Unit - 2

Fourier Series

* Periodic functions
* Dirichlet's condition

Q) Obtain the Fourier series for the function x^2 in $(-\pi, \pi)$ & hence deduce that

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(1)}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{3\pi}\left\{\pi^3 - (-\pi)^3\right\} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left\{x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n^2}\left[x\cos nx\right]_{-\pi}^{\pi} \qquad \sin n\pi = 0$$

$$= \frac{2}{\pi n^2}\left[\pi\cos n\pi - (-\pi)\cos n\pi\right] \qquad \cos(-n\pi) = \cos n\pi$$

$$a_n = \frac{2}{\cancel{\pi} n^2} \times 2\cancel{\pi}\cos n\pi = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left\{-\frac{1}{n}\left[\pi^2\cos n\pi - \pi^2\cos n\pi\right] + 0 + \frac{2}{n^3}\left[\cos n\pi - \cos n\pi\right]\right\}$$

$$b_n = 0$$

Substitute the values a_0, a_n & b_n in (1),

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx \quad \text{—— (2)}$$

Putting $x=0$ in (2), we get

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos 0$$

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \qquad \because \cos 0 = f(0) = 1$$

$$-\frac{\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Putting $x=\pi$ in (2), we get

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos n\pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} \qquad \because f(\pi) = \pi^2$$

$\cos n\pi = (-1)^n$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad (-1)^n\cdot(-1)^n = (-1)^{2n} = 1$$

$$\frac{2\pi^2}{3} = 4\sum \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

8) $f(x) = e^{-x}$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$

$2L = 2\pi \Rightarrow L = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\left[e^{-x}\right]_{-\pi}^{\pi} = \frac{e^{-\pi} - e^{\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1^2+n^2}(-1\cos nx + n\,\cancel{\cos}\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi + n\,\cancel{\sin n\pi}^{0}) - e^{\pi}(-\cos n(-\pi) + n\,\cancel{\sin n(-\pi)}^{0})\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi) + e^{\pi}\cos n\pi\right]$$

$$= \frac{\cos n\pi}{\pi(n^2+1)}\left[-e^{-\pi} + e^{\pi}\right] \quad \because e^{\pi}-e^{-\pi} = 2\sinh\pi$$

$$= 2\cos\frac{\cancel{2}(-1)^n}{\pi(n^2+1)} \quad \text{OR} = \frac{2(-1)^n}{\pi(n^2+1)}(e^{\pi}-e^{-\pi})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\sin nx\,dx \qquad \int e^{ax}\sin nx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin nx - b\cos nx\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-1\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{(1+n^2)\pi}\left[e^{-\pi}(-\cancel{\sin n\pi}^{0} - n\cos n\pi) - e^{\pi}(-\cancel{\sin n(-\pi)}^{0} - n\cos n\pi)\right]$$

$$\because \sin n\pi = 0$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{-\pi}\cos n\pi + ne^{\pi}\cos n\pi\right]$$

$$= \frac{n\cos n\pi}{\pi(1+n^2)}(e^{\pi}-e^{-\pi}) \qquad \because \cos n\pi = (-1)^n$$

$$b_n = \frac{n(-1)^n}{\pi(1+n^2)}(2\sinh\pi)$$

sub in (1), we get

$$f(x) = \frac{e^{\pi}-e^{-\pi}}{2} + \sum \frac{2(-1)^n}{\pi(n^2+1)}\sinh\pi\cos nx +$$

$$\sum \frac{2n(-1)^n}{\pi(1+n^2)}\sinh\pi\cdot\sin nx$$

Q) Obtain the F.S of $f(x)=\frac{\pi-x}{2}$ in $0<x<2\pi$. Hence deduce that

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$

$(0,2\pi) \Rightarrow 2L=2\pi \Rightarrow L=\pi$

$f(x)=\frac{a_0}{2}+\sum a_n \cos nx+\sum b_n \sin nx$ —— (1)

$a_0=\frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$

$=\frac{1}{\pi}\int_{0}^{2\pi}\left(\frac{\pi-x}{2}\right)dx$

$=\frac{1}{2\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{2\pi}=\frac{1}{2\pi}\left[\left(2\pi^2-\frac{4\pi^2}{2}\right)-0\right]=0$

$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)\cos nx\,dx$

$=\frac{1}{2\pi}\left\{(\pi-x)\cancel{\left(\frac{\sin nx}{n}\right)}^{0}-(-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{2\pi}$

