Math 245A Note 2

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1 Selected Exercises in Note 2

Exercise 1

Proof. (1). Let $f = \sum_{i=1}^n c_i 1_{E_i},\ g = \sum_{j=1}^m d_j 1_{F_j} : \mathbf{R}^d \to [0, +\infty]$ be unsigned simple functions. Then $f + g$ and $cf,\ c \in [0, +\infty]$ are both unsigned simple functions, with

$$\begin{aligned} \text{Simp} \int_{\mathbf{R}^d} f(x) + g(x)\ dx &= \sum_{i=1}^n c_i m(E_i) + \sum_{j=1}^m d_j m(F_j) \\ &= \text{Simp} \int_{\mathbf{R}^d} f(x)\ dx + \text{Simp} \int_{\mathbf{R}^d} g(x)\ dx, \end{aligned}$$

and

$$\text{Simp} \int_{\mathbf{R}^d} cf(x)\ dx = \sum_{i=1}^n cc_i m(E_i) = c \sum_{i=1}^n c_i m(E_i) = c \times \text{Simp} \int_{\mathbf{R}^d} f(x)\ dx.$$

(2). From $\text{Simp} \int_{\mathbf{R}^d} f(x)\ dx = \sum_{i=1}^n c_i m(E_i)$, we see that the integral is finite if and only if $c_i m(E_i) < \infty$ for all $1 \le i \le n$, which excludes the only two cases such that $c_i m(E_i) = \infty$, i.e. $c_i > 0$ and $m(E_i) = \infty$, or $m(E_i) > 0$ and $c_i = \infty$. That is, the integral is finite if and only if f is finite almost everywhere, and its support has finite measure.

(3). Clearly $\text{Simp} \int_{\mathbf{R}^d} f(x)\ dx = \sum_{i=1}^n c_i m(E_i) = 0$ if and only if $c_i \neq 0$ only for those i with $m(E_i) = 0$. That is, if and only if f is zero almost everywhere.

(4). Let $N := \{x \in \mathbf{R^d} : f \neq g\}$. On each E_i for $1 \le i \le n$, $g \neq f$ on the null set $E_i \cap N$, which has zero contribution to its integral. On $\mathbf{R^d} \backslash (\bigcup_{i=1}^n E_i)$, again $g \neq f$ on the null set $N \backslash (\bigcup_{i=1}^{n} E_i)$, which has a zero contribution. Hence we have

$$\begin{aligned} \text{Simp} \int_{\mathbf{R}^d} g(x)\ dx &= \text{Simp} \int_{\mathbf{R}^d} \sum_{i=1}^{d} c_i 1_{E_i \backslash N}(x)\ dx \\ &= \sum_{i=1}^{n} c_i m(E_i \backslash N) = \sum_{i=1}^{n} c_i m(E_i) \\ &= \text{Simp} \int_{\mathbf{R}^d} f(x)\ dx. \end{aligned}$$

(5). f and g can be written as the simple functions with respect to the 2^{m+n} disjoint sets formed by intersecting the E_i, F_j and their complements. Furthermore, $g \leq f$ on each of these sets of positive measure, by the definition of simple unsigned integral and Lemma 3, $\text{Simp} \int_{\mathbf{R}^d} f(x)\ dx \leq \text{Simp} \int_{\mathbf{R}^d} g(x)\ dx$.

(6). This follows from the definition of the simple unsigned integral.

Finally, let $\varphi : \text{Simp}^+(\mathbf{R}^d) \to [0, +\infty]$ be a map that obeys all of the above properties. By unsigned linearity and compatibility with Lebesgue measure, $\varphi(f) = \sum_{i=1}^{n} c_i \varphi(1_{E_i}) = \sum_{i=1}^{n} c_i m(E_i) = \text{Simp} \int_{\mathbf{R}^d} f(x)\ dx.$ □

Exercise 2

Proof. (1). First assume that $f,\ g \in \mathbf{R}$. Note that

$$f(x) + g(x) = (f+g)_+(x) - (f+g)_-(x) = f_+(x) - f_-(x) + g_+(x) - g_-(x).$$

Hence $(f+g)_+(x) + f_-(x) + g_-(x) = (f+g)_-(x) + f_+(x) + g_+(x)$, and from unsigned linearity

$$\begin{aligned} &\text{Simp} \int_{\mathbf{R}^d} (f+g)_+(x)\ dx + \text{Simp} \int_{\mathbf{R}^d} f_-(x)\ dx + \text{Simp} \int_{\mathbf{R}^d} g_-(x)\ dx \\ &= \text{Simp} \int_{\mathbf{R}^d} (f+g)_-(x)\ dx + \text{Simp} \int_{\mathbf{R}^d} f_+(x)\ dx + \text{Simp} \int_{\mathbf{R}^d} g_+(x)\ dx, \end{aligned}$$

or equivalently

$$\text{Simp} \int_{\mathbf{R}^d} f(x) + g(x)\ dx = \text{Simp} \int_{\mathbf{R}^d} f(x)\ dx + \text{Simp} \int_{\mathbf{R}^d} g(x)\ dx.$$

by the definition of real convergent simple integral.

For $c < 0$, by the definition of real convergent simple integral and unsigned linearity,

$$\begin{aligned}
\mathrm{Simp}\int_{\mathbf{R}^d} cf(x)\ dx &:= \mathrm{Simp}\int_{\mathbf{R}^d} (-c)f_-(x)\ dx - \mathrm{Simp}\int_{\mathbf{R}^d} (-c)f_+(x)\ dx \\
&= -c \times \mathrm{Simp}\int_{\mathbf{R}^d} f_-(x)\ dx + c \times \mathrm{Simp}\int_{\mathbf{R}^d} f_+(x)\ dx \\
&= c \times (\mathrm{Simp}\int_{\mathbf{R}^d} f_+(x)\ dx - \mathrm{Simp}\int_{\mathbf{R}^d} f_-(x)\ dx) \\
&= c \times \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx.
\end{aligned}$$

Similarly, for $c \geq 0$,

$$\begin{aligned}
\mathrm{Simp}\int_{\mathbf{R}^d} cf(x)\ dx &:= \mathrm{Simp}\int_{\mathbf{R}^d} cf_+(x)\ dx - \mathrm{Simp}\int_{\mathbf{R}^d} cf_+(x)\ dx \\
&= c \times \mathrm{Simp}\int_{\mathbf{R}^d} f_+(x)\ dx - c \times \mathrm{Simp}\int_{\mathbf{R}^d} f_-(x)\ dx \\
&= c \times (\mathrm{Simp}\int_{\mathbf{R}^d} f_+(x)\ dx - \mathrm{Simp}\int_{\mathbf{R}^d} f_-(x)\ dx) \\
&= c \times \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx.
\end{aligned}$$

Hence $\forall c \in \mathbf{C}$, we get

$$\begin{aligned}
\mathrm{Simp}\int_{\mathbf{R}^d} cf(x)\ dx &:= \mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Re} cf(x)\ dx + i\mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Im} cf(x)\ dx \\
&= \mathrm{Re} c \times \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx + i\mathrm{Im} c \times \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx \\
&= (\mathrm{Re} c + i\mathrm{Im} c) \times \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx \\
&= c \times \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx.
\end{aligned}$$

for any $f \in \mathbf{R}$. Now assume that $f, g, c \in \mathbf{C}$. By the definition of complex convergent simple integral, we have

$$\begin{aligned}
\mathrm{Simp}\int_{\mathbf{R}^d} f(x) + g(x)\ dx &:= \mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Re}(f+g)(x)\ dx + i\mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Im}(f+g)(x)\ dx \\
&= \mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Re} f(x)\ dx + \mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Re} f(x)\ dx \\
&+ i(\mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Im} f(x)\ dx + \mathrm{Simp}\int_{\mathbf{R}^d} \mathrm{Im} g(x)\ dx) \\
&= \mathrm{Simp}\int_{\mathbf{R}^d} f(x)\ dx + \mathrm{Simp}\int_{\mathbf{R}^d} g(x)\ dx.
\end{aligned}$$

