# Math 245A Prologue

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## 1 Selected Exercises in the prologue

Exercise 1

*Proof.* By definition of elementary sets, both $E$ and $F$ can be written as a finite union of boxes. Hence both $E \cap F$ and $E \cup F$ are elementary because the intersection of a box with another box is again a box, and a finite union of boxes is elementary. Also note that the difference of two boxes $B_i \setminus B_j^{'}$ is a box, so the the difference $E \setminus F = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} B_i \setminus B_j^{'}$ is elementary, which in turn implies that the symmetric difference $E \Delta F$ is elementary. Finally, the translate $E + x := \{y + x : y \in E\}$ is elementary since the translate of a box remains a box. $\square$

Exercise 2

*Proof.* Let $B_1 \cup \ldots \cup B_k$ and $B_1' \cup \ldots \cup B_{k'}'$ be two partitions of $E$ into disjoint boxes. Then in particular the set $\{B_i \cap B_j' : 1 \leq i \leq k, 1 \leq j \leq k'\}$ is a partition of $E$ into disjoint boxes. By the definition of disjoint partition, every $B_i = \bigcup_{j=1}^{k'} B_i \cap B_j'$ and conversely every $B_j' = \bigcup_{i=1}^{k} B_j' \cap B_i$, so $|B_1| + \cdots + |B_k| = |B_1'| + \cdots + |B_{k'}'| = \sum_{1 \leq i \leq k, 1 \leq j \leq k'} |B_i \cap B_j'| = m(E)$. $\square$

Exercise 3

*Proof.* Set $c := m'([0,1)^d)$, and $A := \{(k_1, \ldots, k_d) \in \mathbf{Z}^d : 0 \leq k_i \leq n-1\}$, then $A$ is a finite set of cardinality $n^d$. As $[0,1)^d = \bigcup_{x \in A} x + [0, \frac{1}{n})^d$, by finite additivity and translation invariance we get $c = n^d m'([0, \frac{1}{n})^d])$, or equivalently $m'([0, \frac{1}{n})^d]) = \frac{c}{n^d} = cm([0, \frac{1}{n})^d)$.

By finite additivity and translation invariance again, it suffices to show the given equality holds for an arbitrary box $B = I_1 \times \cdots \times I_d$ whose left endpoint coincides with the origin of $\mathbf{R}^d$. From non-negativity and finite additivity (and Exercise 1) we conclude the monotonicity property $m'(E) \leq m'(F)$ whenever $E \subset F$ are nested elementary sets. We further conclude that it suffices to show the given equality holds for rational boxes. Since if one denotes the right endpoint of $I_i$ by $a_i$, one can find two sequences of rational numbers $(q_{i,n})_n$ and $(p_{i,n})_n$ converging to $a_i$ from the left and from the right respectively, with $q_{i,n} \leq p_{i,n}$ for all $n$. The sequences of the corresponding rational boxes $(A_n)_n$ and $(C_n)_n$ thus approximate $B$ from within and from without, and by monotonicity we have $cm(A_n) = m'(A_n) \leq m'(B) \leq m'(C_n) = cm(B_n)$, in particular $m'(C_n) - m'(A_n) = c|m(C_n) - m(A_n)| = c\prod_{i=1}^{d} \frac{1}{p_{i,n}-q_{i,n}} \leq \varepsilon$ for sufficiently large $n$ and any $\varepsilon > 0$. By the squeeze test, this implies that $\lim_n m'(A_n) = c \lim_n m(A_n) = cm(B)$.

For such a rational box $B = I_1 \times \cdots \times I_d$, the right endpoint of each $I_i$ can be written as $\frac{q_i}{N_i}$ for some positive integer $N_i$ and some natural number $q_i$. Let $N$ be a common multiple of the $N_i$ for all $1 \leq i \leq d$, then $B$ can be expressed (up to the measure $m$ and $m'$) as the disjoint union of $k$ translates of the box $[0, 1/N)^d$ for some finite $k$. By finite additivity, $m'(B) = km'([0, 1/N)^d) = ckm([0, 1/N)^d) = cm(B)$, as desired. □

Exercise 4

*Proof.* Write $E_1 = \bigcup_{i=1}^{k} B_i^{d_1}$ as a finite union of $d_1$-dimensional boxes and $E_2 = \bigcup_{j=1}^{k'} B_j^{d_2}$ as a finite union of $d_2$-dimensional boxes. Then $_1 \times E_2 = \bigcup_{1\leq i\leq k, 1\leq j\leq k'} (B_i^{d_1} \times B_j^{d_2})$ is elementary, for the products $B_i^{d_1} \times B_j^{d_2}$ are $d_1 + d_2$-dimensional boxes by definition.

By Lemma 2, we can assume that the $B_i^{d_1}$s and the $B_j^{d_2}$s are all disjoint. From the definition of the elementary measure of boxes, it is obvious that $m^{d_1+d_2}(B_i^{d_1} \times B_j^{d_2}) = m^{d_1}(B_i^{d_1}) \times m^{d_2}(B_j^{d_2})$. Hence we obtain

$$\begin{aligned} m^{d_1+d_2}(E_1 \times E_2) &= m^{d_1+d_2}(\bigcup_{i,j} B_i^{d_1} \times B_j^{d_2}) = \sum_{i,j} m^{d_1+d_2}(B_i^{d_1} \times B_j^{d_2}) \\ &= \sum_{i,j} m^{d_1}(B_i^{d_1}) \times m^{d_2}(B_j^{d_2}) = (\sum_i m^{d_1}(B_i^{d_1}))(\sum_j m^{d_2}(B_j^{d_2})) \\ &= m^{d_1}(B_i^{d_1}) \times m^{d_2}(B_j^{d_2}) \end{aligned}$$

and we are done. □

Exercise 5

*Proof.* Assume 2. Then for every $\varepsilon > 0$, $m^{*,(J)}(E) - m_{*,(J)}(E) \leq m(B \backslash A) \leq \varepsilon$, implying $m^{*,(J)}(E) = m_{*,(J)}(E) = m(E)$, so $E$ is Jordan measurable. Clearly 1 implies 2, and 2 implies 3 by monotonicity of the outer Jordan measure. Finally, assume 3. Then one can find an elementary set $A' \supset A\Delta E$ with $m(A') \leq \varepsilon$. The sets $B = A \setminus A'$ and $C = A \cup A'$ are elementary, with $B \subset E \subset C$, and we have $m(C \setminus B) = m(A') \leq \varepsilon$, which is exactly 2. $\square$

Exercise 6

*Proof.* (1). By Exercise 5, For every $\varepsilon > 0$, there exist elementary sets $A \subset E \subset B$ and $A' \subset F \subset B'$ such that $m(B \backslash A) \leq \varepsilon$ and $m(B' \backslash A') \leq \varepsilon$.

