Investigating error propagation in quantum phase estimation

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The quantum phase estimation algorithm estimates the phase that a unitary gate exerts on its eigenvector, and is a key subroutine in influential quantum algorithms such as Shor's prime factorisation algorithm. However, current quantum computers still suffer from significant amounts of quantum error, which propagates and accumulates throughout quantum systems—this error is the primary obstacle for useful quantum computation. This accumulation has an upper bound and thus the error level plateaus when it is reached. Using the depolarisation channel error model and taking measurements in the form of state tomographies, this investigation looked at the accuracy of the quantum state outputs from 2, 3, and 4 qubit quantum phase estimation circuits at different levels of depolarisation error, confirming previous findings about error propagation. This thereby provides further support for this approach in applications to other quantum algorithms.

Keywords: quantum error, quantum phase estimation

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Introduction and literature review

For certain tasks, quantum computers and algorithms are more efficient than their classical counterparts, at least theoretically. A notable example is Shor's namesake period finding algorithm, a quantum algorithm that is theoretically capable of finding prime factors of n-bit integers in polynomial time with a quantum circuit of size $\tilde{O}(n^2)$.[Shor (1996)]

Quantum phase estimation is a core subroutine in Shor's algorithm as well as many other quantum algorithms such as the HHL algorithm (which estimates quadratic functions of the solution vector of a system of linear equations) [Harrow et al. (2009)], or the quantum counting algorithm. The QPE algorithm is generally illustrated for a unitary operator U using its eigenvector $|\psi\rangle$ as an input, outputting an approximation of the number $\theta \in [0, 1)$ such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$.[Nielson and Chuang (2010)]

Unfortunately, currently all quantum computers still suffer from unavoidable noise and error which makes quantum error correction one of the most vital theoretical aspects of quantum computing. [Devitt et al. (2013)] Even with a small error rate, González-García et al. (2022) suggest that a single quantum bit's error rate of lower than $\frac{1}{mD}$ (where m is the number of qubits and D is the circuit depth) is necessary for the possibility of the system being more efficient than its classical counterpart.

On quantum error

Errors in quantum systems can be characterised as departures from the ideal, pure state, and the pure state is mathematically represented as a unit vector within a complex Hilbert space [Sakurai and Napolitano (2021)]. This is a lot easier to visualise in the form of a unit circle known as the Bloch sphere (Figure 1).

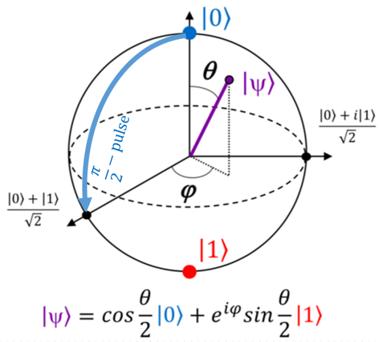

Figure 1: The Bloch sphere—$|0\rangle$ and $|1\rangle$ represent the two computational basis states (that is, 0 and 1 in binary), $|\psi\rangle$ represents any pure quantum state which can be written as a mix of $|0\rangle$ and $|1\rangle$, and θ and φ represent the amount of rotation about the x and y axis respectively.

Error propagation refers to the accumulation and amplification of these deviations over the course of quantum computation, guiding experiments on how large a circuit is able to be given a fixed error level. This study aims to investigate this within the quantum phase estimation algorithm.

There are two primary types of error—bit flip error (state flipped–the computational basis state $|0\rangle$ becomes $|1\rangle$, or the other way around) and phase flip error in which the phase is flipped. [Nielson and Chuang (2010)] Types of complex errors that may arise in the system as a result of these two errors interacting include gate error, when a quantum gate changes the state of a qubit incorrectly; decoherence error, when a quantum system interacts with its environment and becomes an entangled state; measurement error, when the classical basis measurement is wrong; and crosstalk error, when a qubit interacts with a surrounding qubit, interfering with its state. [Chatterjee et al. (2023)]

In the scenario of quantum phase estimation where the input vector for quantum phase estimation is not an eigenvector or the unitary operator is approximated using Trotter or Taylor expansion methods, Li (2022) argues that in order to obtain the phase value with error less or equal to 2^{-n} and probability at least $1-\epsilon$, the required number of qubits is $t \geq n + \log\left(2 + \frac{\delta^2}{2\epsilon\Delta E^2}\right)$ where ϵ quantifies the error and ΔE characterises the spectral gap.

