Math 245A Note 5

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1 Selected Exercises in Note 5

Throughout this section, the notions of measurability and “almost everywhere” are understood to be with respect to Lebesgue measure.

Exercise 2

Proof. By condition, $\forall x \in [a, b]$, and any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|\frac{F(y) - F(x)}{y - x} - F'(x)| \leq \varepsilon$$

whenever $|y - x| \leq \delta,\ y \neq x$. Manipulating the above inequality, one gets

$$|F(y) - F(x)| \leq |y - x|(|F'| + \varepsilon) \leq \delta(|F'(x) + \varepsilon|)$$

whenever $|y - x| \leq \delta,\ y \neq x$. This implies that $\lim_{y \to x, y \neq x} F(y) = F(x)$ as $F'(x) < \infty$. in other words, F is continuous.

Since the inverse image $F^{-1}(U)$ of any open set is open, which in turn is measurable by lemma 10 of note 1, F is measurable by definition. The fact that F' is measurable follows then from (6) and (8) of Exercise 28 in note 3, by writing $F'(x)$ as

$$F'(x) = \lim_{n \to \infty} \frac{|F(x + 1/n) - F(x)|}{1/n}.$$

By modifying the argument for (6) of Exercise 28 in note 3 on a null set, we see the claim there holds for pointwise almost everywhere convergence as well. Hence if F is almost everywhere differentiable, F' is measurable.

Let $F : [0, 1] \to \mathbf{R}$ be such that $F := 1_{[1/2,1]}$, then $F' = 0$ for all $x \in [0, 1]$ except at $1/2$, where F is discontinuous. $\square$

Exercise 3

Proof. As the hint suggests, we consider $F(x) := x^2 \sin(1/x)$ on $[0,1]$, where $F(0) := 0$. Then F' exists for all $x \in [0,1]\backslash\{0\}$, and

$$F'(0) := \lim_{y\to 0; y\in[0,1]\backslash\{0\}} \frac{y^2\sin(1/y)}{y} = 0.$$

So, F is everywhere differentiable on $[0,1]$. But $F'(x) = -\cos(1/x)+2x\sin(1/x)$ is not continuous at 0. $\square$

Exercise 6

Proof. Consider $F : [-1,1] \to \mathbf{R}$ given by $F(x) = \begin{cases} 1+x, & x \in [-1,0) \\ 1-x, & x \in [0,1] \end{cases}$. Then F is continuous with $F(-1) = F(1) = 0$, but only differentiable outside of the null set $\{0\}$, and $F'(x) \neq 0$ for every other x. $\square$

Exercise 10

Proof. Given $F'(x) = G'(x)$ for every $x \in [a,b]$, we see that $\frac{d}{dx}(F(x)-G(x)) = 0$ for every $x \in [a,b]$ by laws of differentiation. Hence $F(x) - C(x) = C$ for some $C \in \mathbf{R}$ and every $x \in [a,b]$. The other direction follows directly from such laws. $\square$

Exercise 16

Proof. Let $x \in \mathbf{R}$ and $y > x$. By linearity of the absolutely convergent integral,

$$|\int_{(-\infty,x]} f(t)\ dt - \int_{(-\infty,y]} f(t)\ dt| = |\int_{(x,y]} f(t)\ dt|,$$

which converges to zero as $|y-x| \to 0$ by DCT. This shows that $\lim_{y\to x^+} F(y) = F(x)$. Similarly one can show that $\lim_{y\to x^-} F(y) = F(x)$, so $\lim_{y\to x} F(y) = F(x)$ and thus F is continuous. $\square$

Exercise 18

Proof. It suffices to show that

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0^-} \frac{F(x+h)-F(x)}{h} = f'(x)$$

for almost every $x \in \mathbf{R}$, which by linearity of the absolutely convergent integral implies that

$$\lim_{h\to 0^+} \frac{1}{h} \int_{[x,x+h]} f(t)\ dt = \lim_{h\to 0^-} \frac{1}{h} \int_{[x-h,x]} f(t)\ dt = f'(x)$$

for almost every $x \in \mathbf{R}$, which follows from theorem 17 by the fact that the union of two null sets is again a null set. □

Exercise 21

Proof. Fix any $x \in \mathbf{R}^d$. Since g is essentially bounded, it stays essentially bounded upon reflection and translation, so $r(y) = g(x-y)$ is essentially bounded. That is, $\exists M > 0$ such that $|g(x-y)| \le M$ for almost every $y \in \mathbf{R}^d$. By monotonicity, one gets

$$\int_{\mathbf{R}^d} |f(y)g(x-y)|\ dy \le M \int_{\mathbf{R}^d} |f(y)|\ dy < \infty.$$

In particular, the integrand is absolutely integrable, and $|f * g(x)| \le M\|f\|_{L^1}$ by the triangle inequality.

To show that $f * g(x)$ is continuous, substitute $z = x - y$ and note that convolution is commutative:

$$\int_{\mathbf{R}^d} g(y)f(x-y)\ dy = -\int_{x-\mathbf{R}^d} g(x-z)f(z)\ dz = \int_{\mathbf{R}^d} f(y)g(x-y)\ dy.$$

That is, $f * g(x) = g * f(x)$ for all $x \in \mathbf{R}^d$. Then for each $h \in \mathbf{R}^d$, we see that

$$\begin{aligned} |f * g(x+h) - f * g(x)| &= |\int_{\mathbf{R}^d} f(y)[g(x-y+h) - g(x-y)]\ dy| \\ &= |\int_{\mathbf{R}^d} g(y)[f(x-y+h) - f(x-y)]\ dy| \\ &\le M \int_{\mathbf{R}^d} |f(x+y+h) - f(x-y)|\ dy, \end{aligned}$$

which converges to zero as the point h approaches the origin by proposition 19. This shows that $f * g$ is continuous. □

Exercise 22

Proof. By intersecting E with a ball, it suffices to show the claim for bounded E. Let $\varepsilon > 0$ be arbitrary, and apply Exercise 21 to the convolution $1_E * 1_{-E}$, then there is a $\delta > 0$ such that $\forall x \in \mathbf{R}^d$ with $|x| \le \delta$,

$$|1_E * 1_{-E}(x) - 1_E * 1_{-E}(0)| \le \varepsilon.$$

Since $1_E * 1_{-E}(0) = m(E)$ is a positive number, it must be that

$$1_E * 1_{-E}(x) = \int_{\mathbf{R}^d} 1_E(y) 1_{-E}(x-y)\, dy > 0$$

for all $x \in \mathbf{R}^d$ such that $|x| \le \delta$ for some sufficiently small δ. In other words, we see that $1_E * 1_{-E}(x) > 0$ on some ball $B(0,\delta) \subset \mathbf{R}^d$. But the integrand on the RHS is positive only if $x \in E - E$, thus $B(0,\delta) \subset E - E$ and the claim follows. $\square$

Exercise 23

Proof. (1). Let $f : \mathbf{R}^d \to \mathbf{C}$ be a measurable homomorphism, and $D \subset \mathbf{C}$ be any complex disk centered at the origin. As $\mathbf{C}$ can be written as

$$\mathbf{C} = \bigcup_{p,q \in \mathbf{Q}} (p,q) + D,$$

and countable union of null sets is again a null set, it must be that $f^{-1}(z+D)$ has positive measure for at least one $z = (p,q) \in \mathbf{C}$ (the fact that f is measurable assures that the $f^{-1}(z+D)$ are measurable).

By Steinhaus theorem, $\forall x_0 \in \mathbf{R}^d$, and a complex disk D of radius at most ε, there exists $\delta > 0$ such that

$$x - x_0 \in f^{-1}(z+D) - f^{-1}(z+D)$$

whenever $|x - x_0| \le \delta$. Since f is a homomorphism, this implies that

$$f(x - x_0) = f(x) - f(x_0) \in D.$$

This suggests that $|f(x) - f(x_0)| \le \varepsilon$ whenever $|x - x_0| \le \delta$. That is, f is continuous.

(2). If f is a measurable homomorphism, and $(x_1, \ldots, x_d) \in \mathbf{Q}^d$, then

$$f(x_1, \ldots, x_d) = f(\sum_{i=1}^{d} x_i e_i) = \sum_{i=1}^{d} f(x_i e_i), \tag{1}$$

where e_i is the i^{th} element in the standard ordered basis for $\mathbf{R}^d$, viewed as a vector space over $\mathbf{R}$. Note that each coordinate x_i can be written as q_i/p_i for some non-negative integer q_i and integer $p_i \neq 0$. Hence we have

$$f(x_i e_i) = f(\frac{q_i}{p_i} e_i) = q_i f(\frac{e_i}{p_i}) \tag{2}$$

for all $1 \leq i \leq d$. Moreover, note that $f(0) = f(x - x) = f(x) - f(x) = 0$, from which we have $f(-x) = f(0 - x) = f(0) - f(x) = -f(x)$ for all $x \in \mathbf{R}^d$. Therefore, from

$$f(\frac{e_i}{p_i} |p_i|) = \frac{|p_i|}{p_i} f(e_i) = |p_i| f(\frac{e_i}{p_i}),$$

we get $f(e_i/p_i) = \frac{1}{p_i} f(e_i)$. Substituting this into (2) and (2) into (1), we obtain

$$f(x_1, \ldots, x_d) = \sum_{i=1}^{d} x_i f(e_i) = \sum_{i=1}^{d} x_i z_i,$$

where $z_i := f(e_i) \in \mathbf{C}$ for all i. Now, let $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, and $\varepsilon > 0$. By the continuity of f, and the denseness of rationals in the reals, there exists $y = (y_1, \ldots, y_d) \in \mathbf{Q}^d$ sufficiently close to x such that

$$|f(x) - f(y)| = |f(x) - \sum_{i=1}^{d} y_i z_i| \leq \varepsilon/2$$

and

$$|\sum_{i=1}^{d} y_i z_i - \sum_{i=1}^{d} x_i z_i| \leq \sum_{i=1}^{d} |z_i||y_i - z_i| \leq \varepsilon/2.$$

The claim follows then from the triangle inequality.