By finite subadditivity and monotonicity of the elementary measure, $A \cup A' \subset E \cup F \subset B \cup B'$ are such that $m(B \cup B' \backslash A \cup A') \leq m(B \backslash A \cup A') + m(B' \backslash A \cup A') \leq m(B \backslash A) + m(B' \backslash A') \leq 2\varepsilon$, so $E \cup F$ is Jordan measurable.

Similarly, by finite additivity and monotonicity of the elementary measure, $A \backslash B' \subset E \backslash F \subset B \backslash A'$ are such that $m((B \backslash A') \backslash (A \backslash B')) \leq m(B' \backslash A') + m(B \backslash A) \leq 2\varepsilon$, so $E \backslash F$ is Jordan measurable, from which follows that $E \cap F = (E \cup F) \backslash (E \backslash F) \cup (F \backslash E)$ and thus $E \Delta F$ are both Jordan measurable.

(2). This follows from non-negativity of the elementary measure.

(3). By finite subadditivity of the elementary measure, $m(B) + m(B') \geq m(B \cup B') \geq m(E \cup F)$, taking the infimum over $B$ and then the infimum over $B'$ gives $m(E) + m(F) \geq m(E \cup F)$. On the other hand, since $E \cap F = \emptyset$, $A \cap A' = \emptyset$. By finite additivity of the elementary measure, we get $m(A \cup A) = m(A) + m(A') \leq m(E \cup F)$, taking the supremum over $A$ and then the supremum over $A'$ gives $m(E) + m(F) \leq m(E \cup F)$, so $m(E) + m(F) = m(E \cup F)$.

(4). This follows from non-negativity and finite additivity.

(5). This follows from monotonicity, finite additivity, and part(1).

(6). $E + x$ is Jordan measurable from Exercise 1 and 5, and since the elementary measure is translation invariant, so is the Jordan measure by definition. $\square$

Exercise 7.

*Proof.* (1). By the Heine-Borel theorem, $B$ is compact. Since on a compact metric space, continuous functions are uniformly continuous, $\forall \varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(y) \leq \varepsilon|$ whenever $|x - y| \leq \delta$.

Let $B = I_1 \times \cdots \times I_d$, where the $I_i$s are closed intervals, one can then partitions each $I_i$ into sub-intervals of length less than $\frac{\delta}{\sqrt{d}}$. Taking Cartesian product, we see that $B$ can be expressed as a disjoint union of finitely many boxes $B_j : 1 \le j \le k$, such that the length of the main diagonal of each box is less than $\delta$. For every $1 \le j \le k$, pick a random point $x_j \in B_j$, then the graph $G = \{(x, f(x)) : x \in B\}$ is contained in the elementary set $A = \bigcup_{j=1}^k (B_j \times [f(x_j) - \varepsilon, f(x_j) + \varepsilon]) \subset \mathbf{R}^{d+1}$. Thus $m(A) = 2\varepsilon m(B)$, $m(B)$ is finite, $G$ is Jordan measurable by part (3) of Exercise 5, with Jordan measure 0 by monotonicity.

(2). This follows from the construction above and part (2) of Exercise 5. □

Exercise 8

*Proof.* (1). The three edges of the triangle determines three continuous functions in the plane, by Boolean closure of the Jordan measure and Exercise 7, the conclusion thus follows.

(2). Denote the solid plane triangle by $\Delta ABC$. We first consider the case when one of the edges, say $AB$, is horizontal. By translation invariance of the Jordan measure, we can put $A$ at the origin with $B = (b, 0)$ and $C = (c, h)$ where $0 < c < b$. Denote by $f : [0, b] \to \mathbf{R}$ the function whose graph agrees with $AC$ on $[0, c]$ and with $CB$ on $(c, b]$. By comparing the definitions of inner/outer Jordan measures with those of upper/lower Riemann integrals, we get $\underline{\int f} \le m(\Delta ABC) \le \overline{\int f}$, implying that the Jordan measure of $\Delta ABC$ equals the area $A(\Delta ABC)$ in the case of a horizontal edge.

The general case then follows from circumscribing any triangle by a rectangular box, and use finite additivity to subtract measures of those triangles with a horizontal edge from the measure of the box. □

Exercise 9

*Proof.* Let $P = \bigcap_{i=1}^n \{x \in \mathbf{R}^d : x \cdot v_i \le c_i\} = \bigcap_{i=1}^n H_i$ be a compact convex polytype. Since $P$ is bounded, there exist a closed box $B \subset \mathbf{R}^d$ such that $P = P \cap B = \bigcap_{i=1}^n H_i \cap B$, it suffices to show that the intersections $H_i \cap B$ are Jordan measurable.

The claim is obvious when $d = 1$, now assume that $d > 1$. Write $x \in \mathbf{R}^d$ as $(x', t)$, where $x' \in \mathbf{R}^{d-1}$ and $t \in \mathbf{R}$. Similarly, write $B = B' \times [a, b]$ and $v_i = (v_i', v)$. Then we have

$$H_i \cap B = \{(x', t) : x' \in B',\ x' \cdot v_i' + tv \le c_i,\ t \in [a, b]\}.$$

For $v \neq 0$, $H_i \cap B = \{(x', t) : x' \in B',\ a \leq t \leq f_i(x')\}$, which is Jordan measurable by Exercise 7, where $f_i(x') = \min(\{b, (c_i - x' \cdot v_i')/v\})$ is continuous. On the other hand, if $v = 0$, $H_i \cap B = \{(x', t) : x' \in B' \cap A,\ t \in [a, b]\}$, where $A = \{x' \in \mathbf{R}^{d-1} : x' \cdot v_i' \leq c_i\}$, which is also Jordan measurable by the fact that continuous functions are uniformly continuous on a compact metric space. $\square$

Exercise 10

*Proof.* (1). By translation invariance, it suffices to show that the open and closed balls centered at the origin are Jordan measurable. By symmetry, it suffices to show that the upper half balls are Jordan measurable. If we define $f : B^{d-1}(0, r) \to \mathbf{R}$ by $f(x) = \sqrt{r^2 - \|x\|^2}$, where the superscript denotes the dimension, then $\{(x, f(x)) : x \in B^{d-1}(0, r)\}$ is the graph of the upper ball in $\mathbf{R}^d$.

