Math 245A Note 1

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1 Selected Exercises in the Note 1

Exercise 1

For any positive integer i, let $I_i := (i-1, i)$. The countable union $\bigcup_{i=1}^{\infty} I_i = (0, \infty)$ is unbounded, and hence not Jordan measurable. As $\mathbf{Q}$ is countable, we order the elements of $\mathbf{Q}^2$ as $\mathbf{Q}^2 = \{q_1, q_2, \dots\}$, the countable intersection $[0,1]^2 \backslash \mathbf{Q}^2 = \bigcap_{i=1}^{\infty} [0,1]^2 \backslash \{q_i\}$ is the bullet-riddled square, which we have shown to be non-Jordan measurable.

Exercise 2

Let $A := \mathbf{Q} \cap [0,1] = \{q_1, q_2, \ldots\}$, and define $f_n(x) := 1_{\{q_n\}}(x),\ f(x) := 1_A(x)$. Clearly, the f_n are bounded uniformly by 1, and $\lim_{n\to\infty} f_n = f$ pointwise. To show that f is not Riemann integrable, refer to Proposition 11.7.1 of "Analysis I", which demonstrates that the upper and lower Darboux integral of f disagree. Alternatively, let $E_+ := \{(x,t) : x \in [a,b]; 0 \le t \le f(x)\}$, then

$$m^{*,(J)}(\partial E_+) \ge m^{*,(J)}(\mathbf{Q} \cap [0,1]) = m^{*,(J)}(\overline{\mathbf{Q} \cap [0,1]}) = m^{*,(J)}([0,1]) = 1.$$

Hence E_+ is not Jordan measurable by Exercise 18 of note 1, and consequently f is not Riemann integrable by Exercise 25 of note 1.

If $f_n \to f$ uniformly, then $\forall \varepsilon > 0,\ \exists N > 0$ such that $|f_n(x) - f(x)| < \varepsilon$ for all x and $n \ge N$. In particular, we have

$$\underline{\int_{[0,1]}} f_n - \varepsilon < \underline{\int_{[0,1]}} f \le \overline{\int_{[0,1]}} f < \overline{\int_{[0,1]}} f_n + \varepsilon$$

for $n \ge N$, from which we see that f must be Riemann integrable.

Exercise 3

Proof. Let $B := \{\alpha \in A : x_\alpha \neq 0\}$. We can partition B as $C \cup (\bigcup_{n=1}^\infty B_n)$, where $C := \{\alpha \in A : x_\alpha \geq 1\}$ and $B_n := \{\alpha \in A : \frac{1}{n+1} \leq x_\alpha < \frac{1}{n}\}$. If B is uncountable, then either C is uncountable or at least one of the B_n is uncountable. In either case, one can find some $\varepsilon > 0$ such that $x_\alpha \geq \varepsilon$ for infinitely many α. In particular, this would suggest that $\sum_{\alpha \in A} x_\alpha = +\infty$, a contradiction. Hence $x_\alpha = 0$ for all but at most countably many $\alpha \in A$. □

Exercise 4

Proof. That $m^*(\emptyset) = 0$ is trivial since no box is needed to cover the empty set. If $E \subset F \subset \mathbf{R}^d$, then any countable union of boxes which cover F also cover E. Since infimum over larger set is smaller, $m^*(E) \leq m^*(F)$.

For countable subadditivity, we use the standard "give yourself an epsilon of room" trick. let $\varepsilon > 0$. Apply the axiom of choice to find for each n a countable covering $\bigcup_{m=1}^\infty B_{n,m}$ of E_n by boxes such that $\sum_{m=1}^\infty |B_{m,n}| \leq m^*(E_n) + \varepsilon/2^n$. Then $(B_{n,m})_{(m,n)\in\mathbf{N}^2}$ is a countable covering of $\bigcup_{n=1}^\infty E_n$ by boxes, and by Tonelli's theorem for series,

$$\sum_{(n,m)\in\mathbf{N}^2} |B_{n,m}| = \sum_{n=1}^\infty \sum_{m=1}^\infty |B_{n,m}| \leq \sum_{n=1}^\infty m^*(E_n) + \varepsilon/2^n = \sum_{n=1}^\infty m^*(E_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\sum_{(n,m)\in\mathbf{N}^2} |B_{n,m}| \leq \sum_{n=1}^\infty m^*(E_n)$, and thus $m^*(\bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty m^*(E_n)$. □

Exercise 5

Proof. Suppose that $\operatorname{dist}(E, F) = 0$ under the given condition. Let E be compact. For any $\varepsilon > 0$ and every $n \geq 1$, we can find $x_n \in E$ and $y_n \in F$ such that $|x_n - y_n| < \varepsilon/n$. By compactness, there is a subsequence x_{n_k} of x_n that converges to some $x \in E$. In particular, $\operatorname{dist}(x, F) = 0$, so $x \in F$, a contradiction.

The claim fails if we drop the compactness hypothesis. For instance, let $E := \{n + 1/n : n \in \mathbf{Z}^+\}$ and $F := \{m + 1/2m : m \in \mathbf{Z}^+\}$. Both E and F are closed, and $n + 1/n = m + 1/2m$ iff $n - m = 1/2m - 1/n \in \mathbf{Z}$, which can not be true, so E and F are disjoint. By definition,

$$\operatorname{dist}(E, F) \leq |(n + \frac{1}{n}) - (n + \frac{1}{2n})| = |\frac{1}{n} - \frac{1}{2n}| = |\frac{1}{2n}|$$

which can be made arbitrarily small by setting n large, so $\operatorname{dist}(E, F) = 0$. □

Exercise 6

Proof. Let $N > 0$, we first show that the elementary measure of N almost disjoint boxes is the sum of their volumes. By Lemma 6, we have

$$m^*(\bigcup_{n=1}^{N} B_n) = m(\bigcup_{n=1}^{N} B_n). \tag{1}$$

Since we can always pad out a finite union of boxes into an infinite union by adding an infinite number of empty boxes, we also have by lemma 7

$$m^*(\bigcup_{n=1}^{N} B_n) = \sum_{n=1}^{N} |B_n|. \tag{2}$$

Combining (1) and (2) gives

$$m(\bigcup_{n=1}^{N} B_n) = \sum_{n=1}^{N} |B_n|.$$

By definition of lower Jordan measure $\sum_{n=1}^{N} |B_n| = m(\bigcup_{n=1}^{N} B_n) \leq m_{*,(J)}(E)$, sending $N \to \infty$ shows $m^*(E) \leq m_{*,(J)}(E)$.

Conversely, $\forall A \subset E,\ A$ elementary, we have by Lemma 6 and monotonicity:

$$m(A) = m^*(A) \leq m^*(E).$$

Taking supremum over all A gives $m_{*,(J)}(E) \leq m^*(E)$. And therefore $m^*(E) = m_{*,(J)}(E)$. □

Exercise 7

Consider $E := [0,1] \backslash \mathbf{Q}$. As no open set is contained in E ($\text{int}(E) = \emptyset$), we have

$$\sup_{U \subset E, U \text{ open}} m^*(U) = 0.$$

On the other hand, for any open U with $E \subset U$, it must be that $\overline{E} = [0,1] \subset U$, taking infimum over U shows that $m^*(E) \geq 1$. Clearly $m^*(E) \leq m^*((0,1)) = 1$ by monotonicity, so $m^*(E) = 1$.