$\square$

Exercise 28

Proof. Let $U \subset \mathbf{R}$ be an open set. For any $x \in U$, we define the open subinterval $I_x = (a, b)$ of U to be the union of all open intervals in U that contains x. Hence I_x is maximal in the sense that $(c, d) \subset I_x$ for any $(c, d) \subset U,\ x \in (c, d)$. Furthermore, if $y \in U$ is a distinct point from x, without loss of generality we may assume $x < y$, then $I_x \cap I_y = \emptyset$, otherwise $I_x \cup I_y$ would have been a maximal interval, which is a contradiction. By the denseness of rationals in the reals, for every $x \in U$, there exists $x' \in \mathbf{Q}$ sufficiently close to x such that $x' \in I_x$. And as the I_x are all disjoint, we can choose the x' to be all distinct. Finally, let $I_{x'} := I_x$ for all $x \in U$. We conclude that

$$U = \bigcup_{x \in U} I_x = \bigcup_{x' \in \mathbf{Q}} I_{x'}.$$

That is, U is the union of at most countably many disjoint non-empty open intervals, whose endpoints lie outside of U (otherwise the intervals won't be disjoint). $\square$

Exercise 29

Proof. Apply right-sided Hardy-Littlewood maximal inequality to the reflection $g(x) := f(-x)$ gives

$$m(\{x \in \mathbf{R} : \sup_{h>0} \frac{1}{h} \int_{[x-h,x]} |f(t)|\ dt \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbf{R}} |f(t)|\ dt;$$

the left-sided Hardy-Littlewood maxima inequality. For every interval I, we denote the left and right end points by $a(I)$ and $b(I)$. By the triangle inequality,

$$\sup_{x \in I} \frac{1}{|I|} \int_I |f(t)|\ dt \leq \sup_{x \in I} \frac{1}{|I|} \int_{[a(I),x]} |f(t)|\ dt + \sup_{x \in I} \frac{1}{|I|} \int_{[x,b(I)]} |f(t)|\ dt,$$

from which we see that $m(\{x : \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)|\ dt \geq \lambda\})$ is bounded by

$$m(\{x : \sup_{x \in I} \frac{1}{|I|} \int_{[a(I),x]} |f(t)|\ dt \geq \frac{\lambda}{2}\}) + m(\{x : \sup_{x \in I} \frac{1}{|I|} \int_{[x,b(I)]} |f(t)|\ dt \geq \frac{\lambda}{2}\}),$$

which in turn is bounded by

$$m(\{x : \sup_{h>0} \frac{1}{h} \int_{[x-h,x]} |f(t)|\ dt \geq \lambda\}) + m(\{x : \sup_{h>0} \frac{1}{h} \int_{[x,x+h]} |f(t)|\ dt \geq \lambda\}),$$

which is then bounded by $\frac{2}{\lambda} \|f\|_{L^1}$, as desired. $\square$

Exercise 30

Proof. For any compact interval $[a,b]$, define $f^* \downharpoonleft_{[a,b]} \colon [a,b] \to \mathbf{R}$ by

$$f^* \downharpoonleft_{[a,b]} (x) := \sup_{h>0; [x,x+h] \subset [a,b]} \frac{1}{h} \int_{[x,x+h]} f(t)\ dt$$

By upwards monotonicity and dominated convergence, it suffices to show that

$$\lambda m(\{x \in [a,b] : f^* \downharpoonleft_{[a,b]} (x) > \lambda\}) \leq \int_{x \in [a,b] : f^* \downharpoonleft_{[a,b]} (x) > \lambda} f(x)\ dx$$

for any such $[a,b]$. Fix $[a,b]$. Also, it suffices to establish the result for $\lambda = 0$, as one can then applies the result to the absolutely integrable function $f - \lambda 1_{[a,b]}$. Hence we aim to show that

$$\int_{x \in [a,b] : f^* \downharpoonleft_{[a,b]} (x) > 0} f(x)\ dx \geq 0.$$

Define the function $F : [a,b] \to \mathbf{R}$ to be such that

$$F(x) := \int_{[a,x]} f(t)\ dt.$$

By Exercise 16, F is continuous. Let U be the set of all $x \in (a,b)$ such that $F(y) > F(x)$ for at least one $x < y < b$. As in the proof of the rising sun lemma, we can find an at most countable sequence of intervals $I_n = (a_n, b_n)$ with the properties given by that lemma, and

$$U = \bigcup_n I_n = \{x \in (a,b] : f^* \downharpoonleft_{[a,b]} (x) > 0\},$$

as the the property $\frac{1}{h} \int_{[x,x+h]} f(t)\ dt > 0$ can be rearranged as $F(x+h) > F(x)$. Since $F(b_n) - F(a_n) \geq 0$, we have $\displaystyle\int_{I_n} f(t)\ dt \geq 0$ for all n. Thus by the dominated convergence theorem, we conclude that

$$\sum_n \int_{I_n} f(t)\ dt = \int_{x \in [a,b] : f^* \downharpoonleft_{[a,b]} (x) > 0} f(x)\ dx \geq 0,$$

as desired. Moreover, the one-sided Hardy-Littlewood maximal inequality can be deduced from the rising sun inequality by applying the latter to $|f|$. $\square$

Exercise 31

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be an absolutely integrable function of compact support, so that $\text{supp}(f) \subset [a,b]$ for some $a < b$, and let $\lambda > 0$. As before, it will suffice to show that

$$\lambda m(\{x \in [a,b] : f^* \downharpoonleft_{[a,b]} (x) > \lambda\}) = \int_{x \in [a,b] : f^* \downharpoonleft_{[a,b]}(x) > \lambda} f(x)\ dx.$$

Define the function $F : [a,b] \to \mathbf{R}$ to be such that

$$F(x) := \int_{[a,x]} f(t)\ dt - (x-a)\lambda.$$

By Exercise 16, F is continuous. Let U be the set of all $x \in (a,b)$ such that $F(y) > F(x)$ for at least one $x < y < b$. As in the proof of the rising sun lemma, we can find an at most countable sequence of intervals $I_n = (a_n, b_n)$ with the properties given by that lemma, and

$$U = \bigcup_n I_n = \{x \in (a,b] : f^* \downharpoonleft_{[a,b]} (x) > \lambda\},$$

as the the property $\frac{1}{h} \int_{[x,x+h]} f(t)\ dt > \lambda$ can be rearranged as $F(x+h) > F(x)$. If $a_n = a$ and $F(b_n) > F(a_n)$, we would have

$$0 < b_n - a < \frac{1}{\lambda} \int_{[a,b_n]} f(t)\ dt \leq \frac{1}{\lambda} \|f\|_{L^1},$$

As a moves well to the left of $\text{supp}(f)$, the distance $b_n - a$ must increases (otherwise the integral over $[a, b_n]$ will be zero), and eventually exceeds $\frac{1}{\lambda}\|f\|_{L^1}$. Fix any such a. From the first property of the lemma, we get $F(a_n) = F(b_n)$ for all n, or equivalently that

$$\int_{I_n} f(t)\ dt = \lambda(b_n - a_n)$$

for all n. By the dominated convergence theorem, and countable additivity, we conclude that

$$\lambda m(\{x \in [a,b] : f^* \downharpoonleft_{[a,b]} (x) > \lambda\}) = \int_{x \in [a,b] : f^* \downharpoonleft_{[a,b]}(x) > \lambda} f(x)\ dx.$$

For some general absolutely integrable f, the claim follows by combining upward monotonicity, dominated convergence theorem, and the compact support case on $f 1_{[-n,n]}$. $\square$

Exercise 33

Proof. (1). Suppose that $f : \mathbf{R}^d \to \mathbf{C}$ is locally integrable, and $r > 0$. The set $\mathbf{Q} \cap B(0,r)$ is countable, whose elements can be listed as $q_1, q_2, \ldots$. By Exercise 24 of note 4, for each $q_i \in \mathbf{Q} \cap B(0,r)$, there exists $r_i > 0$ sufficiently small such that f is absolutely integrable on $B(q_i, r_i)$ with

$$\int_{B(q_i,r_i)} |f(x)| \, dx \le \frac{1}{2^i}.$$

By the denseness of rationals in the reals, $B(0,r) \subset \bigcup_{q_i \in \mathbf{Q} \cap B(0,r)} B(q_i, r_i)$. Hence by monotonicity and countable subadditivity, we have

$$\int_{B(0,r)} f(x) \, dx \le \sum_{q_i \in \mathbf{Q} \cap B(0,r)} \int_{B(q_i,r_i)} |f(x)| \, dx \le \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

showing that $\int_{B(0,r)} f(x) \, dx < \infty$ for all $r > 0$. Conversely, if $\int_{B(0,r)} |f(x)| \, dx$ is finite for all $r > 0$, then $\forall x \in \mathbf{R}^d, \; x \in B(0,r)$ for some $r > 0$, the claim thus follows from the openness of $B(0,r)$ and monotonicity.