And

$$\begin{aligned}
\text{Simp}\int_{\mathbf{R}^d} cf(x)\ dx &:= \text{Simp}\int_{\mathbf{R}^d} c(\text{Re}f(x) + i\text{Im}f(x))\ dx \\
&= \text{Simp}\int_{\mathbf{R}^d} c\text{Re}f(x)\ dx + \text{Simp}\int_{\mathbf{R}^d} ci\text{Im}f(x)\ dx \\
&= c \times \text{Simp}\int_{\mathbf{R}^d} \text{Re}f(x)\ dx + ci \times \text{Simp}\int_{\mathbf{R}^d} \text{Im}f(x)\ dx \\
&= c \times (\text{Simp}\int_{\mathbf{R}^d} \text{Re}f(x)\ dx + i\text{Simp}\int_{\mathbf{R}^d} \text{Im}f(x)\ dx) \\
&= c \times \text{Simp}\int_{\mathbf{R}^d} f(x)\ dx.
\end{aligned}$$

From *-linearity,

$$\begin{aligned}
\text{Simp}\int_{\mathbf{R}^d} \overline{f(x)}\ dx &= \text{Simp}\int_{\mathbf{R}^d} \text{Re}f(x) - i\text{Im}f(x)\ dx \\
&= \text{Simp}\int_{\mathbf{R}^d} \text{Re}(x)\ dx - i\text{Simp}\int_{\mathbf{R}^d} \text{Im}(x)\ dx \\
&= \overline{\text{Simp}\int_{\mathbf{R}^d} f(x)\ dx}
\end{aligned}$$

(2). If $f, g \in \mathbf{R}$, then $f = g$ a.e if and only if $f_+ - f_- = g_+ - g_-$ a.e, or equivalently $f_+ + g_- = g_+ + f_-$ a.e. By equivalence of simple unsigned integral, this implies

$$\text{Simp}\int_{\mathbf{R}^d} f_+(x) + g_-(x)\ dx = \text{Simp}\int_{\mathbf{R}^d} g_+(x) + f_-(x)\ dx.$$

By linearity we thus get

$$\text{Simp}\int_{\mathbf{R}^d} f_+(x) - f_-(x)\ dx = \text{Simp}\int_{\mathbf{R}^d} g_+(x) - g_-(x)\ dx.$$

That is, $\text{Simp}\int_{\mathbf{R}^d} f(x)\ dx = \text{Simp}\int_{\mathbf{R}^d} g(x)\ dx$.

For $f, g \in \mathbf{C}$, $f = g$ a.e if and only if $\text{Re}f = \text{Re}g$ and $\text{Im}f = \text{Im}g$ a.e. By equivalence of real simple integral, we thus have

$$\begin{aligned}
\text{Simp}\int_{\mathbf{R}^d} \text{Re}f(x)\ dx &= \text{Simp}\int_{\mathbf{R}^d} \text{Re}g(x)\ dx, \\
\text{Simp}\int_{\mathbf{R}^d} \text{Im}f(x)\ dx &= \text{Simp}\int_{\mathbf{R}^d} \text{Im}g(x)\ dx,
\end{aligned}$$

which from linearity gives $\text{Simp}\int_{\mathbf{R}^d} f(x)\ dx = \text{Simp}\int_{\mathbf{R}^d} g(x)\ dx$.

(3). This follows from the definition of complex-valued simple integral.

Finally, let $\varphi : \mathrm{Simp}^{abs}(\mathbf{R}^d) \to \mathbf{C}$ be a map that obeys all of the above properties. By *-linearity and compatibility with Lebesgue measure, $\varphi(f) = \sum_{i=1}^{n} c_i \varphi(1_{E_i}) = \sum_{i=1}^{n} c_i m(E_i) = \mathrm{Simp} \int_{\mathbf{R}^d} f(x)\ dx.$ □

Comment on Lemma 7:

$$\bigcap_{N>0} \{x \in \mathbf{R}^d : \sup_{n \geq N} f_n(x) > \lambda + \frac{1}{M}\} = \{x \in \mathbf{R}^d : \inf_{N>0} \sup_{n \geq N} f_n(x) > \lambda + \frac{1}{M}\}$$

as can be seen by directly checking that LHS = RHS.

Basically, taking infimum corresponds to set intersection while taking supremum corresponds to set union.

If $f_n(x) = c$ for some non-zero $c = k2^{-n}$, then $f_n(x)$ equals c on the set $f^{-1}([k2^{-n}, (k+1)2^{-n}))$, or the set $f^{-1}(([k2^{-n}, +\infty))$ if f never attains values larger than c.

Exercise 3

Proof. (1). By Theorem 2.1.5 of "Analysis II" and (10) of Lemma 7, f is measurable.

(2). Taking $f_n := f$ for all n in Definition 6, we see that every unsigned simple function is measurable.

(3). Let $(f_n)_n$ be a sequence of unsigned measurable functions. For every $\lambda \in [0, +\infty]$, we have

$$\{x \in \mathbf{R}^d : \sup_n f_n(x) > \lambda\} = \bigcup_n \{x \in \mathbf{R}^d : f_n(x) > \lambda\},$$
$$\{x \in \mathbf{R}^d : \inf_n f_n(x) > \lambda\} = \bigcap_n \{x \in \mathbf{R}^d : f_n(x) > \lambda\}.$$

Since countable unions or countable intersections of Lebesgue measurable sets are Lebesgue measurable, both sets are Lebesgue measurable, so $\sup_n f_n$ and $\inf_n f_n$ are both measurable by (5) of Lemma 7. Let $f_N^+ := \sup_{n \geq N} f_n$ and $f_N^- := \inf_{n \geq N} f_n$ and apply what we just got to the functions $\inf_N f_N^+$ and $\sup_N f_N^-$, we see that $\limsup_n f_n$ and $\liminf_n f_n$ are both measurable as well.

(4). Outside a null set, f is the pointwise almost everywhere limit of unsigned simple functions, so f is measurable by (3) of Lemma 7.

(5). By (4), we can assume that the sequence f_n of unsigned measurable functions converges everywhere to an unsigned limit f, in particular $f = \limsup_n f_n$, hence f is measurable by (3).

(6). Let $V \subset [0, +\infty]$ be an open set, then $\phi^{-1}(V) \subset [0, +\infty)$ is open by Theorem 2.1.5 of "Analysis II" (bounded as ϕ is continuous), and thus $(\phi \circ f)^{-1}(V) = f^{-1} \circ \phi^{-1}(V)$ is Lebesgue measurable by (10) of Lemma 7, as such, $\phi \circ f : \mathbf{R}^d \to [0, +\infty]$ is measurable.

(7). As both $f_n \times g_n$ and $f_n + g_n$ are unsigned simple functions for sequences $(f_n)_n$ and $(g_n)_n$ of unsigned simple functions converging pointwise everywhere to f and g respectively, $f + g$ and fg are measurable by (2) of Lemma 7. □

Exercise 4

Proof. First suppose that f is the uniform limit of bounded simple functions f_n. Thus $\forall \varepsilon > 0$, $\exists N > 0$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $x \in \mathbf{R}^d$ and all $n \geq N$. By the triangle inequality $|f| \leq |f - f_n| + |f_n|$, f is bounded. Since f is unsigned, we can take the f_n to be unsigned (say by replacing f_n with $-f_n$ where $f_n < 0$). Hence f is the pointwise limit of unsigned simple functions, and thus measurable.

