An open coiled spring carries an axial load $W$, show, that the deflection is related to $W$ by $\delta = \dfrac{8WnD^3}{Gd^u} K$ where $K$ is a corrective factor which allows for the inclination of the coils, $n$ = number of effective coils, $D$ = mean coil diameter and $d$ = wire diameter. (*UPTU : 2007–2008*)

   **[*Ans.* Section 8.13]**

4. A leaf spring is made of plates 50 mm wide and 8 mm thick.The spring has a span of 700 mm. Determine the number of plates required to carry a central load of 45 kN. The maximum allowable stress in the plates is 200 MPa.What is the maximum deflection under this load? (*UPTU : 2007–2008*)

   **[*Ans.* Exampl 8.2]**

5. A cantilever type laminated spring (quarter elleptic) has a span of 0.5 metre. If each leaf be 8 mm thick and 72 mm wide, find the number of leaves so that the spring deflects 60 mm under an end load of 3 kN. Determine maximum bending stress at this load. Also determine the height from which this load may be allowed to fall so that maximum bending stress induced is 700 N/mm$^2$. Take $E$ = 2 ×10 N/mm$^2$. (*UPTU : 2008–2009*)

   **[*Ans.* Example 8.5]**

6. A leaf spring has 12 plates each 50 mm wide and 5 mm thick, the longest plate being 600 mm long. The greatest bending stress is not to exceed 180 N/mm$^2$ and the central deflection is 15 mm. Estimate the magnitude of the greatest central load that can be applied to the spring. $E = 0.206 \times 10^6$ N/mm$^2$. (*UPTU : 2009–2010*)

**[*Ans.* Example 8.3]**

7. A closed coiled helical spring made of 10 mm diameter steel bar 8 coils of l50 mm, mean diameter. Calculate the elongation, torsional stress and strain energy/unit volume when the spring is subjected to an axial load of 130 N. Take $G$ = 80 GPa. If instead of axial twist bending stress and strain energy/volume. Take $E$ = 200 GPa. (*UPTU : 2010–2011*)

**[*Ans.* Example 8.7]**

8. Determine the maximum angle of helix for which the error in calculating the extension of a helical spring under axial load by the "close-coiled" formula is less than 1%. (*UPTU : 2010–2011*)

**[*Ans.* Example 8.9]**

9. A cantilever leaf spring of length 0.43 m has four leaves of thickness 9 mm. If an end load of 2.5 kN causes a deflection of 36 mm find the width of the leaves, $E$ = 200,000 N/mm$^2$. (*UPTU : 2010–2011*)

**[*Ans.* Example 8.3]**

10. A close coiled helical spring made of 10 mm diameter steel bar has 8 coils of 150 mm mean diameter. Calculate the elongation, maximum shear stress and strain energy per unit volume when the spring is subjected to an axial load of 130 N. Take $G$ = 80 GPa. (*UPTU : 2011–2012*)

**[*Ans.* Example 8.6]**

11. An elliptic type of leaf spring is 800 mm long. Static deflection of spring under a load of 3 kN is 100 mm, determine the number of leaves required and maximum stress if the leaves are 75 mm wide and 8 mm thick. Take G = 80 GPa. (*UPTU : 2011– 2012*)

12. Define springs. Name the different types of spring ? (*UPTU : 2012–2013*)

**[*Ans.* Section 8.1 and 8.3]**

13. Find the mean radius of an open coiled spring of helix angle 30˚, to give a vertical displacement of 23 mm and an angular rotation of the load end of 0.02 radians under an axial load of 35 N. The material available is steel rod 6 mm diameter. $E = 2 \times 10^5$ N/mm$^2$, $G = 8.0 \times 10^4$ N/mm$^2$. (*UPTU : 2012–2013*)

**[*Ans.* Example 8.8]**

14. What are leaf springs ? Find proof load and maximum bending stress in a semi-elliptic type leaf spring. (*UPTU : 2012–2013*)

**[*Ans.* Example 8.1 Section: 8.7]**

## PROBLEMS

1. What is a spring? Explain its uses.
2. What are the various types of springs? Distinguish clearly between bending springs and torsion springs.
3. What is a laminated spring? Where is it used?
4. Derive from first principles, making usual assumptions the formula for the maximum bending stress and for the central deflection of a leaf spring consisting of $n$ leaves and subjected to a central load.
5. What are helical springs? Differentiate between a closely coiled helical spring and an open coiled helical spring.
6. A closely coiled helical spring with $D$ as diameter of the coil and $d$ as diameter of the wire is subjected to an axial load $W$. Prove that the maximum shear stress produced is equal to $\dfrac{8WD}{\pi d^3}$.
7. Derive an equation for the deflection of an open coiled helical spring.

CHAPTER

9

# Thin Cylinders and Spheres

## 9.1 ❑ INTRODUCTION

Thin walled shell both cylindrical as well as sphericals, are used as containers for storage of liquids and gases under pressure.

If the ratio of thickness to internal diameter of the shell is less than 1/20 i.e., if $\frac{t}{d} < \frac{1}{20}$ than it is termed as *thin shell*.

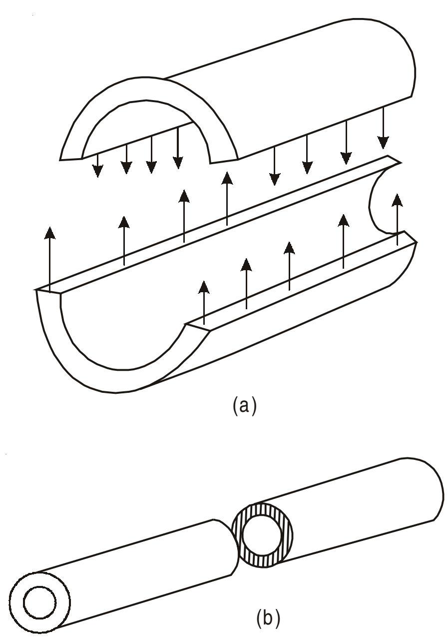



**Fig. 9.1**

Owing to internal fluid (liquid or gas) pressure such shells are subjected to :

(i) Circumferential stress or hoop stress ($\sigma_h$), and

(ii) Longitudinal stress ($\sigma_l$)

Whenever a thin cylinder is subjected to an internal pressure, it is likely to fail either by splitting into cylindrical shell (i.e. circumferentially) or by splitting it up into two troughs (i.e. longitudinally) (Fig. 9.1 (a) and (b))

## 9.2 ❑ FAILURE OF A THIN CYLINDRICAL SHELL DUE TO AN INTERNAL PRESSURE

Whenever a cylindrical shell is subjected to an internal pressure, its walls are subjected to tensile stresses. When these stresses exceed the permissible limit, the cylinder is likely to fail in any one of the following ways (Fig. 9.1 (a) and (b))

1. It may split up into two troughs (a), and
2. It may split up into two cylinders (b).

## 9.3 ❑ DERIVATION OF STRESSES IN THIN CYLINDRICAL SHELL

### 9.3.1 Hoop Stresses : When Thin Cylindrical Shell is Subjected to Internal Fluid Pressure

Consider a thin cylinder/vessel subjected to internal fluid pressure (may be gas or liquid).

Let, $p$ = Intensity of pressure due to fluid

$d$ = Diameter of cylinder

$l$ = Length of cylinder

$t$ = Thickness of cylinder

$\sigma_h$ = Hoop stress or circumferential stress in the cylinder or vessel material

Force due to liquid pressure = Intensity of pressure × Projected area on which $p$ is acting = $p.d.l$ ...(i)

Force due to circumferential stress on the cylinder wall = $\sigma_h.\ 2t.\ l$ ...(ii)

From Eqs. (i) and (ii)

$$\sigma_h.\ 2t.\ l = p.d.l$$

$$\therefore \quad \sigma_h = \frac{p.d.l}{2tl} = \frac{pd}{2t}$$

$$\therefore \text{Hoop stress,} \quad \sigma_h = \frac{pd}{2t} \qquad \text{...(iii)}$$

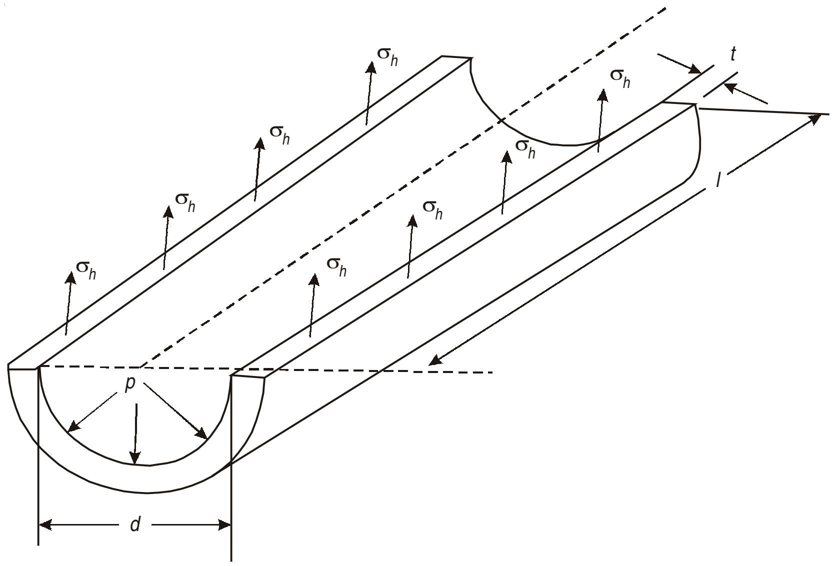



Fig. 9.2

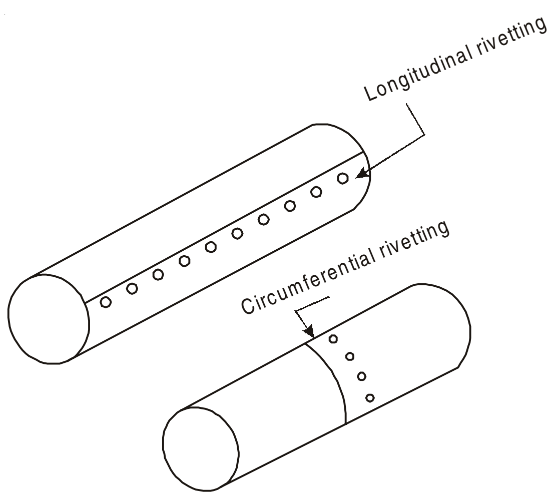



Fig. 9.3

In constructing large pressure vessels or storage tanks such as boilers, air receiver etc., welding or revitting may be required for joining the ends of plate. So while designing the thickness of pressure vessels, we must consider the efficiency of the joints.

If $\eta_l$ is the efficiency of longitudinal joint then,

Hoop stress, $$\sigma_h = \frac{p.d}{2t\eta_l}$$

If $\eta_h$ is the efficiency of the circumferential joint then longitudinal stress, $\sigma_l$ $= \dfrac{p \cdot d}{4t.} \cdot \eta_h$. The thickness of the shell in order that the hoop stress may not exceed the permissible stress is given by, $t = \dfrac{p.d}{2\sigma_h.\eta_l}$.

### 9.3.2 Longitudinal Stress ($\sigma_l$)

Consider the same cylindrical shell, subjected to same internal fluid pressure (Fig. 9.4). Longitudinal stress $\sigma_t$ will set up in the material of the cylinder along the direction of the axis of the cylinder.

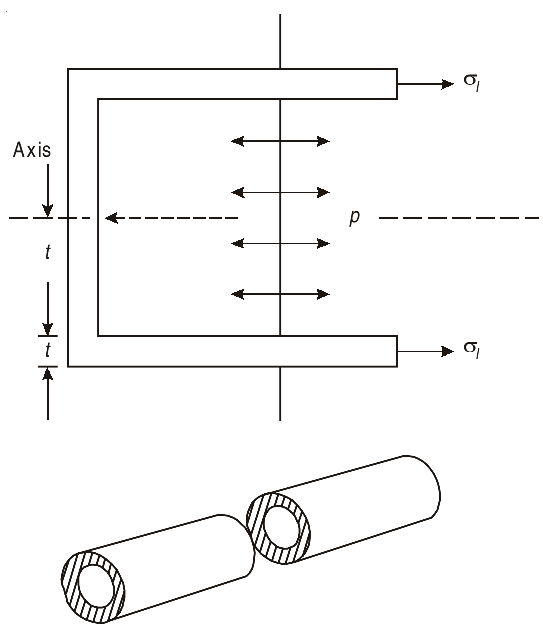



**Fig. 9.4**

Let, $\sigma_l$ = Longitudinal stress.

Force due to fluid pressure = $p$ × Area on which $p$ is acting

$$= p \cdot \frac{\pi}{4} d^2 \quad \text{...(i)}$$

Resisting force offered by the material = $\sigma_l . \pi dt$ ...(ii)

From Eqs. (i) and (ii)

$$\sigma_l . \pi dt = p.\frac{\pi}{4}d^2$$

$$\therefore \qquad \sigma_l = \frac{pd}{4t} \qquad \text{...(iii)}$$

Hence longitudinal stress is half of the hoop stress.

Hence design of pressure vessel must be based on the maximum stress i.e. hoop stress.

So in case of a thin cylinder the tendency to bursting circumferentially is twice that of longitudinal direction.

So in thin cylindrical shell, at any point there are two principal stresses.

(i) Circumferential stress $(\sigma_h) = \dfrac{pd}{2t}$ acting circumferentially, this is a major stress

(ii) Longitudinal stress $(\sigma_l) = \dfrac{pd}{4t}$ acting longitudinally and parallel to the axis of the cylinder, that is a minor stress.

Streses $\sigma_h$ and $\sigma_l$ are tensile in nature and act at right angle to each other.

Since $\sigma_h$ and $\sigma_l$ act perpendicular to each other and are maximum and minimum respectively, we can call them principal stresses also.

**Example 9.1.** *Define thin cylinders.* (*UPTU : 2004–2005*)

**Solution**

(i) Cylindrical and spherical shells are very commonly used in engineering field.

(ii) Boilers, steam pipes, reservoirs, water pipelines, gas tanks, shock absorbers etc. are the example of pressure vessels.

(iii) If the ratio of thickness to internal diameter of shell is less than $\dfrac{1}{20}, \left(\dfrac{t}{d} < \dfrac{1}{20}\right)$ then it is called as thin shell.

e.g., Pressure cooker, LPG cylinder, cylindrical pipe carrying water.

(iv) As the thickness is too small, hoop stress and longitudinal stress induced in the shell is to be assumed uniform over the thickness (In case of thin cylinder).

(v) The stresses developed radially due to internal pressure is known as radial stress. Radial stress is very small and hence neglected.

**Example 9.2.** *Define, Hoop stress, and Longitudinal stress.*

(*UPTU : 2012–2013*)

**Solution** *Stresses in a thin cylindrical shell*

When the cylindrical shell is subjected to an internal fluid pressure, the wall of shell is subjected to the tensile stresses i.e. Hoop stress and longitudinal stress, radial stress.

**(i) Circumferential Stress or Hoop Stress ($\sigma_h$)**

The stresses which are acting in the tangential direction to the perimeter (circumference) of the cylinder are called as hoop stress or circumferential stress. It is denoted by $\sigma_h$ or $\sigma_c$.

Consider a thin cylindrical shell of diameter *d*, thickness *t* and length *L* subjected to an internal fluid pressure *p* as shown in Fig. 9.5.

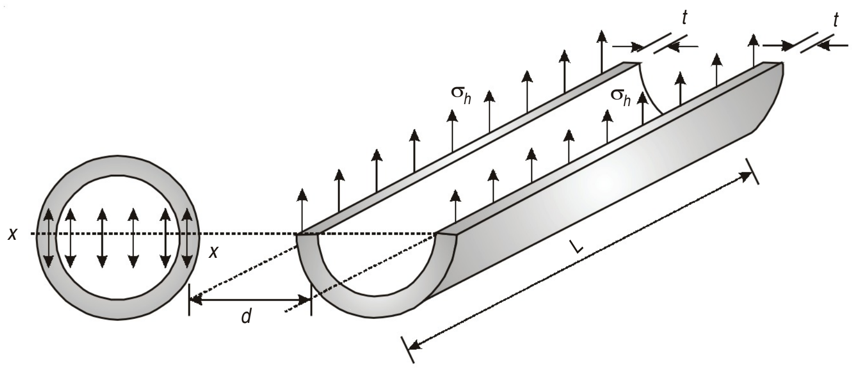



**Fig. 9.5**

The brusting force is produced due to this internal fluid pressure, which split the cylindrical shell into two semi-circular parts.

Consider lower half semi-circular part.

$$\text{Brusting force} = \text{Radial pressure} \times \text{Projected area} = p \times d \times L$$

Hoop stress induced in the material at the cross-section *x–x*, at that time a resisting force developed in the section to balance brusting force.

$$\text{Resisting force} = \text{Hoop stress} \times \text{Resisting area} = \sigma_h \times 2 \times d \times L$$

∴ For equilibrium condition,

$$\text{Resisting force} = \text{Brusting force}$$

$$\sigma_h \, 2tL = pdL \qquad \sigma_h = \frac{pd}{2t}$$

**(ii) Longitudinal Stress or Tangential Stress**

The stress acting along the length of the cylinder is called as longitudinal stress. It is denoted by $\sigma_L$.

Consider a thin cylindrical shell subjected to internal pressure. The longitudinal stress will be induced in the material of cylinder, if the brusting of cylinder takes place along section *x–x*.

If the brusting force is greater than resisting force, then shell will be spilt into two parts as shown in Fig. 9.6.

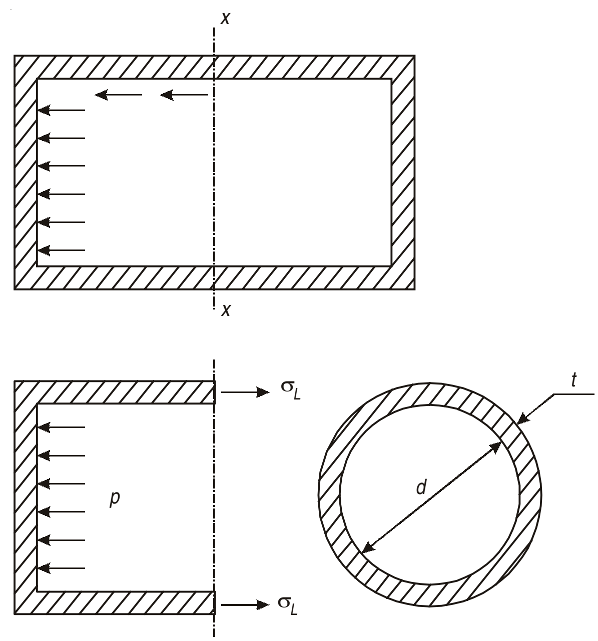



**Fig. 9.6**

$$\text{Brusting force} = p \times \frac{\pi}{4} d^2$$

$$\text{Resisting force} = \sigma_L \times \pi \times d \times t$$

For equilibrium condition,

$$\text{Resisting force} = \text{Brusting force}$$

$$\sigma_L \, \pi dt = p \frac{\pi}{4} d^2$$

$$\sigma_L = \frac{pd}{4t}$$

$$\therefore \qquad \sigma_L = \frac{1}{2} \sigma_h$$

Longitudinal stress = Half of circumferential stress or hoop stress.

### 9.3.3 Maximum Shear Stress

The stress acting on the surface of the wall of thin cylindrical shell consists of hoop stress $\sigma_h$ and longitudinal stress $\sigma_L$ as shown in Fig. 9.7. Since the thickness of the shell is small, the radial stresses are very small compare to $\sigma_h$ and $\sigma_t$. Hence the shell is subjected to bi-axial stress system.

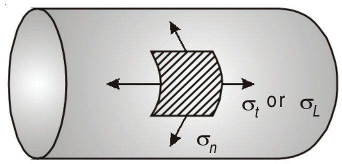



**Fig. 9.7**

$\therefore$ Maximum shear stress

$$\tau_{max} = \frac{\sigma_h - \sigma_t}{2} = \frac{\sigma_h - \frac{\sigma_h}{2}}{2} = \frac{\sigma_h}{4} = \frac{1}{4}\left(\frac{pd}{2t}\right) = \mathbf{\frac{pd}{8t}}$$

**Example 9.3.** *Prove that in case of a thin cylindrical shell, subjected to an internal fluid pressure, the volumetric strain is equal to twice the circumferential strain plus the longitudinal strain.* (*UPTU : 2006–2007*)

**Solution**

**Change in dimension of a thin cylindrical shell due to internal pressure**

We know that, $\sigma_h = \frac{pd}{2t}$ and $\sigma_L = \frac{pd}{4t}$

Let,
$\delta d$ = Change in diameter of shell
$\mu$ = Poisson's ratio
$\delta V$ = Change in volume
$\delta L$ = Change in length of shell
$\delta A$ = Change in area

**Hoop strain or circumferential strain**

$$\varepsilon_h = \frac{1}{E}(\sigma_h - \mu\sigma_L) = \frac{1}{E}\left(\frac{pd}{2t} - \mu\frac{pd}{4t}\right) = \frac{pd}{4tE}(2-\mu) = \frac{\delta d}{d}$$

Longitudinal strain,

$$\varepsilon_L = \frac{1}{E}(\sigma_L - \mu\sigma_h) = \frac{1}{E}\left(\frac{pd}{4t} - \mu\frac{pd}{2t}\right) = \frac{pd}{4tE}(1-2\mu) = \frac{\delta L}{L}$$

Volumetric strain,

$$\text{Volume } V = \frac{\pi}{4}d^2L$$

By partial differentiation,

$$\delta V = \frac{\pi}{4}d^2\delta L + L\times\frac{\pi}{4}2d\ \delta d$$

$$\therefore \qquad \frac{\delta V}{V} = \frac{\frac{\pi}{4}\left(d^2\delta L + 2Ld\,\delta d\right)}{\frac{\pi}{4}d^2 L}$$

$$\varepsilon_v = \frac{\delta L}{L} + \frac{2\delta d}{d} = \boldsymbol{\varepsilon_L + 2\varepsilon_h}$$

$$\varepsilon_v = \frac{pd}{4tE}(1-2\mu) + 2\frac{pd}{4tE}(2-\mu)$$

$$= \frac{pd}{4tE}[1 - 2\,\mu + 4 - 2\,\mu]$$

$$\boldsymbol{\varepsilon_v = \frac{pd}{4tE}[5-4\mu]}$$

Area strain $\varepsilon_A = \dfrac{\delta A}{A}$ $\qquad A = \dfrac{\pi}{4}d^2$

Differentiating area formula

$$\therefore \qquad \delta_A = \frac{\pi}{4}\times 2d \times \delta d$$

$$\frac{\delta A}{A} = \frac{\frac{\pi}{4}\times 2d\times\delta d}{\frac{\pi}{4}d^2} = 2\frac{\delta d}{d} = 2\varepsilon_h$$

$$= 2 \times \frac{pd}{4tE}(2-\mu)$$

$$\boldsymbol{\varepsilon_A = \frac{pd}{2tE}(2-\mu)}$$

$\therefore$ $\left(\dfrac{dA}{A}\right)$ equation i.e. $e_A = \dfrac{\delta A}{A} = \dfrac{pd}{2tE}(2-\mu)$

Thickness strain or radial strain,

$$\frac{\delta t}{t} = \frac{1}{E}(-\mu\sigma_h - \mu\sigma_L)$$

$$\frac{\delta t}{t} = -\mu\frac{\sigma_h}{E} - \mu\frac{\sigma_L}{E} = -\frac{\mu}{E}\left(\frac{pd}{2t}\right) - \frac{\mu}{E}\left(\frac{pd}{4t}\right)$$

$$= -\frac{\mu\, pd}{4tE}(2+1)$$

$$\boldsymbol{\varepsilon_R = \frac{\delta t}{t} = -\frac{3}{4}\frac{pd}{tE}}$$

**Numericals on Thin Cylinder**

Hoop stress $\sigma_h = \dfrac{pd}{2t}$

Longitudinal stress $\sigma_L = \dfrac{pd}{4t} = \dfrac{\sigma_h}{2}$

Maximum shear stress $\tau = \dfrac{\sigma_h - \sigma_L}{2} = \dfrac{pd}{8t}$

Hoop strain $\varepsilon_h = \dfrac{1}{E}(\sigma_h - \mu\sigma_L) = \dfrac{pd}{4tE}(2-\mu) = \dfrac{\delta d}{d}$

Longitudinal strain $\varepsilon_L = \dfrac{1}{E}(\sigma_L - \mu\sigma_h) = \dfrac{pd}{4tE}(1-2\mu) = \dfrac{\delta L}{L}$

Volumetric strain, $\varepsilon_v = 2\,\varepsilon_h + \varepsilon_L$

$$\frac{dV}{V} = \frac{pd}{4tE}(5-4\mu)$$

**Example 9.4.** *In a cylindrical shell of 0.6 m diameter and 0.9 m long is subjected to an internal pressure 1.2 N/mm². Thickness of the cylinder wall is 15 mm. Determine longitudinal stresses, circumferential stress and maximum shear stresses induced and change in diameter, length and volume. Take E = 200 GPa and* $\dfrac{1}{m} = 0.3$. *(UPTU : 2011–2012)*

*Given :*

$$t = 15 \text{ mm}$$
$$d = 0.6 \text{ m} = 600 \text{ mm}$$
$$L = 0.9 \text{ m} = 900 \text{ mm}$$
$$E = 200 \text{ kN/mm}^2$$
$$\mu = \frac{1}{m} = 0.3$$
$$p = 1.2 \text{ N/mm}^2$$

**Solution** $\sigma_L$, $\sigma_h$, $\delta_d$, $\delta_L$ and $\delta_v$

(i) Longitudinal stress $\sigma_L = \dfrac{pd}{4t} = \dfrac{1.2 \times 600}{4 \times 15}$ **= 12 MPa**

(ii) Circumferential stress

$$\sigma_h = \frac{pd}{2t} = \frac{1.2 \times 600}{2 \times 15}$$

$$= \mathbf{24\ MPa}$$

(iii) Maximum shear stress

$$\tau_{max} = \frac{pd}{8t} = \frac{1.2 \times 600}{8 \times 15}$$

$$= \mathbf{6 N/mm^2}$$

(iv) Change in diameter $\delta_d = \varepsilon_h . d = \left[\frac{pd}{4tE}(2 - \mu)\right] \times d$

$$= \left[\frac{1.2 \times 600 (2 - 0.3)}{4 \times 15 \times 2 \times 10^5}\right] \times 600$$

$$= \mathbf{0.0612\ mm}$$

(v) Change in length $\delta L = \varepsilon_L . L = \left[\frac{pd}{4tE}(1 - 2\mu)\right] \times L$

$$= \left[\frac{1.2 \times 600 (1 - 2 \times 0.3)}{4 \times 15 \times 2 \times 10^5}\right] \times 900$$

$$= \mathbf{0.0216\ mm}$$

(vi) Change in volume $\delta V = \varepsilon_v . V = \left[\frac{pd}{4tE}(5 - 4\mu)\right] \times V$

$$= \left[\frac{1.2 \times 600}{4 \times 15 \times 2 \times 10^5}(5 - 4 \times 0.3)\right] \frac{\pi}{4} \times 600^2 \times 900$$

$$= \mathbf{58.02 \times 10^3\ mm^3}$$

**Example 9.5.** *A cylindrical shell 3 m long which is closed at the ends has an internal diameter of 1 m and a wall thickness of 15 mm. Calculate the circumferential and longitudinal stresses induced and also change in the dimensions of the shell, if it is subjected to an internal pressure of 1.5 MN/m². Take E = 200 GN/m² and μ = 0.3.* (*UPTU : 2003–2004*)

*Given :*

$L = 3$ m

$p = 1.5$ MPa

$E = 200 \times 10^3$ MPa

$D = 1$ m

$t = 15$ mm

$\mu = 0.3$

**Solution** $\sigma_h$, $\sigma_L$, $\delta d$, $\delta L$, $\delta V$.

Circumferential stress

$$\sigma_c = \frac{pd}{2t} = \frac{1.5\times1000}{2\times15}$$

$$= \mathbf{50\ MPa\ (Tensile)}$$

Longitudinal stress

$$\sigma_L = \frac{pd}{4t} = \frac{1.5\times1000}{4\times15}$$

$$= \mathbf{25\ MPa\ (Tensile)}$$

Change in diameter

$$\delta d = \frac{pd}{4tE}(2-\mu)d$$

$$= \frac{1.5\times1000}{4\times15\times200\times10^3}(2-0.3)\times1000$$

$$= \mathbf{0.2125\ mm}$$

Change in length

$$\delta L = \frac{pd}{4tE}(1-2\mu)L = \frac{1.5\times1000}{4\times15\times200\times10^3}(1-2\times0.3)\times3000 = \mathbf{0.15\ mm}$$

Change in volume

$$\delta V = \frac{pd}{4tE}(5-4\mu)V = \frac{1.5\times1000}{4\times15\times200\times10^3}(5-4\times0.3)\left(\frac{\pi}{4}\times1000^2\times3000\right)$$

$$= \mathbf{1.119\times10^6\ mm^3}$$

**Example 9.6.** *Wall thickness of a cylindrical shell of 300 mm internal diameter is 10 mm. Length of the cylinder is 2 m. If the shell is subjected to an internal pressure of 1.5 MPa. Determine maximum shear stress induced and change in dimensions of the shell. E = 200 GPa and μ = 0.3.*

*(UPTU : 2011–2012)*

*Given :*

$d$ = 800 mm

$t$ = 10 mm

$L$ = 2 m = 2000 mm

$p$ = 1.5 MPa

$E$ = $200\times10^3$ MPa

$\mu$ = 0.3

**Solution To Find :** $\tau_{max}$, $\delta d$, $\delta L$, $\delta V$

(i) *Maximum shear stress :*

$$\tau = \frac{pd}{8t} = \frac{1.5\times800}{8\times10}$$

$$= \mathbf{15 N/mm^2}$$

(ii) *Change in diameter :*

$$\frac{\delta d}{d} = \left[\frac{pd}{4tE}(2-4)\right]$$

$$d = \left[\frac{1.5\times800}{4\times10\times200\times10^3}(2-0.3)\right]\times10$$

$$= 2.55 \times 10^{-3} \text{ mm}$$

(iii) *Change in length :*

$$\delta L = \left[\frac{pd}{4tE}(1-24)\right]\times L$$

$$= \left[\frac{1.5\times800}{4\times10\times200\times10^3}(1-2\times0.3)\right]\times2000$$

$$= \mathbf{0.12 \ mm}$$

(iv) *Change in volume :*

$$\delta V = \left[\frac{pd}{4tE}(5-4\mu)\right]V$$

$$= \left[\frac{1.5\times800}{4\times10\times200\times10^3}(5-4\times0.3)\right]$$

$$\times\frac{\pi}{4}\times800^2\times2000$$

$$= \mathbf{573.03 \times 10^3 \ mm^3}$$

**Example 9.7.** *A mild steel cylinder has diameter to thickness ratio of 30. Find the internal pressure to which the cylinder should be subjected so that its volume is increased by* $\frac{1}{2000}$ *of its original volume.*

*Take E = 200,000 N/mm².* (UPTU : 2001–2002)

*Given :* $\frac{d}{t} = 30$

$\therefore$ $d = 30\,t$

*Volumetric strain* $\varepsilon_v = \dfrac{\delta V}{v} = \dfrac{1}{2000}$

*Young's modulus* $E = 200{,}000\ N/mm^2$

*Poisson's ratio* $\mu = 0.25$ *(Assumed)*

**Solution To find :** Internal pressure $p$.

$$\text{Volumetric strain} = \frac{\delta V}{v} = \frac{pd}{4tE}(5-4\mu)$$

$$\therefore \quad \frac{1}{2000} = \frac{pd}{4tE}(5-4\mu)$$

$$\therefore \quad \frac{1}{2000} = \frac{P}{4E}\times 30(5-4\times 0.25) = \frac{30p}{E}$$

$$\therefore \quad p = \frac{E}{2000\times 30} = \left(\frac{200000}{2000\times 30}\right)$$

$$\therefore \quad p = \mathbf{3.33\ N/mm^2}$$

$\therefore$ Required internal pressure is **3.33 N/mm²**.

### 9.3.4 Effect of Incompressible Fluid

In case of incompressible fluid, there will be changes in volume of shell only. The volume of fluid is injected or let out is equal to change in volume of cylindrical shell only.

$$dV = (2\,\varepsilon_h + \varepsilon_L)\,V$$

$$\therefore \quad dV = \frac{pd}{4tE}(5-4\mu)V$$

**Example 9.8.** *A cylindrical vessel 1.5 metre in diameter, 2 metre long and 1.5 cm thick is closed at both the ends by rigid plates and this cylinder is filled with water at atmospheric pressure. Find how much additional amount of water is required to be pumped so as to make the final pressure in the cylinder as 70 bar. Take E = 210 GN/m² and μ = 0.3 for the material of the cylinder.*

*Bulk modulus of the water is 2.4 GN/m².* *(UPTU : 2009–2010)*

*Given :*

$d = 1.5\ m = 1500\ mm$

$L = 2\ m = 2000\ mm$

$t = 1.5\ cm = 15\ mm$

$P = 70\ bar = 700\ kN/m^2 = 700\times 10^3\times 10^{-6}$

$= 0.7\ N/mm^2$

$E = 210\ GN/m^2 = 210\times 10^3\ N/mm^2$

$$\mu = 0.3$$

$$K = 2.4\ GN/m^2 = 2.4 \times 10^3\ N/mm^2$$

**Solution To find :** *dv*

Volumetric strain

$$\varepsilon_v = \varepsilon_{v\ \text{shell}} + \varepsilon_{v\ \text{fluid}} = \frac{Pd}{4tE}(s - 4\mu) + \frac{P}{K}$$

$$\varepsilon_v = \frac{0.7 \times 1500}{4 \times 15 \times 210 \times 10^3}(s - 4 \times 0.3) + \frac{0.7}{2.4 \times 10^3}$$

$$= 316.67 \times 10^{-6} + 2.917 \times 10^{-4} = 6.083 \times 10^{-4}$$

$$d_v = \varepsilon_v \cdot V = 6.083 \times 10^{-4} \times \frac{\pi}{4} \times 1500^2 \times 2000$$

$$= \mathbf{215000\ mm^3}$$

**Effect of Compressible Fluids :**

When compressible fluid is injected under pressure, there are two volume changes

(i) Increase in volume of shell

$$\delta v_1 = (2\varepsilon_h + \varepsilon_1)V = \frac{pd}{4tE}(5 - 4\mu)V$$

Volumetric strain, $\varepsilon v_1 = e_{v\ \text{shell}}$ ...(i)

(ii) Decrease in volume,

$$\delta V_2 = \frac{P}{K}V$$

Volumetric strain, $\varepsilon v_2 = \varepsilon_{v\ \text{fluid}} = \dfrac{P}{K}$ ...(ii)

Hence volume of fluid pumped in or let out is equal to the sum of changes in volume of the shell and changes in volume of fluid

$$\delta V = \delta V_1 + \delta V_2$$

Volumetric strain, $\varepsilon_v = \varepsilon_{v\ \text{shell}} + \varepsilon_{v\ \text{fluid}} = \dfrac{pd}{4tE}(5 - 4\mu) + \dfrac{P}{K}$

where, $K$ = Bulk modulus of fluid.

**Example 9.9.** *A thin spherical vessel having diameter of 1.50 m is of uniform thickness. It is filled with water at a gauge pressure of 2.0 MPa. A relief valve attached to the vessel is opened and water is allowed to escape until the pressure falls to atmospheric. If the volume of the water escaped is 4 litre. Find thickness* *of the plate of the vessel. Bulk modulus of water is 2 GPa and Young's modulus of vessel material is 200 GPa and Poisson's ratio is 0.30.*

(*UPTU : 2010–2011*)

*Given :*

$$d = 1.5 \text{ m} = 1500 \text{ mm},$$
$$p = 2 \text{ MPa}$$
$$dv = 4 \text{ liter} = 4 \times 10^{-3} \text{ m}^3 = 4 \times 10^{-3} \times 10^9 = 4 \times 10^6 \text{ mm}^3$$
$$K = 2 \text{ GPa} = 2 \times 10^3 \text{ MPa}.$$
$$E = 200 \text{ GPa} = 200 \times 10^3 \text{ MPa}$$
$$\mu = 0.3$$

**Solution To find :** Thickness of plate

Volume of spherical shell

$$V = \frac{\pi}{6} d^3 = \frac{\pi}{6} \times 1500^2 = 1.767 \times 10^9 \text{ mm}^3$$

Volumetric strain for compressible fluid

$$\varepsilon_v = \varepsilon_{v\text{ shell}} + \varepsilon_{v\text{ fluid}}$$

$$\frac{dV}{V} = \frac{3pd}{4tE}(1-\mu) + \frac{p}{k}$$

$$\frac{4 \times 10^6}{1.769 \times 10^9} = \frac{3 \times 2 \times 1500(1-0.3)}{4 \times t \times 200 \times 10^3} + \frac{2}{2 \times 10^3}$$

$$2.261 \times 10^{-3} = \frac{0.007875}{t} + 1 \times 10^{-3}$$

$$1.26 \times 10^{-3} = 0.007875$$

$$\boldsymbol{t = 6.25 \text{ mm}}$$

### 9.3.5 Efficiency of Joint

When the required shape and size of shell are not available, then plates are bend into the required shape and connected by but *t* joints.

Due to this joint, the resistance area is decreased, which will increase stresses in the material.

Figure 9.8 (a) shows that when the length of cylinder is too long and is not possible to provide single unit, then proper joint is used to increase length. This is longitudinal joint. For example pipelines.

Figure 9.8 (b) shows, when the diameter of tank is too large, then required diameter of cylinder is made by proper joints. This is circumferential joint. For example, steel cylindrical water tank.

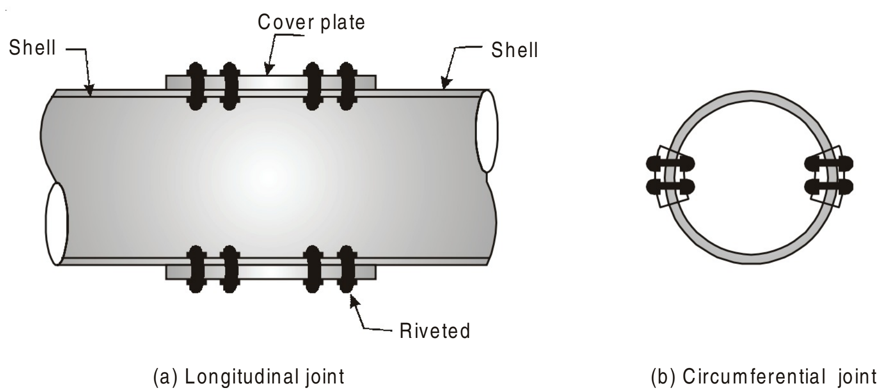



**Fig. 9.8**

If the efficiency of a longitudinal joint $\eta_L$ and circumferential joint $\eta_c$ are known, then

Circumferential stress,

$$\sigma_c = \frac{pd}{2t\,\eta_L}$$

Longitudinal stress, $$\sigma_L = \frac{pd}{4t\,\eta_c}$$

In longitudinal joint, the circumferential stress is developed whereas in circumferential joint, the longitudinal stress is developed.

Efficiency of joint, $$\eta = \frac{\text{Pitch} - \text{Diameter of rivet}}{\text{Pitch}}$$

Where pitch is the distance between two consecutive rivets.

**Example 9.10.** *A cylindrical air drum is 2.25 m in diameter with plates 1.2 cm thick. The efficiencies of the longitudinal and circumferential joints are 0.75 and 0.40 respectively. If the tensile stresses in plating is to be limited to 120 MN/m², determine safe air pressure.* (*UPTU : 2005–2006*)

*Given :*

$$d = 2.25 \text{ m} = 2250 \text{ mm}$$

$$t = 1.2 \text{ cm} = 12 \text{ mm}$$

$$\sigma = 120 \text{ MN/m}^2 = 120 \text{ N/mm}^2$$

$$\eta_L = 0.75,\ \eta_c = 0.4$$

**Solution To find**: Safe air pressure $p$

Maximum tensile stress = Hoop stress = $\sigma_h$ = 120 MPa

$$\sigma_h = \frac{pd}{2t\eta_L}$$

$$120 = \frac{p \times 2250}{2 \times 12 \times 0.75}$$

$$\mathbf{p = 0.96\ N/mm^2}$$

## 9.4 ❑ THIN SPHERICAL SHELL

Consider a thin spherical shell subjected to an internal pressure as shown in Fig. 9.9.

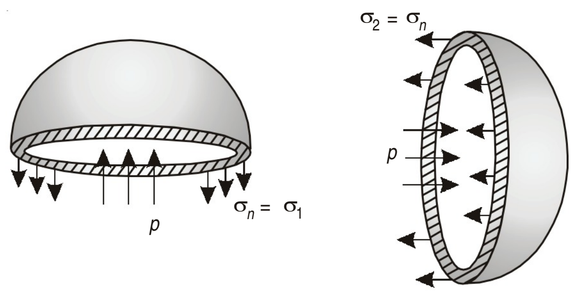



**Fig. 9.9**

Let $p$ = Internal pressure, in N/mm²

$d$ = Diameter of shell, in mm

$t$ = Thickness of shell, in mm

$\sigma$ = Stress in the shell material, in N/mm²

Bursting force acting along the centre of the sphere, which split the sphere along the diameter.

∴ Bursting force = Internal pressure × Projected area = $p \times \frac{\pi}{4} d^2$

Resistance force = Stress in shell × Resisting sectional area

$= \sigma \times \pi\, dt$

∴ For equilibrium,

Resisting force = Bursting force

$$\sigma \pi d t = p \frac{\pi}{4} d^2$$

$$\therefore \quad \boldsymbol{\sigma = \frac{Pd}{4t}}$$

The sphere has only diameter in both mutually perpendicular direction, principal stress on both direction remain same.

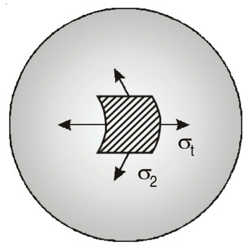



**Fig. 9.10**

$$\boldsymbol{\sigma_1 = \sigma_2 = \frac{pd}{4t}}$$

Change in diameter $\quad \varepsilon_h = \dfrac{\delta d}{d} = \dfrac{1}{E}(\sigma_1 - \mu\sigma_2)$

$\because \quad = \sigma_1 = \sigma_2 = \sigma_h$

$$= \frac{\sigma_h}{E}(1-\mu) = \frac{pd}{4tE}(1-\mu)$$

$$\boldsymbol{\delta d = \frac{pd^2}{4tE}(1-\mu)}$$

Change in thickness, $\quad \dfrac{\delta t}{t} = -\dfrac{\mu\sigma_1}{E} - \dfrac{\mu\sigma_2}{E}$

But, $\quad \sigma_1 = \sigma_2 = \sigma_h = \dfrac{pd}{4t}$

$$\frac{\delta t}{t} = -\frac{2\mu\sigma}{E} = -\frac{2\mu pd}{4tE}$$

$\therefore \quad \dfrac{\delta t}{t} = \dfrac{-\mu pd}{2tE}$ and $\delta t = \dfrac{-\mu pd}{2E}$

Positive sign for $\delta d$ indicates increase in diameter whereas negative sign for $\delta t$ indicates reduction in thickness.

