CLONES, COCLONES, AND THE DUALITY BETWEEN RELATIONS AND OPERATIONS

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Abstract. Abstract clones in universal algebra and Lawvere's algebraic theories in category theory are well-known to be equivalent notions. However, the universal algebraists' notion of a clone of relations, or coclone, has lacked a category theoretic treatment. Relational clones arise as the fixpoints of a certain Galois connection between operations and relations on a given set, with concrete clones forming the fixpoints on the other side. The lattices of fixpoints are dually isomorphic — a duality between clones of operations and clones of relations. It turns out that coclones can be formulated category theoretically as certain finite limit theories, thus we provide semantics for coclone theories under Gabriel-Ulmer duality. Further, we obtain a satisfying account of the duality between clones and coclones as a dual equivalence between categories whose objects on each side are certain classes of finite limit theories; the finite product (Lawvere) theories are recoverable as certain subcategories thereof.

1. Introduction

The notion of single-sorted equational theory is a fundamental one in mathematics, and has various incarnations; two which abstract away from signature are Lawvere's algebraic theories, and the general (universal) algebraists' abstract clones. On the other hand, concrete clones describe algebraic structure defined in terms of the operations available on a fixed base set. Concrete clones have been objects of study since as far back as the 1920s, when Emil Post gave a description of all clones concrete on the 2-element set, calling them 'closed classes' of functions [8]. The abstract notion came some time later, as certain small categories with finite products in Lawvere's now classic thesis of 1963 [9]. Shortly after, general algebraists developed *abstract clones*, finding use in the presentation of equational theories abstracted from signature. Algebraists further developed the notion of a *relational clone*, or *coclone*, which arise as the fixpoints on the relation side of what is known as the Pol-Inv Galois connection between relations and operations on a given set. The name coclone suggests a duality, and indeed the fixpoints on the operation side are clones, and clones and coclones on a given set form dually isomorphic lattices. The Pol-Inv Galois connection was described by McKenzie *et. al.* in [10] as 'the most basic Galois connection in algebra'.

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The Pol-Inv Galois connection was the focus of the 2013 Workshop on Algebra, Coalgebra and Topology at the University of Bath, which was organised as a joint venture between algebraists and categorists. In the preface to the workshop's Proceedings, editors John Power and Cai Wingfield invite a category theoretic treatment of the connection — and thus presumably the duality — between clones and coclones in terms of Lawvere theories. This work provides an answer to that invitation.

However, our interest in this problem originates in its connection with a certain class of concrete dualities, the 'natural dualities' of [4]. In that work, they show that all clones on the 2-element set (with the exception of 8 out of countably many) give the algebraic structure of dualising objects in concrete dualities; relational structure from the coclone along with the discrete topology gives the dualising object its alternate form. Examples of dualities from this list abound, and include Stone duality, Priestley duality, Hofmann-Mislove-Stralka duality for unital meet-semilattices, and Isbell's duality for median algebras. The connection with the present work will be made clear in forthcoming work.

Returning to the matter at hand, it turns out that to capture Pol-Inv duality, we need more than the Lawvere theory alone can provide, in general. Nonetheless, we obtain a dual equivalence between categories whose objects on each side are certain classes of *finite limit theories*, while the finite product (Lawvere) theories are recoverable as full subcategories thereof. We formulate the duality in terms of what we refer to as *abstract coclones*, which we introduce in the lead up to describing the duality in Section 5.2. The description of their dual theories, also in Section 5.2, is based on a characterisation of finite limit theories of quasivarieties given in [1]. The categories involved, along with the duality itself, is the content of Section 6.

2. Background

2.1. Model theory.

This section is based on descriptions given in Chapter 5 of [2].

Definition 1 (Single-sorted signature Σ)**.** A (single-sorted, finitary) *signature* Σ consists of:

(i) Disjoint sets $\Sigma_{\mathcal{O}}$ of *operation symbols* and $\Sigma_{\mathcal{R}}$ of *relation symbols*;
(ii) A function $\mathsf{ar}\colon \Sigma_{\mathcal{O}} \to \mathbb{N}$ assigning each operation symbol τ in $\Sigma_{\mathcal{O}}$ its arity $\tau\colon s^n \to s$, where s is the sort and $n \in \mathbb{N}$;
(iii) A function $\mathsf{ar}\colon \Sigma_{\mathcal{R}} \to \mathbb{N}$ assigning each relation symbol its arity $\rho\colon s^k$, where s is the sort and $k \in \mathbb{N}$.

Definition 2 (Structure for signature Σ)**.** A Σ-*structure* $\mathsf{V} := \langle V; \Sigma\rangle$ of signature Σ is a set V equipped with with an interpretation of each n-ary operation symbol $\tau \in \Sigma_{\mathcal{O}}$ as a function $[\![\tau]\!] : V^n \to V$, and an interpretation of each k-ary relation symbol $\rho \in \Sigma_{\mathcal{R}}$ as a subset $[\![\rho]\!] \subseteq V^k$.

Remark 3. If a signature Σ consists of a single symbol σ, we will write $\langle V; \sigma\rangle$ and not $\langle V; \{\sigma\}\rangle$ to avoid cluttered notation.

Definition 4 (Category of Σ-structures)**.** The *category of Σ-structures* over a signature Σ has

- *Objects* are Σ-structures;
- *Morphisms* $h\colon \mathsf{A} \to \mathsf{B}$ are functions which commute with the operations and preserve the relations. Explicitly:
  For each n-ary operation symbol $\tau \in \Sigma_{\mathcal{O}}$ and $a_1, \dots, a_n \in A$,
  $$h(\llbracket\tau\rrbracket_{\mathsf{A}}(a_1, \dots, a_n)) = \llbracket\tau\rrbracket_{\mathsf{B}}(h(a_1), \dots, h(a_n)),$$
  and for each n-ary relation symbol $\rho \in \Sigma_{\mathcal{R}}$,
  $$(a_1, \dots, a_n) \in \llbracket\rho\rrbracket_{\mathsf{A}} \implies (h(a_1), \dots, h(a_n)) \in \llbracket\rho\rrbracket_{\mathsf{B}} \ .$$

There are a number of useful facts about **Σ-str** which can be found in [2].

(a) **Σ-str** is locally finitely presentable, and so complete and cocomplete;
(b) **Σ-str** has (epi, regular mono) and (regular epi, mono) factorisations of morphisms;
(c) There is a natural forgetful functor $\mathrm{G}\colon$ **Σ-str** $\to$ **Set** which sends a Σ-structure its underlying set and a morphism of Σ-structures to its underlying function;
(d) The forgetful functor has a left adjoint $\mathrm{F}\colon$ **Set** $\to$ **Σ-str**, which sends a set A to the structure whose underlying algebra is freely generated by A over $\Sigma_{\mathcal{O}}$, and with all relations empty.

Remark 5 (Subalgebras vs substructures)**.** An important distinction must be noted here. In the sequel, we will have occasion to consider structures for signatures which are purely operational — containing only operation symbols — and this will mean a distinction must be drawn between a *subalgebra* and *substructure*. If a signature Ω is purely operational the objects of **ΩSt** will be referred to as Ω-algebras. In **ΩSt**, the inclusion of a subalgebra $A \hookrightarrow B$ is a monomorphism. Moreover, for any morphism $h\colon A \to B$ of Ω-algebras, we have the *image factorisation*

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, two heads, "e"'] & & B \\ & h_!(A) \arrow[ur, hook, "m"'] & \end{tikzcd} \tag{2.1}$$

given by the (regular epi, mono) factorisation of h, with $h_!(A)$ a subalgebra of B.

On the other hand, if a signature Ψ has relation symbols, the image factorisation of a morphism $f\colon \mathsf{X} \to \mathsf{Y}$ in **ΨSt** does *not* necessarily give a substructure of Y. However, the inclusion of a substructure is a *regular* monomorphism, and indeed the *regular* image factorisation

$$\begin{tikzcd} \mathsf{X} \arrow[rr, "f"] \arrow[dr, "e"'] & & \mathsf{Y} \\ & f_!(\mathsf{X}) \arrow[ur, tail, "m"'] & \end{tikzcd} \tag{2.2}$$

given by the (epi, regular mono) factorisation of f does give a substructure $f_!(\mathsf{X})$ of Y.

2.2. **Finite limit theories and Gabriel-Ulmer duality.** We base our description of Gabriel-Ulmer duality on that given in [1]. An object A of a category $\mathbf{C}$ is said to be *finitely presentable* if the hom-functor $\mathbf{C}(A, -)\colon \mathbf{C} \to \mathbf{Set}$ preserves filtered colimits (or equivalently directed colimits as shown in [2]).

A category $\mathbf{C}$ is said to be *locally finitely presentable* if it is cocomplete, and has a dense subcategory $\mathbf{C}_{fp}$ of finitely presentable objects, that is, every object in $\mathbf{C}$ is a filtered (or equivalently directed) colimit of objects in $\mathbf{C}_{fp}$. In [1], they refer to $\mathbf{C}_{fp}^{\mathrm{op}}$ as the algebraic theory of $\mathbf{C}$; we use algebraic to refer to structures with operations only, so we will refer to it as the *(finite limit) theory* of $\mathbf{C}$.

Gabriel-Ulmer duality may be described as follows:

(I) For every locally finitely presentable category $\mathbf{C}$, there is an equivalence
$$\mathbf{C} \simeq \mathbf{FL}(\mathbf{C}_{fp}^{\mathrm{op}}, \mathbf{Set});$$
(II) Every essentially small category $\mathbf{K}$ with finite limits, $\mathbf{FL}(\mathbf{K}, \mathbf{Set})$ is locally finitely presentable;

(III) The theory of a locally finitely presentable category is essentially unique, that is, if $\mathbf{FL}(\mathbf{K}, \mathbf{Set}) \simeq \mathbf{FL}(\mathbf{C}_{fp}^{\mathrm{op}}, \mathbf{Set})$, then $\mathbf{K} \simeq \mathbf{C}_{fp}^{\mathrm{op}}$.

