ECE-GY 6263 FINAL PROJECT: STABILITY OF EVOLUTIONARY POPULATION GAMES

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POPULATION GAMES

Define a **population game** with continuous player sets:[1,2]

- $\mathcal{P} = \{1, ..., \bar{p}\}$, set of $\bar{p}$ populations, where $\bar{p} \geq 1$
- $m^p \in \mathbb{Z}$, mass of population p
  - $m = \sum_{p \in \mathcal{P}} m^p$, total mass of all populations
- $\mathcal{S}^p = \{1, ..., n^p\}$, strategy set for member of population p
  - $n = \sum_{p \in \mathcal{P}} n^p$, total number of pure strategies in all populations
- $\Delta^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i \in \mathcal{S}^p} x_i^p = 1\}$, set of probability distributions over strategies in $\mathcal{S}^p$
  - $X^p = m^p \Delta^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i \in \mathcal{S}^p} x_i^p = m^p\}$, set of strategy distributions for population p
  - $X = \{x = (x^1, ..., x^{\bar{p}}) \in \mathbb{R}_+^n : x^p \in X^p\}$, set of overall strategy distributions

POPULATION GAMES

- $\bar{X} = \{x \in \mathbb{R}^n_+ : m^p - \varepsilon \leq \sum_i x^p_i \leq m^p + \varepsilon \forall p \in \mathcal{P}\}$, ε positive constant.
  - payoffs are defined on $\bar{X}$ where population masses vary slightly
- $F^p_i : \bar{X} \to \mathbb{R}$, payoff function for strategy $i \in \mathcal{S}^p$
  - F^p_i continuously differentiable
  - $F^p : \bar{X} \to \mathbb{R}^{n^p}$, payoff functions of strategies for population p
- $F : \bar{X} \to \mathbb{R}^n$, payoff vector field defining a **population game**

POPULATION GAMES, PERTURBED BEST RESPONSE

Define the **choice probability function**,[1] $\mathcal{C}^p : \mathbb{R}^{n^p} \to \Delta^p$:

- $\mathcal{C}_i^p(\pi^p) = \nu^p(\varepsilon^p : i \in \arg_{k \in \mathcal{S}^p} \max \pi_j^p + \varepsilon_j^p)$
- π^p, base payoff vector
- probability payoff perturbation leads to choosing strategy p

Define an **admissible distribution**, ν^p, as one that:

- admits a strictly positive density on $\mathbb{R}^{n^p}$
- is smooth enough that $\mathcal{C}^p$ is continuously differentiable

Then, define the **perturbed best response function**:

- $\tilde{B}^p : X \to \Delta^p$, for the pair (F, ν) by composition $\tilde{B}^p = \mathcal{C}^p \circ F^p$
- $\nu = (\nu^1, \ldots, \nu^{\bar{p}})$ admissible where each component is admissible

Finally, define the **perturbed best response dynamic** for (F, ν):

- $\dot{x}^p = m^p \tilde{B}^p(x) - x^p \quad \forall p \in \mathcal{P}$
- $x \in X$ is **perturbed equilibrium** if fixed point

POPULATION GAMES, RANDOMLY DISTRIBUTED PAYOFFS

Describe a **model of evolution with randomly distributed payoffs**:[1]

- players: members of $\bar{p}$ finite populations of sizes $(Nm^1, \ldots, Nm^{\bar{p}})$
- players recurrently play population game F

Consider **continuous time Markov chain** describing aggregate behavior in the model, as:

- $\{X_t^N\}_{t \geq 0}$, with state space $\mathcal{X}^N = \{x \in X : Nx \in \mathbb{Z}^n\}$

Then, we may describe its **transitions** by:

- $\mathbb{P}[X_{\tau_{r+1}}^N = x + \frac{1}{N}(e_j^p - e_i^p) | X_{\tau_r}^N = x] = \frac{1}{m} x_i^p \tilde{B}_j^p(x)$, transition rule
- $\mathbb{E}[X_{\tau_{r+1}}^{N,p} - X_{\tau_r}^{N,p} | X_{\tau_r}^N = x] = \frac{1}{Nm}(m^p \tilde{B}^p(x) - x^p)$, expected increment

Consider the following:[1]

- $TX = \{z \in \mathbb{R}^n : \sum_{i \in \mathcal{S}^p} z_i^p = 0, \forall p \in \mathcal{P}\}$, the set of directions tangent to the set of population states X

Describe F, for smooth F, as a **stable game** if and only if it satisfies:[1,2]

- $z \cdot DF(x)z \leq 0$ for all $x \in X$ and all $z \in TX$, the **negative semidefiniteness condition**, where $DF(x)$ is the derivative of F at x
- this condition is known as **self-defeating externalities**
- i.e., payoffs of strategies of revising agents are exceeded by payoffs of strategies abandoned by revising agents

POPULATION GAMES, STABILITY

Define the **admissible deterministic perturbation** function $V^p : \text{int}(\Delta^p) \to \mathbb{R}$, differentiably strictly convex and infinitely steep near boundary of Δ^p. If $\mathcal{C}^p$ defined for admissible distribution v^p, $\exists V^p$ s.t.

