Belief Revision

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To attain knowledge, add things everyday.
To attain wisdom, remove things every day.

– Lao Tzu

Belief Change — An example (Gärdenfors & Rott 1995)

- Beliefs

  The bird caught in the trap is a swan
  The bird caught in the trap comes from Sweden
  Sweden is part of Europe
  All European swans are white

- Consequence

  The bird caught in the trap is white

- New information

  The bird caught in the trap is black

Which sentence(s) would you give up?

Problem arises in several areas:

Databases: New entry inconsistent with database.

Robotics: Sensor information inconsistent with plans.

Diagnosis: Device behavior inconsistent with device description.

Ontologies: Concepts or properties added/retracted.

...

What we would like

Keep consistency

Minimal change

Logic is not enough!

Problem is not trivial:

- Choice involved.
- Indirect consequences of revision.
- Representation issues.

AGM Theory

Alchourrón, **G**ärdenfors, and **M**akinson (1985)

“On the Logic of Theory Change: Partial Meet Contraction and Revision Functions”

Preliminaries

- Epistemic States: Sets of formulas (belief sets) $K = Cn(K)$.
- Epistemic Attitudes:
  - $\alpha \in K$ - α accepted
  - $\neg\alpha \in K$ - α rejected
  - otherwise - α undetermined
- Input: Formula

Preliminaries

Three operations:

- Expansion ($K + \alpha = Cn(K \cup \{\alpha\})$)
- Contraction ($K \dot{-} \alpha$)
- Revision ($K * \alpha$)

$K * \alpha = (K \dot{-} \neg\alpha) + \alpha$ (Levi identity)
$K \dot{-} \alpha = K \cap (K * \alpha)$ (Harper identity)

The AGM framework

Five equivalent presentations:

- postulates
- partial meet
- safe
- entrenchment
- spheres/possible worlds

Contraction Postulates

(**K-1**) $K \dot{-} \alpha$ is a belief set (*closure*)
(**K-2**) $K \dot{-} \alpha \subseteq K$ (*inclusion*)
(**K-3**) If $\alpha \notin K$, then $K \dot{-} \alpha = K$ (*vacuity*)
(**K-4**) If not $\vdash \alpha$, then $\alpha \notin K \dot{-} \alpha$ (*success*)
(**K-5**) If $\alpha \in K$, then $K \subseteq (K \dot{-} \alpha) + \alpha$ (*recovery*)
(**K-6**) If $\vdash \alpha \leftrightarrow \beta$, then $K \dot{-} \alpha = K \dot{-} \beta$ (*equivalence*)

Constructions for Contraction

- Partial Meet (AM)
- Safe Contraction (AM)
- Systems of Spheres (Grove)
- Epistemic Entrenchment (MG)

Partial Meet Contraction – 1

Remainder: $X \in K \perp \alpha$ if and only if:

- $X \subseteq K$.
- $X \nvdash \alpha$.
- For all X' such that $X \subset X' \subseteq K$, $X' \vdash \alpha$.

Selection Function: γ such that:

- $\gamma(K \perp \alpha) \subseteq K \perp \alpha$.
- If $K \perp \alpha \neq \emptyset$, then $\gamma(K \perp \alpha) \neq \emptyset$
  otherwise, $\gamma(K \perp \alpha) = \{K\}$

Partial Meet Contraction – 2

Example: Let $K = Cn(p \wedge q)$

- $K \perp p = \{Cn(p \leftrightarrow q), Cn(q)\}$
- $K \perp p \wedge q = \{Cn(p), Cn(q), Cn(p \leftrightarrow q)\}$
- $K \perp p \rightarrow q = \{Cn(p), Cn(q)\}$

We may have:

- $\gamma(K \perp p) = \{Cn(p \leftrightarrow q)\}$
- $\gamma(K \perp p \wedge q) = \{Cn(p), Cn(q)\}$
- $\gamma(K \perp p \rightarrow q) = \{Cn(p), Cn(q)\}$

Partial Meet Contraction – 3

Definition:

$$K \dot{-} \alpha = \bigcap \gamma(K \perp \alpha).$$

Theorem (AGM): An operation $\dot{-}$ on K is a partial meet contraction if and only if it satisfies postulates (**K-1**)-(**K-6**).

Limit Cases – Full meet

- All elements of $K \perp \alpha$ are selected, that is:

$$K \dot{-} \alpha = \bigcap (K \perp \alpha).$$

  No choice involved.

- Problem: revision can lead to a set which is too small.

> *If a revision operation is defined from full meet contraction by means of the Levi identity, then, for any α such that $\neg\alpha \in K$, $K * \alpha = Cn(\alpha)$.*

Limit Cases – Maxichoice

- Only one element of $K \perp \alpha$ is selected, that is:

$$K \dot{-} \alpha = \gamma(K \perp \alpha) \in K \perp \alpha.$$

- Problem: revision may lead to a set which is too big.