2.3. **Clones of operations and clones of relations.** Let $\mathbf{\Delta_0}$ be a skeleton of the category of linearly ordered finite sets and consider its inclusion $[-]\colon \mathbf{\Delta_0} \to \mathbf{Set}_f$ where $\mathbf{Set}_f$ denotes the category of finite sets. We regard A^n as notation for the set $\mathbf{Set}([n], A)$. This section takes from a number of sources, with the biggest contribution coming from [5], with material also from [8]. We note that [3] does give a formulation of clones with nullary operations, though our approach was somewhat different.

Definition 6. A *concrete clone*, or *clone of operations* $\mathscr{C}$ on a set A consists of:

- For each $n \in \mathbb{N}$, a set $\mathscr{C}(n) \subseteq \mathbf{Set}(A^n, A)$ of *n-ary operations*;
- $\mathscr{C}(n)$ contains the projections $\pi_i^n\colon A^n \to A$;
- $\mathscr{C}$ is closed under *composition of operations* in the sense that for any $f \in \mathscr{C}(n)$ and n-tuple $(g_1, \ldots, g_n) \in \mathscr{C}(k)^n$, there is a k-ary operation $f(g_1, \ldots, g_n)$ defined on each $(a_1, \ldots, a_k) \in A^k$ by

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_k) = f(g_1(a_1, \ldots, a_k), \ldots, g_n(a_1, \ldots, a_k)). \tag{2.3}$$

Example 7 (Full concrete clone)**.** Given a set A, take $\mathscr{O}_A(n)$ to be the entirety of $\mathbf{Set}(A^n, A)$ for each n. The set $\mathscr{O}_A$ is known as the *full clone of operations* on A.

Definition 8. Let A be a non-empty set. Considering an n-ary relation on A as a subset of A^n, the set of all finitary relations on A is given by
$$\mathcal{R}_A := \bigcup_{n \in \mathbb{N}} \mathrm{Sub}(A^n)$$
where $\mathrm{Sub}(A^n)$ denotes the set of subsets of A^n.

Definition 9. Given an operation $f\colon A^n \to A$, we have $f^k\colon (A^n)^k \to A^k$, with
$$f^k \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} f(a_{11}, \ldots, a_{1n}) \\ f(a_{21}, \ldots, a_{2n}) \\ \vdots \\ f(a_{k1}, \ldots, a_{kn}) \end{pmatrix} .$$

The operation f is said to *preserve* a relation $S \subseteq A^k$ if for each $k \times n$ matrix over A,

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{pmatrix} \in S \quad (\forall j \in n) \implies \begin{pmatrix} f(a_{11}, \ldots, a_{1n}) \\ f(a_{21}, \ldots, a_{2n}) \\ \vdots \\ f(a_{k1}, \ldots, a_{kn}) \end{pmatrix} \in S \ .$$

Definition 10. A k-ary relation S on a set A is *closed* under an operation f if S is a subalgebra of $\langle A; f\rangle^k$ under the pointwise structure.

Lemma 11. *Given an operation $f \colon A^n \to A$ and a relation $S \subseteq A^k$, the following are equivalent:*

(a) f preserves S;
(b) S is closed *under f, that is, S is a subalgebra of $\langle A; f\rangle^k$;*
(c) f is a morphism of relational structures $\langle A; S\rangle^n \to \langle A; S\rangle$.

Proof. Let $\iota \colon S \to A^k$ be the inclusion of k-ary relation S and let $f^k\restriction$ denote the evident restriction of f^k. Then each of Items (a), (b), and (c) may be interpreted as requiring that the diagram

$$\begin{tikzcd} S^n \arrow[r, "f^k\restriction"] \arrow[d, "\iota^n"'] & S \arrow[d, "\iota"] \\ (A^k)^n \arrow[r, "f^k"'] & A^k \end{tikzcd}$$

commutes. ☐

Definition 12. We say a relation and an operation are *compatible* if any of the items in Lemma 11 hold (and so all of them do).

Remark 13. Algebraists have traditionally opted not to allow an algebra to have empty underlying set, and to exclude nullary operations from clones of operations. However, without their inclusion, the categories we define later will lack necessary (finite) limits and colimits. We resolve this by considering that an empty k-ary relation $\varnothing^k$ is closed under $u \colon A^n \to A$ if the following diagram commutes:

$$\begin{tikzcd} \varnothing^{k\times n} \arrow[r, "u^k\restriction"] \arrow[d, "!^n"'] & \varnothing^k \arrow[d, "!"] \\ (A^k)^n \arrow[r, "u^k"'] & A^k \end{tikzcd} .$$

If $n \neq 0$, the map at the top of the square is simply the identity map $1_{\varnothing^k}$, so to ask that a non-nullary operation u is compatible with an empty relation is no condition at all on u. But if u is nullary, we have that $\varnothing^{k\times 0} = \{1_{\varnothing^k}\}$, so the map at the top of the square does *not exist*.

Thus an empty relation cannot be compatible with any nullary operation but provides no constraint on non-nullary operations, which is what we want.

Remark 14. The definition of a clone of relations on a set A is usually given in terms of being a fix-point of the Pol-Inv Galois connection. If A is not finite, this requires a notion of *local closure*; this ensures that the lattice of clones of operations and the lattice of clones of relations are dually isomorphic.

However, for reasons which will become clear, up until Section 6 we work only with relations and operations on finite sets. Giving the more general version would introduce unnecessary complications. We refer the interested reader to [8] for an overview and a wealth of references.

2.4. **The Pol-Inv Galois connection and the duality between clones of operations and clones of relations.** Definitions 9 and 10 define a binary relation between operations and relations in terms of preservation and closure. That is, we have a binary relation $\mathcal{C} \subseteq \mathcal{O}_A \times \mathcal{R}_A$ defined by $(\tau, S) \in \mathcal{C}$ if τ preserves S, or equivalently, if S is closed under τ (this is indeed equivalent by Lemma 11). As with any binary relation, $\mathcal{C}$ induces a Galois connection between powersets. Known as the Pol-Inv Galois connection[1], it has adjoints

$$\begin{tikzcd} \mathscr{P}(\mathcal{O}_A)^{\mathrm{op}} \arrow[r, shift right, "\mathsf{Inv}"'] \arrow[r, phantom, "\perp"] & \mathscr{P}(\mathcal{R}_A) \arrow[l, shift right, "\mathsf{Pol}"'] \end{tikzcd} .$$

Proposition 15. *Given a finite set A, the fixpoints $\mathsf{Pol}\,\mathsf{Inv}(F) = F$ are precisely the clones of operations on A and the fixpoints $\mathsf{Inv}\,\mathsf{Pol}(R) = R$ are precisely the clones of relations on A. Moreover, the lattices of fixpoints are dually isomorphic.*

Using the equivalent notion of compatibility from Definition 12, we have that the mappings

$$\mathsf{Inv}\colon \mathscr{P}(\mathcal{O}_A)^{\mathrm{op}} \to \mathscr{P}(\mathcal{R}_A) \quad \text{and} \quad \mathsf{Pol}\colon \mathscr{P}(\mathcal{R}_A)^{\mathrm{op}} \to \mathscr{P}(\mathcal{O}_A)$$

may be given by

$$\mathsf{Inv}(F) := \{S \in \mathcal{R}_A : S \text{ is compatible with every } \tau \in F\}$$

and

$$\mathsf{Pol}(R) := \{\tau \in \mathcal{O}_A : \tau \text{ is compatible with every } S \in R\}.$$

Using the characterisations of compatibility in Lemma 11, we obtain:

Corollary 16. *Assume $\mathscr{R} = \mathsf{Inv}(\mathscr{T})$ and $\mathscr{T} = \mathsf{Pol}(\mathscr{R})$ are clones of relations and operations respectively on a set A. The relations in $\mathscr{R}$ are precisely those subsets which correspond to subalgebras of finite powers of A under the algebra structure given by $\mathscr{T}$ on A and extended pointwise.*

Definition 17 (Term-closed sets of relations)**.** A set of relations $\mathscr{R} \subseteq \mathcal{R}_V$ is called *term-closed* when it is closed with respect to the following:

[1]The names of the adjoints come from the traditional terminology of a relation being an 'invariant' of a set of operations and an operation being a 'polymorphism' of a set of relations [8]. These terms are no longer used in this context. Nonetheless, they continue to lend their names to the adjoints.

(i) *Trivial relations*: A *trivial*, or *generalised diagonal* relation on V is one induced by an equivalence relation on indices in the following way: Given an equivalence relation θ on $[k]$, there is a k-ary relation on V given by

$$\Delta_\theta = \{(x_1, \dots x_k) \in V^k : (\forall (i,j) \in \theta) \;\; x_i = x_j\}.$$

(ii) *Projections*: Let $\iota\colon k \to n$ be a monomorphism in $\mathbf{\Delta_0}$. Given an n-ary relation R, the k-ary *projection* of R along $[\iota]$ is given by

$$\iota(R) := \{(x_{\iota(1)}, \dots, x_{\iota(k)}) \in V^k : (x_1, \dots x_n) \in R\}$$

(iii) *Concatenation*: Given an m-ary relation R and a k-ary relation S, their concatenation RS is the $(m+k)$-ary relation given by their cartesian product:

$$\{(x_1, \dots x_m, x_{m+1}, \dots x_{m+k}) \in V^{m+k} : (x_1, \dots x_m) \in R, \; (x_{m+1}, \dots x_{m+k}) \in S\}.$$

(iv) *Intersection*: Given n-ary relations R and S, their intersection is given by

$$R \cap S = \{(x_1, \dots x_n) \in V^n : (x_1, \dots x_n) \in R \text{ and } (x_1, \dots x_n) \in S\}.$$

(v) *Coordinate permutation*: A permutation $s\colon [n] \to [n]$ induces a permutation of variables on an n-ary relation R, giving a relation

$$s[R] = \{(x_{s(1)}, \dots, x_{s(n)}) \in V^n : (x_1, \dots, x_n) \in R\}$$

Remark 18. The items in Definition 17 are essentially the same as those given on p.11 of [5] without providing a name. We used different notation to enforce differentiating between categorical product and the concatenation of relations (which is given by the binary product of the subsets). The terminology term-closed is used in [4] for something different, though related. We use it due to the relationship with closure under terms derived from positive-primitive formulae.

