ECE-GY 6263 Final Project: Stability of Evolutionary Population Games

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***Abstract*—Evolutionary games may be described under various deterministic dynamics, which have various conditions of equilibrium, including evolutionarily stable strategies (ESS) and Nash equilibria. With regard to these equilibria, there are also various stability conditions, which may be characterized in detail. Non-equilibrium behavior within bounds is also known for some games. A review and analysis of the related literature is given, and numerical simulations are performed to demonstrate such equilibria and stability characteristics.**

I. Introduction

In Nash's seminal paper it was showed that for finite n-player games over finite pure strategy sets there exists points where any player deviating from a given pure strategy would produce a lesser payoff (under convention of maximizing payoffs) [1], what would later become known as Nash equilibria. Furthermore he defined equilibrium points across mixed strategies, where players probabilistically varied their pure strategies, which would be maximized for each player against other mixed strategies chosen by the player assuming other players maintained their mixed strategies [2], which would be later known as a Nash equilibrium in mixed strategies [3].

Smith and Price proposed the concept of evolutionary stable strategies (ESS) to explain the phenomenon of "limited war" amongst intraspecific conflicts in the natural world whereas rather than a fight to the death, individuals fought in a limited fashion [4]. They suggested that the population might also be viewed as adopting a strategy set over mutant vs non-mutant behaviors with payoff differences between them. In this model, the mixed strategies could be be considered to be distributions of the population across the strategy set of mutant vs non-mutant behaviors. Thus, the concept of Nash equilibria could be applied to populations [3, 5], in what would later become known as population games [6].

Furthermore it was shown that evolutionarily stable strategies exist for mixed strategies over a population but do not exist for pure strategies, hence, stability of populations under uniform behavior (population choosing a "pure strategy") could not be guaranteed [4]. The payoff or "fitness" to an individual using a given strategy i could be viewed as a function of the state of the population x, i.e. $F(i|x)$ [7].

Evolutionary game theory has been applied to real-world models of biological systems, for example to explain patterns of *E. coli* bacterial growth observed under conditions of constrained resources between two phenotypes, wild type (WT) and growth advantage in stationary phase (GASP) mutants, seemingly approaching to an equilibrium between the two population levels in a laboratory experiment [8]. Additionally, these populations were observed to react to resource conditions (nutrient-rich or nutrient-poor), which had an impact on their growth dynamics and relative fitness.

Thus, evolutionary game theory may be used to model population evolution under various conditions by constructing rules regarding evolutionary dynamics and their application, not limited to evolution in biological systems, but also including economic systems, behavior in groups of individuals, general dynamics of groups, etc.

II. Population Games

Define the set of $\bar{p}$ populations, where $\bar{p} \geq 1$, as $\mathcal{P} = \{1, ..., \bar{p}\}$. Next, define, for $p \in \mathcal{P}$: the integer (for simplicity) mass of population p, m^p; the strategy set for population p, $\mathcal{S}^p = \{1, ..., n^p\}$; the set of probability distributions over strategies in $\mathcal{S}^p$, $\Delta^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i \in \mathcal{S}^p} x_i^p = 1\}$; and the set of strategy distributions for population p, $X^p = m^p \Delta^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i \in \mathcal{S}^p} x_i^p = m^p\}$ [6].

Furthermore define: the total mass of all populations, $m = \sum_{p \in \mathcal{P}} m^p$; the total number of pure strategies in all populations, $n = \sum_{p \in \mathcal{P}} n^p$; and the set of overall strategy distributions, $X = \{x = (x^1, ..., x^{\bar{p}}) \in \mathbb{R}_+^n : x^p \in X^p\}$. Denote x as a population state. Define a set $\bar{X} = \{x \in \mathbb{R}_+^n : m^p - \varepsilon \leq \sum_i x_i^p \leq m^p + \varepsilon \forall p \in \mathcal{P}\}$, where ε is a positive constant, for a population with slightly varied masses.

Then, the payoff function for strategy $i \in \mathcal{S}^p$ is denoted as $F_i^p : \bar{X} \to \mathbb{R}$; payoff function for strategies of population p as $F^p : \bar{X} \to \mathbb{R}^{n^p}$; and finally, the payoff vector field defining a population game as $F : \bar{X} \to \mathbb{R}^n$. Members of a population may choose mixed strategies y^p [6].

A. Population game with randomly distributed payoffs

A model of evolution may be described for population games with randomly distributed payoffs: let the players be members of $\bar{p}$ finite populations of sizes $(Nm^1, ..., Nm^{\bar{p}})$, who recurrently play the population game F [6].

A continuous-time Markov chain is used to described the aggregate behavior in the model, as $\{X_t^N\}_{t \geq 0}$ with state space $\mathcal{X}^N = \{x \in X : Nx \in \mathbb{Z}^n\}$.

Then, the transitions are described by the transition rule,

$$\mathbb{P}[X_{\tau_{r+1}}^N = x + \frac{1}{N}(e_j^p - e_i^p) | X_{\tau_r}^N = x] = \frac{1}{m} x_i^p \tilde{B}_j^p(x) \quad (1)$$

and the expected increment,

$$\mathbb{E}[X_{\tau_{r+1}}^{N,p} - X_{\tau_r}^{N,p} | X_{\tau_r}^N = x] = \frac{1}{Nm}(m^p \tilde{B}^p(x) - x^p) \quad (2)$$

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Revised December 23, 2024.