(2). If $f : \mathbf{R}^d \to \mathbf{C}$ is absolutely integrable, it is absolutely integrable on the open ball $B(x,r)$ for almost every $x \in \mathbf{R}^d$ and all $r > 0$. The claim follows by applying theorem 32 to the restricted function $f 1_{B(x,r)}$.

□

Exercise 34

Proof. By the triangle inequality (and translation invariance), we see that

$$\begin{aligned} \left|\frac{1}{m(E_h)} \int_{x+E_h} f(y) \, dy - f(x)\right| &= \left|\frac{1}{m(E_h)} \int_{x+E_h} f(y) - f(x) \, dy\right| \\ &\le \frac{1}{m(E_h)} \int_{x+E_h} |f(y) - f(x)| \, dy. \end{aligned}$$

By our condition, monotonicity, and translation invariance again, we obtain

$$\begin{aligned} \frac{1}{m(E_h)} \int_{x+E_h} |f(y) - f(x)| \, dy &\le \frac{1}{cm(B(0,r))} \int_{B(x,h)} |f(y) - f(x)| \, dy \\ &= \frac{1}{c} \cdot \frac{1}{m(B(x,h))} \int_{B(x,h)} |f(y) - f(x)| \, dy, \end{aligned}$$

which goes to zero as $h \to 0$, by theorem 32 and the assumption that x is a Lebesgue point of f. In one-dimension, let $E_h := (0,h) \subset (-h,h)$ (so $m(E_h) =$

$\frac{1}{2}m(B(0,h)))$. From what we just proved, for almost every $x \in \mathbf{R}$, one has $\lim_{h\to 0^+} \frac{1}{h}\int_{(x,x+h)} f(y)\, dy = f(x)$, which is half of theorem 17; the other half is obtained by setting $E_h := (-h, 0)$. $\square$

Exercise 35

Proof. The closed balls $\overline{B(x,r)} := \{y \in \mathbf{R}^d : |x-y| \leq r\}$ are compact by the Heine-Borel theorem, hence a continuous $f : \mathbf{R}^d \to \mathbf{C}$ is uniformly continuous on them. Thus for almost every $x \in \mathbf{R}^d$, and $\varepsilon > 0$ arbitrary, $\exists r > 0$ sufficiently small such that $|f(y) - f(x)| \leq \varepsilon$ for all $y \in B(x,r)$. This implies that

$$\frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\, dy \leq \frac{1}{m(B(x,r))}\int_{B(x,r)} \varepsilon\, dy = \varepsilon$$

whenever $|x-y| < r$. Taking $r \to 0$ gives theorem 32 in the case when f is continuous. $\square$

Exercise 38

Proof. Let $f : \mathbf{R}^d \to \mathbf{C}$ be absolutely integrable, and let $\varepsilon, \lambda > 0$ be arbitrary. Then by Littlewood's second principle, we can find a function $g : \mathbf{R}^d \to \mathbf{C}$ which is continuous and compactly supported, with

$$\int_{\mathbf{R}^d} |f(x) - g(x)|\, dx \leq \varepsilon$$

Applying the Hardy-Littlewood maximal inequality, we conclude that

$$m(\{x \in \mathbf{R}^d : \sup_{r>0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-g(y)|\, dy \geq \lambda\}) \leq \frac{\varepsilon C_d}{\lambda}$$

for some constant $C_d > 0$ depending only on d. In a similar spirit, from Markov's inequality we have

$$m(\{x \in \mathbf{R}^d : |f(x) - g(x)| \geq \lambda\}) \leq \frac{\varepsilon}{\lambda}.$$

By subadditivity, we conclude that for all $x \in \mathbf{R}^d$ outside of a set E of measure at most $\frac{\varepsilon(1+C_d)}{\lambda}$, one has both

$$\frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-g(y)|\, dy < \lambda \tag{3}$$

and

$$|f(x) - g(x)| < \lambda \tag{4}$$

for all $r > 0$. Now let $x \in \mathbf{R}^d \backslash E$. From the dense subclass result applied to the continuous function g, we have

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| < \lambda$$

whenever r is sufficiently close to 0. Combining this with (3), (4), and the triangle inequality, we conclude that

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| < 3\lambda$$

for all r sufficiently close to zero. In particular we have

$$\limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| < 3\lambda$$

for all x outside of a set of measure at most $\frac{\varepsilon(1+C_d)}{\lambda}$. Keeping λ fixed and sending ε to zero, we conclude that

$$\limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| < 3\lambda$$

for almost every $x \in \mathbf{R}^d$. If we then let λ go to zero along a countable sequence (e.g. $\lambda := 1/n$ for $n = 1, 2, \ldots$), we conclude that

$$\limsup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| = 0$$

for almost every $x \in \mathbf{R}^d$, and the claim follows. □

Exercise 40

Proof. Introduce a time variable t. For every natural number t, let

$$B'_{1,t}, \ldots, B'_{m_t,t}$$

be the outcome of the algorithm after t iterations (or t_* iterations, if the process ends at some previous time $t_* < t$). For $t = 0$, initialize

$$B'_{1,0} \in B_1, \ldots, B_n$$

to be a random ball in the list. For each $t > 0$, look at all the balls B_j that do not already intersect one of the $B'_{1,t-1}, \ldots, B'_{m_{t-1},t-1}$, if there are no such balls, stop. Otherwise, locate the largest such ball B_j (if there are multiple largest balls with exactly the same radius, break the tie arbitrarily). Add this ball to the collection $B'_{1,t-1}, \ldots, B'_{m_{t-1},t-1}$ by incrementing t by 1, and then set $B_j := B'_{m_t,t}$ with $m_t := m_{t-1} + 1$. □

Exercise 42

Proof. Let $1 \geq \varepsilon > 0$ be arbitrary. As before, it suffices to show that

$$m(K) \leq \frac{2^d}{\lambda} \int_{\mathbf{R}^d} |f(t)|\ dt$$

whenever K is a compact set that is contained in

$$\{x \in \mathbf{R}^d : \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\ dy > \lambda\}.$$

By construction, for every $x \in K$, there exists an open ball $B(x,r)$ such that

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\ dy > \lambda. \tag{5}$$

As $K \subset \bigcup_{x \in K} \varepsilon B(x,r)$ is compact, we can cover K by a finite number

$$\varepsilon B_1, \ldots, \varepsilon B_n$$

of such balls. Clearly this implies that K is covered by $B_1, \ldots, B_n$ as well. By the Vitali-type covering lemma, we can find a subcollection $B'_1, \ldots, B'_m$ of disjoint balls such that the centers of the balls B_i are contained in $\bigcup_{j=1}^m 2B'_j$. In particular, we see that

$$K \subset \bigcup_{i=1}^n \varepsilon B_i \subset \bigcup_{j=1}^m (2+\varepsilon) B'_j,$$

and therefore

$$m(K) \leq \sum_{j=1}^m (2+\varepsilon)^d m(B'_j).$$

By (5), on each ball B'_j we have

$$m(B'_j) < \frac{1}{\lambda} \int_{B'_j} |f(y)|\ dy;$$

summing in j and using the disjointedness of the B'_j we conclude that

$$m(K) \leq \frac{(2+\varepsilon)^d}{\lambda} \int_{\mathbf{R}^d} |f(y)|\ dy.$$

Sending $\varepsilon \to 0$, we obtain Theorem 36 with the improved constant 2^d as desired. $\square$

Exercise 44

Proof. It suffices to verify the claim with strict inequality,

$$m(\{x \in \mathbf{R}^d : \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy > \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbf{R}^d} |f(t)|\, dt$$

as the non-strict case then follows by perturbing λ slightly and then taking limits. Fix f and λ. By inner regularity, it suffices to show that

$$m(K) \leq \frac{1}{\lambda} \int_{\mathbf{R}^d} |f(t)|\, dt$$

whenever K is a compact set that is contained in

$$\{x \in \mathbf{R}^d : \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy > \lambda\}.$$

By construction, for every $x \in K$, there exists an dyadic cube Q such that

$$\frac{1}{|Q|} \int_Q |f(y)|\, dy > \lambda \tag{6}$$

By compactness of K, we can cover K by a finite number $Q_1, \ldots, Q_n$ of such cubes. By the dyadic nesting property, we can find a subcollection $Q'_1, \ldots, Q'_m$ of almost disjoint cubes such that

$$m(\bigcup_{i=1}^n Q_i) = m(\bigcup_{j=1}^m Q'_j) = \sum_{j=1}^m |Q'_j|.$$

By (6), on each cube Q'_j we have

$$|Q'_j| < \frac{1}{\lambda} \int_{Q'_j} |f(y)|\, dy;$$

summing in j and using the disjointedness of the Q'_j we conclude that

$$m(\bigcup_{i=1}^n Q_i) \leq \frac{1}{\lambda} \int_{\mathbf{R}^d} |f(t)|\, dt.$$

