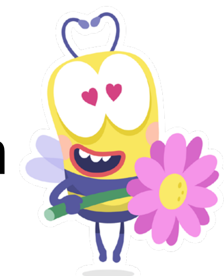
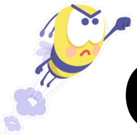
HW1 Tips Office Hours
Tips & Hints for Week 1 Homework

ISyE 6644 Simulation



Georgia Tech H. Milton Stewart School of Industrial and Systems Engineering
College of Engineering



General Homework Guidelines/Tips

- Feel free to work on your homework anytime the submission window is open. No worries if you leave mid-way, your work will be saved.
- Be confident before hitting the 'Submit' button; once submitted, it's final. Consider waiting until after Monday's office hours, where we share helpful tips and advice.
- Please submit your homework on time in Canvas to avoid it being marked as late.
- Encountering any issues with grading or submissions? No problem, just post privately on Piazza for a quick resolution.
- Don't worry about trick questions. They're not intended, though it may seem so at times. Avoid overthinking or making too many assumptions.
- Don’t forget about the additional notes posted in the weekly announcements in Piazza!

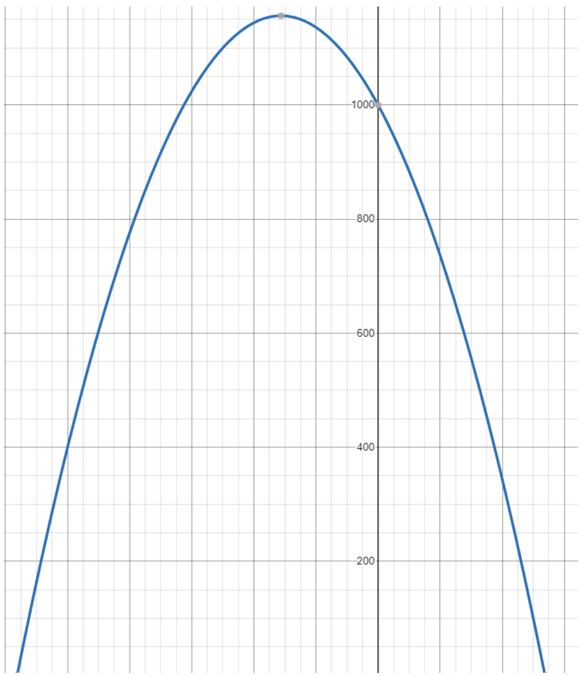
(Lesson 1.3: Deterministic Model.) Suppose you throw a rock off a cliff having height h_0 = 1000 feet. You're a strong bloke, so the initial downward velocity is v_0 = -100 feet/sec (slightly under 70 miles/hr). Further, in this neck of the woods, it turns out there is no friction in the atmosphere - amazing! Now you remember from your Baby Physics class that the height after time t is

$$h(t) = h_0 + v_0 t - 16t^2$$

When does the rock hit the ground?

Problem is an example of a deterministic model (with several assumptions to simplify the problem – e.g., no friction, etc.)

Can use any method (quadratic formula, root solver functions in programming languages, calculator, graphically, Bisection, Newton's method, etc.) to find when the rock (pictured below) hits the ground.



The x-axis has been removed to protect the innocent.

Question 2

(Lesson 1.3: Stochastic Model.) Consider a single-server queueing system where the times between customer arrivals are independent, identically distributed Exp(λ = 2/hr) random variables; and the service times are i.i.d. Exp(μ = 3/hr). Unfortunately, if a potential arriving customer sees that the server is occupied, he gets mad and leaves the system. Thus, the system can have either 0 or 1 customer in it at any time. This is what's known as an M/M/1/1 queue. If $P(t)$ denotes the probability that a customer is being served at time t, trust me that it can be shown that

$$P(t) = \frac{\lambda}{\lambda+\mu} + \left[P(0) - \frac{\lambda}{\lambda+\mu}\right] e^{-(\lambda+\mu)t}.$$

READ THIS QUESTION CAREFULLY AND MAKE SURE YOU ARE ANSWERING WHAT IT IS ASKING!!!