$\because \sin 2n\pi=0=\sin 0$

$=\frac{1}{2\pi}\left\{-\frac{\cos nx}{n^2}\right\}_0^{2\pi}$

$\because \cos 2n\pi=1=\cos 0$

$a_n=\frac{-1}{2\pi n^2}\left[\cos 2n\pi-\cos 0\right]=0$

$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)\sin nx\,dx$

$=\frac{1}{2\pi}\left\{(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\cancel{\left(-\frac{\sin nx}{n^2}\right)}^{0}\right\}_0^{2\pi}$

$=\frac{1}{2\pi n}\left[-(\pi-x)\cos nx\right]_0^{2\pi}$

$=-\frac{1}{2\pi n}\left[-\pi\cos 2n\pi-(\pi\cos 0)\right]=+\frac{2\pi}{2\pi n}=\frac{1}{n}$

Sub in (1), we get

$f(x)=0+\sum 0.\cos nx+\sum\frac{1}{n}\sin nx$ —— (2)

$x=\frac{\pi}{2}$

$\frac{\pi}{4}=\sum\frac{1}{n}\sin\frac{n\pi}{2}$

$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\cdots$

(3)

Q) Obtain the Fourier series to represent

$f(x) = x - x^2$ in $-1 < x < 1$ $(-1, 1)$

$2L = 2 \Rightarrow L = 1$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + \sum b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \quad \text{--- (1)}$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = -\frac{2}{3}.$$

$$a_n = \int_{-1}^{1} f(x) \cdot \cos n\pi x$$

$$= \int_{-1}^{1} (x - x^2) \cos n\pi x\,dx$$

$$= \left\{(x - x^2)\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)}^{\to 0} - (1 - 2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\cancel{\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)}^{\to 0}\right\}_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\Big[(1 - 2x)\cos n\pi x\Big]_{-1}^{1}$$

$$= \frac{-4\cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_{-1}^{1} f(x) \sin n\pi x\,dx$$

$$= \int_{-1}^{1} (x - x^2) \sin n\pi x\,dx.$$

$$= \left\{(x - x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1 - 2x)\cancel{\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)}^{\to 0} + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right\}_{-1}^{1}$$

$$= \frac{-1}{n\pi}\left\{0 - (-2\cos n\pi)\right\} - \frac{2}{n^3\pi^3}(\cos n\pi - \cos n\pi)$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = -\frac{1}{3} - \frac{4}{\pi^2}\sum \frac{(-1)^n}{n^2\pi^2} \cos n\pi x + \frac{2}{n\pi}\sum \frac{(-1)^{n+1}}{n} \sin n\pi x$$

~~Q) Draw the graph of the fn~~

$\cancel{f(x) = \left\{ \pi x \quad \text{in } 0 < x < 1 \right.}$

Q) Obtain the F.S for $f(x) = e^{-x}$ in $(0, 2)$

$$2L = 2 \Rightarrow L = 1$$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \quad \text{—} (1)$$

$$a_0 = \frac{1}{L} \int_0^2 f(x)\,dx$$

$$= \int_0^2 e^{-x}\,dx = -e^{-x}\Big]_0^2 = -(e^{-2} - 1) = 1 - \frac{1}{e^2} = \frac{e^2-1}{e}$$

$$a_n = \frac{1}{L} \int_0^2 f(x) \cos n\pi x\,dx$$

$$= \int_0^2 e^{-x} \cos n\pi x\,dx = \frac{e^{-x}}{1^2 + n^2\pi^2} \Big[-\cos n\pi x + \underbrace{\cancel{n\pi \sin n\pi x}}_{\to 0} \Big]_0^2$$

$$a_n = \frac{-1}{1+n^2\pi^2} \left\{ e^{-2} \cos 2n\pi - 1 \right\} = \frac{e^2 - 1}{e^2(1+n^2\pi^2)}$$

$$b_n = \frac{1}{L} \int_0^2 f(x) \sin n\pi x\,dx$$

$$= \int_0^2 e^{-x} \sin n\pi x\,dx$$

$$= \frac{e^{-x}}{1+n^2\pi^2} \Big[-\sin n\pi x - n\pi \cos n\pi x \Big]_0^2$$

$$= \frac{-n\pi}{1+n^2\pi^2} \Big[e^{-x} \cos n\pi x \Big]_0^2 = \frac{-n\pi}{1+n^2\pi^2} \left(\frac{1}{e^2} - 1 \right)$$

$$b_n = \frac{n\pi(e^2 - 1)}{e^2(1+n^2\pi^2)}$$

Sub. in (1), we get

$$f(x) = \frac{e^2-1}{2e} + \sum \frac{e^2-1}{e^2(1+n^2\pi^2)} \cos n\pi x + \sum \frac{n\pi(e^2-1)}{e^2(1+n^2\pi^2)} \sin n\pi x \quad //$$