Exercise 8

Proof. (1) and (2) are equivalent by definition, clearly they both imply (3). Assume (3), note that $E = (E \cap U) \cup (E \backslash U)$. By monotonicity,

$$m^*(U \backslash (E \cap U)) = m^*(U \backslash E) \leq m^*(U \Delta E) \leq \varepsilon.$$

Similarly $m^*(E \backslash U) \leq m^*(U \Delta E) \leq \varepsilon$. By outer regularity, we can find an open set U' such that $(E \backslash U) \subset U'$ and $m^*(U' \backslash (E \backslash U)) \leq \varepsilon$. Then E is contained in the open set $V = U \cup U'$, by monotonicity again,

$$m^*(V \backslash E) \leq m^*(U \backslash E) + m^*(U' \backslash E) \leq 2\varepsilon.$$

Hence (3) implies (1) and (2).

Now suppose (4). Let $U := \mathbf{R}^d \backslash F$, then U is open and $\overline{E} \subset U$. From $U \backslash \overline{E} = E \backslash F$, we obtain $m^*(U \backslash \overline{E}) \leq \varepsilon$, so $\overline{E}$ is Lebesgue measurable, and E is thus Lebesgue measurable by Lemma 10. Conversely, if E is Lebesgue measurable, then $\overline{E}$ is also Lebesgue measurable by Lemma 10. Hence there is an open set V such that $\overline{E} \subset V$ and $m^*(V \backslash \overline{E}) \leq \varepsilon$. Consequently, the set $F := \overline{V}$ is closed such that $F \subset E$. From $E \backslash \overline{V} = V \backslash \overline{E}$, we obtain $m^*(E \backslash F) \leq \varepsilon$. Thus (1) through (4) are all equivalent.

Clearly (4) implies (5). Suppose (5), and again note that $E = (E \cap F) \cup (E \backslash F)$. By (4) (and Lemma 10), we find a closed set F' contained in F such that $m^*(F \backslash F'') \leq \varepsilon$. As $(E \cap F') \subset (E \cap F)$, we have $m^*(E \cap F') \leq m^*(E \cap F) \leq \varepsilon$. Likewise, we find a closed set F'' contained in $\overline{F}$ such that $m^*(\overline{F} \backslash F'') \leq \varepsilon$. Then $(E \cap F'') \subset (E \cap \overline{F})$ and $m^*(E \cap F'') \leq m^*(E \cap \overline{F}) \leq \varepsilon$. The set $G := F' \cup F''$ is closed with $G \subset E$, and $m^*(E \backslash G) \leq 2\varepsilon$ by monotonicity. i.e. (5) implies (4).

Finally, assume (6). Let $E_{\varepsilon/4}$ be the Lebesgue measurable set which differs from E by a set of outer measure at most $\varepsilon/4$. By outer regularity, there is an open set U that contains $E \backslash E_{\varepsilon/4}$ such that $m^*(U) \leq m^*(E \backslash E_{\varepsilon/4}) + \varepsilon/4 \leq \varepsilon/2$. Let $E'_\varepsilon := U \cup E_{\varepsilon/4}$, then $E \subset E'_\varepsilon$ and

$$m^*(E \Delta E'_\varepsilon) = m^*((U \cup E_{\varepsilon/4}) \backslash E) \leq m^*(U \backslash E) + m^*(E_{\varepsilon/4} \backslash E) \leq \varepsilon/2 + \varepsilon/4 \leq \varepsilon.$$

Let $A := \bigcap_{n=1}^\infty E'_{\varepsilon/n}$, then A is Lebesgue measurable by Lemma 10. In particular, $E \subset A$, and

$$m^*(E \Delta A) = m^*(A \backslash E) = 0.$$

By Lemma 10, the set $A \backslash E$ is Lebesgue measurable, therefore $E = A - (A \backslash E)$ is also Lebesgue measurable. That (1) implies (6) is trivial. □

Exercise 9

Proof. Let E be Jordan measurable. Using the bound

$$m_{*,(J)}(E) \le m^*(E) \le m^{*,(J)}(E),$$

and the definition of Jordan measure, we can find a finite union of boxes $A = \bigcup_{n=1}^N B_n$ such that $A \subset E$ and $m^*(E) - \varepsilon \le m(A)$, as well as a finite union of boxes $B = \bigcup_{n=1}^M B'_n$ such that $E \subset B$ and $m(B) \le m^*(E) + \varepsilon$. By Lemma 6, monotonicity of outer measure, and finite additivity of elementary measure, it follows that

$$m^*(B \backslash E) \le m^*(B \backslash A) = m(B) - m(A) \le 2\varepsilon.$$

By Exercise 8, E is Lebesgue measurable. □

Exercise 10

Proof. I_n is closed for each n, so $C := \bigcap_{n=1}^\infty I_n$ is closed. Clearly, $C \subset [0,1]$ is bounded, and thus compact by the Heine-Borel theorem.

Let $A = \{\sum_{i=1}^\infty a_i/3^i : a_i \in \{0,2\}\}$. Since the sequences of 0 and 2 are uncountable (by diagonalizing), A is uncountable. we show that $A \subset C$, and C must therefore be uncountable. Let $a \in A,\ a = \sum_{i=1}^\infty a_i/3^i$. Observe that

$$a = \sum_{i=1}^n \frac{a_i}{3^i} + \sum_{i=n+1}^\infty \frac{a_i}{3^i} \le \sum_{i=1}^n \frac{a_i}{3^i} + \frac{2}{3^n} \sum_{i=1}^\infty \frac{1}{3^i} = \sum_{i=1}^n \frac{a_i}{3^i} + \frac{1}{3^n}.$$

Hence $a \in I_n$ for every n, and $a \in C$ by construction. It follows that $A \subset C$ and C is uncountable.

Each I_n is the disjoint union of 2^n intervals each of length $1/3^n$, so $m^*(I_n) = m(I_n) = (2/3)^n$ by finite additivity of Jordan (or elementary) measure. By monotonicity of Lebesgue outer measure,

$$m^*(C) \le m^*(I_n) = m(I_n) \le (2/3)^n$$

for each $n \ge 1$. Sending $n \to \infty$ shows that C is a null set. □

Exercise 11

Proof. (1). By condition, we can express $\bigcup_{n=1}^{\infty} E_n$ as the countable disjoint union of measurable sets $\bigcup_{n=1}^{\infty}(E_n \backslash \bigcup_{n'=1}^{n-1} E_{n'})$. By Lemma 11,

$$\begin{aligned} m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty}(E_n \backslash \bigcup_{n'=1}^{n-1} E_{n'})) = \sum_{n=1}^{\infty} m(E_n) - m(\bigcup_{n'=1}^{n-1} E_{n'}) \\ = \sum_{n=1}^{\infty} m(E_n) - m(E_{n-1}) = \lim_{N\to\infty} \sum_{n=1}^{N} m(E_n) - m(E_{n-1}) \\ = \lim_{N\to\infty} m(E_N). \end{aligned}$$

(2). By monotone convergence of sequences and monotonicity,

$$m(\bigcap_{n=1}^{\infty} E_n) \le \inf_{n\ge 1} m(E_n) = \lim_{n\to\infty} m(E_n).$$