The following is Theorem 9.14 (ii) in [5].

Theorem 19. *Given $R \subseteq \mathcal{R}_V$, we have that $R = \mathsf{Inv}\,\mathsf{Pol}(R)$ (and is thus a relational clone on V) if and only if R is term-closed, that is, contains all the trivial relations and is closed under intersection of relations of the same arity, relations induced by permutation of variables, concatenation, and projection.*

Remark 20. We will shift between regarding an element of V^k as a function $\alpha\colon [k] \to V$ or as a k-tuple of elements of V depending on what we find convenient for the matter at hand.

2.5. **Reflective subcategories.** Assume we have an adjunction

$$\mathbf{D} \underset{G}{\overset{F}{\underset{\perp}{\leftrightarrows}}} \mathbf{C} \tag{2.4}$$

with unit $\eta\colon 1_{\mathbf{C}} \to GF$ and counit $\varepsilon\colon FG \to 1_{\mathbf{D}}$.

Lemma 21. *Let $\mathbf{A}$ be the full subcategory of $\mathbf{C}$ whose objects are C such that the unit $\eta_C\colon C \to GFC$ is a monomorphism. If there exists a monomorphism $m\colon C \to GD$, then $C \in \mathbf{A}$.*

Proof. Assume we have the situation as stated above, and morphisms $f, g\colon B \to C$ such that $\eta_C f = \eta_C g$. By adjointness, there exists a unique morphism $\tilde{m}\colon FC \to D$ such that $G\tilde{m} \cdot \eta_C = m$, so $G\tilde{m} \cdot \eta_C \cdot f = mf = G\tilde{m} \cdot \eta_C \cdot g = mg$. As m is monic, $mf = mg$, so $f = g$.

Thence η_C is a monomorphism and so $C \in \mathbf{A}$. □

Proposition 22. *Assume* $\mathbf{C}$ *has a (regular epi, mono) factorisation system and* $\mathbf{A}$ *is closed under the monomorphisms in the manner described in Lemma 21. Then* $\mathbf{A}$ *is a regular epi-reflective subcategory of* $\mathbf{C}$*, with reflector given by the regular epi component of the (regular epi, mono) factorisation of the unit of the adjunction* (2.4)*.*

Proof. Let $X \in \mathbf{C}$ and let

$$\begin{tikzcd} X \arrow[rr, "\eta_X"] \arrow[dr, "e_X"'] & & GFX \\ & \overline{X} \arrow[ur, "m_X"'] & \end{tikzcd} \tag{2.5}$$

be the (regular epi, mono) factorisation of η_X. We will first show that the inclusion of $\mathbf{A}$ has an adjoint by showing that for each $X \in \mathbf{C}$, the component e_X satisfies the universal property of the unit. That is to say, if $Y \in \mathbf{A}$ and there exists $f\colon X \to Y$, there is a unique extension of f along e_X. Consider the following diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "e_X"'] & & Y \arrow[d, "\eta_Y"] \\ \overline{X} \arrow[r, "m_X"'] \arrow[urr, dashed] & GFX \arrow[r, "GFf"'] & GFY \end{tikzcd} \quad . \tag{2.6}$$

Note that the outer rectangle commutes by naturality of η, and that the morphism indicated by the dotted line exists uniquely as η_Y is in $\mathscr{M}$ and e is in $\mathscr{E}$ and are thus orthogonal. We now show there is a family of maps which form the components of the counit, which must invert each $e_{\overline{X}}$. Observe we have a commuting diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "e_{\overline{X}}"] \arrow[d, "1"'] & \overline{\overline{X}} \arrow[d, "m_{\overline{X}}"] \arrow[dl, dashed, "p"'] \\ \overline{X} \arrow[r, "\eta_{\overline{X}}"'] \arrow[d, "e_{\overline{X}}"'] & GF\overline{X} \arrow[d, "1"] \\ \overline{\overline{X}} \arrow[r, "m_{\overline{X}}"'] & GF\overline{X} \end{tikzcd} \quad , \tag{2.7}$$

where again, p exists uniquely by the orthogonality of $\eta_{\overline{X}}$ and $e_{\overline{X}} \in$. So $pe_{\overline{X}} = 1_{\overline{X}}$, and $m_{\overline{X}} e_{\overline{X}} p = m_{\overline{X}}$ and as $m_{\overline{X}}$ is mono, $e_{\overline{X}} p = 1_{\overline{X}}$.

Thus, the inclusion of $\mathbf{A}$ into $\mathbf{C}$ has an adjoint, with unit $e_X\colon X \to \overline{X}$ consisting of the regular epi component of the factorisation of η_X. By construction, each

e_X is regular epi and so indeed, **A** is a regular epi-reflective subcategory of **C** as claimed. □

There are some properties of reflective subcategories we will find useful:

(i) A full subcategory is reflective if and only if its inclusion is monadic (and thus must have a left adjoint);
(ii) In particular, its induced monad is idempotent;
(iii) The inclusion functor creates all limits that exist in its codomain category;
(iv) The domain category inherits colimits from the codomain category by precomposing with the reflector (unit);
(v) If the codomain category is closed under some class of colimits, so is the domain category.

3. Syntactic categories and functorial semantics

3.1. Syntactic category of a concrete clone.

Definition 23 (Syntactic category of a concrete clone)**.** Let $\mathscr{T} \subseteq \mathcal{O}_V$ be a concrete clone. We define a subcategory $\mathbf{T}$ of $\mathbf{Set}_f$ category $\mathbf{T}$ from $\mathscr{T}$ as follows:

- *Objects* $[n, V]$ are sets of the form V^n for some $n \in \mathbb{N}$;
- A *morphism* $g \colon [n, V] \to [m, V]$ is a function $V^n \to V^m$ for which each of the projections $\pi_1 g, \ldots, \pi_m g \colon V^n \to V$ is in $\mathscr{T}(n)$.
- *Composition* of morphisms is given simply by composition of functions.

Proposition 24. *The category* $\mathbf{T}$ *of the preceding definition is closed under finite products in* $\mathbf{Set}_f$*, and hence is equivalent to a Lawvere theory. The faithful inclusion functor* $\mathcal{M} \colon \mathbf{T} \to \mathbf{Set}$ *exhibits* V *as a* $\mathbf{T}$*-algebra.*

3.2. **Syntactic category of a relational clone.** In this section, assume $\mathscr{R}$ is a relational clone on a (finite) set V, that is, $\mathscr{R} \subseteq \mathcal{R}_V$ and $\mathsf{Inv}\,\mathsf{Pol}(\mathscr{R}) = \mathscr{R}$. As clones of operations and clones of relations on V are in bijection, there is a unique clone $\mathscr{T} \subseteq \mathcal{O}_V$ such that $\mathsf{Pol}(\mathscr{R}) = \mathscr{T}$ and $\mathsf{Inv}(\mathscr{T}) = \mathscr{R}$. Also assume that $\mathbf{T}$ is the syntactic category (finite product theory) constructed from $\mathscr{T}$ as described in Subsection 3.1.

Definition 25 (Syntactic category from a relational clone)**.** Given $\mathscr{R}$, we define the category $\mathbf{S}$ as follows:

- *Objects* are pairs (k, S) where $k \in \mathbb{N}$ and $S \in \mathscr{R}(k)$;
- *Morphisms* $f \colon (k, S) \to (n, T)$ is a function $f \colon S \to T$ for which the graph
$$\mathrm{gr}(f) := \{(x, y) \in V^k \times V^n : x \in S, y \in T \text{ and } f(x) = y\}$$
is in $\mathscr{R}(k + n)$.
- *Composition of morphisms* are given by composition of functions; identity morphisms are given by identity functions.