Conversely, suppose that f is a bounded unsigned measurable function. For each positive integer n, we let $f_n(x)$ be defined to be the largest integer multiple of 2^{-n} that is less than or equal to $\min(f(x), n)$. From construction it is easy to see that the $f_n : \mathbf{R}^d \to [0, +\infty]$ are increasing and have f as their supremum. Furthermore, each f_n takes on only finitely many values, and for each non-zero value c it attains, the set $f_n^{-1}(c)$ takes the form $f^{-1}(I_c)$ for some interval or ray I_c, and is thus measurable. As a consequence, f_n is a simple function. As f is bounded, $\exists M > 0$ such that $0 \leq f \leq M$. Let $\varepsilon > 0$. Pick $N \geq M$ with $2^{-N} < \varepsilon$, it holds that $|f_n(x) - f(x)| \leq 2^{-n} < \varepsilon$ for all $x \in \mathbf{R}^d$, and all $n \geq N$. i.e. f is the uniform limit of bounded simple functions. □

Exercise 5

Proof. Suppose that $f : \mathbf{R}^d :\to [0, +\infty]$ is measurable and takes on at most finitely many distinct nonzero values $c_1, \dots, c_k$. As singleton sets are closed,

$f^{-1}(\{c_i\})$ is Lebesgue measurable for all $1 \leq i \leq k$ by (11) of Lemma 7. Hence $f = \sum_{i=1}^k c_i f^{-1}(\{c_i\})$ is a simple function.

Conversely, if $f : \mathbf{R}^d \to [0, +\infty]$ is simple, it is trivially measurable by definition. □

Exercise 6

Proof. It's easy to see that regions under unsigned simple functions are measurable, since they can be expressed as union of boxes. By (4) of Lemma 7, f is the supremum $f(x) = \sup_n f_n(x)$ of an increasing sequence $0 \leq f_1 \leq f_2 \leq \dots$ of unsigned simple functions f_n, each of which are bounded with finite measure support. From this, we have

$$\{(x,t) \in \mathbf{R}^d \times \mathbf{R} : 0 \leq t \leq f(x)\} = \bigcup_{n=1}^{\infty} \{(x,t) \in \mathbf{R}^d \times \mathbf{R} : 0 \leq t \leq f_n(x)\}.$$

As countable union of measurable sets is measurable, the region under f is measurable. □

Exercise 7

Proof. (1) and (2) are equivalent by definition. If f is measurable, it is the pointwise almost everywhere limit of complex-valued simple functions f_n. On the other hand, the simple functions $f_n : \mathbf{R}^d \to \mathbf{C}$ can be decomposed into unsigned simple functions by writing

$$f_n = \mathrm{Re} f_n + i \mathrm{Im} f_n = \mathrm{Re} f_n^+ - \mathrm{Re} f_n^- + i(\mathrm{Im} f_n^+ - \mathrm{Im} f_n^-),$$

where the superscripts denote (the magnitudes of) the positive and negative parts of $\mathrm{Re}(f_n)$ and $\mathrm{Im}(f_n)$. Then

$$\mathrm{Re} f^+ := \lim_{n\to\infty} \mathrm{Re} f_n^+, \ \mathrm{Re} f^- := \lim_{n\to\infty} \mathrm{Re} f_n^-,$$

outside some null sets, and similar for $\mathrm{Im} f^+$ and $\mathrm{Im} f^-$. By (3) of Lemma 7, these are all unsigned measurable functions. i.e. (1) implies (3). Conversely, assuming (3), by (3) of Lemma 7, there are unsigned simple functions $\mathrm{Re} f_n^+$, $\mathrm{Re} f_n^-$, $\mathrm{Im} f_n^+$, $\mathrm{Im} f_n^-$ which have $\mathrm{Re} f^+$, $\mathrm{Re} f^-$, $\mathrm{Im} f^+$, $\mathrm{Im} f^-$ as their pointwise almost everywhere limits respectively. As subtraction of simple functions are simple, we see that f is the pointwise almost everywhere limit of simple functions

$$f_n := \mathrm{Re} f_n + i \mathrm{Im} f_n = \mathrm{Re} f_n^+ - \mathrm{Re} f_n^- + i(\mathrm{Im} f_n^+ - \mathrm{Im} f_n^-)$$

and thus is measurable by definition. i.e. (3) implies (1).

Given $f : \mathbf{R}^d \to \mathbf{C}$ is measurable, let $U \subset \mathbf{C}$ be an open set. Identifying $\mathbf{C}$ with $\mathbf{R}^2$, by Lemma 8 of note 1, U can be expressed as a countable union of almost disjoint boxes $U = \bigcup_{i=1}^{\infty} B_i$, where $B_i = I_i \times J_i$ with I_i and J_i be intervals. Then

$$f^{-1}(U) = \bigcup_{i=1}^{\infty} \mathrm{Re} f^{-1}(I_i) \cap \mathrm{Im} f^{-1}(J_i).$$

Without loss of generality, let $I_i = (a_i, b_i)$. By condition, $\mathrm{Re}f$ is the pointwise almost everywhere limit of simple functions $\mathrm{Re}f_n$. Argue as in the proof for Lemma 7, the set $\{x \in \mathbf{R}^d : \mathrm{Re}f(x) > a_i\}$ is equal to

$$\bigcup_{M>0} \bigcap_{N>0} \bigcup_{n \geq N} \{x \in \mathbf{R}^d : \mathrm{Re}f_n(x) > a_i + \frac{1}{M}\}$$

outside of a set of measure zero, and hence measurable. Similarly, the set $\{x \in \mathbf{R}^d : \mathrm{Re}f(x) < b_i\}$ is equal to

$$\bigcup_{M>0} \bigcap_{N>0} \bigcup_{n \geq N} \{x \in \mathbf{R}^d : \mathrm{Re}f_n(x) < b_i - \frac{1}{M}\}$$

outside of a set of measure zero, which is measurable. Combine these results, we conclude that the set $\mathrm{Re}f^{-1}(I_i) = \{x : \mathrm{Re}f(x) > a_i\} \cap \{x : \mathrm{Re}f(x) < b_i\}$ is measurable. By symmetry, the set $\mathrm{Im}f^{-1}(J_i)$ is also measurable, so the set $\mathrm{Re}f^{-1}(I_i) \cap \mathrm{Im}f^{-1}(J_i)$ is measurable. As the countable union of measurable sets is measurable, $f^{-1}(U)$ is measurable. That is, (1) to (3) implies (4). As $f^{-1}(K)$ is the complement of $f^{-1}(\mathbf{R}^2 \backslash K)$ for any closed set $K \subset \mathbf{C}$, and similarly if one replaces K with the open set U, (4) and (5) are equivalent.

Finally, suppose (4) and (5). First we show that $\mathrm{Re}f^{-1}(I)$ and $\mathrm{Im}f^{-1}(I)$ are measurable for any interval $I \subset \mathbf{R}$. Note that $\mathrm{Re}f^{-1}(I) = f^{-1}(I \times \mathbf{R})$, and $I \times \mathbf{R}$ can always be expressed as $(U \cap K) \times \mathbf{R}$ for some open set $U \subset \mathbf{R}$ and closed set $K \subset \mathbf{R}$ (by writing $I = [a, b] \cap (a, +\infty)$, $[a, b] \cap (-\infty, b)$), $[a, b] \cap \mathbf{R}$ or $(a, b) \cap \mathbf{R}$). It follows that

$$\mathrm{Re}f^{-1}(I) = f^{-1}(U \times \mathbf{R}) \cap f^{-1}(K \times \mathbf{R})$$

which is measurable. Similarly, $\mathrm{Im}f^{-1}(I)$ is measurable. Expressing a ray as the countable union of intervals, we see the claim holds for rays as well.

For each positive integer n, let $f_n(x) := k_1 2^{-n} + i k_2 2^{-n}$ when $|x| \leq n$, and $f_n(x) := 0$ for $|x| > n$. Here k_1 is the largest integer multiple of 2^{-n} that is less than or equal to $\min(|\mathrm{Re}f(x)|, n)$, times $\mathrm{sgn}(\mathrm{Re}f(x))$, and k_2 is the the largest integer multiple of 2^{-n} that is less than or equal to $\min(|\mathrm{Im}f(x)|, n)$, times $\mathrm{sgn}(\mathrm{Im}f(x))$. From construction, we see that the $f_n : \mathbf{R}^d \to \mathbf{C}$ have f as their pointwise limit. Furthermore, each f_n takes on only finitely many values, and for each non-zero value c it attains, the set $f_n^{-1}(c)$ takes the form $\mathrm{Re} f^{-1}(I_c) \cap \mathrm{Im} f^{-1}(J_c) \cap \{x \in \mathbf{R}^d : |x| \leq n\}$ for some intervals or rays I_c and J_c, and is thus measurable. As a consequence, f_n is a simple function, and one gets (2). $\square$

Exercise 8

Proof. (1). By Theorem 2.1.5 of "Analysis II" and (4) of Exercise 7, f is measurable.