Since the Q_i cover K, we obtain the claim as desired. □

Exercise 45

Proof. First, replace each I_i with the maximal interval containing I_i, to get a subfamily $I_1^*, \ldots, I_l^*$ of intervals such that no single one of them contains the other. Then, For every such interval I_i^*, if $I_i^* \subset \bigcup_{j \neq i; 1 \leq j \leq l} I_j^*$, it must be that

$$I_i^* \subset I_j^* \cup I_k^*$$

for some $i \neq j \neq k$. Indeed, if $I_i^* = (a_i, b_i)$ (say), we can take I_j^* to be the interval with the largest right endpoint such that $a_i \in I_j^*$, and similarly I_k^* is the interval with the smallest left endpoint such that $b_i \in I_k^*$. Remove those I_i^* such that $I_j^* \cap I_k^* \neq \emptyset$, we ended up with a subfamily $I_1', \ldots, I_m'$ of intervals. Clearly, $\bigcup_{i=1}^n I_i = \bigcup_{j=1}^m I_j'$. And if $\in I_i' \cap I_j' \cap I_k' \neq \emptyset$, it must be that one the three intervals is contained in the union of the other two, but such intervals are removed already, a contradiction. □

Exercise 46

Proof. It suffices to verify the claim with strict inequality,

$$m(\{x \in \mathbf{R} : \sup_{x \in I} \frac{1}{\mu(I)} \int_I |f(y)| \, d\mu(y) > \lambda\}) \leq \frac{2}{\lambda} \int_{\mathbf{R}} |f(y)| \, d\mu(y)$$

as the non-strict case then follows by perturbing λ slightly and then taking limits. Fix f and λ. By inner regularity, it suffices to show that

$$m(K) \leq \frac{2}{\lambda} \int_I |f(y)| \, d\mu(y)$$

whenever K is a compact set that is contained in

$$\{x \in \mathbf{R} : \sup_{x \in I} \frac{1}{\mu(I)} \int_I |f(y)| \, d\mu(y) > \lambda\}.$$

By construction, for every $x \in K$, there exists an open intervals I that contain x such that

$$\frac{1}{\mu(I)} \int_I |f(y)| \, d\mu(y) > \lambda. \tag{7}$$

By compactness of K, we can cover K by a finite number $I_1, \ldots, I_n$ of such intervals. Applying the Besicovitch covering lemma in one dimension, we can find a subfamily $I_1', \ldots, I_m'$ of intervals such that

$$\mu(\bigcup_{i=1}^n I_i) = \mu(\bigcup_{j=1}^m I_j').$$

By (7), on each interval I_j' we have

$$\mu(I_j') < \frac{1}{\lambda}\int_{I_j'} |f(y)|\ d\mu(y);$$

summing in j and using the fact that point $x \in \mathbf{R}$ is contained in at most two of the I_j' (so that $\sum_{j=1}^m \mu(I_j')$ counts the true size of $\bigcup_{j=1}^m I_j'$ at most twice), we conclude that

$$\mu(\bigcup_{i=1}^n I_i) \leq \frac{2}{\lambda}\int_{I_j'} |f(y)|\ d\mu(y).$$

Since the $I_1, \ldots, I_n$ cover K, we obtain the claim as desired. Note that we need the former lemma here instead of the latter since for general measures μ we do not have a clear relationship between $\mu(3B)$ and $\mu(B)$. $\square$

Exercise 47

Proof. $\forall x \in [a,b],\ \exists n_x > 0$ such that $\delta(x) \geq 1/n_x$. Since

$$[a,b] \subset \bigcup_{x\in[a,b]} (x - 1/2n_x, x + 1/2n_x) = \bigcup_{x\in[a,b]} I_x,$$

by the Heine-Borel theorem, there is a finite subcover $[a,b] \subset \bigcup_{i=1}^n I_i$ of this open cover. By construction, the midpoints t_i of the I_i are such that $|I_i| \leq \delta(t_i)$. By the Besicovitch covering lemma in one dimension, we can find a subfamily $I_1', \ldots, I_m'$ of intervals such that

$$[a,b] \subset \bigcup_{i=1}^n I_i = \bigcup_{j=1}^m I_j'$$

and each point $x \in [a,b]$ is contained in at most two of the I_j'. Without loss of generality, we can assume the I_j' to sit from left to right for $1 \leq j \leq m$, which induces a partition of $[a,b]$ into intervals

$$J_i := I_i' \cap I_{i+1}',\ 1 \leq i \leq m-1$$

(including degenerate intervals of a point), plus the intervals cut out from $[a,b]$ by these sets. That is,

$$K_i := I_i' \backslash (J_i \cup J_{i-1}),\ 1 \leq i \leq m,$$

where $J_0 = J_m := \emptyset$. Hence we get a partition $a = t_0 < t_1 < \ldots < t_k = b$ with $k \geq 1$. For each of the $[t_{j-1}, t_j]$, there are two possibilities, either it contains a midpoint of some I_k', or it does not. In the first case, we can pick t_j^* to be this midpoint, otherwise we can pick any t_j^* of $[t_{j-1}, t_j]$, and shrink the length of the interval so that $t_j - t_{j-1} \leq \delta(t_j^*)$. $\square$

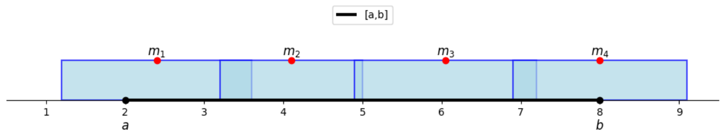

Figure 1: partition induced by the gauge function

Exercise 48

Proof. If $m(E) < \infty$, we can apply Lebesgue differentiation theorem to the integrable function $1_E : \mathbf{R}^d \to \mathbf{C}$:

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} 1_E(y)\ dy = \lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 1_E(x)$$

for almost every $x \in \mathbf{R}^d$. In particular, it holds for almost every $x \in E$, so

$$\lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 1$$

for almost every $x \in E$. By symmetry, this implies that

$$\lim_{r\to 0} \frac{m(E^c \cap B(x,r))}{m(B(x,r))} = \lim_{r\to 0} (1 - \frac{m(E \cap B(x,r))}{m(B(x,r))}) = 0$$

for almost every $x \in \mathbf{R}^d$, and thus it holds for almost every $x \in E^c$. Hence when E has finite measure, almost every point in E is a point of density for E, and almost every point in the complement of E is not.

If $m(E) = \infty$, we can write

$$E = \bigcup_{n=1}^{\infty} E \cap B(0,n) = \bigcup_{n=1}^{\infty} E_n,$$

the claim follows then from the fact that countable union of null sets is again a null set. $\square$

Exercise 49

Proof. (1). Let $\varepsilon > 0$. Since E has positive measure, Exercise 48 ensures that $\exists x \in E, x$ is a density point for E, fix one such x, we claim that x is a Lebesgue point of 1_E. Since $1_E(x) = 1$, we have

$$f(y) = |1_E(y) - 1_E(x)| = |1_E(y) - 1| = 1_{X \setminus E}(y).$$

Substitute into expression (8) in the note, we obtain that

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |1_E(y) - 1_E(x)| \, dy = 1 - \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = 0,$$

hence x is a Lebesgue point of 1_E. $\forall r > 0$, let L be such that $0 < L < r/\sqrt{d}$, then the cube Q_r is such that $Q_r \subset B(0,r)$. Hence by Exercise 34, and translation invariance of the Lebesgue measure, we have

$$\lim_{r\to 0} \frac{1}{m(Q_r)} \int_{x+Q_r} 1_E(y) \, dy = \lim_{r\to 0} \frac{m(E \cap Q_r)}{m(Q_r)} = 1.$$

In particular, for sufficiently small r and $Q = Q_r$, we have

$$m(E \cap Q) > (1 - \varepsilon) m(Q)$$

as desired.

(2). It suffices to establish the claim for bounded E, since we can write:

$$E = \bigcup_{n=1}^{\infty} E \cap B(0,n) = \bigcup_{n=1}^{\infty} E_n.$$

By upwards monotonicity, there exists n such that $m(E_n) > 0$. By monotonicity

$$m(E \cap Q) \geq m(E_n \cap Q)$$

for any cube Q, and the general case follows. Hence we suppose that E is bounded. By Exercise 16 and Lemma 8 of note 1, $\forall \varepsilon > 0$, there is an open set U of finite measure such that

$$U = \bigcup_{i=1}^{\infty} Q_i, \; m(U \setminus E) = \sum_{i=1}^{\infty} m(Q_i \setminus E) < \varepsilon m(U),$$

where the Q_i are almost disjoint closed cubes. We claim that one of the Q_i must sits mostly in E. Suppose for contradiction that $m(Q_i \setminus E) \geq \varepsilon m(Q_i)$ for all i, then

$$\sum_{i=1}^{\infty} m(Q_i \setminus E) \geq \sum_{i=1}^{\infty} \varepsilon m(Q_i) = \varepsilon m(U),$$

a contradiction. Thus $m(Q_i \backslash E) < \varepsilon$ for some i. Fix such i and let $Q = Q_i$, we see that

$$m(E \cap Q) > (1 - \varepsilon)m(Q),$$

and the claim follows.