When $d = 1, 2$, the half balls (both open and closed) are Jordan measurable by Boolean closure and Exercise 7. By induction, suppose that the $d-1$-dimensional balls are Jordan measurable for some $d > 3$. By part (2) of Exercise 5, one can find elementary sets $A \subset \overline{B^{d-1}(0, r)} \subset B$ with $m(B \backslash A) \leq \epsilon$. Thus

$$E = \{(x, f(x)) : x \in A\} \subset \{(x, f(x)) : x \in \overline{B^{d-1}(0, r)}\} \subset \{(x, f(x)) : x \in B\} = F$$

and similarly for the three sets $E' := \{(x, t) : x \in A; 0 \leq t \leq f(x)\}, \{(x, t) : x \in \overline{B^{d-1}(0, r)}; 0 \leq t \leq f(x)\}, F' := \{(x, t) : x \in B; 0 \leq t \leq f(x)\}$. By partitioning $A$ and $B$ into finite union of disjoint boxes, we see by Exercise 7 that $E, E', F, F'$ are all Jordan measurable, with $m(E) = m(F) = 0$ and $m(F' \backslash E') = O(\varepsilon)$. By Exercise 5, this implies that the open and closed half balls in $\mathbf{R}^d$ are Jordan measurable.

Consider the inscribed cube $[-\frac{r}{\sqrt{d}}, \frac{r}{\sqrt{d}}]^d$ and the circumscribed cube $[-r, r]^d$ of a $d$-dimensional ball. It follows by monotonicity that $(\frac{2r}{\sqrt{d}})^d \leq m(B(x, r)) = m(\overline{B(x, r)}) \leq (2r)^d$, so $m(B(x, r)) = c_d r^d$ for some constant $c_d > 0$ depending only on $d$.

(2). This follows from the last paragraph in part (1). $\square$

Exercise 11

*Proof.* (1). We first show that the linear map of elementary sets are Jordan measurable. Let $B := \prod_{i=1}^{d}[a_i, b_i]$ be a closed box. Then we can write

$$B = \bigcap_{i=1}^{d}\{x \in \mathbf{R}^d : x \cdot e_i \in [a_i, b_i]\}$$

Let $[L]$ be the matrix representation of $L$. $\forall x \in B$, $[L]x^T = \begin{pmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_d \cdot x \end{pmatrix}$, where $r_i$ is the $i^{th}$ row of $[L]$. Then $y_i := r_i \cdot x \in [a_i', b_i']$ for some $a_i' \leq b_i' \in \mathbf{R}$ and every $1 \leq i \leq d$. And we have:

$$L(B) = \{[L]x^T : x \in B\} = \bigcap_{i=1}^{d}\{y \in \mathbf{R}^d : y \cdot e_i \in [a_i', b_i']\}$$

which is a compact convex polytope, and thus Jordan measurable by Exercise 9. If $E = \bigcup_{n=1}^{N} B_n$ is elementary, then $L(E) = L(\bigcup_{n=1}^{N} B_n) = \bigcup_{n=1}^{N} L(B_n)$ is Jordan measurable by Boolean closure.

With the notation of Exercise 3, define $m' : \mathcal{E}(\mathbf{R}^d) \to \mathbf{R}^+$ by

$$m'(E) := m(L(E)).$$

If $L$ is singular, then the rank of $[L]$ is smaller than $d$. Particularly, $L(E)$ can be contained in a box with some sides degenerated, and thus

$$m(L(E)) = 0 = 0m(E) = |\det L|m(E).$$

Hence we assume that $L$ is invertible. Let $E_1$, $E_2 \in \mathcal{E}(\mathbf{R}^d)$ with $E_1 \cap E_2 = \emptyset$, then

$$\begin{aligned} m'(E_1 \cup E_2) &= m(L(E_1 \cup E_2)) = m(L(E_1) \cup L(E_2)) = m(L(E_1)) + m(L(E_2)) \\ &= m'(E_1) + m'(E_2). \end{aligned}$$

By Exercise 6, $\forall x \in \mathbf{R}^d$,

$$m'(x + E_1) = m(L(x + E_1)) = m(L(x) + L(E_1)) = m(L(E_1)) = m'(E_1).$$

Also, $m' \geq 0$. So $m'$ obeys non-negativity, finite additivity and translation invariance, which implies by Exercise 3 that

$$\exists D \in \mathbf{R}^+, \ m'(E) = m(L(E)) = Dm(E).$$

(2). Let $E$ be Jordan measurable. By Exercise 2 there exist elementary sets $A \subset E \subset B$ with $m(B \backslash A) = m(B) - m(A) \leq \varepsilon$. Since $L(A) \subset L(E) \subset L(B)$, by monotonicity

$$Dm(A) = m(L(A)) \leq m(L(E)) \leq m(L(B)) = Dm(B).$$

Since $D \geq 0$, we also have

$$Dm(A) \leq Dm(E) \leq Dm(B).$$

Combining the two bounds gives

$$|m(L(E)) - Dm(E)| \leq D(m(B) - m(A)) \leq D\varepsilon$$

Sending $\varepsilon \to 0$ shows that $m(L(E)) = Dm(E)$.

(3). From Exercise 3, we know that $D = m(L([0,1)^d))$. Start with $L$ being an elementary transformation. If $L$ interchanges two rows, then $\det L = -1$ and

$$D = m([0,1)^d) = 1 = |\det L|.$$

If $L$ scales a row by some constant $k$ (assuming without loss of generality that $k > 0$), then $\det L = k$ and

$$D = m([0,1) \times \cdots \times [0,k) \times \cdots \times [0,1)) = |\det L|.$$

