1. Elementary algebra

Problem 1.1

$$\frac{z^{11}}{z^4 z^5} = \frac{z^{11}}{z^9} = z^2$$

Problem 1.2

$$6^2 3^x 2^x = 6^9$$
$$6^x = 6^7$$
$$x = 7$$

Problem 1.3

If $xy = 5$ then
$x^{-3}y^{-3} = (xy)^{-3} = 5^{-3} = 0.2^3 = 0.008$

Problem 1.4

$$\frac{\sqrt{3^{10}}}{\sqrt{9^3}} = \sqrt{\frac{3^{10}}{3^6}} = \sqrt{3^4} = 9$$

Problem 1.5

a) True b) True c) False d) True

Problem 1.6

$\frac{4x-10}{4} \geq 4 \qquad \cdot 2$
$2x - 5 \geq 8 \qquad +5$
$2x \geq 13 \qquad /2$
$x \geq 6.5$

2. Functions of one variable

Problem 2.1

Fahrenheit as a function of Celsius:

$$F = f(c) = 32 + 1.8c$$

Fahrenheit equals Celsius at

$c = 32 + 1.8c \qquad -c - 32$
$0.8c = -32 \qquad /0.8$
$c = -40$

Problem 2.2

$f(x) = 7x + 3$

$f(y) = 52$
$7y + 3 = 52 \qquad -3$
$7y = 49 \qquad /7$
$y = 7$

Problem 2.3

$10^{x^2-2x+2} = 100 \qquad log_{10}$
$x^2 - 2x + 2 = 2 \qquad -2$
$(x-2)x = 0$
$x = 2 \; or \; 0$

Problem 2.4

$$log_{1.02}2 \cong 35$$

Problem 2.5

$$ln\left(\frac{1}{e^3}\right) = lne^{-3} = -3$$

3. Calculus

Problem 3.1

$$\sum_{i=0}^{inf}\left(\frac{1}{8^i} + 0.5^i\right) = \sum_{i=0}^{inf}\frac{1}{8^i} + \sum_{i=0}^{inf}0.5^i = \frac{1}{1-\frac{1}{8}} + \frac{1}{1-0.5} = \frac{8}{7} + 2 = \frac{22}{7}$$

Problem 3.2

$\lim_{x\to3}\frac{x-3}{2} = 0$, since the value of the function at x=3 is 0.

Problem 3.3

$$f(x) = x^2 - 4$$
$$f'(x) = 2x$$

The slope at any z equals f'(z).

$$f'(-1) = -2$$
$$f'(-3) = -6$$

Problem 3.4

$$\frac{d}{dx}\frac{x^2+3}{x+2} = \frac{2x(x+2)-1(x^2+3)}{(x+2)^2} = \frac{x^2+4x-3}{(x+2)^2}$$

Problem 3.5

$$\frac{d^2x}{dx^2}4x^3+4 = \frac{d}{dx}12x^2 = 24x$$

Problem 3.6

No, it is not, since the left limit is –inf, the right limit is inf, and the function takes no value at 0.

Problem 3.7

$$f(x) = 3x^3 - 9x$$
$$f'(x) = 9x^2 - 9$$
$$f''(x) = 18x$$

Stationary points are those where f' equals to 0. These points are -1 and 1

$f''(-1) = -18$ and $f''(1) = 18$, which means that the function has a local maxima at -1 and a local minima at 1.

The second derivate is negative when under 0, zero at 0, and positive above 0, which means that the function is concave under 0, and convex above 0. It has an inflexion point at 0.

Problem 3.8

$$f(x,y) = x^3y^2$$
$$f(2,3) = 2^3 3^2 = 72$$

Problem 3.9

$$f(x,y) = ln(2x-y)$$

The function is defined where $2x - y > 0$, which is satisfied if $y < 2x$.