There are many different ways to model error, but this paper uses the depolarisation channel model. In a circuit, the quantum channel—the communication channel that transmits quantum and classical information—can introduce decoherence or error, by coupling the qubits to other degrees of freedom, usually called the "environment" in a process known as a depolarising channel. The depolarisation channel error model works as such that there is p chance that the qubit will be completely scrambled (replaced by a completely mixed state), and $1-p$ probability that the qubit will remain intact. [Amaral and Temporão (2019)]

A very simple example of error propagation using the quantum error depolarisation channel model is on a circuit made of simply one CX (controlled-not) gate, where it can be seen (Figure 2) that a single bit flip error can be propagated through a CX gate. [Ye (2024)]

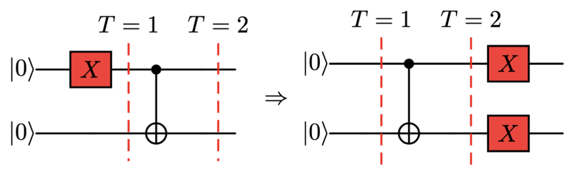

Figure 2: An example of error accumulation/amplification—propagation of a bit flip X error by CX gate. At time step $T=1$, one bit flip error occurs at the first qubit. The error propagates through the CX gate and becomes 2 bit flip errors. [Ye (2024)]

Because quantum error is characterised by discrepancies of Euler angles on the Bloch sphere (or in simpler terms, rotation of a unit vector in three dimensions), the propagation of the error is automatically limited as it is bound to the unit sphere. [Sultanow et al. (2024)]. Further, Yu and Li (2022) prove that the propagation error is upper bounded by $2(1-(1-r)^m)$ where $0 \leq r < 1$ and m is the number of qubits in the circuit, characterising the error growth as plateauing as it reaches the theoretical upper bound; this analysis bound was verified using numerical experiments on a simulator of a quantum machine.

Potential methods of error mitigation include repetition codes, such as used in Google Research's below-threshold quantum processor can be emulated. Using surface codes, which are a type of error correcting code, each group consists of a $d \times d$ square lattice of qubits called a surface code, and each surface code represents a single encoded or "logical" qubit. The expectation is that as the lattice gets bigger, the logical qubit is more and more protected from noise or error, and the logical performance improves—however, this exponential suppression only occurs if the physical error rate is below a critical threshold. Thus instead of surface codes, Google Quantum AI implements repitition codes (a different type of error correction code which focus on bit flip errors) which, despite ignoring other error types, are more efficient and help overcome the threshold accuracy. [Google Quantum AI and collaborators (2024)]

Quantum phase estimation procedure

As mentioned previously, quantum phase estimation takes the eigenvector $|\psi\rangle$ of unitary operator U as an input, and outputs a binary fraction θ in between 0 and 1, with θ representing how much rotation has been done around the bloch sphere such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$. (Remember, $2\pi i\theta$ is the *phase* (or amount of rotation) applied by the unitary operator on $|\psi\rangle$!) In other words, the algorithm takes a quantum state (which, as pictured in Figure 1, is represented by a unit vector in a complex unit sphere) and the unitary operation aka quantum gate which applies a phase upon or rotates the vector as inputs, and then outputs the fraction of the full rotation of the sphere the unitary operation causes the state to rotate!

In order to do this, start by initialising a counting register with t qubits and a target register. With t qubits in the counting register, the result will be accurate to t bits (where θ is a binary fraction that may be expressed exactly in t bits such that $\theta = 0.\theta_1\theta_2...\theta_t$), discounting error. The target register is initialised to $|\psi\rangle$. The first stage of the phase estimation algorithm goes as follows: Hadamard gates are applied to the counting register, one for each bit. Then, for each qubit in the counting register, apply the unitary operation to the qubit controlled by the respective qubit on the target register, with the unitary operation raised to successive powers of 2 for each register. This puts the circuit in this final state

$$\frac{1}{2^{t/2}}(|0\rangle + e^{2\pi i 0.\theta_t}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.\theta_{t-1}\theta_t}|1\rangle) \otimes \ ... \ \otimes (|0\rangle + e^{2\pi i 0.\theta_1\theta_2...\theta_t}|1\rangle) \tag{1}$$

$$= \frac{1}{2^{t/2}} \sum_{k=0}^{2^t-1} e^{2\pi i\theta k}|k\rangle . \tag{2}$$

Now, there is a quantum algorithm called a quantum Fourier transform, which performs this action on the basis states to transform a state into the Fourier or Hadamard basis