Now suppose $L$ adds row $i$ to row $j$. We upper and lower bound the sheared box by a finite union of boxes. $\forall n > 0$, define

$$\overline{B_i} := [\frac{i-1}{n}, \frac{i}{n}) \times [\frac{i-1}{n}, 1 + \frac{i}{n}),\ \underline{B_i} := [\frac{i-1}{n}, \frac{i}{n}) \times [\frac{i}{n}, 1 + \frac{i-1}{n}).$$

and let $E := \bigcup_{i=1}^n \underline{B_i}$ and $F := \bigcup_{i=1}^n \overline{B_i}$. The sets $E$ and $F$ upper and lower bound the sheared box $\{(x, x+y) : x, y \in [0,1)\}$ respectively, with

$$m(E) = 1 - \frac{1}{n},\ m(F) = 1 + \frac{1}{n}.$$

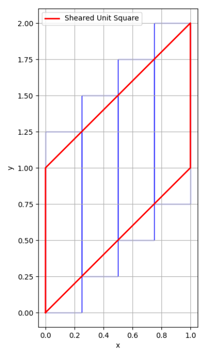
It follows that

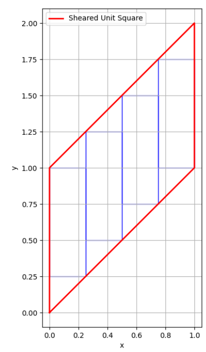
$$E \times B^{d-2} \subset S_{1i} S_{2j} L([0,1)^d) \subset F \times B^{d-2}$$

where $S_{ij}$ is the type I transformation that switches the $i^{th}$ and $j^{th}$ row and $B^{d-2}$ the $d-2$ dimensional unit box. By monotonicity, Exercise 4 and above, we thus have

$$1 - \frac{1}{n} \leq m(L([0,1)^d)) \leq 1 + \frac{1}{n},\ \forall n > 0.$$

Sending $n \to \infty$ implies that $D = 1 = |\det L|$.




Finally, Let $L = \prod_{i=1}^{m} L_i$ be the factorization of $L$ into elementary maps, then $D = m(L([0,1))^d) = m(\prod_{i=1}^{m} L_i([0,1)^d)) = (\prod_{i=1}^{m} |\det L_i|)m([0,1)^d) = |\det L| \cdot 1 = |\det L|$. $\square$

Exercise 12

*Proof.* Let $E$ be a Jordan null set, and $F \subset E$. By Exercise 5 $F$ is Jordan measurable. By monotonicity $m(F) = 0$. $\square$

Exercise 13

*Proof.* Let $A \subset \mathbf{R}^d$ be an Jordan measurable set. By part (2) of Exercise 5, exists $E \subset A \subset F$ with $m(F \backslash E) \leq \varepsilon$, where $E$ and $F$ are elementary sets. By construction, one has

$$\lim_{N\to\infty} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbf{Z}^d) \leq \lim_{N\to\infty} \frac{1}{N^d} \#(A \cap \frac{1}{N}\mathbf{Z}^d), m(A) \leq \lim_{N\to\infty} \frac{1}{N^d} \#(F \cap \frac{1}{N}\mathbf{Z}^d).$$

The LHS and RHS are $m(E)$ and $m(F)$ respectively as shown earlier. It follows that $|m(A) - \lim_{N\to\infty} \frac{1}{N^d} \#(A \cap \frac{1}{N}\mathbf{Z}^d)| \leq m(F \backslash E) \leq \varepsilon$. Sending $\varepsilon \to 0$ gives the desired result. $\square$

Exercise 14

*Proof.* Let $E \subset \mathbf{R}^d$ be bounded and Jordan measurable. By definition, the dyadic cubes are disjoint with measure $2^{-nd}$ each.

We first show that the Jordan measure of a box admits metric entropy formulation. For simplicity, assume all sides of the box are of the form $[a_k, b_k)$ for some $a_k \leq b_k, 1 \leq k \leq d$. Inserting a cube of scale $2^{-n}$ inside the box $\prod_{k=1}^{d}[a_k, b_k)$ requires

$$2^n b_k - 1 > i_k \geq 2^n a_k, 1 \leq k \leq n.$$

which is always possible for sufficiently large $n$. For the smallest $i_k$ with $i_k \geq 2^n a_k$, we have $\frac{i_k}{2^n} - a_k \leq 2^{-n}$, and similarly for the largest $i_k$ with $i_k < 2^n b_k - 1$ we have $b_k - \frac{i_k + 1}{2^n} \leq 2^{-n}$. This implies that

$$m([a_k, b_k)^d) - 2^{-nd}\mathcal{E}_*([a_k, b_k)^d, 2^{-n}) \leq O((2^{-n})^d)$$

On the other hand, Intersecting a cube of scale $2^{-n}$ with the box $\prod_{k=1}^{d}[a_k, b_k)$ requires

$$2^n a_k - 1 \leq i_k < 2^n b_k, 1 \leq k \leq n.$$

For the smallest $i_k$ with $i_k \geq 2^n a_k - 1$ , we have $a_k - \frac{i_k}{2^n} \leq 2^{-n}$, and similarly for the largest $i_k$ with $i_k < 2^n b_k$, we have $\frac{i_k + 1}{2^n} - b_k \leq 2^{-n}$, implying that

$$2^{-nd}\mathcal{E}^*([a_k, b_k)^d, 2^{-n}) - m([a_k, b_k)^d) \leq O((2^{-n})^d)$$

Sending $n \to \infty$ and squeezing one gets

$$m([a_k, b_k)^d) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}_*([a_k, b_k)^d, 2^{-n}) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}^*([a_k, b_k)^d, 2^{-n}).$$

Since elementary sets are finite union of disjoint boxes, it is easy to extend the metric entropy formulation of Jordan measurability to the class of elementary sets. Finally, let $A$ and $B$ be elementary sets such that $A \subset E \subset B$ and $m(B\backslash A) \leq \varepsilon$. By monotonicity and the definition of $\mathcal{E}_*$ and $\mathcal{E}^*$,

$$\lim_{n\to\infty} 2^{-dn}\mathcal{E}_*(A, 2^{-n}) \leq \lim_{n\to\infty} 2^{-dn}\mathcal{E}_*(E, 2^{-n}) \leq \lim_{n\to\infty} 2^{-dn}\mathcal{E}^*(E, 2^{-n}) \leq \lim_{n\to\infty} 2^{-dn}\mathcal{E}^*(B, 2^{-n}).$$

The LHS and RHS equals $m(A)$ and $m(B)$ resp, and are both bounded. Sending $\varepsilon \to 0$ and again by squeezing, we obtain

$$m(E) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}_*(E, 2^{-n}) = \lim_{n\to\infty} 2^{-dn}\mathcal{E}^*(E, 2^{-n}).$$

Conversely, suppose that $\lim_{n\to\infty} 2^{-dn}(\mathcal{E}^*(E, 2^{-n}) - \mathcal{E}_*(E, 2^{-n})) = 0$. As union of dyadic cubes is elementary, $E$ is Jordan measurable from part (2) of Exercise 5. $\square$