Change in volume :

Volume of spherical shell:

$$V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}d^3$$

By differentiation, $\quad \delta V = \dfrac{\pi}{6} \times 3d^2\delta d$

Dividing both sides by volume,

$$\frac{\delta V}{V} = \frac{\frac{\pi}{6}\times 3d^2\delta d}{\frac{\pi}{6}d^3} = \frac{3\delta d}{d}$$

$$\therefore \qquad \varepsilon_v = 3\,\varepsilon_h$$

But, $$\varepsilon_h = \frac{\delta d}{d} = 3\left[\frac{pd}{4tE}(1-\mu)\right]$$

$\therefore$ Change in volume, $$\delta V = \frac{3pd}{4tE}(1-\mu)v = \frac{3pd}{4tE}(1-\mu)\frac{\pi}{6}d^3$$

$$\therefore \qquad \delta V = \frac{\pi p d^4}{8tE}(1-\mu)$$

For numericals on thin shells :

Circumferential stress : $\sigma_c = \dfrac{pd}{4t}$ ;

Change in diameter; $\dfrac{\delta d}{d} = \dfrac{[pd(1-\mu)]}{4tE}$ ;

Change in thickness, $\dfrac{\delta t}{t} = \dfrac{-\mu pd}{2tE}$

Change in voulme, $\dfrac{\delta V}{V} = \dfrac{3pd(1-\mu)}{4tE} = \dfrac{3\delta d}{d}$

$$\therefore \qquad \boldsymbol{\varepsilon_v = 3\,\varepsilon_h}$$

**Example 9.11.** *A seamless spherical shell is of 8 m internal diameter and 4 mm thickness. It is filled with fluid under pressure until its volume increases by 50 C.C. Determine the fluid pressure, taking E = 2 × 10*$^5$ *N/mm*$^2$ *and* $\mu = 0.3$.

*(UPTU : 200 –2006)*

*For the thin spherical shell,*

*Given :*

$d = 8$ m $= 8000$ mm

$t = 4$ mm

$\delta_v = 50$ cm$^3$ $= 50 \times 10^3$ mm$^3$

$E = 2 \times 10^5$ N/mm$^2$

$\mu = 0.3$

**Solution To find :** Fluid pressure $p$

$\therefore$ Hoop stress $\qquad \sigma_h = \dfrac{pd}{4t} = \dfrac{p \times 8000}{4 \times 4} = 500p \text{ N/mm}^2$

Hoop strain $\qquad \varepsilon_h = \dfrac{\sigma}{E} - \dfrac{\mu\sigma}{E} = \dfrac{\sigma}{E}(1-\mu) = \dfrac{500p}{E}(1-0.3)$

$$\varepsilon_h = \frac{350p}{E}$$

$\therefore$ Volumetric strain $\qquad = 3 \times e_h = \dfrac{3 \times 350p}{E} = \dfrac{(1050)p}{E}$ ...(i)

Now, $\qquad$ Volume $V = \dfrac{4}{3}\pi R^3 = \dfrac{\pi}{6} \cdot d^3 = \dfrac{\pi}{6} \times (8000)^3$

$$= 268.1 \times 10^9 \text{ mm}^3$$

$\therefore$ Volumetric strain $\qquad = \dfrac{\delta V}{V} = \dfrac{50 \times 1000}{268.1 \times 10^9}$

$\therefore \qquad \dfrac{\delta V}{V} = 186.5 \times 10^{-9}$ ...(ii)

Equating $\dfrac{(1050)p}{E} = 186.5 \times 10^{-9}$

$\therefore \qquad p = \dfrac{186.5 \times 10^{-9} \times 2 \times 10^5}{1050}$

$$\mathbf{p = 35.5 \times 10^{-6} \ N/mm^2}$$

$\therefore$ The internal fluid pressure is $35.5 \times 10^{-6}$ N/mm$^2$.

**Example 9.12.** *A spherical shell of 1.2 m internal diameter and 6 mm thickness is filled with water under pressure until its volume increases by 400 × $10^3$ mm$^3$. Find the pressure exterted by water on the shell. E = 200 GPa and μ = 0.3.* *(UPTU : 2011–2012)*

*Given :*

*Diameter* $\qquad d = 1.2$ *m = 1200 mm*

*Thickness* $\qquad t = 6$ *mm*

*Increase in volume* $\qquad dV = 400 \times 10^3$ *mm*$^3$

*Young's modulus* $\qquad E = 2 \times 105$ *N/mm*$^2$

*Poisson's ratio* $\qquad m = 0.3$

**Solution To find :** Pressure exerted by fluid $p$

Volume of shell $\qquad V = \dfrac{\pi}{6}d^3 = \dfrac{\pi}{6} \times 1200^3 = 904.78 \times 10^6 \text{ mm}^3$

Volumetric strain $\varepsilon_v = \frac{dV}{V} = \frac{3pd}{4tE}(1-\mu)$

$$\frac{400\times10^3}{904.78\times10^6} = \frac{3p\times1200}{4\times6\times2\times10^5}(1-0.3)$$

$$4.42 \times 10^{-4} = 5.25 \times 10^{-4}\, p$$

$$\mathbf{p = 0.842\ N/mm^2}$$

∴ The pressure exerted by fluid is 0.842 N/mm$^2$.

**Example 9.13.** *A thin spherical shell one metre in diameter with its wall of 1.2 cm thickness is filled with a fluid at atmospheric pressure. What intensity of pressure will be developed in it if 175 cm$^2$ more fluid is pumped into it? Also calculate the circumferential stress at that pressure and increase in diameter and volume of the vessel. Taking E = 200 GN/m$^2$ and Poisson's ratio as 0.3.*

(*MTU : 2012–2013*)

*Given :*

| | |
|---|---|
| *Diameter* | $d = 1\text{ m} = 1000\text{ mm}$ |
| *Thickness* | $t = 1.2\text{ cm} = 12\text{ mm}$ |
| *Increase in volume* | $\delta V = 175 \times 10^3\text{ mm}^3$ |
| *Young's modulus* | $E = 2 \times 10^5\text{ N/mm}^2$ |
| *Poisson's ratio* | $\mu = 0.3$ |

**Solution To find :** Pressure exerted by fluid $p$

Volume of shell $V = \frac{\pi}{6}d^3 = \frac{\pi}{6}\times1000^3 = 523.6\times10^6\,\text{mm}^3$

Volumetric strain $\varepsilon_v = \frac{dV}{V} = \frac{3pd}{4tE}(1-\mu)$

$$\frac{175\times10^3}{523.6\times10^6} = \frac{3p\times1000}{4\times12\times2\times10^5}(1-0.3)$$

$$1.432 \times10^{-3} = 2.1875 \times 10^{-4}$$

$$\mathbf{p = 6.55\ N/mm^2}$$

∴ The pressure exerted by fluid is 6.55 N/mm$^2$.

Circumferential stress $\sigma_c = \frac{P_d}{4t} = \frac{6.55\times1000}{4\times12} = 136.38\text{ MPa}$

Change in diameter $\frac{\delta d}{d} = \frac{pd}{4tE}(1-\mu)$

$$\delta d = \frac{6.55\times1000^2(1-0.3)}{4\times12\times2\times10^5} = 0.478\text{ mm}$$

Change in volume $\varepsilon_v = \frac{dV}{r} = 3\frac{\delta d}{d} = 3\times\frac{0.478}{1000}$

$$dv = 1.433 \times 10^{-3} \times 523.6 \times 10^6$$
$$= 750.22 \text{ mm}^3$$

**Example 9.14.** *A spherical tank has a diameter 20 m and thickness 15 mm induced maximum tensile stress of 120 MPa. Determine change in diameter and change in volume. Take μ = 0.3 and E = 200 GPa.* (*UPTU : 2008–2009*)

*Given :*

$$d = 20 \text{ m} = 20000 \text{ mm}$$
$$t = 15 \text{ mm}$$
$$\sigma_c = 120 \text{ MPa}$$
$$E = 200 \text{ GPa} = 200 \times 10^3 \text{ MPa}$$
$$\mu = 0.3$$

**Solution To find :** $p$, $\delta d$ and $\delta y$.

$$\sigma_c = \frac{pd}{4t}$$

$$120 = \frac{4\times 20000}{4\times 15}$$

$$\mathbf{p = 0.36 \ N/mm^2}$$

Change in diameter

$$\frac{\delta d}{d} = \frac{pd}{4tE}(1-\mu)$$

$$= \frac{0.36\times 20000}{4\times 15\times 200\times 10^3}(1-0.3)\times 2000$$

$$\mathbf{\delta d = 8.4 \ mm}$$

Change in volume $\varepsilon_v = \frac{\delta v}{v} = 3\frac{\delta d}{d} = 3\times\frac{8.4}{2000} = 1.26\times 10^{-3}$

$$\delta_v = 1.26\times 10^{-3}\times\frac{\pi}{6}(20000)^3$$

$$= \mathbf{5.28 \times 10^9 \ mm^3}$$

**Example 9.15.** *A boiler drum consists of a cylindrical portions 2 m long, 1 m diameter and 25 mm thick, closed by hemispherical ends. In a hydraulic test to 10 N/mm², how much additional water will be pumped in after initial filling at atmospheric pressure?*

*Assume the circumferential strain at the junction of cylinder and hemisphere is same for both for the drum material, E = 207000 N/mm², μ = 0.3. For water K = 2100 N/mm².* (*UPTU : 2002–2003, 2013–2014*)

*Given* : For *the cylinder* :

$$d = 1 \text{ m}$$
$$L = 2 \text{ m}$$
$$t = 25 \text{ mm}$$
$$p = 10 \text{ N/mm}^2$$
$$E = 2.07 \times 10^5 \text{ N/mm}^2$$
$$\mu = 0.3$$
$$K = 2100 \text{ N/mm}^2$$

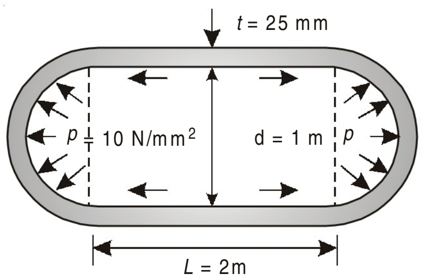



**Fig. 9.11**

**Solution To find :** $\delta V$

$$\text{Volume of cylinder} = V_1 = \frac{\pi}{4} \times d^2 \times L$$

$$V_1 = \frac{\pi}{4} \times (1000)^2 \times (2000) = (1.5708 \times 10^9) \text{mm}^3$$

For the hemispherical ends :

$$\text{Volume } V_2 = \frac{4}{3}\pi R^3 \qquad \ldots\left(R = \frac{d}{2} = 0.5 \text{ m}\right)$$

$$= \frac{4}{3}\pi \times (500)^3 = (523.6 \times 10^6) \text{mm}^3$$

Volume strain in the cylinder is

$$\frac{\delta V_1}{V_1} = \varepsilon_{v_1} = \frac{pd}{4tE}(2 - \mu)$$

$$\therefore \qquad \delta V_1 = \frac{10 \times 1000}{4 \times 25 \times (207 \times 1000)}(2 - 03) \times (1.5708 \times 10^9)$$

$$= (1.29 \times 10^6) \text{ mm}^3$$

$\therefore$ Increase in volume of cylinder

$$= (1.29 \times 10^6) \text{ mm}^3$$

Volume strain in hemispherical portions is

$$\frac{\delta V_2}{V_2} = \varepsilon_{v_2} = \frac{3pd(1-\mu)}{4tE}$$

$$\therefore \qquad \delta V_2 = \frac{3\times10\times1000(1-0.3)}{4\times25\times(207\times1000)}\times(523.6\times10^6)$$

$$= (0.5312 \times 10^6)\text{ mm}^3$$

∴ Increase in volume of hemispherical ends is

$$\delta V_2 = (0.5312 \times 10^6)\text{ mm}^3$$

Total volume of water $V_3 = V_1 + V_2 = 1.5708 \times 10^9 + 523.6 \times 10^6$

$$= (2.0314 \times 10^9)\text{ mm}^3$$

Decrease in volume of water

$$\delta V_3 = \frac{p}{k}\times V_3 = \frac{10}{2100}\times(2.0314\times10^9)$$

$$\delta V_3 = (9.6733 \times 10^6)\text{ mm}^3$$

∴ Additional water that can be pumped is

$$\delta V_1 + \delta V_2 + \delta V_3 = 1.29 \times 10^6 + 0.5312 \times 10^6 + 9.6733 \times 10^6$$

$$= (11.95 \times 10^6)\text{ mm}^3 = 11950 \times 10^3\text{ mm}^3$$

$$= 11950\text{ cm}^3 = \mathbf{11.95\ litres}$$

∴ Additional water $\delta V$ = 11.95 litres or 0.01195 $m^3$

## EXERCISE

**9.1.** A cylindrical shell of 2 m length, 600 mm internal diameter and 20 mm metal thickness is subjected to an internal pressure of 12 MPa. Compute the values of hoop stress, longitudinal strain and change in the diameter of the cylinder. $E$ = 200 GPa and Poisson's ratio = 0.3. **[*Ans.* 180 MPa]**

**9.2.** A thin cylindrical container of diameter 1.5 m and length 4.0 m is subjected to internal fluid pressure of 3 MPa. The maximum shear stress in the material is not to exceed 37.5 MPa. Find the minimum thickness of the container and change in the volume for the same. Take $E$ = 200 GPa and Poisson's ratio = 0.25. **[*Ans.* $t$ = 15 mm; $\delta V$ = 1.0602875 × $10^7$ $mm^3$ (increase)]**

**9.3.** A 5 m diameter cast iron water main carries water under a static head of 80 m. If the ultimate tensile strength of cast iron is 160 MN/$m^2$, find the suitable thickness of the main using a factor of safety 4. **[*Ans.* $t$ = 49.05 mm]**

**9.4.** A water main 1000 mm in diameter contains water at a pressure head of 100 m. Find the thickness of the wall required if the permissible stress in the pipe material is 30 MPa. The unit weight of the water is 10 kN/$m^3$.
**[*Ans.* $t$ = 16.67 mm]**

**9.5.** A boiler made of 12 mm steel plates is 1200 mm in diameter and 2500 mm long. It is subjected to an internal pressure of 3 MPa. Find (a) the hoop stress,

(b) the longitudinal stress, (c) the change in the thickness of the plate and (d) the absolute maximum shear stress.

**[*Ans.* $f_c$ = 150 MPa: $f_L$ = 75 MPa:**
**$\delta$ *t* = 0.00405 mm (decrease). $\tau_{max}$ = 76.5 MPa]**

**9.6.** A cylindrical shell of 2 m length, 600 mm internal diameter and 10 mm metal thickness is subjected to an internal pressure of 10 MPa. Find the change in the diameter and length. Take $E$ = 200 GPa and $\mu$ = 0.30.

**[*Ans.* $\delta d$ = 0.765 mm (increase). $\delta L$ = 0.06 mm (increase)]**

**9.7.** The change in the volume of a thin walled cyinder : 1200 mm long. 150 mm diameter, 5 mm thickness is 15000 mm$^3$, when the internal pressure is $p$. If Poisson's ratio for the material is 0.28 and $E$ = 210 GPa, find the value of $p$.

**[*Ans.* 15.84 MPa]**

**9.8.** A cylindrical shell, 150 mm diameter, thickness of metal 5 mm. 900 mm, long, is filled with an incompressible fluid at atmospheric pressure. If an additional quantity of 10,000 mm$^3$ of fluid is pumped into the cylinder, calculate the pressure exerted by the fluid on the wall of the cylinder. Also calculate the hoop stress induced. Take $E$ = 210 GPa and $\mu$ = 0.25.

**[*Ans.* $p$ = 4.40 MPa, $f_c$ = 66 MPa]**

**9.9.** A thin cylinder 150 mm internal diameter, 2.5 mm thick has its ends closed by rigid plates and is filled with water. When an external pull of 37 kN is applied to the ends, the water pressure is observed to fall by 0.1 N/mm$^2$. Assuming the cylinder remains full of water, determine the value of Poisson's ratio $\mu$, if $E$ = 140 GPa and $K_y$ = 2.2 GPa. **[*And.* $\mu$ = 0.309]**

**9.10.** A spherical steel container has internal diameter of 600 mm and wall thickness of 10 mm. It is being filled with a fluid under a pressure of 5 MPa. The pressure is reduced gradually by letting out some water. To reduce the pressure to atmospheric, find the quantity of fluid required to be let out if $K$, the bulk modulus for the fluid is 2 GPa. Take $E$, the modulus of elasticity of steel as 200 GPa and Poisson's ratio = 0.30. **[*Ans.* $dV$ = 371806.39 mm$^3$]**

**9.11.** A seamless spherical shell 1200 mm in diameter is subjected to an internal pressure of 4 MPa. If the permissible stress in tension for the material of shell is 120 MPa, find the thickness of plate. **[*Ans.* $t$ = 10 mm]**

**9.12.** A spherical shell is 600 mm diameter and has wall thickness 6 mm. If the tensile stress is limited to 60 MPa, calculate the safe working pressure of the inside fluid. Also find the increase in volume of the sphere due to this pressure. $E$ = 210 GPa, $\mu$ = 0.3. **[*Ans.:* $p$ = 2.40 MPa, $dV$ = 6758.40 mm$^3$]**

**9.13.** A spherical shell 1000 mm in diameter is 10 mm thick. It is being filled with water under pressure until its volume increases by 10$^6$ mm$^2$. Calculate the pressure exerted by the water on other shell. $E$ = 200 GPa, $\mu$ = 0.30.

**[*Ans.* $p$ = 7.28 MPa]**

**9.14.** A spherical shell 1000 mm in diameter is 5 mm thick. It is being filled with fluid under a pressure of 1 MPa. If $E$ = 210 GPa and Poisson's ratio = 0.28, find the increase in the capacity of the shell. **[*Ans.* $dV$ = 269279.37 mm$^3$]**

**9.15.** A pipe of 150 mm internal diameter with the thickness of metal 50 mm transmits water under a pressure of 6 MPa. Calculate the maximum and minimum intensities of circumferential stress induced.

**[*Ans.* $\sigma_{max}$ = 12.75 MPa, $\sigma_{min}$ = 6.75 MPa]**

**9.16.** A cylindrical drum 2 m in diameter and closed at ends is subjected to an internal pressure of 12 bar. Find its thickness if

(a) Drum is welded with a 100% joint efficiency and

(b) The drum is riveted with joint efficiency of 84% for the longitudinal joint and 46% for the circumferential joint.

Base your results on an allowable tensile stress of 100 MN/m$^2$.

**[*Ans.* (a) $t$ = 1.2 mm (b) $t$ = 1.43 cm]**

**9.17.** A spherical vessel is 1 m in diameter and 2.5 cm thick. Find the safe internal pressure if

(a) The vessel is welded with 100% joint efficiency, and

(b) The vessel is riveted with 80% joint efficiency

Allowable tensile stress in simple tension is 100 MN/m$^2$.

**[*Ans.* (a) $\sigma$ = 100 bar and (b) $\sigma$ = 80 bar]**

**9.18.** A cylindrical pressure vessel has a diameter of 1m, the ends being hemispherical in shape 10 mm thick. Find the thickness of the cylindrical portion of the pressure vessel if the diametral strains (hoop strains) are to be same at the joint of cylindrical and spherical portions. Take $\mu$ = 0.25.

**[*Ans.* $t$ = 23.33]**

**9.19.** A cylindrical vessel 20 cm internal diameter and 3 mm thick is closed at ends by rigid plates and is filled with fluid pressure. Find what axial pull will make the pressure to drop by 5 bar. Make use of the following data :

$E$ for the cylinder material = 200 GN/m$^2$, $K$ for water = 2.4 GN/m$^2$

And $\mu$ for the cylinder material = 0.25 **[*Ans.* $p$ = 70.69 kN]**

**9.20.** A pipe of 400 mm internal diameter and 600 mm external diameter contains a fluid at a pressure of 10 MPa. Find the maximum and minimum hoop stresses across the section. **[*Ans.* $\sigma_{max}$ = 26 MPa, $\sigma_{min}$ = 16 MPa]**

**9.21.** Find the thickness of the metal of a cylindrical shell of internal diameter 200 mm to withstand an internal pressure of 50 MPa if the maximum hoop stress is limited to 150 MPa. **[*Ans.* 42 MPa]**

**9.22.** A cylindrical shell 2 m long and 1 m internal diameter is made up of 20 mm thick plates. Find the circumferential and longitudinal stresses in the shell material, if it is subjected to an internal pressure of 5 MPa.

**[*Ans.* 125 MPa; 62.5 MPa]**

**9.23.** A steam boiler of 1.25 m in diameter is subjected to an internal pressure of 1.6 MPa. If the steam boiler is made up of 20 mm thick plates, calculate the circumferential and longitudinal stresses. Take efficiency of the circumferential and longitudinal joints as 75% and 60% respectively.

**[*Ans.* 67 MPa; 42 MPa]**

**9.24.** A pipe of 100 mm diameter is carrying a fluid under a pressure of 4 MPa. What should be the minimum thickness of the pipe, if maximum circumferential stress in the pipe material is 12.5 MPa. **[*Ans.* 16 mm]**

**9.25.** A cylindrical shell 3 m long has 1 m internal diameter and 15 mm metal thickness. Calculate the circumferential and longitudinal stresses, if the shell is subjected to an internal pressure of 1.5 MPa. Also calculate the changes in dimensions of the shell. Take $E$ = 200 GPa and Poisson's ratio = 0.3.

**[*Ans.* 50 MPa; 25 MPa; $\delta d$ = 0.21 mm; $\delta t$ = 0.15 mm]**

**9.26.** A cylindrical vessel 1.8 m long 800 mm in diameter is made up of 8 mm thick plates. Find the hoop and longitudinal stresses in the vessel, when it contains fluid under a pressure of 2.5 MPa. Also find the changes in length, diameter and volume of the vessel. Take $E$ = 200 GPa and $\frac{1}{m}$ = 0.3.

**[*Ans.* 125 MPa; 62.5 MPa; 0.42 mm; 0.23 mm; 1074 mm$^3$]**

## UNIVERSITY QUESTIONS

1. A mild steel cylinder has diameter to thickness ratio of 30. Find the internal pressure to which the cylinder should be subjected so that its volume is increased by $\frac{1}{2000}$ of its original volume. Take $E$ = 200,000 N/mm$^2$.

   (*UPTU : 2001–2003*)

   **[*Ans.* Example 9.7]**

2. A cylindrical shell 3 m long which is closed at the ends has an internal diameter of 1m and a wall thickness of 15 mm. Calculate the circumferential and longitudinal stresses induced and also change in the dimensions of the shell, if it is subjected to an internal pressure of 1.5 MN/m$^2$. Take $E$ = 200 GN/m$^2$ and $\mu$ = 0.3. (*UPTU : 2003–04*)

   **[*Ans.* Example 9.5]**

3. Define thin cylinders. Derive an expression for circumferential stress and longitudinal stress for a thin shell subjected to an internal pressure.

   (*UPTU : 2004*)

   **[*Ans.* Example 9.1, 9.2]**

4. Derive the equations for circumferential stress and volumetric strain in a thin spherical shell under internal pressure. (*UPTU : 2005–2006*)

   **[*Ans.* Section 9.4]**

5. A cylindrical air drum is 2.25 m in diameter with plates 1.2 cm thick. The efficiencies of the longitudinal and circumferential joints are 0.75 and 0.40 respectively. If the tensile stresses in plating is to be limited to 120 MN/m$^2$, determine safe air pressure. (*UPTU : 2005–2006*)

   **[*Ans.* Example 9.10]**

6. A seamless spherical shell is of 8 m internal diameter and 4 mm thickness. It is filled with fluid under pressure until its volume increases by 50 CC.

Determine the fluid pressure, taking $E = 2 \times 10^5$ N/mm$^2$ and $v = 0.3$.

(*UPTU : 2005 – 2006*)

**[*Ans.* Example 9.11]**

7. Prove that in case of a thin cylindrical shell, subjected to an internal fluid pressure, the volumetric strain is equal to twice the circumferential strain plus the longitudinal strain. (*UPTU : 2006–2007*)

   **[*Ans.* Example 9.3]**

8. A spherical tank, has a diameter of 20 metre and wall thickness 15 mm if the permissible stress in the material is 120 MPa, determine the maximum pressure at which a gas can be stored in the tank. Determine the increase in diameter and volume of the tank, due to gas pressure. Take $E$ = 200 GPa and Poisson's ratio ($\mu$) = 0.3. (*UPTU : 2008*)

   **[*Ans.* Example 9.14]**

9. A cylindrical vessel 1.5 metre in diameter, 2 metre long and 1.5 cm thick is closed at both the ends by rigid plates and this cylinder is filled with water at atmospheric pressure. Find how much additional amount of water is required to be pumped so as to make the final pressure in the cylinder as 70 bar. Take $E$ = 210 GN/m$^2$ and $\mu$ = 0.3 for the material of the cylinder. Bulk modulus of the water is 2.4 GN/m$^2$. (*UPTU : 2009–2010*)

   **[*Ans.* Example 9.8]**

10. A thin spherical vessel having diameter of 1.50 m is of uniform thickness. It is filled with water at a gauge pressure of 2.0 MPa. A relief valve attached to the vessel is opened and water is allowed to escape until the pressure falls to atmospheric. If the volume of the water escaped is 4 litre. Find thickness of the plate of the vessel. Bulk modulus of water is 2 GPa and Young's modulus of vessel material is 200 GPa and Poisson's ratio is 0.30.

    (*UPTU : 2010–2011*)

    **[*Ans.* Example 9.13]**

11. In a cylindrical shell of 0.6 m diameter and 0.9 m long is subjected to an internal pressure 1.2 N/mm$^2$. Thickness of the cylinder wall is 15 mm. Determine longitudinal stresses, circumferential stress and maximum shear stresses induced and change in diameter, length and volume. Take $E$ = 200 GPa and $\dfrac{1}{\text{m}} = 0.3$. (*UPTU : 2011–2012*)

    **[*Ans*. Example 9.4]**

12. Wall thickness of a cylindrical shell of 800 mm internal diameter is 10 mm. Length of the cylinder is 2 m. If the shell is subjected to an internal pressure of 1.5 MPa, determine maximum shear stress induced and change in dimensions of the shell. $E$ = 200 GPa and $\mu$ = 0.3. (*UPTU : 2012–2013*)

    **[*Ans.* Example 9.6]**

13. A spherical shell of 1.2 m internal diameter and 6 mm thickness is filled with water under pressure until its volume increases by $400 \times 10^3$ mm$^3$. Find the pressure exterted by water on the shell. $E$ = 200 GPa and $\mu$ = 0.3.

    (*UPTU : 2012–2013*)

    **[*Ans.* Example 9.12]**

14. Define hoop stress and longitudinal stress. (*UPTU : 2012–2013*)

**[*Ans.* Example 9.2]**

15. A thin spherical shell one metre in diameter with its wall of 1.2 cm thickness is filled with a fluid at atmospheric pressure. What intensity of pressure will be developed in it if 175 cm more fluid is pumped into it? Also calculate the circumferential stress at that pressure and increase in diameter and volume of the vessel. Taking $E$ = 200 GN/m and Poisson's ratio as 0.3.

(*UPTU : 2012–2013*)

**[*Ans.* Example 9.9]**

16. A boiler drum consists of a cylindrical portions 2 m long, 1 m diameter and 25 mm thick, closed by hemispherical ends in a hydraulic test to 10 N/mm$^2$. How much additional water will be pumped in after initial filling at atmosphere pressure?

Assume the circumferential strain at the junction of cylinder and hemisphere is same for both for the drum material. Take $E$ = 207000 N/mm$^2$, $\mu$ = 0.3, for water, $K$ = 2100 N/mm$^2$. (*UPTU : 2013–2014*)

**[*Ans.* Example 9.15]**

17. The cylinder of a hydraulic ram is of 6 cm internal diameter. Find the thickness required to withstand an internal pressure of 40 N/mm$^2$, if the maximum tensile stress is limited to 60 N/mm$^2$ and the maximum shear stress to 50 N/mm$^2$. (*UPTU : 2010*)

**[*Ans.* Example 9.14]**

18. Derive Lame's equations to find out the stresses in thick spherical shells. (*UPTU : 2002–2003*)

**[*Ans.* Example 9.6]**

19. The maximum stress permitted in a thick cylinder, radii 8 cm and 10 cm, is 20 N/mm$^2$, the external pressure is 6 N/mm$^2$, what internal pressure can be applied ? Plot curves showing the variation of hoop and radial stresses through the material. (*UPTU : 20023–2004*)

**[*Ans.* Example 9.19]**

20. A thick spherical shell having internal radius of 75 mm is subjected to an internal pressure of 25 N/mm$^2$. If the maximum hoop stress 100 N/mm$^2$. Find the thickness of the shell. (*UPTU : 2004–2005*)

**[*Ans.* Example 9.13]**

21. What do you mean by Lame's equations ? How will you derive these equations? (*UPTU : 2004–2005*)

**[*Ans.* Example 9.5]**

22. Explain the following (*UPTU : 2004–2005*)

(i) What is compound cylinder ? What is its advantage over a single cylinder?

**[*Ans.* Example 9.23]**

(ii) Shrinkage allowance **[*Ans.* Example 9.24]**

(iii) State assumptions made in Lame's Theory. **[*Ans.* Example 9.5]**

23. A thickj cylinder with closed ends has 100 mm internal radius and 150 mm external radius. It is subjected to an internal pressure of 60 MN/m$^2$ and

external pressure of 30 MN/m$^2$. Determine the hoop and radial stresses at the inside and outside of the cylinder together with longitudinal stress.

(*UPTU : 2005–2006*)

**[*Ans.* Example 9.9]**

24. How thick and thin cylinder are classfied ? Derive the equation for Hoop stress and radial stress in thick cylinder ? (*UPTU : 2005–2006*)

**[*Ans.* Example 9.7]**

25. Derive the equation to obtain the radial and circumferential streeses in thick shell subjected to external and internal pressure both. (*UPTU : 2005–2006*)

**[*Ans.* Example 9.15]**

26. A thick cylinder of 150 mm outside and 100 mm inside radius is subjected to an external pressure of 30 MN/m$^2$. Calculate the maximum sgear stress in the material of the cylinder at inner radius. (*UPTU : 2005–2006*)

**[*Ans.* Example 9.8]**

27. Calculate the thickness of metal necessary for a cylinderical shell of internal diameter 160 to withstand an internalpressure of 25 MN/m$^2$, if maximum permissible tensile stress is 125 MN/m$^2$. (*UPTU : 2005–2006*)

**[*Ans.* Example 9.10]**

28. Derive an expression for maximu pricnipal stress on thick cylindrical shell subjected to external pressure. (*UPTU : 2006–2007*)

**[*Ans.* Example 9.10]**

29. A hollow cylinder of 45 cm internal diameter and 10 cm thickness contains the fluid under pressure of 850 N/cm$^2$. Find the maximum and minimum hoop stress across the section. (*UPTU : 2006–2007*)

**[*Ans.* Example 9.2]**

30. Derive the Lame equations for the hoop and radial stresses in a thick cylinder subjected to an internal and external pressure and show how these may be expressed in graphical form. (*UPTU : 2007–2008*)

**[*Ans.* Example 9.6]**

31. Derive an expression to determne stresses in thick-walled cylinder subjected to internal pressure only. (*UPTU : 2008–2009*)

**[*Ans.* Example 9.16]**

32. Calculate the thickness of metal necessary for a cylindrical shell of internal diameter of 80 mm to withstand on internal pressure of 25 N/mm$^2$, if the maximum permissible tensile stress if 125 N/mm$^2$. (*UPTU : 2009–2010*)

**[*Ans.* Example 9.10]**

33. Write short notes on any *two* of the following : (*UPTU : 2009–2010*)

    (i) Lame's theory of thick cylinders. **[*Ans.* Example 9.5]**

    (ii) Compound cylinders : **[*Ans.* Section 10.3]**

    (iii) Radial, axial and circulferential stresses in thick cylinders :

    **[*Ans.* Section 10.2]**

34. Derive expressions for radial and hoop stresses in athick cylinder with internal and external radii of $a$ and $b$ subjected to an inetnal pressure of $p_1$.

(*UPTU : 2010*)

**[*Ans.* Example 9.16]**

35. The cylinder of a hydraulic ram is of 6 cm internal diameter. Find the thickness required to withstand and internal pressure of 40 $N/mm^2$, if the maximum tensile stress is limited to 60 $N/mm^2$ and the maximum shear stress to 50 $N/mm^2$. (*UPTU : 2010*)

**[*Ans.* Example 9.14]**

36. A thick cylinder of 160 mm internal 240 mm external diamter is subjected to an external pressure 12 MPa. Determine the maximum internal pressure that can be applied if the maximum allowable normal stress is 36 MPa. Plot the variation of radial and hoop stresses. (*UPTU : 2010–2011*)

**[*Ans.* Example 9.20]**

37. A compound cylinder is to be made by shrinking one tube on to another so that the radial compression stress at the friction is 28.5 $N/mm^2$. If the outside diameter is 26.5 cm, and the bore 12.5 cm, calculate the allowace for shrinkage at the common diameter, which is 20 cm. $E$ = 210,000 $N/mm^2$.

(*UPTU : 2010–2011*)

**[*Ans.* Example 9.4]**

38. What are the assumptions made in Lame's equation ? (*UPTU : 2011–2012*)

**[*Ans.* Example 9.5]**

39. A thick hollow cylinder 200 mm internal and 300 mm external diameter is subjected to an internal pressure 50 MPa and external pressure 25 MPa. Find the maximum shear stress at the inner surface of the cylinder.

(*UPTU : 2012–2013*)

**[*Ans.* Example 9.???]**

40. Derive the expressions for circumferential and radial stress in the wall of thick cylinder (Lame's equation). (*UPTU : 2012–2013*)

**[*Ans.* Example 9.6]**

41. An external pressure of 10 $MN/m^2$ is applied to thick cylinder of internal diameter 150 mm and external diameter 200 mm. If the maximum hoop stress permitted on the inside wall is 35 $MN/m^2$, calculate the maximum internal pressure that can be applied. (*UPTU : 2012–2013*)

**[*Ans.* Example 9.23]**

42. The maximum stress permitted in a thicj cylinder of inner and outer radius of 10 cm and 15 cm is 20 $N/mm^2$. The external pressure is 8 $N/mm^2$, what internal pressure can be applied? (*UPTU : 2012–2013*)

**[*Ans.* Example 9.18]**

CHAPTER 10

# Thick Cylinders

## 10.1 ❑ INTRODUCTION

If the ratio of thickness to the internal diameter is more than $\frac{1}{20}$ i.e., $\frac{t}{di} > \frac{1}{20}$ then such cylinder is known as *thick cylinder*. These cylinders are used where fluid is stored or transferred under very high pressure. We will denote circumferential stress by ($\sigma_c$) and hoop stress by $\sigma_h$ . Infact both are same, but we will use separate notation for each.

## 10.2 ❑ STRESSES IN THICK CYLINDERS

Consider thick cylinder having its :

(i) Length = $l$

(ii) Inner Radius = $r_i$, and

(iii) Outer Radius = $r_0$

Consider that the cylinder is subjected to : (i) Internal pressure, $p_i$ acting on its inner surface and (ii) Outer pressure $p_0$ acting on its outer surface. We have to evaluate the stress developed due to internal pressure $p_i$ .

Due to internal fluid pressure, following types of stresses will be developed in the cylinder:

(i) Longitudinal stress, $\sigma_l$, (ii) Radial stress, $\sigma_r$, and (iii) circumferential stress, $\sigma_c$ or Hoop stress, $\sigma_h$ circumferential stress $\sigma_c$ Hoop stress $\sigma_h$ is same, but we should remember both and not to get confused.

$$\sigma_r = \frac{B}{r^2} - A \qquad ...(1)$$

(Radial stress ($\sigma_r$) is always compressive)

$$\sigma_c = \frac{B}{r^2} + A \qquad \text{...(2)}$$

Equations (1) and (2) are known as *Lame's Formula* or *Lame's Theorem* which give radial stress and circumferential stress or Hoop stress between $r = r_i$ and $r = r_0$

At inner radius, $r_i$, $\sigma_r = p_i$ or $p_i = \frac{B}{r_i^2} - A$

where, $p_i$ = Internal pressure.

At outer radius $r_0$, $\sigma_r = p_0$

$$\therefore \qquad p_0 = \frac{B}{r_0^2} - A$$

In most cases $p_0 = 0$

Thus the constants $A$ and $B$ can be calculated from these equations.

## 10.3 ❑ COMPOUND CYLINDERS

These cylinders are formed by shrinking one cylinder on to the other. Due to this, the inner cylinder is selected to initial hoop compression, which enables the compound cylinder to withstand much higher working pressure. In addition to withstand high internal pressure, a compound cylinder results in appreciable saving in weight as compared to thick cylinder.

### 10.3.1 Stresses in Compound Cylinders

Let

$r_0$ = Outer radius of compound cylinder

$r_i$ = Internal radius

$r_j$ = Radius of the *junction* of the two cylinders

$p_j$ = Radial pressure at the junction of two cylinders

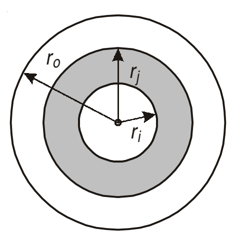



**Fig. 10.1**

**Case I :** *Hoop or circumferential stresses when fluid is not admitted in the cylinder:*

(i) For outer cylinder, Lame's equation at radius $x$

$$p_x = \frac{B_1}{x^2} - A_1 \quad ...(1)$$

$$\sigma_x = \frac{B_1}{x^2} + A_1 \quad ...(2)$$

$A_1$ and $B_1$ are constants for outer cylinder

At $x = r_0;\ p_x = 0$

$$\therefore \quad 0 = \frac{B_1}{r_0^2} - A_1 \quad ...(3)$$

At $x = r_j\ ;\ p_x = p_j$

$$\therefore \quad p_j = \frac{B_1}{r_j^2} - A_1 \quad ...(4)$$

From Eqs. (3) and (4) the constants $A_1$ and $B_1$ can be determined and substituted in Eq. (2) to get *hoop stresses* in outer cylinder due to *shrinkage*.

(ii) For Inner Cylinder : Lame's Equation for Inner Cylinder

At radius $x$ :

$$p_x = \frac{B_2}{x^2} - A_2 \quad ...(5)$$

and

$$\sigma_x = \frac{B_2}{x^2} + A_2 \quad ...(6)$$

$A_2$ and $B_2$ are other constants

At $x = r_i\,;\ p_x = 0$

As fluid pressure is not inside the inner cylinder

At $x = r_j\,;\ p_x = p_j$

Hence above equation reduces to

$$0 = \frac{B_2}{r_i^2} - A_2 \quad ...(7)$$

and

$$p_j = \frac{B_2}{r_j^2} - A_2 \quad ...(8)$$

From Eqs. (7) and (8) the constants $A_2$ and $B_2$ can be determined. These values are substituted in Eq. (6) to get the *hoop stress in inner cylinder*.

**Case II :** *Hoop or circumferential stresses when fluid is admitted in the compound cylinder.*

In order to find stresses, the inner cylinder and outer cylinder together will be considered as *thick shell*. Let, $p_x$ = internal fluid pressure. Hence by *Lame's theorem*

$$p_x = \frac{B}{x^2} - A \quad ...(9)$$

and
$$\sigma_x = \frac{B}{x^2} + A \qquad ...(10)$$

where $A$ and $B$ are constants for a single *thick shell due to internal fluid pressure*

At $x = r_0 ; p_x = 0$

Substituting these values in Eq. 9, we get

$$0 = \frac{B}{r_0^2} - A \qquad ...(11)$$

and at $x = r_i ; p_x = p_i$ (Fluid pressure)

$$p_i = \frac{B}{r_i^2} - A \qquad ...(12)$$

From Eqs. (11) and (12), the constants $A$ and $B$ can be determined.

The resultant *hoop stress* will be the algebraic sum of the *hoop stresses* caused due to *shrinkage* and *internal fluid pressure.*

Shrinkage allowance

$$= \frac{r}{E}(\sigma_{h_0} - \sigma_{h_i}) \qquad ...(13)$$

Since *hoop stress* $\sigma_h$ and *circumferential stress* $\sigma_c$ are same, so we can write $\sigma_c$ in above Eq. (13).

Hence shrinkage allowance

$$= \frac{r}{E}(\sigma_{c_0} - \sigma_{c_i})$$

## 10.4 ❑ INTERFERENCE

Let there be a shaft of diameter $D$ and a thin hollow *sleeve* of thickness $t$ and internal diameter $d$ is to be fitted on the shaft of outer diameter $D$. $D > d$. There is very small difference between $D$ and $d$. The difference between the two diameters $(D - d)$ is called *interference*.

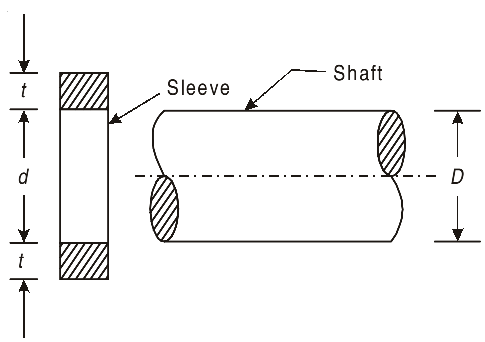



**Fig. 10.2**

To fit the sleeve over the shaft it is heated and made to slide over the shaft ($D$). When the temperature of both sleeve and shaft becomes normal, sleeve grips the shaft firmly and interference pressure is developed between the surfaces in contact.

The interface pressure ($p_j$) tries to compress the shaft and to expand the sleeve (Fig. 10.3).

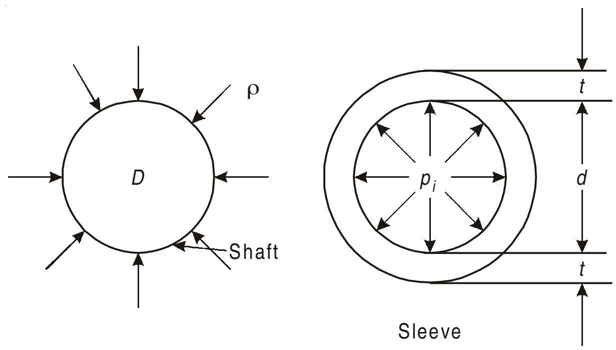



**Fig. 10.3**

Shrinkage allowance = Increase in diameter of outer cylinder + decrease in diameter of inner cylinder.