(2). Suppose that $f : \mathbf{R}^d :\to \mathbf{C}$ is measurable and takes on at most finitely many distinct nonzero values $c_1, \ldots, c_k$. As singleton sets are closed, $f^{-1}(\{c_i\})$ is Lebesgue measurable for all $1 \leq i \leq k$ by (5) of Exercise 7. Hence $f = \sum_{i=1}^k c_i f^{-1}(\{c_i\})$ is a simple function.

Conversely, if $f : \mathbf{R}^d \to \mathbf{C}$ is simple, it is trivially measurable by definition.

(3). The claim follows from (4) of Exercise 3 and (3) of Exercise 7.

(4). The claim follows from (5) of Exercise 3 and (3) of Exercise 7.

(5). Let $U \subset \mathbf{C}$ be an open set. From $(\phi \circ f)^{-1}(U) = f^{-1}(\phi^{-1}(U))$, (4) of Exercise 7, and the fact that the inverse image of open sets under continuous map is open, we see that $\phi \circ f : \mathbf{R}^d \to \mathbf{C}$ is measurable.

(6). By (2) of Exercise 7, f and g are pointwise almost everywhere limits of sequences of complex-valued simple functions f_n and g_n respectively. As the space $\mathrm{Simp}(\mathbf{R}^d)$ of complex-valued simple functions forms a complex vector space and is closed under pointwise product, $f + g$ and fg are also measurable. $\square$

Exercise 9

Proof. By condition, f is bounded, by Exercise 22 of Note 0 and the axiom of choice, we can find sequences $\underline{f_n} \leq f \leq \overline{f_n}$ of simple functions such that $\mathrm{Simp} \int_{\mathbf{R}^d} \overline{f_n} - \underline{f_n}\, dx < 1/n$. In particular, $\lim_{n\to\infty} \mathrm{Simp} \int_{\mathbf{R}^d} \overline{f_n} - \underline{f_n}\, dx = 0$. Without loss of generality, we can take $\overline{f_n},\ \underline{f_n} := 0$ outside $[a, b]$. By the definition of absolutely convergent simple integral, this implies that $\lim_{n\to\infty}(\overline{f_n} - \underline{f_n}) = 0$ almost everywhere, i.e. $\lim_{n\to\infty} \overline{f_n} = \lim_{n\to\infty} \underline{f_n} = f$ almost everywhere. By definition, $f : \mathbf{R}^d \to \mathbf{R}$ is thus measurable. $\square$

Exercise 10

Proof. (1). This follows trivially from the definition.

(2). For any simple function $0 \leq h \leq f,\ h \leq g$ a.e, and similarly for any simple function $h \geq g,\ h \geq f$ a.e, thus by the definition of lower and upper Lebesgue integral,

$$\underline{\int_{\mathbf{R}^d}} f(x)\ dx \leq \underline{\int_{\mathbf{R}^d}} g(x)\ dx \text{ and } \overline{\int_{\mathbf{R}^d}} f(x)\ dx \leq \overline{\int_{\mathbf{R}^d}} g(x)\ dx.$$

(3). First assume that $c > 0$, note that

$$\{0 \leq g \leq cf : g \text{ simple}\} = c \times \{0 \leq g' \leq f : g' \text{ simple}\}.$$

Taking supremum on both sides, by unsigned linearity of the simple unsigned integral, we have

$$\begin{aligned}
\underline{\int_{\mathbf{R}^d}} cf(x)\ dx :&= \sup_{0 \leq g \leq cf; g \text{ simple}} \text{Simp} \int_{\mathbf{R}^d} g(x)\ dx \\
&= \sup_{0 \leq g' \leq f; g' \text{ simple}} \text{Simp} \int_{\mathbf{R}^d} cg'(x)\ dx \\
&= c \sup_{0 \leq g' \leq f; g' \text{ simple}} \text{Simp} \int_{\mathbf{R}^d} g'(x)\ dx \\
&= c \underline{\int_{\mathbf{R^d}}} f(x)\ dx.
\end{aligned}$$

If $c = 0$, then we have $0 = 0$, so the claim holds for all $c \in [0, +\infty)$.

(4). This follows from the definition, and the fact that simple integral is not affected by modifications on sets of measure zero.

(5). This follows from the fact that the set

$$\{0 \leq h_1 \leq f, 0 \leq h_2 \leq g, h_1, h_2 \text{ simple} :\ h_1 + h_2\}$$

is a subset of $\{0 \leq h \leq f + g, h \text{ simple} : h\}$, the definition of lower Lebesgue integral, and unsigned linearity of the simple unsigned integral.

(6). Again, this follows from the fact that the set

$$\{h_1 \geq f, h_2 \geq g, h_1, h_2 \text{ simple} : h_1 + h_2\}$$

is a subset of $\{h \geq f + g, h \text{ simple} : h\}$, the definition of upper Lebesgue integral, and unsigned linearity of the simple unsigned integral.

(7). For any measurable set $E \subset \mathbf{R}^d$, and any simple function $0 \leq h \leq f$, the functions $h1_E$ and $h1_{\mathbf{R}^d \setminus E}$ are both simple with $h1_E \leq f1_E$ and $h1_{\mathbf{R}^d \setminus E} \leq f1_{\mathbf{R}^d \setminus E}$. Hence the set

$$\{0 \leq h_1 \leq f1_E, 0 \leq h_2 \leq f1_{\mathbf{R}^d \setminus E}, h_1, h_2 \text{ simple} : h_1 + h_2\}$$

is equal to the set $\{0 \leq h \leq f : h \text{ simple}\}$. The claim then follows from the definition of lower Lebesgue integral and unsigned linearity of the simple unsigned integral.

(8). If $f = +\infty$ on some set E with $m(E) > 0$, then by monotonicity and compatibility with the simple integral, we have

$$\underline{\int_{\mathbf{R}^d}} f(x)\ dx \geq \underline{\int_{\mathbf{R}^d}} f1_E(x)\ dx \geq \underline{\int_{\mathbf{R}^d}} n1_E(x)\ dx = nm(E).$$

Likewise we get $\underline{\int_{\mathbf{R}^d}} \min(f(x), n)\ dx \geq nm(E)$.

Otherwise, f is bounded a.e on every set of positive measure, so f is bounded a.e. In particular, $\exists N > 0$ such that $\min(f_n, n) = f(x)$ a.e for all $n \geq N$. By equivalence, this implies that $\underline{\int_{\mathbf{R}^d}} \min(f(x), n)\ dx = \underline{\int_{\mathbf{R}^d}} f(x)\ dx$ for all $n \geq N$. Sending $n \to \infty$ gives the claim in both cases.