> *If a revision operation is defined from a maxichoice contraction by means of the Levi identity, then, for any α such that $\neg\alpha \in K$, $K * \alpha$ will be maximal, i.e., for every formula β, either $\beta \in K * \alpha$ or $\neg\beta \in K * \alpha$.*

Limit Cases – Examples

- Full meet: Suppose I believe that p (Buenos Aires is the capital of Brazil) and that q (there is no King of France). When I learn $\neg p$ and revise my belief set using a revision operation based on full meet contraction, I give up the belief that there is no King of France.
- Maxichoice: Suppose I believe p (that Buenos Aires is the capital of Brazil) and have no idea about q (that the King of France is bald). Finding out that $\neg p$ is the case and revising my belief set using a revision based on maxichoice contraction means that I will make a decision as to q or $\neg q$.

Safe Contraction – 1

Kernel: $X \in K \perp\!\!\!\perp \alpha$ if and only if:

- $X \subseteq K$.
- $X \vdash \alpha$.
- For all X' such that $X' \subset X \subseteq K$, $X' \nvdash \alpha$.

Incision Function: Let $<$ be an acyclical hierarchy over K. An incision function σ selects from each element of $K \perp\!\!\!\perp \alpha$ the minimal elment w.r.t. $<$:

- $\sigma(K \perp\!\!\!\perp \alpha) \subseteq \bigcup(K \perp\!\!\!\perp \alpha)$.
- If $X \in K \perp\!\!\!\perp \alpha$, $X \neq \emptyset$, β minimal in X, then $\beta \in X \cap \sigma(K \perp\!\!\!\perp \alpha)$.

Safe Contraction – 2

Definition:

$$K \dot{-} \alpha = K \setminus \sigma(K \perp\!\!\!\perp \alpha).$$

Theorem (AM85): An operation $\dot{-}$ on K is a safe contraction if and only if it satisfies postulates (**K-1**)-(**K-6**).

Criticisms

- Recovery
- Success
- Iteration!

Recovery – Example (Hansson)

I believe that "Cleopatra had a son" *(α) and that* "Cleopatra had a daughter" *(β), and thus also that* "Cleopatra had a child" *($\alpha \vee \beta$, briefly δ). Then I receive information that makes me give up my belief in δ, and contract my belief set accordingly, forming $K \dot{-} \delta$. Soon afterwards I learn from a reliable source that* "Cleopatra had a child"*. It seems perfectly reasonable for me to then add δ (i.e. $\alpha \vee \beta$) to my set of beliefs without also reintroducing either α or β.*

Success: Examples (Fermé)

“Yesterday, the Pope called to wish me good luck in the tutorial.”
“Yesterday, my mother called to wish me good luck in the tutorial.”

One day when you return back from work, your son tells you, as soon as you see him: “A dinosaur has broken our Ming’s dynasty vase in the living-room”.

Problems with the use of logically closed belief sets

- Infinite sets.
- Inconsistency leads to trivialization.
- No distinction between explicit and implicit beliefs.

Reasons for Using Belief Sets

- Syntax independence
  - what matters is the content, not the form.
- Knowledge level in AI
  - coexists with other levels of description.
- Logical elegance.

Bases – Two traditions

- **Dalal:** Base is just a representation, syntax independence.
- **Hansson:** Syntax matters.

Belief Bases (à la Hansson)

- Belief base B finite set of formulas.
- Expansion: $B + \alpha = B \cup \{\alpha\}$.
- Epistemic attitudes:
  - $\alpha \in Cn(B)$: α (implicitly) believed.
  - $\alpha \in B$: α explicitly believed.
  - $\alpha \in Cn(B) \setminus B$: α merely derived.

Example (Hansson)

- α: Paris is the capital of France.
- β: There is milk in the fridge.
- $\alpha, \beta \in B \Rightarrow \alpha \leftrightarrow \beta \in Cn(B)$

When we revise by $\neg\beta$, we must choose between giving up α and $\alpha \leftrightarrow \beta$.

*In the belief base approach, $\alpha \leftrightarrow \beta$ is automatically chosen and α remains in the revised base (***"Disbelief Propagation"***).*

More advantages of the use of bases

Expressivity $B_1 = \{\alpha, \beta\}$, $B_2 = \{\alpha, \alpha \leftrightarrow \beta\}$.
$Cn(B_1) = Cn(B_2)$
$B_1 * \neg\alpha = \{\neg\alpha, \beta\}$
$B_2 * \neg\alpha = \{\neg\alpha, \alpha \leftrightarrow \beta\}$
$\beta \in Cn(B_1 * \neg\alpha)$, but $\beta \notin Cn(B_2 * \neg\alpha)$.