- $\mathcal{C}^p(\pi^p) = \arg\max_{y^p \in \text{int}(\Delta^p)} (y^p \cdot \pi^p - V^p(y^p))$

Consider function $\Lambda : X \to \mathbb{R}_+$,[1]

$$\Lambda(x) = \sum_{p \in \mathcal{P}} m^p [\max_{y^p \in \text{int}(\Delta^p)} (y^p \cdot F^p(x) - V^p(y^p)) - (\frac{1}{m^p} x^p \cdot F^p(x) - V^p(\frac{1}{m^p} x^p))]$$

Theorem: Suppose F is a stable game and v is admissible, then,

- function Λ is a **strict Lyapunov function** for the perturbed best response dynamic, i.e. its value decreases strictly along each non-constant solution trajectory
- (F, v) admits unique and globally asymptotically stable perturbed equilibrium, the lone state at which $\Lambda(x) = 0$.

MEAN DYNAMICS

Define the **revision protocol**[3,4] as a Lipschitz continuous map $\rho : \mathbb{R}^n \times X \to \mathbb{R}_+^{n\times n}$ with payoff vector π and population states x as arguments, returning nonnegative matrices as outputs:

- $\rho_{ij}(\pi, x)$, scalar **conditional switch rate** from strategy i to j

Consider revision protocol ρ, population game F, population size N, and Markov process $\{X_t^N\}$ on space $\mathcal{X}^N$.[4]

Define the **mean dynamic** generated by ρ and F, as an ODE on state space X:[3]

- $\dot{x}_i = \sum_{j\in S} x_j \rho_{ji}(F(x), x) - x_i \sum_{j\in S} \rho_{ij}(F(x), x)$

MEAN DYNAMICS

Compare deterministic **mean dynamics** which are well-known which characterize the expected motion of the Markov process:[2]

, Table 1 Five basic deterministic dynamics

Revision protocol	Mean dynamic	Name and source
$\rho_{ij} = x_j[\pi_j - \pi_i]_+$	$\dot{x}_i = x_i \hat{F}_i(x)$	Replicator (Taylor and Jonker 1978)
$\rho_{ij} = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}$	$\dot{x}_i = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{k \in S} \exp(\eta^{-1}F_k(x))} - x_i$	Logit (Fudenberg and Levine 1998)
$\rho_{ij} = 1_{\{j = \operatorname{argmax}_{k \in S} \pi_k\}}$	$\dot{x} \in B^F(x) - x$	Best response (Gilboa and Matsui 1991)
$\rho_{ij} = [\pi_j - \sum_{k \in S} x_k \pi_k]_+$	$\dot{x}_i = [\hat{F}_i(x)]_+ - x_i \sum_{j \in S} [\hat{F}_j(x)]_+$	BNN (Brown and von Neumann 1950)
$\rho_{ij} = [\pi_j - \pi_i]_+$	$\dot{x}_i = \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+$	Smith (1984)
	$-\dot{x}_i = \sum_{j \in S} x_i [F_j(x) - F_i(x)]_+$	

Consider **Lyapunov functions** which may be constructed on these mean dynamics to show that they converge to Nash equilibria from all initial conditions in all stable games.[2]

Evolutionary Game Theory, Table 3 Lyapunov functions for five basic deterministic dynamics in stable games

Dynamic	Lyapunov function for stable games
Replicator	$H_{x^*}(x) = \sum_{i \in S(x^*)} x_i^* \log \frac{x_i^*}{x_i}$
Logit	$\tilde{G}(x) = \max_{y \in \mathrm{int}(X)} \left(y'\hat{F}(x) - \eta \sum_{i \in S} y_i \log y_i \right) + \eta \sum_{i \in S} x_i \log x_i$
Best response	$G(x) = \max_{i \in S} \hat{F}_i(x)$
BNN	$\Gamma(x) = \frac{1}{2} \sum_{i \in S} \left[\hat{F}_i(x)\right]_+^2$
Smith	$\Psi(x) = \frac{1}{2} \sum_{i \in S} \sum_{j \in S} x_i \left[F_j(x) - F_i(s)\right]_+^2$

EVOLUTIONARILY STABLE STRATEGY, ECOLOGICAL DYNAMICS

Consider the **classical dynamical game**:[5]