III. Evolutionary Games

The classic evolutionary game is defined in a simpler manner. It can be considered a special case of the population game where $\bar{p} = 1$. Rather, in the evolutionary game there is a single set of pure strategies, using the previous notation, $\mathcal{S}^p$, which may be considered a compact subset of the Euclidean space $\mathbb{R}^d$ [5]. Thus the mixed strategy simplex (i.e. set of probability distributions over the pure strategy set), Δ^p, is a probability measure on the Borel σ-algebra of $\mathcal{S}^p$, i.e. $\mathcal{B}(\mathcal{S}^p)$, and the set of probability distributions over strategies in $\mathcal{S}^p$ is Δ^p. The mixed strategy is as a distribution of behaviors in a population where individuals use fixed pure strategies. Finally, the fitness mapping $F_i^p : \mathcal{S}^p \times \mathcal{S}^p \rightarrow \mathbb{R}$ is a continuous function on pure strategies i against an opposing strategy i' [5].

For sake of simplicity, denote, in the context of evolutionary games, the strategy set $\mathcal{S}^p$ of size n for this population p as $\mathcal{S}$, where $\mathcal{S} = \{1, ..., n\}$. Similarly, denote Δ^p as Δ. Furthermore consider the set of population states X (in population game notation, X^p for population p) containing population state x. Note that here, $\Delta = X$. The payoff function here is characterized as $F : X \rightarrow \mathbb{R}^n$ (i.e. F^p), with scalar $F_i(x)$ (i.e. F_i^p) for payoff to strategy i on population state x [9].

A. Static Games

Static games are described for n players maximizing payoffs $F_i(\mathbf{u})$, where $\mathbf{u} = [u_1, ..., u_n]$, denoting player i's strategy as u_i [10].

B. Dynamic Games

Consider the dynamic game as a set of ODEs

$$\dot{x}_i = f_j(\mathbf{u}, \mathbf{x}) \tag{3}$$

where f_i is the instantaneous payoff function for player i, $i = 1, ..., n$, $\mathbf{u}$ and $\mathbf{x}$ are state and control vectors, and $\mathbf{u} = [u_1, ..., u_n]$, where u_i is player i's strategy [10]. Here, $u_i \in \mathcal{S}$.

IV. Mean Dynamics

Consider the population games played by a single population, i.e. the evolutionary game.

A revision protocol is defined as a Lipschitz continuous mapping $\rho : \mathbb{R}^n \times X \rightarrow \mathbb{R}_+^{n\times n}$ on arguments π (payoff vector) and population states x, and returns a nonnegative matrix output [9, 11, 12], where $\rho_{ij}(\pi, x)$ is the scalar conditional switch rate from strategy i to j [9, 11].

Imitation protocols are revision protocols of the form [9]:

$$\rho_{ij}(\pi, x) = x_j \hat{\rho}_{ij}(\pi, x) \tag{4}$$

A mean dynamic generated by a revision protocol ρ and population game F is an ordinary differential equation (ODE) on the state space X [11],

$$\dot{x}_i = \sum_{j\in\mathcal{S}} x_j \rho_{ji}(F(x), x) - x_i \sum_{j\in\mathcal{S}} \rho_{ij}(F(x), x) \tag{5}$$

For population size N, revision protocol ρ, and population game F, the mean dynamic, a deterministic process, describes the expected motion of the Markov process $\{X_t^N\}$ [9].

A. Replicator Dynamic

Consider the pairwise proportional imitation protocol,

$$\rho_{ij}(\pi, x) = x_j [\pi_j - \pi_i]_+ \tag{6}$$

which suggests that the agent imitates a selected opponent when the opponent's payoff is higher [9].

The pairwise proportional imitation protocol generates the replicator dynamic [9], defined as

$$\dot{x}_i = x_i \hat{F}_i(x) \tag{7}$$

where $\hat{F}_i(x) = F_i(x) - \bar{F}(x)$ is the excess payoff to strategy i over the population's average payoff [7, 9, 11].

The Maynard Smith replicator dynamic is defined as

$$\dot{x}_i = \frac{x_i F_i(x)}{\sum_{k\in\mathcal{S}} x_k F_k(x)} - x_i = \frac{x_i \hat{F}_i(x)}{\bar{F}(x)} \tag{8}$$

differing from the standard replicator dynamics in its more rapid change under relatively low average payoffs [9].

B. Brown-von Neumann-Nash Dynamic

Consider a protocol where at an agent's revision opportunity, the agent switches to a randomly chosen strategy if its payoff is above average, with probability proportional to excess payoff, defined as [9]:

$$\rho_{ij}(\pi, x) = [\pi_j - \sum_{k\in\mathcal{S}} x_k \pi_k] \tag{9}$$

The Brown-von Neumann-Nash Dynamic (BNN) dynamic is therefore defined as [9, 12]

$$\dot{x}_i = [\hat{F}_i(x)]_+ - x_i \sum_{k\in\mathcal{S}} [\hat{F}_k(x)]_+ \tag{10}$$

C. Smith Dynamic

Consider a revision protocol where at an agent's revision opportunity, the agent switches to a newer strategy if its payoff is higher at probability proportionate to the difference in payoffs [9], i.e.

$$\rho_{ij}(\pi, x) = [\pi_j - \pi_i]_+ \tag{11}$$

Then, the Smith dynamic may be defined as

$$\dot{x}_i = \sum_{j\in\mathcal{S}} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j\in\mathcal{S}} [F_i(x) - F_j(x)]_+ \tag{12}$$

D. Logit Dynamic

The logit choice rule $\rho_{ij}(\pi, x) \propto \exp(\eta^{-1}\pi_j)$ is given by [9]:

$$\rho_{ij}(\pi, x) = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k\in\mathcal{S}} \exp(\eta^{-1}\pi_k)} \tag{13}$$

Then, the logit dynamic is given by

$$\dot{x}_j = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{k\in\mathcal{S}} \exp(\eta^{-1}F_k(x))} \tag{14}$$

E. Best Response Dynamic

Define the best response as the differential inclusion (which generalizes ODEs), as [9]:

$$\dot{x} \in B^F(x) - x$$
$$B^F(x) = \arg\max_{y \in X} y' F(x) \tag{15}$$

F. Imitative Dynamic

Suppose a revision protocol of the imitative form,

$$\rho_{ij}(\pi, x) = x_j r_{ij}(\pi, x) \tag{16}$$

where for all $i, j, k \in S$, $\pi_j \geq \pi_i$ if and only if $r_{kj}(\pi, x) - r_{jk}(\pi, x) \geq r_{ki}(\pi, x) - r_{ik}(\pi, x)$ [11].