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N}|k\rangle . \tag{3}$$

A product representation with j having a binary representation of $j_1 j_2 ... j_n$ and $N = 2^n$ would be

$$|j_1 j_2 ... j_n\rangle \to \frac{1}{2^{n/2}} \otimes (|0\rangle + e^{2\pi i 0.j_n}|1\rangle) \otimes (|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle) \otimes ... \otimes (|0\rangle + e^{2\pi i 0.j_1 j_2 ... j_n}|1\rangle \tag{4}$$

$$= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi ijk/2^n}|k\rangle . \tag{5}$$

Back to the phase estimation algorithm, the second stage of the QPE algorithm is simply to apply the *inverse* quantum Fourier transform to the circuit. Now, by comparing the equation of the final state after the Hadamard and controlled-unitary operations (1, 2) to the product form of the quantum Fourier transformation algorithm (4, 5), we can see that the output from the second stage is the product state $|\theta_1\theta_2...\theta_t\rangle$. The third and final stage of the algorithm is to measure all qubits from the counting register. [Nielson and Chuang (2010)]

A circuit diagram of the entire operation can be viewed as such in Figure 3.

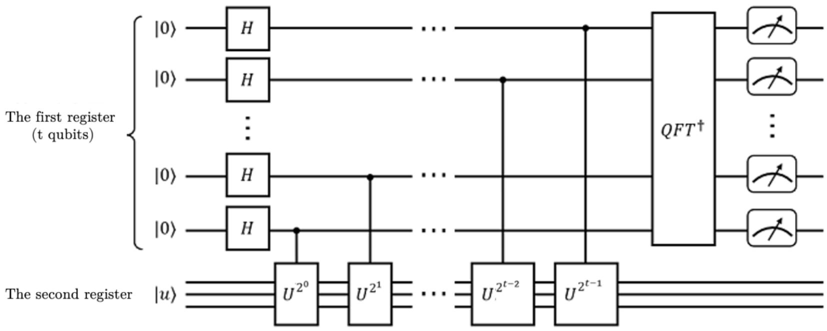

Figure 3: The quantum phase estimation circuit composition where $|u\rangle$ is the eigenvector where a phase is applied, or the quantum state in the form of a unit vector being rotated. Hadamard gates are applied to each bit of the first register, then the unitary operation (the quantum gate applying a phase on the quantum state) U is applied in consecutive powers of 2, controlling each bit on the first register and with the target being the second register, applying the phase to the control register. Then the inverse Fourier transform ($QFT^\dagger$) is applied to transform the circuit back into the computational basis, where the first register is then measured to determine the phase. [Nielson and Chuang (2010)]

Methodology

In this paper, 2, 3, and 4 qubit quantum phase estimation algorithms (that is, QPE circuits with 2, 3 and 4 bits in their counting registers) are simulated with varying levels of depolarisation channel error, essentially conducting a trial very similar to that of Woordum (2023).

As explained in the introduction, the depolarisation channel model of error means that for each gate there is probability p of each qubit's quantum state getting completely randomised and probability $1 - p$ of the quantum state remaining the same as before. The amount of error at different levels of depolarisation error will be compared, from p equals 0.01 (almost no error), 0.05, 0.1, to 0.2 (near complete randomness).

```
depol_amt = 0.01 ## this initialises level of error
dp_errormodel = NoiseModel() ## creates an error model
error = depolarizing_error(depol_amt, 1) ## init error for 1 qubit gates
error2 = depolarizing_error(depol_amt, 2) ## same but for 2 qubits, eg CX

dp_errormodel.add_all_qubit_quantum_error(error,["u1", "u2", "u3"])
dp_errormodel.add_all_qubit_quantum_error(error,"reset")
dp_errormodel.add_all_qubit_quantum_error(error,"measure")
dp_errormodel.add_all_qubit_quantum_error(error2,["cx"])
```

Qiskit's given reference code for quantum state estimation is used with its inbuilt `phase_estimation` function. The reference code (with 3 qubits), which should give a θ of 0.5, is as follows.

```
from qiskit.circuit import QuantumCircuit
from qiskit.circuit.library import phase_estimation
unitary = QuantumCircuit(2)
unitary.x(0)
unitary.y(1)
circuit = phase_estimation(3, unitary)
```

State tomography measurement circuits (which will provide us with an estimation of the quantum state of the circuit that is to be measured) are used to extract the quantum state from the measurement results in the form of density matrices, which can represent both pure and mixed states. A pure state is essentially a noise-free quantum state, whereas a mixed state has been affected by quantum noise or error. Each circuit is run on 2048 times each on a noisy and ideal simulator to extract tomography results. Each tomography result is an *estimation* of the true quantum state of the qubit.