It left to show that $\lim_{n\to\infty} m(E_n) \le m(\bigcap_{n=1}^{\infty} E_n)$. Since $m(E_n) < \infty$ for some n, either $\lim_{n\to\infty} m(E_n) = 0$ or $\lim_{n\to\infty} m(E_n) = m(E_N)$ for some N. In the former case the above inequality trivially holds, so we assume the latter case. By finite additivity, it must be that $m(E_m \backslash E_{m+1}) = 0$ for all $m \ge N$. By Lemma 11, we obtain

$$\begin{aligned} m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=N}^{\infty} E_n) = m(E_N \backslash \bigcup_{n=N}^{\infty}(E_n \backslash E_{n+1})) \\ = m(E_N) - m(\bigcup_{n=N}^{\infty} E_n \backslash E_{n+1}) \\ = m(E_N) - \sum_{n=N}^{\infty} m(E_n \backslash E_{n+1}) \\ = m(E_N), \end{aligned}$$

as desired. A counterexample showing that the hypothesis that at least one of the $m(E_n)$ is finite cannot be dropped is $\mathbf{R}\backslash\{0\} \supset \mathbf{R}\backslash\{0,1\} \supset \ldots$. □

Exercise 12

Proof. Since Lebesgue measure of a set is defined to equal its Lebesgue outer measure, monotonicity for Lebesgue measurable sets follows from Exercise 4. Similarly, as countable union of Lebesgue measurable sets is measurable by Lemma 10, we get countable subadditivity for Lebesgue measurable sets from its' counterpart in Exercise 4 as well. □

Exercise 13

Proof. (1). By definition, we have the identity $1_E(x) = \liminf_{n\to\infty} 1_{E_n}(x)$. In particular, Let $a_N^-(x) := \inf_{n\geq N} 1_{E_n}(x)$, then

$$1_E(x) = \sup_{N\geq 1} a_N^-(x) = \sup_{N\geq 1} 1_{\cap_{n=N}^{\infty} E_n}(x) = 1_{\cup_{N\geq 1}\cap_{n\geq N} E_n}(x).$$

In particular, we see that $E = \bigcup_{N\geq 1}\bigcap_{n\geq N} E_n$, which is Lebesgue measurable by Lemma 10.

(2). As $\bigcap_{n\geq 1} E_n \subset \bigcap_{n\geq 2} E_n \subset \ldots$, we have by upward monotone convergence theorem that

$$m(E) = m(\bigcup_{N\geq 1}\bigcap_{n\geq N} E_n) = \lim_{N\to\infty} m(\bigcap_{n\geq N} E_n).$$

Using the identity $1_E(x) = \limsup_{n\to\infty} 1_{E_n}(x)$ instead, one can similarly derive that $E = \bigcap_{N\geq 1}\bigcup_{n\geq N} E_n$. As $\bigcup_{n\geq 1} E_n \supset \bigcup_{n\geq 2} E_n \supset \ldots$, and $m(\bigcup_n E_n) \leq m(F) < \infty$, we have by downward monotone convergence theorem that

$$m(E) = m(\bigcap_{N\geq 1}\bigcup_{n\geq N} E_n) = \lim_{N\to\infty} m(\bigcup_{n\geq N} E_n).$$

By monotonicity, we see that

$$\lim_{N\to\infty} m(\bigcap_{n\geq N} E_n) \leq \lim_{N\to\infty} m(E_N), \quad \lim_{N\to\infty} m(\bigcup_{n\geq N} E_n) \geq \lim_{N\to\infty} m(E_N),$$

so $\lim_{N\to\infty} m(E_N) = m(E)$.

(3). For a counterexample, let $E_n := [n, n+1],\ E := \emptyset$. By definition $E_n \to E$ pointwise, and $m(E_n) \leq 1$ for all n, but $\lim_n m(E_n) = 1 \neq 0 = m(\emptyset)$. □

Exercise 14

Proof. If $m^*(E) = \infty$, we can take the measurable set to be $\mathbf{R}^d$. Hence we assume that $m^*(E) < \infty$. By outer regularity, for each $n \geq 1$, we can find an open set $V_n \subset \mathbf{R}^d$ such that $E \subset V_n$ and $m(V_n) \leq m^*(E) + \varepsilon/n$. Take $V := \bigcap_{n=1}^{\infty} V_n$, then $V \supset E$ is Lebesgue measurable by Lemma 10. Without loss of generality, we can assume that $V_1 \supset V_2 \supset \ldots$ (taking intersection if necessary), by downward monotone convergence,

$$m(V) = \lim_{n\to\infty} m(V_n) \leq m^*(E) + \lim_{n\to\infty} \varepsilon/n = m^*(E).$$

On the other hand, $m^*(E) \leq m(V)$ by monotonicity, so $m(V) = m^*(E)$. □

Exercise 15

Proof. First suppose that $E \subset \mathbf{R}^d$ is bounded. By inner approximation by closed and finite additivity, for any $\varepsilon > 0$, there exists a compact $K \subset E$ such that $m(E) - \varepsilon \leq m(K) \leq m(E)$. Taking supremum over K shows that

$$m(E) = \sup_{K \subset E, K \text{ compact}} m(K).$$

If E is unbounded, we express E as the countable union of bounded sets (by intersecting E with, say, the closed balls $\overline{B(0,n)}$ of radius n for $n = 1, 2, 3, \ldots$). Let $F_n := E \cap \overline{B(0,n)}$, then $F_1 \subset F_2 \subset \ldots$, and from upward monotone convergence,

$$m(E) = \lim_{n \to \infty} m(F_n).$$

By the bounded case, we can find compact sets $K_n \subset F_n$ such that $m(F_n) - \varepsilon \leq m(K_n) \leq m(F_n)$. And similarly we can take the K_n to be non-decreasing. Taking the limit as $n \to \infty$, we obtain the unbounded case. $\square$

Exercise 16

Proof. Assume (1). By Exercise 8, for every $\epsilon > 0$, one can contain E in an open set U such that $m^*(U \backslash E) \leq \epsilon$. By finite additivity,

$$m^*(U) = m(U) \leq m(U \backslash E) + m(E) < \infty.$$

This gives (2) and (3). Given (2), from monotonicity,

$$\infty < m^*(U) \leq m^*(U \backslash E) + m^*(E) = m(U \backslash E) + m(E),$$

and we get (1). Given (3), E is measurable by Exercise 8. Again by finite additivity,

$$m^*(E \Delta U) = m(E \backslash U) + m(U \backslash E) \leq \varepsilon,$$

which implies that $m(E \backslash U) < \infty$. In particular,

$$m(E) = m(E \backslash U) + m(E \cap U) < \infty.$$

Hence (1) to (3) are all equivalent.

Clearly (1) implies (4) by the Heine-Borel theorem. Conversely, (4) implies (1) by finite additivity. Also, (4) trivially implies (5), hence we suppose (5). In the same way we show (3) $\to$ (1), it can be shown that (5) $\to$ (1). Again (5) trivially implies (6), and (6) $\to$ (1) the same way (5) $\to$ (1).

As bounded sets have finite measure, (6) $\to$ (7) automatically. If (7) holds, the same argument used to show that $(3), (5), (6) \to (1)$ gives (1) as well. Hence (1) through (7) are all equivalent.