Define monotone percentage growth rates for continuously differentiable (C^1) mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ as

$$G_i(x) \geq G_j(x) \iff F_i(x) \geq F_j(x) \tag{17}$$

Furthermore define forward invariance condition,

$$x' G(x) = 0 \tag{18}$$

Then, a family of dynamics, the imitative dynamic, may be defined, taking the form [11]

$$\dot{x}_i = x_i G_i(x) \tag{19}$$

for continuously differentiable (C^1) mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfying monotone percentage growth rates and the forward invariance condition.

Define the general imitative dynamic,

$$\dot{x} = V(x) = \text{diag}(x) G(x) \tag{20}$$

where G satisfies monotone percentage growth rates of (17) [11].

Note that the replicator dynamic (7) may be considered a special case of the general imitative dynamic where $G = \hat{F}$.

V. Perturbed Best Response Dynamic

Now reconsider the population game allowing for more than one population each with mixed strategies. Consider a game where players choose best responses at occasional revision opportunities with regard to current population state, after a random perturbation occurs on the payoffs at the revision opportunity, i.e. a model of evolution with randomly distributed payoffs [6].

In such a game, players are members of $\bar{p}$ finite populations of sizes $(Nm^1, ..., Nm^{\bar{p}})$, who recurrently play the population game F. Revision (strategy switching) opportunities occur as independent rate 1 Poisson processes for each of the Nm players, and thus occur as a Poisson process for the society as a whole at a rate Nm, where τ_k describes the time of the k-th revision opportunity.

The choice probability function $\mathcal{C}^p : \mathbb{R}^{n^p} \to \Delta^p$ describes the probability a realized payoff perturbations leads the player to choose a strategy i, defined as

$$\mathcal{C}_i^p(\pi^p) = v^p(\varepsilon^p : i \in \arg_{k \in \mathcal{S}^p} \max \pi_j^p + \varepsilon_j^p) \tag{21}$$

Define the admissible distribution v^p which describes random shocks on assessments of expected payoffs, as a distribution admitting a strictly positive density on $\mathbb{R}^{n^p}$ and is sufficiently smooth such that $\mathcal{C}^p$ is continuously differentiable. Define $v = (v^1, ..., v^{\bar{p}})$ as admissible where each component is admissible.

The choice probability function, $\mathcal{C}^p : \mathbb{R}^{n^p} \to \Delta^p$, is defined as

$$\mathcal{C}_i^p(\pi^p) = v^p(\varepsilon^p : i \in \arg_{k \in \mathcal{S}^p} \max \pi_j^p + \varepsilon_j^p) \tag{22}$$

where π^p is the base payoff vector [6].

Then, the perturbed best response function $\tilde{B}^p : X \to \Delta^p$ for the pair (F, v) is defined by composition $\tilde{B}^p = \mathcal{C}^p \circ F^p$.

Finally, the perturbed best response dynamic for (F, v) is defined as

$$\dot{x}^p = m^p \tilde{B}^p(x) - x^p \quad \forall p \in \mathcal{P} \tag{23}$$

VI. Ecological Dynamics

Modify (3) to the following ecological dynamic [10]

$$\dot{x}_i = x_i F_i(\mathbf{u}, \mathbf{x}) \tag{24}$$

where F_i is player i's payoff, x_i is the density of a group of individuals or species, $\mathbf{x} = [x_1, ..., x_n]$ for n species, and $\dot{x}_i$ is the per capita growth rate.

Then, the G-function or fitness-generating function of the population dynamics may be denoted as $G(v, \mathbf{u}, \mathbf{x})$, where [10]:

$$G(v, \mathbf{u}, \mathbf{x})|_{v=u_i} = F_i(\mathbf{u}, \mathbf{x}) \tag{25}$$

thereby giving the population dynamics

$$\dot{x}_i = x_i G(v, \mathbf{u}, \mathbf{x})|_{v=u_i} \tag{26}$$

Define an **ecological equilibrium** point for the ecological dynamics as $\mathbf{x}^* \in \mathcal{O}$, $\mathcal{O}$ as the non-negative orthant, as a point where $\exists n_{s^*}$, $1 \leq n_{s^*} \leq n_s$ such that

$$F_i(\mathbf{u}, \mathbf{x}^*) = 0, \quad x_i^* = 0 \quad \forall i \in \{1, ..., n_{s^*}\} \tag{27}$$

where $x_i^* = 0 \quad \forall i \in \{n_{s^*} + 1, ..., n_s\}$ [13].

Then define the **ecologically stable equilibrium** (ESE) as an ecological equilibrium point $\mathbf{x}^* \in \mathcal{O}$ where if $\exists \mathcal{B}$ (ball) s.t. $\forall \mathbf{x}(0) \in \mathcal{O} \cap \mathcal{B}$ the solution generated by ecological dynamics satisfies $\mathbf{x}(t) \in \mathcal{O} \quad \forall t > 0$, asymptomatically approaches x^* as $t \to \infty$.