(9). We first establish the claim for simple functions. Let $f = \sum_{i=1}^k c_i 1_{E_i}$ be an unsigned simple function. By compatibility with simple integral and Exercise 11 of Note 1,

$$\begin{aligned} \lim_{n\to\infty} \underline{\int_{\mathbf{R}^d}} f(x) 1_{|x| \leq n}\ dx &= \lim_{n\to\infty} \sum_{i=1}^k c_i m(E_i \cap \{x : |x| \leq n\}) = \sum_{i=1}^k c_i m(E_i) \\ &= \underline{\int_{\mathbf{R}^d}} f(x)\ dx. \end{aligned}$$

For an arbitrary $f : \mathbf{R}^d \to [0, +\infty]$, if $\underline{\int_{\mathbf{R}^d}} f(x)\ dx < \infty$, by the definition of lower Lebesgue integral, $\exists 0 \leq g \leq f,\ g$ simple, such that

$$|\underline{\int_{\mathbf{R}^d}} f(x)\ dx - \underline{\int_{\mathbf{R}^d}} g(x)\ dx| < \varepsilon$$

$\forall \varepsilon > 0$. Furthermore, from monotonicity and the simple case,

$$\underline{\int_{\mathbf{R}^d}} g(x)\ dx = \lim_{n\to\infty} \underline{\int_{\mathbf{R}^d}} g(x) 1_{|x| \leq n}\ dx \leq \lim_{n\to\infty} \underline{\int_{\mathbf{R}^d}} f(x) 1_{|x| \leq n}\ dx \leq \underline{\int_{\mathbf{R}^d}} f(x)\ dx,$$

from which we see that

$$|\underline{\int_{\mathbf{R}^d}} f(x)\ dx - \lim_{n\to\infty} \underline{\int_{\mathbf{R}^d}} f(x) 1_{|x|\le n}\ dx| < \varepsilon$$

$\forall \varepsilon > 0$. And the claim follows. If $\underline{\int_{\mathbf{R}^d}} f(x)\ dx = \infty$, by the definition of lower Lebesgue integral, $\exists 0 \le g \le f,\ g$ simple, such that

$$\underline{\int_{\mathbf{R}^d}} g(x)\ dx > M,\ \forall M > 0.$$

Again, from monotonicity and the simple case,

$$\underline{\int_{\mathbf{R}^d}} f(x)\ dx \ge \lim_{n\to\infty} \underline{\int_{\mathbf{R}^d}} f(x) 1_{|x|\le n}\ dx \ge \lim_{n\to\infty} \underline{\int_{\mathbf{R}^d}} g(x) 1_{|x|\le n}\ dx > M$$

$\forall M > 0$. The claim follows.

The horizontal truncation property fails for upper integral: Let $\sum_n a_n$ be a convergent series of positive terms, and define $f := \sum_{k=1}^{\infty} k 1_{E_k}$, where the E_k are measurable disjoint sets with $m(E_k) = a_k/k$. Any simple function that majorizes f will be infinite on a set of positive measure, forcing $\overline{\int_{\mathbf{R}^d}} f(x)\ dx = \infty$, while the horizontal truncations are the partial sums and of course converge to the sum of the series as $n \to \infty$.

Likewise, the vertical truncation property fails for upper integral: Let E_k be the annulus given by $E_0 := \emptyset,\ E_k := \{x : |x| \le n\} \backslash E_{k-1}$ for $k = 1, 2, \ldots,$ and $f := \sum_{k=1}^{\infty} \frac{a_k}{m(E_k)} 1_{E_k}$. Any simple function that majorizes f will be bounded away from zero on a set of infinite measure, forcing $\overline{\int_{\mathbf{R}^d}} f(x)\ dx = \infty$, and the vertical truncations are the partial sums, which converge to the sum of the series as $n \to \infty$.

(10). By definition of the lower and upper Lebesgue integral, there exist simple function r with $g \le r$ such that

$$\overline{\int_{\mathbf{R}^d}} g(x)\ dx \le \mathrm{Simp} \int_{\mathbf{R}^d} r(x)\ dx \le \overline{\int_{\mathbf{R}^d}} g(x)\ dx + \varepsilon$$

and simple function s with $0 \le s \le f$ such that

$$\underline{\int_{\mathbf{R}^d}} f(x)\ dx - \varepsilon \le \mathrm{Simp} \int_{\mathbf{R}^d} s(x)\ dx \le \underline{\int_{\mathbf{R}^d}} f(x)\ dx.$$

for any $\varepsilon > 0$. We can further take r obeying $r \le f + g$ (as $f + g$ is absolutely integrable, all quantities above are finite). Define the simple function h by $h := \max(f + g - r, s)$. Then $s \le h \le f$ and $g \le f + g - h \le r$, in particular

$$|\mathrm{Simp} \int_{\mathbf{R}^d} h(x)\ dx - \underline{\int_{\mathbf{R}^d}} f(x)\ dx| \le \varepsilon,$$

and

$$|\mathrm{Simp}\int_{\mathbf{R}^d}(f+g-h)(x)\ dx - \overline{\int_{\mathbf{R}^d}} g(x)\ dx| \leq \varepsilon.$$

From linearity of the simple integral

$$\mathrm{Simp}\int_{\mathbf{R}^d}(f+g)(x)\ dx = \mathrm{Simp}\int_{\mathbf{R}^d}(f+g-h)(x)\ dx + \mathrm{Simp}\int_{\mathbf{R}^d} h(x)\ dx,$$

we see that $|\mathrm{Simp}\int_{\mathbf{R}^d}(f+g)(x)\ dx - (\underline{\int_{\mathbf{R}^d}} f(x)\ dx + \overline{\int_{\mathbf{R}^d}} g(x)\ dx)| \leq 2\varepsilon$, as desired. □

Exercise 11

Proof. By Exercise 4, f is the uniform limit of bounded simple functions f_n, which we can take to be supported on $E := \mathrm{spprt}(f)$. If $m(E) = 0$, then $\overline{\int_{\mathbf{R}^d}} f(x)\ dx = \underline{\int_{\mathbf{R}^d}} f(x)\ dx = 0$ by monotonicity. Hence we assume $m(E) > 0$.

For any $\varepsilon > 0$, we can pick a simple function $f_n := \sum_{i=1}^m c_i 1_{E_i}$ such that $|f_n(x) - f(x)| \leq \varepsilon/m(E)$ for all $x \in \mathbf{R}^d$. By modifying c_i for $1 \leq i \leq m$, we get simple functions $0 \leq g_n \leq f$ and $h_n \geq f$ such that $f(x) - g_n(x) \leq \varepsilon/m(E)$ and $h_n(x) - f(x) \leq \varepsilon/m(E)$ for all $x \in \mathbf{R}^d$. It follows by the definition of upper and lower Lebesgue integral,

$$\overline{\int_{\mathbf{R}^d}} f(x)\ dx - \underline{\int_{\mathbf{R}^d}} f(x)\ dx \leq m(E) \cdot 2\varepsilon/m(E) = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\overline{\int_{\mathbf{R}^d}} f(x)\ dx = \underline{\int_{\mathbf{R}^d}} f(x)\ dx$.

Finally, if f is unbounded and supported inside a set of finite measure, or bounded but supported on a set of infinite measure, we can take the same counterexamples used to show that horizontal and vertical truncation property fail for upper Lebesgue integral in Exercise 10. Hence we see that f will have to be absolutely integrable for its' upper and lower integral to match. □

Exercise 12

Proof. By definition of the upper Lebesgue integral, and compatibility with the simple integral, we have

$$\overline{\int_{\mathbf{R}^d}} 1_E(x)\ dx = \inf_{F \supset E; F \text{ measurable}} m(F).$$

Hence we want to show that

$$\inf_{F \supset E; F \text{ measurable}} m(F) = m^*(E)$$

By outer regularity, and the fact that open sets are Lebesgue measurable, LHS $\leq m^*(E)$. That $m^*(E) \leq$ LHS is trivial for LHS infinite, so we may assume that LHS is finite. Let $\varepsilon > 0$, there exists a measurable set $F \supset E$ such that $m(F) \leq \text{LHS} + \varepsilon$. By outer approximation by open sets, there exists an open set $U \supset F$ such that $m(U) \leq \text{LHS}+2\varepsilon$. Taking infimum over U, by monotonicity of the outer measure, we get

$$m^*(E) \leq m^*(F) \leq \text{LHS} + 2\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, this implies that $m^*(E) \leq$ LHS, as desired.