(3). Let $E \subset \mathbf{R}^d$ be a measurable set of positive measure. For contradiction, if $E - E$ contains no neighborhood of zero, then there is a sequence $x_n \to 0$ such that $x_n \notin E - E$. Equivalently, we have

$$(x_n + E) \cap E = \emptyset \text{ for all } n.$$

By (1), we can pick some non-degenerate cube Q with much of its mass concentrated in E. That is, $\forall \varepsilon > 0,\ \exists Q \subset \mathbf{R}^d$ such that

$$m(E \cap Q) > (1 - \varepsilon)m(Q).$$

The idea is to use these two facts to "shift" most of the mass of Q outside E, creating the desired contradiction.

Fix an $\varepsilon > 0$. For x_n sufficiently close to zero, we can control the proportion of Q that got shifted outside Q:

$$m((x_n + Q) \backslash Q) \le \varepsilon m(Q).$$

This gives a control over the proportion of $E \cap Q$ that got shifted outside Q:

$$m((x_n + E \cap Q) \backslash Q) \le \varepsilon m(Q) < (\frac{\varepsilon}{1 - \varepsilon})m(E \cap Q).$$

By subadditivity and translation invariance, we thus see that

$$m((x_n + E \cap Q) \cap Q) > (1 - \frac{\varepsilon}{1 - \varepsilon})m(E \cap Q)$$

By our construction of x_n, this implies by monotonicity that

$$m(Q \backslash E) > (1 - \frac{\varepsilon}{1 - \varepsilon})m(E \cap Q).$$

But we know that $m(Q \backslash E) < \varepsilon m(E \cap Q)$, so we must have

$$1 - \frac{\varepsilon}{1 - \varepsilon} < \varepsilon,$$

which can be easily made to fail for sufficiently small ε.

$\square$

Remark: For a cube $Q = \prod_{i=1}^{d}[a_i, b_i] \subset \mathbf{R}^d$ of sidelength $s > 0$, and $x = (x_1, \ldots, x_d)$, if a point $y = (y_1, \ldots, y_d) \in Q$ is such that

$$x + y \notin Q,$$

then it must be that for at least one i (say $i = d$), we have

$$y_d \in [a_d, a_d + |x_d|] \text{ or } y_d \in [b_d - |x_d|, b_d]$$

depending on the sign of x_d. Hence if we bound the l^∞ metric

$$d_{l^\infty}(x, 0) = \sup\{|x_i| : 1 \leq i \leq n\} \leq rs$$

with some $0 < r < 1$, then we see that

$$(x + Q)\backslash Q \subset \prod_{i=1}^{d-1}[a_i, b_i] \times I_d,$$

where $|I_d| \leq rs$. And therefore $m((x + Q)\backslash Q) \leq rs^d = rm(Q)$.

Exercise 50

Proof. (1). Let $0 < \varepsilon < 1$, and define

$$U := \bigcup_{q_i \in \mathbf{Q} \cap (0,1)} B(q_i, \varepsilon/2^{i+1}) \cap (0, 1).$$

Then U is an open subset of $[0, 1]$ of measure at most $\sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$, and $\forall x \in [0, 1]$, every neighborhood $B(x, r)$ contains some rationals of $(0, 1)$, thus $U \cap B(x, r) \neq \emptyset$. i.e. U is dense in $[0, 1]$.

Take $K := [0, 1]\backslash U$, which is closed and bounded, hence compact. For any interval I of positive length, if $m(K \cap I) = 0$ then we are done. Otherwise, we have

$$m(K \cap I) = m([0, 1] \cap I\backslash U) > 0.$$

In particular, $[0, 1] \cap I \neq \emptyset$. Since intersection of intervals are intervals, $[0, 1] \cap I$ contains infinitely many rationals of $(0, 1)$, and thus contains some $B(q_i, \varepsilon/2^{\varepsilon+1})$ of positive measure $\varepsilon/2^i$ to be removed. i.e. $m(K \cap I) < |I|$.

(2). It suffices to find a measurable subset E of some bounded interval (e.g. $(-1,2)$) such that

$$0 < m(E \cap I) < |I|$$

for every subinterval $I \subset (-1,2)$ of positive length. Since the union $\bigcup_{i \in \mathbf{Z}} 3i + E$ of the translates then obeys the given condition on $\mathbf{R}$. Let K be the compact set of (1). By Exercise 28, we see that

$$(-1,2)\backslash K := I_0 \backslash K_0 = \bigcup_{n=1}^{\infty} I_n^{(1)}$$

is a countable union of disjoint open intervals. For each n, there exists a compact set $K_n^{(1)} \subset I_n^{(1)}$ of positive measure that obeys the condition of (1), and similarly,

$$\bigcup_{n=1}^{\infty} I_n^{(1)} \backslash K_n^{(1)} = \bigcup_{n=1}^{\infty} I_n^{(2)}$$

is a countable union of disjoint open intervals. For each n, there exists a compact set $K_n^{(2)} \subset I_n^{(2)}$ of positive measure that obeys the condition of (1), so we get $\bigcup_{n=1}^{\infty} I_n^{(3)} \ldots$ and so on. Let us denote

$$K_1 := \bigcup_{n=1}^{\infty} K_n^{(1)},\ K_2 := \bigcup_{n=1}^{\infty} K_n^{(2)}, \ldots, K_j := \bigcup_{n=1}^{\infty} K_n^{(j)},$$

and set $E := \bigcup_{j=0}^{\infty} K_j$. Let $I \subset (-1,2)$ be an interval. By construction, there exists $j > 0$ such that

$$|I_n^{(j)}| < |I|, \text{ for all } n.$$

Thus, $I \not\subset I_n^{(j)}$ for any n, so if $I \cap K_j = \emptyset$, we must have $x, y \in I,\ x \neq y$, and

$$x \in I_m^{(j)},\ y \in I_n^{(j)}$$

for some $m \neq n$. Assume that $x < y$, then the interval (x,y) must contain some of the compact sets of our constructions. By the connectedness of $I,\ (x,y) \subset I$, and the claim follows.

$\square$

Remark: The argument in part (2) of Exercise 50 is analogous to dropping increasingly smaller stones into a glass jar to fill it up, and take the limit of that process. Then, as one fixes a resolution (no matter how fine it is) on a microscope, and take a look at any portion of the jar through it, he will find both stones as well as empty spaces between these stones.

Exercise 51

Proof. (1). Substituting $t = 1$, we see the base kernel of the heat kernels is

$$P(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}.$$

P is measurable since it is continuous, and clearly P is non-negative, radial, and radially non-increasing as well. By Fubini's theorem,

$$\int_{\mathbf{R}^d} P(x)\ dx = \frac{1}{(4\pi)^{d/2}} \int_{\mathbf{R}^d} \prod_{i=1}^{d} e^{-x_i^2}\ dx = \frac{1}{(4\pi)^{d/2}} \prod_{i=1}^{d} \int_{\mathbf{R}} e^{-x_i^2/4}\ dx_i = 1$$

by the change of variable $u_i = x_i/2$. Likewise, rescaling $t = 1$, we get the base kernel of the Poisson kernels

$$P(x) = c_d \frac{1}{(1 + |x|^2)^{(d+1)/2}}.$$

As before, P is measurable since it is continuous. Clearly P is also non-negative, radial, and radially non-increasing. Let $I_d = \int_{\mathbf{R}^d} \frac{1}{(1+|x|^2)^{(d+1)/2}}\ dx$, we declare c_d by fiat by $c_d := \frac{1}{I_d}$ to normalize the mass of the Poisson kernel.

(2). We compare P with such "horizontal wedding cake" function as

$$f(x) = \sum_{n=-\infty}^{\infty} 1_{2^{n-1} < |x| \le 2^n} \tilde{P}(2^n).$$

By Exercise 10 in the prologue, the annuli $\{x : 2^{n-1} < |x| \le 2^n\}$ has measure

$$m(B(0, 2^n)) - m(B(0, 2^{n-1})) = k_d (2^n)^d - k_d (2^{n-1})^d = k_d 2^{nd} (1 - 2^{-d})$$

for some constant $(\frac{2}{\sqrt{d}})^d \le k_d \le 2^d$, and are all disjoint. By the monotone convergence theorem and monotonicity, we see that

$$\int_{\mathbf{R}^d} f(x)\ dx = k_d (1 - 2^{-d}) \sum_{n=-\infty}^{\infty} 2^{nd} \tilde{P}(2^n) \le \int_{\mathbf{R}^d} P(x)\ dx = 1,$$

which implies half of the bounds

$$\sum_{n=-\infty}^{\infty} 2^{nd} \tilde{P}(2^n) \le C_d := \frac{1}{k_d (1 - 2^{-d})}.$$

Similarly, by comparing P with the dilated function $f(x/2)$, we obtain that

$$\int_{\mathbf{R}^d} f(x/2)\ dx = k_d(2^d - 1) \sum_{n=-\infty}^{\infty} 2^{nd} \tilde{P}(2^n) \geq \int_{\mathbf{R}^d} P(x)\ dx = 1,$$

which gives the other half of the bounds

$$\sum_{n=-\infty}^{\infty} 2^{nd} \tilde{P}(2^n) \geq c_d := \frac{1}{k_d(2^d - 1)}.$$

Combine these inequalities, we get the pointwise upper and lower bounds:

$$c_d \leq \sum_{n=-\infty}^{\infty} 2^{nd} \tilde{P}(2^n) \leq C_d$$

for constants $0 < c_d < C_d$ (replace c_d with cc_d for some $0 < c < 1$ to get a strict lower bound) depending only on d, as desired.

