

TTK4250 Sensor Fusion

Assignment 2

Hand in: Monday 13. September 16.00 on Blackboard.

Tasks are to be solved on paper if you are not told otherwise, and you are supposed to show how you got to a particular answer. It is, however, encouraged to use Python, MATLAB, Maple, etc. to verify your answers. Rottmann's mathematical formula collection is allowed at both the exercises and the exam.

Task 1: Transformation of Gaussian random variables

Let $x \in \mathbb{R}^n$ be $\mathcal{N}(\mu, \Sigma)$. Find the distribution and see if you recognize it:

Hint: they are all given in the book.

(a) $z = \Sigma^{-\frac{1}{2}}(x - \mu)$, where $\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^T = \Sigma$

Hint: If you are using theorem 2.4.1, you might need $\det(A^{\frac{1}{2}}) = \det(A)^{\frac{1}{2}}$, $(A^{-1})^T = (A^T)^{-1}$, and $\det(A^T) = \det(A)$ whenever A has full rank.

(b) Use transformation of random variables to find $y_i = z_i^2$, where z_i is the i 'th variable in the vector z .

(c) $y = (x - \mu)^T \Sigma^{-1} (x - \mu) = z^T z = \sum z_i^2 = \sum y_i$.

Hint: The MGF of y_i is given in the book through example 2.8 and 2.10. Example 2.6 might also be handy.

Task 2: Sensor fusion

In this task we want to find out if a boat is above the line $y = x + 2$. In order to do this we will fuse measurements from two sensors with our prior belief: A drone-mounted camera, and a maritime surveillance radar. You have some prior knowledge of the state of the boat. You get 1 measurement from each sensor that are processed so that you know them to be (approximately) Gaussian¹ conditioned on the position.

To be more specific, let us denote the state by x and our prior Gaussian by $\mathcal{N}(x; \bar{x}, P)$. The measurement from the camera is given by $z^c = H^c x + v^c$ and the measurement from the radar by $z^r = H^r x + v^r$, where v^j , $j \in \{c, r\}$ denotes the measurement noise and is distributed according to $\mathcal{N}(0, R^c)$ and $\mathcal{N}(0, R^r)$, respectively.

The boat is assumed to move around according to the model $x^+ = Fx + w$, where w is $\mathcal{N}(0, Q)$. Note the similarity to the measurement models.

Only insert the numbers when asked to. The needed values are given by

$$\begin{aligned} \bar{x} &= \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & P &= 25I_2, & H^c &= H^r = I_2, \\ R^c &= \begin{bmatrix} 79 & 36 \\ 36 & 36 \end{bmatrix}, & R^r &= \begin{bmatrix} 28 & 4 \\ 4 & 22 \end{bmatrix}, & z_c &= \begin{bmatrix} 2 & 14 \end{bmatrix}^T, & z_r &= \begin{bmatrix} -4 & 6 \end{bmatrix}^T \end{aligned}$$

(a) What is $p(z^c|x)$.

¹In reality, a camera measures a bearing from a point while a radar measures in polar coordinates. However, with some knowledge of which plane/distance something is operating in, we can extract an approximate 3d cartesian measurement and approximate it as Gaussian (more on that later in the course). At a certain distance a Gaussian in polar coordinates with small enough covariance can safely be approximated by a Gaussian in cartesian coordinates.

- (b) Show that the joint $p(x, z^c)$ can be written as a Gaussian distribution.

Hint: conditional probability and the proof of theorem 3.3.1.

- (c) Find the marginal $p(z^c)$ and the conditional $p(x|z^c)$, using the above and either theorems from the book or calculations.
- (d) Can what was found above be reused to find the marginal $p(x^+)$ and/or $p(x|z^r)$? If so, state them.
- (e) What is the MMSE and MAP estimate of x given z^c ? You do not need to do calculations to find the answer, but briefly state what you would do if you had to.
- (f) Using Python. Use what you have found to condition x on z for each sensor to find the conditional mean and covariance, and plot and comment. I.e., insert the values to find the parameters of $p(x|z^c)$ and $p(x|z^r)$. You don't have to write anything on paper.