Exercise 15

*Proof.* Let $E$ be a Jordan measurable set. By Exercise 5 there exist elementary sets $A$ and $B$ such that $A \subset E \subset B$ and $m(B \backslash A) \leq \varepsilon$. Furthermore, from finite additivity of $m'$ we also get monotonicity of $m'$. Thus if $c = m'([0,1)^d])$, by Exercise 3, one gets

$$cm(A) = m'(A) \leq m'(E), cm(E) \leq m'(B) = cm(B).$$

And thus $|cm(E) - m'(E)| \leq c\varepsilon$. Sending $\varepsilon \to 0$ gives the result. $\square$

Exercise 16

*Proof.* As before, we find elementary sets $A_1, A_2 \subset \mathbf{R}^{d_1}, B_1, B_2 \subset \mathbf{R}^{d_2}$ such that $A_1 \subset E_1 \subset B_1$ and $A_2 \subset E_2 \subset B_2$ with $m(B_1 \backslash A_1), m(B_2 \backslash A_2) \leq \varepsilon$. Then in particular $A_1 \times A_2 \subset E_1 \times E_2 \subset B_1 \times B_2$, and by Exercise 4, one gets

$$\begin{aligned} m((B_1 \times B_2) \backslash (A_1 \times A_2)) &= m^{d_1}(B_1) m^{d_2}(B_2) - m^{d_1}(A_1) m^{d_2}(A_2) \\ &\leq (m^{d_1}(A_1) + \varepsilon)(m^{d_2}(A_2) + \varepsilon) - m^{d_1}(A_1) m^{d_2}(A_2) \\ &= \varepsilon(m^{d_1}(A_1) + m^{d_2}(A_2) + \varepsilon) = O(\varepsilon). \end{aligned}$$

This implies that $E_1 \times E_2$ is Jordan measurable, and since $m(A_1 \times A_2) \leq m^{d_1}(E_1) m^{d_2}(E_2) \leq m(B_1 \times B_2)$, we get $m^{d_1+d_2}(E_1 \times E_2) = m^{d_1}(E_1) \times m^{d_2}(E_2)$ by squeezing. $\square$

Exercise 17

*Proof.* By Exercise 11, rigid motions preserve Jordan measurability, and the Jordan measure is invariant after the transformation. Let $\{P_1, \ldots, P_n\}$ be the partition of $P$ into sub-polytopes, with $Q_i$ be the transformed $P_i$ for all $i$. It is easy to show that the boundaries of these sub-polytopes have Jordan measure zero, thus by finite additivity $m(Q) = \sum_{i=1}^n m(Q_i) = \sum_{i=1}^n m(P_i) = m(P)$. $\square$

Exercise 18

*Proof.* (1). First note that a box $\prod_{i=1}^d I_i$ and its' closure $\prod_{i=1}^d \overline{I_i}$ have the same Jordan measure $\prod_{i=1}^d |I_i|$, which in turn implies that an elementary set and its' closure have the same Jordan measure, as the closure of the union is the union of the closure. For any elementary set $B \supset E$, it holds that $\overline{B} \supset \overline{E}$ since $\overline{E}$ is the smallest closed set containing $E$. Hence

$$m^{*,(J)}(\overline{E}) := \inf_{A \supset E, A \text{ elementary}} m(A) \leq m(\overline{B}) = m(B), \text{ for any elementary } B \supset E,$$

taking the infimum over all elementary $B \supset E$ on the RHS gives $m^{*,(J)}(\overline{E}) \leq m^{*,(J)}(E)$. On the other hand, $m^{*,(J)}(E) \leq m^{*,(J)}(\overline{E})$ naturally by definition, as the former is taking infimum over a larger set, and so $m^{*,(J)}(\overline{E}) = m^{*,(J)}(E)$.

(2). Similarly, one can show that an elementary set and its' interior have the same Jordan measure. For any elementary set $B \subset E$, it holds that $B^\circ \subset E^\circ$ since $E^\circ$ is the largest open set contained in $E$. Hence

$$m_{*,(J)}(E^\circ) := \sup_{A \subset E^\circ, A \text{ elementary}} m(A) \geq m(B^\circ) = m(B), \text{ for any elementary } B \subset E,$$

taking the supremum over all elementary $B \subset E$ on the RHS gives $m_{*,(J)}(E^\circ) \geq m_{*,(J)}(E)$. Clearly one has $m_{*,(J)}(E^\circ) \leq m_{*,(J)}(E)$ as the former is taking supremum over a smaller set, and so $m_{*,(J)}(E^\circ) = m_{*,(J)}(E)$.

(3). Assume first $m^{*,(J)}(\partial E) = 0$. Let $C$ be any elementary set containing $E$, which is possible since $E$ is bounded, and $B$ a small elementary set containing $\partial E$ such that $m^{*,(J)}(B) \leq \varepsilon$. Since boxes are topologically connected, the elementary set $C \backslash B$ can be written as a finite union of disjoint boxes $(\bigcup_{i \in I} B_i) \cup (\bigcup_{j \in J} B_j)$, where $I$ and $J$ are index sets such that $B_i \subset E$ for any $i \in I$ and $B_j \cap E = \emptyset$ for any $j \in J$. Let $A := \bigcup_{i \in I} B_i$. It then follows that

$$m^{*,(J)}(A \Delta E) = m^{*,(J)}(E \backslash A) \leq m^{*,(J)}(B) \leq \varepsilon.$$

Hence $E$ is Jordan measurable by Exercise 5.

Next, let $E$ be Jordan measurable, then $m(E) := m_{*,(J)}(E) = m^{*,(J)}(E)$. We choose elementary sets $B_1$ and $B_2$ with $B_1 \supset \overline{E}$ and $B_2 \subset E^\circ$ such that

$$|m(B_1) - m^{*,(J)}(\overline{E})| \leq \varepsilon \text{ and } |m_{*,(J)}(E^\circ) - m(B_2)| \leq \varepsilon.$$

The elementary set $B_1 \backslash B_2$ contains $\partial E$, and by part (1) and (2) we conclude that

$$m^{*,(J)}(\partial E) \leq m(B_1 \backslash B_2) \leq |m^{*,(J)}(\overline{E}) + \varepsilon - (m_{*,(J)}(E^\circ) - \varepsilon)| = 2\varepsilon.$$

Sending $\varepsilon \to 0$, we see that $m^{*,(J)}(\partial E) = 0$.