Consider a coalition vector $\mathbf{u}_c$ made up of the first $n_{s^*} < n_s$ components of $\mathbf{u}$.

Then the ESS maximum principle [10, 13] may be defined as

$$\max_{v \in \mathcal{U}} G(v, \mathbf{u}, \mathbf{x}^*) = G(v, \mathbf{u}, \mathbf{x}^*)|_{v=u_i} = 0 \tag{28}$$

Theorem: For given $u \in \mathcal{U}$ assume $\exists \mathbf{x}^*$ which is an ESE. If coalition vector $\mathbf{u}_c$ is an ESS for $\mathbf{x}^*$ then the ESS maximum principle is satisfied [13].

A. Lotka-Volterra Competition Model

Consider the Lotka-Volterra competition model with n species [10]. Let the carrying capacity be assumed to be identical for all species, and be a function of u_i,

$$K(u_i) = K_m \exp(-\frac{u_i^2}{2\sigma_k^2}) \tag{29}$$

where σ_k is the phenotypic plasticity of u_i with regard to competition, and K_m a positive constant.

A competition function between species i and species j is denoted as $\alpha(u_i, u_j)$. The G-function for evolutionarily identical species is given as [10]

$$G(v, \mathbf{u}, \mathbf{x}) = r - \frac{r}{K(v)} \sum_{j=1}^{n} \alpha(v, u_j) x_j \tag{30}$$

where r is an intrinsic growth rate [13].

The competition function is chosen as [10]

$$\alpha(v, u_j) = \exp\left(-\frac{(v-u_j)^2}{2\sigma_a^2}\right) \tag{31}$$

where σ_a is the phenotypic plasticity with regard to the carrying capacity.

VII. Nash Equilibrium

Under the assumption of rational players, players are assumed to choose the best actions given beliefs of other players' actions [14]. Note that here the convention is that players will maximize playoffs (some literature uses minimizers instead).

The mixed strategy $x^* \in X$ is defined formally as as Nash equilibrium if it fulfills condition [11]

$$(x - x^*)' F(x^*) \leq 0 \quad \forall x \in X \tag{32}$$

VIII. Evolutionarily Stable Strategies

Define a mixed strategy $x^* \in X$ as an evolutionarily stable strategy (ESS) if for any mutant strategy $x \neq x^*$, perturbed state $x' = (1-\varepsilon)x^* + \varepsilon x$, then

$$F(x|x') \leq F(x^*|x) \tag{33}$$

for sufficiently small $\varepsilon > 0$ where $F(x|x')$ denotes strategy x competing against strategy x' [5, 7, 11].

Every evolutionarily stable strategy must be a Nash equilibrium [9, 14]. But, not every Nash equilibrium is an ESS [10].

This may be noted by defining ESS as $x^* \in X$ fulfilling (32) and the condition that there exists a neighborhood $O \subset X$ of x^* such that, for all $x \in O - \{x^*\}$ (i.e. nearby points) [11],

$$(x - x^*)' F(x^*) = 0 \implies (x - x^*)' F(x) < 0 \tag{34}$$

An ESS is convergent stable, but not all convergent stable equilibrium points necessarily correspond to evolutionarily stable strategies [13].

Non-equilibrium Darwinian dynamics may still generate bounded solutions following periodic orbits, limit cycles, or n-cycles [9, 13].

A. Evolutionarily Stable Strategies of Ecological Dynamics

Consider classical dynamical game given in (3). This is modified to yield the ecological dynamics, as in (24).

A generalization of evolutionarily stable strategies is proposed in terms of ecological dynamics, as follows: define a strategy vector $\mathbf{u}_c \in \mathcal{U}$ as an evolutionarily stable strategy (ESS) [10, 13] if and only if for every initial condition $\mathbf{x}_0 > \mathbf{0}$, the solution of the system tends to equilibrium $\mathbf{x}^* = [\mathbf{x}_c^*, \mathbf{0}]$, $\mathbf{x}_c^* > \mathbf{0}$.

B. Regular Evolutionarily Stable Strategy

A regular ESS has slightly stronger conditions than an ESS, it is defined as fulfilling the conditions [11]:

$$F_i(x^*) = \bar{F}(x^*) > F_j(x^*), \quad \text{when } x_i^* > 0, x_j^* = 0 \tag{35}$$

$$\forall y \in X - \{x^*\}, (y - x^*)' F(x^*) = 0$$
$$\implies (y - x^*)' DF(x^*)(y - x^*) < 0 \tag{36}$$

where (35) is called the quasistrict equilibrium condition.

C. Evolutionarily Robust Strategy

Define a value $\sigma(i, x)$ for a pure strategy i as [5]

$$\sigma(i, x) = F(i|x) - F(x|x) \tag{37}$$

Consider X as a subset of linear space $\mathcal{M}$ of measures on $\mathcal{S}$. Define variational norm as

$$\|\mu\|_{\text{var}} = \sup_{|f| \leq 1} \left| \int_{\mathcal{S}} f(x) \mu dx \right| \tag{38}$$

where f is measurable. Then the topology generated by $\|\cdot\|_{\text{var}}$ isa strong (variational) topology.

Define the weak topology as one where for probability measures R_n, R, then $R_n \to R$ if and only if $\int \mathcal{S} f(x) R_n dx$ converges to $\int \mathcal{S} f(x) R dx$ for a measurable f which is continuous on $\mathcal{S}$ [5].