(3). The aim is to show that

$$\forall x \in \mathbf{R}^d, |f * P_t(x)| \leq C'_d Mf(x),$$

where $Mf(x)$ is the Hardy-Littlewood maximal function of f, and $C'_d > 0$ depends only on d. On the annuli $A_n := \{y : 2^n < |\frac{x-y}{t}| \leq 2^{n+1}\}$, we have

$$P(\frac{x-y}{t}) \leq \tilde{P}(2^n).$$

Hence, by the triangle inequality and the DCT, it follows that

$$|\int_{\mathbf{R}^d} f(y) P_t(x-y)\ dy| \leq t^{-d} \sum_{n \in \mathbf{Z}} \int_{A_n} |f(y)| \tilde{P}(2^n)\ dy$$

The integral $\int_{A_n} |f(y)|\ dy$ is bounded by $\int_{B(x,t2^{n+1})} |f(y)|\ dy$, or

$$\int_{B(x,t2^{n+1})} |f(y)|\ dy/|B(x,t2^{n+1})| \cdot |B(x,t2^{n+1})|.$$

Use the definition of $Mf(x)$, we thus obtain that

$$|\int_{\mathbf{R}^d} f(y) P_t(x-y)\ dy| \leq t^{-d} \sum_{n \in \mathbf{Z}} Mf(x) |B(x,t2^{n+1})| \tilde{P}(2^n).$$

Inserting the formula for the volume of a ball, we then get

$$|\int_{\mathbf{R}^d} f(y) P_t(x-y)\ dy| \leq 2^d \sum_{n \in \mathbf{Z}} 2^{dn} \tilde{P}(2^n) Mf(x).$$

Finally, we have from (2) that

$$\sum_{n\in\mathbf{Z}} 2^{dn}\tilde{P}(2^n) \le C_d$$

for some $C_d > 0$, insert this into the above inequality, we conclude that

$$|f * P_t(x)| \le C_d' Mf(x),\ C_d' := 2^d C_d > 0,$$

as desired.

Alternatively, by Exercise 20 and 21 of note 1, and Exercise 36 of note 3,

$$\frac{1}{t^d}\int_{\mathbf{R}^d} P(\frac{x-y}{t})\ dt = 1.$$

In particular, we see that $P(\frac{x-y}{t}) \le t^d/|B(x,rt)|$ for all $y \in B(x,rt)$, therefore,

$$\int_{B(x,rt)} |f(y)|P_t(x-y)\ dy \le \frac{1}{|B(x,rt)|}\int_{B(x,rt)} |f(y)|\ dy$$

for every $r > 0$. By definition, the RHS is bounded by $Mf(x)$, sending $r \to \infty$, the claim follows from the DCT.

(4). Split $f(y)$ as $f(x) + f(y) - f(x)$, then correspondingly we get

$$f * P_t(x) = f(x)\int_{\mathbf{R}^d} P_t(x-y)\ dy + \int_{\mathbf{R}^d} (f(y) - f(x))P_t(x-y)\ dx$$

The first term on the RHS evaluates to $f(x)$, so we show the second term vanishes as $t \to 0$. By (2), we see that

$$\int_{B(x,rt)} |f(y) - f(x)|P_t(x-y)\ dy \le \frac{1}{|B(x,rt)|}\int_{B(x,rt)} |f(y) - f(x)|\ dy$$

for any fixed $r > 0$. Since x is a Lebesgue point of f, Lebesgue differentiation theorem implies that the RHS vanishes as $t \to 0$, and we get the claim by the triangle inequality.

□

Exercise 52

Proof. (1). By the Weierstrass M-test, the series of functions is uniformly convergent, and since each individual function $4n\cos(16^n\pi x)$ is continuous, the function F is also continuous. Clearly F is bounded by $\sum_{n=1}^{\infty} 4^{-n}$, and is absolutely convergent for every x.

(2). By definition of the Weierstrass function, we have

$$|F(\frac{j+1}{16^m}) - F(\frac{j}{16^m})| = |\sum_{n=1}^{\infty} 4^{-n}[\cos(16^{n-m}(j+1)\pi) - \cos(16^{n-m}j\pi)]|,$$

which vanishes for $n > m$, and the expression above becomes

$$|\sum_{n<m} 4^{-n}[\cos(\frac{(j+1)\pi}{16^{m-n}}) - \cos(\frac{j\pi}{16^{m-n}})] + 4^{-m}[\cos((j+1)\pi) - \cos(j\pi)]|.$$

Denote $a_n := \cos(\frac{(j+1)\pi}{16^{m-n}}) - \cos(\frac{j\pi}{16^{m-n}})$. By the mean value theorem,

$$|\cos(x) - \cos(y)| \leq |x - y|$$

for any real numbers x and y. Hence we see that

$$|\sum_{n<m} 4^{-n} a_n| \leq \sum_{n<m} 4^{-n} \frac{\pi}{16^{m-n}} = \sum_{n<m} \frac{\pi}{4^{2m-n}} = \frac{\pi}{4^{2m}} \sum_{n<m} 4^n,$$

which is less than $\pi/3 \cdot 4^{-m}$ by the geometric series formula. Also,

$$|4^{-m}[\cos((j+1)\pi) - \cos(j\pi)]| = 2 \cdot 4^{-m}$$

Combine these estimates, we obtain via the triangle inequality that

$$|F(\frac{j+1}{16^m}) - F(\frac{j}{16^m})| \geq (2 - \frac{\pi}{3})4^{-m},$$

and the claim follows.

(3). By (2), we see that for every $m \geq 1$,

$$\frac{|F(\frac{j+1}{16^m}) - F(\frac{j}{16^m})|}{1/16^m} \geq c4^{-m}/16^{-m} = c4^m.$$

For every $x \in \mathbf{R}$ and every $m \geq 1$, $\exists j \in \mathbf{Z}$ such that

$$j \leq 16^m x \leq j+1.$$

Hence for every $m \geq 1$, we have

$$|\frac{F(\frac{j+1}{16^m}) - F(x)}{1/16^m} + \frac{F(x) - F(\frac{j}{16^m})}{1/16^m}| \geq c4^m.$$

Without loss of generality, this implies that

$$|\frac{F(x_m) - F(x)}{x_m - x}| \geq \frac{1}{2} \cdot 4^m$$

where $x_m := (j+1)/16^m$. But x_m can be made arbitrarily close to x for large m, while the RHS far exceeds $|F'(x)|$, a contradiction.

Note that it is not enough to formally differentiate the series term by term and observe that the resulting series is divergent, since we can have differentiable functions with a divergent series of derivatives. $\square$

Exercise 54

Proof. Let $F : \mathbf{R} \to \mathbf{R}$ be a monotone function, and $U \subset \mathbf{R}$ be an open set. By Exercise 28, $U = \bigcup_{n=1}^{\infty} I_n$, where the I_n are intervals. By monotonicity of F, $F^{-1}(I)$ is an interval for any interval $I \subset \mathbf{R}$, which is measurable. Consequently, F is measurable by Exercise 7 of note 2. $\square$

Exercise 55

Proof. Let $F : \mathbf{R} \to \mathbf{R}$ be monotone non-decreasing, which is measurable by Exercise 54. By definition,

$$\overline{D^+}F(x) := \inf_{\delta>0} \sup_{h:h\in(0,\delta)} \frac{F(x+h)-F(x)}{h}.$$

The two free variables δ and h can be handled separately. By the property of infimum,

$$\overline{D^+}F(x) = \lim_{n\to\infty} \sup_{h:h\in(0,1/n)} \frac{F(x+h)-F(x)}{h}.$$

We claim that $\forall x \in \mathbf{R}$,

$$\overline{D^+}F(x) = \lim_{n\to\infty} \sup_{h:h\in\mathbf{Q}\cap(0,1/n)} \frac{F(x+h)-F(x)}{h}. \tag{1}$$

Indeed, fix an $h \in (0, 1/n)$. Since $\mathbf{Q} \cap (0, 1/n)$ is dense in $(0, 1/n)$, there is a monotone sequence $(h_k)_k$ of rationals in $(0, 1/n)$ with $\lim_k h_k = h$. From this, we get

$$\lim_{k\to\infty} F(x+h_k) = F(x+h)$$

by the fact that bounded monotone sequence converges. In particular, we see that

$$\lim_{k\to\infty} \frac{F(x+h_k)-F(x)}{h_k} = \frac{F(x+h)-F(x)}{h},$$

which in turns gives (1). Now, if we define $F_n : \mathbf{R} \to \mathbf{R}$ by

$$F_n(x) := \sup_{h: h \in \mathbf{Q} \cap (0,1/n)} \frac{F(x+h) - F(x)}{h},$$

then F_n is measurable by Exercise 3 of note 2. Finally, as

$$\overline{D^+} F(x) = \lim_{n \to \infty} F_n(x)$$

is the pointwise limit of measurable functions, it is itself measurable. The measurability of the other three Dini derivatives follows similarly.