**Example 10.1.** *A hollow forged boiler drum 1.9 m outside diameter and 130 mm thick, was tested before entering service upto 20 bar. Calculate the maximum and minimum hoop stress under test.*

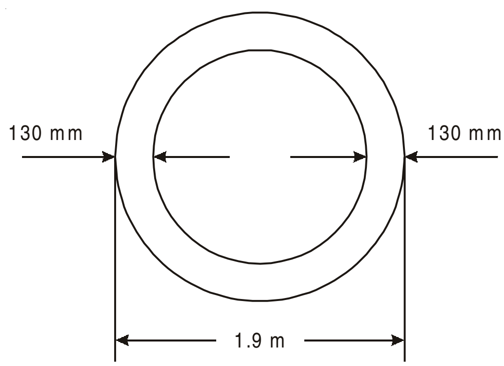



**Fig. 10.4**

**Solution**

$$1 \text{ Bar} = 1 \text{ N/m}^2$$

At inner surface:
$$r_i = \frac{d_i}{2} = \frac{D_0 - 2t}{2} = \frac{1.9 - 2 \times 0.130}{2}$$

$$\therefore \qquad r_i = \frac{1.9 - 0.260}{2} = 0.82 \text{ m} = 820 \text{ mm} \qquad ...(1)$$

At outer surface

$$R_0 = \frac{D_0}{2} = \frac{1.9 \times 1000}{2} = \frac{1900}{2} = 950 \text{ mm} \qquad ...(2)$$

$\because$ From Eqs. (1) and (2) we have

$$\sigma_r = \frac{B}{r^2} - A$$

and

$$\sigma_c = \frac{B}{r^2} + A \qquad \text{(Lame's equation)}$$

$$\therefore \qquad \sigma_r = 20 = \frac{B}{(820)^2} - A \qquad ...(i)$$

and

$$\sigma_r = 0 = \frac{B}{(950)^2} - A \qquad ...(ii)$$

$$\therefore \qquad 20 - 0 = \frac{B}{(820)^2} - \frac{B}{(950)^2}$$

$$\therefore \qquad B = 52.746 \times 10^6, \quad A = 58.444$$

$$\sigma_c \min = \frac{B}{R_0^2} + A = \frac{52.746 \times 10^6}{(950)^2} + 58.444$$

$$= \mathbf{116.89 \ N/mm^2}$$

$$\sigma_c \max = \frac{B}{R_2^2} + A = \frac{52.746 \times 10^6}{(820)^2} + 58.444$$

$$= \mathbf{136.89 \ N/mm^2}$$

**Example 10.2.** *A hollow cylinder of 45 cm internal diameter and 10 cm thickness contains fluid under pressure of 850 N/cm². Find the maximum and minimum hoop stress across the section.* (*UPTU : 2006–2007*)

*Given :* $r_i = 225$ mm, $t = 100$ mm

$r_0 = 325$ mm, $p_i = 850$ N/cm² $= 8.50$ N/mm²

**Solution** At $r = r_i = 225$ mm; $\sigma_r = p_i = 850$ N/mm²

At $r = r_0 = 325$ mm; $\sigma_r = p_0 = 0$

Therefore,

$$8.5 = \frac{B}{(225)^2} - A \qquad ...(i)$$

and $$0 = \frac{B}{(325)^2} - A \qquad \text{...(ii)}$$

Substracting from (ii) and (i)

$$8.5 = \frac{B}{(225)^2} - \frac{B}{(325)^2}$$

$$B = 826.44 \times 10^3$$

and $$A = 7.824$$

Maximum *hoop stress* will develop at inside surface of the cylinder and minimum *hoop stress* will develop at the outer cylinder.

So, at $$r = r_i = 225 \text{ mm}, \sigma_c \text{ or } \sigma_h = \frac{b}{r_i^2} + A$$

$$\therefore \quad \sigma_h = \frac{826.44 \times 10^3}{(225)^2} + 7.824$$

$$= \mathbf{24.14 \ N/mm^2}$$

At $$r = r_0 = 325; \ \sigma_h = \frac{b}{r_0^2} + A$$

$$= \frac{826.44 \times 10^3}{(325)^2} + 7.824$$

$\therefore$ Hoop stress $(\sigma_c) = \mathbf{15.64 \ N/mm^2}$

**Example 10.3.** *For a tube having $E = 2 \times 10^5$ N/mm² and $v = 0.3$, the hoop stress at the inner face is twice the internal pressure. Find the thickness of the wall if internal radius is 60 cm.* (*UPTU : 2008–2009*)

*Given :*

$$r_i = 60 \text{ cm}$$
$$v = 0.3$$
$$E = 2 \times 10^5 \text{ N/mm}^2$$

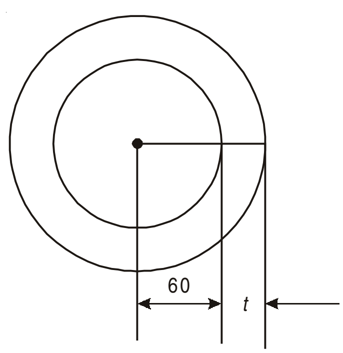



**Fig. 10.5**

*Also, $\sigma_c$ at inner surface = 2 × Internal pressure*

**Solution**

$$\frac{B}{r_i^2} + A = 2\left(\frac{B}{r_i^2} - A\right)$$

or
$$\frac{\left(\frac{B}{r_i^2} + A\right)}{\left(\frac{B}{r_i^2} - A\right)} = \frac{2}{1}$$

or
$$\frac{\frac{2B}{r_i^2}}{(2A)} = \frac{3}{1}$$

or
$$\frac{2B}{r_0^2} \times \frac{1}{2A} = \frac{3}{1}$$

or
$$\frac{B}{A} = 3r_i^2$$

or
$$\frac{B}{A} = 3(60)^2$$

$$= 3 \times 3600 = 10800$$

At the outer surface $\sigma_r = 0$

∴
$$0 = \frac{B}{(r_i + t)^2} - A$$

∴
$$A = \frac{B}{(r_i + t)^2}$$

or
$$(r_i + t)^2 = \frac{B}{A}$$

or
$$(60 + t)^2 = 10800$$

or
$$60 + t = \sqrt{10800} = 103.92$$

∴
$$t = 103.92 - 60 = 43.92 \text{ cm},$$ **Say 44 cm**

**Example 10.4.** *A compound cylinder is to be made by shrinking one tube on to the another so that the radial compressive stress at the friction is 28.5 N/mm$^2$. If outside diameter is 26.5 cm, and the aore 12.5 cm, calculate the allowance for shrinkage at the common diameter, which is 20 cm. Take E = 210 ×10$^3$ N/mm$^2$.* *(UPTU : 2011–2012)*

*Given :* $d_0 = 26.5\ cm = 265\ mm$
$d_i = 12.5\ cm = 125\ mm$
$d_j = 20\ cm = 200\ cm$
$\sigma_r\ Comp. = 28.5\ N/mm^2$
$E = 210 \times 10^3\ N/mm^2$

**Solution** We know that by Lame's equations

$$p_x = \frac{B_1}{x^2} - A_1 \quad \text{...(i)}$$

and
$$\sigma_x = \frac{B_1}{x^2} + A_1 \quad \text{...(ii)}$$

$$r_0 = \frac{265}{2} = 132.5 \text{ mm}$$

$$r_i = \frac{125}{2} = 62.50 \text{ mm}$$

and
$$r_j = \frac{200}{2} = 100 \text{ mm}$$

(i) *For inner cylinder : Applying boundary conditions*

$p_x = 0$

At $x = r_i = 62.50$ mm

$$0 = \frac{B_1}{(62.5)^2} - A_1$$

$$\therefore \quad A_1 = \frac{B_1}{62.5^2} \quad \text{...(1)}$$

and $p_x = 28.5$ N/mm²

At $x = r_j = 100$ mm

$$28.5 = \frac{B_1}{100^2} - A_1$$

$$= \frac{B_1}{100^2} - \frac{B_1}{62.5^2}$$

$\therefore \quad B_1 = -182.7 \times 10^3$

$A_1 = -46.77$

Hoop stress at inner cylinder at junction i.e.,

$r_j = 100$ mm

$$\sigma_{c_1} = \frac{B_1}{x^2} + A_1$$

$$= \frac{-182.7 \times 10^3}{(100)^2} - 46.77$$

$= \mathbf{-65\ N/mm^2 = 65\ N/mm^2\ \ Comp.}$

(ii) *For outer cylinder:*

$$p_x = \frac{B_2}{x^2} - A_2,$$

and

$$\sigma_x = \frac{B_2}{x^2} + A_2$$

Applying boundary conditions,

At $x = r_j = 100$ mm

$p_x = 28.5$ N/mm²

$$28.5 = \frac{B_2}{100^2} - A_2$$

and at $x = r_0 = 132.50$ mm,

$p_x = 0$

$$0 = \frac{B_2}{(132.5)^2} - A_2$$

or

$$A_2 = \frac{B_2}{132.5^2}$$

$\therefore$ $B_2 = 662.17 \times 10^3$

$$28.5 = \frac{B_2}{100^2} - \frac{B_2}{132.5^2}$$

$A_2 = 37.72$

$\therefore$ *Hoop stress in outer cylinder at junction*

$$\sigma_{oj} = \frac{B_2}{x^2} + A_2$$

$$= \frac{662.17 \times 10^3}{(100)^2} + 37.72$$

$= \mathbf{103.93\ \ N/mm^2}$

$\therefore$ *Shrinkage allowance*

$$= \frac{r_j}{E}\left(\sigma_{h_0} - \sigma_{h_i}\right)$$

$$= \frac{100}{210 \times 10^3}\left[(103.93 - (-65)\right]$$

$$= \frac{100}{210 \times 10^3}(168.93)$$

$$\text{S.A.} = \mathbf{0.08044\ mm}$$

Hence shrinkage allowance at the common diameter (junction)

$$= \mathbf{0.0804\ mm}$$

**Example 10.5.** *what are the assumptions made in Lame's equation?*

(*UPTU : 2011–2012*)

**Solution** Stresses in thick cylindrical shell is derived from Lame's theorem.

It is based on following assumptions :

1. The cylinder is made of homogenous and isotropic material.
2. Plane section perpendicular to the longitudinal axis remain plane i.e. longitudinal strain is constant and is independent of radius under the effect of pressure.
3. The material is stressed within the elastic limit.
4. Longitudinal stress remains constant over its thickness.

**Example 10.6.** *Derive the expressions for circumferential and radius stress in the wall of thick cylinder (Lame's equation).*

(*UPTU : 2011–2012, MTU : 2012–2013*)

**Solution**

**Derivation for Lame's Theorem**

Consider a thick cylinder subjected to internal fluid pressure. The thick cylinder may consist of a number of concentric elementary rings. Consider an elementary ring of the cylinder of radius $x$ and thickness $dx$ as shown in Fig. 10.6.

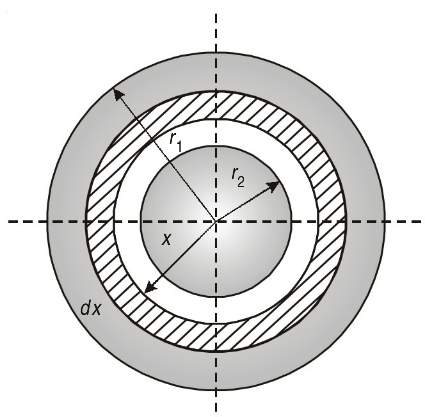



**Fig. 10.6**

Let
$r_1$ = External radius of the cylinder, mm
$r_2$ = Internal radius of the cylinder, mm
$L$ = Length of cylinder, mm

Let $p_x$ = Radial pressure on the inner surface of the ring, MPa

$p_x + dp_x$ = Radial pressure on the outer surface of the ring, MPa

Now consider $\sigma_x$ is the hoop stress induced in the ring as shown in Fig. 10.6. Consider a longitudinal section $x$–$x$.

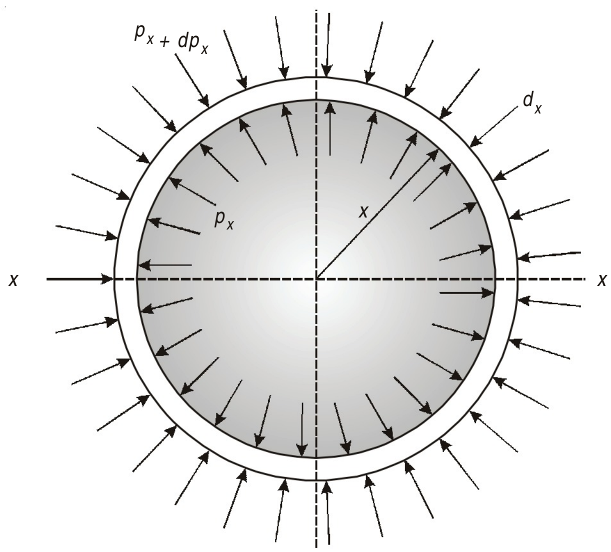



**Fig. 10.7**

$$\text{Bursting force} = p_x (2 \times L) - (p_x + dp_x)\, 2\,(x + d_x)L$$
$$= 2\,L\,[p_x \,.\, x - (p_x\, x + p_x\, d_x + x\, dp_x + dp_x)]$$

Neglecting $dp_x \,.\, dx$

$$\text{Bursting force} = -\,2L(p_x dx + x\, dp_x) \quad \text{...(i)}$$

$$\text{Resisting force} = \sigma_x \times 2\, dx\, L \quad \text{...(ii)}$$

For equilibrium

$$\text{Resisting force} = \text{Bursting force}$$
$$\sigma_x\, 2\, dx\, L = -2L\,(p_x dx + x\, dp_x)$$
$$\sigma_x = -\,p_x - x\,\frac{dp_x}{dx} \quad \text{...(iii)}$$

For thick shell, longitudinal strain at any point in the section is same, the relation between the radial pressure and hoop stress is obtained for this condition.

Force due to longitudinal stress = Force due to pressure

$$\sigma_L \times \pi\left(r_1^2 - r_2^2\right) = p\pi\, r_2^2$$

$$\sigma_L = \frac{p\pi\, r_2^2}{\pi\left(r_1^2 - r_2^2\right)}$$

$$\boldsymbol{\sigma_L = \frac{p\, r_2^2}{r_1^2 - r_2^2}} \qquad \text{...(iv)}$$

At any point on the elementary ring, the following three principal stresses exist

(i) Radial pressure $p_x$

(ii) Hoop stress $\sigma_x$

(iii) Longitudinal tensile stress $\sigma_L$

Since the longitudinal strain $\varepsilon_L$ is constant, we have,

$$\varepsilon_L = \frac{\sigma_L}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu(-p_x)}{E} = \text{Constant}$$

Since $\sigma_L$, $\mu$, $E$ are constant.

$$\therefore \qquad \sigma_x = p_x = \text{Constant}$$

$$\sigma_x - p_x = 2A$$

$$\sigma_x = p_x + 2A \qquad \text{...(v)}$$

Putting value of Eq. (v) in Eq. (iii)

$$p_x + 2A = -p_x - x\frac{dp_x}{dx}$$

$$\frac{dp_x}{dx} = -\frac{2(p_x + A)}{x}$$

$$\frac{dp_x}{p_x + A} = -\frac{2dx}{x}$$

Integrating, $\log_e = (p_x + A) = -2\log_e x + \log_e B$

where $\log_e B$ = Constant of integration

$$\log_e (p_x + A) = \log_e \left(\frac{B}{x^2}\right)$$

$$(p_x + A) = \frac{B}{x^2}$$

$$\boldsymbol{p_x = \frac{B}{x^2} - A} \qquad \text{...(vi)}$$

Substituting the value of Eq. (vi) in Eq. (v), we get,

$$\sigma_x = \left(\frac{B}{x^2} - A\right) + 2A$$

$$\therefore \qquad \boldsymbol{\sigma_x = \frac{B}{x^2} + A} \qquad \text{...(vii)}$$

The Eqs. (vi) and (vii) are Lame's equations. The constant $A$ and $B$ are calculated, by applying boundary condition, i.e. at $x = r_2$, $p_x = p$ and at $x = r_1$, $p_x = 0$.

The variation of radial pressure and hoop stress across the thickness of the shell is shown in Fig. 10.8.

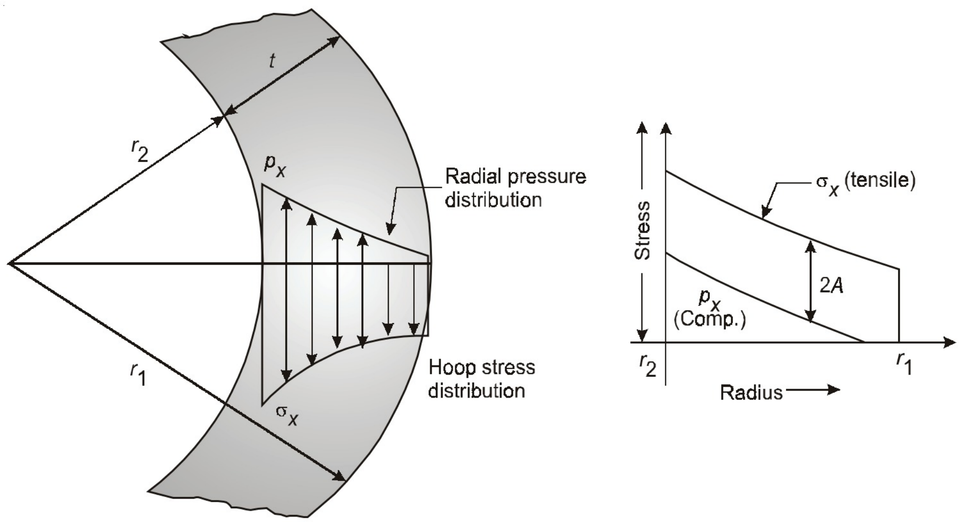



Fig. 10.8

**Example 10.7.** *What is the difference between thin and thick cylinders?*

(*UPTU : 2001–2002, 2005–2006*)

**Solution** *Comparison between Thin Shell and Thick Shell*

| *S.No.* | *Thin shell* | *Thick shell* |
|---|---|---|
| 1. | Thickness is less than $\frac{d}{20}$, where $d$ = inner diameter $\therefore \leq \frac{d}{20}$ | Thickness is greater than $\frac{d}{20}$, where $d = 2r_1 \;\therefore t > \frac{d}{20}$ |
| 2. | Hoop stress is assumed uniform over the thickness. | Hoop stress is variable. It is maximum at inner surface, minimum at outer surface. |

*Contd.*

*Contd.*

| | |
|---|---|
| For cylinder : $\sigma_c = \dfrac{pd}{2t}$ tensile<br>For sphere : $\sigma_c = \dfrac{pd}{4t}$ tensile. | For cylinder :<br>$\sigma_x = \dfrac{B}{x^2} + A$<br>For sphere : $\sigma_r = \dfrac{B}{x^3} + A$<br>Variation is hyperbolic. |
| 3. Longitudinal stress is assumed to be uniform over the cross section.<br>For cylinder : $\sigma_L = \dfrac{pd}{4t}$<br>For sphere : $\sigma_L = \dfrac{pd}{2t}$ tensile | It is assumed to be uniform over the cross section.<br>For cylinder : $\sigma_L = \dfrac{PR_1^2}{\left(r_2^2 - r_1^2\right)}$<br>For sphere : $\sigma_L = \dfrac{Pr_1^2}{\left(r_2^2 - r_1^2\right)}$ tensile |
| 4. Radial stress is very small in case of thin shells and hence, it is neglected. | Radial stress varies over thickness. It is calculated from Lame's formula:<br>For cylinder : $\sigma_x = \dfrac{B}{x^2} + A$<br>For sphere : $\sigma_r = \dfrac{2B}{x^3} + A$<br>Both compressive. |
| 5. Internal pressure is less | Internal pressure is more. |

**For Numerical on thick cylinder shell :**

$$\text{Hoop stress } \sigma_x = \frac{B}{x^2} + A$$

$$\text{Radial stress } p_x = \frac{B}{x^2} - A$$

$$\text{Longitudinal stress } \sigma_L = \frac{Pr_2^2}{r_1^2 - r_2^2}$$

**Example 10.8.** *A thick cylinder of 150 mm outside and 100 mm inside radius is subjected to an external pressure of 30 MN/m*$^2$. *Calculate the maximum shear stress in the material of the cylinder at inner radius.*

(*UPTU : 2005–2006*)

*Given :*

*Inside radius* $r_1$ = *150 mm, Outside radius* $r_2$ = *100 mm*

*Inside pressure* $p_1$ = *30* $MN/m^2$ = *30* $N/mm^2$

**Solution To find :** Maximum shear stress $\tau_{max}$ at $r_1$

By Lame's equations

$$\text{Hoop stress } \sigma_x = \frac{B}{x^2} + A \text{ (Tensile)}$$

$$\text{Radial stress } p_x = \frac{B}{x^2} - A \text{ (Compressive)}$$

Applying boundary conditions

At $x = r_1 = 150$ mm, $p_1 = 30$ MPa

At $x = r_2 = 100$ mm, $p_2 = 0$

Putting these values

$$0 = \frac{B}{(100)^2} - A$$

$$30 = \frac{B}{(150)^2} - A$$

$$\therefore \quad -30 = B\left(\frac{1}{100^2} - \frac{1}{150^2}\right)$$

$$\therefore \quad B = -540000$$

$$\therefore \quad A = \frac{B}{(100^2)} = \frac{-540000}{(100)^2} = -54$$

$$\therefore \quad \sigma_x = \frac{-540000}{x^2} - 54$$

$$p_x = \frac{-540000}{x^2} + 54$$

∴ Hoop stress at inner surface is

$$\sigma_1 = \frac{-540000}{(100)^2} - 54 = -108 \text{ N/mm}^2$$

$$= 108 \text{N/mm}^2 \quad \text{(Compressive)}$$

∴ Maximum shear stress at the inner surface is

$$\tau_{max} = \frac{\sigma_1}{2} = \mathbf{54\ N/mm^2}$$

**Example 10.9.** *A thick cylinder with closed ends has 100 mm internal radius and 150 mm external radius. It is subjected to an internal pressure of 60 MN/m² and external pressure of 30 MN/m². Determine the hoop and radial stresses at the inside and outside of the cylinder together with longitudinal stress.*

(*UPTU : 2005–2006*)

*Given :*

$$r_2 = 100 \text{ mm} \qquad r_1 = 150 \text{ mm}$$

*Pressure* $p_2 = 60 \text{ MN/m} = 60 \text{N/mm (external)}$

$$p_1 = 30 \text{ MN/m}^2 = 30 \text{ N/mm}^2 \text{ (external)}$$

**Solution To find :** $\sigma_x$ and $\sigma_L$

By Lame's equations. At distance '$x$' from the centre

Hoop stress
$$\sigma_x = \frac{B}{x^2} + A \qquad ...(i)$$

Radial stress
$$p_x = \frac{B}{x^2} - A \qquad ...(ii)$$

Applying boundary conditions

At $x = r_2 = 100$ mm, $p_x = 60$ N/mm²

$x = r_1 = 150$ mm, $p_x = 30$ N/mm²

Putting these values in Eq. (ii)

$$60 = \frac{B}{(100)^2} - A$$

$$30 = \frac{B}{(150)^2} - A$$

$$\therefore \qquad (60 - 30) = B\left[\frac{1}{(100)^2} - \frac{1}{(150)^2}\right]$$

$$B = 540000$$

Now
$$A = \frac{B}{(100)^2} - 60 = \frac{540000}{(100)^2} - 60 = -6$$

$\therefore$ Lame's equation is; $\sigma_x = \dfrac{(540000)}{x^2} - 6$

To find hoop stress at inner surface

$$x = 100 \text{ mm}$$

$$\therefore \qquad \sigma_x = \frac{540000}{(100)^2} - 6$$

$\therefore$ $\boldsymbol{\sigma_x = 48}$ **N/mm², (tensile) at** $\mathbf{r_2}$

At the outer face, $x = 150$ mm

$$\therefore \quad \sigma_x = \frac{540000}{(150)^2} - 6$$

$$\boldsymbol{\sigma_x = 18\ \text{N/mm}^2,\ \text{tensile at } R_1}$$

Longitudinal stress $\sigma_L = \dfrac{r_1^2 p_1 - r_2^2 p_2}{r_1^2 - r_2^2} = \dfrac{(100)^2 \times 60 - (150)^2 \times 30}{(150)^2 - (100)^2} = -6$

$\therefore$ Longitudinal stress $= \mathbf{-\ 6\ N/mm^2}$

$= 6\ \text{N/mm}^2$ (Compressive)

**Example 10.10.** *Calculate the thickness of metal necessary for a cylindrical shell of internal diameter of 80 mm to withstand an internal pressure of 25 N/mm², if the maximum permissible tensile stress is 125 N/mm².* (*UPTU : 2009–2010*)

*Given :* $d = 80$ mm, $P_x = 25\ \text{N/mm}^2$

$\sigma_{max} = \sigma_x = 125\ \text{N/mm}^2$

**Solution To find** : $t$

$$r_2 = \frac{80}{2} = 40 \text{ mm}$$

$$P_1 = 25\ \text{N/mm}^2$$

$$P_2 = 0$$

By Lame's equation

Hoop stress $\sigma_x = \dfrac{B}{x^2} + A$ (Tensile) ...(i)

Radial stress $p_x = \dfrac{B}{x^2} - A$ (Compressive) ...(ii)

At $x = r_2 = 40$ mm $\quad P_x = P_1 = 25$ MPa

$x = r_1 \quad P_x = 0$

Substituting value of $P_x$ in above equation is

$$25 = \frac{B}{40^2} - A \quad \text{...(iii)}$$

$$0 = \frac{B}{r_1^2} - A$$

$$A = \frac{B}{r_1^2} \quad \text{...(iv)}$$

Also at $r_2 = 40$ mm, $\sigma_x = 125\ \text{N/mm}^2$

$$125 = \frac{B}{40^2} + A \qquad \text{...(v)}$$

Adding Eq. (iii) and (v), we get

$$150 = \frac{2B}{40^2}$$

$$B = 120{,}000$$

Putting value of $B$ in Eq. (v), we get

$$125 = \frac{120000}{40^2} + B$$

$$B = 50$$

From Eq. (iv), we get

$$A = \frac{B}{r_1^2}$$

$$\therefore \qquad 50 = \frac{120000}{r_1^2}$$

$$r_1 = 48.99 \approx 49 \text{ mm}$$

Thickness of metal

$$t = r_1 - r_2$$

$$= 49 - 40 = \mathbf{9\ mm}$$

**Example 10.11.** *Calculate the thickness of metal necessary for a cylindrical shell of internal diameter 160 mm to withstand an internal pressure of 25 MN/m², if maximum permissible tensile stress is 125 MN/m.*

(*UPTU : 2005–2006*)

*Given :* $d_2$ = *160 mm,* $\sigma_x$ = *125 Mpa* $P_x$ = *25 MPa*

**Solution To find :** Thickens of metal '$t$'

$$r_2 = \frac{160}{2} = 80 \text{ mm}$$

$$p_1 = 25 \text{ MN/m}^2 = 25 \text{ N/mm}^2$$

$$p_2 = 0$$

By Lame's equation

Hoop stress $$\sigma_x = \frac{B}{x^2} + A \qquad \text{...(Tensile)}$$

Radial stress $$p_x = \frac{B}{x^2} - A \qquad \text{...(Compressive)}$$

At $x = r_2 = 80$ mm, $p_x = P_1 = 25$ MPa

$x = r_1 \qquad p_x = 0$

Putting these values

$$\therefore \qquad 25 = \frac{B}{(80)^2} - A \qquad \text{...(i)}$$

$$0 = \frac{B}{r_1^2} - A \qquad \text{...(ii)}$$

Also at $x = r_2 = 80$ mm, $\sigma_x = 125$ N/mm$^2$

$$\therefore \qquad 125 = \frac{B}{(80)^2} + A \qquad \text{...(iii)}$$

From Eq. (i) $$25 = \frac{B}{(80)^2} - A \qquad \text{...(iv)}$$

$$\therefore \qquad 100 = 2A \quad \therefore A = 50$$

$$\therefore \qquad 25 = \frac{B}{(80)^2} - 50$$

$$B = 75 \times (80)^2 = 480000$$

$\therefore$ From Eq. (iii) $$0 = \frac{B}{r_1^2} - A = \frac{480000}{r_1}$$

$$r_1 = 97.97$$

$$r_1 \approx 98 \text{ mm}$$

$\therefore \qquad r_1 = 98$ mm (outer radius)

and $\qquad r_2 = 80$ mm (inner radius)

$\therefore$ Thickness $t = (r_1 - r_2) = 18$ mm

$\therefore$ **Required thickness = 18 mm**

**Example 10.12.** *The cylinder of a hydraulic ram is of 160 mm internal diameter. Find the thickness required to withstand an internal pressure of 60 N/mm$^2$. If the maximum tensile stress is limited to 90 N/mm$^2$, and the maximum shear stress to 80 N/mm$^2$.* (*UPTU : 2001–2002*)

*Given:* *Internal diameter* = *160 mm*

*Internal radius $r_2$* = *80 mm*

*Internal pressure p* = *60 N/mm$^2$*

*Permissible tensile stress* = *90 N/mm$^2$*

*Permissible shear stress* = *80 N/mm$^2$*

**Solution To find :** Thickness $t$

By Lame's equation :

Hoop stress $\sigma_x = \dfrac{B}{x^2} + A$ (Tensile)

Radial stress $p_x = \dfrac{B}{x^2} - A$ (Compressive)

Applying boundary conditions

At $r_2 = 80$ mm, $\sigma_x = 90$ N/mm²

$$\therefore \quad 90 = \frac{B}{(80)^2} + A \quad \text{...(i)}$$

At $r_2 = 80$ mm, $p_x = 60$ N/mm²

$$\therefore \quad 60 = \frac{B}{(80)^2} - A \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii),

$$150 = \frac{2B}{(80)^2}$$

$$\therefore \quad B = 480000$$

and $A = 15$

Now radial stress, $p_r = \dfrac{B}{x^2} - A$ (Compressive)

$\sigma_r$ is zero at $x = r_1$ (i.e., external surface)

$$\therefore \quad 0 = \frac{480000}{(r_1)^2} - 15$$

$$\therefore \quad 480000 = 15 \times (r_1)^2$$

$$r_1 = 178.88$$

$$\therefore \quad r_1 \approx 179 \text{ mm}$$

$$\text{Thickness} = (r_1 - r_2) = (179 - 80) = 99 \text{ mm}$$

To check the maximum shear stress

$$q_{max} = \left(\frac{\sigma_x + p_x}{2}\right) \quad \text{as } \sigma_r \text{ is compressive}$$

$$= \frac{90 + 60}{2} = 75 \text{ N/mm}^2 < 80 \text{ N/mm}^2$$

Also consider the longitudinal stress,

$$\sigma_L = \frac{pR_1^2}{\left(r_2^2 - r_1^2\right)} = \frac{60 \times (80)^2}{(179^2 - 80^2)} = 14.97 \text{ N/mm}^2$$

$$\approx 15 \text{ N/mm} \quad \text{(Tensile)}$$

$$\therefore \quad q_{max} = \frac{\sigma_x + p_x}{2} = \frac{15+60}{2}$$

$$= 37.5 \text{ N/mm less than permissible limit}$$

$\therefore$ Required thickness = **99 mm**

**Example 10.13.** A *thick spherical shell having internal radius of 75 mm is subjected to an internal pressure of 25 N/mm². If the maximum hoop stress 100 N/mm². Find the thickness of the shell.* (UPTU : 2004–2005)

*Given :* *Internal radius $r_2$ = 75 mm, $p_x$ = 25 N/mm , $\sigma_x$ = 100 N/mm²*

**Solution To find :** Thickness of shell

For a thick spherical shell, the Lame's equations are :

$$\text{Hoop stress } \sigma_x = \frac{B}{x^3} + A \quad \text{(Tensile)} \qquad \text{...(i)}$$

$$\text{Radial stress } p_x = \frac{B}{x^3} - A \quad \text{(Compressive)} \qquad \text{...(ii)}$$

Here '$x$' is distance of the point from centre of the sphere.

Applying boundary conditions

At $x$ = 75 mm, $\sigma_x$ = 100 N/mm²

At $x$ = 75 mm, $p_r$ = 25 N/mm²

Putting these values in Eqs. (i) and (ii),

$$100 = \frac{B}{(75)^3} + A \qquad \text{...(iii)}$$

$$25 = \frac{2B}{(75)^3} - A \qquad \text{...(iv)}$$

Adding Eqs. (i) and (ii)

$$125 = \frac{3B}{(75)^3}$$

$$\therefore \quad B = 17.58 \times 10^6$$

$$\text{Also} \quad 100 = \frac{B}{(75)^3} + A$$

$$\therefore \quad A = 58.33$$

$\therefore$ Lame's equations are

$$\sigma_x = \frac{(17.58 \times 10^6)}{x^3} + 58.33$$

$$p_x = \frac{2 \times 17.58 \times 10^6}{(r^1)} - 58.33$$

Radial stress i.e., pressure is zero at the external surface where $x = r_1$

$$0 = \frac{2 \times 17.58 \times 10^6}{(r_1)^3} - 58.33$$

$\therefore$ $r_1$ = 84.5 mm external radius

$r_2$ = 75.00 mm internal radius

$\therefore$ Thickness $t = (r_1 - r_2)$ = (84.5 – 75.00) = **9.5 mm**

$\therefore$ Thickness required is 9.5 mm.

**Example 10.14.** *The cylinder of a hydraulic ram is of 6 cm internal diameter. Find the thickness required to withstand an internal pressure of 40 N/mm*$^2$ *. If the maximum tensile stress is limited to 60 N/mm and the maximum shear stress to 50 N/mm*$^2$*.* (*UPTU : 2001–2002, 2002–2003, 2010–2011*)

*Given : Internal diameter*

$d_2$ = *6 cm = 60 mm*

*Internal pressure =* $p$ = *40 N/mm*$^2$

*Maximum tensile stress = 60 N/mm*$^2$ *and Max shear stress = 50 N/mm*$^2$

**Solution** By Lame's equations : At a distance '$x$' from the centre :

Circumferential stress $\sigma_x = \dfrac{B}{x^2} + A$ (Tensile)

Radial stress $p_x = \dfrac{B}{x^2} - A$ (Compressive)

$\sigma_x$ is maximum at inner face i.e. at $x$ = 30 mm

$p_x$ is equal to $p$ = 40 N/mm

At $x$ = 30 mm

$$\therefore \quad 60 = \frac{B}{(30)^2} + A \quad \text{...(i)}$$

$$40 = \frac{B}{(30)^2} - A \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii) we get,

$$100 = \frac{2B}{(30)^2}$$

$\therefore \quad B = 50 \times 900 = (45000)$

$$A = 60 - \frac{B}{(30)^2} = 10$$

$\therefore$ Radial stress $\quad p_x = \dfrac{B}{x^2} - A$

$$\therefore \quad p_x = \frac{45000}{x^2} - 10$$

$p_x = 0$ at the outer face i.e. at $x = r_1$

$$\therefore \quad 0 = \frac{5000}{r_1^2} - 10$$

$\therefore \quad r_1 = 67$ mm

$\therefore$ External radius $= r_1 = 67$ mm

Internal radius $= r_2 = 30$ mm

$\therefore$ Thickness $t = (r_1 - r_2) = (67 - 30) =$ **37 mm**

$\therefore$ The thickness of cylinder is 37 mm

Longitudinal stress,

$$\sigma_L = \frac{pr_2^2}{\left(r_1^2 - r_2^2\right)} = \frac{40 \times (30)^2}{67^2 - 30^2}$$

$=$ **10 N/mm (Tensile)**

$\therefore$ The three principal stresses are:

$\sigma_1 = 60$ N/mm$^2$ (Tensile)

$\sigma_2 = 10$ N/mm$^2$ (Tensile)

$\sigma_3 = 40$ N/mm$^2$ (Compressive)

From these the maximum stress is,

$$\frac{\sigma_1 + \sigma_3}{2} = \frac{60 + 40}{2}$$

$= 50$ N/mm$^2 < 60$ N/mm$^2$

**Example 10.15.** *Derive an equation to obtain the radial and circumferential stresses in thick shell subjected to external and internal pressure both.*

*(UPTU : 2005–2006)*

**Solution** Lame's equation

$$\sigma_x = \frac{B}{x^2} + A \qquad \text{....(i)}$$

$$p_x = \frac{B}{x^2} - A \quad \text{...(ii)}$$

Let, external pressure = $p_1$ ; Internal pressure = $p_2$ ;

At $r = r_2$, $p_x, = p_2$ and $r = r_d$, $\sigma_x = p_1$

Substituting in Eq. (ii), we have,

$$p_2 = \frac{B}{r_2^2} - A \quad \text{...(iii)}$$

$$p_1 = \frac{B}{r_1^2} - A \quad \text{...(iv)}$$

Now from Eqs. (iii) and (iv), we get

$$B = \frac{r_1^2\, r_2^2}{\left(r_1^2 - r_2^2\right)}\left(p_2 - p_1\right) \quad \text{...(v)}$$

$$A = \frac{p_2\, r_2^2 - p_1\, r_1^2}{r_1^2 - r_2^2}$$

Substituting the value of Eq. (v) in Eq. (vii), we get

$$\text{Radial stress, } p_x = \frac{1}{(r_1^2 - r_2^2)}\left[(p_1\, r_1^2 - p_2\, r_2^2) + \frac{r_1^2\, r_2^2}{r^2}(p_2 - p_1)\right] \quad \text{...(vii)}$$

and

$$\text{Hoop stress, } \sigma_x = \frac{1}{r_1^2 - r_2^2}\left[(p_2\, r_2^2 - p_1\, r_1^2) + \frac{r_1^2\, r_2^2}{r^2}(p_2 - p_1)\right] \quad \text{...(viii)}$$

**Example 10.16.** *Derive an expression to determine stresses in thick walled cylinder subjected to internal pressure only.* (*UPTU : 2008–2009*)

**Solution**

Lame's equation

$$\sigma_x = \frac{B}{x^2} + A \quad \text{...(i)}$$

$$p_x = \frac{B}{x^2} - A \quad \text{...(ii)}$$

Internal pressure = $p_1$ ; external pressure = zero

When there is only internal pressure and outer surface of the cylinder is exposed to atmospheric pressure, then

At $r = r_1$, $\sigma_r = p_1$ and $r = r_2$, $\sigma_r = p_2 = 0$

Substituting in Eq. (ii).

We have

$$p_2 = \frac{b}{r_2^2} - A \qquad \text{...(iii)}$$

$$p_1 = \frac{B}{r_1^2} - A = 0 \qquad \text{...(iv)}$$

From Eqs. (i) and (ii), we get

$$B = p_1 \left[\frac{r_1^2 - r_2^2}{r_2^2 - r_1^2}\right]$$

and

$$A = p_2 \left[\frac{r_1^2}{r_1^2 - r_2^2}\right]$$

Substituting above Eqs. (ii) and (iii) we get,

$$p_x = \frac{p_2\, r_2^2}{r_1^2 - r_2^2}\left[\frac{r_1^2}{r^2} - 1\right]$$

$$\sigma_x = \frac{p_2\, r_2^2}{r_1^2 - r_2^2}\left[\frac{r_1^2}{r^2} + 1\right]$$

At inner face, $r = r_2$ ; $(\sigma_c)_{r2} = p_2 \cdot \dfrac{r_1^2 + r_2^2}{r_1^2 - r_2^2}$

At outer face, $r = r_1$ ; $(\sigma_c)_{r1} = p_2 \cdot \dfrac{2r_2^2}{r_1^2 - r_2^2}$

Further for the closed ends cylinder, the longitudinal stress is

$$\sigma_L = \frac{p_2 \pi\, r_2^2}{\pi\left(r_1^2 - r_2^2\right)} = \frac{p_2 r_1^2}{r_1^2 - r_2^2} = A$$

**Example 10.17.** *Derive an expression for maximum principal stress on thick cylindrical shell subjected to external pressure.* (*UPTU : 2006–2007*)

**Solution** Lame's Equation

$$p_x = \frac{B}{x^2} - A \qquad \text{...(i)}$$

$$\sigma_x = \frac{B}{x^2} + A \qquad \text{...(ii)}$$

External pressure = $p_1$ ; internal pressure = zero;

At, $r = r_2$, $p_r = 0$ and at $r = r_1$, $p_r = p_1$

Substituting in Eq. (ii), we get

$$0 = \frac{B}{r_2^2} - A \quad \text{...(iii)}$$

$$p_1 = \frac{B}{r_1^2} - A \quad \text{...(iv)}$$

From Eq. (iii) and (iv), we have

$$B = \frac{p_1 r_1^2 r_2^2}{\left(r_1^2 - r_2^2\right)}$$

$$A = \frac{p_1 r_1^2}{r_1^2 - r_2^2}$$

The maximum circumferential (or hoop) stress at radius $r_1$,

$$\sigma_x = p_1\left(\frac{2r_1^2}{r_1^2 - r_2^2}\right) \quad \text{...(v)}$$

**Example 10.18.** *The maximum stress permitted in a thick cylinder of inner and outer radius of 10 cm and 15 cm is 20 N/mm². The external pressure is 8 N/mm², what internal pressure can be applied ?* (*MTU : 2012–2013*)

*Given :* *For this thick cylinder*

*Radius* $r_1$ = *10 cm = 100 mm*

$r_2$ = *15 cm = 150 mm*

*Maximum hoop stress* $\sigma_x$ = *20 N/mm²*

*External pressure* = *6 N/mm²*

**Solution To find** : Internal pressure

By Lame's equations

At a point distance '$x$' from the centre :

$$\text{Hoop stress } \sigma_x = \frac{B}{x^2} + A \text{ (Tensile)}$$

$$\text{Radial stress } p_x = \frac{B}{x^2} - A \text{ (Compressive)}$$

Applying boundary conditions

At the inner face : $x = 100$ mm, $\sigma_x = 20$ N/mm²

$$\therefore \quad 20 = \frac{B}{(100)^2} + A \quad \text{...(i)}$$

At the outer face : $x = 120$ mm, $p_x = 6$ N/mm²

$$\therefore \qquad 8 = \frac{B}{(150)^2} - A \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii)

$$28 = B\left(\frac{1}{100} + \frac{1}{150^2}\right)$$

$$\therefore \qquad B = 193846.15$$

Now from Eqs. (i)

$$20 = \frac{B}{(100)^2} + A = \frac{19384.615}{100^2} + A$$

$$\therefore \qquad A = 0.615$$

$\therefore$ Lame's equations are: $\sigma_x = \dfrac{19384.615}{x^2} + 0.615$

$$p_x = \frac{19384.615}{x^2} - 0.615$$

At the inner face $x$ = 100 mm

$$\therefore \qquad p_x = \frac{19384.615}{(100)^2} - 2$$

$$= \mathbf{17.38 \ N/mm^2 \ (Compressive)}$$

$\therefore$ This is internal pressure.

$\therefore$ Maximum permissible internal pressure is 17.38 N/mm$^2$.

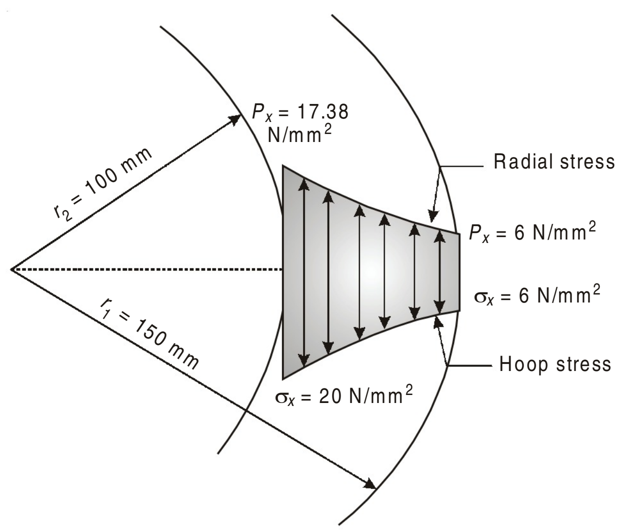



**Fig. 10.9** *Pressure distribution across the thickness of cylinder*

**Example 10.19.** *The maximum stress permitted in a thick cylinder, radii 8 cm and 12 cm, is 20 N/mm$^2$, the external pressure is 6 N/mm, what internal pressure can be applied ? Plot curves showing the variation of hoop and radial stresses through the material.* (*UPTU : 2003–2004*)

*Given :* Internal radius $r_2$ = 8 cm = 80 mm

External radius $r_1$ = 12 cm = 120 mm

External pressure $p_1$ = 6 N/mm$^2$

Maximum hoop stress = $\sigma_{x2}$ = 20 N/mm$^2$

**Solution To find :** Internal pressure $p_2$, hoop stress at outer face $\sigma_{x1}$

Figure 10.10 shows stress distribution across the thickness of cylinder.