Similarly, as elementary sets are measurable with finite Lebesgue measure, (8) $\to$ (7). Conversely, let E be Lebesgue measurable with finite measure, and $U \supset E$ be an open set of finite measure with $m^*(U \backslash E) \leq \varepsilon$. By Lemma 8, U can be expressed as a countable union of almost disjoint boxes $\bigcup_{n=1}^{\infty} B_n$. Furthermore, since $\sum_{n=1}^{\infty} |B_n| < \infty$, $\exists N > 0$ such that $\sum_{n=1}^{N} |B_n| \geq m(U) - \varepsilon$. Let $U_\varepsilon := \bigcup_{n=1}^{N} B_n$. From

$$m(U_\varepsilon \Delta U) \leq \varepsilon,\ m(U \Delta E) \leq \varepsilon,\ \forall \varepsilon > 0,$$

we get $m(U_\varepsilon \Delta E) \leq \varepsilon$. This gives (1) $\to$ (8).

Clearly (9) $\to$ (8). Given (8), and the elementary set U_ε of (8), we have

$$m(U_\varepsilon) = \lim_{n \to \infty} 2^{-dn} \mathcal{E}_*(U_\varepsilon, 2^{-n})$$

by Exercise 14 of note 0. Here $\mathcal{E}_*(U_\varepsilon, 2^{-n})$ denote the number of dyadic cubes of sidelength 2^{-n} that are contained in U_ε. Then for n large enough, we have $m(U_\varepsilon) - 2^{-dn} \mathcal{E}_*(U_\varepsilon, 2^{-n}) \leq \varepsilon$. Let A_ε be the union of dyadic cubes of sidelength 2^{-n} contained in U_ε, then the fact that

$$m(U_\varepsilon \Delta E) \leq \varepsilon,\ m(U_\varepsilon \Delta A_\varepsilon) \leq \varepsilon,\ \forall \varepsilon > 0$$

again shows $m(E \Delta A_\varepsilon) \leq \varepsilon$. Alternatively, let n be large enough so that 2^{-n} is smaller than any sidelength of the cube in U_ε, as any dyadic cube of sidelength 2^{-m} is a finite union of cubes of sidelength smaller than 2^{-m}, we see that U_ε is a finite union of cubes of sidelength 2^{-n}. $\square$

Exercise 17

Proof. Clearly (1) implies (2). Assume (2), and that E is bounded. By outer regularity, there exists an open set $U \supset E$ such that $m(U) \leq m^*(E) + \varepsilon$. By lemma 8, $U = \bigcup_{n=1}^{\infty} B_n$ is a countable union of disjoint boxes. We replace each box B_n with a larger open box B_n' such that $m(B_n' \backslash B_n) \leq \varepsilon / 2^n$. In particular,

$$m(\bigcup_{n=1}^{\infty} B_n') \leq \sum_{n=1}^{\infty} |B_n| + \varepsilon.$$

As $\bigcup_{n=1}^{\infty} B_n' \supset \overline{E}$ and $\overline{E}$ is compact, we get a finite subcover $A := \bigcup_{n=1}^{N} B_n' \supset E$ (after re-index) by the Heine-Borel theorem. By our hypothesis, it follows that

$$m^*(E) \leq m^*(A \cap E) + m^*(A \backslash E) \leq m^*(E) + 2\varepsilon,$$

or equivalently that

$$m^*(E) - m^*(A \cap E) \leq m^*(A \backslash E) \leq m^*(E) - m^*(A \cap E) + 2\varepsilon.$$

But $m^*(E) - m^*(A \cap E) = 0$, so $0 \leq m^*(A \backslash E) \leq 2\varepsilon$, and thus E is Lebesgue measurable by Exercise 16.

Now suppose that E is unbounded. For any $k \geq 1$, let $E_k := E \cap B(0,k)$, and $A_k := A \cap B(0,k)$. Note that

$$A \backslash E_k = A \backslash (E \cap B(0,k)) = A \backslash B(0,k) \cup (A_k \backslash E).$$

By subadditivity and hypothesis

$$\begin{aligned} m^*(A \backslash E_k) + m^*(A \cap E_k) &\leq m^*(A \backslash B(0,k)) + m^*(A_k \backslash E) + m^*(A_k \cap E) \\ &= m^*(A \backslash B(0,k)) + m(A_k) \\ &= m(A). \end{aligned}$$

By subadditivity we also have $m(A) \leq m^*(A \backslash E_k) + m^*(A \cap E_k)$. Hence $m(A) = m^*(A \backslash E_k) + m^*(A \cap E_k)$ and E_k is Lebesgue measurable by the finite case. It turns out that $E = \bigcup_{k=1}^{\infty} E_k$ is Lebesgue measurable by Lemma 10.

Finally, boxes are elementary, so (2) $\to$ (3). As every elementary set A can be written as a finite union of disjoint boxes, we have (3) $\to$ (2). □

Exercise 18

Proof. (1). We show that

$$m_*(E) = \sup_{F \subset E, F \text{ measurable}} m(F),$$

and thus showing that the inner measure is independent of the choice of A.

Let $F \subset E$ be a measurable set. Then $A \backslash E \subset A \backslash F$ (and $m^*(A \backslash E) \leq m^*(A \backslash F)$). By outer regularity, exists an open set $U \supset A \backslash E$ such that

$$m(U) \leq m(A \backslash F) = m(A) - m(F).$$

Or equivalently $m(F) \leq m(A) - m(U)$, which implies that $m(F) \leq m(A) - m^*(A \backslash E)$, and thus

$$\sup_{F \subset E, F \text{ measurable}} m(F) \leq m(A) - m^*(A \backslash E).$$

On the other hand, for any open set $U \supset A \backslash E$, the set $A \backslash U$ is a measurable set contained in A, from which we see

$$m(A) - m(U) \leq \sup_{F \subset E, F \text{ measurable}} m(F).$$

Taking infimum over U gives

$$m(A) - m^*(A \backslash E) \leq \sup_{F \subset E, F \text{ measurable}} m(F).$$

(2). From our proof of (1), it's clear that $m_*(E) \leq m^*(E)$ by monotonicity. Suppose the equality holds, and denote the common value by $m(E)$, then we can find an open set $U \supset E$, and a measurable set $F \subset E$, such that

$$m(E) - \varepsilon \leq m(F) \leq m(U) \leq m(E) + \varepsilon$$

for any $\varepsilon > 0$. In particular, by monotonicity we have

$$m^*(U \backslash E) \leq m^*(U \backslash F) \leq 2\varepsilon,$$

and hence E is Lebesgue measurable by Exercise 16. Conversely, if $E \subset \mathbf{R}^d$ is bounded and Lebesgue measurable, then by definition

$$m_*(E) = \sup_{F \subset E, F \text{ measurable}} m(F) = m(E) = m^*(E).$$

□

Exercise 19

Proof. If E is a G_δ set with a null set removed or the union of a F_σ set and a null set, then E is Lebesgue measurable by Lemma 10. Conversely, suppose that E is Lebesgue measurable. By Exercise 8, we can find for any $\varepsilon > 0$ and each $n \geq 1$ an open set $U_n \supset E$ such that $m(U_n \backslash E) \leq \varepsilon / n$. Let $G := \bigcap_{n=1}^\infty U_n$, then $G \supset E$ is a G_δ set, and $m(G \backslash E) \leq \varepsilon / n$ for all n. Hence $m(G \backslash E) = 0$. Similarly, we can find for any $\varepsilon > 0$ and each $n \geq 1$ a closed set $F_n \subset E$ such that $m(E \backslash F_n) \leq \varepsilon / n$. Let $F := \bigcup_{n=1}^\infty F_n$, then $F \subset E$ is an F_σ set, and $m(E \backslash F) \leq \varepsilon / n$ for all n. Consequently, $m(E \backslash F) = 0$. □