Consider an evolutionary game. Define a strategy $x \in X$ as locally superior with regard to ε if a ε-neighborhood O of x exists such that $\sigma(i, x')$ for all $x' \neq x$, $x' \in O$ [5]. Define the strategy as strongly uninvadable, when it is locally superior on the strong (variational) topology, and evolutionarily robust if the strategy is locally superior on the weak topology.

IX. Equilibrium Stabilities

Define TX as the tangent space of the set of population states X [6, 11] i.e.

$$TX = \{z \in \mathbb{R}^n : \sum_{i \in \mathcal{S}^p} z_i^p = 0, \forall p \in \mathcal{P}\} \tag{39}$$

A. Lyapunov Stability

Define x^*, a rest point of a mean dynamic (5) as Lyapunov stable, if, given a neighborhood O of x^*, there also exists neighborhood O' of x^* where for every solution over time $\{x_t\}_{t \geq 0}$ of (5) starting in O', it is contained in O [11, 15].

Any Lyapunov stable state x^* within the set of stationary points of replicator dynamics is a Nash equilibrium [5].

Consider a differential inclusion, for $x(t) \in \mathbb{R}^n$ [16]:

$$\dot{x}(t) \in F(x(t)) \tag{40}$$

Suppose the following conditions on F hold:

1) for all $x \in \mathbb{R}^n$,

$$F(x) \subseteq \mathbb{R}^n \text{ is nonempty, compact, convex} \tag{41}$$

2) F is upper semicontinuous, i.e. for open ball B, $x \in \mathbb{R}^n$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|x - \tilde{x}| < \delta \implies F(\tilde{x}) \subseteq F(x) + \varepsilon B \tag{42}$$

When these conditions (41) and (42) are satisfied, it is known that local existence of solutions of (40) holds i.e. $\forall x_0 \in \mathbb{R}^n$, $\exists x(\cdot)$ such that there is a solution $x(0) = x_0$ for (40) on some interval $[0, T)$, where $T > 0$.

Note that the classical dynamic as described in (3) is a special case of the differential inclusion in (40). Thus properties applicable to (40) apply to classical dynamics (3).

Define a pair of continuous functions (V, W) on $\mathbb{R}^n$ where $V \in C^\infty(\mathbb{R}^n)$ and $W \in C^\infty(\mathbb{R}^n \setminus \{0\})$ as a C^∞-smooth strong Lyapunov pair for F if the conditions are satisfied:

1) Positive definiteness:

$$\forall x \neq 0, \quad V(x) > 0, W(x) > 0 \tag{43}$$

2) Properness: sublevel sets bounded for all $a \geq 0$, where the sublevel sets are given by

$$\{x \in \mathbb{R}^n : V(x) \leq a\} \tag{44}$$

3) Strong infinitesimal decrease,

$$\max_{v \in F(x)} \langle \nabla V(x), v \rangle \leq -W(x) \quad \forall x \neq 0 \tag{45}$$

B. Asymptotic stability

Define a rest point x^* of a mean dynamic as asymptotically stable if Lyapunov stability holds and there exists $\varepsilon > 0$ such that for distance $\mathrm{dist}(x, x^*) < \varepsilon$, $\mathrm{dist}(\{x_t\}_{t\geq 0}, x^*) \to 0$ [15].

Consider a point $x^* \in X$ for F as weakly asymptotically stable on strategy set X if it fulfills asymptotic controllability i.e. for some x there is a strategy set X where $\{x_t\}_{t\geq 0} \to 0$ in a stable and uniform manner [16].

Theorem: If the evolutionary game is doubly symmetric, then an evolutionarily robust strategy (ERS) is weakly asymptotically stable.

Theorem: For multifunction F that satisfies (41) and (42), F is strongly asymptomatically stable if and only $\exists (V, W)$ which is a C^∞-smooth strong Lyapunov pair (i.e. satisfying (43), (44), and (45)).

Theorem: For any x^* which is a regular ESS (fulfills (35) and (36)) of F, x^* is asymptotically stable for the following dynamics: impartial pairwise comparison dynamics for F, separable excess payoff dynamics for F, and best response dynamics for F [11].

C. Stability of Replicator Dynamics

By constructing suitable Lyapunov functions, any regular ESS is asymptotically stable under the replicator dynamic (7) [11].

The replicator dynamic is also known to converge to Nash equilibrium from any interior initial condition (where $x_0 \in TX$) [9].

A local Lyapunov function may be constructed for the ESS x^* for the replicator dynamic, as [9, 11]:

$$H_{x^*}(x) = \sum_{i \in \mathcal{S}(x^*)} x_i^* \log \frac{x_i^*}{x_i} \tag{46}$$

The replicator function can be considered a case of the differential inclusion (40).

Suppose normal-form matrix A describes the game such that

$$F(x) = xA \tag{47}$$

Then, the replicator dynamic (7) resolves to

$$\dot{x}_i = (xA - (xAx')1_{1\times n})\,\mathrm{diag}(x) = x'\tilde{A} \tag{48}$$

where $\tilde{A}$ is a constant matrix of the form

$$\tilde{A} = \left[(xA - (xAx')1_{1\times n})\right] \tag{49}$$

Note that for a normal-form matrix A, where A is time-invariant, then this system is linear, thus convex, defined on all values of x (nonempty), and compact (as it yields a single value), thus fulfilling (41). Furthermore, as it is a single-valued function, then as the linear function is continuous, it is upper semicontinuous.

Thus, it can be demonstrated that for this normal form game, the replicator function satisfies the local existence of solutions, i.e. there is some equilibrium solution for the system.

D. Stability of Perturbed Best Response

For the perturbed best response dynamic, described in (23), the perturbed equilibrium is defined as a value $x \in X$ where x is a rest point of (23) [6]. The set of perturbed equilibria is denoted $PE(F, v)$.