$= \frac{1}{\pi} f$

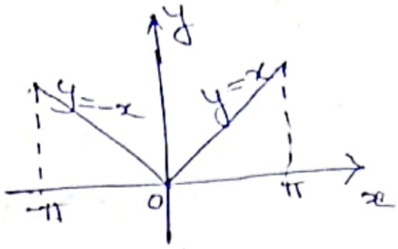
Ex:- An alternating current after passing through a rectifier has the form

$$I = \begin{cases} I_0 \sin\theta & \text{for } 0 < \theta \le \pi \\ 0 & \text{for } \pi < \theta \le 2\pi \end{cases}$$

where I_0 is the maximum current. Express I as a Fourier series in $(0, 2\pi)$

Even & odd f^n

Q) Sketch the graph of the function $f(x) = |x|$ in $-\pi \le x \le \pi$ & obtain its Fourier series. Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$



$f(x) = |x|$

$f(-x) = |-x| = |x|$ in $(-\pi, \pi)$

$\therefore f(x)$ is an even f^n

$\therefore \boxed{b_n = 0}$

$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$ —— (1)

$a_0 = \frac{2}{\pi} \int_0^\pi f(x)\,dx = \frac{2}{\pi} \int_0^\pi |x|$

$= \frac{2}{\pi} \int_0^\pi x\,dx = \frac{\cancel{2}}{\pi} \left[\frac{x^2}{\cancel{2}} \right]_0^\pi = \frac{[\pi^2 - 0]}{\pi} = \pi$

$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx\,dx$

$= \frac{2}{\pi} \int_0^\pi x \cdot \cos nx\,dx$

$= \frac{2}{\pi} \left\{ \cancel{x\left(\frac{\sin nx}{n}\right)} - 1\left(-\frac{\cos nx}{n^2}\right) \right\}_0^\pi$

$= \frac{2}{\pi n^2} [\cos n\pi - \cos 0] = \frac{2}{\pi n^2} [(-1)^n - 1]$

sub in (1), we get

$f(x) = \frac{\pi}{2} + \sum \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx$ —— (2)

put $x = 0$

$0 = \frac{\pi}{2} + \frac{2}{\pi} \sum \frac{(-1)^n - 1}{n^2}$

$1-(-1)^n = 0$, if n is even
$= 2$ if n is odd

$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum \frac{2}{n^2}$ if n is odd

$\frac{\pi^2}{4} = 2\sum \frac{1}{n^2}$

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Q) Obtain the Fourier Series expansion of the function $f(x) = \begin{cases} x & \text{in } 0 < x < \pi \\ x - 2\pi & \text{in } \pi < x < 2\pi \end{cases}$

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

$f(2\pi - x) = 2\pi - x = -(x - 2\pi) = -f(x)$.

$\therefore f(x)$ is odd.

$a_0 = a_n = 0$

$f(x) = \sum b_n \sin nx$ — (1)

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx$

$= \frac{2}{\pi}\int_0^{\pi} x \sin nx$

$= \frac{2}{\pi}\left\{ x\left(\frac{-\cos nx}{n}\right) - 1\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{0} \right\}_0^{\pi}$

$= \frac{-2}{\pi n}\left[\pi \cos n\pi - 0\right] = \frac{-2(-1)^n}{n}$

$f(x) = \sum \frac{-2(-1)^n}{n}\sin nx$ — (2)

put $x = \frac{\pi}{2}$

$\frac{\pi}{2} = -2\sum \frac{(-1)^n}{n}\sin\frac{n\pi}{2}$

$-\frac{\pi}{4} = \frac{-1}{1} + \frac{1}{3} - \frac{1}{5}$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

Q) $f(x) = |x|$ in $(-l, l)$

Q) $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2-x) & \text{in } 1 \le x \le 2 \end{cases}$