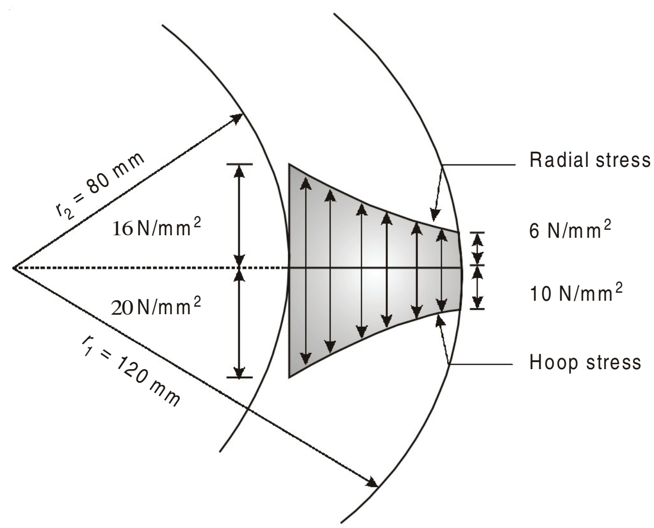



**Fig. 10.10** *Stress distribution across the thickness of cylinder*

Hoop stress at inner face

$$\sigma_{x_2} = \frac{1}{r_1^2 - r_2^2}[p_2 r_2^2 - p_1 r_1^2 + r_1^2(p_2 - p_1)]$$

$$20 = \frac{1}{120^2 - 80^2}\ [p_2 \times 80^2 - 6 \times 120^2 + 120^2\ (p_2 - 6)\ ]$$

$$160 \times 10^3 = p_2 \times 80^2 - 6 \times 120^2 + 120^2\ p_2 - 6 \times 120^2$$

$$(80^2 + 120^2)\ p_2 = 160 \times 10^3 + 6 \times 120^2 + 6 \times 120^2$$

$$p_2 = \frac{332.8 \times 10^3}{(80^2 + 120^2)} = \mathbf{16\ N/mm^2}$$

Hoop stress at outer face $\sigma_{x_1} = \dfrac{1}{r_1^2 - r_2^2}[p_2 r_2^2 - p_1 r_2^2 + r_2^2(p_2 - p_1)]$

$$\sigma_{x_1} = \frac{1}{120^2 - 80^2}[16 \times 80^2 - 6 \times 120^2 + 80^2(16-6)]$$

$$\sigma_{x_1} = \mathbf{10\ N/mm^2 \quad (Tensile)}$$

**Example 10.20.** *A thick cylinder of 160 mm internal 240 mm external diameter is subjected to an external pressure 12 MPa. Determine the maximum internal pressure that can be applied if the maximum allowable normal stress is 36 MPa. Plot the variation of radial and hoop stresses.* (*UPTU : 2010–2011*)

*Given :* *External diameter* $d_1 = 240$ *mm*

*Internal diameter* $d_2 = 160$ *mm*

*External pressure* $p_1 = 12$ *MPa*

*Maximum allowable normal stress* $\sigma_{x_2} = 36$ *MPa*

**Solution To find :** Internal pressure $p_2$.

$$r_1 = 120 \text{ mm}, \quad r_2 = \frac{160}{2} = 80 \text{ mm}$$

Hoop stress at inner face,

$$\sigma_{x_2} = \frac{1}{r_1^2 - r_2^2}[p_2 r_2^2 - P_1 r_1^2 + r_1^2(P_2 - P_1)]$$

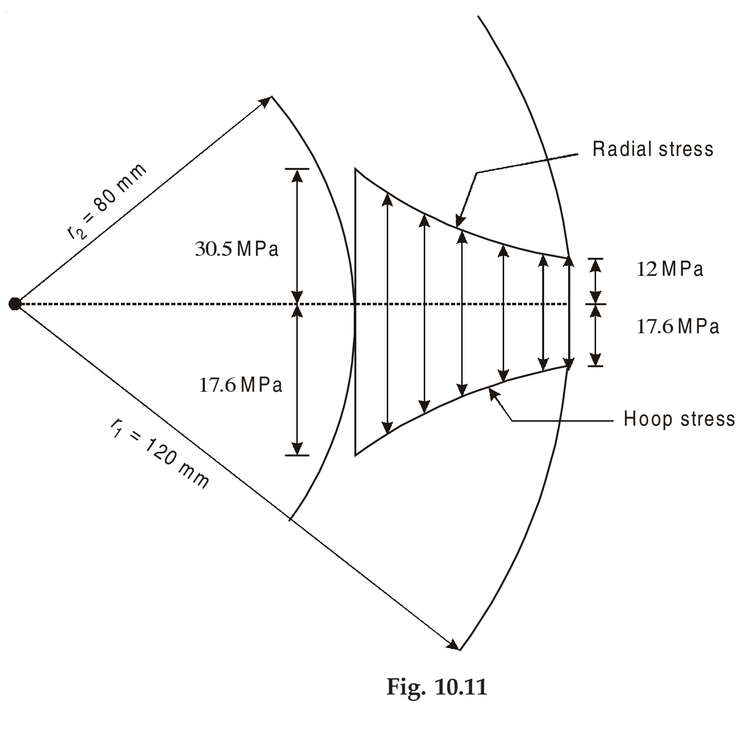



**Fig. 10.11**

$$36 = \frac{1}{120^2 - 80^2} \; [P_2 \times 80^2 - 12 \times 120^2 + 120^2 (P_2 - 12)]$$

$$288 \times 10^3 = (80^2 + 120^2)\, P_2 - 345.6 \times 10^3$$

$$P_2 = \frac{633.6 \times 10^3}{(80^2 + 120^2)} = \mathbf{30.5\ MPa}$$

Hoop stress at outer face

$$\sigma_{x_1} = \frac{1}{r_1^2 - r_2^2} [p_2 r_2^2 - P_1 r_1^2 + r_1^2 (P_2 - P_1)]$$

$$= \frac{1}{120^2 - 80^2} \; [30.5 \times 80^2 - 12 \times 120^2 + 80^2 (30^5 - 12)]$$

$$\sigma_{x_1} = \frac{140800}{120^2 - 80^2} = 17.6 \text{ MPa}$$

**Example 10.21.** *An external pressure of 10 MN/m² is applied to thick cylinder of intemal diameter 150 mm and external diameter 300 mm. If the maximum hoop stress permitted on the inside wall is 35 MN/m², calculate the maximum internal pressure that can be applied.* (*MTU : 2012–2013*)

*Given :* *External diameter* $d_1$ *= 300 mm*

*Internal diameter* $d_2$ *= 150 mm*

*External pressure* $p_1$ *= 10 MPa*

*Maximum allowable normal stress* $\sigma_{x_2}$ *= 35 MPa*

**Solution To find :** Internal pressure $p_2$.

$$r_1 = 150 \text{ mm}, \quad r_2 = \frac{150}{2} = 75 \text{ mm}$$

Hoop stress at inner face,

$$\sigma_{x_2} = \frac{1}{r_1^2 - r_2^2} [P_2 r_2^2 - P_1 r_1^2 + r_1^2 (P_2 - P_1)]$$

$$35 = \frac{1}{150^2 - 75^2} [P_2 \times 75^2 - 10 \times 150^2 + 150^2 (P_2 - 10)]$$

$$590.625 \times 10^3 = (75^2 + 150^2)\, P_2 - 450 \times 10^3$$

$$P_2 = \frac{1.041 \times 10^6}{75^2 + 150^2} = \mathbf{37\ MPa}$$

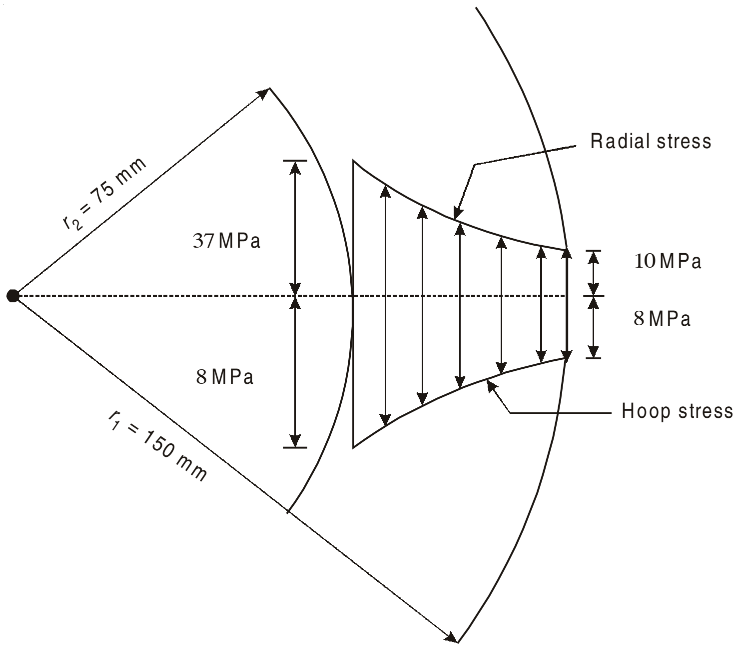



**Fig. 10.12**

Hoop stress at outer face

$$\sigma_{x_1} = \frac{1}{r_1^2 - r_2^2}[P_2 r_2^2 - P_1 r_1^2 + r_1^2 (P_2 - P_1)]$$

$$= \frac{1}{150^2 - 75^2}[37 \times 75^2 - 10 \times 150^2 + 75^2 (37 - 10)]$$

$$\sigma_{x_1} = 8 \text{ MPa}$$

**Example 10.22.** *A thick hollow cylinder 200 mm internal and 300 mm external diameter is subjected to an internal pressure 50 MPa and external pressure 25 MPa. Find the maximum shear stress at the inner surface of the cylinder.* (*UPTU : 2011–2012*)

*Given :* $d_1 = 200$ mm, $d_2 = 300$ mm

$r_1 = 100$ mm $= 0.1$ m

$r_2 = 150$ mm $= 0.15$ m

*External pressure,* $p_2 = 25$ MPa

*Internal pressure* $p_1 = 50$ MPa

**Solution To find :** Maximum shear stress $(\tau_{max})_{r1}$

$$\text{Hoop stress } \sigma_x = \frac{1}{r_1^2 - r_2^2}\left[p_2 r_2^2 - p_1 r_1^2 + \frac{r_1^2 r_2^2}{r^2} - (p_2 - p_1)\right]$$

$$(\sigma_c)_{r2} = \frac{1}{r_1^2 - r_2^2}\left[p_2 r_2^2 - P_1 r_1^2 + r_1^2 - (p_2 - p_1)\right]$$

(Substituting $r = r_2$)

$$= \frac{1}{0.15^2 - 0.1^2}[50 \times 0.1^2 - 25 \times 0.15^2 + 0.15^2\,(50 - 25)]$$

$$= \frac{1}{0.0125}(0.5) = \mathbf{40\ MN/m^2\ \ (Tensile)}$$

$$\text{Radial stress} \quad \sigma_r = \frac{1}{r_1^2 - r_2^2}\left[p_1 r_1^2 - p_2 r_2^2 + \frac{r_1^2 r_2^2}{r^2} - (p_2 - p_1)\right]$$

(Substituting $r = r_2$) $$\sigma_r = \frac{1}{r_1^2 - r_2^2}\left[p_1 r_1^2 - p_1 r_1^2 + r_2^2 - (p_2 - p_1)\right]$$

$$= \frac{1}{0.15^2 - 0.1^2}[25 \times 0.15^2 - 50 \times 0.1^2 + 0.15^2\,(50 - 25)]$$

$$= \frac{1}{0.0125}(0.625) = \mathbf{50\ MN/m^2\ (Compressive)}$$

Maximum shear stress

$$(\tau_{max})_{r2} = \frac{(\sigma_c)_{r2} - (\sigma_r)_{r2}}{2} = \frac{40 - (-50)}{2}$$

$$= \mathbf{45\ MN/mm^2}$$

[$\because$ $(\sigma_r)_{r1}$ is compressive]

**Example 10.23.** *What is compound cylinder? What is its advantage over a single cylinder?* (*UPTU : 2004–2005*)

**Solution**

(i) The thick walled cylinder example as shown in Fig. 10.13, that there is a considerable variation in stress intensity in the wall of a thick cylinder subjected to internal pressure, with the tensile hoop stress being highest at the inner surface.

(ii) As a consequence of this stress variation, much of the material forming a standard thick cylinder is not efficiently used.

(iii) A more uniform stress distribution can be achieved if one tube is shrunk fit into the outside of another to form a compound cylinder.

(iv) The contraction of the external tube puts the inner tube into compression and the outer tube into tension.

(v) As a consequence, when internal pressure is applied, the inner surface tensile hoop stress that results is lower than that generated in a similar sized standard thick cylinder, whilst the outer surface stress is higher i.e. the stress distribution is changed and the material can be more efficiently used.

In practice, this technique is only used when internal pressure is very high e.g. gum barrels.

Let $r_1$ and $r_2$ be the outer and inner of the compound tube. Let the radius at the junction of the two tubes be $r_3$.

Let $p_j$ be the radial pressure intensity at the junction of the two tubes after shrinking the outer tube over the inner tube.

Let Lame's equation for the outer tube,

$$p_x = \frac{B_1}{x^2} - A_1$$

and

$$\sigma_x = \frac{B_1}{x^2} + A_1$$

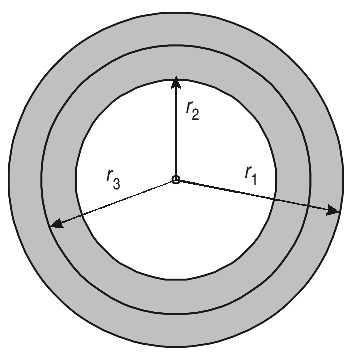



**Fig. 10.13**

At $x = r_1, \quad p_x = 0$

$$\therefore \quad 0 = \frac{B_1}{r_1^2} - A_1 \qquad ...(i)$$

and at $x = r_3$

$$\therefore \quad p_1 = \frac{B_1}{r_3^2} - A_1 \qquad ...(ii)$$

The constants $A_1$ and $B_1$ can be determined from Eqs. (i) and (ii).

Let Lame's equation for the inner tube

$$p_x = \frac{B_2}{x^2} - A_2$$

and $$\sigma_x = \frac{B_2}{x^2} + A_2 \qquad \text{...(iii)}$$

at $$x = r_2, \quad p_x = 0$$

$$0 = \frac{B}{r_2^2} - A_2$$

and at $$x = r_3, \quad p_x = p_j$$

$\therefore$ $$p_j = \frac{B_2}{r_3^2} - A_2 \qquad \text{...(iv)}$$

The constants $A_2$ and $B_2$ can be determind from Eqs. (ii) and (iv).

The hoop stresses for the outer and the inner tube can be easily determined.

Now consider a compound tube is subjected to an internal fluid pressure $p_0$. For this analysis, the inner and the outer tubes will together be considered as one thick shell. The stresses due to internal fluid pressure alone can now be determined. For this condition Lame's equation.

$$p_x = \frac{B}{x^2} - A$$

and $$\sigma_x = \frac{B}{x^2} + A$$

at $$x = r_1, \quad p_x = 0$$

$$0 = \frac{B}{r_1^2} - A \qquad \text{...(v)}$$

$$x = r_2, \quad p_x = p_0$$

$\therefore$ $$p_0 = \frac{B}{r_2^2} - A \qquad \text{...(vi)}$$

The constant $A$ and $B$ can now be evaluated. The final hoop stress is calculated by adding, the hoop stresses caused due to shrinking and the hoop stresses caused by internal fluid pressure.

**Example 10.24.** *Explain shrinkage allowance.* (*UPTU : 2004–2005*)

**Solution**

In order to shrink outer tube over the inner tube, it is necessary to keep the radius of inner face of the outer tube should be slightly less than the radius of the outer face of inner tube, so that after the tube is shrunk-on, the required radial pressure at junction of tube is achieved.

Let $r_3$ = Radius of the tube at junction

$\delta r'$ = Difference between the radius of the outer face of the inner tube and $r_3$

$\delta r''$ = Difference between the radius of the inner face of the outer tube and $r_3$

∴ Original difference of the radii of the two tubes at the junction

$$\delta r_3 = \delta r' + \delta r''$$

Now, for the outer tube let Lame's relations be

$$p_x = \frac{B_1}{x^2} - A_1$$

and
$$f = \frac{B_1}{x^2} + A_1$$

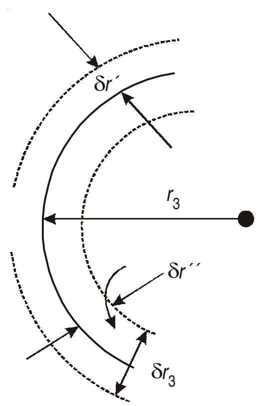



**Fig. 10.14**

∴ Circumferential tensile strain for the outer tube at the junction

$$\frac{\sigma r''}{r_3} = \frac{1}{E}\left[\left(\frac{B_1}{r_3^2} + A_1\right) + \frac{p}{m}\right] \qquad ...(i)$$

where $p'$ = Radial pressure at the junction

and $\frac{1}{m}$ = Poisson's ratio

Similarly, for the inner tube, let Lame's relations be

$$p_x = \frac{B_2}{x^2} - A_2$$

and $$f_x = \frac{B_2}{x^2} + a_2$$

$\therefore$ Circumferential compressive strain for the inner tube at the junction

$$= \frac{\delta r'}{r_3} = \frac{1}{E}\left[\left(\frac{b_1}{r_3^2} + a_2\right) + \frac{p}{m}\right] \quad \text{...(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$\frac{\delta r' + \delta r''}{r_3} = \frac{\delta r_3}{r_3} = \frac{1}{E}\left[\left(\frac{b_3}{r_3^2} + a_2\right) - \left(\frac{b_3}{r_3^2} + a_2\right)\right] \quad \text{...(iii)}$$

$$\therefore \quad \frac{\delta r_3}{r_3} = \frac{\begin{pmatrix}\text{Algebraic difference between the hoop} \\ \text{stresses in the tubes at the junction}\end{pmatrix}}{E}$$

## EXERCISE

**10.1.** Determine the maximum hoop stress across the section of a pipe of external diameter 600 mm and internal diameter 440 mm, when the pipe is subjected **to an internal fluid pressure of 10 N/mm².** [*Ans.* 99.9 N/mm²]

**10.2.** The internal fluid pressure in a pipe is 6 MPa. The internal and external radii of the pipe are respectively 250 mm and 350 mm. Find the maximum and minimum hoop stresses across the section.

**[*Ans.* 15.24 N/mm²; 5.98 N/mm²]**

**10.3.** The internal fluid pressure in a cylindrical shell is 6 MPa. The external and internal diameters of the shell are respectively 400 mm and 200 mm. Find the maximum and minimum hoop stress in the cylinder material.

**[*Ans.* 10 N/mm²; 4 N/mm²]**

**10.4.** Find the thickness of metal necessary for a cylindrical shell of internal diameter 150 mm to withstand an internal pressure of 50 N/mm². The maximum hoop stress in the section is not to exceed 150 N/mm². **[*Ans.* 31 mm]**

**10.5.** The internal radius of a thick walled cylindrical shell is 75 mm. The internal fluid pressure in the shell is 8 N/mm². The maximum allowable tensile stress in the shell material is 20 N/mm³. Find the thickness of the shell.

**[*Ans.* 40 mm]**

**10.6.** A compound cylinder is made by shrinking a cylinder of external radius 150 mm over another cylinder of external radius 125 mm and internal radius 75 mm. After shrinking, the radial compression at the common junction is 28 N/mm². If $E = 200$ kN/mm², find the original difference in radii at the junction.

**[*Ans.* 1.5 mm]**

**10.7.** A compound cylinder consists of a tube of 200 mm external radius, shrunk on another tube of 100 mm internal radius such that the radius at the junction is 150 mm. After shrinking, the radial compressive stress at the junction is

21 N/mm$^2$. Finally, the cylinder is subjected to an internal fluid pressure of 9 0 N/mm$^2$. Determine the resultant hoop stresses in the cylinder.

**[*Ans.* For outer tube ; $f_{outer}$ = 114 N/mm$^2$ (tensile); $f_{junctjon}$ = 158.33 N/mm$^2$ (tensile). For inner tube; $f_{junction}$ = 28.73 N/mm$^2$ (tensile); $f_{inner}$ = 74.4 N/mm$^2$ (tensile)]**

**10.8.** A compound cylinder is made by shrinking on a cylinder of external diameter 200 mm and internal diameter 160 mm over another cylinder of external diameter 160 mm and internal diameter 120 mm. The radial pressure at the junction after shrinking is 8 N/mm$^2$. Find the final stresses set up across the section, when the compound cylinder is subjected to an internal fluid pressure of 60 N/mm$^2$.

**[*Ans.* $F_{60}$ = 90.9 and $F$80 = 57.9 N/mm$^2$, outer $F_{80}$ = 122.9 and $F_{100}$ = 25.9 N/mm$^2$]**

**10.9.** A thick walled spherical shell 75 mm internal radius is subjected to an internal fluid pressure of 10 MPa. If the maximum allowable tensile stress in the shell material be 100 N/mm$^2$. Find the external radius of the shell.

**[*Ans.* 82.55 mm]**

## UNIVERSITY QUESTIONS

1. What is the difference between thin and thick cylinders ? State the assumptions made in the analysis of stress in thick cylinders. Derive Lames' equations to find the stresses in thick cylinders. (*UPTU : 2002, 2005, 2001, 2011, 2013*)

   **[*Ans.* Examples 10.7, 10.5, 10.6)**

2. The cylinder of a hydraulic ram is of 160 mm internal diameter. Find the thickness required to withstand an internal pressure of 60 N/mm$^2$, If the maximum tensile stress is limited to 90 N/mm$^2$, and the maximum shear stress to 80 N/mm$^2$. (*UPTU : 2001*)

   **[*Ans.* Example 10.12)**

3. The cylinder of a hydraulic ram is of 6 cm internal diameter. Find the thickness required to withstand an internal pressure of 40 N/mm$^2$, if the maximum tensile stress is limited to 60 N/mm$^2$ and the maximum shear stress to 50 N/mm$^2$. (*UPTU : 2001*)

   **[*Ans.* Example 10.14)**

4. The cylinder of a hydraulic ram is of 6 cm internal diameter. Find the thickness required to withstand an internal pressure of 40 N/mm$^2$, if the maximum tensile stress is limited to 60 N/mm$^2$ and the maximum shear stress to 50 N/mm$^2$. (*UPTU : 2010*)

   **[*Ans.* Example 10.14)**

5. Derive Lame's equations to find out the stresses in thick spherical shells. (*UPTU : 2002–2003*)

   **[*Ans.* Example 10.6)**

6. The maximum stress permitted in a thick cylinder, radii 8 cm and 12 cm, is 20 N/mm$^2$, the external pressure is 6 N/mm$^2$, what internal pressure can be applied ? Plot curves showing the variation of hoop and radial stresses through the material. (*UPTU : 2003–04*)

**[*Ans.* Example 10.19]**

7. A thick spherical shell having internal radius of 75 mm is subjected to an internal pressure of 25 N/mm$^2$. If the maximum hoop stress is 100 N/mm$^2$, find the thickness of the shell. (*UPTU : 2004–05*)

**[*Ans.* Example 10.1]**

8. What do you mean by Lame's equations ? How will you derive these equations ? (*UPTU : 2004–05*)

**[*Ans.* Example 10.5]**

9. Explain the following :
   (i) What is compound cylinder ? What is its advantages over a single cylinder?
   (ii) Shrinkage allowance
   (iii) State assumptions made in Lame's Theory. (*UPTU : 2004–05*)

**[*Ans.* (i) Example 10.23; (ii) Example 10.24; (iii) Example 10.5]**

10. A thick cylinder with closed ends has 100 mm internal radius and 150 mm external radius. It is subjected to an internal pressure of 60 MN/m$^2$ and external pressure of 30 MN/m$^2$. Determine the hoop and radial stresses at the inside and outside of the cylinder together with longitudinal stress. (*UPTU : 2005–2006*)

**[*Ans.* Example 10.9]**

11. How thick and thin cylinders are classified ? Derive the equation for Hoop stress and radial stress in thick cylinder ? (*UPTU : 2005–06*)

**[*Ans.* Example 10.7]**

12. Derive the equation to obtain the radial and circumferential stresses in a thick shell subjected to external and internal pressure both. (*UPTU : 2005–06*)

**[*Ans.* Example 10.15]**

13. A thick cylinder of 150 mm outside and 100 mm inside radius is subjected to an external pressure of 30 MN/m$^2$. Calculate the maximum shear stress in the material of the cylinder at inner radius. (*UPTU : 2005–06*)

**[*Ans.* Example 10.8]**

14. Calculate the thickness of metal necessary for a cylindrical shell of internal diameter 160 mm to withstand an internal pressure of 25 MN/m , if maximum permissible tensile stress is 125 MN/m$^2$. (*UPTU : 2006–07*)

**[*Ans.* Example 10.10]**

15. Derive an expression for maximum principal stress on thick cylindrical shell subjected to external pressure. (*UPTU : 2006–07*)

**[*Ans.* Example 10.17]**

16. A hollow cylinder of 45 cm internal diameter and 10 cm thickness contains the fluid under pressure of 850 N/cm$^2$. Find the maximum and minimum hoop stress across the section. (*UPTU : 2006–07*)

**[*Ans.* Example 10.2]**

17. Derive the Lame's equations for the hoop and radial stresses in a thick cylinder subjected to an internal pressure and show how these may be expressed in graphical form. (*UPTU : 2007–08*)

**[*Ans.* Example 10.6]**

18. Derive an expression to determine stresses in a thick-walled cylinder subjected to internal pressure only. (*UPTU : 2008–09*)

**[*Ans.* Example 10.16]**

19. Calculate the thickness of metal necessary for a cylindrical shell of internal diameter of 80 mm to withstand an internal pressure of 25 N/mm$^2$, if the maximum permissible tensile stress is 125 N/mm$^2$. (*UPTU : 2009–10*)

**[*Ans.* Example 10.10]**

20. Write short notes on any two of the following

(i) Lame's theory of thick cylinders

(ii) Compound cylinders

(iii) Radial, axial and circumferential stresses in thick cylinder

(*UPTU : 2009–10*)

**[*Ans.* (i) Example 10.5; (ii) Section 10.3; (iii) Section 10.2]**

21. Derive expressions for radial and hoop stresses in a thick cylinder with internal and external radii of $a$ and $b$ subjected to an internal pressure of $p_1$.

(*UPTU : 2009–10*)

**[*Ans.* Example 10.16]**

22. The cylinder of a hydraulic ram is of 6 cm internal diameter. Find the thickness required to withstand an internal pressure of 40 N/mm$^2$, if the maximum tensile stress is limited to 60 N/mm$^2$ and the maximum shear stress to 50 N/mm$^2$. (*UPTU : 2010*)

**[*Ans.* Example 10.14]**

23. A thick cylinder of 160 mm internal, 240 mm external diameter, is subjected to an external pressure 12 MPa. Determine the maximum internal pressure that can be applied if the maximum allowable normal stress is 36 MPa. Plot the variation of radial and hoop stresses. (*UPTU : 2010–2011*)

**[*Ans.* Example 10.20]**

24. A compound cylinder is to be made by shrinking one tube on to another so that the radial compression stress at the friction is 28.5 N/mm$^2$. If the outside diameter is 26.5 cm, and the bore 12.5 cm, calculate the allowance for shrinkage at the common diameter, which is 20 cm. (*UPTU : 2010–2011*)

**[*Ans.* Example 10.4]**

25. What are the assumptions made in Lame's equation ? (*UPTU : 2011–2012*)

**[*Ans.* Example 10.5]**

26. A thick hollow cylinder 200 mm internal and 300 mm external diameter is subjected to an internal pressure 50 MPa and external pressure 25 MPa. Find the maximum shear stress at the inner surface of the cylinder.

(*UPTU : 2011–2012*)

27. Derive the expressions for circumferential and radial streses in the wall of thick cylinder (Lame's equation). (*UPTU :. 2011–2012*)

**[*Ans.* Example 10.5]**

28. An external pressure of 10 MN/m$^2$ is applied to thick cylinder of internal diameter 150 mm and external diameter 300 mm. If the maximum hoop stress permitted on the inside wall is 35 MN/m$^2$, calculate the maximum internal pressure that can be applied. (*UPTU : 2012 –13*)

**[*Ans.* Example 10.21]**

29. The maximum stress permitted in a thick cylinder of inner and outer radius of 10 cm and 15 cm is 20 N/mm$^2$. The external pressure is 8 N/mm$^2$, what internal pressure can be applied? (*UPTU : 2012–13*)

**[*Ans.* Example 10.18]**

# Curved Beams

CHAPTER 11

## 11.1 ❑ INTRODUCTION

In 'simple bending' the relations between the straining actions, stresses and strains were established for a *straight beam*. The well-known formula : $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$ is called *straight beam formula* : *Bending Formula or Flexure Formula.*

The results of simple bending can be applied, with sufficient accuracy, to the beams or bars having small initial curvature.

It is common practice to distinguish rods or bars of small and large initial curvature.

The chief characteristics of such a division is the ratio of the depth of the section $h$ = depth of beam in the plane of curvature to the radius of curvature $R_0$ of the rod axis.

If this ratio $\frac{h}{R_0}$ is 0.2 and less, the rod has a small curvature $R_0 < 5\,h$.

For the rod of large curvature, the ratio $\frac{h}{R_0}$ radius of curvature, is comparable with unity. This division is purely conventional.

The practical cases of beams with large initial curvature are *Hooks, Links, Rings* etc.

## 11.2 ❑ ASSUMPTIONS MADE IN THE THEORY OF BENDING OF CURVED BARS

The stresses in the bending of curved bars are determined on the following assumptions :

1. Material of the bar is stressed within elastic limit and thus obeys Hook's Law.

2. The value of Young's modulus of elasticity ($E$) is same in *Tension* and *Compression.*

3. The transverse sections which weve plane before bending remains plane after bending.

4. Longitudinal fibres of the bar, parallel to the centroidal axis exert no pressure on each other.

5. The transverse cross-section has atleast one axis of symmetry and B.M. lies on this plane.

## 11.3 ❑ BARS WITH LARGE INITIAL CURVATURE : WINKLER BATCH FORMULA

Bending stress is given by the expression :

$$\sigma = \frac{M}{AR}\left(1+\frac{1}{m}\times\frac{y}{R+y}\right)$$

$M$ = Moment or couple applied at the ends

$A$ = Area of cross-section of curved beam

$R$ = Distance of centroidal axis from centre of curvature

$y$ = Distance from centroidal axis

$m$ = Pure number depending upon shape of cross-section of the curved beam

Consider a curved beam of constant cross-section, subjected to pure bending produced by couples $M$ applied at the ends. Let $AB$ and $CD$ be two adjacent cross-sections of the beam subtending a small angle $d\theta$ at the centre of curvature, before bending. Let the bending moment $M$ cause the plane $CD$ to rotate through $\Delta d\theta$ (Fig. 11.1), changing the centre of curvaute from $O$ to $O'$. Let the distance of the centroidal axis from the centre of curvature be changed from initial value of $R$ to $\rho$. Consider any fibre $PQ$ distant $y$ from the centroidal axis. The section $CD$ rotates about the point $H$; hence the layer $GH$ is the neutral layer. It should be noted that the quantities $R$, $\rho$ and $y$ are measured from the centroidal axis; and not from the neutral axis.

It will be assumed, for simplicity, that the section of the beam is symmetrical about the plane of curvature. The $y$-axis is then the axis of symmetry of the section (Fig. 11.1 b) and the moment of elementry force $\sigma.dA$ with respect to this axis is zero.

The expression, $\sigma = \dfrac{M}{A.R}\left(1+\dfrac{1}{m}.\dfrac{y}{R+y}\right)$ is known as *winkler batch formula*.

The distribution of $\sigma$, given by above equation is hyperbolic (Fig. 11.1 c).

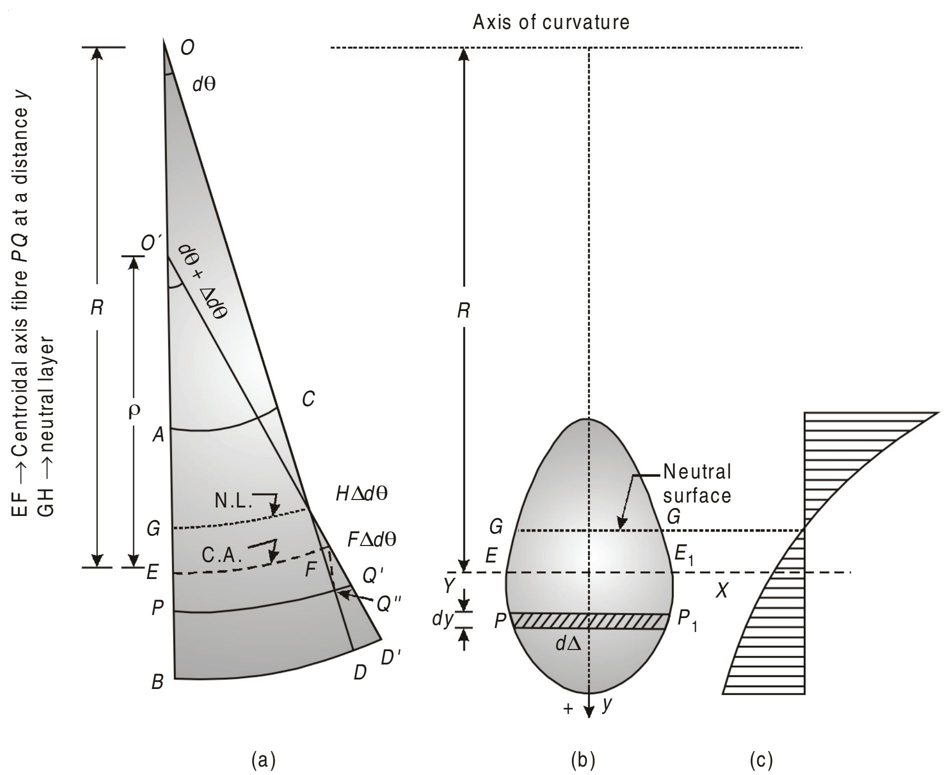



**Fig. 11.1** *Beam with large initial curvature*

### 11.3.1 Sign Convention

Following are the sign convention.

1. Distance $y$ is considered positive when measured towards the convex side of the beam and negative when measured towards concave inside i.e. towards centre of curvature.

2. The B.M. is considered positive if it decreases the radius of curvature and negative if it increases the radius of curvature.

3. The stress $\sigma$ if positive it is *tensile stress* and if negative it is *compressive stress*.

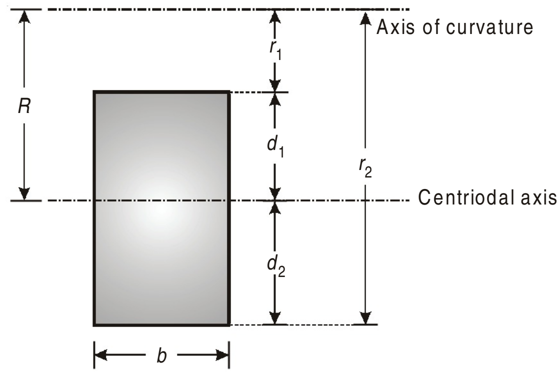
### 11.3.2 Application of Curved Beam with Large Initial Curvature

It is used for analysis and design of :

(i) Crane hooks, (ii) Curved beams, (iii) Rings, and (iv) Chain links

## 11.4 ❑ FACTOR 'M' FOR VARIOUS CROSS-SECTIONS

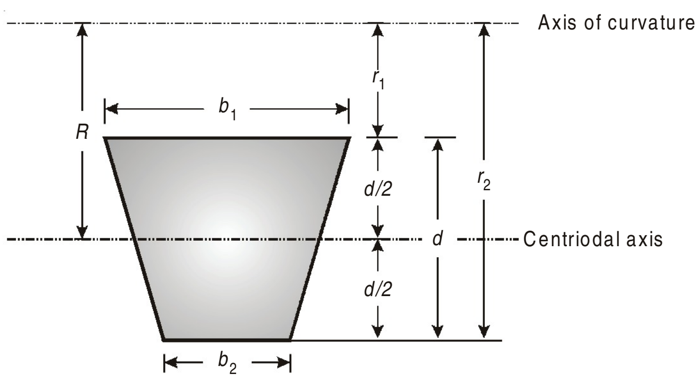
**1. Rectangular cross-section :** Refer Fig. 11.2




$$m = \frac{R}{d}\log_e\left(\frac{r_2}{r_1}\right) - 1$$

**Fig. 11.2**

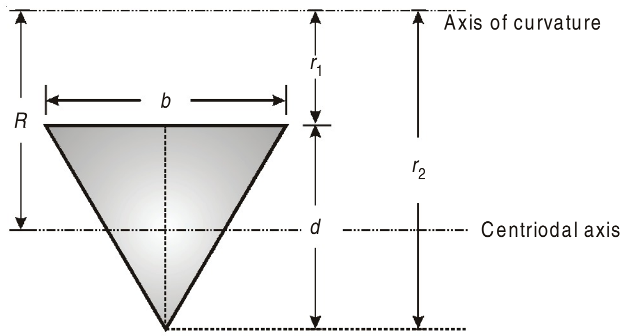
**2. Trapezoidal cross-section :** Refer Fig. 11.3




$$m = -1 + \frac{R}{A}\left\{\left[b_2 + (b_1 - b_2)\frac{r_2}{d}\log_e\frac{r_2}{r_1} - (b_1 - b_2)\right]\right\}$$

**Fig. 11.3**

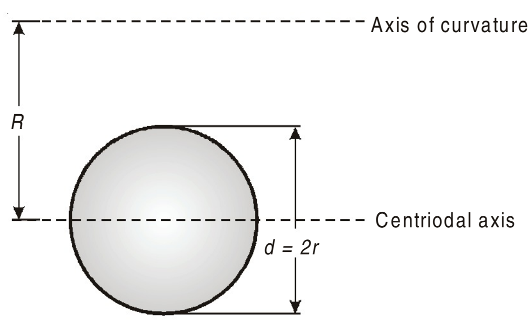
**3. Triangular cross-section :** Refer Fig. 11.4




$$m = -1 + \frac{R}{A}\left[\left(\frac{br_2}{d}\log_e \cdot \frac{r_2}{r_1}\right) - b\right]$$

**Fig. 11.4**

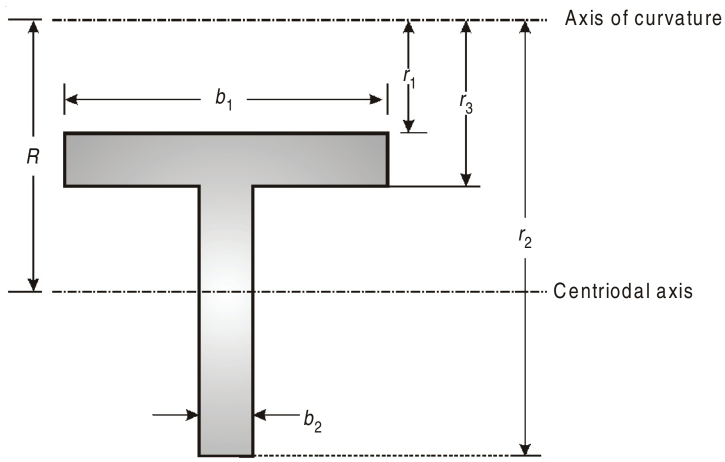
**4. Circular cross-section :** Refer Fig. 11.5




$$m = -1 + 2\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)\left[\left(\frac{R}{r}\right)^2 - 1\right]^{1/2}$$

**Fig. 11.5**

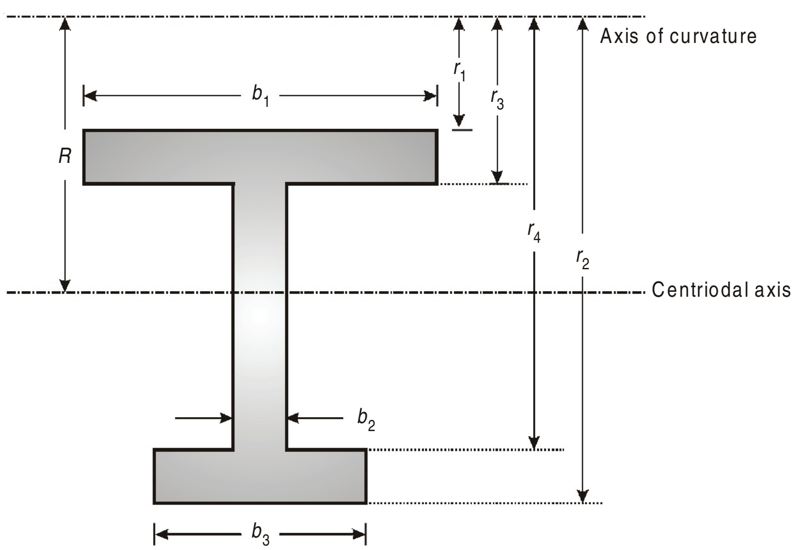
**5. T-cross-section :** Refer Fig. 11.6




$$m = \frac{R}{A}\left[b_1 \log_e \frac{r_3}{r_1} + b_2 \log_e \frac{r_2}{r_3}\right] - 1$$

**Fig. 11.6**

**6. I–cross-section :** Refer Fig. 11.7




$$m = \frac{R}{A}\left[b_1 \log_e\left(\frac{r_3}{r_1}\right) + b_2 \log_e\left(\frac{r_4}{r_3}\right) + b_3 \log_e\left(\frac{r_2}{r_4}\right) - 1\right.$$

**Fig. 11.7**

### 11.4.1 Comparison : Bending of Straight Members and Bending of Curved Members

The comparism between bending of straight members and bending of curved members is given in the following Table.

| S.No. | *Straight member* | *Curved member* |
|---|---|---|
| (i) | Neutral axis passes through c.g. | N. A. does not pass through c.g. It is shifted towards centre of curvature |
| (ii) | There is no eccentricity of N.A. | Eccentricity of N.A. is '$e$'. |
| (iii) | Bending stress is $\sigma = \dfrac{My}{I}$ | Bending stress is $\sigma = \dfrac{My}{A \times e \times Z}$ |
| (iv) | $y$ is measured from N.A. | $y$ is measured from N.A. |
| (v) | Distribution of bending stress is linear. | Distribution of bending stress is hyperbolic. |

For identical cross-sections subjected to same B.M. maximum bending stress is more in curved number than in straight member.