The following two lemmas show that this is indeed well-defined:

Lemma 26. *Given any* k*-ary relation* $T \in \mathscr{R}(n)$*, the diagonal relation* $\Delta_T = \{(x, x) \in V^n \times V^n : x \in T\}$ *is in* $\mathscr{R}(n + n)$*.*

Proof. We use Theorem 19: Closure under the trivial relations and binary product means that Δ_{V^n} and TT are $(n+n)$-ary relations in $\mathscr{R}$. Then the diagonal relation on T can be written as

$$\Delta_T = TT \cap \Delta_{V^n}, \tag{3.1}$$

which is in $\mathscr{R}$ as it is closed under intersection of relations of the same arity. □

Lemma 27. *Assume $S \in \mathscr{R}(k)$, $T \in \mathscr{R}(n)$, $R \in \mathscr{R}(m)$, and that the graphs* $\mathrm{gr}(g)$ *and* $\mathrm{gr}(f)$ *of functions $g\colon S \to T$ and $f\colon T \to R$ are relations in $\mathscr{R}(k+n)$ and $\mathscr{R}(n+m)$ respectively. Then* $\mathrm{gr}(f \cdot g)$ *is in $\mathscr{R}(k+m)$.*

Proof. We use the closure of $\mathscr{R}$ under concatenation, the diagonal relations, intersection, and projection (Theorem 19). We have that $(S\Delta_T R) \cap (\mathrm{gr}(g)\,\mathrm{gr}(f))$ is in $\mathscr{R}(k+2n+m)$, and is equal to

$$\{(a,b,b,c) \in V^k \times V^n \times V^n \times V^m : (a,b) \in \mathrm{gr}(g), (b,c) \in \mathrm{gr}(f)\}$$

Projection onto the first and fourth coordinates give the graph of $f \cdot g$. □

Lemma 28. *Let (k,S) and (k,T) be objects in* $\mathbf{S}$*. If $T \subseteq S$, then the inclusion map lifts to a morphism $(k,T) \to (k,S)$ in* $\mathbf{S}$*.*

Proof. The graph of an inclusion is simply the diagonal of the domain, so this is a corollary of Lemma 26. □

Lemma 29. *Let $g\colon [k] \to [n]$ be a function. Then the graph of $V^g\colon V^n \to V^k$ is in $\mathscr{R}$.*

Proof. A subset of V^m defined only by equality of coordinates corresponds to an equivalence relation on m, which by Theorem 19 is in $\mathscr{R}$ as it is closed under the generalised diagonal relations. Explicitly, there is an equivalence relation θ on $[n]+[k]$ whose equivalence classes are given by $g^*(j) + \{j\}$ for each $j \in [n]$. We have

$$\begin{aligned}
\mathrm{gr}(V^g) &= \{(\alpha,\beta) \in V^n \times V^k : \beta = \alpha \cdot g\} \\
&= \{\gamma \in V^{n+k} : \gamma(i) = \gamma(g(i))\} \\
&= \{\gamma \in V^{n+k} : (\forall (i,j) \in \theta)\ \gamma(i) = \gamma(j)\} \\
&= \Delta_\theta^{n+k},
\end{aligned}$$

which is a generalised diagonal relation, and so is in $\mathscr{R}$. □

Lemma 30. *Let (k,S) and (n,T) be objects in* $\mathbf{S}$*. The graphs of the projections $\pi_1\colon T \times S \to T$ and $\pi_2\colon T \times S \to S$ are in $\mathscr{R}$.*

Proof. Let $i_n\colon [n] \to [n+k]$ be the coprojection map in $\mathbf{Set}_f$. We have that $V^{i_n}\colon V^{n+k} \to V^n$ is the same as the evaluation map $V^n \times V^k \to V^n$, and π_1 is the restriction of this evaluation map to $T \times S$. Explicit calculations give

$$\begin{aligned}
\mathrm{gr}(\pi_1) &= \mathrm{gr}(V^{i_n}) \cap TST \\
&= \mathrm{gr}(V^{i_n} \restriction_{T \times S \times T}),
\end{aligned}$$

which is in $\mathscr{R}$ by Theorem 19. Taking the coprojection i_k instead of i_n gives the desired result for π_2. □

Lemma 31. *Any morphism $(g, n+k)\colon (n,T) \to (k,S)$ in* $\mathbf{S}$ *admits a factorisation into three morphisms, whose underlying functions consist of a bijection, followed by an inclusion, followed by a projection, as in the diagram*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "(\pi_1 i)^{-1}"'] & T \\ \mathrm{gr}(g) \arrow[r, "i"] & S \times T \arrow[u, "\pi_2"'] \end{tikzcd} \qquad (3.2)$$

Proof. Consider the following diagram in $\mathbf{Set}_f$:

$$\begin{tikzcd} & S \arrow[ddl, "1_S"'] \arrow[d] \arrow[ddr, "g"] & \\ & \mathrm{gr}(g) \arrow[d, "i"] & \\ S & S \times T \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & T \end{tikzcd} \qquad (3.3)$$

Note that the inclusion i composed with the projection π_1 is a bijection and so has an inverse $(\pi_1 i)^{-1}$, appearing as the unlabelled morphism in the diagram above. Also, $\mathrm{gr}(\pi_1 i)$ is in $\mathscr{R}(n+k+k)$ as inclusions are in $\mathscr{R}$ by Lemma 28, and (binary) projections are in $\mathscr{R}$ by Lemma 30. Since $\mathrm{gr}(\pi_1 i^{-1})$ may be obtained from $\mathrm{gr}(\pi_1 i)$ by applying a permutation of coordinates, it is in $\mathscr{R}$ by Theorem 19. It is easily seen that diagram (3.3) commutes, thus giving the underlying diagram of functions of a diagram in $\mathbf{S}$. The right hand triangle gives the factorisation of $(g, n+k)\colon (n,T) \to (k,S)$ we sought. □

Notice that for each (k,S) in $\mathbf{S}$, the set S has canonical structure of a $\mathscr{T}$-algebra as a subalgebra of $\mathscr{T}$-algebra V^k.

Lemma 32. *Given objects (k,S) and (n,T) in* $\mathbf{S}$ *and a function $h\colon S \to T$ which preserves the operations in $\mathsf{Pol}(\mathscr{R}) = \mathscr{T}$, the image of h, denoted $h_!(S)$, is the underlying set of an object $(n, h_!(S))$ in* $\mathbf{S}$.

Proof. We have $h_!(S) \subseteq T \subseteq V^n$ and $h_!(S)$ is closed under the operations of $\mathsf{Pol}(\mathscr{R})$ on T. By Corollary 16, $h_!(S)$ is an n-ary relation in $\mathscr{R}$. □

Lemma 33. *Given objects (k,S) and (n,T) in* $\mathbf{S}$*, a function $h\colon S \to T$ is the underlying function of a morphism $(k,S) \to (n,T)$ in* $\mathbf{S}$ *if and only if it preserves the operations in $\mathsf{Pol}(\mathscr{R})$.*

Proof. Let (k,S) and (n,T) be objects in $\mathbf{S}$ and assume the graph of the function $g\colon S \to T$ is in $\mathscr{R}$. We will use Lemma 31.

Recall that Lemma 11 says that a relation $S \subseteq V^k$ is compatible with an operation on V just when it is a subset closed under that operation, that is, a subalgebra. So an object (k,S) in $\mathbf{S}$ admits an inclusion $r\colon S \to V^k$ in $\mathbf{Set}_f$ which preserves the operations in $\mathsf{Pol}(\mathscr{R})$. For any $R \subseteq S \subseteq V^k$, the inclusion $R \to S$ commutes with the inclusions of S and R into V^k, and so must also preserve the operations in $\mathsf{Pol}(\mathscr{R})$ (by Lemma 28 all such inclusions are morphisms in $\mathbf{S}$).

Also, given $r\colon S \to V^k$ and $q\colon T \to V^n$ are the canonical inclusions, then $r \times q\colon S \times T \to V^k \times V^n$ is also the canonical inclusion of the concatenation ST, meaning the algebra structure on $S \times T$ is pointwise. So the projections $\pi_1\colon T \times S \to T$ and $\pi_2\colon T \times S \to S$ preserve the operations in $\mathsf{Pol}(\mathscr{R})$.

Consider diagram (3.3). We have shown that the inclusion i and projections preserve the operations in $\mathsf{Pol}(\mathscr{R})$, so it remains to show that $\pi_1 i^{-1}$ does. But the inverse of a function that preserves the operations in $\mathsf{Pol}(\mathscr{R})$ must also preserve the operations in $\mathsf{Pol}(\mathscr{R})$. We may now conclude that g does also.

Conversely, assume $g\colon S \to T$ is a function preserving the operations in $\mathsf{Pol}(\mathscr{R})$. We need to show that $\mathrm{gr}(g)$ is in $\mathscr{R}$. Note that we have a span in $\mathbf{Set}_f$ whose right leg is g and whose left leg is 1_S, and from the universal property of the product, a unique function $h\colon S \to S \times T$ which commutes with the projections. Now, the projections preserve the operations in $\mathsf{Pol}(\mathscr{R})$, as do $\pi_1 h = 1_S$ and $\pi_2 h = g$ by assumption. As we have algebra structure (from $\mathsf{Pol}(\mathscr{R})$) on $S \times T$ which is pointwise, h must preserve the operations in $\mathsf{Pol}(\mathscr{R})$. The image of h is in $\mathscr{R}$ by Lemma 32, and is equal to the graph of g. We conclude that $(k+n, \mathrm{gr}(g))$ is in $\mathbf{S}$ as required. ☐

3.3. Pol-Inv **and categories of models.** Recall we have a finite product theory $\mathbf{T}$ defined from the clone of operations $\mathscr{T} = \mathsf{Pol}(\mathscr{R})$, and a model $\mathcal{M}$ endowing V with its canonical $\mathbf{T}$-algebra structure. We now characterise $\mathbf{S}$ as equivalent to a certain subcategory of the category of finite product preserving functors $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$, that is, the category of $\mathbf{T}$-models.

Proposition 34. *Let* G *be the forgetful functor from the category of* $\mathbf{T}$*-models to* $\mathbf{Set}$ *and* U *the forgetful functor* $\mathbf{S}$ *to* $\mathbf{Set}$*. There is a full and faithful functor* I *making the following diagram of functors commute:*

$$\begin{tikzcd} \mathbf{S} \arrow[d, "\mathrm{I}"'] \arrow[dr, "\mathrm{U}"] & \\ \mathbf{FP}(\mathbf{T}, \mathbf{Set}) \arrow[r, "\mathrm{G}"'] & \mathbf{Set} \end{tikzcd} \qquad . \tag{3.4}$$

Proof. It suffices to prove that given $S \subseteq V^k$, $T \subseteq V^n$ and a function $h\colon S \to T$ we have:

- S is the underlying set of an object (k, S) in $\mathbf{S}$ implies it is the underlying set of a subalgebra of $\mathcal{M}^k$ (and likewise for (n, T));
- Given the above holds, the function h is the underlying function of a morphism in $\mathbf{S}$ if and only if it is homomorphism of $\mathbf{T}$-algebras.