If the system is empty at time 0, i.e., $P(0) = 0$, what is the probability that there will be no people in the system at time 1 hr?

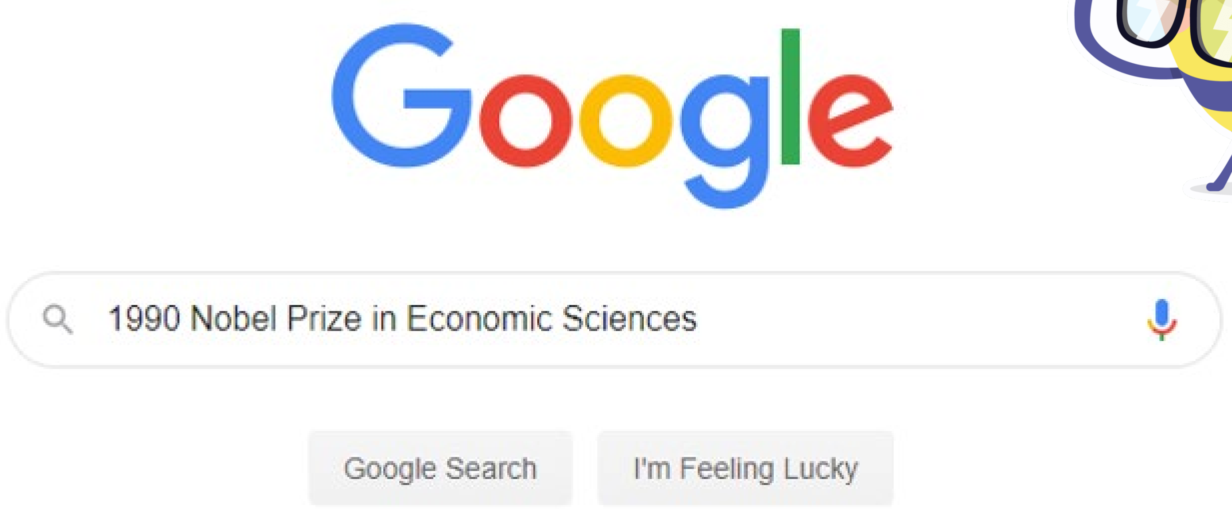
An example of a stochastic model (an example from queueing theory). The equation is derived based on properties of this M/M/1/1 queueing model by solving differential equations.

For reference, the "A/B/C/D/E" notation is known as Kendall's Notation. A is the probability distribution of the interarrival time. B is the probability distribution of the service time. C is the number of servers in the system. D is the maximum number of customers allowed in the system. E would be the queue discipline (FIFO, LIFO, etc.). In this case, there is no queue, so E is omitted. "M" stands for Markovian or memoryless and refers to a Poisson process (e.g., exponential inter-arrival / exponential service times). <u>This is extra information for your own edification that I thought would be interesting to include – knowing this notation is not critical to answer this question.</u>

See https://en.wikipedia.org/wiki/Kendall%27s_notation

Question 3

(Lesson 1.4: History.) Harry Markowitz (one of the big wheels in simulation language development) won his Nobel Prize for portfolio theory in 1990, though the work that earned him the award was conducted much earlier in the 1950s. Who won the 1990 Prize with him? You are allowed to look this one up.



(Lesson 1.5: Applications.) Which of the following situations might be good candidates to use simulation? (There may be more than one correct answer.)