Problem 3.10

$$\frac{\partial^2}{\partial x^2}(x^5 + x^2y^3) = \frac{\partial}{\partial x}(5x^4 + 2xy^3) = 20x^3 + 2y^3$$

Problem 3.11

$f(x,y) = \sqrt{xy} - 0.25x - 0.25y$

The derivatives:

$f'_x(x,y) = 0.5x^{-0.5}y^{0.5} - 0.25$
$f'_y(x,y) = 0.5x^{0.5}y^{-0.5} - 0.25$
$f''_{xx}(x,y) = -0.25x^{-1.5}y^{0.5}$

$f''_{yy}(x,y) = -0.25x^{0.5}y^{-1.5}$

Solve first derivatives for 0. It takes zero at the local maximas / minimas and inflection points.

$$
\begin{array}{ll}
f'_x = 0 & \\
0.5x^{-0.5}y^{0.5} - 0.25 = 0 & +0.25; \cdot 2 \\
x^{-0.5}y^{0.5} = 0.5 & \cdot 2x^{0.5} \\
x^{0.5} = 2y^{0.5} & \text{^}2 \\
x = \pm 4y &
\end{array}
$$

$$
\begin{array}{ll}
f'_y = 0 & \\
0.5x^{0.5}y^{-0.5} - 0.25 = 0 & +0.25; \cdot 2 \\
x^{0.5}y^{-0.5} = 0.5 & \cdot 2y^{0.5} \\
y^{0.5} = 2x^{0.5} & \text{^}2 \\
y = \pm 4x &
\end{array}
$$

x and y can’t be zero due to the negative exponent. Given the result of the two equations, the only possible solution would have been (x,y)=0, which is not allowed. Hence, the function has no local minima or maxima.

Problem 3.12

$$\max x^2y^2$$
$$x + y = 5$$

The Lagrange equation and it’s derivatives:

$$L(x,y) = x^2y^2 - \lambda(x + y - c)$$
$$L'_x = 2xy^2 - \lambda$$
$$L'_y = 2x^2y - \lambda$$

Solve derivatives to zero:

$$L'_x = 0$$
$$x = \frac{\lambda}{2y^2}$$

$$L'_y = 0$$
$$y = \frac{\lambda}{2x^2}$$

The three equation to solve is:

1) $x = \frac{\lambda}{2y^2}$
2) $y = \frac{\lambda}{2x^2}$
3) $x + y = 5$

The solution:

$\lambda = 2xy^2$ from 1)
$y = \frac{2xy^2}{2x^2}$ from 2) and the previous equation
$2x^2y = 2xy^2$ from the previous equation
$x = y$ from the previous equation
$2x = 5$ from 3) and the previous equation
$\underline{\underline{x = y = 2.5}}$ from the previous equation

4. Linear algebra

Problem 4.1

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 8 \\ 6 & 17 & 6 \\ 5 & 6 & 5 \end{bmatrix}$$

Problem 4.2

$$\begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 9 \\ 10 & 11 \end{bmatrix}$$

Problem 4.3

$$\begin{bmatrix} 3.3 & 5.1 & 4.7 \\ 2 & 6.1 & 1.23 \\ 4 & 5.76 & 0 \end{bmatrix}^T = \begin{bmatrix} 3.3 & 2 & 4 \\ 5.1 & 6.1 & 5.76 \\ 4.7 & 1.23 & 0 \end{bmatrix}$$

Problem 4.4

$$det\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = 2 \cdot 5 - 3 \cdot 4 = -2$$

5. Probability theory

Problem 5.1

The sample space is.

$$\begin{aligned}\Omega = \{&(H,H,H,H),(H,H,H,T),(H,H,T,H),(H,H,T,T),\\ &(H,T,H,H),(H,T,H,T),(H,T,T,H),(H,T,T,T),\\ &(T,H,H,H),(T,H,H,T),(T,H,T,H),(T,H,T,T),\\ &(T,T,H,H),(T,T,H,T),(T,T,T,H),(T,T,T,T)\}\end{aligned}$$

Problem 5.2

$$P(X = 1) = 0.01$$

$$P(X=0)=0.99$$
$$P(Y=1\,|X=1)=0.99$$
$$P(Y=0\,|X=0)=0.995$$

$$P(Y=1\,|X=0)=0.005$$
$$P(X=1\,|Y=1)=\frac{P(X=1)\cdot P(Y=1\,|X=1)}{P(X=1)\cdot P(Y=1\,|X=1)+P(X=0)\cdot P(Y=1\,|X=0)}$$
$$=\frac{0.01\cdot 0.99}{0.01\cdot 0.99+0.99\cdot 0.005}=\frac{2}{3}$$

Problem 5.3

$$E(x)=\frac{1+2+3+4+5+6}{6}=3.5$$
$E(x_1)+E(x_2)=E(2x)=2E(x)$, because the tosses are independent.

$2E(x)=7$