The underlying set of $\mathcal{M}^k$ is V^k for all $k \in \mathbb{N}$, thus the first point is equivalent to the characterisation of relations in $\mathscr{R}$ as subalgebras of finite powers of V under the pointwise algebra structure given by $\mathscr{T}$ — this is Corollary 16. The second point is Lemma 33. ☐

Lemma 35. *An object lies in the essential image of* I *if and only if it admits a monomorphism into a finite power of* $\mathcal{M}$*.*

Proof. The image factorisation of a monomorphism into a finite power of $\mathcal{M}$ gives an isomorphism with an object in the image of I. ☐

Taking our inspiration from this result, we define the following:

Definition 36. Let $\mathcal{M}$ be an object in a category $\mathbf{C}$ with finite products. We define $\mathbf{Coclo}(\mathcal{M})$ as the full subcategory of $\mathbf{C}$ whose objects consist of subobjects of finite powers of $\mathcal{M}$.

The equivalence $\mathbf{S} \simeq \mathbf{Coclo}(\mathcal{M})$ makes proving the next lemma more straightforward than using the definition of $\mathbf{S}$ alone.

Lemma 37. *We have that* $\mathbf{S} \simeq \mathbf{Coclo}(\mathcal{M})$ *has finite limits.*

Proof. The algebra $\mathcal{M}^0$ is the terminal object in $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$, and is in the essential image of I. If $\iota_A \colon A \hookrightarrow \mathcal{M}^k$ and $\iota_B \colon B \hookrightarrow \mathcal{M}^n$ are inclusions of subalgebras in $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$, then we have an inclusion $\iota_A \times \iota_B \colon A \times B \hookrightarrow \mathcal{M}^k \times \mathcal{M}^n$ in $\mathbf{Coclo}(\mathcal{M})$ (This corresponds to the fact that $\mathscr{R}$ is closed under concatenation). Since $\mathbf{Coclo}(\mathcal{M})$ is a full subcategory, this means it has finite products. Also, any equaliser of a parallel pair $A \rightrightarrows B$ with ι_A and ι_B (as given above) will admit a monomorphism into $\mathcal{M}^k$ by composing with ι_A. $\square$

From Gabriel-Ulmer duality, we obtain a characterisation of the category of models of $\mathbf{S}$:

Corollary 38. *Given the syntactic category* $\mathbf{S}$ *defined from a relational clone* $\mathscr{R}$*, the restricted presheaf category* $\mathbf{FL}(\mathbf{S}, \mathbf{Set})$ *of finite limit preserving functors is locally finitely presentable.*

We now consider how to recover the finite product theory $\mathbf{T}$. Note that for each $n \in \mathbb{N}$, the free algebra $\mathbf{T}([n, V], -)$ is a subalgebra of $\mathcal{M}^{V^n}$ as $\mathscr{T}(n)$ is a subset of V^{V^n} compatible with each operation in $\mathscr{T}$. Indeed, the free algebras on finitely many generators form a full subcategory of $\mathbf{Coclo}(\mathcal{M})$. Thus there is an object $\mathcal{W}$ in $\mathbf{FL}(\mathbf{S}, \mathbf{Set})$ corresponding to the free algebra on one generator in $\mathbf{Coclo}(\mathcal{M})$, and $[1, V]$ in $\mathbf{T}$. It is now natural to make the following definition.

Definition 39. Let $\mathcal{W}$ be an object in some category $\mathbf{C}$ with finite products. We define $\mathbf{Clo}(\mathcal{W})$ as the full subcategory of $\mathbf{C}$ whose objects are finite powers of $\mathcal{W}$.

Proposition 40. *Given a complementary clone and coclone* $\mathscr{T} = \mathsf{Pol}(\mathscr{R})$ *and* $\mathscr{R} = \mathsf{Inv}(\mathscr{T})$ *on a finite set* V*, we have a finite product theory* $\mathbf{T}$ *and a finite limit theory* $\mathbf{S}$*; Moreover, there is an object* $\mathcal{W}$ *in* $\mathbf{FL}(\mathbf{S}, \mathbf{Set})$ *and an object* $\mathcal{M}$ *in* $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$ *such that we have equivalences* $\mathbf{Clo}(\mathcal{W}) \simeq \mathbf{T}$ *and* $\mathbf{Coclo}(\mathcal{M}) \simeq \mathbf{S}$*.*

3.4. **Categorical formulation.** We now consider our setting entirely in categorical terms.

Definition 41 (Locally finite category)**.** A category is called *locally finite* if it has finite homsets.

This terminology is entirely consistent with the definition of locally finite varieties in universal algebra, which are varieties in which finitely generated free algebras are finite. Indeed, locally finite varieties are precisely those which are models of a locally finite (finite product) theory. We therefore have:

Proposition 42. *A finite product theory is equivalent to the syntactic category of a concrete clone on a finite set if and only if it is a locally finite category.*

Proposition 43. *Let* $\mathbf{T}$ *be a finite product (Lawvere) theory which is a locally finite category. Assume we have a faithful finite product preserving functor* $\mathcal{M}\colon \mathbf{T} \to \mathbf{Set}$*. Then* $\mathbf{T}$ *is equivalent to* $\mathbf{Clo}(\mathsf{W})$ *for some* W *in the category of models of* $\mathbf{Coclo}(\mathcal{M})$*. Moreover, we have* $\mathbf{FL}(\mathbf{Coclo}(\mathcal{M}), \mathbf{Set})(\mathrm{Y}\mathcal{M}, \iota(-)) \cong \mathcal{M}$ *where* ι *denotes the inclusion and* Y *is the Yoneda embedding functor.*

To say we have a concrete clone on a finite set V is to say we have a finite product (Lawvere) theory $\mathbf{T}$ which is a locally finite category, and a faithful finite product preserving functor $\mathcal{M}\colon \mathbf{T} \to \mathbf{Set}$.

Take the full subcategory $\mathbf{Coclo}(\mathcal{M})$ of $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$; it has finite limits. For any $A \in \mathbf{FP}(\mathbf{T}, \mathbf{Set})$, we can precompose the hom-functor with the inclusion, as in the diagram:

$$\begin{tikzcd} \mathbf{Coclo}(\mathcal{M}) \arrow[d, "\mathrm{I}"'] \arrow[dr] & \\ \mathbf{FP}(\mathbf{T}, \mathbf{Set}) \arrow[r, "\mathbf{FP}(\mathbf{T},\mathbf{Set})(A,-)"'] & \mathbf{Set} \end{tikzcd} \quad . \tag{3.5}$$

This functor is a model of $\mathbf{Coclo}(\mathcal{M})$; as a full and faithful functor, I preserves limits, as do covariant hom-functors, so their composition certainly preserves finite limits. Let $\mathrm{Y}\colon \mathbf{T} \to \mathbf{FP}(\mathbf{T}, \mathbf{Set})$ be the Yoneda embedding, with $\mathrm{Y}\mathbf{1}$ thus denoting the free algebra on one generator (following the usual convention with finite product theories). Take diagram (3.5) with $A = \mathsf{Y}\mathbf{1}$.

We now write $\mathsf{W} := \mathbf{FP}(\mathbf{T}, \mathbf{Set})(\mathrm{Y}\mathbf{1}, \mathrm{I}(-))$. Take $\mathbf{Clo}(\mathsf{W})$ in $\mathbf{FL}(\mathbf{Coclo}(\mathcal{M}), \mathbf{Set})$. In particular, limit preserving functors, preserve finite products, so for any $\mathsf{X} \in \mathbf{FL}(\mathbf{Coclo}(\mathcal{M}), \mathbf{Set})$, we can precompose the hom-functor with the inclusion and obtain a model of the finite product theory $\mathbf{Clo}(\mathsf{W})$:

$$\begin{tikzcd} \mathbf{Clo}(\mathsf{W}) \arrow[d, "\iota"'] \arrow[dr] & \\ \mathbf{FL}(\mathbf{Coclo}(\mathcal{M}), \mathbf{Set}) \arrow[r, "\mathbf{FL}(\mathbf{Coclo}(\mathcal{M}),\mathbf{Set})(\mathsf{X},-)"'] & \mathbf{Set} \end{tikzcd} \quad . \tag{3.6}$$

Take X in diagram (3.6) to be $\mathrm{Y}(\mathcal{M})$, where we use Y again to denote the Yoneda embedding. We have $\mathbf{Clo}(\mathsf{W}) \simeq \mathbf{T}$. Moreover,

$$\mathbf{FL}(\mathbf{Coclo}(\mathcal{M}), \mathbf{Set})(\mathrm{Y}\mathcal{M}, \iota(-)) \cong \mathcal{M} \tag{3.7}$$

and

$$\mathbf{FP}(\mathbf{Clo}(\mathsf{W}), \mathbf{Set})(\mathrm{Y}\mathsf{W}, \mathrm{I}(-)) \cong \mathsf{W} \ . \tag{3.8}$$

Let N be equal to the expression on the left-hand side of (3.7). It is straightforward that $\mathbf{Coclo}(N) \simeq \mathbf{Coclo}(\mathcal{M})$. Therefore we can go from 'categorical concrete clone' to 'categorical coclone' back to 'categorical concrete clone', and get the same thing up to isomorphism. It is less clear with what we have so far how to go from 'categorical coclone' back to 'categorical concrete clone' then back again and get essentially the same thing. How we can have this work will be made clear in Section 6. We will first find a presentation of the finitely presentable objects in the category of models of $\mathbf{Coclo}(\mathcal{M})$. This involves looking at the category of structures over the relational signature obtained from the relational clone $\mathscr{R}$.