Hint: In task2.py, edit `def condition_mean`, `def condition_cov` and `def get_task_2f`.

- (g) Using Python. Perform the update of the other sensor (for both) to get $p(x|z^r, z^c)$ and investigate. Are the distributions the same? does it matter which order we condition?

Hint: In task2.py, edit `def get_task_2g`

- (h) You now want to know the probability that the boat is above the line, $\Pr(x_2 - x_1 > 5)$. Find this probability using the appropriate linear transform and the CDF.

Hint: In task2.py, edit `def get_task_2h`. To get the cdf you can use `from scipy.stats import norm`, and then used as `norm.cdf(value, mean, std)`.

Task 3: Working with the canonical form

In Section 3.3 the fundamental product identity was studied using a moment-based parametrization. Clearly, it must also be possible to establish an equivalent result using the canonical representation. In this exercise we shall therefore consider the product

$$\mathcal{N}^{-1}(\mathbf{x}; \mathbf{a}, \mathbf{B})\mathcal{N}^{-1}(\mathbf{y}; \mathbf{C}\mathbf{x}, \mathbf{D}). \quad (6)$$

- (a) Show that (6) is identical to the Gaussian

$$\mathcal{N}^{-1}\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}; \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{B} + \mathbf{C}^\top \mathbf{D}^{-1} \mathbf{C} & -\mathbf{C}^\top \\ -\mathbf{C} & \mathbf{D} \end{bmatrix}\right) \quad (7)$$

Hint: Taking the logarithm of the form (3.17) in the book with (3.20) inserted give a relatively simple way to the goal, after the terms constant in \mathbf{x} and \mathbf{y} are subtracted. Also

$$\mathbf{a}^\top \mathbf{A} \mathbf{a} + \mathbf{b}^\top \mathbf{B} \mathbf{b} + 2\mathbf{a}^\top \mathbf{C} \mathbf{b} = \mathbf{a}^\top \mathbf{A} \mathbf{a} + \mathbf{b}^\top \mathbf{B} \mathbf{b} + \mathbf{a}^\top \mathbf{C} \mathbf{b} + \mathbf{b}^\top \mathbf{C}^\top \mathbf{a} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}^\top \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix},$$

is handy, and valid for any vectors \mathbf{a} and \mathbf{b} and matrices \mathbf{A} , \mathbf{B} and \mathbf{C} of interest (variable names are not related to the task).

- (b) Show that the marginal distribution of \mathbf{y} , from the joint density (7), is

$$\mathcal{N}^{-1}(\mathbf{y}; \mathbf{C}^\top (\mathbf{B} + \mathbf{C}^\top \mathbf{D}^{-1} \mathbf{C})^{-1} \mathbf{a}, \mathbf{D} - \mathbf{C} (\mathbf{B} + \mathbf{C}^\top \mathbf{D}^{-1} \mathbf{C})^{-1} \mathbf{C}^\top). \quad (8)$$

Hint: Theorem 3.4.1

- (c) Show that the conditional distribution of \mathbf{x} given \mathbf{y} is

$$\mathcal{N}^{-1}(\mathbf{x}; \mathbf{a} + \mathbf{C}^\top \mathbf{y}, \mathbf{B} + \mathbf{C}^\top \mathbf{D}^{-1} \mathbf{C}). \quad (9)$$

Hint: Theorem 3.4.1

- (d) Let us now return to the original formulation of the product identity in Theorem 3.3.1. Use the result from c) to show that

$$\hat{\mathbf{P}}^{-1} = \bar{\mathbf{P}}^{-1} + \mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}. \quad (10)$$

Hint: Match variables in (3.21) with (3.10) in the book.