- [ ] a. We put $5000 into a savings account paying 2% continuously compounded interest per year, and we are interested in determining the account's value in 5 years.
- [ ] b. We are interested in investing one half of our portfolio in fixed-interest U.S. bonds and the remaining half in a stock market equity index. We have some information concerning the distribution of stock market returns, but we do not really know what will happen in the market with certainty.
- [ ] c. We have a new strategy for baseball batting orders, and we would like to know if this strategy beats other commonly used batting orders (e.g., a fast guy bats first, a big, strong guy bats fourth, etc.). We have information on the performance of the various team members, but there's a lot of randomness in baseball.
- [ ] d. We have an assembly station in which "customers" (for instance, parts to be manufactured) arrive every 5 minutes exactly and are processed in precisely 4 minutes by a single server. We would like to know how many parts the server can produce in a hour.
- [ ] e. Consider an assembly station in which parts arrive randomly, with independent exponential interarrival times. There is a single server who can process the parts in a random amount of time that is normally distributed. Moreover, the server takes random breaks every once in a while. We would like to know how big any line is likely to get.
- [ ] f. Suppose we are interested in determining the number of doctors needed on Friday night at a local emergency room. We need to insure that 90% of patients get treatment within one hour.

Module 1, Lesson 3 – Stochastic vs. Deterministic

Examine each scenario and if it is deterministic, then it is probably not a good candidate for simulation.

Scenarios that are too tough for analytical methods and include some randomness are probably good candidates for a simulation approach.

Question 5 & 6

(Lessons 1.6 and 1.7: Baby Examples.) The planet Glubnor has 50-day years.

Suppose there are 2 Glubnorians in the room. What's the probability that they'll have the same birthday?

Different version of "the birthday problem" below (for humans). I'm not going through the derivation for questions 5 and 6 – the explanation in M1L6 Baby Examples Additional Notes can be modified for planet Glubnor.

(Lessons 1.6 and 1.7: Baby Examples.) The planet Glubnor has 50-day years.

Suppose there are 2 Glubnorians in the room. What's the probability that they'll have the same birthday?

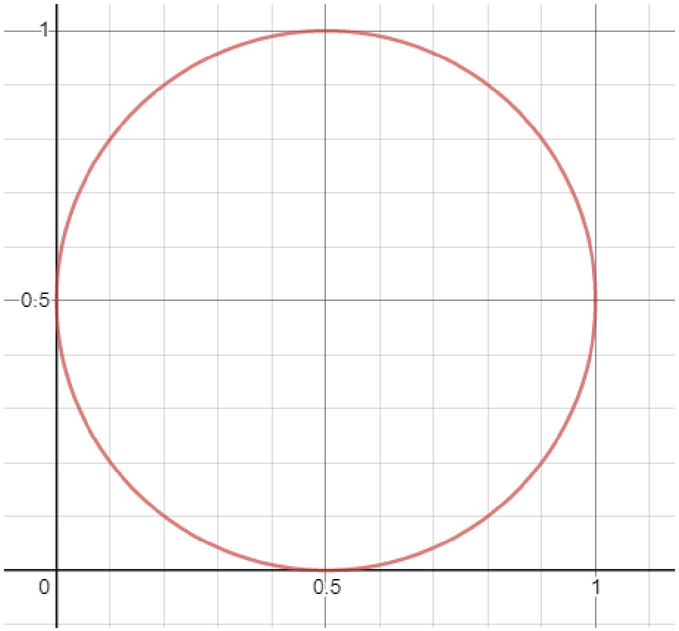
See file "HW1 Tips Office Hours.R"

```r
# Questions 5 and 6 ####

# R has a built in function, pbirthday
# Check ?pbirthday for documentation

pbirthday(n = 2, classes = 365)

1 / 365

1 - (365/365)*(364/365)

# Custom function to calculate probability there are no matches
# with d days and n people
p_no_match <- function(d, n) {
  prod(d:(d - n + 1)) / d^n
}

1 - p_no_match(365, 2)

# Simulation approach
# Generate a sample of 2 days and check if they match
set.seed(1)
(birthdays <- sort(sample(365, size = 2, replace = TRUE)))
any(duplicated(birthdays))

# Do this many times to approximate the probability calculated above
check_sample <- function(d, n) {
  birthdays <- sample(d, size = n, replace = TRUE)
  any(duplicated(birthdays))
}

results <- sapply(1:1e5, function(x) check_sample(365, 2))
mean(results)
```

(Lessons 1.6 and 1.7: Baby Examples.) Inscribe a circle in a unit square and toss $n = 500$ random darts at the square.