Half Range series

Q) Expand $f(x)=2x-1$ as a cosine half range Fourier series in $0<x<1$

$l=1$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(n\pi x) \quad \text{—} (1)$$

$$a_0=\frac{2}{L}\int_0^L f(x)\,dx$$

$$=2\int_0^1 (2x-1)\,dx$$

$$=2\left[\frac{\cancel{2}x^2}{\cancel{2}}-x\right]_0^1=0$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$=2\int_0^1 (2x-1)\cos n\pi x\,dx$$

$$=2\left\{(2x-1)\cancel{\left[\frac{\sin n\pi x}{n\pi}\right]}^{\to 0}-(2)\left[\frac{-\cos n\pi x}{n^2\pi^2}\right]\right\}_0^1$$

$$=\frac{4}{n^2\pi^2}\left[\cos n\pi-\cos 0\right]=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]$$

sub in (1)

$$f(x)=\sum\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]\cdot\cos n\pi x$$

Q) Obtain the sine half range series of

$$f(x)=\begin{cases}\frac{1}{4}-x & \text{in } 0<x<\frac{1}{2}\\ x-\frac{3}{4} & \text{in } \frac{1}{2}<x<1\end{cases}$$

$L=1$

$$f(x)=\cancel{\frac{a_0}{2}+}\sum b_n\sin n\pi x \quad \text{—} (1)$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin n\pi x\,dx$$

$$=2\int_0^1 f(x)\sin n\pi x\,dx$$

$$=2\left\{\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx+\int_{1/2}^1\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right\}$$

$$= 2\left\{\left[\left(\frac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right)-(-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2}+\left[\left(x-\frac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right)\right.\right.$$

$$\left.\left.+1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}\right\}$$

$$= 2\left\{-\frac{1}{n\pi}\left(-\frac{1}{4}\cos\frac{n\pi}{2}-\frac{1}{4}\right)-\frac{1}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)-\frac{1}{n\pi}\left(\frac{1}{4}\cos n\pi+\frac{1}{4}\cos\frac{n\pi}{2}\right)\right.$$

$$\left.+\frac{1}{n^2\pi^2}\left(0-\sin\frac{n\pi}{2}\right)\right\}$$

$$= 2\left\{\frac{1}{4n\pi}\left(\cos\frac{n\pi}{2}+1-\cos n\pi-\cos\frac{n\pi}{2}\right)-\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= 2\left\{\frac{1}{4n\pi}(1-\cos n\pi)-\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$b_n = \frac{1}{2n\pi}\{1-(-1)^n\}-\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\{1-(-1)^n\}-\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x$$

Q) Find If $f(x) = \begin{cases} x & \text{in } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{in } \frac{\pi}{2} < x < \pi \end{cases}$ S.T

i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$

ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$

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Practical Harmonic Analysis

Q) The turning moment T on the crankshaft of a steam engine for the crank angle θ' is given as follows

θ°	0	30	60	90	120	150	180	210	240	270	300	330
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

Expand T as a Fourier series upto to First harmonic

— $0 \le \theta < 2\pi$ $2L = 2\pi \Rightarrow L = \pi$

$N = 12$

θ	T	cos θ	T cos θ	Sin θ	T Sin θ
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	−0.5	−4.05	0.866	7.0146
150	8.3	−0.866	−7.1878	0.5	4.15
180	7.9	−1	−7.9	0	0
210	6.8	−0.866	−5.8888	−0.5	−3.4
240	5.5	−0.5	−2.75	−0.866	−4.763
270	4.1	0	0	−1	−4.1
300	2.6	0.5	1.3	−0.866	−2.2516
360	1.2	0.866	1.0392	−0.5	−0.6
Total	59.4		−20.4992		8.9032

$a_0 = \frac{1}{6}\sum T = \frac{1}{6}(59.4) = 9.9$

$a_1 = \frac{1}{6}\sum T\cos\theta = \frac{1}{6}(-20.4992) = -3.4165$

$b_1 = \frac{1}{6}\sum T\sin\theta = \frac{1}{6}(8.9032) = 1.4839$

$\therefore T = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta)$

$T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta$

8) Express y as a Fourier series upto to the third harmonic given the following values.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