Exercise 20

Proof. We first show that if $x+E$ is Lebesgue measurable, then it must be that

$$m(E) = m(x+E).$$

Note that every countable covering of E by boxes B_n corresponds to a countable covering of $x+E$ by boxes $x+B_n$, and conversely every countable covering of $x+E$ by boxes B'_n corresponds to a countable covering of E by boxes $-x+B'_n$. Furthermore, the translation of a box is a box with the same measure by Exercise 6 of note 0, therefore

$$\inf_{\bigcup_{n=1}^{\infty} B'_n \supset x+E; B'_n \text{ boxes}} \sum_{n=1}^{\infty} |B^{'}_n| = \inf_{\bigcup_{n=1}^{\infty} B_n \supset E; B_n \text{ boxes}} \sum_{n=1}^{\infty} |B_n|$$

That is, $m(x+E) = m(E)$.

We now show that the translation of a null set is null. Let N be a null set, there is an open set $U \supset N$ such that $m(U) \leq \varepsilon$. Write $U = \sum_{n=1}^{\infty} B_n$ as a countable union of almost disjoint boxes, then $x+U$ is an open set containing $x+N$ and $m(x+U) = m(U) \leq \varepsilon$ by Exercise 6 of note 0 and countable additivity, which implies that $x+N$ is a null set.

Finally, by Exercise 19, $E = G \backslash N$, where G is a G_δ set and N a null set. It's straightforward to show that $x+E = x+G\backslash N = (x+G)\backslash(x+N)$ is the difference of a G_δ set and a null set, so $x+E$ is Lebesgue measurable. By the above argument, $m(x+E) = m(E)$. ☐

Exercise 21

Proof. By Exercise 8, there exists an open set U with $m(U\Delta E) \leq \varepsilon$. Let $\bigcup_{n=1}^{\infty} B_n \supset U\Delta E$ be a countable covering by boxes such that

$$\sum_{n=1}^{\infty} |B_n| \leq m(U\Delta E) + \varepsilon \leq 2\varepsilon.$$

By Exercise 11 of note 0, $T(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} T(B_n)$ is Lebesgue measurable, and by countable subadditivity,

$$m^*(T(U\Delta E)) \leq m(T(\bigcup_{n=1}^{\infty} B_n)) \leq \sum_{n=1}^{\infty} m(T(B_n)) = \sum_{n=1}^{\infty} |\det T||B_n| \leq 2|\det T|\varepsilon.$$

It follows that $m^*(T(U\Delta E)) = m^*(T(U)\Delta T(E)) \leq \varepsilon$ for any $\varepsilon > 0$. From Lemma 8, U can be expressed as a countable union of boxes $\bigcup_{n=1}^{\infty} B'_n$. Hence

$T(U) = \bigcup_{n=1}^{\infty} T(B_n')$ is Lebesgue measurable by Exercise 11 of note 0. Therefore, $T(E)$ is Lebesgue measurable by Exercise 8.

Another way to proceed is to use the following results:

Lemma: Let $T : V \to V$ be a linear operator on a d-dimensional inner product space, then T is Lipschitz continuous.

We want to show that $\|T(x-y)\| \leq \lambda \|x-y\|$ for any $x,\ y \in V$ and some $\lambda > 0$, which holds if and only if

$$\frac{\|T(x-y)\|}{\|x-y\|} = \|T(\frac{x-y}{\|x-y\|})\| \leq \lambda$$

by linearity. Let $z := \dfrac{x-y}{\|x-y\|}$, then it suffices to bound the operator norm $\sup\limits_{z:\|z\|=1} \|T(z)\|$ by λ. By Theorem 6.5 of "Linear algebra" by Friedberg, V has an orthonormal basis $\beta = \{v_1, \ldots, v_d\}$. Moreover, if $z = \sum_{i=1}^{d} a_i v_i$, then $\|z\| = 1$ implies

$$\langle \sum_{i=1}^{d} a_i v_i, \sum_{j=1}^{d} a_j v_j \rangle = \sum_{i=1}^{d} a_i \langle v_i, \sum_{j=1}^{d} a_j v_j \rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} a_i \overline{a_j} \langle v_i, v_j \rangle = \sum_{i=1}^{d} |a_i|^2 = 1$$

by properties of the inner product. In particular, $\sum_{i=1}^{d} |a_i| \leq C$ for some absolute constant C. consequently, we have

$$\|T(z)\| = \|\sum_{i=1}^{d} a_i v_i\| \leq \sum_{i=1}^{d} |a_i| \|T(v)\| \leq (\sum_{i=1}^{d} |a_i|) \max_{1 \leq i \leq d} \|T(v_i)\| \leq C \max_{1 \leq i \leq d} \|T(v_i)\|.$$

by Theorem 6.2 of "Linear Algebra". Take $\lambda := C \max\limits_{1 \leq i \leq d} \|T(v_i)\|$, we are done.

Now, since E is measurable, E can be written as the union of an F_σ set and a null set N by Exercise 19. Denote $E = \bigcup_{n=1}^{\infty} F_n \cup N$, where the F_n are closed sets. We can further express E as a union of countable compact sets and a null set by writing $F_n = \bigcup_{m=1}^{\infty} F_n \cap \overline{B(0,m)}$ for each n. Hence $E = \bigcup_{n=1}^{\infty} K_n \cup N$, where the K_n are compact sets. Thus we have

$$T(E) = T(\bigcup_{n=1}^{\infty} K_n \cup N) = T(\bigcup_{n=1}^{\infty} K_n) \cup T(N) = \bigcup_{n=1}^{\infty} T(K_n) \cup T(N).$$

Let $\bigcup_{n=1}^{\infty} B_n \supset N$ be a countable box covering of N with $\sum_{n=1}^{\infty} |B_n| \leq \varepsilon$. By Exercise 11 of note 0, $T(B_n) = |\det T||B_n|$, so

$$m^*(T(N)) \leq |\det T| \sum_{n=1}^{\infty} |B_n| \leq \varepsilon,$$

which implies that $T(N)$ is a null set. From the Lemma, T is continuous, so the $T(K_n)$ are compact by Theorem 2.3.1 of "Analysis II". It follows that $T(E)$ is the union of an F_σ set and a null set, which is Lebesgue measurable by Exercise 19.