Suppose that 380 of those darts land in the circle. Using the technology developed in this lesson, what is the resulting estimate for π?

Review M1L6 Baby Examples Additional Notes



$\boldsymbol{X \sim Uniform(0,1)}$
$\boldsymbol{Y \sim Uniform(0,1)}$

When would (x, y) fall inside the circle?

The equation of the circle is $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

We want points that are inside, so this would be when the equation above is not an equality, but $\leq$

Question 7

(Lessons 1.6 and 1.7: Baby Examples.) Inscribe a circle in a unit square and toss $n = 500$ random darts at the square.

Suppose that 380 of those darts land in the circle. Using the technology developed in this lesson, what is the resulting estimate for π?

See file “HW1 Tips Office Hours.R”

```r
# Questions 7 and 8 ####
# If you don't have the following packages, then install them by uncommenting
# the following two lines.
# install.packages("ggplot2")
# install.packages("ggforce")
library(ggplot2)
library(ggforce)

# Generate many uniform xs and ys between 0 and 1
x <- runif(1e3)
y <- runif(1e3)

# Logical indicating whether or not the point "landed" inside the circle
inside_circle <- (x - 0.5)^2 + (y - 0.5)^2 < 0.5^2

# Graph to show the points and whether or not they landed inside the circle
ggplot(data.frame(x = x, y = y, inside_circle = inside_circle)) +
  geom_point(aes(x = x, y = y, color = inside_circle)) +
  geom_circle(aes(x0 = 0.5, y0 = 0.5, r = 0.5)) +
  scale_color_manual(values = c("red", "green"))

# Find proportion by dividing number inside circle divided by total number
sum(inside_circle == TRUE) / length(inside_circle)
# Alternative is to take the mean of the logical vectors (TRUE = 1, FALSE = 0)
mean(inside_circle)

# Approximation of pi is proportion times 4
mean(inside_circle) * 4
```

(Lessons 1.6 and 1.7: Baby Examples.) Again inscribe a circle in a unit square and toss n random darts at the square.

What would our estimate be if we let $n \to \infty$ and we applied the same ratio strategy to estimate π ?

The Law of Large Numbers is critical to this problem. This problem can fall under the category of people thinking "too much" about it and talking themselves out of the correct answer.

The key is $n \to \infty$

If it were possible to run the simulation an infinite number of times, what result would we get?

Since M2L13 Limit Theorems Additional Notes are available, you can look at the section on the Law of Large Numbers.

(Lessons 1.6 and 1.7: Baby Examples.) Suppose customers arrive at a single-server ice cream parlor times 3, 6, 15, and 17. Further suppose that it takes the server 7, 9, 6, and 8 minutes, respectively, to serve the four customers. When does customer 4 leave the shoppe?

Arrive	Start	Service	Leave	Wait
3		7		
6		9		
15		6		
17		8		

Arrive: The arrival times of the customers

Start: When service starts for a customer. Can only start when the server is not busy. Will start at the arrival time if the server is not busy when customer arrives or at the leave time of the prior customer if the customer arrives and the server is busy.

Service: The service times for each customer

Leave: When the customer has completed service (Start + Service)

Wait: How long the customer waited (Start – Arrive)

Will learn more about the ins and outs of how the arrival times and service times are generated and how different queue disciplines (FIFO, LIFO, etc.) work in future modules.

(Lesson 1.8: Generating Randomness.) Suppose we are using the (awful) pseudo-random number generator

$$X_i = (5X_{i-1} + 1)\text{mod}(8),$$

with starting value ("seed") $X_0 = 1$. Find the second PRN, $U_2 = X_2/m = X_2/8$.

Able to "plug and chug" using the formula.

Think about why this is an "awful" pseudo-random number generator.