1) Stable Game: Define population game F as a stable game if it satisfies

$$(y - x)'(F(y) - F(x)) \leq 0 \quad \forall x, y \in X \tag{50}$$

and a strictly stable game if this inequality holds strictly [9].

Define population game F, if F is smooth, as a stable game if and only if it satisfies the condition of self-defeating externalities, or equivalently, the negative semidefinite condition,

$$z \cdot DF(x)z \leq 0, \quad \forall x \in X, \forall z \in TX \tag{51}$$

where $DF(x)$ is the derivative of $F : X \to \mathbb{R}^n$ at x [6, 9].

Theorem: Suppose population game F is a stable game and v is admissible. Then, a strict Lyapunov function $\Lambda : X \to \mathbb{R}_+$ exists for (23), in the form

$$\Lambda(x) = \sum_{p \in \mathcal{P}} m^p [\max_{y^p \in \mathrm{int}(\Delta^p)} (y^p \cdot F^p(x) - V^p(y^p)) - (\frac{1}{m^p} x^p \cdot F^p(x) - V^p(\frac{1}{m^p} \tag{52}$$

	D	E
D	$\frac{1}{2}v$	0
E	v	$\frac{1}{2}(v-c)$

TABLE I
HAWK-DOVE GAME PAYOFFS

where, as a a strict Lyapunov function, its value strictly decreases along each non-constant solution trajectory. Furthermore, (F, v) admits then a unique and globally asymptotically stable perturbed equilibrium which is the lone state where $\Lambda(x) = 0$.

E. Potential Game

Define the population game F as a potential game if it satisfies externality symmetry, given by

$$\frac{\partial F_i^p}{\partial x_j^q}(x) = \frac{\partial F_i^q}{\partial x_j^p}(x) \quad \forall i \in \mathcal{S}^p, j \in \mathcal{S}^p, p, q \in \mathcal{P}, x \in X \tag{53}$$

or in other words, if $DF(x)$ is symmetric for all $x \in X$ [6].

Theorem: For population game F, if it is a potential game, and v is admissible, then there is a strict Lyapunov function for (23), given by

$$\Pi(x) = f(x) - \sum_{p \in \mathcal{P}} m^p V^p(\frac{1}{m^p}x^p) \tag{54}$$

and all solution trajectories of the perturbed best response dynamic (23) converge to connected subsets of $PE(F, v)$, where $PE(F, v) = \{x \in X : x \text{ critical point of } \Pi\}$. Global asymptotical stability results if the set $PE(F, v)$ is a singleton.

1) Stochastic evolution: Consider the Markov process $\{X_t^N\}$.

Theorem: Under the admissible disturbance distribution v^p, for stochastic evolution in population game F, [6]

1) If F is a stable game, then the Markov process $\{X_t^N\}$ converges from every initial condition $x_0 \in X$ in the medium run (in a long but finite time span) to singleton $PE(F, v)$.
2) If F is a potential game, then the Markov process $\{X_t^N\}$ converges from every initial condition $x_0 \in X$ in the medium run to a connected subset of $PE(F, v)$.

X. Numerical Simulation

A. Hawk-Dove game

The behavior of evolutionary dynamics were simulated for a classic game. Consider the dove-hawk game, as defined in table I. The payoffs of the hawk-dove game may be given by, for a resource of value $v > 0$, where options are between display (D) and escalate (E) [9]:

Various dynamics were modeled using various starting parameters. For example, setting $v = 1$ and $c = 2$. Various initial values for x were tested: $x(0) = [1 \quad 0]$ (all doves, fig. 2), $x(0) = [1/2 \quad 1/2]$ (half doves, half hawks, fig. 1), and $x(0) = [0 \quad 1]$ (all hawks, fig. 3). It was found that the fourth-order Runge-Kutta method [17, 18] was not sufficient to avoid numerical error based on naive implementation in MATLAB, so the `ode89` function in MATLAB was used to numerically solve the initial value problem (IVP) [18, 19].

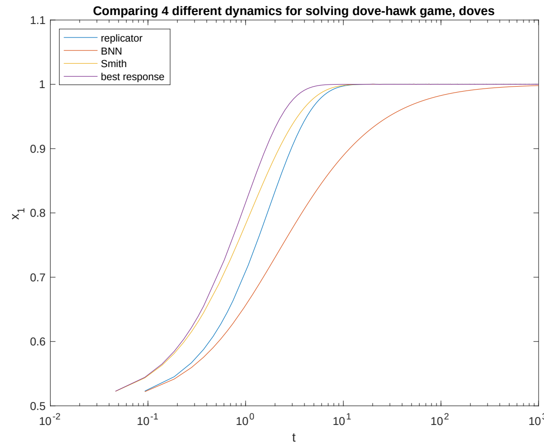

Fig. 1. Comparing the replicator (7), BNN (10), Smith dynamics (12), and best response dynamic (15) for the dove-hawk game with initial value of $x = [0.5 \quad 0.5]$ "half dove half hawk", $v = 1, c = 2$. Horizontal time axis plotted on a logarithmic scale to better distinguish plots.