(4). By part (1) and (2), we have

$$m^{*,(J)}([0,1]^2\backslash\mathbf{Q}^2) = m^{*,(J)}(\overline{[0,1]^2\backslash\mathbf{Q}^2}) = m^{*,(J)}([0,1]^2) = 1$$

and

$$m_{*,(J)}([0,1]^2\backslash\mathbf{Q}^2) = m_{*,(J)}(([0,1]^2\backslash\mathbf{Q}^2)^\circ) = m_{*,(J)}(\emptyset) = 0.$$

Similar for the set of bullets $[0,1]^2 \cap \mathbf{Q}^2$. $\square$

Exercise 19

*Proof.* Note that Jordan outer measure is subadditive from the fact that elementary measure is subadditive. Thus $m^{*,(J)}(E) \le m^{*,(J)}(E\cap F) + m^{*,(J)}(E\backslash F)$.

On the other hand, let $B \supset E$ for some elementary set $B$. The elementary sets $B\cap F$ and $B\backslash F$ contain $E\cap F$ and $E\backslash F$ resp. In particular,

$$m(B) = m(B\cap F) + m(B\backslash F) \ge m^{*,(J)}(E\cap F) + m^{*,(J)}(E\backslash F)$$

Taking infimum over all such $B$ shows $m^{*,(J)}(E) \ge m^{*,(J)}(E\cap F) + m^{*,(J)}(E\backslash F)$, and we are done. $\square$

Exercise 20

*Proof.* Let $f : [a,b] \to \mathbf{R}$ be a p.c function with respect to some partition $\mathbf{P} := \{I_i : 1 \le i \le n\}$ such that $f$ is equal to a constant $c_i$ on each of the intervals $I_i$. If $f$ is to be p.c with respect to another partition $\mathbf{P'} := \{I'_j : 1 \le j \le m\}$, then it must be that $\mathbf{P'}$ is finer than $\mathbf{P}$ or $\mathbf{P}$ is finer than $\mathbf{P'}$, as length is finitely additive, either way we have

$$\sum_{i=1}^n c_i|I_i| = \sum_{1\le k\le n: c_k \text{distinct}} c_k(\sum_{i\in B_k}|I_i|); \; \sum_{j=1}^m c_j|I'_j| = \sum_{1\le k\le n: c_k \text{distinct}} c_k(\sum_{j\in B'_k}|I'_j|),$$

where $B_k := \{i : 1 \le i \le n, f(x) = c_k \text{ on } I_i\}$, and similarly $B'_k := \{j : 1 \le j \le m, f(x) = c_k \text{ on } I'_j\}$. By finite additivity of length, $\sum_{i\in B_k}|I_i| = \sum_{j\in B'_k}|I'_j|$, so $\sum_{i=1}^n c_i|I_i| = \sum_{j=1}^m c_j|I'_j|$, as desired. $\square$

Exercise 21

*Proof.* (1). Clearly $cf$ is p.c since it is just $f$ scaled by $c$, and by definition we have h p.c. $\int_a^b cf(x)\ dx = c$p.c. $\int_a^b f(x)\ dx$. Let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of $[a,b]$ such that $f$ is p.c with respect to $\mathbf{P}$ and $g$ is p.c with respect to $\mathbf{P}'$. Then $f+g$ is p.c with respect to the common refinement $\mathbf{P}\#\mathbf{P}' := \{K \cap J : K \in \mathbf{P} \text{ and } J \in \mathbf{P}'\}$, with $f + g(x) = c_k + c_j$ for $x \in K \cap J$, where $c_k$ is the constant value of $f$ on $K$ and $c_j$ the constant value of $f$ on $J$. It follows that p.c. $\int_a^b f(x) + g(x)\ dx =$ p.c. $\int_a^b f(x)\ dx +$ p.c. $\int_a^b g(x)\ dx$.

(2). This follows from the definition of p.c integral.

(3). As $E \subset [a,b]$ is elementary, it can be expressed as a finite union of disjoint subintervals of $[a,b]$. That is, $E = \bigcup_{i=1}^n I_i$ for some positive integer $n$. But the elementary set $[a,b] \backslash \bigcup_{i=1}^n I_i$ can also be expressed as a finite union of $m$ disjoint intervals $\bigcup_{j=1}^m I'_j$ for some positive integer $m$, rearranging indices, we obtain $[a,b] = \bigcup_{i=1}^{m+n} J_i$ as a finite union of disjoint intervals, where each $J_i$ is either $I_k$ for some $1 \le k \le n$ or $I'_j$ for some $1 \le j \le m$. We conclude that $1_E$ is a p.c function with respect to the partition $\{J_i : 1 \le i \le m+n\}$, and by finite additivity of length, p.c. $\int_a^b 1_E(x)\ dx = \sum_{i=1}^n |I_i| = m(E)$. □

Exercise 22

*Proof.* Assume that $f : [a,b] \to \mathbf{R}$ is Riemann integrable, with Riemann integral $\int_a^b f(x)\ dx$. For any tagged partition $\mathcal{P}$, we pick $x_i^*$ for each $i = 1, \dots, n$ in the tagged partition to be such that $f(x_i^*) = \sup_{x \in [x_{i-1}, x_i]} f(x)$, then the Riemann sum is the upper Riemann sum $U(f, \mathcal{P})$. Likewise, by picking $x_i^*$ for each $i = 1, \dots, n$ to be such that $f(x_i^*) = \inf_{x \in [x_{i-1}, x_i]} f(x)$, one gets the lower Riemann sum $L(f, \mathcal{P})$. In particular,

$$L(f,\mathcal{P}) \le \underline{\int_a^b} f(x)\ dx \le \overline{\int_a^b} f(x)\ dx \le U(f,\mathcal{P}).$$

As the norm $\Delta(\mathcal{P}) \to 0$, or equivalently as $n \to \infty$, the right-most term and the left-most term in the above inequality both tend to $\int_a^b f(x)\ dx$, forcing the two middle terms to agree on the same value.