Then Qiskit's `state_fidelity` function is used to see how close the quantum states (a value from 0 to 1, can be thought of as percentage accuracy) of the outputs are to that of the ideal output. A value of 1 would be a perfect match of two quantum states, and 0 would mean the states are completely unlike each other.

The code for this project can be found at `https://github.com/synthesis0x42/error-study-qpe/blob/main/qpe.ipynb`.

Results and discussion

The state fidelity values for each number of qubits in the counting register, as well as for each amount of depolarisation error are collected.

Table 1: State fidelity values for each value of no. qubits in counting register used in QPE algorithm versus depolarisation channel error probability p.

no. of qubits / p in depolarisation channel	0.01	0.05	0.1	0.2
2	0.80895	0.34853	0.16417	0.09647
3	0.66034	0.15086	0.05554	0.04548
4	0.45516	0.04857	0.02627	0.02361

As seen in Table 1, the state fidelity (or accuracy) of the final quantum state decreases with the addition of more and more error. Further, the more qubits (and thus more controlled-unitary operations) there are, the lower the fidelity is. This is due to accumulation and amplification of error throughout the circuit, which occurs more with higher circuit depth (more gates in the circuit). The example given in Figure 3 depicts how a single controlled-not gate causes one bit flip error to propagate and become 2 bit flip errors—with the sheer number of controlled unitary operations that are applied at higher numbers of counting registers, it is almost certain that error accumulates and is amplified very quickly, and that the amount accumulates more and faster with higher levels of error.

Plotted on a smooth line graph, the results look as in Figure 4.

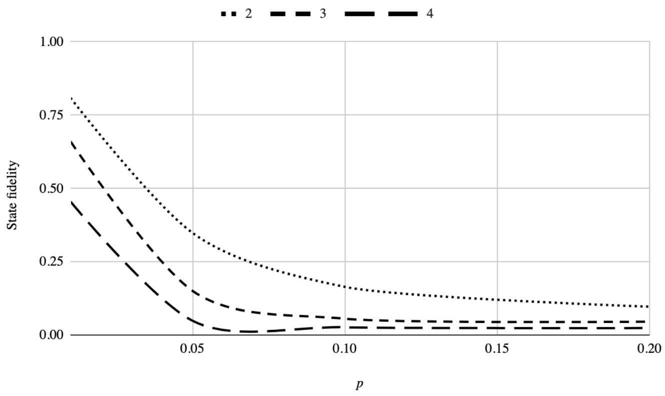

Figure 4: State fidelities of final quantum state after the QPE algorithm against depolarisation channel probability p, for 2, 3, and 4 qubits in the counting register, plotted against a linear scale.

The state fidelity rapidly drops, then its rate of decrease slows as the amount of error reaches the upper bound.

Plotted with the x-axis being a logarithmic scale, it looks more like this (Figure 5).

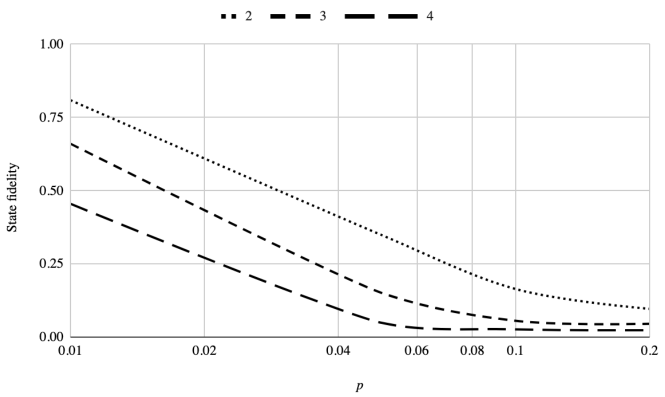

Figure 5: State fidelities of final quantum state after the QPE algorithm against depolarisation channel probability p, for 2, 3, and 4 qubits in the counting register, but plotted against a logarithmic scale. Here it is easier to see the relationship between the level of depolarisation channel error and the fidelity.