$\Rightarrow \; 0 \le x < 6 \Rightarrow [0,6)$

$2L = 6 \Rightarrow L = 3 \;, \; N = 6$

$$f(y) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

x	y	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$	$y\cos\pi x$	$y\sin\pi x$
0	4	4	0	4	0	4	0
1	8	4	6.928	−4	6.928	−8	0
2	15	−7.5	12.99	−7.5	−12.99	15	0
3	7	−7	0	7	0	−7	0
4	6	−3	−5.196	−3	5.196	6	0
5	2	1	−1.732	−1	−1.732	−2	0
	42	−8.5	12.99	−4.5	−2.598	+8	0

$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(42) = 14$

$a_1 = \frac{2}{N}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-8.5) = -2.833$

$b_1 = \frac{2}{N}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}(12.99) = 4.33$

$a_2 = \frac{2}{N}\sum y\cos\frac{2\pi x}{3} = \frac{2}{6}(-4.5) = -1.5$

$b_2 = \frac{2}{N}\sum y\sin\frac{2\pi x}{3} = \frac{2}{6}(-2.598) = -0.866$

$a_3 = \frac{2}{N}\sum y\cos\pi x = \frac{2}{6}(8) = 2.667 \;, \; b_3 = 0$

$$y = 7 + \left(-2.833\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3}\right) + \left(-1.5\cos\frac{2\pi x}{3} - 0.866\sin\frac{2\pi x}{3}\right) + \left(2.667\cos\pi x\right)$$

Q) Obtain the constant term & the first three co-efficient in the fourier cosine series for y using the foll. table

x	0	1	2	3	4	5
y	4	8	15	7	6	2

— $0 \le x < 6 \Rightarrow [0,6) \Rightarrow 2L = 6 \Rightarrow \cancel{L=3}$

$N = 6$

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{6} + a_2 \cos\frac{2\pi x}{6} + a_3 \cos\frac{3\pi x}{6} \quad \text{—(1)}$$

x	y	$y\cos\frac{\pi x}{6}$	$y\cos\frac{\pi x}{3}$	$y\cos\frac{\pi x}{2}$
0	4	4	4	4
1	8	6.928	4	0
2	15	7.5	−7.5	−15
3	7	0	−7	0
4	6	−3	−3	6
5	2	−1.732	1	0
Total	42	13.696	−8.5	−5

$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(42) = 14$

$a_1 = \frac{2}{N}\sum y\cos\frac{\pi x}{6} = \frac{2}{6}(13.696) = 4.565$

$a_2 = \frac{2}{N}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-8.5) = -2.833$

$a_3 = \frac{2}{N}\sum y\cos\frac{\pi x}{2} = \frac{2}{6}(-5) = -1.667$

$$y = 7 + 4.565\cos\frac{\pi x}{6} + (-2.833)\cos\frac{\pi x}{3} - 1.667\cos\frac{\pi x}{2}$$

Q) The following table give the variation of a periodic current A over a period T

t sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Show that there is a constant part of 0.75 amp in the current A & also obtain the amplitude of the first harmonic

t	A	$A\cos\frac{2\pi t}{T}$	$A\sin\frac{2\pi t}{T}$
0	1.98	1.98	0
$T/6$	1.3	0.65	1.125
$T/3$	1.05	−0.52	1.29
$T/2$	1.3	−1.30	0
$2T/3$	−0.88	0.44	0.76
$5T/6$	−0.25	−0.125	0.216
~~T~~			
	4.5	1.125	3.391

$\Rightarrow$ ~~$x \in$~~ $0 \le x \le T$ $\quad N=6$

$2L = T \Rightarrow L = T/2$

$y\ A = \frac{a_0}{2} + a_1\cos\frac{\pi t}{L} + b_1\sin\frac{\pi t}{L}$

$A = \frac{a_0}{2} + a_1\cos\frac{2\pi t}{T} + b_1\sin\frac{2\pi t}{T}$

$a_0 = \frac{2}{N}\sum A = \frac{2}{6}(4.5) = 1.5$

$a_1 = \frac{2}{N}\sum A\cos\frac{2\pi t}{T} = \frac{2}{6}(1.125) = 0.375$

$b_1 = \frac{2}{N}\sum A\sin\frac{2\pi t}{T} = \frac{2}{6}(3.391) = 1.13$

$A = 0.75 + \left(0.375\cos\frac{2\pi t}{T} + 1.13\sin\frac{2\pi t}{T}\right)$

$\therefore$ Direct current $= 0.75$

Amplitude $= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.375)^2 + (1.13)^2} = 1.19$