4. Presentations via signatures

4.1. **Setting: Ω-algebras and Ψ-structures.** We will continue to consider a clone of operations $\mathscr{T} \subseteq \mathcal{O}_V$ and a clone of relations $\mathscr{R} \subseteq \mathcal{R}_V$ on a finite set V such that $\mathsf{Pol}(\mathscr{R}) = \mathscr{T}$ and $\mathsf{Inv}(\mathscr{T}) = \mathscr{R}$. From the previous section, we also have the finite product theory $\mathbf{T}$ constructed from $\mathscr{T}$, and finite limit theory $\mathbf{S}$ constructed from $\mathscr{R}$.

In addition, we have a (purely operational) signature Ω consisting of operation symbols and arities assigned from $\mathscr{T}$. Similarly, we have a (purely relational) signature Ψ consisting of relation symbols and arities from $\mathscr{R}$.

Definition 44 (The category $\mathbf{\Omega St}$)**.** We define $\mathbf{\Omega St}$ as the category whose objects A are sets A endowed with an operation $[\![\tau]\!] : A^n \to A$ for each $\tau \in \mathscr{T}(n)$, and whose morphisms are maps commuting with the interpretations. We write $\mathbf{\Omega St}_f$ for the category of finite Ω-structures (Ω-algebras). Note that $\mathbf{\Omega St}_f$ has finite products, where the interpretations of operations are given pointwise.

Definition 45 (The category $\mathbf{\Psi St}$)**.** We define $\mathbf{\Psi St}$ as the category whose objects X are sets X endowed with a relation $[\![S]\!] \subseteq X^k$ for each $S \in \mathscr{R}(k)$, and whose morphisms are maps which preserve the relations. Note that we write $\mathbf{\Psi St}_f$ for the category of Ψ-structures with finite underlying sets, and that this category has finite products where the interpretation of relations is given (pointwise) by the cartesian product.

Definition 46 ($\mathscr{R}$ inside $\mathbf{\Omega St}$)**.** We write M for the Ω-structure $\langle V; \Omega\rangle$, that is, the object with underlying set V and interpretation $[\![\tau]\!] = \tau$. For any relation $S \in \mathscr{R}(k)$, its compatibility with each operation in $\mathscr{T}$ means that we have a Ω-subalgebra

$$\mathsf{S} \xhookrightarrow{\iota_S} \mathsf{M}^k \ .$$

Definition 47 ($\mathscr{T}$ inside $\mathbf{\Psi St}$)**.** We write W for the Ψ-structure $\langle V; \Psi\rangle$, that is, the object with underlying set V and interpretations $[\![S]\!] = S$. For any n-ary operation $\tau \in \mathscr{T}$, its compatibility with each relation in $\mathscr{R}$ means that we have a Ψ-structure morphism

$$\mathsf{W}^n \xrightarrow{\tau} \mathsf{W} \ .$$

It follows from Definition 46 that M is an internal Ψ-structure in $\mathbf{\Omega St}$, and it follows from Definition 47 that W is an internal Ω-structure (Ω-algebra) in $\mathbf{\Psi St}$. Thus, we can lift the hom-functors

$$\mathbf{\Omega St}_f(-, \mathsf{M})\colon \mathbf{\Omega St}_f^{\mathrm{op}} \to \mathbf{Set}_f \qquad \mathbf{\Psi St}_f(-, \mathsf{W})\colon \mathbf{\Psi St}_f \to \mathbf{Set}_f^{\mathrm{op}}$$

to functors

$$\mathsf{M}^{(-)}\colon \mathbf{\Omega St}_f^{\mathrm{op}} \to \mathbf{\Psi St}_f \qquad \mathsf{W}^{(-)}\colon \mathbf{\Psi St}_f \to \mathbf{\Omega St}_f^{\mathrm{op}} \ .$$

Lemma 48. *We have* $\mathsf{W}^{(-)} \dashv \mathsf{M}^{(-)}$.

Proof. For an Ω-algebra A and Ψ-structure X, to give a map of Ω-algebras

$$\mathsf{A} \to \mathsf{W}^{\mathsf{X}}$$

is to give a function $\theta\colon A \times X \to V$ such that:

- For any $a \in A$, the function $\theta(a, -)\colon X \to V$ is a Ψ-structure map $\mathsf{X} \to \mathsf{W}$;
- For any $x \in X$, the function $\theta(-, x)\colon A \to V$ is an Ω-algebra map $\mathsf{A} \to \mathsf{M}$.

Symmetrically, to give a map of Ψ-structures

$$\mathsf{X} \to \mathsf{M}^{\mathsf{A}}$$

is to give a function $\theta\colon A \times X \to V$ such that:

- For any $a \in A$, the function $\theta(a, -)\colon X \to V$ is a Ψ-structure map $\mathsf{X} \to \mathsf{W}$;
- For any $x \in X$, the function $\theta(-, x)\colon A \to V$ is an Ω-algebra map $\mathsf{A} \to \mathsf{M}$.

These correspondences are easily seen to be natural in A and X. □

Lemma 49. *For any* $\mathsf{X} \in \mathbf{\Psi St}_f$*, the object* $\mathsf{W}^{\mathsf{X}} \in \mathbf{\Omega St}$ *lands in* $\mathbf{Coclo}(\mathsf{M})$*.*

Proof. Recall we have free and forgetful functors

$$\mathbf{\Psi St} \underset{\mathrm{G}^{\Psi}}{\overset{\mathrm{F}^{\Psi}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Set} \ . \tag{4.1}$$

Let $\mathsf{X} \in \mathbf{\Psi St}_f$, and consider the free Ψ-structure $\mathrm{F}^{\Psi}X$ on the underlying (finite) set of X. Note that it has the same underlying set as X, but all relations are empty. The counit $\mathrm{F}^{\Psi}\mathrm{G}^{\Psi}\mathsf{X} \to \mathsf{X}$ is an epimorphism whose image under $\mathsf{W}^{(-)}$ is a monomorphism $\mathsf{W}^{\mathsf{X}} \hookrightarrow \mathsf{M}^{\mathrm{G}^{\Psi}(\mathsf{X})}$, as desired. □

Proposition 50. *The counit of the adjunction* $\mathsf{W}^{(-)} \dashv \mathsf{M}^{(-)}$ *is invertible at any* $\mathsf{A} \in \mathbf{Coclo}(\mathsf{M})$*.*

Proof. Let $\mathsf{A} \in \mathbf{\Omega St}_f$. As Ω consists only of operations, it suffices to show that the counit $\varepsilon_{\mathsf{A}}\colon \mathsf{A} \to \mathsf{W}^{\mathsf{M}^{\mathsf{A}}}$ is an invertible function. The object $\mathsf{W}^{\mathsf{M}^{\mathsf{A}}} \in \mathbf{\Omega St}$ is given by the set of functions $\theta\colon \mathbf{\Omega St}(\mathsf{A}, \mathsf{M}) \to V$ such that, for all $S \in \mathscr{R}(k)$ and $f_1, \ldots, f_k \in \mathbf{\Omega St}(\mathsf{A}, \mathsf{M})$, we have

$$(f_1(a), \ldots, f_k(a)) \in S \text{ for all } a \in A \quad \Longrightarrow \quad (\theta(f_1), \ldots, \theta(f_k)) \in S \ . \tag{4.2}$$

In particular, for all $a \in A$, the evaluation function

$$\mathsf{ev}_a\colon f \in \mathbf{\Omega St}(\mathsf{A}, \mathsf{M}) \quad \mapsto \quad f(a) \in V$$

is in $\mathsf{W}^{\mathsf{M}^{\mathsf{A}}}$, and the counit ε_{A} sends a to ev_a.

To see ε_{A} is injective, let us begin by expressing A as a subobject

$$\mathsf{A} \overset{\iota}{\rightarrowtail} \mathsf{M}^n$$

for some $n \in \mathbb{N}$, and write $\iota_1, \ldots, \iota_n\colon \mathsf{A} \to \mathsf{M}$ for the composites of ι with the n projection maps. Now, if $\mathsf{ev}_a = \mathsf{ev}_b \in \mathsf{W}^{\mathsf{M}^{\mathsf{A}}}$, then $\iota_k(a) = \mathsf{ev}_a(\iota_k) = \mathsf{ev}_b(\iota_k) = \iota_k(b)$ for all $k \in [n]$, and so, since ι is injective, $a = b$.

This proves ε_{A} is injective; we now show surjectivity. Without loss of generality, we may assume ι is a subset inclusion; whence, since $\mathscr{R} = \mathsf{Inv}(\mathscr{T})$, we have $A \subseteq V^n$ is a relation in $\mathscr{R}(n)$. Taking S in (4.2) to be A, and the f_i's to be $\iota_1, \ldots, \iota_n$, it is obvious that $(\iota_1(a), \ldots, \iota_n(a)) \in A$ for all $a \in A$, whence $a_\theta := (\theta(\iota_1), \ldots, \theta(\iota_n)) \in A$ also.