Alternatively, since F is monotone, there are at most countably many discontinuities of F, which is a null set. In view of Exercise 3 of note 2, we can assume that F is continuous everywhere. Then, one obtains (1) by the fact that the function

$$G_x(h) := \frac{F(x+h) - F(x)}{h},$$

as a composition of continuous functions, is continuous on $(0, \delta)$. $\square$

Exercise 58

Proof. Let $F : [a, b] \to \mathbf{R}$ be a discontinuous monotone non-decreasing function, and let $\lambda > 0$. It suffices to prove the claim for $\overline{D^+}F$; by reflection (replacing $F(x)$ with $-F(-x)$, and $[a, b]$ with $[-b, -a]$), the same argument works for $\overline{D^-}F$, and then this trivially implies the same inequalities for $\underline{D^+}F$ and $\underline{D^-}F$. By modifying λ by an epsilon, and dropping the endpoints from [a,b] as they have measure zero, it suffices to show that

$$m(\{x \in (a, b) : \overline{D^+}F(x) > \lambda\}) \leq C \frac{F(b) - F(a)}{\lambda}$$

for some absolute constant $C > 0$. By inner regularity, it suffices to show that

$$m(K) \leq C \frac{F(b) - F(a)}{\lambda}$$

whenever K is a compact set that is contained in $\{x \in (a, b) : \overline{D^+}F(x) > \lambda\}$. By construction and the openness of (a, b), for every $x \in K$, there exists $h_x > 0$ such that

$$\frac{F(x + h_x) - F(x)}{h_x} > \lambda, \ (x - h_x, x + h_x) \in (a, b). \tag{2}$$

We create an epsilon of room here and enlarge each $(x, x+h_x)$ by a tiny amount to generate an open cover without degrading the lower bound on the difference quotient by too much, and then take limits at the end of the argument to remove the loss: Let $0 < \varepsilon < 1$ be arbitrary. The intervals $(x - \varepsilon h_x, x + h_x)$ cover K. By compactness of K, we can cover K by a finite number $I_1, \ldots, I_n$ of such intervals. Applying the Vitali-type covering lemma, we can find a subcollection $I'_1, \ldots, I'_m$ of disjoint intervals such that

$$m(\bigcup_{i=1}^{n} I_i) \leq 3 \sum_{j=1}^{m} |I'_j|.$$

By (2), on each interval I'_j, with $x_j \in I'_j$ we have

$$|I'_j| = (1+\varepsilon) h_{x_j} < (1+\varepsilon) \frac{F(x_j + h_{x_j}) - F(x_j)}{\lambda};$$

summing in j and using the disjointness of the I'_j we conclude that

$$m(\bigcup_{i=1}^{n} I_i) < 3(1+\varepsilon) \frac{F(b) - F(a)}{\lambda}.$$

Sending $\varepsilon \to 0$ gives the claim for $C = 3$. □

Remark on the jump function: From the absolute convergence of the c_n,

$$|F(x) - \sum_{n=1}^{N} c_n J_n(x)| = |\sum_{n=N+1}^{\infty} c_n J_n(x)| \leq \sum_{N+1}^{\infty} c_n$$

which can be made arbitrarily small independent of x by letting N large, so every jump function is the uniform limit of p.c jump functions.

Since uniform limits preserve continuity (c.f. Theorem 3.3.1 of "Analysis II"), F is continuous outside of the x_n, on which F is clearly discontinuous. Hence the points of discontinuity of a jump function $\sum_{n=1}^{\infty} c_n J_n$ are precisely those of the individual summands $c_n J_n$.

Exercise 64

Proof. Suppose we have another decomposition $F = F'_c + F'_{pp}$ into continuous F'_c and jump component F'_{pp}. Then

$$F_c - F'_c = F_{pp} - F'_{pp}.$$

By construction, the LHS is continuous everywhere, so both F_{pp} and F'_{pp} must jump at exactly where F jumps, and vice versa. This implies that the RHS is constant between any two points of A, and so it must be that $F_{pp} - F'_{pp} = c$ for some $c \in \mathbf{R}$.

Here is the subtle point: By our analysis above, $F_{pp} - F'_{pp}$ is a jump function that equals a constant c. It forces $c = 0$ since the only constant jump function on the real line is the zero function, which is the "stay-on-the-ground" member in the family of jump functions, the claim thus follows. □

Exercise 65

Proof. Let $F : \mathbf{R} \to \mathbf{R}$ be a monotone non-decreasing function, and A the set of discontinuities of F, which is at most countable. We define a locally jump function to be a function that is a jump function on any compact interval $[a, b]$, and claim that F can be expressed as the sum of a continuous monotone non-decreasing function F_c and a locally jump function F_{pp}.

For each $x \in A$, we define the jump $c_x := F_+(x) - F_-(x) > 0$, and the fraction $\theta_x := \frac{F(x)-F_-(x)}{F_+(x)-F_-(x)} \in [0, 1]$. Thus

$$F_+(x) = F_-(x) + c_x \text{ and } F(x) = F_-(x) + \theta_x c_x.$$

Note that c_x is the measure of the interval $(F_-(x), F_+(x))$. By monotonicity, these intervals are disjoint. Since F is bounded on $[a, b]$, the union of these intervals for $x \in A \cap [a, b]$ is bounded. By countable additivity, we thus have $\sum_{x \in A \cap [a,b]} c_x < \infty$, and so if we let J_x be the basic jump function with point of discontinuity x and fraction θ_x, then the function

$$F_{pp} := \sum_{x \in A} c_x J_x$$

is a locally jump function. F is discontinuous only at A, and for each $x \in A$ one easily checks that

$$(F_{pp})_+(x) = (F_{pp})_-(x) + c_x \text{ and } F_{pp}(x) = (F_{pp})_-(x) + \theta_x c_x$$

where $(F_{pp})_-(x) := \lim_{y \to x^-} F_{pp}(y)$, and $(F_{pp})_+(x) := \lim_{y \to x^+} F_{pp}(y)$. We thus see that the difference $F_c := F - F_{pp}$ is continuous. Finally, we verify that F_c is monotone non-decreasing. By continuity it suffices to verify this away from the (countably many) jump discontinuities, thus we need

$$F_{pp}(b) - F_{pp}(a) \le F(b) - F(a)$$

for all $a < b$ that are not jump discontinuities. But the LHS can be rewritten as $\sum_{x \in A \cap [a,b]} c_x$, while the RHS is $F_-(b) - F_+(a)$. As each c_x is the measure of the interval $(F_-(x), F_+(x))$, and these intervals for $x \in A \cap [a, b]$ are disjoint and lie in $(F_+(a), F_-(b))$, the claim follows from countable additivity. □

Exercise 67

Proof. Replacing F with $-F$, it suffices to establish the claim for monotone non-decreasing function F. For any finite sequence $a \leq x_0 < \ldots < x_n \leq b$, monotonicity gives

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(x_n) - F(x_0) \leq F(b) - F(a),$$

so $\|F\|_{TV([a,b])} \leq |F(b) - F(a)|$. Pick the sequence $x_0 = a, x_1 = b$, we see that

$$|F(b) - F(a)| \leq \|F\|_{TV([a,b])},$$

thus the two quantities are equal. $\square$

Exercise 68

Proof. The triangle property comes from the triangle inequality $|F + G| \leq |F| + |G|$. The homogeneity property comes from the fact that $|cF(x) - cF(y)| = |c||F(x) - F(y)|$ for any $x, y \in \mathbf{R}$. If the total variation $\|F\|_{TV} = 0$, and $\exists x < y \in \mathbf{R}$ such that $F(x) \neq F(y)$, then $\|F\|_{TV} \geq |F(y) - F(x)| > 0$, a contradiction. Conversely, if $F = c$ for some constant c, then $\|F\|_{TV} = 0$ by definition. $\square$

Exercise 69

Proof. For any sequences $a \leq x_0 < \ldots < x_n \leq b$ and $b \leq y_0 < \ldots < y_m \leq c$, we can combine them to make a sequence $a \leq x_0 < \ldots < x_p \leq c$, where $p = n + m$ if $x_n = y_0 = b$ and $p = n + m + 1$ otherwise. It follows that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{j=1}^{m} |F(x_j) - F(x_{j-1})| \leq \sum_{k=1}^{p} |F(x_k) - F(x_{k-1})|$$

On taking the sup over the sequence x_i, followed by taking the sup over the sequence y_j, we observe that

$$\|F\|_{TV([a,b])} + \|F\|_{TV([b,c])} \leq \|F\|_{TV([a,c])}.$$

Conversely, for any sequence $a \leq z_0 < \ldots < z_r \leq c$, let s be the largest index i such that $z_i \leq b$. By adding the point b to the pair (z_s, z_{s+1}) (or add nothing if $z_s = b$), and using the triangle inequality

$$|F(z_s) - F(z_{s-1})| \leq |F(b) - F(z_s)| + |F(z_{s+1}) - F(b)|,$$

we get two sequences $a \le z_0 < \ldots < z_n = b$ and $b < z_{n+1} < \ldots < z_t \le c$; where $n = s$ and $t = r$ if $z_s = b$, and $n = s + 1$ and $t = r + 1$ if $z_s < b$, such that

$$\sum_{k=1}^{r} |F(z_k) - F(z_{k-1})| \le \sum_{k=1}^{n} |F(z_k) - F(z_{k-1})| + \sum_{k=n+2}^{t} |F(z_k) - F(z_{k-1})|.$$