Finally, if T is singular, then $\text{rank}(T) < d$ and $T(E)$ is a null set for every measurable E. Hence we assume that T is invertible. For each $n \geq 1$, let $U_n \supset E$ be an open set such that $m(U_n \backslash E) \leq \varepsilon/n$. By the proof of Exercise 19, $E = \bigcap_{n=1}^{\infty} U_n \backslash N$, where $N = \bigcap_{n=1}^{\infty} U_n \backslash E$ is a null set. Writing U_n as a countable union of almost disjoint boxes and using Exercise 11 of note 0, we see that $m(T(U_n)) = |\det T| m(U_n)$ for every n. By downward monotone convergence, we get

$$\begin{aligned} m(T(E)) &= m(\bigcap_{n=1}^{\infty} T(U_n) \backslash T(N)) = m(\bigcap_{n=1}^{\infty} T(U_n)) - m(T(N)) \\ &= \lim_{n \to \infty} |\det T| m(U_n) = |\det T| m(E) \end{aligned}$$

given that $m(E) < \infty$. If $m(E) = \infty$, we decompose $E = \bigcup_{n=1}^{\infty} E \cap A_n$ as a countable disjoint union of bounded Lebesgue measurable sets, with A_n taken to be the annuli $A_n := \{x \in \mathbf{R}^d : n - 1 \leq |x| < n\}$ (say). By finite subadditivity, the finite case, and the fact that T is a bijection,

$$\begin{aligned} m(T(E)) &= m(T(\bigcup_{n=1}^{\infty} E \cap A_n)) = \sum_{n=1}^{\infty} m(T(E \cap A_n)) \\ &= |\det T| \sum_{n=1}^{\infty} m(E \cap A_n) = |\det T| m(E). \end{aligned}$$

This completes the proof. $\square$

Exercise 22

Proof. (1). We first show that the statement holds when both E and F are open sets. First, observe that $E \times F \subset \mathbf{R}^{d+d'}$ is open: Let $x_0 = (a_1, \ldots, a_d) \in V$, and $y_0 = (b_1, \ldots, b_{d'}) \in F$, there exist open balls $B(x_0, r_1) \subset E$, and $B(y_0, r_2) \subset F$.

Denote $r := \min(r_1, r_2)$, and $z := (x_0, y_0)$. For any $x = (x_i)_{1\le i\le d+d'} \in B(z, r)$, we have

$$\sum_{i=1}^{d}(x_i - a_i)^2 + \sum_{i=d+1}^{d+d'}(x_i - b_i)^2 < r^2,$$

In particular, this implies $\sum_{i=1}^{d}(x_i - a_i)^2 < r_1^2$ and $\sum_{i=d+1}^{d+d'}(x_i - b_i)^2 < r_2^2$. That is, $B(z, r) \subset B(x_0, r_1) \times B(y_0, r_2) \subset E \times F$. By definition, $E \times F$ is open.

By Lemma 8, $E = \bigcup_{n=1}^{\infty} B_n$ and $F = \bigcup_{m=1}^{\infty} B'_m$ are countable union of almost disjoint boxes. By Lemma 10, countable additivity, and Tonelli's theorem for series,

$$\begin{aligned} m^{d+d'}(E \times F) &= m^{d+d'}((\bigcup_n B_n) \times (\bigcup_m B'_m)) \\ &= m^{d+d'}(\bigcup_{n,m} B_n \times B'_m) \\ &= \sum_{(n,m)\in\mathbf{N}^2} |B_n| \times |B'_m| = (\sum_{n=1}^{\infty} |B_n|)(\sum_{m=1}^{\infty} |B'_m|) \\ &= m^d(E) \cdot m^{d'}(F). \end{aligned}$$

If E and F are Lebesgue measurable, $U \supset E$ and $V \supset F$ are open, then

$$(m^{d+d'})^*(E \times F) \le (m^{d+d'})^*(U \times V) = m^d(U) \cdot m^{d'}(V).$$

Taking infimum over $U \subset \mathbf{R}^d$ and then over $V \subset \mathbf{R}^{d'}$, by outer regularity,

$$(m^{d+d'})^*(E \times F) \le (m^d)^*(E)(m^{d'})^*(F).$$

(2). Assume that $m(E) < \infty$ and $m(F) < \infty$. By outer approximation by open, exist open sets $U \supset E$ and $V \supset F$ such that $m^d(U\backslash E) \le \varepsilon$ and $m^{d'}(V\backslash F) \le \varepsilon$. By (1) and subadditivity,

$$\begin{aligned} &(m^{d+d'})^*((U \times V)\backslash(E \times F)) \\ &\le m^*(E \times (V\backslash F)) + m^*((U\backslash E) \times F) + m^*(U\backslash E \times (V\backslash F)) \\ &\le m^d(E)m^{d'}(V\backslash F) + m^d(U\backslash E)m^{d'}(F) + m^d(U\backslash E)m^{d'}(V\backslash F) \\ &\le \varepsilon(m^d(E) + m^{d'}(F) + \varepsilon) \\ &= O(\varepsilon). \end{aligned}$$

By outer approximation by open, $E \times F$ is Lebesgue measurable. Furthermore, $(U \times V)\backslash(E \times F)$ is the disjoint union

$$E \times (V\backslash F) \cup (U\backslash E) \times F \cup (U\backslash E) \times (V\backslash F).$$

By what we have shown and finite additivity,

$$\begin{aligned} m^{d+d'}(E \times F) &= m^{d+d'}(U \times V) - m^{d+d'}((U \times V)\backslash(E \times F)) \\ &\geq m^d(U) \cdot m^{d'}(V) - O(\varepsilon) \\ &\geq m^d(E) m^{d'}(F) - O(\varepsilon) \end{aligned}$$

Consequently, we have

$$m^d(E) m^{d'}(F) - O(\varepsilon) \leq m^{d+d'}(E \times F) \leq m^d(E) m^{d'}(F).$$

Sending $\varepsilon \to 0$ gives the result.

Now we remove the hypothesis that both E and F have finite measure. For each $n \geq 1$, define $E_n := E \cap B^d(0, n)$, and $F_m := F \cap B^{d'}(0, m)$ for each $m \geq 1$. Then $E \times F = \bigcup_{(n,m) \in \mathbf{N}^2} E_n \times F_m$ is Lebesgue measurable as a countable of measurable sets. By construction, for each pair of (n, m), $E_n \times F_m \subset E_k \times F_k$, where $k := \max(n, m)$. Hence we can write $E \times F = \bigcup_{k=1}^{\infty} E_k \times F_k$. By two applications of the monotone convergence theorem and the finite measure case,

$$m^{d+d'}(E \times F) = m^{d+d'}(\bigcup_{k=1}^{\infty} E_k \times F_k) = \lim_{k \to \infty} m^d(E_k) m^{d'}(F_k) = m^d(E) m^{d'}(F).$$

□

Exercise 23

Proof. Let m be a map obeying the given axioms. By Exercise 15 of note 0 (and Lemma 6), m matches elementary measure on elementary sets. By countable additivity of m, we easily get monotonicity and finite additivity as well. It follows that

$$m(E) \leq m^{*,(J)}(E) = m^*(E)$$

for any Lebesgue measurable set E, where m^* denote the Lebesgue outer measure. Let E be a bounded Lebesgue measurable set, and $A \supset E$ an elementary set. By the Carathéodory criterion, we have

$$m(A) = m^*(E) + m^*(A \backslash E).$$

In particular, this implies that

$$m^*(E) = m(A) - m^*(A \backslash E) \leq m(A) - m(A \backslash E) = m(E).$$

Hence m matches Lebesgue measure on bounded measurable sets.