- $\dot{x}_i = f_i(\mathbf{u}, \mathbf{x})$
- f_i, instantaneous payoff function for player i, $i = 1, \ldots, n$
- $\mathbf{u}$ and $\mathbf{x}$, state and control vectors
- $\mathbf{u} = [u_1, \ldots, u_n]$, where u_i is player i's strategy

Modify to yield the **ecological dynamics**:

- $\dot{x}_i = x_i H_i(\mathbf{u}, \mathbf{x})$
- H_i, player i's payoff
- x_i, density of group of individuals (species)
- $\mathbf{x} = [x_1, \ldots, x_n]$, for n species
- $\dot{x}_i$, instantaneous per capita growth rate

EVOLUTIONARILY STABLE STRATEGY, G-FUNCTION

Define the ***G*-function**:[5]

- $G(v, \mathbf{u}, \mathbf{x})$, **fitness-generating function** (G-function) of population dynamics, with virtual variable v
- $G(v, \mathbf{u}, \mathbf{x})|_{v=u_i} = H_i(\mathbf{u}, \mathbf{x})$, fitness of individual in population of individuals defined by same G-function
- $\dot{x}_i = x_i G(v, \mathbf{u}, \mathbf{x})|_{v=u_i}$, population dynamics

EVOLUTIONARILY STABLE STRATEGY

Recall the system describing **ecological dynamics**,[5]

- $\dot{x}_i = x_i H_i(\mathbf{u}, \mathbf{x})$

Define a strategy vector $\mathbf{u}_c \in \mathcal{U}$ as an **evolutionarily stable strategy** (ESS)[5,6] if and only if for every initial condition $\mathbf{x}_0 > \mathbf{0}$, the solution of the system tends to equilibrium $\mathbf{x}^* = [\mathbf{x}_c^*, \mathbf{0}]$, $\mathbf{x}_c^* > \mathbf{0}$.

- i.e., the system is resistant to invasion
- $\mathbf{u}_c$ is a **local ESS** if $\mathbf{x}_0$ in neighborhood of $\mathbf{x}_c^*$
- original definition of an ESS by Smith[7] is a **local ESS** with scalar $\mathbf{u}_c, \mathbf{u}_m$

EVOLUTIONARILY STABLE STRATEGY, ECOLOGICALLY STABLE EQUILIBRIUM

Define an **ecological equilibrium** point for the ecological dynamics as $\mathbf{x}^* \in \mathcal{O}$, $\mathcal{O}$ as the non-negative orthant, as a point where $\exists n_{s^*}, 1 \leq n_{s^*} \leq n_s$ such that:[6]

- $H_i(\mathbf{u}, \mathbf{x}^*) = 0, \quad x_i^* = 0 \quad \forall i \in \{1, ..., n_{s^*}\}$
- $x_i^* = 0 \quad \forall i \in \{n_{s^*} + 1, ..., n_s\}$

Define the **ecologically stable equilibrium** (ESE) as an ecological equilibrium point $\mathbf{x}^* \in \mathcal{O}$ where if $\exists \mathcal{B}$ (ball) s.t. $\forall \mathbf{x}(0) \in \mathcal{O} \cap \mathcal{B}$ the solution generated by ecological dynamics satisfies $\mathbf{x}(t) \in \mathcal{O} \quad \forall t > 0$, asymptomatically approaches x^* as $t \to \infty$.[6]

Define coalition vector $\mathbf{u}_c$ made up of the first $n_{s^*} < n_s$ components of $\mathbf{u}$.[6]

Define the **ESS maximum principle**[5,6] as:

- $\max_{v \in \mathcal{U}} G(v, \mathbf{u}, \mathbf{x}^*) = G(v, \mathbf{u}, \mathbf{x}^*)|_{v=u_i} = 0$

Theorem: For given $u \in \mathcal{U}$ assume $\exists \mathbf{x}^*$ which is an ESE. If coalition vector $\mathbf{u}_c$ is an ESS for $\mathbf{x}^*$ then the ESS maximum principle is satisfied.[6]

STABILITY ANALYSIS

- An ESS is convergent stable, but not all convergent stable equilibrium points necessarily correspond to evolutionarily stable strategies.[6]
- Non-equilibrium Darwinian dynamics may still generate bounded solutions following periodic orbits, limit cycles, or n-cycles.[2,6]

CONCLUSION

- Stability of evolutionary games is not guaranteed
- Evolutionary games may display non-equilibrium behavior, cycles, etc.
- Equilibrium points may not necessarily be ESS
- For a stable population game F with admissible perturbed best response, Λ is a strict Lyanpunov function
- Deterministic mean dynamics such as replicator, logit, best response, BNN, and Smith may be shown to converge to Nash equilibria based on Lyapunov functions on stable games

FUTURE DIRECTIONS

- Consider continuous strategy sets[8]
- Consider robust games and incomplete information[9,10]
- Further consideration of spatial dynamics[2]

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