See file "HW1 Tips Office Hours.R"

(Lesson 1.8: Generating Randomness.) Suppose we are using the "decent" pseudo-random number generator

$$X_i = 16807 X_{i-1} \,\mathrm{mod}(2^{31} - 1),$$

with seed X_0 = 12345678. Find the resulting integer X_1. Feel free to use something like Excel if you need to.

Able to "plug and chug" using the formula.

Think about why this is a "decent" pseudo-random number generator.

See file "HW1 Tips Office Hours.R"

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

The Inverse Transform Theorem is an "acquired taste". You will see it several more times throughout the course. It can take some time to fully understand what is going on. For now, it is perfectly fine to "just trust" that it works. Please review the [Inverse Transform Method Supplemental Notes.]

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

If you're willing to trust that the random variable $F(X) \sim Uniform(0,1)$, then this leads to the method of setting $F(X) = U$ and finding $X = F^{-1}(U)$ (fancy way of saying to solve for X). $F(x)$ is the cumulative distribution function. For an exponential random variable this is $F(x) = 1 - e^{-\lambda x}$

Set $F(X) = U$:

$$F(X) = 1 - e^{-\lambda X} = U \Rightarrow 1 - U = e^{-\lambda X}$$

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Set $F(X) = U$:

$F(X) = 1 - e^{-\lambda X} = U \Longrightarrow 1 - U = e^{-\lambda X}$

Take the natural log of both sides:

$\ln(1 - U) = \ln e^{-\lambda X} \Longrightarrow \ln(1 - U) = -\lambda X$

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

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Set $F(X) = U$:

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Take the natural log of both sides:

$$\ln(1 - U) = \ln e^{-\lambda X} \Longrightarrow \ln(1 - U) = -\lambda X$$

Solve for X:

$$X = -\frac{1}{\lambda}\ln(1 - U)$$

Question 12

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

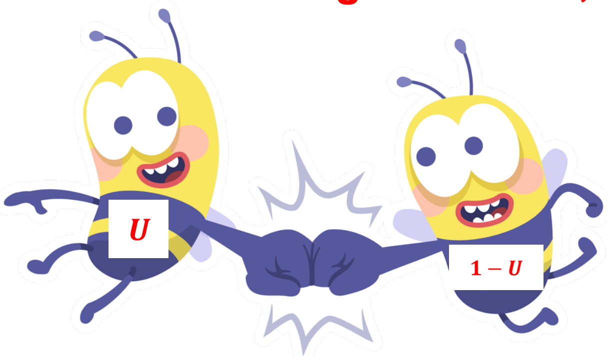
$$X = -\frac{1}{\lambda}\ln(1-U)$$

However, since $U \sim Uniform(0,1)$ then $(1-U) \sim Uniform(0,1)$.

To save the extra step of doing the subtraction:

$$X = -\frac{1}{\lambda}\ln(U)$$

For any given value of U, the value of X obtained by using $1-U$ will not be the same. However, in the long run, if you simulate using U or $1-U$, both will result in the desired distribution.



Beware – interchanging U and $1-U$ does not always work depending on what you are trying to simulate. We will see a few examples later in the course.

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

Here is some support for showing the similarities between U and $1 - U$ using properties of expected values and variance (if you don't remember these, never fear – they are part of the stats review in upcoming modules).

Let $U \sim Uniform(0,1)$.

The expected value of U, $E[U] = \frac{1}{2}(0+1) = \frac{1}{2}$

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number $U = 0.128$. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

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Let $U \sim Uniform(0, 1)$.

The expected value of U, $E[U] = \frac{1}{2}(0 + 1) = \frac{1}{2}$

The variance of U, $VAR(U) = \frac{1}{12}(1 - 0)^2 = \frac{1}{12}$

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

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Let $Y = 1 - U$.

$$E[Y] = E[1 - U] = E[1] - E[U] = 1 - \frac{1}{2} = \frac{1}{2}$$

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

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Let $Y = 1 - U$.

$$E[Y] = E[1 - U] = E[1] - E[U] = 1 - \frac{1}{2} = \frac{1}{2}$$

$$VAR(Y) = VAR(1 - U) = VAR(1) + VAR(U) = 0 + \frac{1}{12} = \frac{1}{12}$$

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

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Let $U \sim Uniform(0,1)$.