**Example 11.1.** *A curved bar of square section 3 cm sides and mean radius of curvature 4.5 cm is initially unstressed. If a bending moment of 300 N-m is applied to the bar tending to straighten it, find the stresses at the inner and outer faces.* (*UPTU : 2012 – 2013*)

*Given :*

*Side,* $a = 3 \text{ cm} = 30 \text{ mm}$

*Mean radius,* $R = 4.5 \text{ cm} = 45 \text{ mm}$

*Bending moment,* $M = 300 \text{ N/M} = 300 \times 10^3 \text{ N/mm}$

**Solution To find :** $\sigma_i$ and $\sigma_o$

$$r_1 = R - \frac{a}{2} = 45 - \frac{30}{2} = 30 \text{ mm}$$

$$r_2 = R + \frac{a}{2} = 45 + \frac{30}{2} = 60 \text{ mm}$$

*Factor m for rectangular section*

$$m = \frac{R}{d}\log_e\left(\frac{r_2}{r_1}\right) - 1$$

$$= \frac{45}{30}\log_e\left(\frac{60}{30}\right) - 1 = 0.0397$$

Area, $a = 30^2 = 900 \text{ mm}^2$

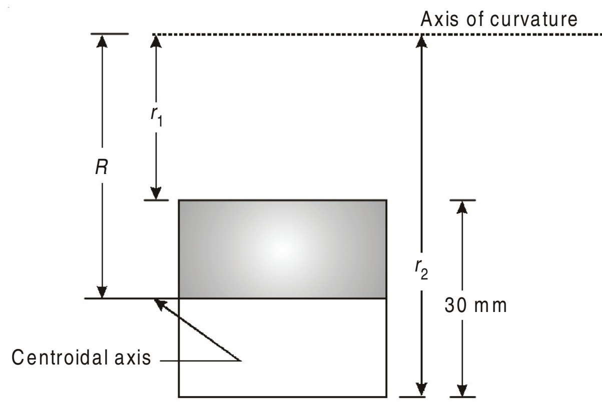



**Fig. 11.8**

Stress in the beam

$$\sigma = \frac{M}{AR}\left[1+\frac{1}{m}\left(\frac{y}{R+y}\right)\right]$$

$$= \frac{300\times10^3}{900\times45}\left[1+\frac{1}{0.0397}\left(\frac{y}{45+y}\right)\right]$$

Maximum stress occur at the innermost fiber i.e. $y = -25$ mm and minimum stress occur at outermost fiber i.e. $y = +25$ mm, substituting in above equation, we get,

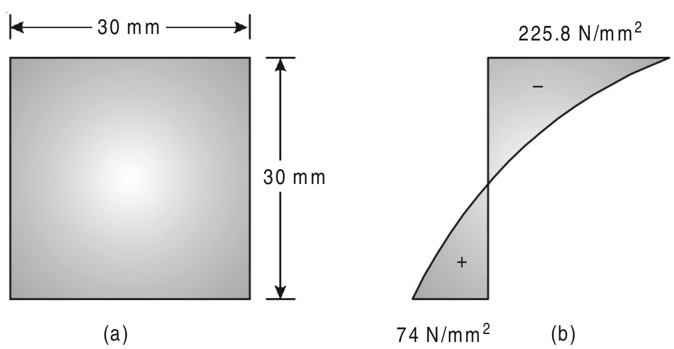



**Fig. 11.9**

$$\sigma_1 = 7.407\left[1+\frac{1}{0.0397}\left(\frac{-25}{45-25}\right)\right]$$

$$\sigma_1 = \mathbf{-225.8\ N/mm^2\ (Comp.)}$$

$$\sigma_2 = 7.407\left[1+\frac{1}{0.0397}\left(\frac{25}{45+25}\right)\right]$$

$$= \mathbf{74\ N/mm^2\ (Tensile)}$$

**Example 11.2.** *A curved beam, rectangular in cross-section is subjected to pure bending with a couple of + 40 kN cm. The beam has width of 2 cm and depth of 4 cm and is curved in plane parallel to width. The mean radius of curvature is 5 cm. Find the position of the neutral axis, and the ratio of the maximum to the minimum stress.* (*UPTU : 2006 – 2007*)

*Given :*

$$b = 2 \text{ cm} = 20 \text{ mm}$$
$$d = 4 \text{ cm} = 40 \text{ mm}$$
$$R = 5 \text{ cm} = 50 \text{ mm}$$
$$M = 40 \text{ kN-cm} = 400 \times 10^3 \text{ N-mm}$$

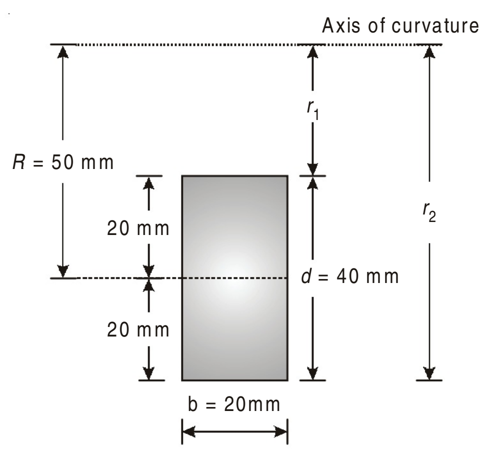
**Solution**

$$r_1 = R-\frac{d}{2} = 50-\frac{40}{2} = 30 \text{ mm}$$

$$r_2 = R+\frac{d}{2} = 50+\frac{40}{2} = 70 \text{ mm}$$

$$A = b \times d = 20 \times 40 = 800 \text{ mm}^2$$

$$m = \frac{R}{d}\log_e\left(\frac{r_2}{r_1}\right)-1 = 1.5$$

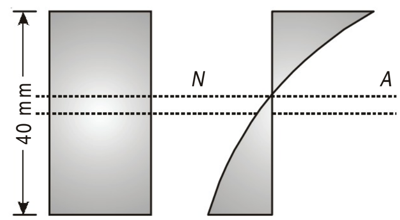


(a)

$$1.25 \log_e (2.33)-1 = \frac{50}{40}\log_e\left(\frac{70}{30}\right)-1 = 0.05912$$

Stresses in the beam $\sigma = \frac{M}{AR}\left[1+\frac{1}{m}\times\frac{y}{R+y}\right]$

Maximum stress occurs at the innermost fibre i.e., at $y$ = –20 mm and minimum stress occurs at outermost fibre i.e. at $y$ = + 20 mm




(b) Stress distribution across the section

**Fig. 11.9**

$$\therefore \quad \sigma_{max} = \frac{400\times10^3}{800\times50}\left[1+\frac{1}{0.05912}\times\frac{(-20)}{50-20}\right]$$

$$= \mathbf{102.765\ N/mm^2\ (Comp.)}$$

$$\sigma_{min} = \frac{400\times10^3}{800\times50}\left[1+\frac{1}{0.05912}\times\frac{20}{50+20}\right]$$

$$= \mathbf{58.328\ N/mm^2}$$

Ratio of maximum to minimum stress

$$\frac{\sigma_{max}}{\sigma_{min}} = \frac{102.765}{58.328} = \mathbf{1.762}$$

**Example 11.3.** *A curved beam of rectangular cross-section 30 mm × 50 mm has its centre line curved to a radius of 60 mm. The beam is subjected to a bending moment of 5 kNm. Find stresses developed in the beam and plot bending stress variation for the section.*

*Given : b = 30 mm, d = 50 mm, R = 60 mm, M = 5 kN.m = 5 × 10⁶ N-mm*

**Solution To find** : $\sigma_{max}$ and $\sigma_n$.

Refer Fig. 11.10 (a).

$$r_1 = R-\frac{d}{2} = 60-\frac{50}{2} = 35\text{ mm};$$

$$r_2 = r_1+\frac{d}{2} = 60+\frac{50}{2} = 85\text{ mm};$$

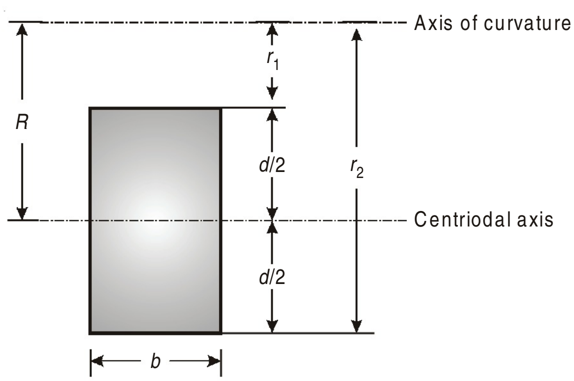
$$A = 30\times50 = 1500\text{ mm}^2.$$

$$m = \frac{R}{d}\log_e\left(\frac{r_2}{1_1}\right)-1$$

$$= \frac{60}{50}\log_e\left(\frac{85}{35}\right) - 1 = 0.0647638$$

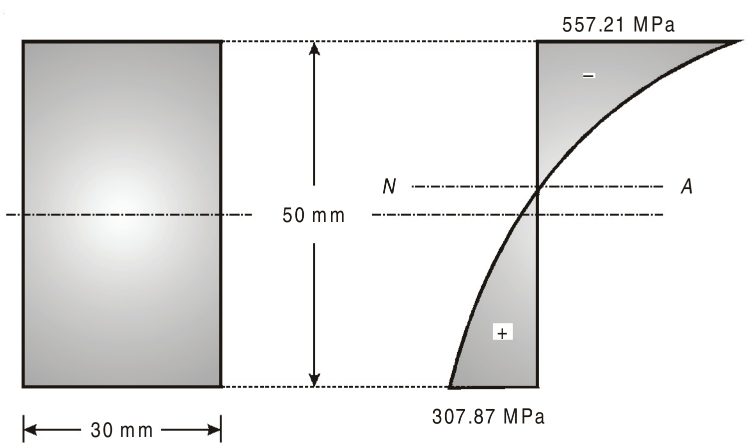
Stresses in the beam $\sigma = \frac{M}{AR}\left[1 + \frac{1}{m}\frac{y}{R+y}\right]$

$$= \frac{5\times10^6}{1500\times60}\left[1 + \frac{1}{0.06476}\times\frac{y}{60+y}\right]$$




(a)

Maximum stress occurs at the innermost fibre.




(b) Stress distribution across the cross-section

**Fig. 11.10**

i.e., at $y$ = –25 mm and minimum stress occurs at outermost fibre, i.e. at $y$ = + 25 mm; substituting in above expression we get,

$$\sigma_1 = -\mathbf{557.21\ MPa\ (Comp.)\ and}$$
$$\sigma_2 = +\mathbf{307.87\ MPa\ (Tensile)}$$

## 11.5 ❑ STRESSES IN HOOKES

Consider a hook carrying a vertical load $P$, as shown in Fig. 11.11

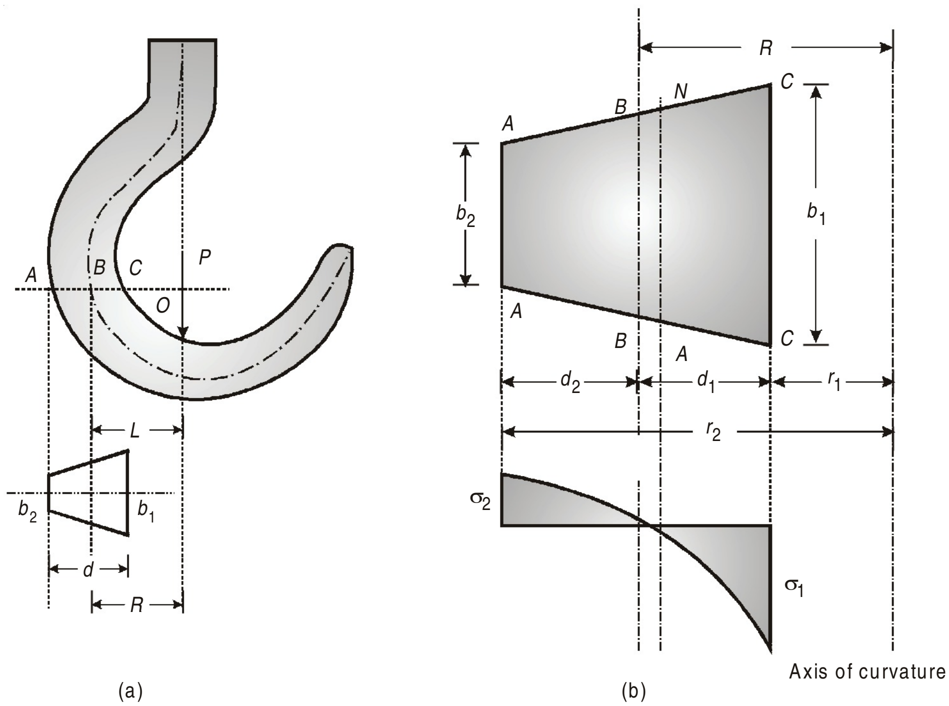



**Fig. 11.11**

A little consideration shows that the horizontal section through the centre of curvature is the most stressed section.

Stresses at points $C$ and $A$ are given by,

$$\sigma_1 = \frac{P}{A} + \frac{M}{AR}\left[1 + \frac{1}{m} \times \frac{y}{R+y}\right]$$

$$\sigma_2 = \frac{P}{A} - \frac{M}{AR}\left[1 + \frac{1}{m} \times \frac{y}{R+y}\right]$$

where $M = -P.L$

Assume $y$ as positive when measured towards the convex side and negative when measured towards concave side.

(i) Stress at $C$, here $y = -d_1$ is

$$\sigma_1 = \frac{P}{A} - \frac{PL}{AR}\left[1 - \frac{1}{m} \times \frac{d_1}{R + d_1}\right] \quad \text{...(Tensile stress)}$$

(ii) Stress at $A$, here $y = +d_2$ is

$$\sigma_2 = \frac{P}{A} - \frac{PL}{AR}\left[1 + \frac{1}{m} \times \frac{d_2}{R - d_2}\right] \quad \text{...(Comp. stress)}$$

In a well designed hook, the centre of load passes through the centre of curvature. Above equation can be written as ($L = R$),

$$\sigma_1 = \frac{P}{A} - \frac{PL}{AR}\left[1 - \frac{1}{m} \times \frac{d_1}{R - d_1}\right] = \frac{P}{Am} \times \frac{d_1}{(R - d_1)}$$

$$\sigma_2 = \frac{P}{A} - \frac{PL}{AR}\left[1 + \frac{1}{m} \times \frac{d_2}{R + d_2}\right] = \frac{P}{Am} \times \frac{d_2}{(R + d_2)}$$

**Example 11.4.** *A chain link (Fig. 11.12) is made of round steel of 15 mm diameter. If, R = 45 mm, L = 75 mm and load applied is 1.5 kN. Determine the maximum compressive stress in the link and tensile stresses at the same section.*

(*UPTU : 2005–2006*)

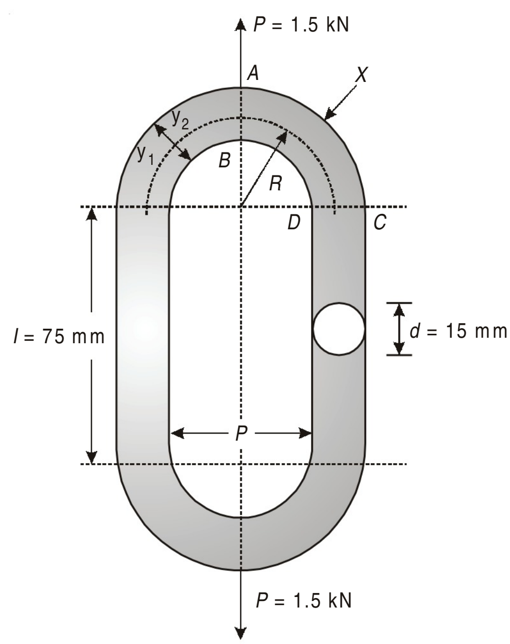



**Fig. 11.12**

*Given :*
$$P = 1.5\ kN = 15000\ N$$
$$d = 15\ mm$$
$$r = 7.5\ m$$
$$R = 45\ mm$$
$$L = 75\ mm$$

**Solution To find :** Maximum compressive stress and tensile stress.

$$\frac{r}{R} = \frac{7.5}{45} = 0.167$$

Area
$$A = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 15^2$$
$$= 176.72\ \text{mm}^2$$

Moment of inertia
$$I = \frac{\pi}{64} d^4 = \frac{\pi \times 15^4}{64} = 2485.05\ \text{mm}^4$$

The pure number $m$ is given by,

$$m = \frac{1}{4}\left(\frac{r}{R}\right)^2 + \frac{1}{8}\left(\frac{r}{R}\right)^4 = \frac{1}{4}(0.167)^2 + \frac{1}{8}(0.167)^4$$
$$m = 0.00707$$

$\therefore$ $d_1 = 7.5$ mm, $d_2 = 7.5$ m

**Stress in the link :**

**(i) Section AB :**

Stress at intrados

$$\sigma_1 = \frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left(1 - \frac{1}{m} \times \frac{d_1}{R-d_1}\right)$$

$$= \frac{1.5\times 10^3}{2\times 176.72}\left[\frac{75+2\times 45}{75+\pi\times 45}\right]\left(1 - \frac{1}{0.00707} \times \frac{7.5}{45-7.5}\right)$$

$$= \mathbf{-\ 88.32\ MPa\ (Comp.)}$$

Stress at extrados

$$\sigma_e = \frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left(1 + \frac{1}{m} \times \frac{d_2}{R+d_2}\right)$$

$$= \frac{1.5\times10^3}{2\times176.72}\left[\frac{75+2\times45}{75+\pi\times45}\right]\left(1+\frac{1}{0.00707}\times\frac{7.5}{45+7.5}\right)$$

$$\sigma_e = \mathbf{68.63\ MPa\ (Tensile)}$$

**(ii) Section DC :**

Stress at intrados

$$\sigma_i = \frac{P}{2A}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left(1-\frac{1}{m}\times\frac{d_1}{R-d_1}\right)$$

$$= \frac{1.5\times10^3}{2\times176.72}-\frac{1.5\times10^3\times45}{2\times176.72}\left[\frac{\pi-2}{75+\pi\times45}\right]\left(1-\frac{1}{0.00707}\times\frac{7.5}{45-7.5}\right)$$

$$= \mathbf{31.74\ N/mm^2\ (Tensile)}$$

Stress at extrados

$$\sigma_e = \frac{P}{2A}-\frac{PR}{2A}\left[\frac{\pi-2}{L+\pi R}\right]\left(1+\frac{1}{m}\times\frac{d_2}{R+d_2}\right)$$

$$= \frac{1.5\times10^3}{2\times176.72}-\frac{1.5\times10^3\times45}{2\times176.72}\left[\frac{\pi-2}{75+\pi\times45}\right]\left(1+\frac{1}{0.00707}\times\frac{7.5}{45+7.5}\right)$$

$$= \mathbf{-17.123\ N/mm^2\ (Compressive)}$$

**(iii) Straight portion :**

Stress at intrados $\sigma_i = \dfrac{P}{2A}+\dfrac{PR^2}{2I}\left[\dfrac{\pi-2}{L+\pi R}\right]\times d_1$

$$= \frac{1.5\times10^3}{2\times176.72}+\frac{1.5\times10^3\times45^2}{2\times2485.05}\left[\frac{\pi-2}{75+\pi\times45}\right]7.5$$

$$= \mathbf{28.43\ N/mm^2\ (Tensile)}$$

Stress at extrados $\sigma_e = \dfrac{P}{2A}-\dfrac{PR^2}{2I}\left[\dfrac{\pi-2}{L+\pi R}\right]\times d_2$

$$= \frac{1.5\times10^3}{2\times176.72}-\frac{1.5\times10^3\times45^2}{2\times2485.05}\left[\frac{\pi-2}{75+\pi\times45}\right]7.5$$

$$= 4.2325 - 24.184$$

$$= \mathbf{-19.94\ N/mm^2\ (Compressive)}$$

∴ Maximum tensile stress in the link is 68.63 MPa in the section $AB$ and maximum compressive stress in the link is 88.32 N/mm$^2$ in the section $AB$.

## 11.6 ❑ STRESSES IN RING

Consider a ring subjected to a pull (or push) through its centre as shown in Fig. 11.12.

Stress at any point is given by

$$\sigma = \sigma_d + \sigma_b = \frac{P\sin\theta}{2A} + \frac{M}{AR}\left[1 + \frac{1}{m} \times \frac{y}{R+y}\right]$$

Let $d_1$, $d_2$ = Distance of extreme inside edge and extreme outside edge from centre line.

Stress at the intrados ($y = -d_1$)

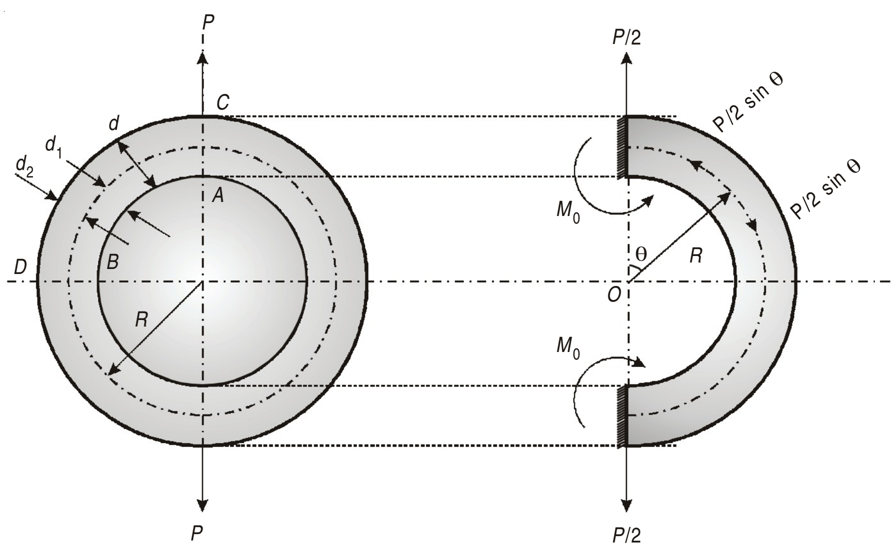



**Fig. 11.12**

$$\sigma_i = \frac{P\sin\theta}{2A} + \frac{M}{2R}\left[1 - \frac{1}{m} \times \frac{d_1}{R-d_1}\right]$$

where $$M = PR\left(\frac{1}{\pi} - \frac{1}{2}\sin\theta\right)$$

(i) At θ = 0; substituting in above expression,

$$\sigma_i = 0 + \frac{PR\left(\frac{1}{\pi} - 0\right)}{2R}\left(1 - \frac{1}{m} \times \frac{d_1}{R-d_1}\right)$$

$$= \frac{P}{2\pi}\left(1 - \frac{1}{m} \times \frac{d_1}{R - d_1}\right)$$

Here $\left(\frac{1}{m} \times \frac{d_1}{R - d_1}\right) > 1$ i.e., $\sigma_i$ is negative, maximum compressive stress is at $\theta = 0$.

(ii) At $\theta = \frac{\pi}{2}$; substituting in above expression,

$$\sigma_i = \frac{P \times 1}{2A} + \frac{PR\left(\frac{1}{\pi} - \frac{1}{2}\right)}{AR} \times \left(1 - \frac{1}{m} \times \frac{d_1}{R - d_1}\right)$$

$$= \frac{P}{2A} - \frac{0.1816P}{A}\left(1 - \frac{1}{m} \times \frac{d_1}{R - d_1}\right)$$

Here $\frac{1}{m} \times \frac{d_1}{R - d_1} > 1$ i.e., $\sigma_i$ is positive, tensile stress at $\theta = \frac{\pi}{2}$.

Stress at the extrados ($y = + d_2$)

$$\sigma_c = \frac{P\sin\theta}{2A} + \frac{M}{AR} \times \left[1 - \frac{1}{m} \times \frac{d_2}{R + d_2}\right]$$

where $$M = PR\left(\frac{1}{\pi} - \frac{1}{2}\sin\theta\right)$$

(i) At $\theta = 0°$; substituting in above expression.

$$\sigma_c = 0 + \frac{PR\left(\frac{1}{\pi} - 0\right)}{AR}\left(1 - \frac{1}{m} \times \frac{d_2}{R + d_2}\right)$$

$$= \frac{P}{\pi A}\left(1 + \frac{1}{m} \times \frac{d_2}{R + d_2}\right)$$

$\sigma_e$ is wholly tensile at $\theta = 0°$.

(ii) At $\theta = \frac{\pi}{2}$; substituting in above expression,

$$\sigma_e = \frac{P \times 1}{2A} + \frac{PR\left(\frac{1}{\pi} - \frac{1}{2}\right)}{AR} \times \left(1 + \frac{1}{m} \times \frac{d_2}{R + d_2}\right)$$

$$= \frac{P}{2A} - \frac{0.1816P}{A}\left(1 + \frac{1}{M} \times \frac{d_2}{R + d_2}\right)$$

$\sigma_e$ is compressive at $\theta = \dfrac{\pi}{2}$.

**Example 11.5.** *A ring made of 20 mm diameter steel bar has a mean radius of 160 mm. Two loads of 5 kN each are applied along a diameter of the ring. Determine the maximum stress in the ring.* (*UPTU : 2011 –2012*)

*Given :*

*Diameter of steel ring,* $d = 20$ *mm*

$$A = \frac{\pi}{4} \times 20^2 = 314.15 \text{ mm}^2$$

$$P = 5000 \text{ N}, R = 160 \text{ mm}, r = 10 \text{ mm}$$

**Solution To find :** Maximum tensile and compressive stresses.

The pure number $m$ is given by,

$$m = -1 + 2\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)\left[\left(\frac{R}{r}\right)^2 - 1\right]^{1/2} = -1 + 2\,(16)^2 - 2\,(16)\,[(16)^2 - 1]^{1/2}$$

$$= -1 + 512 - 32\,[256 - 1^{1/2}$$

$$= 511 - 32\,(255)^{1/2}$$

$$= 511 - 32\,(15.9687)$$

$$= 511 - 510.999$$

$$= 0.0978 \times 10^{-3}$$

i.e. $$\frac{1}{m} = 1022.5$$

Also $$m = \frac{1}{4}\left(\frac{r}{R}\right)^2 + \frac{1}{8}\left(\frac{r}{R}\right)^4 + \frac{5}{64}\left(\frac{r}{R}\right)^6 + \ldots$$

$$= \frac{1}{4}\left(\frac{1}{16}\right)^2 + \frac{1}{8}\left(\frac{1}{16}\right)^4 + \frac{5}{64}\left(\frac{1}{16}\right)^6 + \ldots$$

$$= 0.09785 \times 10^{-3}$$

Stresses in the ring (Fig. 11.14)

(i) At $\theta = 0°$; the stress at intrados

$$\sigma_i = \frac{P}{\pi A}\left[1 - \frac{1}{m} \times \frac{d_1}{R - d_1}\right]$$

$$= \frac{5000}{\pi \times 314}\left[1 - 1022.5 \times \frac{10}{160 - 10}\right]$$

$$= 5.07\ [1 - 68.13]$$

$$= \mathbf{-\ 340\ MPa\ (Comp.)}$$

Stress at extrados $\quad \sigma_c = \frac{P}{\pi A}\left[1 + \frac{1}{m} \times \frac{d_2}{R + d_2}\right]$

$$= \frac{5000}{\pi \times 314}\left[1 + 1022.5 \times \frac{10}{160 + 10}\right]$$

$$= 5.07[1 + 68.13]$$

$$= \mathbf{+\ 350.5\ MPa\ (Tensile)}$$

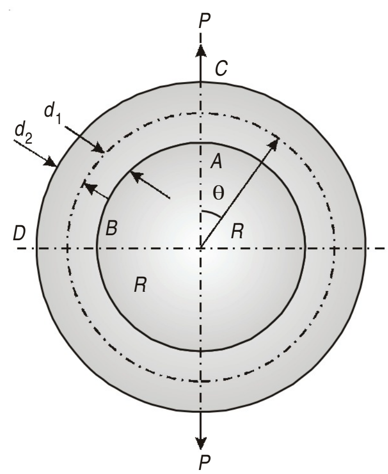



**Fig. 11.14**

(ii) At θ = 90°; the stress at intrados

$$\sigma_i = \frac{P}{2A} - \frac{0.1816P}{A}\left[1 - \frac{1}{m} \times \frac{d_1}{R - d_1}\right]$$

$$= \frac{5000}{2 \times 314.15} - \frac{0.1816 \times 5000}{314.15}\left[1 - 1022.5 \times \frac{10}{(160 - 10)}\right]$$

$$= 7.96 - 2.89\ [1 - 68.17]$$

$$= \mathbf{202.08\ MPa\ (Tensile)}$$

Stress at extrados

$$\sigma_e = \frac{P}{2A} - \frac{0.1816P}{A}\left[1 + \frac{1}{m} \times \frac{d_2}{R + d_2}\right]$$

$$= \frac{5000}{2 \times 314.15} - \frac{0.1816 \times 5000}{314.15}\left[1 + 1022.5 \times \frac{10}{(160 + 10)}\right]$$

$$= 7.96 - 2.89\ [1 + 68.17]$$

$$= \mathbf{-191.94\ MPa\ (Comp.)}$$

Thus, the maximum tensile stress = 202.08 MPa and maximum compressive stress = 191.94 MPa.

**Example 11.6.** *A steel ring has a rectangular cross-section, 75 mm in the radial direction and 45 mm perpendicular to the radial direction. If the mean radius of the ring is 150 mm and maximum tensile stress is limited to 180 MN/m² calculate the tensile load the ring can carry.* (*UPTU : 2005–2006*)

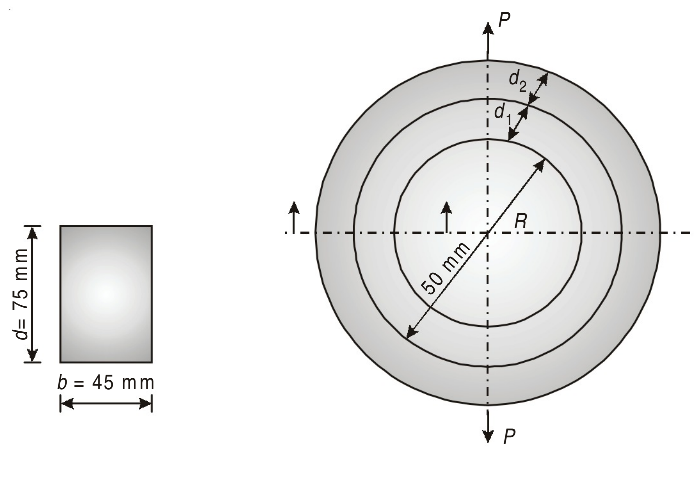
*Given :*

$$d = 75 \text{ mm}$$

$$b = 45 \text{ mm}$$

$$\sigma_{max} = 180 \text{ MN/m}^2 \text{ (tensile)}$$

$$R = 150 \text{ mm}$$




(a) (b)

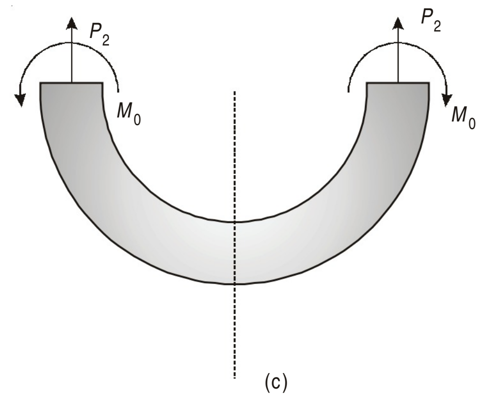



**Fig. 11.15**

**Solution To find :** Tensile load $P$

Forces and moments acting on the half portion are shown in Fig. 11.15.

Here
$$M_0 = \frac{PR}{2}\left(1-\frac{2}{\pi}\right) = \frac{P\times 150}{2}\left(1-\frac{2}{\pi}\right)$$
$$= (27.25)P \text{ Nmm}$$

Direct stress
$$\sigma_0 = \frac{P}{2A}$$
$$\frac{P}{(2\times 75\times 45)} = (148 \times 10^{-6})\, P \text{ N/mm}^2 \text{ tensile}$$

Moment of inertia
$$I = \frac{bd^3}{12} = \frac{1}{12}\times 45\times (75)^3$$
$\therefore$
$$I = 1.58 \times 10^6 \text{ mm}^4$$
$$y = \frac{d}{2} = \frac{75}{2} = 37.5 \text{ mm}$$

$\therefore$ Bending stress
$$\sigma_b = \frac{M_0 \times y}{I} = \frac{(27.25\times 37.5)}{(1.58\times 10^6)}P$$
$$\sigma_b = (646.76 \times 10^{-6})\, P \text{ N/mm}^2$$

$\therefore$ Maximum tensile stress
$$\sigma_t = \sigma_0 + \sigma_b = (148 + 646.76) \times 10^{-6} \times P$$
$$\sigma_t = (794.76) \times 10^{-6}\, P$$

Equating this with permissible tensile stress

$$(794.76 \times 10^{-6})\, P = 180$$

$$\therefore \quad P = 0.2265 \times 10^6 \text{ N} = 226.5 \text{ kN}$$

∴ The tensile load that the ring can carry is

$$\mathbf{P = 226.5\ kN}$$

## 11.7 ❑ STRESSES IN CHAIN LINKS

A simple chain link consists of semi-circular ends and straight sides connecting them, as shown in Fig. 11.16.

Stresses in curved portion

$$\sigma = \sigma_d + \sigma_b$$

$$= \frac{P}{2A}\sin\theta + \frac{M}{AR}\left(1 + \frac{1}{m} \times \frac{y}{R+y}\right)$$

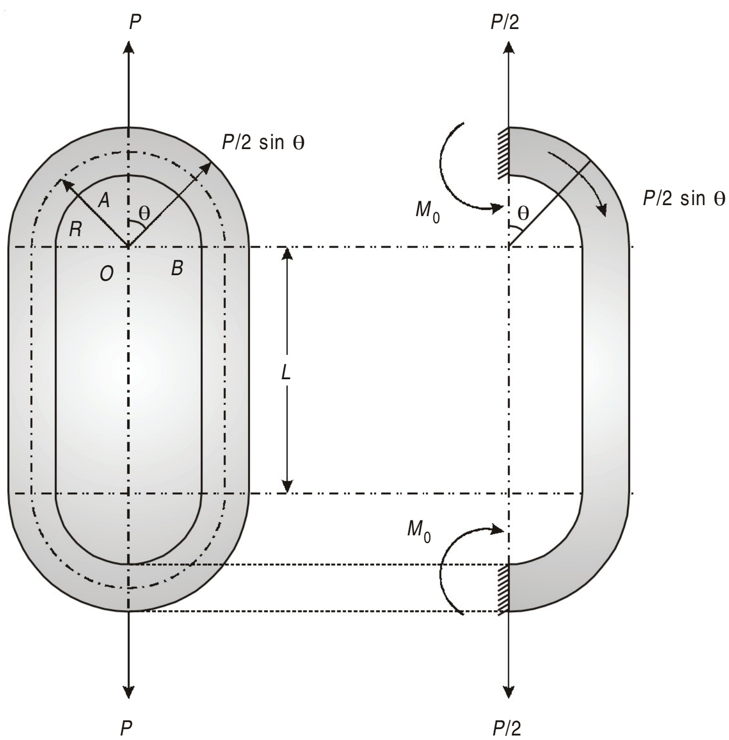



**Fig. 11.16**

$$= \frac{P}{2A}\sin\theta + \frac{P}{2A}\left[\frac{L+2R}{L+\pi R} - \sin\theta\right]\left[1 + \frac{1}{m} \times \frac{y}{R+y}\right]$$

(i) Stress at $A$ ($\theta = 0°$)

$$\sigma_i = 0 + \frac{P}{2A}\left[\frac{L+2R}{L+\pi R} - 0\right]\left[1 + \frac{1}{m} \times \frac{y}{R+y}\right]$$

$$= \frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1 + \frac{1}{m} \times \frac{y}{R+y}\right]$$

Stress at intrados, $y = -d_1$

$$\therefore \quad \sigma_i = \frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1 - \frac{1}{m} \times \frac{d_1}{R+d_1}\right]$$

...Compressive stress as $\frac{1}{m} \times \frac{d_1}{R-d_1} > 1.$

Stress at extrados, $y = +d_2$

$$\sigma_e = \frac{P}{2A}\left[\frac{L+2R}{L+\pi R}\right]\left[1 + \frac{1}{m} \times \frac{d_2}{R+d_2}\right] \quad \text{...Tensile stress}$$

(ii) Stress at $B$, $\theta = \frac{\pi}{2}$

$$\sigma = \frac{P}{2A} + \frac{P}{2A}\left[\frac{L+2R}{L+\pi R} - 1\right]\left(1 + \frac{1}{m} \times \frac{y}{R+y}\right)$$

$$= \frac{P}{2R} - \frac{PR}{2A}\left[\frac{\pi - 2}{L+\pi R}\right]\left(1 + \frac{1}{m} \times \frac{y}{R+y}\right)$$

Stress at intrados, $y = -d_1$

$$\sigma_i = \frac{P}{2A} - \frac{PR}{2A}\left[\frac{\pi - 2}{L+\pi R}\right]\left(1 - \frac{1}{m} \times \frac{d_1}{R+d_1}\right)$$

...Tensile stress

Stress at extrados, $y = +d_2$

$$\sigma_e = \frac{P}{2A} - \frac{PR}{2A}\left[\frac{\pi - 2}{L+\pi R}\right]\left(1 + \frac{1}{m} \times \frac{d_2}{R+d_2}\right)$$

Stresses in straight portion

$$\sigma = \frac{P}{2A} - \frac{PR}{2I}\left[\frac{2R - \pi R}{L + \pi R}\right].y$$

$$= \frac{P}{2A} - \frac{PR^2}{2I}\left(\frac{\pi - 2}{L + \pi R}\right).y$$

Stress at intrados, $y = -d_1$

$$\therefore \qquad \sigma_i = \frac{P}{2A} + \frac{PR^2}{2I}\left(\frac{\pi - 2}{L + \pi R}\right)\times d_1 \qquad \text{...Tensile stress}$$

Stress at extrados, $y = +d_2$

$$\sigma_e = \frac{P}{2A} - \frac{PR^2}{2I}\left(\frac{\pi - 2}{L + \pi R}\right)\times d_2$$

**Example 11.7.** *Derive the equation to find the position of neutral axis for the following cross-sections of curved beam :*

(*1*) *Rectangular section*

(*2*) *Circular cross-section* (*UPTU : 2007–2008*)

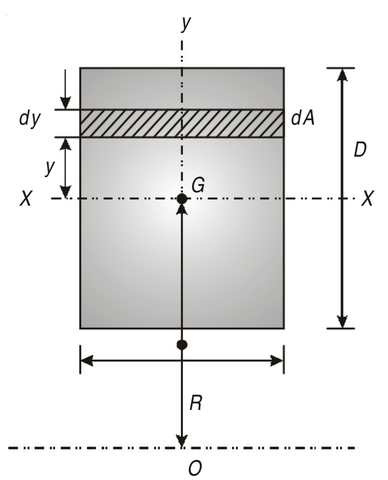
**Solution**

**1. Rectangular section :**

Consider rectangular normal section of a curved beam.

Let $R$ be the distance of the centroidal axis from the axis of curvature about $O$.

Consider an elementary strip of width $B$ and depth $dy$ at a distance $y$ from centroidal layer.




(a)

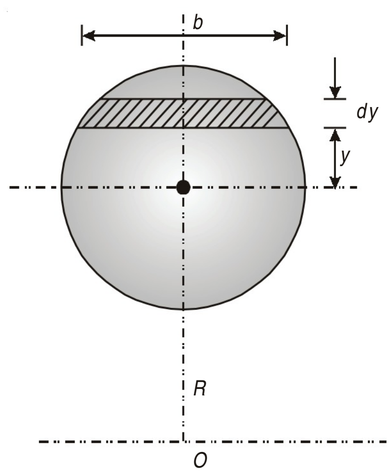
Area of strip, $dA = B\,dy$

Area of section $A = B \times D$

$$h^2 = \frac{R^3}{B \times D} \int_{-D/2}^{+D/2} \frac{B\,dy}{R+y} - R^2$$

$$= \frac{R^3}{D}\left[\log_e (R+y)\right]_{-D/2}^{D/2} - R^2$$

$$h^2 = \frac{R^3}{D}\log_e\left[\frac{2R+D}{2R-D}\right] - R^2$$




(b)

**Fig. 11.17**

**2. Circular cross-section :**

Consider a circular normal section of a curved beam. Let $R$ be the distance of centroidal axis from the axis of curvature about $O$.

Consider an elementary strip of width $B$ and depth $dy$ at a distance $y$ from centroidal layer.

Area of cross section $A = \frac{\pi}{4} d^2$

Area of strip, $dA = b \cdot dy$

$\therefore$ $b = 2\sqrt{\left(\frac{d}{2}\right)^2 - y^2}$

$$dA = \left[2\sqrt{\left(\frac{d}{2}\right)^2 - y^2}\right] dy$$

$$\therefore \qquad h^2 = \frac{R^3}{A}\int_{-d/2}^{+d/2} \frac{2\times\sqrt{\left(\frac{d}{2}\right)^2 - y^2}}{R+y}.\,dy - R^2$$

$$= \frac{8R^3}{\pi d^2}\int_{-d/2}^{+d/2} \frac{\sqrt{\left(\frac{d}{4}\right)^2 - y^2}}{R+y}.\,dy - R^2$$

Expanding the integral by binomial expression and then integrating, we get,

$$\mathbf{h^2 = \frac{d^2}{16} + \frac{1}{128}.\frac{d^4}{R^2} + ..........}$$

**Example 11.8.** *Determine the numerical value of the ratio $\sigma_{max}/\sigma_{min}$ for the case ot plane bending of a curved beam having 2.5 × 2.5 cm. Square cross section if the radius of curvature of the centroidal axis is R = 3.75 cm.*

(*UPTU : 2007–2008*)

*Given :*

*Cross-section = 2.5 × 2.5 cm*

*Radius of curvature = R = 3.75 cm*

**Solution To Find :** $\sigma_{max}/\sigma_{min}$

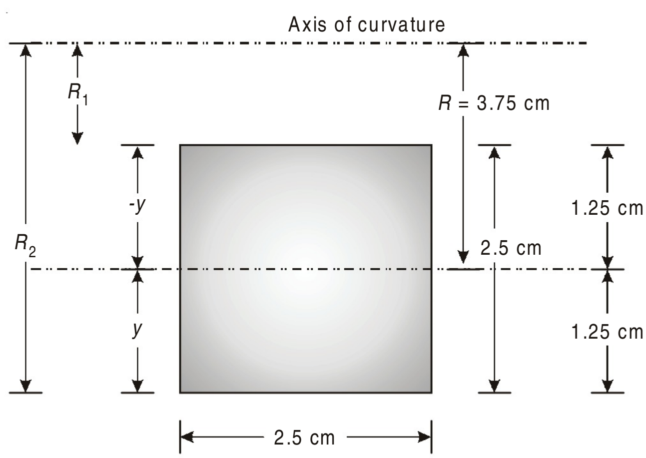



**Fig. 11.18**

$$R = 3.75 \text{ cm}$$
$$R_1 = 3.75 - 1.25 = 2.5 \text{ cm}$$
$$R_2 = 3.75 + 1.25 = 5 \text{ cm}$$
$$d = 2.5 \text{ cm}$$

$$\therefore \quad m = \frac{R}{d}\log_e\left(\frac{R_2}{R_1}\right) - 1 = \frac{3.75}{2.5}\log_e\left(\frac{5}{2.5}\right) - 1$$
$$= 0.0397$$

Stress

$$\sigma = \frac{M}{AR}\left(1 + \frac{1}{m}\frac{y}{R+y}\right)$$

For maximum stress $y = -1.25$ cm

For minimum stress $y = +1.25$ cm

$$\therefore \quad \text{Ratio} = \frac{\sigma_{max}}{\sigma_{min}} = \frac{\frac{M}{AR}\left(1 + \frac{1}{0.0397} \times \frac{-1.25}{3.75 - 1.25}\right)}{\frac{M}{AR}\left(1 + \frac{1}{0.0397} \times \frac{1.25}{3.75 + 1.25}\right)}$$

$$= \frac{-11.594}{7.297}$$

$$\frac{\boldsymbol{\sigma_{max}}}{\boldsymbol{\sigma_{min}}} = \mathbf{-1.589}$$

**Example 11.9.** *An open ring has T section as shown in the Fig. 11.19. Determine the stress at points P and Q.* *(UPTU : 2010–2011)*

**Solution**

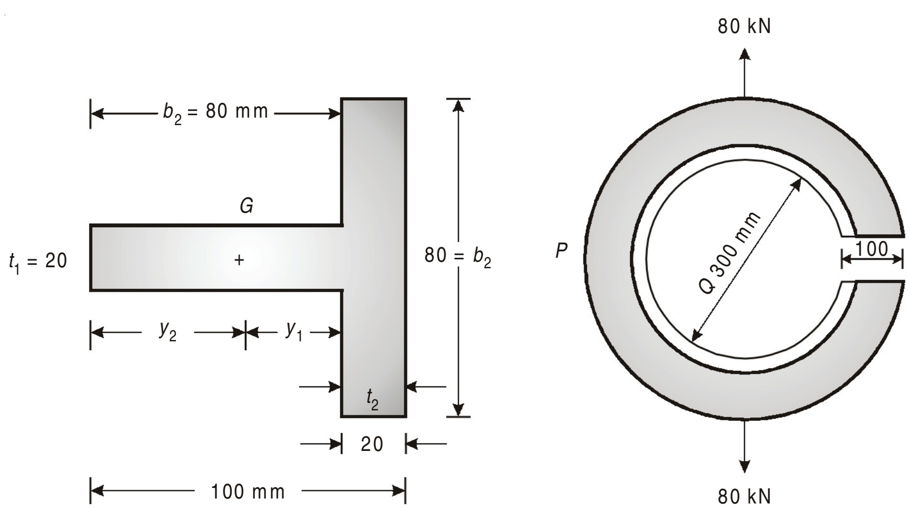



Fig. 11.19

Area of $T$ section $\quad A = b_1 t_1 + b_2 t_2 = 80 \times 20 + 80 \times 20$

$= 3200 \text{ mm}^2$

Centroidal position from face of web

$$y_2 = \frac{(80\times20)\times40+(80\times20)\times90}{3200}$$

$= 65$ mm

$y_1 = 100 - 65 = 35$ mm

Radius of curvature $\quad R_1 = \dfrac{300}{2} = 150$ mm

$R_2 = R_1 + 20 = 150 + 20 = 170$ mm

$R_3 = R_1 + 100 = 150 + 100 = 250$ mm

$R = R_1 + y_1 = 150 + 35 = 185$ mm

$$h^2 = \frac{R^3}{A}\left[b_2 \log_e \frac{R_2}{R_1} + t_1 \log_e \frac{R_3}{R_2}\right] - R^2$$

$$= \frac{185^3}{3200}\left[80 \log_e \frac{170}{150} + 20 \log_e \frac{250}{170}\right] - 185^2$$

$h^2 = 848.84 \text{ mm}^2$

Direct stress $\quad \sigma_d = \dfrac{P}{A} = \dfrac{80\times10^3}{3200} = 25 \text{ N/mm}^2$

Bending moment $\quad M = -P \times R$

Negative sign of BM because of decrease in curvature

$$\text{Bending stress } (\sigma_b)_p = \frac{M}{AR}\left[1+\frac{R^2}{h^2}\left(\frac{y_2}{R+y_2}\right)\right]$$

$$= \frac{-P\times R}{AR}\left[1+\frac{185^2}{848.84}\left(\frac{65}{185+65}\right)\right]$$

$= -25[1 + 11.483] = -312.07 \text{ N/mm}^2$

$=$ **312.07 N/mm² (Compressive)**

Bending stress at $Q$,

$$(\sigma_b)_a = \frac{M}{AR}\left[1-\frac{R^2}{h^2}\left(\frac{y_2}{R-y_1}\right)\right]$$

$$(\sigma_b)_a = -25\left[1 - \frac{185^2}{848.84}\left(\frac{65}{185-35}\right)\right]$$

$$= -25\,[1 - 17.47]$$

$$= 411.8 \text{ N/mm}^2 \text{ (Tensile)}$$

Resultant stress at $P$, $\sigma_p = \sigma_d + (\sigma_b)_p = 25 - 312.07 = -287.08 \text{ N/mm}^2$

$= 282.08 \text{ N/mm}^2$ (Comp.)