By Proposition 12 of Note 1, There exists a subset $E \subset [0, 1]$ which is not Lebesgue measurable. Thus the function 1_E is not measurable by (5) of Exercise 7. Let $A \supset E$ be any elementary set. By the Carathéodory criterion, $m(A) < m^*(A \backslash E) + m^*(E)$. It follows that

$$\overline{\int_{\mathbf{R}^d}} 1_{A\backslash E}(x) + 1_E(x)\ dx < \overline{\int_{\mathbf{R}^d}} 1_{A\backslash E}(x)\ dx + \overline{\int_{\mathbf{R}^d}} 1_E(x)\ dx.$$

We claim that $m^*(E) > \underline{\int_{\mathbf{R}^d}} 1_E(x)\ dx$. By definition of the lower Lebesgue integral, it suffices to show

$$m^*(E) > \sup_{F \subset E; F \text{measurable}} m(F).$$

Suppose for contradiction that equality holds. Let $\varepsilon > 0$. By definition, there is a measurable set $F \subset E$ such that $m^*(E) - \varepsilon \leq m(F)$, and an open set $U \supset E$ such that $m(U) \leq m^*(E) + \varepsilon$. By monotonicity,

$$m^*(E \backslash F) \leq m^*(U \backslash F) \leq 2\varepsilon,$$

which implies that E is Lebesgue measurable, a contradiction. By the reflection property (and compatibility with the simple integral),

$$\begin{aligned} \underline{\int_{\mathbf{R}^d}} 1_{A\backslash E}(x) + 1_E(x)\ dx &= \overline{\int_{\mathbf{R}^d}} 1_E(x)\ dx + \underline{\int_{\mathbf{R}^d}} 1_{A\backslash E}(x)\ dx \\ &> \underline{\int_{\mathbf{R}^d}} 1_E(x)\ dx + \underline{\int_{\mathbf{R}^d}} 1_{A\backslash E}(x)\ dx. \end{aligned}$$

In conclusion, the upper and lower Lebesgue integrals are not necessarily additive if no measurability hypotheses are assumed. $\square$

Exercise 13

Proof. From the horizontal truncation property and a limiting argument, we may assume that f is bounded. From the vertical truncation property and another limiting argument, we may assume that f is supported inside a bounded set.

Denote $A_f := \{(x,t) \in \mathbf{R}^d \times \mathbf{R} : 0 \le t \le f(x)\}$. We first establish the result for simple functions. Let $f = \sum_{i=1}^k c_i 1_{E_i}$ be an unsigned simple function, where the E_i are disjoint measurable sets. Then $A_f = \bigcup_{i=1}^k E_i \times [0, c_i]$, by Exercise 22 of Note 1, $m(A_f) = \sum_{i=1}^k c_i m(E_i) = \int_{\mathbf{R}^d} f(x)\ dx$.

Let $\varepsilon > 0$. By Exercise 4, there is a bounded simple function g_1 such that $|f(x) - g_1(x)| \le \varepsilon$ for all $x \in \mathbf{R}^d$. Without loss of generality, we can take g_1 with $0 \le g_1 \le f$. Then $A_g \subset A_f$, and

$$m(A_{g_1}) \le m(A_f) \le m(A_{g_1}) + 2r\varepsilon$$

where $r := m(\text{spprt}(f))$. By definition of the Lebesgue integral, there is a simple function $0 \le g_2 \le f$ such that $m(A_{g_2}) \ge \int_{\mathbf{R}^d} f(x)\ dx - 2r\varepsilon$. Hence

$$m(A_{g_2}) \le \int_{\mathbf{R}^d} f(x)\ dx \le m(A_{g_2}) + 2r\varepsilon.$$

Define the simple function g as $g := \max(g_1, g_2)$. By monotonicity, we then get

$$m(A_g) \le m(A_f), \int_{\mathbf{R}^d} f(x)\ dx \le m(A_g) + 2r\varepsilon.$$

Thus $|m(A_f) - \displaystyle\int_{\mathbf{R}^d} f(x)\ dx| \le 2r\varepsilon$, the claim is obtained. $\square$

Exercise 14

Proof. Denote the map obeying the given properties by φ. By the horizontal and vertical truncation properties, it suffices to prove the claim for f bounded with finite measure support.

First get monotonicity. Let $f \le g$. By (5) of Exercise 8, $-f$ is (complex) measurable, so $g - f$ is unsigned measurable, and by finite additivity it follows that $\varphi(g) - \varphi(f) = \varphi(g - f) \ge 0$. i.e. $\varphi(g) \ge \varphi(f)$.

Let $\varepsilon > 0$. By Exercise 4, exists a bounded simple function g_1 such that $|f(x) - g_1(x)| \leq \varepsilon$ for all $x \in \mathbf{R}^d$. We can further take g_1 to be such that $0 \leq g_1 \leq f$. By finite additivity and monotonicity, we get

$$\varphi(f) - \varphi(g_1) \leq r\varepsilon,$$

where $r := m(\text{spprt}(f))$. By definition of the Lebesgue measure, exists a simple function $0 \leq g_2 \leq f$ such that

$$\int_{\mathbf{R}^d} f(x)\ dx - \varphi(g_2) \leq r\varepsilon.$$

Define $g := \max(g_1, g_2)$, from monotonicity and the triangle inequality, we obtain

$$|\int_{\mathbf{R}^d} f(x)\ dx - \varphi(f)| \leq |\int_{\mathbf{R}^d} f(x)\ dx - \varphi(g)| + |\varphi(g) - \varphi(f)| \leq 2r\varepsilon.$$

Sending $\varepsilon \to 0$ gives the result. □

Exercise 15

Proof. Note that

$$\{(x,t) \in \mathbf{R}^d \times \mathbf{R} : 0 \leq t \leq f(x+y)\} = (-y, 0) + \{(x,t) \in \mathbf{R}^d \times \mathbf{R} : 0 \leq t \leq f(x)\}$$

The conclusion then follows from the area interpretation of Lebesgue integral, and translation invariance of the Lebesgue measure. □

Exercise 16

Proof. With the same notation as in Exercise 13, it suffices to show that

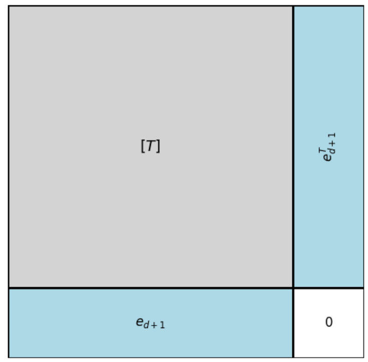
$$m(A_f) = |\det T| m(A_{f \circ T}).$$

By Exercise 21 of Note 1, it suffices to show that

$$A_f = T'(A_{f \circ T})$$

for some linear transformation $T' : \mathbf{R}^{d+1} \to \mathbf{R}^{d+1}$ with $|\det T'| = |\det T|$.

Define $T' : \mathbf{R}^{d+1} \to \mathbf{R}^{d+1}$ by $T'((x,t)) = (Tx, t),\ \forall (x,t) \in \mathbf{R}^d \times \mathbf{R}$. Clearly T' is a linear map. By cofactor expansion along any row of $[T']$ - the matrix representation of T', and observing that $[T']$ is $[T]$ with an additional row vector e_{d+1} appended to the last row, and an additional column vector e_{d+1}^T to the last column, we get $|\det T'| = |\det T|$.



Matrix representation of $[T']$

Let $(x,t) \in A_f$, with $x \in \mathbf{R}^d$ and $0 \le t \le f(x)$. Then we have

$$(x,t) = (T(T^{-1}x), t) = T'((T^{-1}x, t)) \in T'(A_{f\circ T}).$$

Conversely, let $(x,t) \in T'(A_{f\circ T})$, then we have

$$(x,t) = T'((y,t)) = (Ty, t),\ 0 \le t \le f \circ T(y)$$

for some $y \in \mathbf{R}^d$. Hence $(x,t) \in A_f$. We conclude that $A_f = T'(A_{f\circ T})$ and the result follows. $\square$

Exercise 17

Proof. First we show that any unsigned Riemann integrable function is Lebesgue measurable, and the result follows from the area interpretations of the Riemann and Lebesgue integral.