We now show that $\theta = \mathsf{ev}_{a_\theta}$. To this end, let $f\colon \mathsf{A} \to \mathsf{M}$ be a homomorphism of Ω-structures. We have $\mathrm{gr}(f) \in \mathscr{R}(n+1)$ by Lemma 33, and evidently

$$(\iota_1(a), \ldots, \iota_n(a), f(a)) \in \mathrm{gr}(f) \quad \text{for all } a \in A.$$

Thus by (4.2) we have

$$(\theta(\iota_1), \ldots, \theta(\iota_n), \theta(f)) \in \mathrm{gr}(f)$$

which is to say that $f(x_\theta) = \theta(f)$, and so $\theta = \mathrm{ev}_{x_\theta}$ as required. □

From Lemma 49 and Proposition 50, it follows that each counit map

$$\mathsf{W}^{\mathsf{X}} \to \mathsf{W}^{\mathsf{M}^{\mathsf{W}^{\mathsf{X}}}}$$

is invertible; so the adjunction $\mathsf{W}^{(-)} \dashv \mathsf{M}^{(-)}$ is *idempotent*. In particular, this means that the unit of the adjunction is invertible at precisely those objects in the essential image of $\mathsf{M}^{(-)}$. We now characterise those.

Lemma 51. *An object $\mathsf{X} \in \mathbf{\Psi St}_f$ is in the essential image of $\mathsf{M}^{(-)}$ if and only if it can be expressed as an equaliser of maps between finite powers of W:*

$$\mathsf{X} \xrightarrow{\iota} \mathsf{W}^n \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{W}^m \tag{4.3}$$

Proof. Suppose first that $\mathsf{X} \cong \mathsf{M}^{\mathsf{A}}$ for some $\mathsf{A} \in \mathbf{\Omega St}_f$. Consider the free Ω-structure $FU(\mathsf{A}) \in \mathbf{\Omega St}_f$ on the underlying set of A. We have a coequaliser diagram of the form

$$FUFU(\mathsf{A}) \rightrightarrows FU(\mathsf{A}) \twoheadrightarrow \mathsf{A}\ .$$

Note that all of these Ω-structures are indeed finite. Applying the functor $\mathsf{M}^{(-)}$, we get an equaliser diagram in $\mathbf{\Psi St}_f$ of the form (4.3), where $n = U(\mathsf{A})$ and $m = UFU(\mathsf{A})$.

Conversely, assume that $\mathsf{X} \in \mathbf{\Psi St}$ has an expression as an equaliser (4.3). Since $\mathscr{T} = \mathsf{Pol}(\mathscr{R})$, the components $f_1, \ldots, f_m, g_1, \ldots, g_m\colon \mathsf{W}^n \to \mathsf{W}$ of f and g are operations in $\mathscr{T}(n)$. Thus we have a parallel pair of maps in $\mathbf{\Omega St}_f$

$$F(m) \underset{\gamma}{\overset{\varphi}{\rightrightarrows}} F(n)\ ,$$

where $\varphi(\pi_i) = f_i$ and $\gamma(\pi_i) = g_i$, whose image under $\mathsf{M}^{(-)}$ is the parallel pair (f, g). Thus, if we let $\mathsf{Q} \in \mathbf{\Omega St}$ be the coequaliser of φ and γ, then its image under $\mathsf{M}^{(-)}$ will be isomorphic to X, as claimed. □

Corollary 52. *The adjunction $\mathsf{W}^{(-)} \dashv \mathsf{M}^{(-)}$ restricts to an equivalence of categories between $\mathbf{Coclo}(\mathsf{M})^{\mathrm{op}}$ and the full subcategory of $\mathbf{\Psi St}_f$ on those X which can be expressed as an equaliser diagram (4.3).*

It is easily observed that $\mathbf{Coclo}(\mathsf{M}) \simeq \mathbf{Coclo}(\mathcal{M})$, so we also have that:

Corollary 53. *The full subcategory of $\mathbf{\Psi St}_f$ on those X which can be expressed as an equaliser diagram (4.3) is dually equivalent to $\mathbf{Coclo}(\mathcal{M}) \simeq \mathbf{S}$.*

Lemma 54. *The full subcategory of* $\mathbf{\Psi St}_f$ *on those* X *which can be expressed as an equaliser diagram* (4.3) *is a reflective subcategory.*

Proof. From Corollary 52, we know that the subcategory described is equivalent to $\mathbf{Coclo}(\mathsf{M})$, and so its inclusion into $\mathbf{\Psi St}_f$ has a left adjoint. In particular, the reflector is given by the unit $\mathsf{X} \to \mathsf{M}^{\mathsf{W}^{\mathsf{X}}}$ of the adjunction $\mathsf{W}^{(-)} \dashv \mathsf{M}^{(-)}$. □

Recall we have syntactic category $\mathbf{S}$ defined from the relational clone $\mathscr{R}$. An object Y of $\mathbf{FL}(\mathbf{S}, \mathbf{Set})$ admits a presentation as a structure for the signature Ψ, with the interpretation of a relation $S \in \mathscr{R}(k)$ as a relation

$$[\![S]\!] = \mathbf{FL}(\mathbf{S}, \mathbf{Set})(\mathrm{Y}(k, S), Y)$$

where Y denotes the restricted Yoneda embedding. Naturally, the morphisms in $\mathbf{FL}(\mathbf{S}, \mathbf{Set})$ preserve the relations in Ψ, and so we obtain a forgetful functor $\mathbf{FL}(\mathbf{S}, \mathbf{Set}) \to \mathbf{\Psi St}$.

5. Interlude

We continue to assume we are given a clone of operations $\mathscr{T} \subseteq \mathcal{O}_V$ and a clone of relations $\mathscr{R} \subseteq \mathcal{R}_V$ on a finite set V, with $\mathsf{Pol}(\mathscr{R}) = \mathscr{T}$ and $\mathsf{Inv}(\mathscr{T}) = \mathscr{R}$. We also have the finite product theory $\mathbf{T}$ constructed from $\mathscr{T}$, and finite limit theory $\mathbf{S}$ constructed from $\mathscr{R}$ in Section 3.3. In the same section, we introduced the canonical model $\mathcal{M}$ in $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$ endowing V with $\mathbf{T}$-algebra structure, and $\mathcal{W}$ in $\mathbf{FL}(\mathbf{S}, \mathbf{Set})$ endowing V with $\mathbf{S}$-model structure.

5.1. The story so far.

(a) The fixed points on each side of the Pol-Inv Galois connection are clones of operations $\langle V; \Omega\rangle$ and clones of relations $\langle V; \Psi\rangle$, and these are in bijection when the base set V is finite;

(b) We can construct syntactic categories $\mathbf{T}$ from $\mathscr{T}$ and $\mathbf{S}$ from $\mathscr{R}$, with equivalences $\mathbf{T} \simeq \mathbf{Clo}(\mathcal{W})$ and $\mathbf{S} \simeq \mathbf{Coclo}(\mathcal{M})$ where concrete clone $\mathscr{T} = \mathsf{Pol}(\mathscr{R})$ and relational clone $\mathscr{R} = \mathsf{Inv}(\mathscr{T})$ on V, and these are equipped with faithful functors into $\mathbf{Set}_f$ endowing V with $\mathbf{T}$-model structure corresponding to model $\mathcal{M}$ and $\mathbf{S}$-model structure corresponding to model $\mathcal{W}$;

(c) Given any of the four listed above, we can recover the other three;

(d) We can construct $\mathbf{T} \simeq \mathbf{Clo}(\mathcal{W})$ and $\mathbf{S} \simeq \mathbf{Coclo}(\mathcal{M})$ up to equivalence as subcategories of the categories of Ω-structures and Ψ-structures respectively and their models as subcategories of the categories of Ψ-structures and Ω-structures, respectively;

(e) The coclone theory (syntactic category) $\mathbf{S}$ has finite limits and so has a well developed theory of functorial semantics under Gabriel-Ulmer duality — its models form a locally finitely presentable category;

(f) Although we always have a full and faithful functor $\mathbf{T}^{\mathrm{op}} \to \mathbf{S}$, the category $\mathbf{T}$ is *not*, in general, enough to reconstruct $\mathbf{S}$ — the situation is asymmetric;

This is not altogether satisfying from a categorical perspective. We don't have an equivalence of categories capturing the duality between clones and coclones as apparent in the setting of the Pol-Inv Galois connection.

It turns out we can rectify this situation; this will be the topic of Section 6. First, we will need to know more about **Coclo**($\mathcal{M}$), and a particular class of finite limit theories and their categories of models, namely *quasivarieties*.

5.2. Quasivarieties, quasivarietal theories, and the dual of a coclone.

Definition 55 (Quasivariety)**.** A (finitary, single-sorted) *quasivariety* is a class of algebras of some fixed (purely operational) signature Ω, defined by *quasiequations*, that is, implications of the following form:

$$\bigwedge_{i\in[n]} (x_i = y_i) \implies x = y \tag{5.1}$$

where x_i, y_i, x, y are all Ω-terms and $n \in \mathbb{N}$.

Example 56. Every equation may be considered a quasiequation in which $n = 0$. Therefore any variety is a quasivariety.

Example 57. Left-cancellative semigroups are semigroups which satisfy the following quasiequation:

$$(x \cdot y = x \cdot z) \implies y = z.$$

Various characterisations of quasivarieties in categorical terms have been given, with the first attributed to Isbell [7]. The formulation we present below is Corollary 4.6 in [11].

Proposition 58. *For any category* $\mathbf{A}$*, the following are equivalent:*

(a) $\mathbf{A}$ *is a regular category with a finitely presentable, regular projective, regular generator;*
(b) $\mathbf{A}$ *is equivalent to a regular epi-reflective subcategory of a variety, and the inclusion functor preserves filtered colimits;*
(c) $\mathbf{A}$ *is equivalent to a quasivariety.*

As we might expect from Example 56, $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$ automatically satisfies the conditions to be a quasivariety, most obviously Item (b). Let $\mathbf{A}_f$ denote the full subcategory of $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$ whose objects are algebras with finite underlying sets. As a locally finite variety, the algebras with finite underlying set are precisely the finitely presentable objects in the category of $\mathbf{T}$-models $\mathbf{FP}(\mathbf{T}, \mathbf{Set})$ (which is indeed locally finitely presentable). Now, the free and forgetful adjunction may be written as

$$\mathbf{FL}(\mathbf{A}_f^{\mathrm{op}}, \mathbf{Set}) \underset{\mathbf{FL}(\mathbf{A}_f^{\mathrm{op}}, \mathbf{Set})(\mathcal{G}, -)}{\overset{\mathcal{G} * (-)}{\underset{\perp}{\leftrightarrows}}} \mathbf{Set}\ , \tag{5.2}$$

where $\mathcal{G}$ denotes the free algebra on one generator, and $*$ denotes copower. We have that $\mathcal{G}$ is finitely presentable and every object in $\mathbf{A} := \mathbf{FL}(\mathbf{A}_f^{\mathrm{op}}, \mathbf{Set})$. is a regular quotient of a copower of $\mathcal{G}$ — this is the object of Item (a).