Resultant stress at $Q$, $\sigma_a = \sigma_d + (\sigma_b)_Q = 25 + 411.8$

$= 436.8 \text{ N/mm}^2$ (Tensile)

**Example 11.10.** *A ring carrying a load of 30 kN is shown in Fig. 11.20. Calculate the stress at positions 1 and 2.* *(MTU : 2012–2013)*

*Given :* $d = 12$ cm

$R = 7.5 + 6 = 13.5 \text{ cm} = 0.135 \text{ m}$

$$y = \frac{d}{2} = \frac{12}{2} = 6 \text{ cm} = 0.06 \text{ m}$$

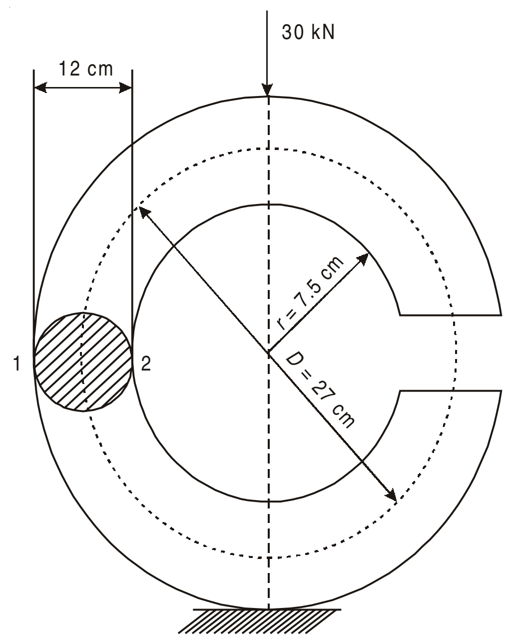



**Fig. 11.20**

**Solution**

Area of c/s $A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} 12^2 = 113.1 \text{ cm}^2$

$$= 0.01131 \text{ m}^2$$

$h^2$ for circular section

$$h^2 = \frac{d^2}{16} + \frac{1}{128}\frac{d^4}{R^2} + \ldots$$

$$= \frac{12^2}{16} + \frac{1}{128} \times \frac{12^4}{13.5^2} = 9.89 \text{ cm}^2$$

$$= 9.86 \times 10^{-4} \text{ m}^2$$

Direct stress $\sigma_d = \dfrac{P}{A} = \dfrac{30 \times 10^3}{0.01131} = 2.65 \times 10^6 \text{ N/m}^2$

$$= 2.65 \text{ N/mm}^2 \text{ (Compressive)}$$

Bending moment $M = P \times R = 30 \times 10^3 \times 0.135 = 4050$ N.m

Bending stress at point 1

$$\sigma_{b_1} = \frac{M}{AR}\left[\frac{R^2}{h^2}\left(\frac{y}{R+y}\right)+1\right]$$

$$= \frac{4050}{0.01131 \times 0.135}\left[\frac{0.135^2}{9.89 \times 10^{-4}}\left(\frac{0.06}{0.135+0.06}\right)+1\right]$$

$$= 17.815 \text{ N/m}^2$$

$$= 17.815 \text{ N/mm}^2 \text{ (Tensile)}$$

Bending stress at point 2

$$\sigma_{b_2} = \frac{M}{AR}\left[\frac{R^2}{h^2}\left(\frac{y}{R-y}\right)-1\right]$$

$$= \frac{4050}{0.01131 \times 0.135}\left[\frac{0.135^2}{9.89 \times 10^{-4}}\left(\frac{0.06}{0.135-0.06}\right)-1\right]$$

$$= 36.41 \times 10^6 \text{ N/m}^2$$

$$= 36.41 \text{ N/mm}^2 \text{ (Compressive)}$$

$\sigma_1 = \sigma_d + \sigma_{b_1} = -2.65 + 17.815$ **= 15.025 N/mm²(T)**

$\sigma_2 = \sigma_d + \sigma_{b_2} = -2.65 + 36.41$ **= –39.06 N/mm² (C)**

## EXERCISE

**11.1.** State the assumptions made in the theory of bending of curved beams.

**11.2.** Find the ratio of maximum bending stress to minimum bending stres in case of a curved beam of solid circular cross-section in pure bending if $R$ = 120 mm and diameter of cross-section as 100 mm. **[*Ans.* 1; 97]**

**11.3.** A crane hook curved to an internal diameter of 50 mm carries a load of 60 kN. The cross-section of the hook is a T-section with a flange 100 mm wide by 20 mm thick and a web is 80 mm deep by 20 mm thick. The flange is on the concave side.

Determine the maximum stress in the cross-section.

**[*Ans.* $\sigma_{max}$ = 68.22 MPa (Tensile), $\sigma_{min}$ = 49.59 MPa (Comp.)]**

**11.4.** A crane hook curved to an internal diameter of 50 mm carries a load of 60 kN. The cross-section of the hook is a symmetrical trapezium with top width 75 mm (concave side), bottom width 25 mm and depth 75 mm.

Determine the maximum stresses in the cross-section.

**[Ans. : $\sigma_{\text{outer face}}$ = 80.17 MPa (Comp.); $\sigma_{\text{inner face}}$ = 129.72 MPa (Tensile)]**

**11.5.** A curved bar of circular cross-section having diameter 100 mm is loaded as shown in Fig. 11.21 Find the maximum and minimum stress intensities at the critical section.

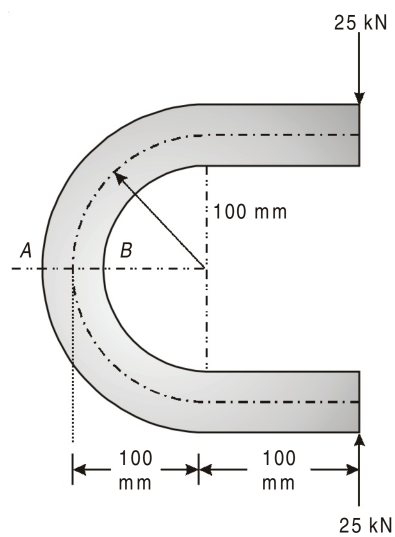



**Fig. 11.21**

**[*Ans.* $\sigma_A$ = 33.37 MPa (Tensile), $\sigma_B$ = 87.36 MPa (Comp.)]**

**11.6.** A chain link made up of 20 mm diameter round steel bar. The length of straight portion is 20 mm and mean radius of circular ends is 25 mm. If the link is subjected to a pull of 20 kN, find the maximum stresses induced in the link.

**[*Ans.* 172.1 MPa (Tensile) and 326.2 MPa (Comp.)]**

## UNIVERSITY QUESTIONS

1. A steel ring has a rectangular cross-section, 75 mm in the radial direction and 45 mm perpendicular to the radial direction. If the mean radius of the ring is 150 mm and maximum tensile stress is limited to 180 MN/m$^2$ calculate the tensile load the ring can carry. (*UPTU : 2005–06*)

**[*Ans.* Example 11.5]**

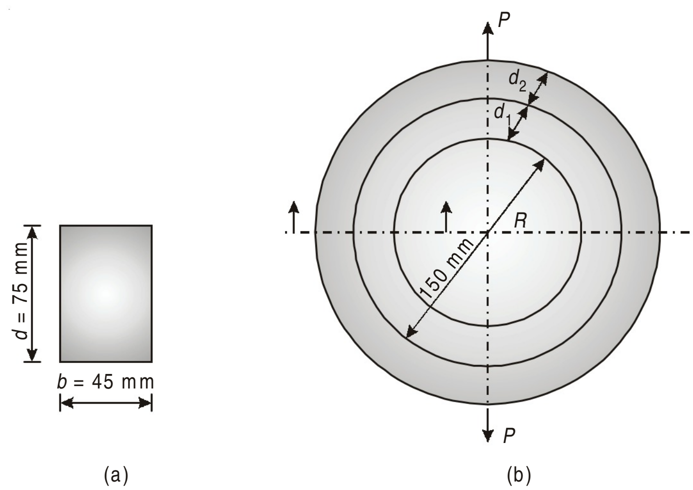



**Fig. 11.22**

2. A chain link (Fig. 11.23) is made of round steel of 15 mm diameter. If $R$ = 45 mm, $L$ = 75 mm load applied is 1.5 kN. Determine the maximum compressive stress in the link and tensile stresses at the same section. (*UPTU : 2005 – 2006*)

**[*Ans.* Example 11.4]**

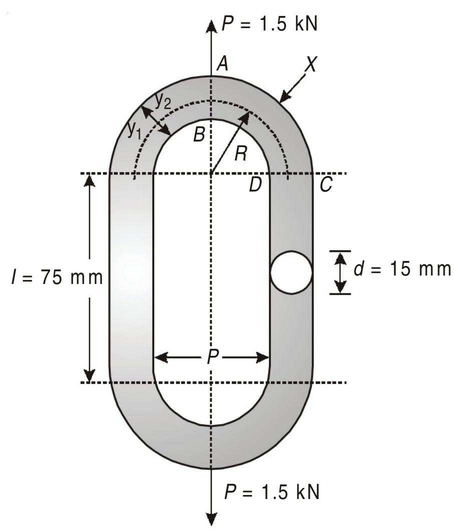



**Fig. 11.23**

3. A curved beam, rectangular in cross-section is subjected to pure bending with a couple of + 40 kN-cm. The beam has width of 2 cm and depth of 4 cm and is curved in plane parallel to width. The mean radius of curvature is 5 cm. Find the position of the neutral axis, and the ratio of the maximum to the minimum stress. (*UPTU : 2006–2007*)

**[*Ans.* Example 11.2]**

4. Explain the following :
   (i) In crane hooks, trapezoidal section is very commonly used.
   (ii) Some application of curved beam with large initial curvature.

(*UPTU : 2006–07*)

[*Ans.* (i) In crane hooks, trapezoidal section is very commonly used : In trapezoidal section, at inner face tensile stress is more than the compressive stress at outer face of crane hook. To achieve economical section with less difference between extreme stresses at end fibres, trapezoidal section is commonly used in crane hooks

(ii) Some application of curved beam with large initial curvature : Rings, links and crane hooks are common engineering application of curved beam with large initial curvature]

5. Derive the equation to find the position of neutral axis for the following cross sections of curved beam :
   (1) Rectangular section
   (2) Circular cross section (*UPTU : 2013–14*)

**[*Ans.* Example 11.6]**

6. Determine the numerical value of the ratio $\sigma_{max} / \sigma_{min}$ for the case of plane bending of a curved beam having 2.5 × 2.5 cm. Square cross section if the radius of curvature of the centroidal axis $R$ = 3.75 cm. (*UPTU : 2007–2008*)

**[*Ans.* Example 11.7]**

7. A curved bar of square section 3 cm sides and mean radius of curvature 4.5 cm is initially unstressed. If a bending moment of 300 N-m is applied to the bar tending to straighten it, find the stresses at the inner and outer faces.

(*UPTU : 2009–2010*)

**[*Ans.* Example 11.1]**

8. Write short notes on :
   (i) Assumptions for the theory of curved beams (*UPTU : 2009–2010*)

**[*Ans.* Section 11.2]**

   (ii) Application of curved beams with large initial curvature

(*UPTU : 2009–2010*)

**[*Ans.* Section 11.1 and 11.2]**

9. A curved bar of square section 3 cm sides and mean radius of curvature 4.5 cm is initially unstressed. If a bending moment of 300 N-m is applied to the bar tending to straighten it, find the stresses at the inner and outer faces.

(*UPTU : 2010–2011*)

**[*Ans.* Example 11.1]**

10. An open ring has *T* section as shown in the Fig. 11.24. Determine the stress at points *P* and *Q*. (*UPTU : 2010–2011*)

**[*Ans.* Example 11.8]**

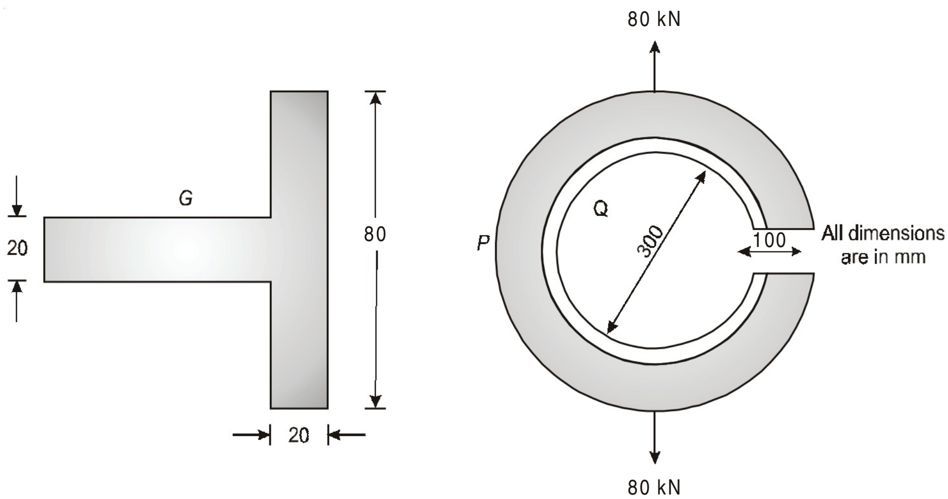



**Fig. 11.24**

11. A curved bar of square section 3 cm sides and mean radius of curvature 4.5 cm is initially unstressed. If a bending moment of 300 N-m is applied to the bar tending to straighten it, find the stresses at the inner and outer faces.

(*UPTU : 2011–2012*)

**[*Ans.* Example 11.1]**

12. A ring made of 20 mm diameter steel bar has a mean radius of 160 mm. Two loads of 5 kN each are applied along a diameter of the ring. Determine maximum stress in the ring. (*UPTU : 2011–2012*)

**[*Ans.* Example 11.4]**

13. Write down the expression for Winkler-Bach formula. (*UPTU : 2011–12*)

**[*Ans.* Section 11.3]**

14. Differentiate between straight and curved beam. (*UPTU : 2012–13*)

**[*Ans.* Section 11.4.1]**

15. A ring carrying a load of 30 kN is shown in Fig. 11.25 Calculate the stress at positions 1 and 2. (*UPTU : 2012–2013*)

**[*Ans.* Example 11.9]**

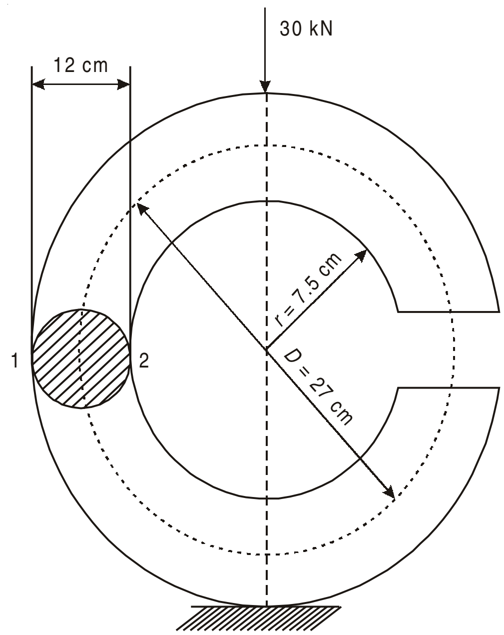



**Fig. 11.25**

16. A curved bar of square section 4 cm sides and mean radius of curvature 5 cm is initially unstressed. If bending moment of 300 Nm is applied to bar to straighten it, find the stresses at the inner and outer faces.

(*UPTU : 2012–2013*)

**[*Ans.* Example 11.1]**

# CHAPTER 12

# Unsymmetrical Bending

## 12.1 ❑ SYMMETRICAL BENDING

In simple bending $\left(\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}\right)$ theory the *Neutral Axis* of the cross-section of the beam is perpendicular to the plane of loading (Fig. 12.1) or bending.

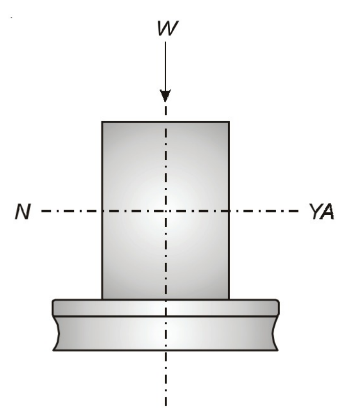



**Fig. 12.1**

## 12.2 UNSYMMETRICAL BENDING

In unsymmetrical bending, the direction of *Neutral Axis* will not be perpendicular to the plane of bending. In this case the plane of bending does not coincide or parallel to the plane containing the principal centroidal axis of cross-section.

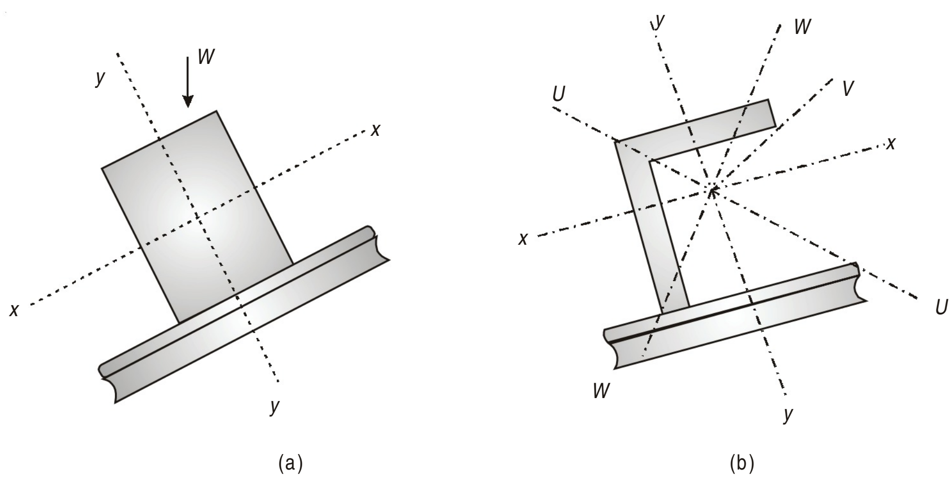



**Fig. 12.2** *Unsymmetrical bending*

## 12.3 ❑ CENTROIDAL PRINCIPAL AXIS OF A SECTION

The centroidal principal axes of a section are defined as a pair of rectangular axes through the centre of gravity of a plane area such that the product of inertia is zero.

Let $U–U$, $V–V$ = Principal centroidal axes

$X–X$, $Y–Y$ = Any pair of centroidal rectangular axes

$\alpha$ = Angle between $U–U$ and $X–X$ axes

If a plane area has an axis of symmetry (principal axis) then that axis has to satisfy the condition $\Sigma\, uv\, \delta a = 0$ about it.

Let $x, y$ be the co-ordinate of an elementary area $\delta a$ with respective to $X–Y$ axis and $u, v$ be the corresponding co-ordinate w.r.t principal axes $U–V$.

From Fig. 12.3 the relationship between $x, y$ and $u, v$ co-ordinates are

$$u = x \cos \alpha + y \sin \alpha$$

$$y = y \cos \alpha – x \sin \alpha$$

By definition,

$$I_{UU} = \Sigma\, v^2\, \delta a = \Sigma\, (y \cos \alpha - x \sin \alpha)^2\, \delta a$$

$$= \cos^2 \alpha\, \Sigma\, y^2\, \delta a + \sin^2 \alpha\, \Sigma\, x^2\, \delta a - 2 \sin \alpha \cos \alpha\, \Sigma\, xy\, \delta a$$

$$= I_{xx} \cos^2 \alpha + I_{yy} \sin^2 \alpha - 2 \sin \alpha \cos \alpha\, \Sigma\, xy\, \delta a \quad \text{...(i)}$$

$$I_{VV} = \Sigma\, u^2\, \delta a = \Sigma\, (x \cos \alpha + y \sin \alpha)^2\, \delta a$$

$$= \sin^2 \alpha\, \Sigma y^2\, \delta a + \cos^2 \alpha\, \Sigma\, x^2\, \delta x + 2 \sin \alpha \cos \alpha\, \Sigma\, xy\, \delta a \quad \text{...(ii)}$$

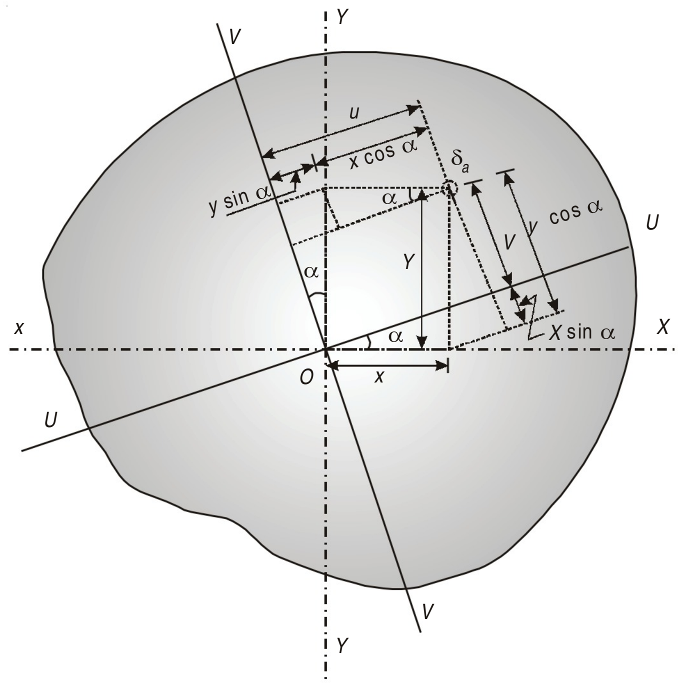



**Fig. 12.3** *Principal axes*

$$I_{VV} = \Sigma u^2 \delta a = \Sigma (x \cos \alpha + y \sin \alpha)^2 \delta a$$

$$= \sin^2 \alpha \, \Sigma y^2 \, \delta a + \cos^2 \alpha \, \Sigma x^2 \, \delta x + 2 \sin \alpha \cos \alpha \, \Sigma x \quad \text{...(ii)}$$

$$Iu_v = \Sigma uv \, \delta a = \Sigma (x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha)$$

$$= \cos^2 \alpha \, \Sigma xy \, \delta a - \sin^2 \alpha \, \Sigma xy \delta a + \sin \alpha \cos \alpha \, (\Sigma y^2 \, \delta a - \Sigma x^2 \, \delta a)$$

$$= \cos^2 \alpha \, I_{xy} - \sin^2 \alpha \, I_{xy} + \sin \alpha \cos \alpha \, (I_{xx} - I_{yy})$$

$$= \left(\frac{I_{xx} - I_{yy}}{2}\right) \sin 2\alpha + I_{xy} \cos 2\alpha \quad \text{...(iii)}$$

Since $U$–$U$ and $V$–$V$ are principal axes,

$$I_{uv} = 0 = \left(\frac{I_{xx} - I_{yy}}{2}\right) \sin 2\alpha + I_{xy} \cos 2\alpha$$

$$\frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-1_{xy}}{\left(\frac{I_{xx} - I_{xy}}{2}\right)}$$

$$\tan 2\alpha = \frac{-2\, I_{xy}}{I_{xx} - I_{yy}} \qquad \text{...(iv)}$$

Now Principal moment of inertia can be calculated as follows.

$$I_{UU} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\,\alpha - I_{xy} \sin 2\,\alpha \qquad \text{...(v)}$$

$$I_{VV} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\,\alpha - I_{xy} \sin 2\,\alpha \qquad \text{...(vi)}$$

From Eq. (iv) we get

$$\sin 2\,\alpha = \frac{-I_{xy}}{\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + \left(I_{xy}\right)^2}}$$

$$\cos 2\,\alpha = \frac{I_{xx} - I_{yy}}{\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + \left(I_{xy}\right)^2}}$$

Substituting the value of sin 2 α and cos 2 α in Eq. (v) and (vi), we get

$$I_{uy} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$I_{yy} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

### 12.3.1 Moment of Inertia about any Set of Rectangular Axis

Consider an elementary area $\delta a$, having co-ordinates be $u$, $v$ with respect to $U$–$V$ axes and $x' - y'$ with respect to $X' - Y'$ axes.

From figure,

$$\begin{aligned}
x' &= u \cos \beta + v \sin \beta \\
y' &= v \cos \beta - u \sin \beta \\
Ix'x' &= \Sigma\, y'^2\, \delta a = \Sigma\, (v \cos \beta - u \sin \beta)^2\, \delta a \\
&= \cos^2 \beta\, \Sigma\, v^2\, \delta a + \sin^2 \beta\, \Sigma\, u^2\, \delta a - 2 \sin \beta \cos \beta\, \Sigma\, uv\, \delta a \\
&= I_{UU} \cos^2 \beta + I_{vv} \sin^2 \beta \ \because\ I_{uv} = \Sigma uv \delta a = 0 \\
&= I_{UU} \qquad \text{...(vii)}
\end{aligned}$$

$$I_{y'y'} = \Sigma x'^2\, \delta a = \Sigma (u \cos \beta + v \sin \beta)^2\, \delta a$$
$$= \cos^2 \beta\, \Sigma u^2\, \delta a + \sin^2 \beta\, \Sigma v^2\, \delta a + 2 \sin \beta \cos \beta\, uv\, \delta a$$
$$= I_{UU} \sin^2 \beta + I_{vv} \cos^2 \beta \qquad \text{...(viii)}$$

Adding Eq. (vii) and (viii), we get,

$$I'_{xx} + I'_{yy} = I_{UU} + I_{YY}$$

Also, adding from Eqs. (i) and (ii), we get

$$I_{UU} + I_{VV} = I_{XX} + I_{YY} \qquad \text{...(ix)}$$

$$\therefore \quad I_{XX} + I_{YY} = I_{X'X'} + I_{Y'Y'} = I_{UU} + I_{vv} \qquad \text{...(x)}$$

Hence, the sum of moment of inertia about any set of rectangular axes is constant.

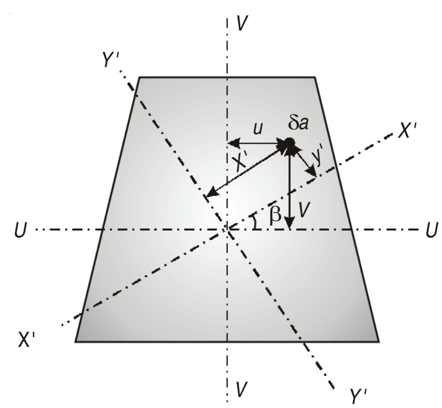



**Fig. 12.4**

### 12.3.2 Bending Stress in Beam Subjected to Unsymmetricai Bending

In unsymmetrical bending, neutral axis is not perpendicular to the plane of bending. The bending stress at any point in the beam subjected to unsymmetrical beading can be determined by the following methods.

1. Resolution of bending moment into two components along principal axes through the centroid.

2. Resolution of bending moment into two components along any rectangular axis through the centroid.

3. Locating neutral axis of the section.

Consider the plane of bending ($M$) be inclined at an angle θ w.r.t. one of the principal planes. Resolve the moment $M$ along the principal axis $U$–$U$ and $V$–$V$.

$$M_v = M \cos \theta$$
$$M_u = M \sin \theta$$

Now apply the bending formula for finding bending stress at any point $(u,v)$.

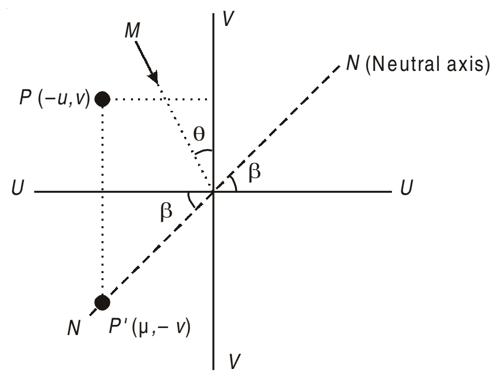



**Fig. 12.5**

$$\sigma_b = \frac{M_y}{I_{UU}} \cdot v + \frac{M_U}{I_{UU}} \cdot u$$

$$= \frac{M\cos\theta}{I_{UU}} \cdot V + \frac{M\sin\theta}{I_{VV}} \cdot u \qquad \text{...(xi)}$$

This method is suitable to those sections which have at least one axis of symmetry which is also of the principal axes.

### 12.3.3 Location of Neutral Axis

At any point $P\,(u, v)$. on neutral axis, the bending stress is equal to zero i.e. $\sigma_b = 0$

$$\therefore \qquad 0 = \frac{M\cos\theta}{I_{UU}} v + \frac{M\sin\theta}{I_{VV}} u$$

$$\frac{M\cos\theta}{I_{UU}} \cdot v = -\frac{M\sin\theta}{I_{VV}} . u$$

$$v = -u\tan\theta \frac{I_{UU}}{I_{vv}} \qquad \text{...(xii)}$$

Equation (xii) is the equation of neutral axis $N$–$N$ which is a straight line.

We know that when $u = v = 0$, the neutral axis passes through the centroidal axis of section. From Eq. (x), we get

$$\therefore \qquad \tan\beta = -\frac{v}{u} \qquad \text{...(xiii)}$$

From Eq. (xii),
$$-\frac{v}{u} = \tan\theta \frac{I_{UU}}{I_{VV}} \quad \text{...(xiv)}$$

Equate Eqs. (xiii) and (xiv),

$$\tan\beta = \tan\theta \frac{I_{UU}}{I_{VV}} \quad \text{...(xv)}$$

Let $I_{NN}$ is the moment of inertia of the beam about the neutral axis.

Thus from Eq. (vii), $x' - x'$ axis is treated as the neutral axis, we have,

$$Ix'x' = I_{uu}\cos^2\beta + I_{vv}\sin^2\beta$$
$$I_{NN} = I_{uu}\cos^2\beta + I_{vv}\sin^2\beta \quad \text{...(xvi)}$$

The neutral axis is inclined at β with the $U$ axis, while the plane of loading is inclined at angle θ with respect to $V$ axis, hence the plane of loading is inclined at (90 – θ + β) w.r.t neutral axis.

The plane of bending will be inclined at (β – θ), if the line is drawn perpendicular to the neutral axis.

∴ Component of BM along Neutral axis

$$M_{NN} = M\cos(\beta - \theta)$$

∴
$$\sigma_b = \frac{M\cos(\beta-\theta)}{I_{NN}} Y_N$$

where

$M_{NN}$ = B.M about N.A.

$Y_N$ = Perpendicular distance of any point from N.A.

## 12.4 ❑ NUMERICAL ON UNSYMMETRICAL BENDING

Co-ordinate of any point w.r.t Neutral axis.

$$u = x\cos\alpha + y\sin\alpha$$
$$v = y\cos\alpha - x\sin\alpha$$

where α = angle between $U$–$U$ and $x$-$x$ axis.

$$I_{xy} = A_1x_1y_1 + A_2x_2y_2$$

where $(x_1, y_1)$ are the co-ordinates of C.G. of area $A$, position of principal axis.

$$\tan 2\alpha = \frac{-2I_{xy}}{I_{xx} - I_{yy}}$$

Principal moment of inertia

$$I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$I_{vv} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$I_{xx} + I_{yy} = I_{uu} + I_{vv}$$

Bending stress

$$\sigma_b = \frac{M\cos\theta}{I_{uu}} . v + \frac{M\sin\theta}{I_{yy}} . u$$

where $\theta$ = angle of moment w.r.t $v$–$v$ axis.

Location of neutral axis

$$\tan\beta = \frac{I_{uu}}{I_{vv}}\tan\theta$$

where $\beta$ = Inclination of N.A w.r.t $u$–$u$ axis

Moment of inertia about N.A.

$$I_{NN} = I_{uu}\cos^2\beta + I_{vv}\sin^2\beta$$

Bending stress

$$\sigma_b = \frac{M_{NN}}{I_{NN}} . y_N = \frac{M\cos(\beta - \theta)}{I_{NN}} . y_N$$

**Example 12.1.** *A beam of rectangular section 80 mm wide and 120 mm deep is subjected to a B.M. of 12 kN.m. The trace of the plane of loading is inclined at 45° to the Y–Y axis of the section. Locate the neutral axis of the section and calculate the maximum bending stress induced in the section.*

*Given :* Width $b$ = 80 mm

Depth $d$ = 120 mm

Bending Moment $M$ = 12 kN.m = $12 \times 10^6$ N.mm

$\theta$ = 45°

**Solution To find :** $\beta$ and $\sigma_b$

Moment of Inertia

$$I_{xx} = I_{uu} = \frac{bd^3}{12} = \frac{80 \times 120^3}{12} = 11.52 \times 10^6 \text{ mm}^4$$

$$I_{yy} = I_{vv} = \frac{db^3}{12} = \frac{120 \times 80^3}{12} = 5.12 \times 10^6 \text{ mm}^4$$

The inclination β of the Neutral Axis

$$\tan \beta = \frac{I_{uu}}{I_{yy}} \tan \theta = \frac{11.52 \times 10^6}{5.12 \times 10^6} \tan 45° = 2.25$$

$$\beta = \tan^{-1}(2.25) = 66°$$

∴ Location of N.A of the section is at 66°

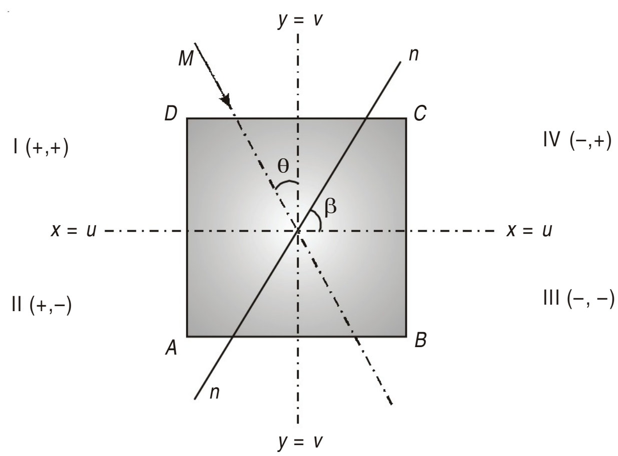



**Fig. 12.6**

Maximum stress will occur either at $B$ or $D$ whichever is more distant from the N.A.

$$\sigma_b = \frac{M \cos\theta}{I_{uu}} . v + \frac{M \sin\theta}{I_{vv}} . u = \frac{M \cos\theta}{I_{xx}} . y + \frac{M \sin\theta}{I_{yy}} . x$$

co-ordinates of points $B$ and $D$ are as follows

for point $B$, $x_B = -40$ mm and $y_B = -60$ mm

for point $D$, $x_D = +40$ mm and $y_D = 60$ mm

bending stress at point $B$.

$$\therefore \qquad \sigma_b = \frac{M \cos\theta}{I_{xx}} . y_B + \frac{M \sin\theta}{I_{yy}} . x_B$$

$$= \frac{12 \times 10^6 \cos 45°}{11.52 \times 10^6} \times (-60) + \frac{12 \times 10^6 \sin 45°}{5.12 \times 10^6} (-40)$$

$$\mathbf{(\sigma_b)_B = -110.5 \ N/mm^2 \ (Tensile)}$$

Bending stress at point $D$

$$\sigma_b = \frac{M\cos\theta}{I_{xx}} \cdot y_D + \frac{M\sin\theta}{I_{yy}} \cdot x_D$$

$$\sigma_b = \frac{12\times10^6\cos 45°}{11.52\times10^6}\times 60 + \frac{12\times10^6\sin 45°}{5.12\times10^6}\times 40$$

$$(\sigma_b)_B = \mathbf{110.5 \ \ N/mm^2} \ \ \textbf{(Compressive)}$$

**Example 12.2.** *A 60 mm × 40 mm × 6 mm unequal angle is placed with the longer leg, vertical, and is used as a beam. It is subjected to a bending moment 12 kN.cm acting in the vertical plane through the centroid of the section. Determine the maximum bending stress induced in the section.* (*UPTU : 2009–2010*)

*Given : BM = 12 kN.cm = 12 × $10^5$ N.mm*

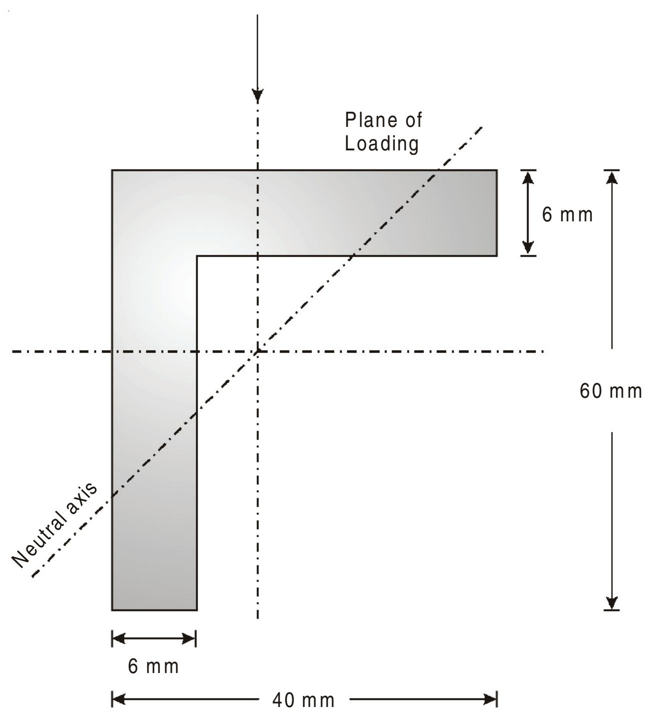



**Fig. 12.7**

**Solution To find** : $\sigma_{max}$

$A_1$ = 6 × 10 = 240 mm$^2$ $\qquad$ $A_2$ = 54 × 6 = 324 mm$^2$

$A$ = $A_1 + A_2$ = 564 mm$^2$

$$Y_1 = 60 - \frac{6}{2} = 57 \text{ mm} \qquad y_2 = \frac{54}{2} = 27 \text{ mm}$$

$$x_1 = 20 \text{ mm} \qquad x_2 = 3 \text{ mm}$$

$$\therefore \quad \bar{x} = \frac{A_1x_1 + A_2x_2}{A} = \frac{240 \times 20 + 324 \times 3}{564}$$

$$= 10.23 \text{ mm}$$

$$\bar{y} = \frac{A_1y_1 + A_2y_2}{A} = \frac{240 \times 57 + 324 \times 27}{564}$$

$$= 39.8 \text{ mm}$$

$$I_{x1} = I_G + A_1h_1^2 = \frac{40 \times 6^3}{12} + 240(39.8 - 57)^2$$

$$= 71721.6 \text{ mm}$$

$$I_{y2} = \frac{6 \times 54^3}{12} + 324(39.8 - 27)^2 = 131816.16 \text{ mm}^4$$

$$I_{xx} = 203537.7 \text{ mm}^4$$

$$I_{yy} = I_{yy1} + I_{yy2} = \frac{6 \times 40^3}{12} + 240(10.23 - 20)^2 + \frac{54 \times 6^3}{12} + 324(10.23 - 3)^2$$

$$= 72817.12 \text{ mm}^4$$

$$I_{xy} = A_1x_1y_1 + A_2x_2y_2$$

$$= 240\ (20 - 10.23)\ (39.8 - 57) + 324\ (10.23 - 3)\ (27 - 39.8)$$

$$= -70314.8 \text{ mm}^4$$

Position of principle axis

$$\tan 2\alpha = -\frac{2I_{xy}}{I_{xx} - I_{yy}} = \frac{-2 \times (-70314.8)}{203537.7 - 72817.12} = 1.076$$

$$\alpha = 23.55° \text{ (anticlockwise)}$$

$$I_{uu} = \frac{1}{2}(I_{xx} + I_{yy}) + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{203537.7 + 72817.12}{2} + \sqrt{\left(\frac{203537.7 - 72817.2}{2}\right)^2 + (-70314.8)^2}$$

$= 1318177.42 + 96000 = 234178.11 \text{ mm}^4$

$$l_{vv} = \left(\frac{I_{xx} + I_{yy}}{2}\right) - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$= 138177.42 - 96000$

$= 42177.42 \text{ mm}^4$

The plane of loading is vertical hence $Y'$ axis and $Y$ axis is coincide.