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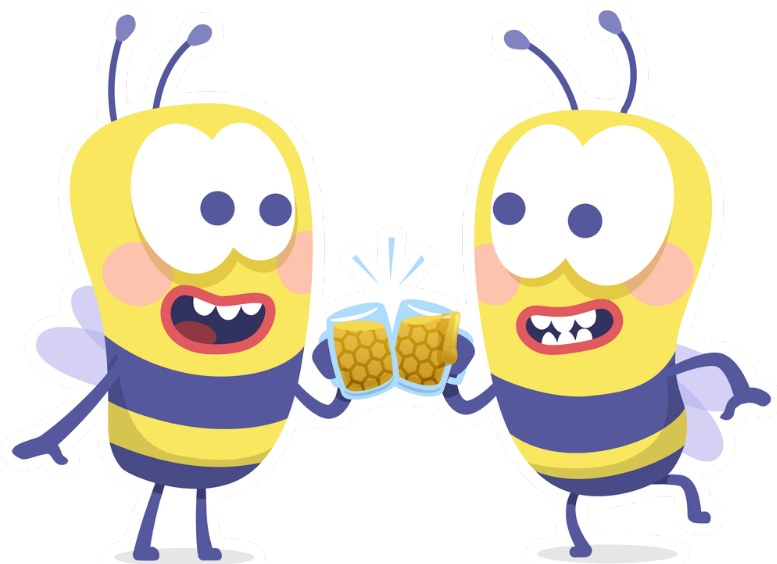

Let $Y = 1 - U$.

$E[Y] = E[1-U] = E[1] - E[U] = 1 - \frac{1}{2} = \frac{1}{2}$

$VAR(Y) = VAR(1-U) = VAR(1) + VAR(U) = 0 + \frac{1}{12} = \frac{1}{12}$

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

If you really want to prove that U and $1 - U$ have the same distribution, then one could show that they have the same cumulative distribution function.

The cumulative distribution function for $U \sim Uniform(0,1)$ is $\mathrm{P}(X \leq x) = x$

Let $Y = 1 - U$.

$$F_Y(y) = P(Y \leq y) = P(1 - U \leq y) = P(U \geq 1 - y) = 1 - P(U \leq 1 - y) = 1 - (1 - y) = y$$

Would also need to show that $F_y(y) = 0$ if $y < 0$ and $F_y(y) = 1$ if $y > 1$, but I will leave that as an exercise for the reader. ☺

Question 12

(Lesson 1.8: Generating Randomness.) Suppose that we generate a pseudo-random number U = 0.128. Use this to generate an Exponential $(\lambda = 1/3)$ random variate.

See file “HW1 Tips Office Hours.R”

```r
# Question 12 ####
U <- runif(1e5)
x_1 <- -(1/5) * log(U)
x_2 <- -(1/5) * log(1 - U)

x_3 <- rexp(1e5, rate = 5)

mean(x_1)
mean(x_2)
mean(x_3)

var(x_1)
var(x_2)
var(x_3)

summary(x_1)
summary(x_2)
summary(x_3)

par(mfrow = c(3, 1))
hist(x_1, 100)
hist(x_2, 100)
hist(x_3, 100)
```

(Lesson 1.9: Output Analysis.) BONUS: Which scenarios are most apt for a steady-state analysis? (More than one answer may be right.)

- [ ] a. We simulate a bank from noon till 1:00 pm.
- [ ] b. We investigate a production line that runs 24/7.
- [ ] c. We are interested in seeing what our portfolio is likely to be 3 months from now.
- [ ] d. We try to estimate the unemployment rate 30 years from now.

Terminating vs. Steady-State Simulations

Short-term vs. Long-term behavior

Review [M1L9 Simulation Output Analysis Additional Notes]