Thus we can find the state fidelity as a function of the probability of depolarization channel error p, with a stronger relationship at lower error values and with fewer registers used, due to the error plateauing as it nears the upper bound, confirming Yu and Li (2022)'s characterisation of propagation error plateauing as it reaches a theoretical upper bound.

Plotting the values on a scatterplot, the trendlines' equations are as follows (Figure 6).

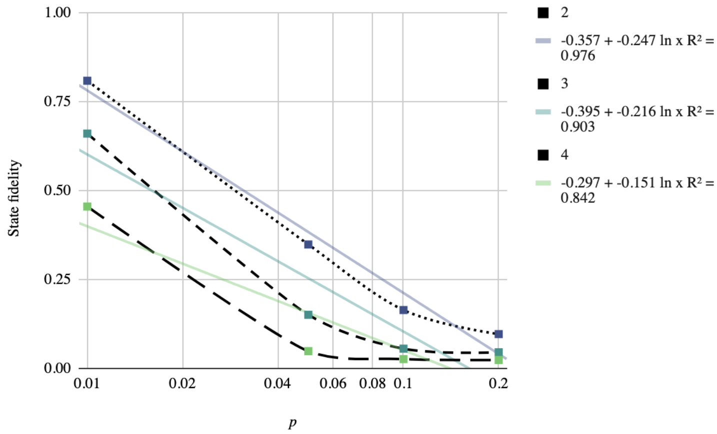

Figure 6: Trendlines, trendline equations and coefficients of determination of state fidelity against depolarisation channel error, plotted on a logarithmic scale. Trendlines can be found as a logarithmic function of amount of depolarisation channel error, although relationship is more accurate for lower levels of error and lower number of qubits in counting register or lower circuit depth).

The coefficient of determination (the proportion of the variance in state fidelity that is predictable from the value of p) is higher for the 2-qubit register, as the more controlled unitary gates the more error propagates. However, it is still reasonably high (0.903 and 0.842) when there are 3 or 4 qubits. Although the trendlines follow similarly for lower values of p, they stop becoming as accurate for higher error levels. This is to be expected, as it simply shows that there exists an upper bound to the error propagation.

Conclusion

The results of this study confirm that higher levels of error, as well as more qubits/gates used in the quantum phase estimation, lead to higher levels of accumulated error in the output. The error propagation behaves as expected, with state fidelity decreasing logarithmically as the rate of decrease slows as the amount of propagation error nears its upper bound (proven by Yu and Li (2022) to be bounded by $2(1-(1-r)^m)$, where $0 \leq r < 1$ and m is the total number of qubits in the circuit).

This research has useful applications in error analysis in Shor's period-finding algorithm and many other quantum algorithms that contain QPE as a subroutine, as one is able to isolate the error output of the quantum phase estimation circuit when analysing the whole algorithm.

Unfortunately, a slight issue with getting the state fidelity from state tomographies is that even with the exact same quantum state, the state fidelity output is unable to reach 1, which means that the fidelity that is output by the function is likely to be lower than the actual fidelity, and will suggest that the error is higher than it actually is. However, as the quantum state is collapsed upon measurement, tomography is the best option in determining the quantum states. A potential alternative way to measure accuracy is to use the mean squared error (MSE) like in Woordum (2023)'s thesis or the root mean squared error (rMSE). MSE is useful in punishing outliers more than small errors, as the square function amplifies large magnitudes more. [Jaradat et al. (2025)]

There were also only measurements of state fidelity for 4 values of p in this experiment; more experiments at different levels of error still should be completed to validate the relationship between the state fidelity and probability of the qubit scrambling or amount of depolarisation channel error p. Thus, potential next steps may involve collecting data for more values of p, alternative ways of calculating accuracy and/or error output, and potentially running on a real noisy quantum machine which by its nature as a real machine would give more accurate (albeit less predictable) results.

As an application of this research into error output of the QPE algorithm at different circuit depths and levels of error is to better determine how error propagates within Shor's period-finding algorithm (which utilises quantum phase estimation as a key subroutine) it may also be beneficial to perform error analysis directly on Shor's algorithm, or the HHL algorithm, or any other algorithm in which the QPE algorithm exists as a core subroutine.

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