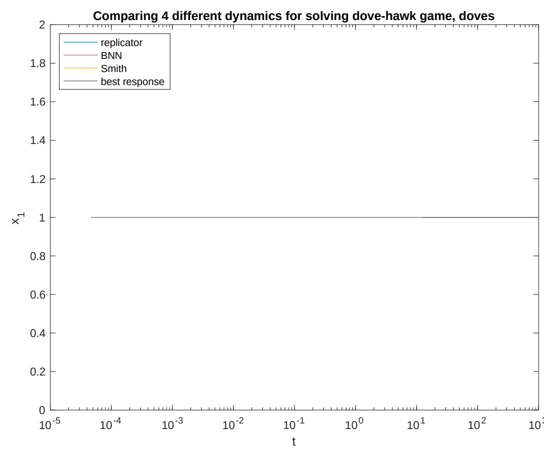

Fig. 2. Comparing the replicator (7), BNN (10), Smith dynamics (12), and best response dynamic (15) for the dove-hawk game with initial value of $x = [1 \quad 0]$ "all dove", $v = 1, c = 2$. Horizontal time axis plotted on a logarithmic scale to better distinguish plots.

For each of these starting conditions, it was found that the equilibrium eventually converged on the "all dove" solution, $x = [1 \quad 0]$. This numerically demonstrates that the $x = [1 \quad 0]$ equilibrium is asymptomatically stable for the dove-hawk game under each of the replicator (10), BNN (10), Smith (12), and best response (15) dynamics. When starting with an "all dove" population (fig. 2), the population remains as "all dove".

The dove-hawk game appears to exhibit asymptotic stability under each of replicator, BNN, Smith, and best response dynamics towards the solution "all doves" under the parameters $v = 1, c = 2$, for the starting conditions of "all doves", "all hawks", and "half hawks half doves".

This is a normal-form game described by a single payoff matrix (symmetric-form). It is known that as a doubly-symmetric game, the game is weakly asymptotically stable.

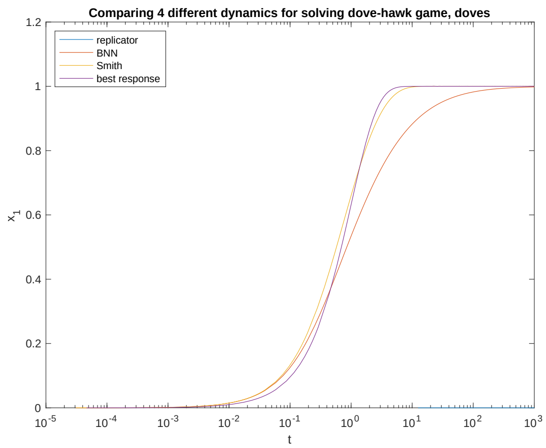

Fig. 3. Comparing the replicator (7), BNN (10), Smith dynamics (12), and best response dynamic (15) for the dove-hawk game with initial value of $x = [0 \quad 1]$ "all hawk", $v = 1, c = 2$. Horizontal time axis plotted on a logarithmic scale to better distinguish plots.

	R	P	S
R	0	-l	w
P	w	0	-l
S	-l	w	0

TABLE II
HAWK-DOVE GAME PAYOFFS

B. Rock-Paper-Scissors game

The classic game of rock-paper-scissors (RPS) is given in table II [9].

Here, a "bad game" is defined as starting when $w < l$, "standard game" when $w = l$, "good game" when $w > l$.

When starting from "all rock" (WLOG similar to "all paper", "all scissors") i.e. initial $x = [1 \quad 0 \quad 0]$, the different w vs l game types are compared (fig. 4, 5, 6). The `ode89` function of MATLAB is again used to solve the IVP [19].

In the "bad RPS" game, when starting from "all rock", the game seems to converge to Lyapunov stability under the BNN and Smith dynamics, while not changing under the replicator dynamic (this can also be considered a Lyapunov stability). The best response dynamic approaches pure strategy. In the

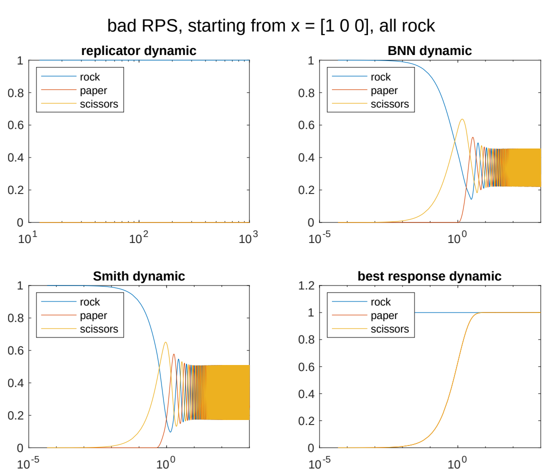

Fig. 4. "Bad game" ($w = 1, l = 2$) starting from $x = [1 \quad 0 \quad 0]$ "all rock"

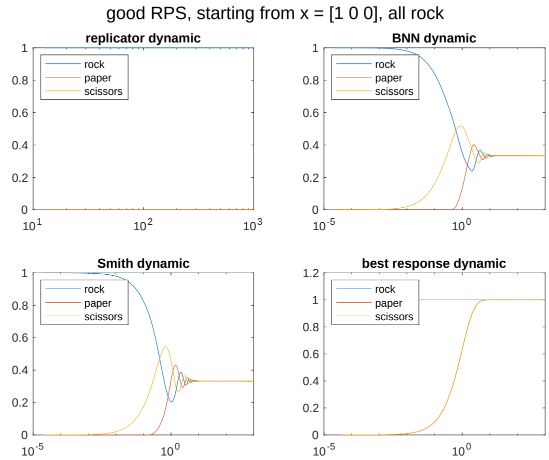

Fig. 5. "Good game" ($w = 2, l = 1$) starting from $x = [1 \quad 0 \quad 0]$ "all rock"