$\therefore \quad \theta = \alpha = 23°40'$

Bending stress

$$\sigma_{bs} = \frac{M\cos\theta}{I_{uu}} \cdot v + \frac{M\sin\theta}{I_{vv}} \cdot u$$

$x = -10.2 + 6 = -4.2 \text{ mm}$

$y = -60 + 20.2 = -39.8 \text{ mm}$

$\therefore \quad u = y\sin\alpha - x\cos\alpha = 39.8\sin 23.55° - (-4.2\cos 23.55°)$

$= -12.2 \text{ mm}$

$v = y\cos\alpha + x\sin\alpha = -39.8\cos 23.55° - 4.2\sin 23.55°$

$= -38.2 \text{ mm}$

$$\therefore \quad \sigma_{bs} = \frac{12\times10^4\cos 23.55}{234178.11}\times(-38.2)$$

$$+\frac{12\times10^4\sin 23.55}{42177.42}(-12.2)$$

$= -17.95 - 13.87$

$\mathbf{= -31.8 \ N/mm^2}$

**Example 12.3.** *An angle section 9 cm × 6 cm × 1.2 cm is used as a cantilever 1 m long with 6 cm leg. horizontal. A vertical load 1200 N is applied at free end. Determine the position of the neutral axis and the maximum stress set up.*

*$I_{uu} = 147.5 \text{ cm}^4$, $I_{vv} = 26.8 \text{ cm}^4$, $\tan\alpha = 0.42$*

*Given :* $P = 1200 \text{ N}$

$L = 1\text{m} = 1000 \text{ mm}$

$I_{uu} = 1.475 \times 10^6 \text{ cm}^4 \qquad C_{xx} = 16$

$I_{vv} = 0.268 \times 10^6 \text{ cm}^4 \qquad C_{yy} = 22.78$

$\alpha = \tan^{-1}(0.42) = 22.78°$

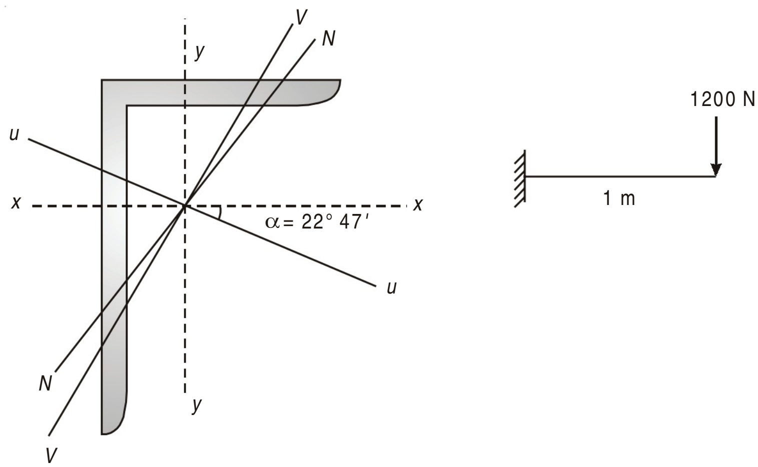



**Fig. 12.8**

**Solution To find :** $\beta$ and $\sigma_b$

Bending Moment $M = P \times L = 1200 \times 1000 = 1.2 \times 10^6$ N-mm

The Inclination $\beta$ of N.A

$$\tan\beta = \frac{I_u}{I_v}\tan\alpha = \frac{1.475\times10^6}{0.268\times10^6}\times\tan 22.78 = 2.312$$

$$\beta = 66.60°$$

Maximum stress will be at the toe of the vertical leg.

Its co-ordinate are :

$$u = x\sin\alpha - y\cos\alpha = (90 - 31.2)\sin 22.78 - 16.3\cos 22.78$$
$$= 22.78 - 15.02 = 7.75 \text{ mm}$$
$$v = -[x\cos\alpha + y\sin\alpha]$$
$$= -[(90 - 31.2)\cos 22.78 + 16.3\cos 22.78]$$
$$= -[54.21 + 6.31]$$
$$= -60.52 \text{ mm}$$

Stress at toe, $$\sigma = \frac{M_u}{I_{\mu\mu}}.v + \frac{M_v}{I_{yy}}.u, \text{ from Fig. 12.8}$$

$$M_u = M\cos\alpha = 1.26\times10^6\cos 22.78$$
$$= 1.1617\times10^6 \text{ N-mm}$$
$$M_v = M\sin\alpha = 1.26\times10^6\sin 22.78$$
$$M_v = 0.49\times10^6 \text{ N-mm}$$

$$= \frac{1.1617\times10^6}{147.5\times10^4}\times60.52+\frac{0.49\times10^6\times7.75}{26.3\times10^4}$$

$$= \mathbf{47.66 + 14.44 = 62.1\ N/mm^2}$$

**Example 12.4.** *A simply supported I section beam of span 2 m carries a concentrated load of 4.0 kN at an angle of 20° form vertical as shown in Fig. 12.9. The load passes though CG of the section. Determine the maximum bending stress in the beam.* (*UPTU : 2010–2011*)

*Given :* $L = 2\ m,\ W = 4\ kN$

$\theta = 20°$

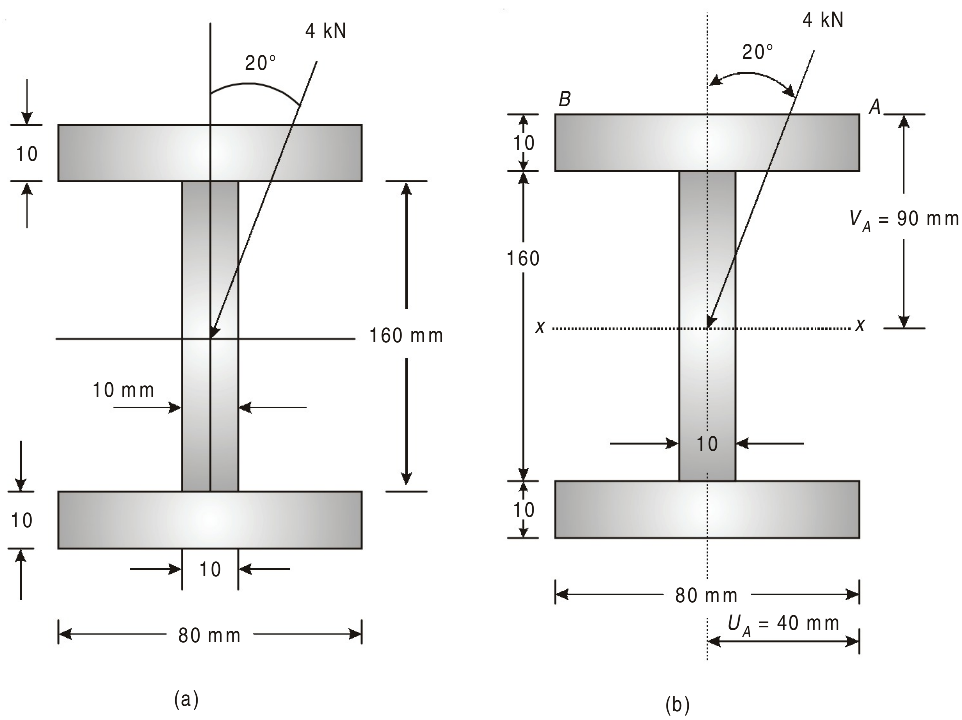



**Fig. 12.9**

**Solution To find :** $\sigma_{max}$

$$\bar{y} = \frac{160}{2}+10 = 90\text{ mm}$$

$$I_{xx} = \frac{BD^3 - bd^3}{12}$$

$$= \frac{80 \times 180^3 - 70 \times 160^3}{12}$$

$$= 14.987 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{10 \times 80^3}{12} + \frac{160 \times 10^3}{12} + \frac{10 \times 80^3}{12}$$

$$= 0.866 \times 10^6 \text{ mm}^4$$

The symmetric axes are the principle axes

$$I_{uu} = I_{xx} = 14.987 \times 10^6 \text{ mm}^4$$
$$I_{vv} = I_{yy} = 0.8667 \times 10^6 \text{ mm}^4$$

Bending moment

$$M_u = \frac{W_x L}{4} = \frac{4 \cos 20° \times 2}{4} = 1.879 \text{ kN.m}$$

$$M_u = M \cos \theta$$

$$M_v = \frac{W_y L}{4} = \frac{4 \sin 20° \times 2}{4} = 0.684 \text{ kN.m}$$

$$M_v = M \sin \theta$$

Maximum stress

$$\sigma_4 = \frac{M_u}{I_u} V_A + \frac{M_v}{I_v} U_A$$

$$= \frac{1.879 \times 10^6}{14.987 \times 10^6} \times 90 + \frac{0.684 \times 10^6}{0.8667 \times 10^6} \times 40$$

$$= \mathbf{42.85 \ N/mm^2 \ (Compressive)}$$

**Example 12.5.** *A 6 cm × 4 cm × 0.6 cm unequal angle is placed with the longer leg vertical and is used as beam. It is subjected to a bending moment of 150 N-m acting in a vertical plane through the centroid of the section. Determine the maximum bending stress induced in the section.* (*UPTU : 2006 – 2007*)

*Given :* $M = 150$ *N-m*

**Solution To find :** $\sigma_{\max}$

Location of c.g. (Fig. 12.10)

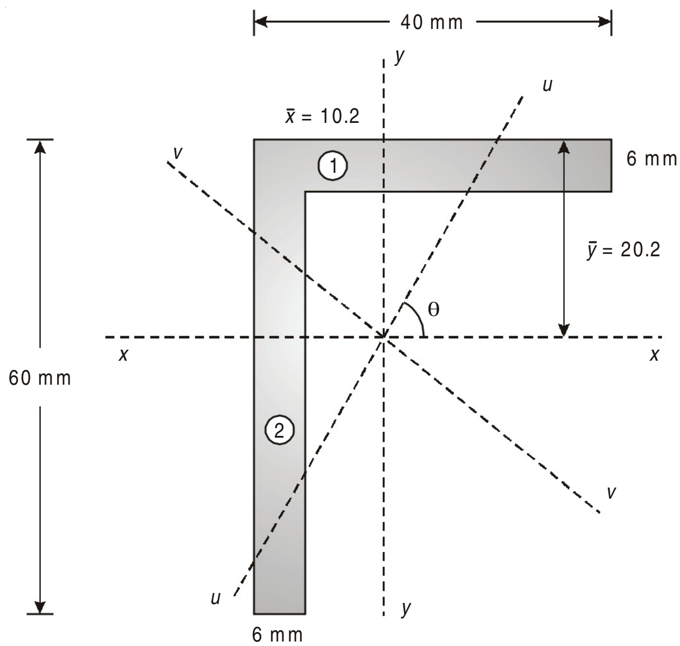



**Fig. 12.10**

$$\text{Area } A_1 = 40 \times 6 = 240 \text{ mm}^2$$

$$A_2 = (60 - 6) \times 6 = 324 \text{ mm}^2$$

$$\text{Total } A = 564 \text{ mm}^2$$

$$\therefore \quad 564\bar{x} = 240 \times 20 + 324 \times 3$$

$$\mathbf{\bar{x} = 10.2 \text{ mm}}$$

Also

$$564\,\bar{y} = 40 \times 6 \times 3 + 6 \times 54 \times \left(6 + \frac{54}{2}\right)$$

$$\therefore \quad \mathbf{\bar{y} = 20.2 \text{ mm}}$$

$$\therefore I_{xx} = \frac{1}{12} \times 40 \times (6)^3 + 240 \times (17.2)^2 + \frac{1}{12} \times 6 \times (54)^3 + 324 \times (33 - 20.2)^2$$

$$= (204 \times 10^3) \text{ mm}^4$$

$$\therefore I_{yy} = \frac{1}{12} \times 6 \times (40)^3 + 240 \times (20 - 10.2)^2 + \frac{1}{12} \times 54 \times (6)^3 + 324 \times (7.2)^2$$

$$= (72.8 \times 10^3) \text{ mm}^4$$

Product of inertia

$$\therefore \quad I_{xy} = 240 \times 17.2 \times 9.8 + 324 \times (-7.2) \times (-6.8)$$

$$= (56.32 \times 10^3) \text{ mm}^4$$

Now $\tan 2\theta = \dfrac{2I_{xy}}{(I_{yy} - I_{xx})} = \dfrac{2 \times 56.32}{(72.8 - 204.0)}$

$\therefore$ $\boldsymbol{\theta = 69.7°}$ inclination of $uu$ axis with $x$ axis

Now
$$I_{uu} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin 2\theta$$
$$= (204 \times \cos^2\theta + 72.8 \times \sin^2\theta - 56.32 \times \sin 2\theta) \times 10^3$$
$$I_{uu} = (51.94 \times 10^3)\ \text{mm}^4$$
$$I_{vv} = I_{xx}\sin^2\theta + I_{yy}\cos 2\,\theta + I_{xy}\sin^2\theta$$
$$= (224.86 \times 10^3)\ \text{mm}^4$$

Check $(I_{xx} + I_{yy}) = (I_{uu} + I_{vv})$

The given moment is 150 Nm = (150 × 1000) Nmm. It is in vertical plane. Resolving it along $uu$ and $vv$ axes :

$$M_u = M\cos\theta = (150 \times 1000)\cos 69.7$$
$$= (52.04 \times 1000)$$
$$M_v = M\sin\theta = (150 \times 1000)\sin 69.7$$
$$= (140.68 \times 1000)$$

$\therefore$ Bending stress is

$$\sigma = \frac{M_v \times u}{I_{vv}} + \frac{M_u \times v}{I_{uu}}$$

$$= \left(\frac{140.68 \times 1000}{224.86 \times 1000}\right)u + \left(\frac{52.04 \times 1000}{51.94 \times 1000}\right)v$$

$\therefore$ $\sigma = (0.6256)\,u + (1)\,v$

$\therefore$ $(0.6256)\,u + v = 0$ is the equation of the N.A

$\therefore$ $\tan\alpha = \dfrac{-1}{0.6256}$

$\therefore$ $\boldsymbol{\alpha = -58°}$ **with $\boldsymbol{uu}$-axis**

**Example 12.6.** *A cantilever of length 1.2 m is of the cross-section as shown in Fig. 12.11. It carries a vertical load of 10 kN at its outer end, the line of action being parallel with the longer leg and arranged to pass through the shear centre of the section (i.e. there is no twisting of the section). Working from first principles, find the stress setup in the section at points A, B and C given that the centroid is located as shown. Determine the angle of inclination of the N.A.*

$I_{xx} = 4 \times 10^{-6}\ m^4$; $I_{yy} = 1.08 \times 10^{-6}\ m^4$ *(UPTU : 2005 – 2006)*

*Given : Load* $W = 10\ kN$

*Length* $L = 1.2\ m$

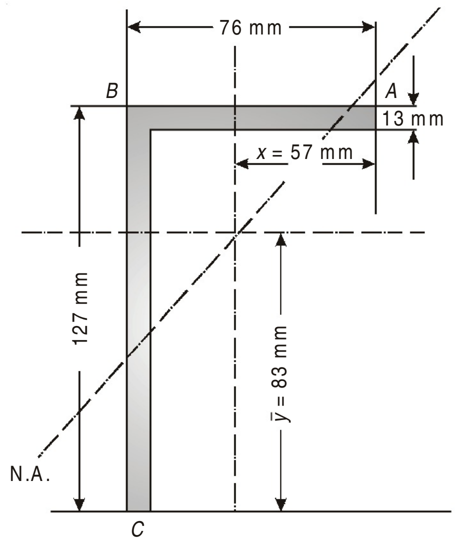



**Fig. 12.11**

**Solution To find :** Angle of inclination θ, $\sigma_A$ and $\sigma_B$.

$$\therefore \text{ B.M. } \quad M_0 = W \times L = 12 \text{ kNm}$$

$$I_x = (4 \times 10^{-6}) \text{ m}^4$$

$$I_y = (1.08 \times 10^{-6}) \text{ m}^4$$

Product of inertia

$$I_{xy} = 76 \times 13 \times (44 - 6.5) \times (57 - 38) + 13 \times 114 \times (-19 + 6.5) \times (-83 + 57)$$

$$= 1.186 \times 10^{+6} \text{ mm}^4$$

$$I_{xy} = 1.186 \times 10^{-6} \text{ m}^4$$

Now
$$\tan 2\theta = \frac{2I_{xy}}{(I_y - I_x)} = \frac{2 \times 1.186}{(1.08 - 4)} = -0.8123$$

$$\therefore \quad 2\theta = 140.9$$

$$\therefore \quad \theta = 70.5°$$

$$I_u = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y) \times \sec 2\theta$$

$$= \frac{1}{2}(4 + 1.08) + \frac{1}{2}(4 - 1.08) \times \sec 2\theta$$

$$I_u = (0.6613 \times 10^{-6}) \text{ m}^4$$

Now
$$I_u + I_v = I_x + I_y$$

$$\therefore \quad 0.6613 + I_v = 4 + 1.08$$

$\therefore \qquad I_v = (4.4187 \times 10^{-6})\ m^4$

Now $\qquad M_v = M_0 \sin\theta = 12 \times \sin 70.5° = 11.31$ kNm

$M_u = M_0 \sin\theta = 12 \times \cos 70.5° = 4$ kNm

$\therefore$ Bending stress is

$$\sigma = \frac{M_y \times u}{I_v} + \frac{M_u \times v}{I_u}$$

$$= \frac{(11.31)\,u}{4.4187 \times 10^{-6}} + \frac{4v}{0.6613 \times 10^{-6}}$$

$\therefore \qquad \sigma = (2.56 \times 10^{+6})\,u + (6.05 \times 10^{+6})\,v$ kN/m$^2$

$\therefore$ Equation of N.A. is

$$(2.56)\,u + (6.05)\,v = 0$$

$\therefore$ Slope of N.A. $\qquad \tan\alpha = \dfrac{-2.56}{6.05}$

$\therefore \qquad \boldsymbol{\alpha = -\,22.94°}$

$\therefore$ Inclination of N.A. with $uu$ axis is 22.94° clockwise

Now $\qquad \theta = 70.5°$

To find bending stress at $A$

**At 'A' :** $\qquad x = -19$ mm, $y = +44$ mm

$\therefore \qquad u = x\cos\theta + y\sin\theta = +\,19 \cos 70.5 + 44 \sin 70.5$

$= 47.1$ mm $= (47.1 \times 10^{-3})$ m

$v = y\cos\theta - x\sin\theta = 44\cos 70.5 - 19 \sin 70.5$

$= -3.22 = (-3.22 \times 10^{-3})$ m

$$\therefore \qquad \sigma_A = \frac{M_y \times u}{I_v} + \frac{M_u \times v}{I_u}$$

$$= 2.56 \times 10^{+6} \times u + \left(6.05 \times 10^{+6}\right) v$$

$= [2.56 \times 10^{+6} \times 47.1 + 6.05 \times 10^{+6} \times (-3.22)] \times 10^{-3}$

$= 101 \times 10^3$ kN/m$^2$ $= 101 \times 10^6$ N/m$^2$

$\boldsymbol{\sigma_A = 101}$ **N/mm$^2$ (Tensile)**

At point $B$ :

$\theta = 70.5°$ ; $x = 76 - 19 = 57$ mm ; $y = 44$ mm

$U = x\cos\theta + y\sin\theta = 60.5$ mm

$V = y\cos\theta - x\sin\theta = 44\cos\theta - 57\sin\theta$

$= -\ 39$ mm

$$= (-39 \times 10^{-3}) \text{ m}$$

$$\therefore \qquad \sigma_B = 2.56 \times 10^6 \times u + 60.5 \times 10^6\, v$$

$$= 2.56 \times 10^6 \times 60.5 \times 10^{-3} + 6.05\,(-39)\,(10^{-3})$$

$$= -81 \text{ N/mm}^2$$

$$= -81 \times 10^6 \text{ N/m}^2$$

Hence, stress at $B$, $\sigma_B$ = **81 N/mm² (Compressive)**

## 12.5 ❑ SHEAR CENTRE

The shear centre is explained below.

(i) It is the point in the cross-section of the beam where the load should be applied so that it causes only bending without any twisting.

(ii) It is also known as centre of twist.

(iii) Shear centre is the point of intersection of the bending axis and the plane if transverse section.

(iv) In case of beam having two axis of symmetry the shear centre coincides with the centroid.

(v) In case of beam having only one axis of symmetry, the shear centre does not coincide with the centriod but lies on axis of symmetry.

## 12.6 ❑ SHEAR FLOW

The variation of shear per unit length is known as shear flow.

Longitudinal shear per unit length $\theta = \tau \times b = \dfrac{VA\bar{Y}}{I}$

Figure 12.12 shows shear force variation across the section.

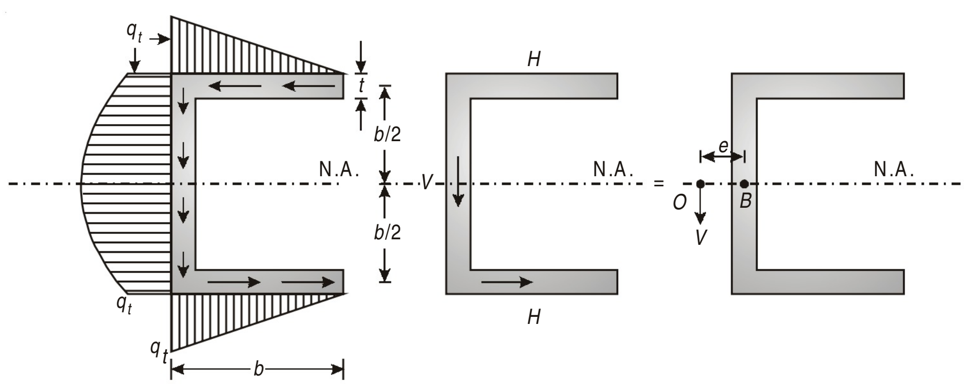



**Fig. 12.12**

Consider a small portion of a flange of width $x$ and thickness '$t$' as shown in Fig. 12.13.

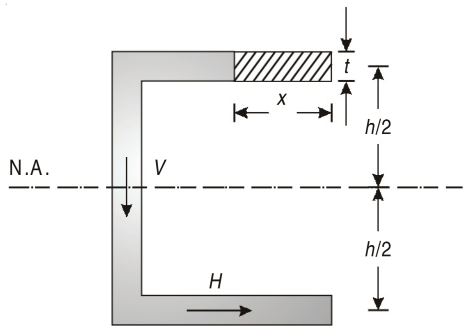



**Fig. 12.13**

Let a cantilever beam consisting of a channel carrying a point load $P$ at a eccentricity $e$ as shows in Fig. 12.14.


**Fig. 12.14**

$\therefore$ Torsional moment $\quad T = P \times e \quad$ ...(1)

Maximum shear flow $\quad q = \dfrac{VA\bar{Y}}{I} = \dfrac{V}{I}(t\,x)\dfrac{h}{2} = \dfrac{Vth}{2I}x$

This shows that shear force varies linearly with the distance from the free edges.

$\therefore$ Average shear flow in the flange ($x = b$)

$$q_{\text{ave}} = \frac{1}{2}\left(\frac{Vth}{2I}\right)b = \frac{Vth}{4I}b$$

Now, to prevent the beam from twisting, the torsional moment ($P \times e$) must be balanced by the couple formed by shearing forces $H$ ($h \times y$)

$\therefore \quad P.e = h.y \quad (\because h = q_{\text{ave}})$

but $H$ is equal to average shear flow in the flange multiplied by the length of flange $b$.

$$\therefore \quad V.e = \left[\left(\frac{Vthb}{4I}b\right)\right]h \quad (\because P = V)$$

$$e = \frac{h^2b^2t}{4I}$$

where

$h$ = height of the section

$b$ = width of the flange

$t$ = thickness of flange

**Example 12.7.** *A thin channel section has outside flange and web dimensions of 102 mm and 204 mm respectively. The thickness of flanges and web is uniform and equal to 4 mm. Draw the shear stress flow distribution for the section and find the position of the shear centre. Take value of shear force at section = 50 kN.*

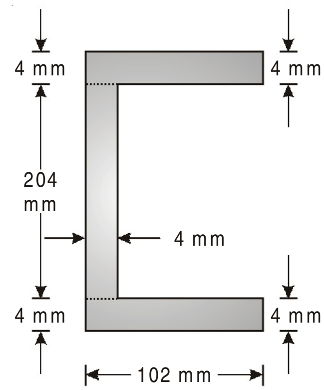



**Fig. 12.15**

*Given :* $S = 50\ kN$

**Solution To find :** (i) Shear stress (ii) Position of shear centre

(i) Shear stress :

Position of neutral axis $\bar{y} = \dfrac{D}{2} = \dfrac{4+204+4}{2} = \dfrac{212}{2} = 106$ mm

Moment of inertia $I = I_{xx1} + I_{xx2} + I_{xx3} \qquad \because I_{xx2} = I_{xx3}$

$$= [I_{G1}] + \left[I_{G_2} + A_2\, h_2^2\right] \times 2$$

$$= \left[\frac{4 \times 204^3}{12}\right] + \left[\frac{102 \times 4^3}{12} + (102 \times 4)\left(106 - \frac{4}{2}\right)^2\right]$$

$$= 2.83 \times 10^6 + (544 + 4.413 \times 10^6) = 7.24 \times 10^6$$

mm$^4$

Shear stress at extreme top and bottom fibre is zero.

Shear stress at junction of flange and web

$b_1$ = 102 mm $\quad b_2$ = 4 mm

$A$ = (102 × 4) = 408 mm

$$\bar{y}_2 = 106 - \frac{4}{2} = 104 \text{ mm}$$

$$\tau_1 = \frac{SAY}{b_1 I} = \frac{50 \times 10^3 \times 408 \times 104}{102 \times 7.24 \times 10^6} = \mathbf{2.87\ N/mm^2}$$

$$\tau_2 = \tau_1 \times \frac{b_1}{b_2} = 2.87 \times \frac{102}{4}$$

$$= \mathbf{73.26\ N/mm^2}$$

Shear stress at neutral axis

$$b = 4 \text{ mm}$$

$$A\bar{Y} = A_1 Y_1 + A_2 Y_2 = (102 \times 4 \times 104 + 102 \times 4 \times 104) = 84864 \text{ mm}^3$$

$$\tau_{NA} = \frac{SA\bar{Y}}{bI} = \frac{50 \times 10^3 \times 84864}{4 \times 7.24 \times 10^6} = 146.52 \text{ N/mm}^2$$

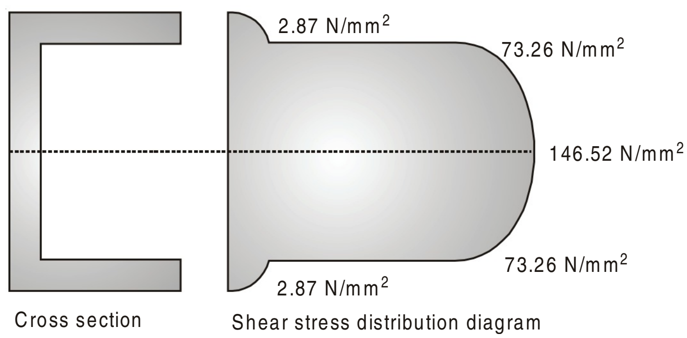



**Fig. 12.16**

(ii) Distance of shear centre :

$$e = \frac{h^2 b^2 t_f}{4I} = \frac{212^2 \times 102^2 \times 4}{4 \times 7.24 \times 10^6}$$

$$= \mathbf{64.58\ \ mm}$$

**Example 12.8.** *A thin channel section has outside flange and web dimensions of 100 mm and 200 mm respectively. The thickness of flanges and web is uniform and equal to 4 mm. Draw the shear stress and shear flow distribution for the section and find the position of the shear center. Take value of shear force at section = 60 kN.*

*Given :* $t_w = t_f = 4$ *mm*

$S = 60$ *kN*

**Solution To find :** Shear stress, shear flow distribution and *e*

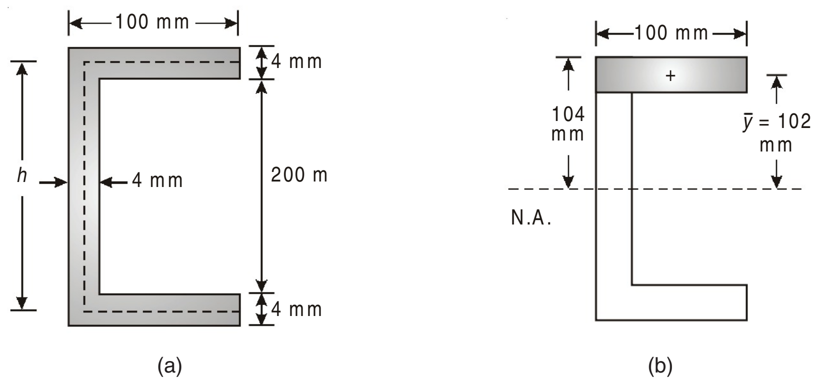



**Fig. 12.17**

Moment of inertia

$$I = I_{xx1} - I_{xx2}$$

$$= \frac{100 \times 208^3}{12} - \frac{96 \times 200^3}{12}$$

$$= 10.99 \times 10^6 \text{ mm}^4$$

Position of shear centre

$$b = 100 - \frac{4}{2} = 98 \text{ mm}$$

$$h = 200 + \frac{4}{2} + \frac{4}{2} = 204 \text{ mm}$$

$$e = \frac{h^2 b^2 t_f}{4I} = \frac{98^2 \times 204^2 \times 4}{4 \times 10.99 \times 10^6}$$

$$= \mathbf{36.36 \text{ mm}}$$

**Shear stress distribution**

Shear stress at extreme top and bottom fibre is always zero

Shear stress at junction of web and flange

$$\tau = \frac{SA\bar{Y}}{bI}$$

$$A = 100 \times 4 = 400 \text{ mm}^2$$

$$b_1 = 100 \text{ mm}$$

$$b_2 = 4 \text{ mm} \qquad \bar{y} = 104 - \frac{4}{2} = 102 \text{ mm}$$

$$\therefore \qquad \tau_1 = \frac{60 \times 10^3 \times 400^2 \times 102}{100 \times 10.99 \times 10^6} = 2.23 \text{ mm}$$

$$\tau_2 = \tau_1 \times \frac{b_1}{b_2} = 2.23 \times \frac{100}{4} = 55.68 \text{ MPa}$$

Shear stress at Neutral axis

$$\tau_{NA} = \frac{SA\bar{y}}{bI} = \frac{S}{bI}(A_1\, y_1 + A_2\, y_2)$$

$$A_1 = 100 \times 4 = 400 \text{ mm}^2$$

$$y_1 = 102 \text{ mm}$$

$$A_2 = 4 \times 100 = 400 \text{ mm}^2$$

$$y_2 = \frac{100}{2} = 50 \text{ mm}$$

$$\tau_{NA} = \frac{60 \times 10^3 (400 \times 102 + 400 \times 50)}{4 \times 10.99 \times 10^6} = 82.98 \text{ MPa}$$

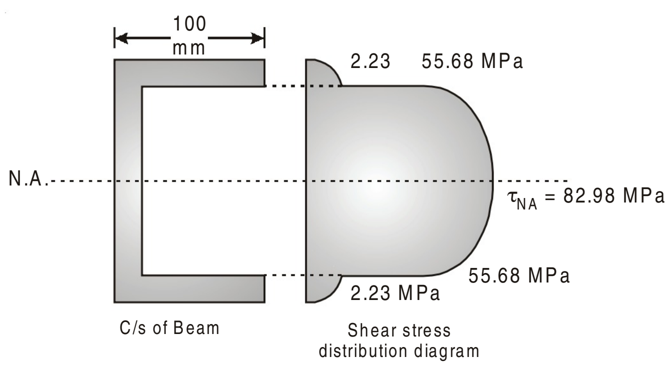



**Fig. 12.18**

**Example 12.9.** *Locate the position of shear centre for a thin channel section having total depth of 400 mm, width of flanges 120 mm, thickness of flange 10 mm and thickness of web equal to 6 mm. When will the distance of the shear centre from the centre line of the web be minimum ?*

*Given :*

$$h = \left(400 - \frac{10}{2} - \frac{10}{2}\right) = 390 \text{ mm}$$

$$b = 120 - \frac{6}{2} = 117 \text{ mm}$$

$$t_f = 10 \text{ mm}, \; t_w = 6 \text{ mm}$$

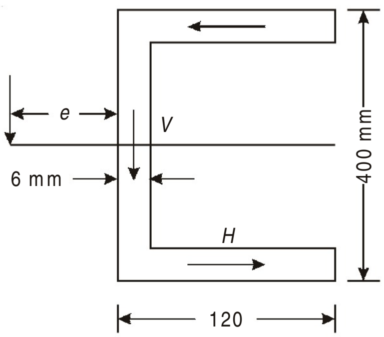



Fig. 12.19

**Solution To find :** Distance of shear centre '$e$'

Moment of inertia

$$I = I_{xx1} - I_{xx2}$$

$$= \frac{120 \times 400^3}{12} - \left[\frac{(120-6)(400-20)^3}{12}\right]$$

$$= 118.716 \times 10^6$$

Distance of shear centre ($e$)

$$e = \frac{h^2 b^2 t_f}{4I} = \frac{390^2 \times 117^2 \times 10}{4 \times 118.716 \times 10^6}$$

$$= \mathbf{43.85 \ mm}$$

**Example 12.10.** *Locate the shear centre with sketch for the section as shown in Fig. 12.20.* (*UPTU : 2013–2014*)

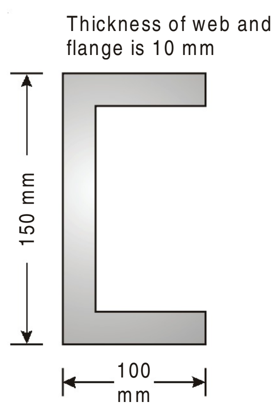



Fig. 12.20

**Solution**

$$h = 150 - \frac{10}{2} - \frac{10}{2} = 140 \text{ mm}$$

$$b = 100 - \frac{10}{2} = 95 \text{ mm}$$

$$t_f = t_w = 10 \text{ mm}$$

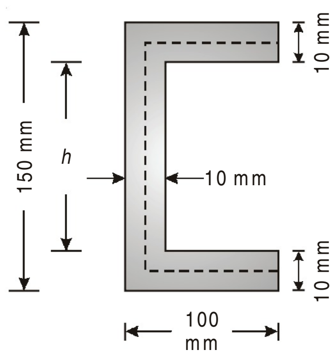



**Fig. 12.21**

Moment of inertia $I = I_{xx1} - I_{xx2}$

$$= \frac{100 \times 150^3}{12} - \frac{(100-10)(150-20)^3}{12}$$

$$= 11.6475 \times 10^6 \text{ mm}^4$$

Distance of shear centre ($e$)

$$e = \frac{h^2 b^2 t_f}{4I} = \frac{140^2 \times 95^2 \times 10}{4 \times 11.6475 \times 10^6} = \mathbf{37.967 \text{ mm}}$$

**Example 12.11.** *Locate the position of shear centre for thin channel section as shown in Fig. 12.22.*

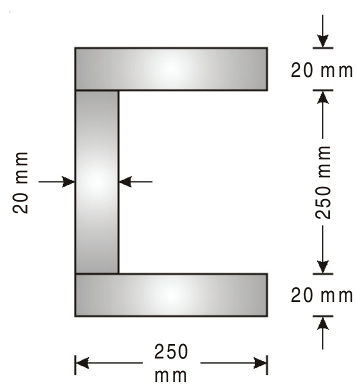



**Fig. 12.22**

Given : $h = 250 + \frac{20}{2} + \frac{20}{2} = 270 \text{ mm}$

$b = 120 - \frac{20}{2} = 110 \text{ mm}$

$t_f = t_b = 20 \text{ mm}$

**Solution To find :** Distance of shear centre

$$I = I_{xx1} + I_{xx2}$$

$$= \frac{20 \times 250^3}{12} + \left[\frac{120 \times 20^3}{12} + (120 \times 20)135^2\right]$$

$$= 26.04 \times 10^6 + 44.54 \times 10^6$$

$$= 70.58 \times 10^6 \text{ mm}^4$$

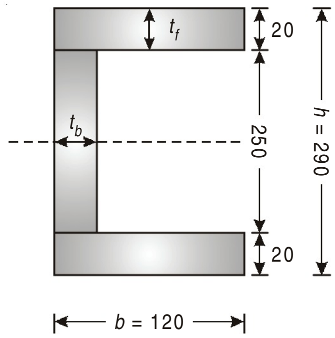



Fig. 12.23

Position of shear centre,

$$e = \frac{h^2 b^2 t_f}{4I} = \frac{270^2 \times 110^2 \times 20}{4 \times 70.58 \times 10^6}$$

$$= \mathbf{62.49 \text{ mm}}$$

## EXERCISE

**12.1.** Calculate the distance $e$ location of the shear center $O$. Assume constant thickness of the section. (Fig. 12.24)

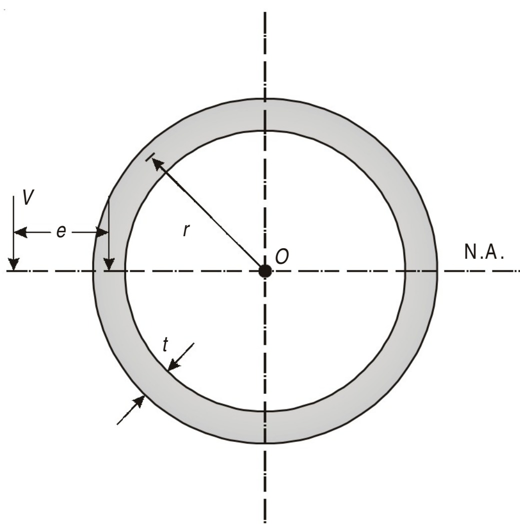



Fig. 12.24

**[*Ans.* $e$ = 35.3 mm]**

**12.2.** Determine the distance of shear center for section shown in Fig. 12.25.

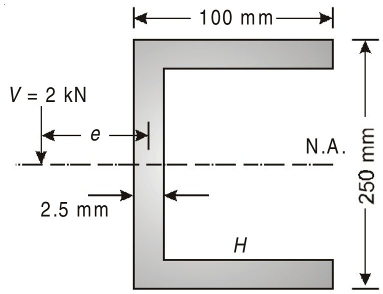



**Fig. 12.25**

**12.3.** The cross section of the beam shown has a uniform wall thickness. Determine the location for shear relation to point *O*. (Fig. 12.26)

**[*Ans.* *e* = 36 mm from *O*]**

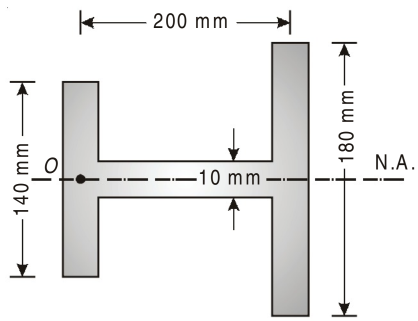



**Fig. 12.26**

**12.4.** The cross-section of the beam shown in Fig. 12.27 has a uniform wall thickness. Determine the location for shear relation to point *O*.

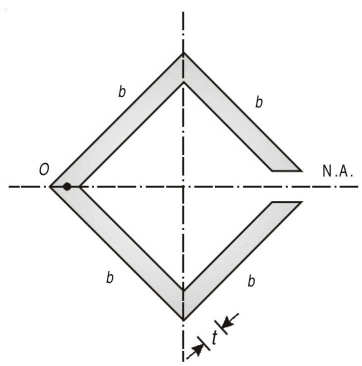



**Fig. 12.27**

## UNIVERSITY QUESTIONS

1. A cantilever of length 1.2 m is of the cross-section as shown in Fig. 12.28, It carries a vertical load of 10 kN at its outer end, the line of action being parallel with the longer leg and arranged to pass through the shear centre of the section (i.e. there is no twisting of the section). Working from first principles, find the stress setup in the section at points *A*, *B* and *C* given that the centroid is located as shown. Determine the angle of inclination of the N.A.

   $I_{xx} = 4 \times 10^{-6}$ m$^4$; $I_{yy} = 1.08 \times 10^{-6}$ m$^4$ (*UPTU : 2006–2007*)

   **[*Ans.* Example 12.6]**

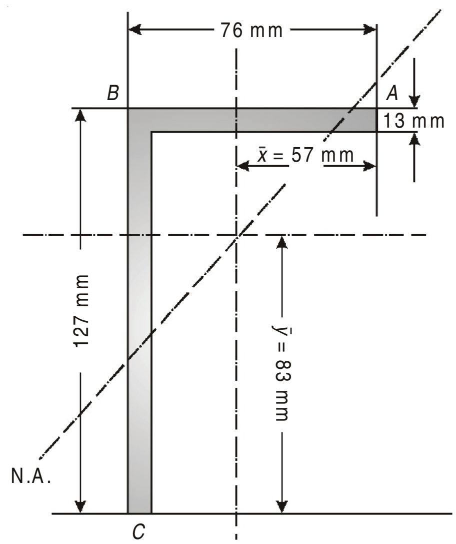



**Fig. 12.28**

2. Explain : Why is the knowledge of shear center of a beam important?

   **[*Ans.* Section 12.4]**

3. What is shear center? Prove that the shear center for a thin-walled balanced *z* section coincides with its centroid. (*UPTU : 2006–2007*)

   **[*Ans.* Section 12.4]**

4. A 6 cm × 4 cm × 0.6 cm unequal angle is placed with the longer leg vertical and is used as beam. It is subjected to a bending moment of 150 N-m acting in the vertical plane through the centroid of the section. Determine the maximum bending stress induced in the section. (*UPTU : 2006–2007*)

   **[*Ans.* Example 12.5]**

5. What is shear center? Prove that the shear center for a thin-walled balanced *z* section coincides with its centroid. (*UPTU : 2007–2008*)

   **[*Ans.* Section 12.4]**

6. A 60 mm × 40 mm × 6 mm unequal angle is placed with the longer leg. vertical, and is used as a beam. It is subjected to a bending moment 12 kN.cm acting in the vertical plane through the centroid of the section. Determine the maximum bending stress induced in the section.

(*UPTU : 2009–2010*)

**[*Ans.* Example 12.2]**

Write short note on : Centroidal Principal Axes. **[*Ans.* Section 12.3]**

7. A simply supported *I* section beam of span 2 m carriers as concentrated load of 4.0 kN at an angle of 20° form vertical as shown in Fig. 12.29. The load passes though CG of the section. Determine the maximum bending stress in the beam. (*UPTU : 2010*)

**[*Ans.* Example 12.4]**

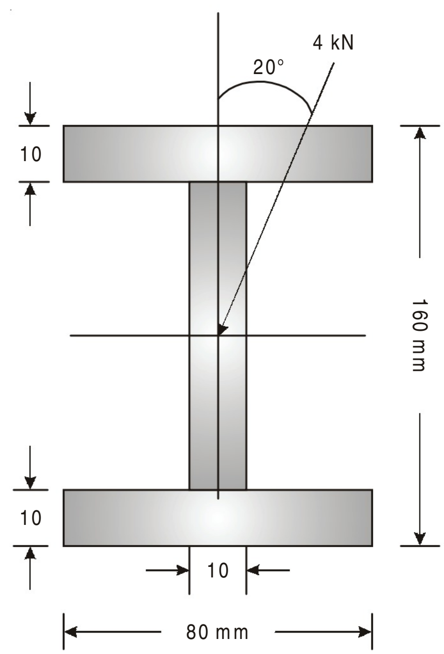



**Fig. 12.29**

8. Develop a general theory of bending of prismatic beam of arbitrary cross-section. (not having an axial plane of symmetry).