By condition, $f : [a,b] \to [0,+\infty]$ is bounded, and $\forall n > 0$, there is an unsigned p.c function $g_n \le f$ such that $\int_a^b f(x)\ dx - 1/n \le \int_a^b g_n(x)\ dx$. By linearity, and monotonicity of the upper Riemann integral,

$$\overline{\int_a^b} f(x) - \sup_n g_n(x)\ dx = 0.$$

Extend f (and the g_n) to $\mathbf{R}$ as directed, by the definitions of upper Lebesgue integral and upper Riemann integral, and the fact that p.c functions are simple,

$$\overline{\int_{\mathbf{R}}} f(x) - \sup_n g_n(x)\ dx \le \overline{\int_a^b} f 1_{[a,b]}(x) - \sup_n g_n 1_{[a,b]}(x)\ dx = 0.$$

This in turn implies that

$$f(x) = \sup_n g_n(x)$$

almost everywhere. Since the g_n are unsigned measurable functions, we get the claim by (3) Exercise 3. $\square$

Exercise 18

Proof. (1). By monotonicity and the Markov inequality, $\forall \lambda \in (0, \infty)$,

$$m(\{x \in \mathbf{R}^d : f(x) = +\infty\}) \leq m(\{x \in \mathbf{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbf{R}^d} f(x)\ dx.$$

Sending $\lambda \to \infty$, and using the fact that $\int_{\mathbf{R}^d} f(x)\ dx < \infty$, we obtain the claim.

(2). Given $\int_{\mathbf{R}} f(x)\ dx = 0$, from the Markov inequality we get

$$m(\{x \in \mathbf{R}^d : f(x) \geq \lambda\}) = 0,\ \forall \lambda \in (0, \infty).$$

Therefore,

$$m(\{x \in \mathbf{R}^d : f(x) > 0\}) = m(\bigcup_{n=1} \{x \in \mathbf{R}^d : f(x) \geq 1/n\}) = 0$$

since the countable union of null sets is a null set. That is, $f = 0$ almost everywhere. Conversely, given $f = 0$ almost everywhere, we have $\int_{\mathbf{R}} f(x)\ dx = 0$ by definition. $\square$

Exercise 19

Proof. By definition, it suffices to establish linearity for real absolutely integrable $f, g : \mathbf{R}^d \to \mathbf{R}$ and real numbers c. From the identity

$$f + g = (f+g)_+ - (f+g)_- = f_+ - f_- + g_+ - g_-,$$

one obtains

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+,$$

which by linearity of the unsigned integral implies

$$\begin{aligned} &\int_{\mathbf{R}^d} (f+g)_+(x)\ dx + \int_{\mathbf{R}^d} f_-(x)\ dx + \int_{\mathbf{R}^d} g_-(x)\ dx \\ &= \int_{\mathbf{R}^d} (f+g)_-(x)\ dx + \int_{\mathbf{R}^d} f_+(x)\ dx + \int_{\mathbf{R}^d} g_+(x)\ dx. \end{aligned}$$

Rearrange the terms, we get $\int_{\mathbf{R}^d} f(x)+g(x)\ dx = \int_{\mathbf{R}^d} f(x)\ dx + \int_{\mathbf{R}^d} g(x)\ dx$. If $c=0$, then clearly $\int_{\mathbf{R}^d} cf(x)\ dx = c\int_{\mathbf{R}^d} f(x)\ dx = 0$. If $c<0$, then

$$\begin{aligned}\int_{\mathbf{R}^d} cf(x)\ dx &= \int_{\mathbf{R}^d} (-c)f_-(x)\ dx - \int_{\mathbf{R}^d} (-c)f_+(x)\ dx \\ &= -c(\int_{\mathbf{R}^d} f_-(x)\ dx - \int_{\mathbf{R}^d} f_+(x)\ dx) \\ &= c\int_{\mathbf{R}^d} f(x)\ dx\end{aligned}$$

by homogeneity ((3) of Exercise 10). Similar for $c>0$. Hence $\int_{\mathbf{R}^d} cf(x)\ dx = c\int_{\mathbf{R}^d} f(x)\ dx$ for any $c \in \mathbf{R}$.

By linearity, we see that

$$\begin{aligned}\int_{\mathbf{R}^d} \overline{f(x)}\ dx &= \int_{\mathbf{R}^d} \mathrm{Re}\overline{f(x)}\ dx + i\int_{\mathbf{R}^d} \mathrm{Im}\overline{f(x)}\ dx \\ &= \int_{\mathbf{R}^d} \mathrm{Re}f(x)\ dx + i\int_{\mathbf{R}^d} -\mathrm{Im}f(x)\ dx \\ &= \int_{\mathbf{R}^d} \mathrm{Re}f(x)\ dx - i\int_{\mathbf{R}^d} \mathrm{Im}f(x)\ dx \\ &= \overline{\int_{\mathbf{R}^d} f(x)\ dx}.\end{aligned}$$

We are done. □

Exercise 20

Proof. By definition, it suffices to establish the results for $f \in L^1(\mathbf{R}^d \to \mathbf{R})$. Let $y \in \mathbf{R}^d$. From Exercise 15 we get

$$\begin{aligned}\int_{\mathbf{R}^d} f(x+y)\ dx &= \int_{\mathbf{R}^d} f_+(x+y)\ dx - \int_{\mathbf{R}^d} f_-(x+y)\ dx \\ &= \int_{\mathbf{R}^d} f_+(x)\ dx - \int_{\mathbf{R}^d} f_-(x)\ dx \\ &= \int_{\mathbf{R}^d} f(x)\ dx.\end{aligned}$$

Similarly, from Exercise 16 we get

$$\begin{aligned}\int_{\mathbf{R}^d} f(Tx)\ dx &= \int_{\mathbf{R}^d} f_+(Tx)\ dx - \int_{\mathbf{R}^d} f_-(Tx)\ dx \\ &= \frac{1}{|\det T|}(\int_{\mathbf{R}^d} f_+(x)\ dx - \int_{\mathbf{R}^d} f_-(x)\ dx) \\ &= \frac{1}{|\det T|}\int_{\mathbf{R}^d} f(x)\ dx.\end{aligned}$$

Finally, let $f : [a, b] \to \mathbf{R}$ be Riemann integrable. If we extend f to $\mathbf{R}$ by declaring f to equal zero outside of $[a, b]$, then $f_+(x)$ and $f_-(x)$ are both Lebesgue measurable. Thus by Exercise 17, and linearity of the Riemann integral,

$$\begin{aligned}\int_{\mathbf{R}} f(x)\ dx &= \int_{\mathbf{R}} f_+(x)\ dx - \int_{\mathbf{R}} f_-(x)\ dx \\ &= \int_a^b f_+1_{[a,b]}(x)\ dx - \int_a^b f_-1_{[a,b]}(x)\ dx \\ &= \int_a^b f_+1_{[a,b]}(x) - f_-1_{[a,b]}(x)\ dx \\ &= \int_a^b f(x)\ dx.\end{aligned}$$

These conclude the proof. □

Exercise 21

Proof. Note that the function $f(x)$ is the pointwise limit of complex simple functions $f1_{|x|\le n}(x) = \sum_{i=-n}^{n} c_i 1_{[i,i+1)}(x)$, and thus measurable by definition.

Clearly $|f|$ is also measurable. By the vertical truncation property (and the definition of simple integral),

$$\begin{aligned}\int_{\mathbf{R}} |f(x)|\ dx = \lim_{n\to\infty}\int_{\mathbf{R}} |f(x)|1_{|x|\le n}\ dx &= \lim_{n\to\infty}\int_{\mathbf{R}} |\sum_{i=-n}^{n} c_i 1_{[i,i+1)}(x)|\ dx \\ &= \lim_{n\to\infty}\int_{\mathbf{R}} \sum_{i=-n}^{n} |c_i| 1_{[i,i+1)}(x)\ dx \\ &= \lim_{n\to\infty}\sum_{i=-n}^{n} |c_i| = \sum_{n\in\mathbf{Z}} |c_n|.\end{aligned}$$

The third equality holds since for each $x \in \mathbf{R}$, $x \in [n, n+1)$ for exactly one integer n. The claim then follows.