The following characterisation of quasivarietal theories will play an important role. It is Corollary 2 in [1].

Proposition 59. *An essentially small category is equivalent to the theory of a (single-sorted) quasivariety if it has finite limits and contains a single object I such that*

(i) I is regular injective (and so every power I^n is regular injective);
(ii) Every object is a regular subobject of I^n for some $n \in \mathbb{N}$.

Note that $\mathbf{A}_f$ is finitely complete, once again, as it is a locally finite variety, so we have restricted hom-adjunctions

$$\begin{tikzcd} \mathbf{Set}_f^{\mathrm{op}} \arrow[r, shift right, "\mathcal{M}^{(-)}"'] \arrow[r, phantom, "\perp"] & \mathbf{A}_f \arrow[l, shift right, "\mathbf{A}_f(-{,}\mathcal{M})"'] \arrow[r, shift right, "\mathbf{A}_f(\mathcal{G}{,}-)"'] \arrow[r, phantom, "\perp"] & \mathbf{Set}_f \arrow[l, shift right, "\mathcal{G}*(-)"'] \end{tikzcd} \tag{5.3}$$

Lemma 60. *Consider the full subcategory of $\mathbf{A}_f$ whose objects are those algebras A for which the unit $\eta_A\colon A \to \mathcal{M}^{\mathbf{A}_f(A,\mathcal{M})}$ is a monomorphism. This category is a regular epi-reflective subcategory of $\mathbf{A}_f$, and is equivalent to* $\mathbf{Coclo}(\mathcal{M})$.

Proof. Observe that $\mathbf{A}_f$ has a (regular epi, mono) factorisation system, so by Proposition 22 the full subcategory described in the statement of the lemma is a regular epi-reflective subcategory thereof. Now, the (regular epi, mono) factorisation gives the factorisation of a morphism $h\colon A \to \mathcal{M}^k$ through its image.

By Lemma 21, the unit of the left hom-adjunction (5.3) is mono at the object $h_!(A)$, and every subalgebra of a finite power of $\mathcal{M}$ is isomorphic to $h_!(A)$ for some $h\colon A \to \mathcal{M}^k$. We may conclude that $\mathbf{Coclo}(\mathcal{M})$ is equivalent to a regular epi-reflective subcategory of $\mathbf{A}_f$. ☐

As $\mathbf{A}_f$ is finitely *cocomplete*, referring to Subsection 2.5, Items (iii)–(v), we also obtain:

Corollary 61. *The category* $\mathbf{Coclo}(\mathcal{M})$ *has finite colimits.*

In light of the results leading up to here, we will reformulate our setting in a way that suits our purposes. First, we need one last component:

Proposition 62. *The object $\mathcal{M}$ is a cogenerator in* $\mathbf{Coclo}(\mathcal{M})$.

Proof. Consider a parallel pair of morphisms in $\mathbf{Coclo}(\mathcal{M})$; their codomain is a subobject of $\mathcal{M}^k$ for some $k \in \mathbb{N}$. There is a projection which separates them. ☐

Consider the free and forgetful adjunction and the adjunction given by the inclusion ι and reflector ρ:

$$\begin{tikzcd} \mathbf{Coclo}(\mathcal{M}) \arrow[r, shift right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathbf{A}_f \arrow[l, shift right, "\rho"'] \arrow[r, shift right, "\mathbf{A}_f(\mathcal{G}{,}-)"'] \arrow[r, phantom, "\perp"] & \mathbf{Set}_f \arrow[l, shift right, "\mathcal{G}*(-)"'] \end{tikzcd} \tag{5.4}$$

From the results of this section thus far, we have the following.

Lemma 63. *The category* $\mathbf{Coclo}(\mathcal{M})$ *is small, has finite colimits, and contains an object $\rho\mathcal{G}$ such that:*

- *$\rho\mathcal{G}$ is a regularly generating object;*
- *$\rho\mathcal{G}$ is regular projective (and so every copower $\rho(\mathcal{G}) * n$ is regular projective);*
- *Every object is a regular quotient of $\rho(\mathcal{G}) * n$ for some $n \in \mathbb{N}$.*

Moreover, it contains an object $\mathcal{M}$ *such that*
- *Every object is a subobject of* $\mathcal{M}^k$ *for some* $k \in \mathbb{N}$*;*
- $\mathcal{M}$ *is a cogenerator.*

Corollary 64. *The category* $\mathbf{Coclo}(\mathcal{M})^{\mathrm{op}} \simeq \mathbf{S}^{\mathrm{op}}$ *is equivalent to the theory of a quasivariety.*

6. Duality

6.1. **Abstract coclones and their duals.** The results of Section 5.2 invite the following definitions.

Definition 65 (Abstract coclone)**.** An *abstract coclone* is an essentially small category $\mathbf{K}$ which has finite limits and finite colimits, and contains objects $\mathcal{M}$ and $\mathcal{P}$ such that:
(a) $\mathcal{M}$ is a cogenerator;
(b) Every object is a subobject of $\mathcal{M}^k$ for some $k \in \mathbb{N}$;
(c) $\mathcal{P}$ is a regular generator;
(d) $\mathcal{P}$ is regularly projective (and so every copower $\mathcal{P} * n$ is regularly projective);
(e) Every object is a regular quotient of $\mathcal{P} * n$ for some $n \in \mathbb{N}$.

We resisted the temptation to call the following theories cococlones.

Definition 66. A *quasiclone* is simply a category whose dual is an abstract coclone. That is to say, it satisfies Items (i) and (ii) in Proposition 59; has finite colimits; and contains an object J which is a generator and such that every object admits an epimorphism from a finite copower $\mathsf{J} * n$ for some $n \in \mathbb{N}$.

Example 67 (Locally finite abstract coclones)**.** Locally finite abstract relational clones are exactly those which arise as concrete on finite sets.

Example 68. An abstract coclone which does not arise as concrete on a finite set is given by the category of finite-dimensional vector spaces over the reals. As this category is self-dual, the category of models of the coclone and its dual quasivarietal theory are the same: real vector spaces.

Definition 69. The category $\mathbf{RAdj}$ has small categories as objects and right adjoints as morphisms.

Definition 70. There is a functor $(-)^\circ \colon \mathbf{RAdj}^{\mathrm{op}} \to \mathbf{RAdj}$ whose action on objects is simply given by sending a small category to its opposite (dual) category and a right adjoint to its left adjoint.

Lemma 71. *The functor* $(-)^\circ$ *gives a dual equivalence of* $\mathbf{RAdj}$ *with itself.*

Definition 72. We write $\mathbf{Coclo}$ for the full subcategory of $\mathbf{RAdj}$ whose objects are abstract coclones. Similarly, we write $\mathbf{QClo}$ for the full subcategory of $\mathbf{RAdj}$ whose objects are quasiclones.

Proposition 73. *The dual equivalence of* $\mathbf{RAdj}$ *with itself restricts to a dual equivalence* $\mathbf{Coclo}^{\mathrm{op}} \to \mathbf{QClo}$.

Proposition 74. *The dual equivalence between* $\mathbf{Coclo}$ *and* $\mathbf{QClo}$ *restricts to a dual equivalence between their locally finite members, that is, between* ${\mathbf{Coclo}_{\mathbf{Loc}}}^{\mathrm{op}} \to \mathbf{QClo}_{\mathbf{Loc}}$.

7. Concluding remarks

The duality presented in this work is a *syntactic* duality, which will never extend to the entire categories of semantics as the opposite of a locally finitely presentable category is never itself locally finitely presentable (unless it is a partially ordered set) [6]. However, there are many examples of dualities where the objects with finite underlying sets are the finitely presentable models of a coclone theory equipped with the discrete topology (given at length in [4]). In these examples, the dualising objects are a finite algebra and a finite relational structure with the discrete topology. These correspond to our objects M and W, where W is equipped with the discrete topology. The insights gained from work for this paper will be applied to concrete dualities in forthcoming work.

An interesting line of inquiry following the work of Section 4.1 would be to establish whether the category of *all* models of $\mathbf{Coclo}(\mathsf{M})$ is equivalent to a reflective subcategory of $\mathbf{\Psi St}$ — not just those with finite underlying sets. An obstacle was presented by the fact that having a finite underlying set is not sufficient to be finitely presentable in $\mathbf{\Psi St}$. Consequently, we leave open whether $\mathbf{FL}(\mathbf{Coclo}(\mathsf{M}, \mathbf{Set})$ is equivalent to a full reflective subcategory of $\mathbf{\Psi St}$.

We stated at the outset that we would only be considering relational clones on finite sets. It should now be clear that this ensured that the finitely presentable objects in our categories were *precisely the objects with finite underlying sets*. However, our duality in Section 6 did not require the imposition of any finiteness conditions. The reader may recall we began this paper by saying we provided an answer to an invitation. We end the paper with our own invitation: can Pol-Inv duality for non-finite sets be reconciled with ours?

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