If E is an unbounded Lebesgue measurable set, we decompose E as a countable disjoint union of bounded Lebesgue measurable sets, by writing E as $\bigcup_{n=1}^{\infty} E \cap A_n$, where A_n is the annuli $A_n := \{x \in \mathbf{R}^d : m-1 \leq |x| < m\}$, the general case then follows from the bounded case and countable additivity. $\square$

Exercise 24

Proof. (1). Suppose that $E, F, G \in 2^A$. Clearly $E\Delta E = \emptyset$ is a null set. If $E\Delta F$ is a null set, then by definition $F\Delta E = E\Delta F$ is also a null set. Let $E\Delta F$ and $F\Delta G$ both be null sets. By monotonicity and subadditivity of the Lebesgue outer measure,

$$m^*(E\backslash G) \leq m^*((E \cap F)\backslash G) + m^*((E\backslash F)\backslash G) \leq m^*(F\backslash G) + m^*(E\backslash F) = 0.$$

Hence $E\backslash G$ is a null set. By symmetry $G\backslash E$ is also a null set, and thus $E\Delta G$ is a null set. As the given relation is reflexive, symmetric and transitive, it is an equivalence relation.

(2). We first show that symmetric difference is associative. That is,

$$A\Delta(B\Delta C) = (A\Delta B)\Delta C.$$

By definition, we have

$$\begin{aligned} A\backslash(B\Delta C) &= A\backslash((B\backslash C) \cup (C\backslash B)) = (A\backslash(B\backslash C)) \cap (A\backslash(C\backslash B)) \\ &= ((A\backslash B) \cup (A \cap C)) \cap ((A\backslash C) \cup (A \cap B)) \\ &= (A\backslash(B \cup C)) \cup (A \cap B \cap C). \end{aligned}$$

Also, $(B\Delta C)\backslash A = ((B\backslash C) \cup (C\backslash B))\backslash A = (B\backslash(C \cup A)) \cup (C\backslash(B \cup A))$. Hence we get

$$A\Delta(B\Delta C) = (A\backslash(B \cup C)) \cup (A \cap B \cap C) \cup (B\backslash(C \cup A)) \cup (C\backslash(B \cup A)). \quad (3)$$

Similarly, it can be shown that

$$C\backslash(A\Delta B) = (C\backslash(A \cup B)) \cup (C \cap B \cap A),$$

and that $(A\Delta B)\backslash C = (A\backslash(B \cup C)) \cup (B\backslash(A \cup C))$. If follows that

$$(A\Delta B)\Delta C = (A\backslash(B \cup C)) \cup (B\backslash(A \cup C)) \cup (C\backslash(A \cup B)) \cup (C \cap B \cap A). \quad (4)$$

Comparing (3) and (4) gives $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

By associativity and commutativity of symmetric difference, we then get

$$\begin{aligned}
m^*(E\Delta E') &= m^*((E\Delta F)\Delta(F\Delta E')) \\
&= m^*((E\Delta F)\Delta((F\Delta F')\Delta(F'\Delta E'))) \\
&= m^*(((E\Delta F)\Delta(F'\Delta E'))\Delta(F\Delta F')) \\
&= m^*(F\Delta F')
\end{aligned}$$

where we use the fact that $E\Delta F$ and $F'\Delta E'$ are Lebesgue null sets. Hence the distance $d : 2^A/\sim \times 2^A/\sim \to \mathbf{R}^+$ is well-defined.

To show the induced metric space is complete, let $[E_1],\ [E_2], \dots$ be a Cauchy sequence in $2^A/\sim$. Let $([E_{n_j}])_{j=1}^{\infty}$ be a fast Cauchy subsequence such that $\sum_{j=1}^{\infty} m^*(E_{n_j}\Delta E_{n_{j+1}}) < \varepsilon$ for a given $\varepsilon > 0$. Define $E := \bigcup_{k=1}^{\infty}\bigcap_{j=k}^{\infty} E_{n_j}$. That is, $E := \liminf_j E_{n_j}$. By Lemma 1.4.9 of "Analysis II", it suffices to show that $[E_{n_j}]$ converges to $[E]$. By subadditivity,

$$m^*(E_{n_j}\Delta E) \leq m^*(E_{n_j}\backslash E) + m^*(E\backslash E_{n_j}).$$

We control the two terms separately. Let N be such that $m^*(E_m\Delta E_n) \leq \varepsilon$ for $m,\ n > N$, and choose $n_j > N$, we see that

$$m^*(E\backslash E_{n_j}) \leq \varepsilon.$$

Now we control the first term. By construction,

$$E_{n_j}\backslash E \subset E_{n_j}\backslash \bigcap_{i=j}^{\infty} E_{n_i} = \bigcup_{i=j}^{\infty} E_{n_j}\backslash E_{n_i} \subset \bigcup_{i=j}^{\infty} E_{n_j}\Delta E_{n_i}.$$

By iteration of the formula

$$A\Delta C = (A\Delta B)\Delta(B\Delta C)$$

we see that

$$\bigcup_{i=j}^{\infty} E_{n_j}\Delta E_{n_i} \subset \bigcup_{i=j}^{\infty} E_{n_i}\Delta E_{n_{i+1}}.$$

Hence by countable subadditivity,

$$m^*(E_{n_j}\backslash E) \leq \sum_{i=j}^{\infty} m^*(E_{n_i}\Delta E_{n_{i+1}}) < \varepsilon.$$

Combine these results, we see that $m^*(E_{n_j}\Delta E) \leq 2\varepsilon$ for n_j large enough, and thus $[E_{n_j}]$ converges to $[E]$.

(3). Let $[F] \in 2^A/\sim$ be an adherent point of $\mathcal{E}/\sim$. i.e, $\forall r > 0$, there is an elementary set $E \in 2^A$ such that $m^*(E\Delta F) < r$. By Exercise 16, $F \in \mathcal{L}$, and thus $[F] \in \mathcal{L}/\sim$ (since any set that differs from a Lebesgue measurable set by a null set is Lebesgue measurable). Consequently, $\overline{\mathcal{E}/\sim} \subset \mathcal{L}/\sim$.

On the other hand, if $[F] \in \mathcal{L}/\sim$, there is an elementary set E such that $m^*(E\Delta F) < r$ for any $r > 0$. As intersection of elementary sets are elementary, we can take $E \in 2^A$ by replacing E with $E \cap A$. Hence $[F]$ is in the closure of $\mathcal{E}/\sim$, and $\overline{\mathcal{E}/\sim} = \mathcal{L}/\sim$. By Proposition 1.4.12(b) of "Analysis II", $\mathcal{L}/\sim$ is complete.

(4). Define the function $m : \mathcal{L}/\sim \to \mathbf{R}^+$ by setting $m([E]) := m(E)$, which is well-defined because different representatives for the same equivalence class of $\mathcal{L}/\sim$ have the same measure. If $d([E_2], [E_2]) < \varepsilon$ for some $[E_1],\ [E_2] \in \mathcal{L}/\sim$, then $m(E_1) - m(E_2) \leq m(E_1 \backslash E_2) < \varepsilon$ by subadditivity, so m is continuous.