Inconsistency Tolerance $B_1 = \{p, \neg p, q_1, q_2, q_3\}$
$B_2 = \{p, \neg p, \neg q_1, \neg q_2, \neg q_3\}$
$Cn(B_1) = Cn(B_2)$, but $Cn(B_1 \dot{-} p) \neq Cn(B_2 \dot{-} p)$

Partial Meet Base Contraction — Construction

- $B \perp \alpha$: maximal subsets of B that fail to imply α
- γ: function that selects some elements of $B \perp \alpha$
- $B \dot{-}_{\gamma} \alpha = \bigcap \gamma(B \perp \alpha)$

Partial Meet Base Contraction — Postulates

- If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(B \dot{-} \alpha)$ (success)
- $B \dot{-} \alpha \subseteq B$ (inclusion)
- If $\beta \in B \setminus (B \dot{-} \alpha)$, then there is some B' such that $B \dot{-} \alpha \subseteq B' \subseteq B$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$ (relevance)
- If for all subsets B' of B, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B \dot{-} \alpha = B \dot{-} \beta$ (uniformity)

Partial Meet Base Contraction — Results

- Representation Theorem.
- Postulates not as intuitive as AGM.
- Recovery does not hold: $B = \{p \wedge q\}$, $(B \dot{-} p) + p = \{p\}$.
- Operation is really different from AGM. For example, maxichoice is not a problem for belief bases. It is actually often used in practice.

Kernel Contraction — Idea

- Generalization of safe contraction.
- No hierarchy of formulas.
- Unlike AGM partial meet/safe contraction, kernel contraction is more general than partial meet.

Kernel Base Contraction — Construction

- $B \perp\!\!\!\perp \alpha$: minimal subsets of B that imply α
- σ: function that selects at least one element of each set in $B \perp\!\!\!\perp \alpha$
- $B \dot{-}_{\sigma} \alpha = B \setminus \sigma(B \perp\!\!\!\perp \alpha)$

Kernel Base Contraction — Postulates

- If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(B \dot{-} \alpha)$ (success)
- $B \dot{-} \alpha \subseteq B$ (inclusion)
- If $\beta \in B \setminus B \dot{-} \alpha$, then there is some $B' \subseteq B$ such that $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$ (core-retainment)
- If for all subsets B' of B, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B \dot{-} \alpha = B \dot{-} \beta$ (uniformity)

Contraction - Example

$B = \{p, p \vee q, p \leftrightarrow q\}$

$B \perp\!\!\!\perp (p \wedge q) = \{\{p, p \leftrightarrow q\}, \{p \vee q, p \leftrightarrow q\}\}$
$B \perp (p \wedge q) = \{\{p, p \vee q\}, \{p \leftrightarrow q\}\}$

$B \dot{-} (p \wedge q) = \{p\}$ can be constructed as kernel but not partial meet contraction.

Revision of Belief Bases

- Levi Identity
  Internal revision:
  $B \mp \alpha = (B \dot{-} \neg\alpha) + \alpha$
- Reversed Levi Identity (Hansson)
  External revision:
  $B \pm \alpha = (B + \alpha) \dot{-} \neg\alpha$
    - Intermediate state may be inconsistent.

Revision of Belief Bases – Examples

Let $B = \{p \to r, p \to \neg r\}$.

(a) internal revision: $B \perp \neg p = \{\{p \to r\}, \{p \to \neg r\}\}$, hence,
for $\gamma_1(B \perp \neg p) = \{\{p \to r\}\}$, we have
$B \mp_{\gamma_1} p = \{p \to r, p\}$.

(b) external revision: $B + p \perp \neg p = \{\{p \to r, p\}, \{p \to \neg r, p\}\}$,
hence,
for $\gamma_2(B + p \perp \neg p) = \{\{p \to \neg r, p\}\}$, we have
$B \pm_{\gamma_2} p = \{p \to \neg r, p\}$.

Partial Meet Revision

(B*1) If $\neg\alpha \notin Cn(\emptyset)$, then $\neg\alpha \notin Cn(B \mp \alpha)$ (non-contradiction)
(B*2) $B \mp \alpha \subseteq B \cup \{\alpha\}$ (inclusion)
(B*3) If $\beta \in B \setminus B \mp \alpha$, then there is some B' such that $B \mp \alpha \subseteq B' \subseteq B \cup \{\alpha\}$, $\neg\alpha \notin Cn(B')$ but $\neg\alpha \in Cn(B \cup \{\beta\})$ (relevance)
(B*4) $\alpha \in B \mp \alpha$ (success)
(B*5) If for all $B' \subseteq B$, $\neg\alpha \in Cn(B')$ if and only if $\neg\beta \in Cn(B')$, then $B \cap (B \mp \alpha) = B \cap (B \mp \beta)$ (uniformity)