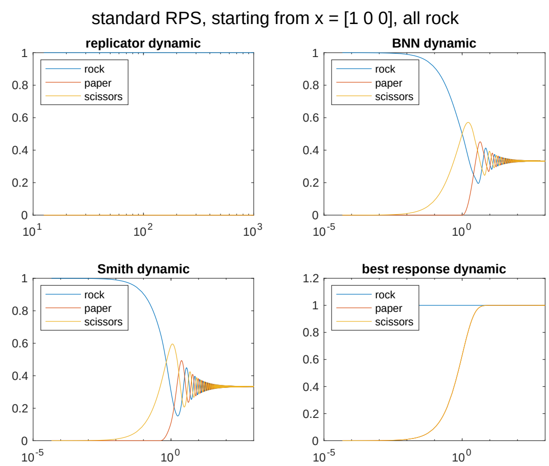

Fig. 6. "Standard game" ($w = 1, l = 1$) starting from $x = [1 \quad 0 \quad 0]$ "all rock"

"good RPS" and "standard RPS", when starting from "all rock", the game exhibits asymptotic stability in the BNN and Smith dynamics, to the equilibrium point of the simplex at $x = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$.

As expected, when starting from the equilibrium point $x = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$, the plots under the "good RPS" (fig. 8), "bad RPS" (fig. 7), and "standard RPS" (fig. 9) maintain the equilibrium point under the replicator, BNN, and Smith dynamics. The best response dynamic chooses one strategy and converges to a pure strategy.

The RPS game is stable (50) but not strictly stable, as a symmetric zero-sum game [9].

XI. Discussion and Conclusion

Various evolutionary dynamics are summarized here, including mean dynamics such as the replicator dynamic, Brown-von Neumann-Nash dynamic, Smith dynamic, logit dynamic, best response dynamic, and the family of imitative dynamics.

Stability analysis for these dynamics reveals various conditions for equilibrium stabilities. Deterministic mean dynamics such as replicator, logit, best response, BNN, and Smith may be shown to converge to Nash equilibria based on Lyapunov

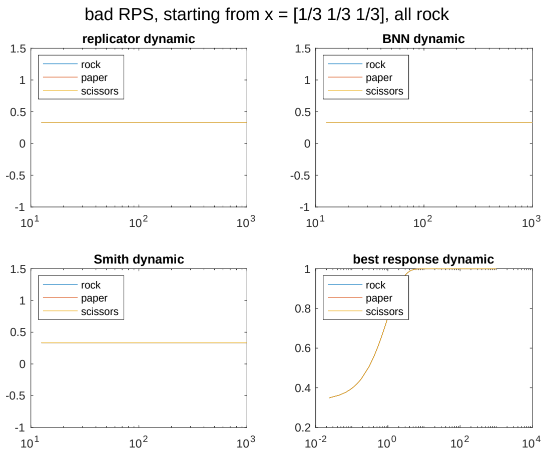

Fig. 7. "Bad game" ($w = 1, l = 2$) starting from $x = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$ "equal RPS"

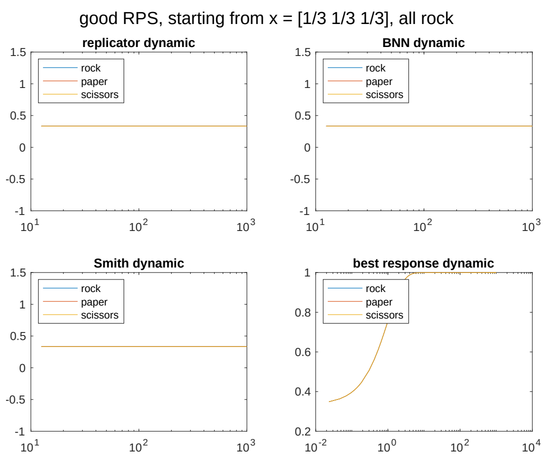

Fig. 8. "Good game" ($w = 2, l = 1$) starting from $x = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$ "equal RPS"

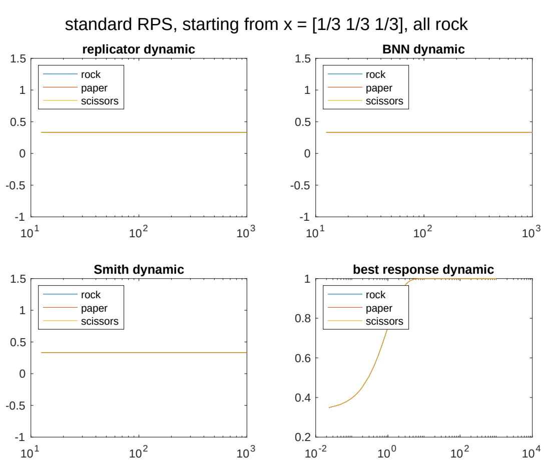

Fig. 9. "Standard game" ($w = 1, l = 1$) starting from $x = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$ "equal RPS"

functions on stable games. Evolutionarily stable strategies, as Nash equilibria, are subject to stability conditions which apply to Nash equilibria.

The stability of evolutionary games is not guaranteed. Evolutionary games may display non-equilibrium behavior including bounded behavior such as cycles, etc. Equilibrium points may also not necessarily be ESS.

XII. Future Directions

Future studies could further consider incomplete information and robust games [5, 20, 21] especially with regard to evolutionary games and population games.

Learning in continuous strategy/action sets e.g. [22] could also be applied to evolutionary games.

Agents and reinforcement learning over evolutionary games [14] could also be adapted to population games and robust games.

Spatial dynamics of games have also been considered [9], it follows that robust games could be considered spatially.

The convexity of the payoff space could be considered in analysis of the stability, and optimization techniques could be applied. Alternatively, for non-convex spaces, stochastic methods could be utilized to find local and global equilibria.

Acknowledgment

Thank you to Professor Quanyuan Zhu for teaching us about game theory this semester.

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