By definition of the complex-valued integral, we can assume that $c_n \in \mathbf{R}$. Then

$$\begin{aligned}
\int_{\mathbf{R}} f(x)\ dx &= \int_{\mathbf{R}} \sum_{n\in\mathbf{Z}} c_{n+} 1_{[n,n+1)}(x)\ dx - \sum_{n\in\mathbf{Z}} c_{n-} 1_{[n,n+1)}(x)\ dx \\
&= \sum_{n\in\mathbf{Z}} c_{n+} - \sum_{n\in\mathbf{Z}} c_{n-} \\
&= \sum_{n\in\mathbf{Z}} (c_{n+} - c_{n-}) = \sum_{n\in\mathbf{Z}} c_n.
\end{aligned}$$

The second to last equality holds since the sums are all convergent. □

Exercise 22

Proof. By condition, $f(x)1_E(x) : E \cup U \to \mathbf{C}$ is absolutely integrable. Therefore:

$$\int_{E\cup F} f(x)1_E(x)\ dx := \int_{\mathbf{R}^d} \widetilde{f1_E}(x)\ dx = \int_E f(x)\ dx.$$

And similarly,

$$\begin{aligned}
\int_{E\cup F} f(x)\ dx &:= \int_{\mathbf{R}^d} \tilde{f}1_E(x) + \tilde{f}1_F(x)\ dx = \int_{\mathbf{R}^d} \tilde{f}1_E(x)\ dx + \int_{\mathbf{R}^d} \tilde{f}1_F(x)\ dx \\
&= \int_E f(x)\ dx + \int_F f(x)\ dx
\end{aligned}$$

by linearity of the absolutely convergent integral. □

Exercise 23

Proof. Let $f : \mathbf{R}^d \to \mathbf{C}$ be locally absolutely integrable. Write $f := \sum_{n=1}^{\infty} f1_{B_n}$, where B_n is the annulus given by $B_1 := B(0,1)$, $B_n = B(0,n)\backslash B_{n-1}$ for any $n > 1$. By Lusin's theorem, for any $n \geq 1$ there is a Lebesgue measurable set $E_n \subset \mathbf{R}^d$ of measure at most $\varepsilon/2^n$ such that the restriction of $f1_{B_n}$ to the complementary set $\mathbf{R}^d\backslash E_n$ is continuous on that set. Letting $E := \bigcup_{n=1}^{\infty} E_n$, we conclude that E is Lebesgue measurable with measure at most ε, and the restriction of f to the complementary set $\mathbf{R}^d\backslash E$ is continuous on that set.

Now suppose that f is only measurable (but still finite almost everywhere). For every $n \geq 1$ let $f_n := f1_{|f|\leq n}$, which is locally absolutely integrable. By the last paragraph, there is a Lebesgue measurable set $E_n \subset \mathbf{R}^d$ of measure at most $\varepsilon/2^{n+1}$ such that the restriction of f_n to the complementary set $\mathbf{R}^d\backslash E_n$ is continuous on that set. Letting $E := \bigcup_{n=1}^{\infty} E_n$, we conclude that E is Lebesgue measurable with measure at most $\varepsilon/2$, and $\forall n \geq 1$, the restriction of f_n to the complementary set $\mathbf{R}^d\backslash E$ is continuous on that set. By construction, $f_n \to f$ pointwise everywhere, thus by Egorov's theorem, there is a Lebesgue measurable set F of measure at most $\varepsilon/2$, such that f_n converges locally uniformly to f outside of F. In particular, we see that $f_n|_{\mathbf{R}^d\backslash(E\cup F)} \to f|_{\mathbf{R}^d\backslash(E\cup F)}$ locally uniformly, with $m(E \cup U) \leq \varepsilon$. It's easy to show that locally uniform limit of continuous functions is continuous, we conclude that the restriction f to $\mathbf{R}^d\backslash(E \cup F)$ is continuous, as required. $\square$

Exercise 24

Proof. Let $f : \mathbf{R}^d \to \mathbf{C}$ be measurable, and $\varepsilon > 0$. By Exercise 25, there is a measurable set $E \subset \mathbf{R}^d$ of measure at most ε outside of which $f1_{B(0,n)}$ is locally bounded. In particular, $f1_{B(0,n)\backslash E}$ is bounded and thus absolutely integrable. By Theorem 15, there exists a continuous, compactly supported f_n such that $\|f1_{B(0,n)\backslash E} - f_n\|_{L^1(\mathbf{R}^d)} \leq \varepsilon$. By monotonicity we have

$$\begin{aligned}
&\frac{1}{n} \cdot m(\{x \in B(0,n) : |f(x) - f_n(x)| \geq 1/n\}) \\
&\leq \frac{1}{n} \cdot m(\{x \in B(0,n)\backslash E : |f(x) - f_n(x)| \geq 1/n\}) + \frac{1}{n}m(E) \\
&\leq \|f1_{B(0,n)\backslash E} - f_n\|_{L^1(\mathbf{R}^d)} + \frac{1}{n}m(E) \\
&\leq \varepsilon(1 + \frac{1}{n}).
\end{aligned}$$

By choosing $\varepsilon \leq \frac{1}{(n+1)2^n}$, we get a continuous function $f_n : \mathbf{R}^d \to \mathbf{C}$ for which the set $A_n := \{x \in B(0,n) : |f(x) - f_n(x)| \geq 1/n\}$ has measure at most $1/2^n$. Let $F := \{x \in \mathbf{R}^d : f_n(x) \not\to f(x)\}$. For every $x \in F$, there exists $N > 0$ such that $x \in A_n$ for all $n \geq N$. Hence $F \subset \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} A_n$, and we get $m(F) = 0$.

Conversely, if $f : \mathbf{R}^d \to \mathbf{C}$ is the pointwise almost everywhere limit of continuous functions $f_n : \mathbf{R}^d \to \mathbf{C}$, then f is measurable by part (1) and part (4) of Exercise 8. $\square$

Alternate proof of Lusin's theorem for arbitrary measurable functions:

Proof. Let $f : \mathbf{R}^d \to \mathbf{C}$ be measurable. By Exercise 24, there is a sequence of continuous functions $f_n : \mathbf{R}^d \to \mathbf{C}$ such that $f_n(x) \to f(x)$ almost everywhere. By Egorov's theorem, there exists a Lebesgue measurable set E of measure at most ε, such that f_n converges locally uniformly to f outside of E. Since locally uniform limit of continuous functions is continuous, the restriction of f to the complementary set $\mathbf{R}^d \backslash E$ is continuous on that set. $\square$

Exercise 25

Proof. (1). By (1) of Theorem 15, there exists a continuous, compactly supported g such that $\|f - g\|_{L^1(\mathbf{R}^d)} \le \varepsilon$. Pick $R > 0$ such that $\mathrm{supp}(g) \subset B(0, R)$. By divisibility, we have

$$\int_{\mathbf{R}^d \backslash B(0,R)} |f(x)|\ dx + \int_{B(0,R)} |f(x) - g(x)|\ dx = \int_{\mathbf{R}^d} |f(x) - g(x)|\ dx \le \varepsilon.$$

In particular, $\displaystyle\int_{\mathbf{R}^d \backslash B(0,R)} |f(x)|\ dx \le \varepsilon.$

(2). We assume that $f < \infty$ almost everywhere. As shown above, there exists a Lebesgue measurable set $E \subset \mathbf{R}^d$ of measure at most ε such that f_n converges locally uniformly to f outside of E, where $f_n : \mathbf{R}^d \to \mathbf{C}$ is a sequence of continuous functions. In particular, $\exists N > 0$ such that $|f_n(x) - f(x)| \le \varepsilon$ for all $x \in B(0, R) \backslash E, n \ge N$, and $R > 0$. By the maximum principle, $\exists C > 0$ such that $|f_N(x)| \le C$ on $\overline{B(0, R)}$. By the triangle inequality, we get

$$|f(x)| \le |f_N(x)| + \varepsilon \le C + \varepsilon.$$

for all $x \in B(0, R) \backslash E$. Setting $M := C + \varepsilon$ completes the proof. $\square$