Now suppose $f : [a,b] \to \mathbf{R}$ is Darboux integrable. Define

$$\int_a^b f(x)\ dx := \underline{\int_a^b} f(x)\ dx = \overline{\int_a^b} f(x)\ dx.$$

By Proposition 11.3.12 of "Analysis I",

$$\underline{\int_a^b} f(x)\ dx = \sup\{L(f,\mathcal{P}) : \mathcal{P} \text{ a partition of } [a,b]\},$$

$$\overline{\int_a^b} f(x)\ dx = \inf\{U(f,\mathcal{P}) : \mathcal{P} \text{ a partition of } [a,b]\}.$$

By construction, for any tagged partition $\mathcal{P}$ of $[a,b]$, it holds that

$$L(f,\mathcal{P}) \leq \int_a^b f(x)\ dx,\ \mathcal{R}(f,\mathcal{P}) \leq U(f,\mathcal{P}).$$

Now, note that if $\mathcal{P}'$ and $\mathcal{P}$ are two partitions of $[a,b]$ with $\Delta(\mathcal{P}') \leq \Delta(\mathcal{P})$, then $\mathcal{P}'$ is finer than $\mathcal{P}$, and in particular it holds that $U(f,\mathcal{P}') \leq U(f,\mathcal{P}),\ L(f,\mathcal{P}') \geq L(f,\mathcal{P})$. Thus it follows that $\lim_{\Delta(\mathcal{P})\to 0} \mathcal{R}(f,\mathcal{P}) = \int_a^b f(x)\ dx$. □

Exercise 23

*Proof.* Also see Exercise 11.5.1 of "Analysis I". Let $f : [a,b] \to \mathbf{R}$ be a function bounded and piecewise continuous with respect to some partition $\mathcal{P} = \{I_i : 1 \leq i \leq n\}$ of $[a,b]$. By Exercise 22 and linearity of p.c integral, it suffices to show that $f|_{I_i} : I_i \to \mathbf{R}$ is Riemann integrable for any interval $I_i \subset [a,b]$ in the partition, but this follows from Proposition 11.5.3 of "Analysis I". □

Exercise 24

*Proof.* (1). $\forall \varepsilon > 0$, we can find p.c functions $u, v : [a,b] \to \mathbf{R}$ which majorize $f$ and $g$ resp, and p.c functions $r, s : [a,b] \to \mathbf{R}$ which minorize $f$ and $g$ resp, such that

$$\text{p.c.} \int_a^b u(x)\ dx \leq \int_a^b f(x)\ dx + \varepsilon,\ \text{p.c} \int_a^b v(x)\ dx \leq \int_a^b g(x)\ dx + \varepsilon,$$

$$\text{p.c.} \int_a^b r(x)\ dx \geq \int_a^b f(x)\ dx - \varepsilon,\ \text{p.c} \int_a^b s(x)\ dx \geq \int_a^b g(x)\ dx - \varepsilon.$$

Clearly $cu$ majorizes $f$, and $u+v$ majorizes $f+g$. Similar for $cr$ and $r+s$. By Exercise 21, it follows that

$$\overline{\int_a^b} cf(x)\ dx - \underline{\int_a^b} cf(x)\ dx \leq 2c\varepsilon,\ \overline{\int_a^b} (f+g)(x)\ dx - \underline{\int_a^b} (f+g)(x)\ dx \leq 4\varepsilon.$$

By Exercise 22, this implies that $cf$ and $f+g$ are Riemann integrable, with

$$\int_a^b cf(x)\ dx = c \cdot \int_a^b f(x)\ dx, \quad \int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$$

(2). Since $f \leq g$ on $[a,b]$, any p.c function $h : [a,b] \to \mathbf{R}$ which minorizes $f$ also minorizes $g$, so p.c$\displaystyle\int_a^b h(x)\ dx \leq \underline{\int_a^b} g(x)\ dx$. Taking the supremum over all such $h$ on the LHS, and using Exercise 22, $\displaystyle\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$.

(3). By the characterization of Jordan measurability, there exist elementary sets $A$ and $B$ with $A \subset E \subset B$ such that $m(A) \geq m(E) - \varepsilon$ and $m(B) \leq m(E) + \varepsilon$. Note that $1_A$ minorizes $1_E$ and $1_B$ majorizes $1_E$, combined with Exercise 21, we obtain

$$\underline{\int_a^b} 1_E(x)\ dx \geq \int_a^b 1_A(x)\ dx = m(A) \geq m(E) - \varepsilon,$$

$$\overline{\int_a^b} 1_E(x)\ dx \leq \int_a^b 1_B(x)\ dx = m(B) \leq m(E) + \varepsilon.$$

Or equivalently, $m(E) - \varepsilon \leq \underline{\displaystyle\int_a^b} 1_E(x)\ dx \leq \overline{\displaystyle\int_a^b} 1_E(x)\ dx \leq m(E) + \varepsilon$. Sending $\varepsilon \to 0$ finished the argument.

Finally, let $G$ be a function from the space of Riemann integrable functions on $[a,b]$ to $\mathbf{R}$ which obeys all three of the above properties. We first show that $G(f) = \int_a^b f(x)\ dx$ for piecewise constant $f = \sum_{i=1}^n c_i 1_{I_i}$.

By linearity, $G(f) = \sum_{i=1}^n c_i G(1_{I_i})$ and $\int_a^b f(x)\ dx = \sum_{i=1}^n c_i \int_a^b 1_{I_i}(x)\ dx$, both equal $\sum_{i=1}^n c_i m(E_i)$ by the indicator property. Hence $G$ agrees with the Riemann integral on p.c functions on $[a,b]$.

For an arbitrary Riemann integrable $f : [a,b] \to \mathbf{R}$, we can find p.c functions $h, g : [a,b] \to \mathbf{R}$ with $h \leq f \leq g$ and

$$\int_a^b f(x)\ dx - \varepsilon \leq \int_a^b h(x)\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx \leq \int_a^b f(x)\ dx + \varepsilon,$$

therefore by what we just showed it holds that

$$\int_a^b f(x)\ dx - \varepsilon \le G(h) \le \int_a^b f(x)\ dx \le G(g) \le \int_a^b f(x)\ dx + \varepsilon.$$

Let $\varepsilon \to 0$, we established that $G(f) = \int_a^b f(x)\ dx$ for any Riemann integrable $f : a, b \to \mathbf{R}$. $\square$

Exercise 25

*Proof.* First suppose that $f : [a, b] \to \mathbf{R}$ is Riemann integrable and non-negative. Immediately one gets $E_- = \emptyset$, which is trivially measurable. As before, we find p.c functions $h, g : [a, b] \to \mathbf{R}$ with $0 \le h \le f \le g$ (If $h(x) < 0$ at some point $x \in [a, b]$, we simply replace it with $\min(-h(x), f(x))$) such that

$$\int_a^b f(x)\ dx - \varepsilon \le \int_a^b h(x)\ dx \le \int_a^b g(x)\ dx \le \int_a^b f(x)\ dx + \varepsilon.$$

Write $h := \sum_{i=1}^n c_i 1_{I_i}$ and $g := \sum_{j=1}^m d_j 1_{J_j}$, where $c_i, d_j \ge 0$. From Exercise 21, one gets $\displaystyle\int_a^b h(x)\ dx = m(A)$, where $A := \bigcup_{i=1}^n I_i \times [0, c_i] \subset E_+$, and likewise one gets $\displaystyle\int_a^b g(x)\ dx = m(B)$, where $B := \bigcup_{j=1}^m J_j \times [0, d_j] \supset E_+$, as $m(B) - m(A) \le 2\varepsilon$, we conclude by Exercise 5 that $E_+$ is Jordan measurable, and that $\displaystyle\int_a^b f(x)\ dx = m^2(E_+) - m^2(E_-) = m^2(E_+)$.