Let $m' : \mathcal{L}/\sim \to \mathbf{R}^+$ be a continuous map that agrees with the elementary measure function $m : \mathcal{E}/\sim \to \mathbf{R}^+$ on $\mathcal{E}/\sim$, and let $[F] \in \mathcal{L}/\sim$. By Exercise 16, $\forall \varepsilon > 0$, $\exists E_\varepsilon \in \mathcal{E}$ such that $m^*(E_\varepsilon \Delta F) < \varepsilon$. By continuity of m', this implies that $|m'([F]) - m(E_\varepsilon)| < \varepsilon'$ for any $\varepsilon' = \varepsilon'(\varepsilon) > 0$. As before, we have $|m(E_\varepsilon) - m(F)| < \varepsilon$ by subadditivity. Hence by the triangle inequality,

$$|m'([F]) - m(F)| \leq \varepsilon + \varepsilon'.$$

Sending $\varepsilon,\ \varepsilon' \to 0$ shows that $m'([F]) = m(F) = m([F])$. □

Exercise 25

Proof. Let E be the set constructed in Proposition 12. If $m^*(E) = 0$ then E is a null set, and thus measurable by (3) of Lemma 10, a contradiction. Hence $m^*(E) > 0$. Suppose for contradiction that the Lebesgue outer measure is finitely additive. Then for any natural number $N \geq 1$, and $\forall q_1, \ldots q_N \in \mathbf{Q} \cap [-1, 1]$, we have

$$m^*(\bigcup_{i=1}^{N}(E + q_i)) = \sum_{i=1}^{N} m^*(E + q_i).$$

Observe that outer measure is translation invariant, so the right hand side equals $Nm^*(E)$, which exceeds 3 for sufficiently large N, contradicting (6) in the proof of Proposition 12. □

Exercise 26

Proof. Define $F := E \times \{0\} \subset \mathbf{R}^2$, where E is the non-measurable set of Proposition 12. F is measurable since it is a null set (it can be contained in a 2-d degenerate box). But $\pi(F) = E$ is not measurable. □

Exercise 27

Proof. (1). Note that $\gamma \in C^1([a,b])$ is Lipschitz continuous. Let $t_1 < t_2 \in [a,b]$, by the mean value theorem, $\exists t \in (t_1,t_2)$ such that $\gamma'(t) = \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}$. As γ' is continuous on $[a,b]$, it attains a maximum of $\sup_{t\in[a,b]} \gamma'(t)$ on $[a,b]$. It follows that $|\gamma(t_1)-\gamma(t_2)| \leq K|t_1-t_2|$, where $K := \sup_{t\in[a,b]} |\gamma'(t)|$.

Let $\varepsilon := (b-a)/N$ for some natural number $N \geq 1$. For each $i = 0,\dots,N-1$, pick a random $t_i \in [a+i\varepsilon, a+(i+1)\varepsilon]$, by Lipschitz continuity,

$$\{\gamma(t) : a \leq t \leq b\} \subset \bigcup_{i=0}^{N-1} B(\gamma(t_i), K\varepsilon).$$

Thus by countable subadditivity, we have

$$\begin{aligned} m^*(\{\gamma(t) : a \leq t \leq b\}) &\leq \sum_{i=0}^{N-1} m^*(B(\gamma(t_i), K\varepsilon)) \\ &= Nc_d(K\varepsilon)^d \\ &= (b-a)C_d K^d \varepsilon^{d-1} \end{aligned}$$

by Exercise 10 of note 0. As $d \geq 2$, $\varepsilon^{d-1} \to 0$ as $\varepsilon \to 0$, and thus the curve $\{\gamma(t) : a \leq t \leq b\}$ is a null set. From the proof we also see that the condition $d \geq 2$ is necessary to control the term $A_d\varepsilon^{d-1}$, where $A_d := (b-a)C_dK^d$. If $d = 1$, then $\{\gamma(t) = t : a \leq t \leq b\}$ is the interval $[a,b]$, which has positive measure.

(2). From (1), if $d \geq 2$, the unit cube $[0,1]^d$ cannot be covered by countably many continuously differentiable curves, for otherwise the unit cube will have measure zero by countable additivity. $\square$

Exercise 28

Proof. First we suppose that $\sum_{(n,m)\in A\times B} x_{n,m} < \infty$. Define

$$S := \{(n,m) \in A \times B : x_{n,m} > 0\},$$

then S is at most countable by Exercise 3, and $\sum_{(n,m)\in A\times B} x_{n,m} = \sum_{(n,m)\in S} x_{n,m}$. Let

$$\begin{aligned} A' &:= \{n \in A : \exists m \in B \text{ such that } (n,m) \in S\}, \\ B' &:= \{m \in B : \exists n \in A \text{ such that } (n,m) \in S\} \end{aligned}$$

be the projections of S onto A and B respectively. Then A' and B' are at most countable with $S \subset A' \times B'$. In particular, we have

$$\sum_{(n,m)\in A\times B} x_{n,m} = \sum_{(n',m')\in A'\times B'} x_{n',m'}.$$

The case where A' and B' are both finite is obvious, hence we assume that either A' or B' is countable. Then $A' \times B'$ is countable, and there is a bijection $\phi : \mathbf{N} \times \mathbf{N} \to A' \times B'$ given by $\phi(n, m) = (n', m')$. By the change of variables formula, we get

$$\sum_{(n',m')\in A'\times B'} x_{n',m'} = \sum_{\phi(n,m)\in A'\times B'} x_{\phi(n,m)} = \sum_{(n,m)\in \mathbf{N}\times\mathbf{N}} x_{n,m}.$$

By Theorem 2 and our construction,

$$\sum_{(n,m)\in \mathbf{N}\times\mathbf{N}} x_{n,m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{n,m} = \sum_{n'\in A'}\sum_{m'\in B'} x_{n',m'} = \sum_{n\in A}\sum_{m\in B} x_{n,m}.$$

Similarly,

$$\sum_{(n,m)\in \mathbf{N}\times\mathbf{N}} x_{n,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{n,m} = \sum_{m'\in B'}\sum_{n'\in A'} x_{n',m'} = \sum_{m\in B}\sum_{n\in A} x_{n,m}.$$

Hence

$$\sum_{(n,m)\in A\times B} x_{n,m} = \sum_{n\in A}\sum_{m\in B} x_{n,m} = \sum_{m\in B}\sum_{n\in A} x_{n,m}.$$

Now we suppose that $\sum_{(n,m)\in A\times B} x_{n,m} = \infty$. Let $M > 0$. By definition, there is a finite subset $F \subset A \times B$ such that $\sum_{(n,m)\in F} x_{n,m} > M$. Then there are finite subsets $A' \subset A$ and $B' \subset B$ such that $F \subset A' \times B'$. By the non-negativity of the $x_{n,m}$,

$$M < \sum_{(n,m)\in F} x_{n,m} \le \sum_{(n,m)\in A'\times B'} x_{n,m} = \sum_{n\in A'}\sum_{m\in B'} x_{n,m} \le \sum_{n\in A}\sum_{m\in B} x_{n,m}.$$

It turns out that $\sum_{n\in A}\sum_{m\in B} x_{n,m} = \infty$. In similar fashion we can show that $\sum_{m\in B}\sum_{n\in A} x_{n,m} = \infty$ as well. $\square$

Some ending remark: A bounded Lebesgue measurable set has finite measure, but a Lebesgue measurable set of finite measure is not necessarily bounded. For example, let $\sum_{n=1}^{\infty} a_n,\ a_n > 0$ be a convergent series with sum s. The set $E := \bigcup_{n=1}^{\infty}(n, n+a_n)$ is measurable with measure at most s, but is not bounded.