(*UPTU : 2010–2011*)

**[*Ans.* Sections 12.2 and 12.3.2]**

9. (a) Write brief note of Shear Centre. (*UPTU : 2010–2011*)
   (b) Define shear centre (*UPTU : 2013–2014*)
10. Why the shear centre is called the centre of twist? (*UPTU : 2011–2012*)

**[*Ans.* Section 12.4]**

11. Define unsymmetrical bending. (*UPTU : 2012–2013*)

**[*Ans.* Section 12.2]**

12. Explain stress due to unsymmetrical bending. (*UPTU : 2012–2013*)

**[*Ans.* Section 12.3.2]**

13. Locate the shear centre with sketch for the section as shown in Fig. 12.30

(*UPTU : 2013–2014*)

**[*Ans.* Example 12.10]**

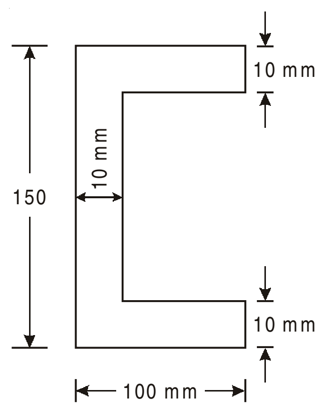



**Fig. 12.30**

# Important Questions for Practice and University Papers

CHAPTER 13

## SECTION A

1. What do you understand by strain energy absorbed by the system, complimentary strain energy and elastic strain energy? Explain these with the help of diagrams. (2005 – 2006)
2. Define principal stresses and principal plane. (2013 – 14)
3. What do you mean by strength of a shaft.
4. Write down the expression for power transmitted by shaft.
5. Define springs. Name different types of springs.
6. What are the limitations of Euler's formula?
7. Define hoop stress and longitudinal stress.
8. Write down the assumptions made in Lame's equation.
9. Write down the expression for Winkler-Batch formula.
10. Differentiate between straight and curved beam.
11. Define unsymmetrical bending.
12. Why the shear centre is called the centre of twist? (*UPTU : 2013–2014*)
13. What is stress? In what way does the shear stress differ from direct stress? Explain.
14. Define the principle of superposition. What is its utility?
15. Define Bulk Modulus. Also write relation between Young's modulus, Bulk modulus and Poisson's Ratio.
16. Depending upon the types of supports, classify the beams.
17. Define bending stress and neutral axis.
18. Define equivalent torque.
19. Define crippling load and safe load.
20. Define interference.
21. Define thin and thick cylinders. (*UPTU : 2013–2014*)

22. Explain complementary shear stress. (*UPTU : 2013–14*)
23. Draw the Mohr's circle for pure shear. (*UPTU : 2013–14*)
24. Define neutral axis. (*UPTU : 2013–2014*)
24. What do you understand by section modulus? (*UPTU : 2013–14*)
25. Explain point of contraflexure in a beam. (*UPTU : 2013–14*)
26. What do you understand by effective length of the column? (*UPTU : 2013–2014*)
27. Explain torsional stiffness and Torsional flexibility. (*UPTU : 2013–2014*)

**SECTION B**

1. While testing on a metallic rod, it is observed that the diameter of rod is 15 mm. If rigidity modulus for the rod metal be 50 kN/mm$^2$. Find the Young's modulus and Bulk modulus. (*UPTU : 2004 – 2005*)

   **[*Ans.* $E$ = 117.3 GPa, $K$ = 59.73 GPa]**
2. The bulk modulus for a material is $0.5 \times 10^5$ N/mm$^2$. A 12 mm diameter rod of the material was subjected to an axial pull of 14 kN and the change in diameter was observed to be $3.6 \times 10^{-3}$ mm. Calculate Poisson's ratio and modulus of elasticity. (*UPTU : 2005 – 2006*)

   **[*Ans.* $\mu$ = 0.21, $E$ = 86.86 GPa]**
3. Show that if $E$ is assumed correct, an error of 1% in the determination of $G$ will involve an error of about 5% in the calculation of Poisson's ratio when its correct value is 0.25. (*UPTU : 2002–2003, 2013–2014*)
4. The stresses in the three principal directions are +65 MN/m$^2$, +20 MN/m$^2$, and –85 MN/m$^2$. Find the principal strain. Take $\mu$ = 0.3 and $E$ = 200 GN/m$^2$. (*UPTU : 2006 – 2007*)

   **[*Ans.* $\varepsilon_x = 0.4225 \times 10^{-3}$, $\varepsilon_y = 0.13 \times 10^{-3}$, $\varepsilon_z = -0.5525 \times 10^{-3}$]**
5. The principal stresses at a point in an elastic material are 60 N/mm$^2$ tensile, 20 N/mm$^2$ tensile, and 50 N/mm$^2$ compressive. Calculate volumetric strain. Take $E$ = 100 kN/mm$^2$ and $\mu$ = 0.3. (*UPTU : 2010–2011*)

   **[*Ans.* $\varepsilon_v = 1.2 \times 10^{-3}$]**
6. A weight $W$ = 5 kN attached to the end of a steel wire rope moves downward with constant velocity 1 m/sec. What stresses are produced in the rope when the upper end is suddenly stripped? The free length of the rope at the moment of impact is 20 m, its net cross-sectional area is 10 sq. cm. and $E = 2 \times 10^5$ N/mm$^2$. (*UPTU : 2012–2013*)

   **[*Ans.* $\sigma$ = 71.39 N/mm$^2$]**
7. A vertical rod 2 m long fixed at the upper end, is 13 cm$^2$ in area for 1 m and 20 cm$^2$ in area in remaining 1 m. A collar is attached to the free end. Through what height can a load of 100 kg fall on the collar to cause a maximum stress of 50 N/mm$^2$? Take $E$ = 200 kN/mm$^2$. (*UPTU : 2011–2012*)

   **[*Ans.* $h$ = 13.25 mm]**

8. A point in a strained material is subjected to a tensile stress of 65 N/mm$^2$ and a compressive stress of 45 N/mm$^2$, acting on two mutually perpendicular planes and a shear stress of 10 N/mm$^2$ are acting on these planes. Find the normal stress, tangential stress and resultant stress on a plane inclined to 30° with the plane of compressive stress. (*UPTU : 2006–2007, 2013–2014*)

**[*Ans.* $\sigma_R$ = 53 MPa; $\sigma_n$ = –8.9 MPa; $\sigma_t$ = 52.5 MPa]**

9. (i) Construct Mohr's circle for the case of plane stress $\sigma_x$ = 360 kg/cm$^2$, $\sigma_y$ = 200 kg/cm$^2$ and $\tau_{xy}$ = 60 kg/cm$^2$ and determine the magnitudes of two principal stresses $\sigma_1$ and $\sigma_2$ and the angle $\phi$ between the direction $\sigma_x$ and $\sigma_1$. (*UPTU : 2010–2011*)

**[*Ans.* $\sigma_1$ = 380 kg/cm$^2$; $\sigma_2$ = 175 kg/cm$^2$; $\phi$ = 17.5°]**

(ii) At a point in a body the normal and shear stresses on two perpendicular planes are given as $\sigma_x$ = –100 MN/m$^2$, $\sigma_y$ = 40 MN/m$^2$, $\tau_{xy}$ = 50 MN/m$^2$. Using Mohr's circle determine the principal stresses and their planes. (*UPTU : 2008–2009*)

**[*Ans.* $\sigma_{n1}$ = –116 N/mm$^2$; $\sigma_{n2}$ = 56 N/mm$^2$]**

10. (i) At a point in a material there are normal stresses of 30 N/mm$^2$ and 60 N/mm$^2$ tensile together with a shearing stress of 22.5 N/mm$^2$. Find the value of principal stresses and the inclination of the principal planes to the direction of the 60 N/mm$^2$ stress. (*UPTU : 2012–2013*)

**[*Ans.* θ = 152°; $\sigma_{n1}$ = 72.04 N/mm$^2$; 17.96 N/mm$^2$]**

11. A two dimensional state of stress is given by $\sigma_{xx}$ = 10 MPa, $\sigma_{yy}$ = 5 $\sigma_{xy}$ MPa, and $\sigma_{xy}$ = 2.5 MPa. Determine the following on a plane inclined at an angle of 30° from $x$-plane in clockwise direction.

12. Normal stress, (ii) Shear stress, (iii) Resultant stress and (iv) Principal stresses at the point. (*UPTU : 2010 – 2011*)

**[*Ans.* $\sigma_n$ = 14.04 N/mm$^2$; τ = –0.912 N/mm$^2$;**
**$\sigma_{n1}$ =14.05 MPa; $\sigma_{n2}$ = 8.55 MPa]**

13. Direct stress of 120 MN/m$^3$ in tension and 90 MN/m$^2$ in compression are applied to an elastic material at a certain point on a plane at 25° with the tensile stress. If the maximum principal stress is not to exceed 150 MN/m$^2$ in tension to what sheaving stress can the material be subjected? What is then the maximum resulting shearing stress in the material and also find the magnitude of the other principal stress and its inclination to plane of 120 MN/m$^2$ stress. (*UPTU : 2012–2013*)

**[*Ans.* $\theta_{s1}$ = 64.47°, $\theta_{s2}$ = 154°, τ = 84.85 N/mm$^2$, $\tau_{max}$ = 135 N/mm$^2$]**

14. At a point in strained material, there are normal stresses of 30 N/mm$^2$, tension and 20 N/mm$^2$, compression on two planes at right angles to one another together with shearing stresses of 15 N/mm$^2$ on the same planes. If the loading on the material is increased so that the decreases reach values of $k$ times those given, find the maximum permissible value of $k$ if

the maximum direct stress in the material is not to exceed 80 N/mm$^2$, and maximum shear stress is not to exceed 50 N/mm$^2$. (*UPTU : 2009–2010*)

**[*Ans.* $K$ = 715]**

15. An element of material in plane strain undergoes the following strains : $\varepsilon_x = 340 \times 10^{-6}$, $\varepsilon_y = 110 \times 10^{-6}$, $\gamma_{xy} = 180 \times 10^{-6}$.

    Determine (i) strain of a line inclined at an angle of 30° from $x$-axis, (ii) Principal strains and, (iii) Maximum shear strain. (*UPTU : 2011–2012*)

    **[*Ans.* $\varepsilon = 3.234 \times 10^{-4}$; $\varepsilon_{n1} = 4.386 \times 10^{-4}$, $\varepsilon_{n2} = 0.114 \times 10^{-4}$, $\gamma_{max} = 2.13 \times 10^{-4}$]**

16. Derive an expression for deformation of a conical bar hung to a ceiling having diameter $D$ and height $L$, weight density of bar $\rho$ and Young's modulus is $E$. (*UPTU : 2013–2014*)

**SECTION C**

1. A beam having T-section (Fig. 1) with flanges (180 mm × 10 mm) is subjected to sagging bending moment 15 kN-m. Determine maximum tensile stress and maximum compressive stress. (*UPTU : 2012–2013*)

   [*Ans.* $\sigma_{\in}$ = 7.56 MPa, $\sigma_c$ = 3.19 MPa)

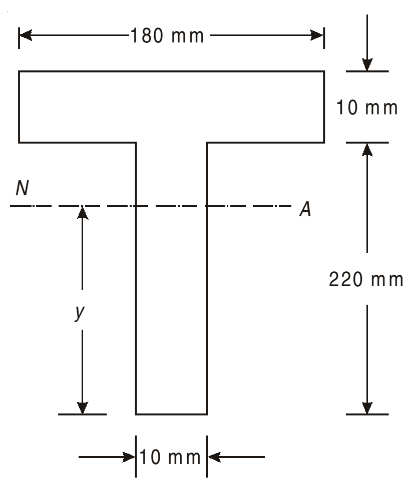



**Fig. 1**

2. A cast iron water pipe of 500 mm inside diameter and 20 mm thick is supported over a span of 10 m. Determine the maximum stress in the pipe material, when the pipe is running full. Take density of C.I. as 70.6 kN/m$^2$ and that of water as 9.8 kN/m$^3$. (*UPTU : 2005–2006*)

   **[*Ans.* $\sigma_{max}$ = 12.9 N/mm$^2$ tensile at bottom, comp. at top]**

3. A cantilever, length $L$ having square cross-section of side $a$ is subjected to transverse load of $w$ per unit length. The maximum bending stress in

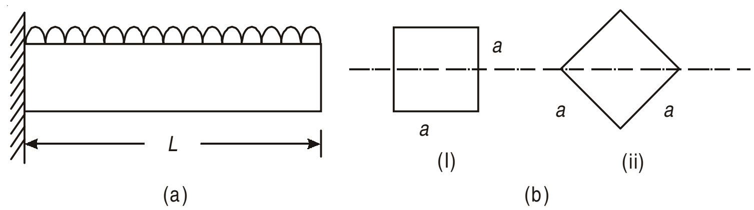



**Fig. 2**

the beam, caused for section shown in Fig. IQ. 2 (a) is $f$ find the maximum bending stress in the beam if the section is placed as shown in Fig. IQ. 2 (b).
(*UPTU : 2001–2002*)

**[*Ans.* $\sigma = \sqrt{2}f$ ]**

4. A timber joist of 6 m span has to carry a load of 15 kN/m. Find the dimensions of the joist if the maximum permissible stress is limited to 8 N/mm$^2$. The depth of the joist has to be twice the width.
(*UPTU : 2009–2010*)
**[*Ans.* $b$ = 233 mm, $d$ = 466 mm]**
5. A timber beam of 3 m span carries a uniformly distributed load of 5 kN/m and a point load 1 kN at centre of the span. If the permissible bending stress be 100 N/mm$^2$, determine the section taking depth as twice the breadth. (*UPTU : 2009 – 2010*)
[*Ans.* 46 mm × 92 mm]
6. A cantilever 2.5 m long carries a U.D.L. of 20 kN/m run. The breadth of the section remains constant and is equal to 100 mm. Determine the depth of section at middle of the length of the cantilever and at fixed end if stress remains the same throughout and equal to 120 MN/m$^2$.
(*UPTU : 2003–2004*)
**[*Ans.* $d_m$ = 88.32 mm, $d_{FE}$ = 161.4 mm]**
7. Calculate the dimensions of the strongest section, that can be cut, of a circular log of wood 25 cm in diameter. (*UPTU : 2002–2003*)
**[*Ans.* Depth = 20.4 cm, width = 14.5 cm]**
8. A timber beam of 140 mm width and 180 mm depth is reinforced by 140 mm × 10 mm steel plates at top and bottom. The beam is subjected to a bending moment of 24 kNm. Determine the maximum bending stress in the steel and wood. Given that the Young's modulus of steel and wood are 210 GPa and 15 GPa respectively. (*UPTU : 2011–2012*)
**[*Ans.* $\sigma_s$ = 79.59 MPa, $\sigma_w$ = 3.58 MPa]**
9. A flitched beam consists of two 50 mm × 200 mm wooden beam and a 12 mm × 80 mm steel plate. The plate is placed centrally between the wooden beams and recessed into each so that, when rigidly joined, the three units form a 100 mm × 200 mm section (Fig. 3). Determine the moment of

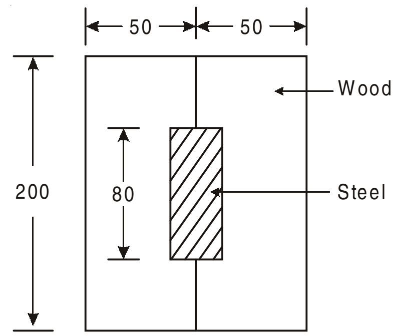



**Fig. 3**

resistance of the flitched beam when the maximum bending stress in the timber is 12 MN/m$^2$. What will then be the maximum bending stress in the steel? For steel, $E$ = 200 GPa, for wood, $E$ = 10 GPa.

(*UPTU : 2005–2006*)

**[*Ans.* M.R. = 9.1668 kN-m, $\sigma_{max.steel}$ = 96 N/mm$^2$]**

10. Deduce the formula for shear stress at the junction of flange and web in the I-section of a beam. (*UPTU : 2004–2005*)

$$\left(\textbf{\textit{Ans.}}\ \tau = \frac{\left[SB\left(D^2 - d^2\right)\right]}{8\,bI}\right)$$

11. What assumptions are made in simple theory of bending?

(*UPTU : 2005–2006*)

12. A 100 mm × 150 mm wooden bar is to be symmetrically loaded with two equal forces, $P$ (Fig. 4). Determine the position of loads and their

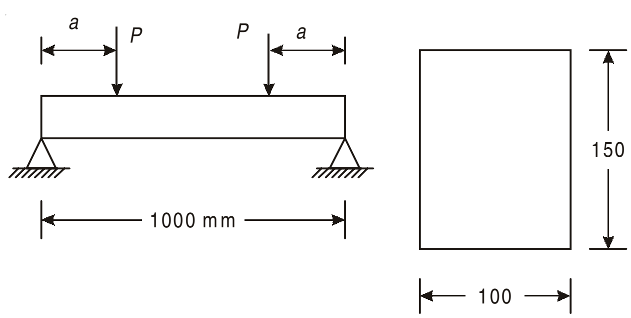



**Fig. 4**

magnitudes when a bending stress of 10 MPa and shearing stress of 2.5

MPa are just reached. Neglect the weight of the beam.

(*UPTU : 2010–2011*)

**[*Ans.* $p$ = 25 kN, $a$ = 150 mm]**

13. In a thin circular tube show that the maximum shear stress is twice the average shear stress over the cross-section. (*UPTU : 2007–2008*)

**[*Ans.* $\tau_{max} = 2\ \tau_{ave}$]**

14. What is Macaulay's method? Where it is used? Find an expression for simply supported beam with an eccentric point load, using Macaulay's method. (*UPTU : 2011–2012*)

15. (i) Explain in brief Macaulay's method. (*UPTU : 2012–2013*)

(ii) A cantilever of span $L$ carries a point load $w$ at free end. Determine the maximum slope and deflection. (*UPTU : 2012–2013*)

16. A simply supported beam with point load $w$ at a distance $a$ from support $A$. Determine slope at supports, deflection under load and also find maximum deflection. (*UPTU : 2011–2012*)

17. A beam of length $L$ and flexural rigidity $EI$ is fixed at both ends at the same level and carries a U.D.L. of $w$ intensity per unit length over whole span. Obtain expressions for maximum deflection of the beam.

(*UPTU : 2008–2009*)

18. A simply supported beam of span '$L$' carrying two equal point loads $W$ at $\dfrac{L}{4}$ from each support. Find slope at support and maximum deflection at centre. (*UPTU : 2004 – 2005*)

19. A simply supported beam subjected to a U.D.L. $w$/unit length over its entire span. Determine maximum slope and deflection.

(*UPTU : 2003–2004*)

20. A beam of uniform section, 10 m long is simply supported at the ends. It carries point loads of 150 kN and 65 kN at distances 2.5 m and 5.5 m respectively from the left end. Calculate (i) Deflection under each load, (ii) Maximum deflection. Take $E$ = 200 GN/m$^2$ and $I = 118 \times 10^{-4}$ mm$^4$.

(*UPTU : 2006–2007*)

**[*Ans.* $y_c$ = 1.1244 mm, $y_D$ = 1.2476 mm, $y_{max}$, = 1.48 mm]**

21. A beam simply supported at ends $A$ and $B$ is loaded with two point loads of 60 kN and 50 kN at distance 1 m and 3 m respectively from end $A$. Determine the position and magnitude of maximum deflection. Take $E$ = $2 \times 10^5$ N/mm$^2$ and $I$ = 8500 cm$^4$. (*UPTU : 2009–2010*)

**[*Ans.* $x$ = 1.98 m, $y_{max}$ = 5.932 mm↓)**

22. A simply supported beam of length 8 m carries two concentrated forces of magnitude 64 kN and 48 kN in downward direction at distances of 1 m and 4 m from left end. Find the deflection below the 48 kN load. Take $E$ = 210 GPa and $I = 180 \times 10^6$ mm$^4$. (*UPTU : 2011–2012*)

**[*Ans.* $y_D$ = 20.18 mm↓]**

23. Derive an expression for the slope and deflection of a simply supported beam, span $L$ carrying a U.D.L. $w$ per unit length and a point load $P$ at the mid span. Hence find slope and deflection at a point $\frac{L}{4}$ from the left support. (*UPTU : 2004–2005*)

$$\left\{ \textbf{Ans.}\ \frac{dy}{dx} = -\left[\frac{11wL^3}{384EI} + \frac{5}{192}\cdot\frac{PL^2}{EI}\right],\ y = -\frac{5}{384}\cdot\frac{wL^4}{EI} - \frac{7}{768}\cdot\frac{PL^3}{EI} \right\}$$

(ii) Simply supported beam of span $L$ is subjected to a clockwise couple $M$ at a distance $a$ from left support. Determine the slope at supports and deflection under couple.

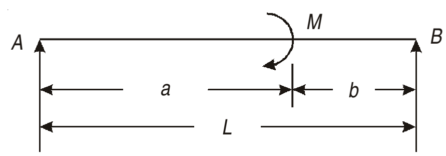



**Fig. 5**

$$\theta_A = \frac{\left[-M\left(2L^2 - 6La + 3a^2\right)\right]}{6LEI}$$

$$\theta_B = \frac{\left[-M\left(3a^2 - L^2\right)\right]}{6LEI}$$

$$y_c = \frac{\left[-Ma\left(L-a\right)\left(L-2a\right)\right]}{3LEI}$$

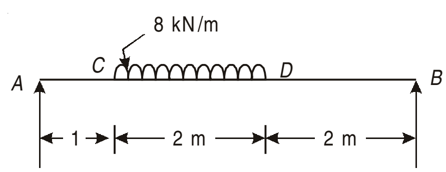



**Fig. 6**

24. A simply supported beam has a flexural rigidity of 24 MN/m$^2$ and is loaded as shown in Fig. 7. Determine deflection at mid point.

$$\frac{(y = (-36.687)}{EI}$$ (*UPTU : 2008–2009*)

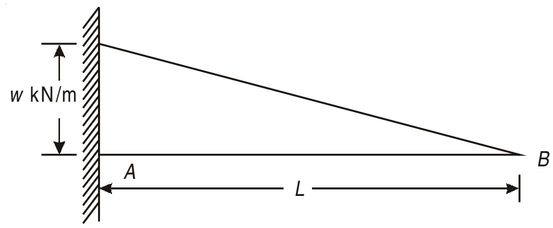



**Fig. 7**

25. A cantilever subjected to uniformly varying load in such a way that zero at free end and maximum load intensity $w$ kN/m at fixed end. Determine maximum slope and deflection. (*UPTU : 2005–2006*)

$$\left[\textbf{\textit{Ans.}}\ y_{\max} = \frac{-wL^4}{30EI};\frac{dy}{dx} = \frac{wL^3}{24EI}\right]$$

26. A beam of uniform section 9 m long is carried on three supports at the same level, one at each end and one at 6 m from the left end. A U.D.L. of 16 kN/m is carried across the whole span, and a point load of 20 kN at 4.5 m from the end. Draw the S.F. and B.M. diagrams.

(*UPTU : 2010–2011*)

27. A solid shaft rotating at 500 r.p.m. transmits 300 kW. The maximum torque is 20% more than mean torque. Material of the shaft has the allowable shear stress of 65 MPa and modulus of rigidity of 81 GPa, the angle of twist in the shaft should not to exceed 1° in 1 metre length. Determine diameter of the shaft. (Safe diameter, $D$ = 83.84 mm)

(*UPTU : 2012–2013*)

28. Find the internal and external diameters required for a hollow shaft, which is to transmit 40 kW of power at 240 rev/minute. The shear stress is to be limited to 100 MN/m$^2$. Take outside diameter to be twice the inside diameter. (*UPTU : 2009–2010*)

**[*Ans.* $d$ = 22.11, $D$ = 44.22 mm]**

29. In a shaft of hollow circular section has external diameter 100 mm and internal diameter is 60 mm. The allowable shear stress in the shaft material is 55 N/mm$^2$. Determine the angle of twist in a length of twenty times the external diameter of the shaft. (*UPTU : 2006–2007*)

**[*Ans.* $\theta$ = 1.483°]**

30. Determine the dimensions of hollow shaft with a diameter ratio of 3 : 4, which is to transmit 60 kN at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m$^2$ and the angle of twist to 3.8° in a length of 4 m. Take $G$ = 80 GPa. **[*Ans.* $d$ = 56.45 mm, $D$ = 75.32 mm]**

31. A close coiled helical spring made of 10 mm diameter steel bar has 8 coils

of 150 cmm mean diameter. Calculate the elongation, maximum shear stress and strain energy per unit volume when the spring is subjected to an axial load of 130 N. Take $G$ = 80 GPa. (*UPTU : 2011–2012*)

**[*Ans.* $\delta$ = 35.1 mm, $s$ = 3.704 N/mm, $\mu/v$ = 7.705 × $10^{-3}$ N/mm$^2$]**

32. A leaf spring has 12 plates each 50 mm wide and 5 mm thick, the longest plate being 600 mm long. The greatest bending stress is not to exceed 180 N/mm$^2$ and the central deflection is 15 mm. Estimate the magnitude of greatest central load that can be applied to the spring. Take $E$ = 0.206 × $10^6$ N/mm$^2$. (*UPTU : 2009–2010*)

**[*Ans.* $W$ = 2861.11 N]**

33. A close coiled helical spring of stiffness 100 N/m in compression with a maximum load of 45 N and a maximum shearing stress of 120 N/mm$^2$. The solid length of spring (i.e. coils touching) is 45. Find :

(i) Wire diameter, (ii) Mean coil radius, and (iii) Number of coils. Take $G$ = 0.4 × $10^5$ N/mm$^2$. **[*Ans.* $d$ = 5.61 mm, $R$ = 91.28 mm, $n$ = 8]**

34. A mild steel hollow column, having 100 mm external diameter and 60 mm internal diameter and 4 m length is used as a column. Determine the crippling load by Rankine's formula, when both ends are hinged. Take,

$\sigma_c$ = 320 N/mm$^2$ and $a = \dfrac{1}{7500}$. (*UPTU : 2006–2007*)

**[*Ans.* $P_{CR}$ = 458.36 kN]**

35. A 1.5 m long column has a circular cross-section of 5 cm diameter. One end of the column is fixed and the other end is free. Considering factor of safety as 3, find the safe load on column by using :

(a) Rankine – Gordon formula; using yield stress as 560 MN/mm$^2$ and $a = \dfrac{1}{1600}$ for pinned ends.

(b) Euler's formula; assume Young's modulus for C.I. as 120 GN/m$^2$ (*UPTU : 2012–2013*)

**[*Ans.* $P_e$ = 40.20 kN, $P_R$ = 29.71 kN, S.L. = 1.34 kN]**

36. Find the expression for crippling load for a long column when one end of the column is fixed and other end is hinged. (*UPTU : 2011–2012*)

$$\left[\textbf{\textit{Ans.}}\ P = \frac{2\pi^2 EI}{L^2}\right]$$

37. In a cylindrical shell of 0.6 m diameter and 0.9 m long is subjected to an internal pressure 1.2 N/mm$^2$. Thickness of the cylinder wall is 15 mm. Determine longitudinal stress, circumferential stress and maximum shear

stresses induced and change in diameter, length and volume. Take $E$ = 200 GPa and $\frac{1}{m}$ = 0.3. (*UPTU : 2011–2012*)

**[*Ans* $\sigma_h$ = 24 MPa, $\tau_{max}$ = 6 N/mm$^2$, $\delta d$ = 0.0612 mm, $\delta L$ = 0.0216 mm, $\delta v$ = 58.02 × 10$^3$ mm$^3$)**

38. Wall thickness of a cylindrical shell of 300 mm internal diameter is 10 mm. Length of the cylinder is 2 m. If the shell is subjected to an internal pressure of 1.5 MPa. Determine maximum shear stress induced and change in dimensions of the shell. Take $E$ = 200 GPa and $\mu$ = 0.3. (*UPTU : 2011–12*)

**[*Ans.* $\tau$ = 15 N/mm$^2$; $\delta d$ = 2.55 × 10$^{-3}$ mm; $\delta L$ = 0.12 mm, $\delta V$ = 573.03 × 10$^3$ mm$^3$]**

39. A cylindrical vessel 1.5 m in diameter, 2 m long and 1.5 cm thick is closed at both the ends by rigid plates and this cylinder is filled with water at atmospheric pressure. Find how much additional amount of water is required to be pumped so as to make the final pressure in the cylinder as 70 bar. Take $E$ = 210 GN/m$^2$ and $\mu$ = 0.3 for the material of the cylinder, Bulk modulus of the water = 2.4 GN/m$^2$. (*UPTU : 2009–2010*)

**[*Ans.* $dv$ = 215000 mm$^3$]**

40. A spherical tank is of 20 m diameter and wall thickness 15 mm. If the permissible stress in the material is 120 MPa, determine maximum pressure at which gas can be stored in the tank. Determine the increase in diameter and volume due to gas pressure. Take $E$ = 200 GPa and Poisson's Ratio = 0.3. (*UPTU : 2008–2009*)

**[*Ans.* $\delta d$ = 8.4 mm, $\delta v$ = 5.275 m$^3$]**

41. A spherical shell of 1.2 m internal diameter and 6 mm thickness is filled with water under pressure until its volume increases by 400 × 10$^3$ mm$^3$. Find the pressure exerted by water on the shell. Take $E$ = 200 GPa and $\mu$ = 0.3. (*UPTU : 2011–12*)

**[*Ans.* $P$ = 0.842 N/mm$^2$]**

42. A thin spherical shell one metre in diameter with its wall of 1.2 cm thickness is filled with a fluid at atmospheric pressure. What intensity of pressure will be developed in it if 175 cm$^2$ more fluid is pumped into it? Also calculate the circumferential stress at that pressure and increase in diameter and volume of the vessel. Take $E$ = 200 GN/m$^2$ and Poisson's ratio as 0.3. (*UPTU : 2012–2013*)

**[*Ans.* $p$ = 6.55 N/mm$^2$. $\sigma_c$ = 136.38 MPa; $\delta d$ = 0.478 mm; $dv$ = 750.22 mm$^3$]**

43. The maximum stress permitted in a thick cylinder of inner and outer radius of 10 cm and 15 cm is 20 N/mm$^2$. The external pressure is 8 N/mm$^2$, what internal pressure can be applied? (*UPTU : 2012–2013*)

**[*Ans.* $p_x$ = 17.38 N/mm$^2$]**

44. A hollow cylinder of 45 cm internal diameter and 10 cm thickness contains the fluid under pressure of 850 N/cm$^2$. Find the maximum and minimum hoop stress across the section. (*UPTU : 2006–2007*)

**[*Ans.* $\sigma_h$ = 24.144 N/mm$^2$, $\sigma_c$ = 15.64 N/mm$^2$]**

45. A compound cylinder is to be made by shrinking one tube on to another so that the radial compressive stress at the friction is 28.5 N/mm$^2$. If outside diameter is 26.5 cm, and the bore 12.5 cm, calculate the allowance for shrinkage at common diameter, which is 20 cm. Take $E = 210 \times 10^3$ N/mm$^2$. (*UPTU : 2011–2012, 2013–2014*)

**[*Ans.* Shrinkage allowance = 0.0804]**

46. For a tube having $E = 2 \times 10^5$ N/mm$^2$ and $\upsilon = 0.3$, the hoop stress at the inner face is twice the internal pressure. Find the thickness of the wall if internal radius is 60 cm. (*UPTU : 2009–2010*)

**[*Ans.* $t$ = 44 cm]**

47. Write short notes on :
(i) Assumptions for the theory of curved beams
(ii) Application of curved beams with large initial curvature
(iii) In crane hooks, trapezoidal section is very commonly used.

48. A curved beam rectangular in cross-section is subjected to pure bending with copule of 40 kN-cm. The beam has width of 2 cm and depth 4 cm and is curved in plane parallel to width. The mean radius of curvature is 5 cm. Find the position of the neutral axis, and the ratio of the maximum to minimum stress.

**[*Ans.* $\sigma_{max}$ = –102.765 N/mm$^2$, $\sigma_{min}$ = 58.328 N/mm$^2$, $\dfrac{\sigma_{max}}{\sigma_{min}}$ = 1.762]**

49. A ring made of 20 mm diameter steel bar has a mean radius of 160 mm. Two loads of 5 kN each are applied along a diameter of the ring. Determine maximum stress in the ring. (*UPTU : 2011–2012*)

**[*Ans.* Maximum stress = 202.08 MPa]**

50. Wrire short notes on :
(i) Shear centre, (ii) Unsymmetrical bending, and (iii) Bending stress in beam subjected to unsymmetrical bending. (*UPTU : 2010–2011, 2011–2012*)

51. Calculate the shear centre for the section shown in Fig. 8. (*UPTU : 2013–2014*)

**[*Ans.* $e$ = 37.97 mm]**

52. A simply supported I-section beam of span 2 m carries a concentrated load of 4 kN at an angle of 20° from the vertical (as shown in Fig. 9. The load passes through C.G. of the section. Determine the maximum bending stress in the beam. **[*Ans.* $\sigma_{max}$ = 42.85 N/mm$^2$ comp.]**

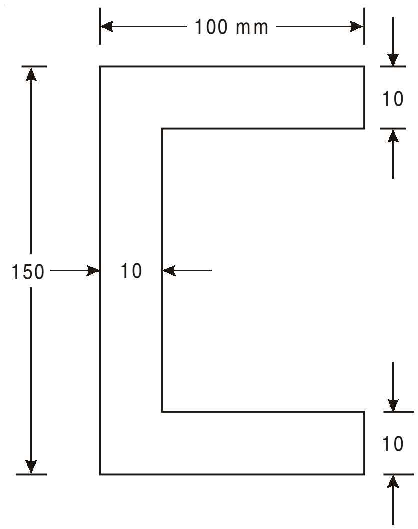



**Fig. 8**

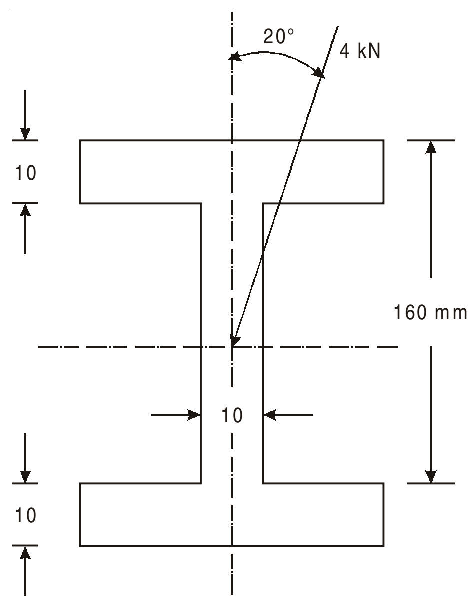



**Fig. 9**

53. Draw S.F. and B.M. Diagrams for the cantilever and simple supported beam shown in Fig. 10 and 11.

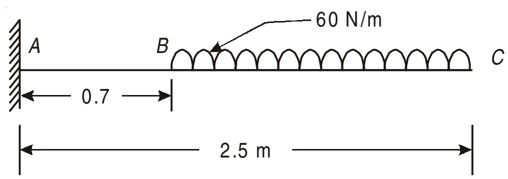



**Fig. 10**

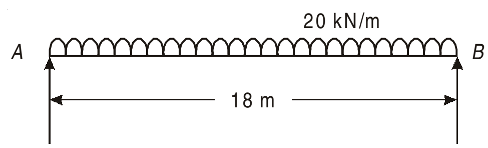



**Fig. 11**

54. An overhanging beam is shown in Fig. 12. Draw S.F. and B.M. diagrams and find point of centraflexure.

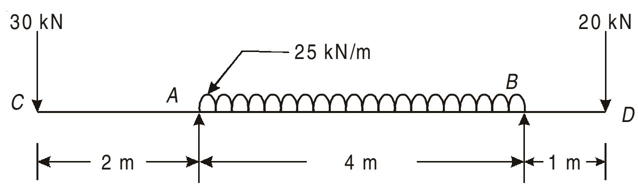



**Fig. 12**

55. Draw S.F. and B.M. diagrams and find the position of point of contraflexure for the overhanging beam shown in Fig. 13.

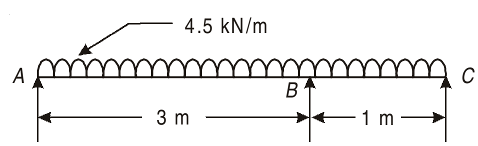



**Fig. 13**

56. Derive the equation to find the position of neutral axis for the following cross-sections of curved beam :

(i) Rectangular $x$-section, and (ii) Circular $x$-section (*UPTU : 2013–2014*)

57. A boiler drum consists of a cylinder 2 m long, 1 m diameter and 25 mm thick closed by hemisherical ends. In a hydraulic test 10 N/mm$^2$, how much additional water will be pumped in after initial filling at atmospheric pressure? Assume the circumferential strain at junction of cylinder and

hemisphere is same for both drum material. Take $E$ = 207000 N/mm$^2$, $\mu$ = 0.3, $w$ = 2100 N/mm$^2$. (*UPTU : 2002–2003, 2013–2014*)

58. Write the assumptions for pure bending and also derive the bending equation.

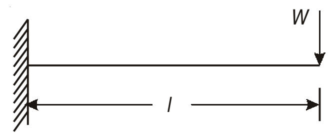



**Fig. 14**

59. Find the deflection of a cantilever of length, $l$ at free end by area moment method. (*UPTU : 2013–2014*)
60. What is leaf spring? Find maximum deflection and maximum bending stress in semi-elliptical type leaf spring (*UPTU : 2013–2014*)
61. Write the assumptions for Lami's equation and also derive the expression for Lami's equation. (*UPTU : 2013–2014*)

    What do you understand by unsymmetrical bending? Prove that the sum of moment of inertia about any rectangular axis is constant.
62. Draw the shear force and bending moment diagram for the beam given below.

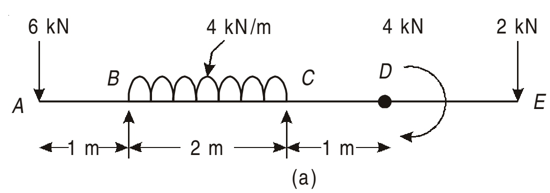



**Fig. 15**

63. Find deflection at point $B$ and $C$ of beam given below. $E$ = 200 GPa, $I$ = 19802.8 cm$^4$.

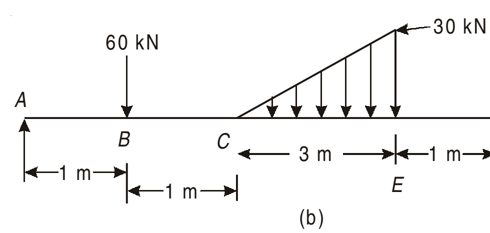



**Fig. 16**

64. Deduce an expression for the extension of an open coiled helical spring carrying an axial load $W$. Take $\alpha$ as the inclination of coils, $d$ as the diameter of the wire and $R$ as mass radius of the coil. Find by what percentage the

axial deflection of the coil is neglected for spring in which $\alpha = 25°$. Assume $n$ and $R$ remain constant.

65. Write the assumption for Euler's Theory and derive the expression for critical load for a column having both end fixed.
66. A hollow C.I. column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both end. Calculate the safe load of Rankine formula using

    Factor of Safety 4, $\sigma_c = 550$ MPa, $\alpha = \dfrac{1}{1600}$.

# University Questions (UPTU)
# B. Tech.
## (SEM. III) ODD SEMESTER THEORY EXAMINATION 2013–14
## STRENGTH OF MATERIALS

*Time : 3 Hours* *Total Marks : 100*

### SECTION A

Attempt all questions :

1. (i) Define principal plane and principal stress.
   (ii) Explain the complementary shear stress.
   (iii) Draw the Mohr's circle for pure shear.
   (iv) Define neutral axis.
   (v) What do you understand by section modulus?
   (vi) Explain point of contraflexure in a beam.
   (vii) What do you understand by effective length of the column?
   (viii) Explain Torsional stiffness and Torsional flexibility.
   (ix) Differentiate between thin cylinder and thick cylinder.
   (x) Define shear centre.

### SECTION B

Attempt any three questions :

1. Derive an expression for deformation of conical bar hung to a ceiling having diameter $D$ and height $L$, weight density of bar $p$ and Young's modulus is $E$.

2. (a) Write the assumption for pure bending and also derive the bending equation.
   (b) Find the deflection of cantilever of $l$ at free end by Area Moment Method.

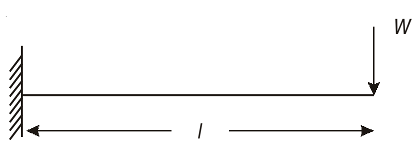



**Fig. 1**

3. What are leaf springs? Find maximum deflection and maximum bending stress in semi-elliptical type leaf spring.

4. Write the assumptions for Lame's equation and also derive the expression for Lame's equation.

5. What do you understand by unsymmetrical bending? Prove that the sum of moment of inertia about any rectangular axis is constant.

**SECTION C**

Attempt all questions :

1. Show that if $E$ is assumed correct an error of 1% in the determination of $G$ will involve an error of about 5% in the calculation of Poisson's ratio when its correct value is 0.25.

*Or*

A point in a strained material is subjected to a tensile stress 65 N/mm$^2$ and compressive stress of 45 N/mm$^2$, acting on two mutually perpendicular planes and shear stress of 10 N/mm$^2$ are acting on these planes. Find the normal stress, tangential stress and resultant stresses on a plane inclined 30° with the plane of compressive stress.

2. (a) Draw the shear force and bending moment diagram for the beam given below.

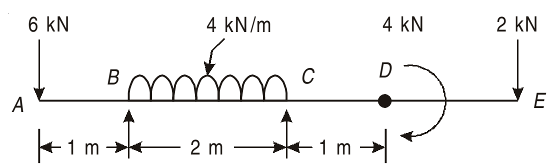



**Fig. 2 (a)**

*Or*

(b) Find deflection at point $B$ and $C$ of beam given below. $E$ = 200 GPa, $I$ =19802.8 cm$^4$.

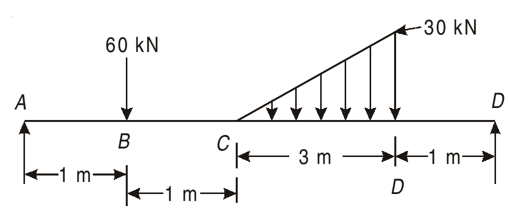



**Fig. 2 (b)**

3. Deduce an expression for the extension of an open coiled helical spring carrying an axial load $W$. Take '$a$' as the inclination of coils, $d$ as the diameter of the wire and '$R$' as mass radius of the coil. Find by what percentage the axial deflection of the coil is neglected for spring in which $a$ = 25°. Assume $n$ and $R$ remain constant.

*Or*

(i) Write the assumptions for Euler's Theory and derive the expression for critical load for column having both end fixed.

(ii) A hollow C.I. column whose outside diameter is 200 mm has a thickness of 20 mm. It is 4.5 m long and is fixed at both end. Calculate the safe load of Rankine formula using

Factor of safety 4, $\sigma_c = 550$ MPa, $\alpha = \dfrac{1}{1600}$.

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Assume the circumferential strain at junction of cylinder and hemisphere is same for both drum material.

$E = 207000$ N/mm$^2$, $\mu = 0.3$, $W = 2100$ N/mm$^2$.

*Or*

A compound cylinder is to be made by shrinking one tube on to another so that the radial compressive stress at the junction is 28.5 N/mm$^2$. If the outside diameter is 26.5 cm and the bore 12.5 cm, calculate the allowance for shrinkage at common diameter which is 20 cm. $E = 210000$ N/mm$^2$.

5. Locate the shear centre with sketch for the section as shown in Fig. 4.

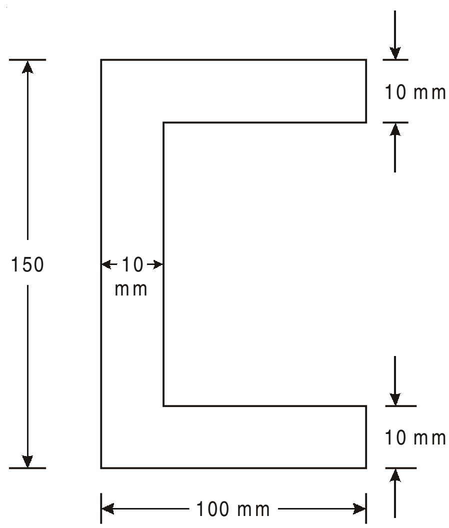


**Fig. 4**

*Or*

Derive the equation to find the position of neutral axis for the following cross-section of curved beam :

(i) Rectangular $X$-section

(ii) Circular $X$-scction.

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