Now we remove the condition that $f \ge 0$, assuming only Riemann integrability. Note that by the reflection identity of Darboux upper and lower integral, if $r : [a, b] \to \mathbf{R}$ is Riemann integrable then so is $-r$, with $\int_a^b -r(x)\ dx = -\int_a^b r(x)\ dx$. And if $h$ and $g$ are p.c functions with $h \le f \le g$, then $h1_{f<0}$ and $g1_{f<0}$ are p.c functions with $h1_{f<0} \le f1_{f<0} \le g1_{f<0}$ (Same with $h1_{f\ge0}$ and $g1_{f\ge0}$). By linearity of p.c integral, we see that $f1_{f<0}$ (and $f1_{f\ge0}$) are both Riemann integrable. By linearity of Riemann integrable, we have

$$\int_a^b f = \int_a^b (f1_{f\ge0} + f1_{f<0}) = \int_a^b f1_{f\ge0} + \int_a^b f1_{f<0} = m^2(E_+) - m^2(E_-),$$

where the last step follows from the non-negative case.

Conversely, suppose that the sets $E_+$ and $E_-$ are Jordan measurable for a bounded function $f : [a, b] \to \mathbf{R}$. Find elementary sets $A, B, C, D$ with $A \subset E_+ \subset B$ and $C \subset E_- \subset D$, such that $m^2(B \backslash A),\ m^2(D \backslash C) \le \varepsilon$. We can further assume that $B \cap C = \emptyset$ by setting $B := B \cap E_+$. Expressing $B \cup C$ as a finite union of disjoint boxes induces a p.c function $g : [a, b] \to \mathbf{R}$ which majorize $f$ with p.c$\int_a^b g(x)\ dx = m^2(B \cup C)$. Likewise, setting $D := D \cap E_-$ and expressing $A \cup D$ as a finite union of disjoint boxes induces a p.c function $h : [a, b] \to \mathbf{R}$ which minorizes $f$ with p.c$\int_a^b h(x)\ dx = m^2(A \cup D)$. Consequently,

$$\int_a^b g - \int_a^b h = m^2(A) + m^2(D) - m^2(B) - m^2(C) \leq 2\varepsilon,$$

and $f$ is Riemann integrable. Finally, by construction, we have

$$m^2(E_+) - m^2(E_-) - 2\varepsilon \leq \int_a^b h \leq \int_a^b f \leq \int_a^b g \leq m^2(E_+) - m^2(E_-) + 2\varepsilon,$$

and thus $\displaystyle\int_a^b f(x)\ dx = m^2(E_+) - m^2(E_-)$. □

Exercise 26

**(Riemann integrability)**: Let $B \subset \mathbf{R}^d$ be a non-degenerate box, and $f : B \to \mathbf{R}$ be a function. A tagged partition $\mathcal{P} = \{(B_i)_{1 \leq i \leq n},\ (x_i^*)_{1 \leq i \leq n}\}$ of $B$ is a partition $B = \biguplus_{i=1}^n B_i$ of $B$ into finitely many boxes, together with additional points $x_i^* \in B_i$ for each $i = 1, \ldots, n$. The quantity $\Delta(\mathcal{P}) := \sup_{1 \leq i \leq n} m(B_i)$ will be called the norm of the tagged partition. The Riemann sum $\mathcal{R}(f, \mathcal{P})$ of $f$ with respect to the tagged partition $\mathcal{P}$ is defined as

$$\mathcal{R}(f, \mathcal{P}) := \sum_{i=1}^n f(x_i^*) m(B_i).$$

We say that $f$ is Riemann integrable on $B$ if there exists a real number, denoted $\int_B f(x)\ dx$ and referred to as the Riemann integral of $f$ on $B$, for which we have

$$\int_B f(x)\ dx = \lim_{\Delta(\mathcal{P}) \to 0} \mathcal{R}(f, \mathcal{P})$$

by which we mean that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\mathcal{R}(f, \mathcal{P}) - \int_B f(x)\ dx| \leq \epsilon$$

for every tagged partition $\mathcal{P}$ with$\Delta(\mathcal{P}) \leq \delta$. If $B$ is a degenerate box with at least one side of length zero, we adopt the convention that every function $f : B \to \mathbf{R}$ is Riemann integrable, with a Riemann integral of zero.

**(Piecewise constant functions)**: Let $B \subset \mathbf{R}^d$ be a box. A piecewise constant function $f : B \to \mathbf{R}$ is a function for which there exists a partition of $B$ into finitely many boxes $B_1, \ldots, B_n$, such that $f$ is equal to a constant $c_i$ on each of the boxes $B_i$. Denote the quantity $\sum_{i=1}^{n} c_i |B_i|$ by p.c. $\int_B f(x)\, dx$, and refer to it as the piecewise constant integral of $f$ on $B$.

**(Darboux integral)**: Let $B \subset \mathbf{R}^d$ be a box, and $f : B \to \mathbf{R}$ be a bounded function. The lower Darboux integral $\underline{\int_B} f(x)\, dx$ of $f$ on $B$ is defined as

$$\underline{\int_B} f(x)\, dx := \sup_{g \leq f, \text{ piecewise constant}} \text{p.c.} \int_B g(x)\, dx,$$

where $g$ ranges over all piecewise constant functions that are pointwise bounded above by $f$. Similarly, we define the upper Darboux integral $\overline{\int_B} f(x)\, dx$ of $f$ on $B$ by the formula

$$\overline{\int_B} f(x)\, dx := \inf_{h \geq f, \text{ piecewise constant}} \text{p.c.} \int_B h(x)\, dx.$$

Clearly $\underline{\int_B} f(x)\, dx \leq \overline{\int_B} f(x)\, dx$. If these two quantities are equal, we say that $f$ is Darboux integrable, and refer to this quantity as the Darboux integral of $f$ on $B$.