Taking the sup over the sequence z_k, we see that

$$\|F\|_{TV([a,c])} \le \|F\|_{TV([a,b])} + \|F\|_{TV([b,c])},$$

and we are done. □

Exercise 70

Proof. (1). Let $f : \mathbf{R} \to \mathbf{R}$ be a function of bounded variation. Suppose for contradiction that f is unbounded, and pick any point $x_0 \in \mathbf{R}$. By condition, $\forall M > 0$, $\exists x_1 \in \mathbf{R}$ such that $|f(x_1)| > |f(x_0)| + M$. Without loss of generality, assume that $x_0 < x_1$. Then in particular,

$$\|f\|_{TV(\mathbf{R})} \ge |f(x_1) - f(x_0)| > M.$$

Since $M > 0$ is arbitrary, this implies that $\|f\|_{TV(\mathbf{R})} = +\infty$, a contradiction. As f has bounded variation, its' oscillation tends to zero as $x \to \pm\infty$. Specifically, $\forall \varepsilon > 0$, it must be that $|f(x) - f(y)| < \varepsilon$ for all sufficiently large (or small) $x < y$. As f is bounded, the limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are well-defined.

(2). Let $F : \mathbf{R} \to \mathbf{R}$ be the Weierstrass function

$$F(x) := \sum_{n=1}^{\infty} 4^{-n} \cos(16^n \pi x).$$

For any compact interval $[a, b]$, the function $F1_{[a,b]}$ is bounded, continuous, compactly supported by Exercise 52. Let $K > 0$ be arbitrary. Pick $m \ge 1$ sufficiently large such that

$$\lceil \frac{K \cdot 4^m}{c} \rceil \cdot \frac{1}{16^m} < |b - a|$$

where $c > 0$ is the absolute constant in (2) of Exercise 52. With $n := \lceil \frac{K \cdot 4^m}{c} \rceil$,

$$\sum_{a=x_0<\ldots<x_n\le b} |F(\frac{j+1}{16^m}) - F(\frac{j}{16^m})| \ge K,$$

where $x_0 := a$, and $x_j := x_{j-1} + 1/16^m$ for all $j > 1$. Hence $F1_{[a,b]}$ is not of bounded variation.

□

Exercise 71

Proof. By linearity of integration and the triangle inequality, we see that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} |\int_{[x_{i-1},x_i]} f(t)\ dt| \leq \int_{[x_0,x_n]} |f(t)|\ dt.$$

Since $f \in L^1(\mathbf{R})$, RHS is bounded by $\|f\|_{L^1(\mathbf{R})}$. Taking supremum over all finite sequences $x_0 < \ldots < x_n$, we see $\|F\|_{TV(\mathbf{R})} \leq \|f\|_{L^1(\mathbf{R})}$. Naturally, F is of bounded variation.

To show that $\|F\|_{TV(\mathbf{R})} \geq \|f\|_{L^1(\mathbf{R})}$, we use the density argument, with the continuous, compactly supported function as the dense subclass. Let $[a,b],\ a < b$ be the support of f. Express $f = f_+ - f_-$, with

$$f_+ := \max(f,0),\ \ f_- := \min(-f,0);$$

be the positive and negative parts of f. As $f_+,\ f_-$ do not change signs, we have

$$\int_{[a,b]} |f(t)|\ dt = \int_{[a,b]} f_+(t)\ dt + \int_{[a,b]} f_-(t)\ dt \leq \|F^+\|_{TV([a,b])} + \|F^-\|_{TV([a,b])},$$

where $F^+(x) := \int_{[-\infty,x]} f_+(t)\, dt$ and $F^-(x) := \int_{[-\infty,x]} f_-(t)\, dt$ are the indefinite integrals for f_+ and f_- respectively. We can show further that

$$\|F^+\|_{TV([a,b])} + \|F^-\|_{TV([a,b])} = \|F\|_{TV([a,b])}.$$

by linearity of integration. Hence we get $\|f\|_{L^1(\mathbf{R})} \leq \|F\|_{TV(\mathbf{R})}$ for continuous function of compact support.

For the quantitative estimate, we use the triangle property that for any two functions $F, F' : \mathbf{R} \to \mathbf{R}$,

$$\|F\|_{TV(\mathbf{R})} \leq \|F - F'\|_{TV(\mathbf{R})} + \|F'\|_{TV(\mathbf{R})}.$$

Let $\varepsilon > 0$ be arbitrary. By Littlewood's second principle, we can find a function $g : \mathbf{R} \to \mathbf{R}$ which is continuous and compactly supported, with

$$\int_{\mathbf{R}} |f(t) - g(t)|\ dt \leq \varepsilon.$$

Let $G(x) := \int_{[-\infty,x]} g(t)\ dt$. Applying the quantitative estimate and the dense subclass result,

$$\int_{\mathbf{R}} |g(t)|\ dt \leq \|G\|_{TV(\mathbf{R})} \leq \|G - F\|_{TV(\mathbf{R})} + \|F\|_{TV(\mathbf{R})}.$$

And thus we conclude that

$$\int_{\mathbf{R}} |g(t)|\ dt \leq \varepsilon + \|F\|_{TV(\mathbf{R})}.$$

From the triangle inequality,

$$\int_{\mathbf{R}} |f(t)|\ dt \leq \int_{\mathbf{R}} |f(t) - g(t)|\ dt + \int_{\mathbf{R}} |g(t)|\ dt \leq 2\varepsilon + \|F\|_{TV(\mathbf{R})},$$

from which we get $\|f\|_{L^1(\mathbf{R})} \leq \|F\|_{TV(\mathbf{R})}$ for all $f \in L^1(\mathbf{R})$. Consequently, we see that $\|F\|_{TV(\mathbf{R})} = \|f\|_{L^1(\mathbf{R})}$, as desired. □

Exercise 73

Proof. For a given partition $P = x_0 < \ldots < x_n \leq x$, denote the quantities

$$\sum_{i=1}^{n} \max(F(x_i) - F(x_{i-1}), 0),\ \sum_{i=1}^{n} \max(-F(x_i) + F(x_{i-1}), 0)$$

by $S^+(P)$ and $S^-(P)$ respectively. First note that

$$S^+(P) - S^-(P) = F(x_n) - F(x_0)$$

for any such partition P. i.e. Net change in height equals net rise minus net fall. Next, if $P = x_0 < \ldots < x_n \leq x$ and $P' = x'_0 < \ldots < x'_m \leq x$ are two partitions up to x, we claim that the common refinement $P\#P'$ is such that

$$S^+(P) \leq S^-(P\#P'),\ S^-(P) \leq S^-(P\#P'). \tag{1}$$

$\forall y \in P\#P'$, either $y \leq x_0,\ x_n \leq y$, or $x_{i-1} \leq y \leq x_i$ with $1 \leq i \leq n$. If an inequality is ever an equality, the claim is trivial. Otherwise, the first two cases do not decrease $S^+(P)$ and $S^-(P)$, as the intervals $[y, x_0]$ and $[x_n, y]$ contribute non-negatively to their values. When $x_{i-1} < y < x_i$ with $F(x_{i-1}) \leq F(x_i)$, either

$$F(y) \geq \max(F(x_{i-1}), F(x_i)),\ F(y) \leq \min(F(x_{i-1}), F(x_i)),$$

or $F(x_{i-1}) \leq F(y) \leq F(x_i)$. In the first two cases, both $S^+(P)$ and $S^-(P)$ increase by at least

$$\min(|F(y) - F(x_{i-1})|, |F(x_i) - F(y)|) \geq 0$$

over the interval $[x_{i-1}, x_i]$. In the third case, $S^+(P)$ and $S^-(P)$ do not change over the interval $[x_{i-1}, x_i]$. The same holds with $F(x_{i-1}) \geq F(x_i)$. Hence we obtain (1). By definition, $\exists P := x_0 < \ldots < x_n \leq x,\ P' := x'_0 < \ldots < x'_m \leq x$, with

$$S^+(P) > F^+(x) - \varepsilon,\ S^-(P) > F^-(x) - \varepsilon. \tag{2}$$

Assume w.lo.g that $x_n = x'_m = x$. By Exercise 70, exists $y \in \mathbf{R}$ such that $|F(y) - F(-\infty)| < \varepsilon$. Form the common refinement $P\#P'$, and replace the starting point of $P\#P'$ by $a := \min(x_0, y_0, y)$. By (1) and the triangle inequality,

$$|F(-\infty) + F^+(x) - F^-(x) - (F(a) + S^+(P\#P') - S^-(P\#P'))| < 3\varepsilon.$$

On the other hand, we have

$$F(a) + S^+(P\#P') - S^-(P\#P') = F(a) + F(x) - F(a) = F(x)$$

as initially shown, and we get

$$|F(-\infty) + F^+(x) - F^-(x) - F(x)| \leq 3\varepsilon.$$

Sending $\varepsilon \to 0$ gives the first identity.

Fix some $a < b$. Taking common refinement, we can find partitions

$$P_1 = x_0 < \ldots < x_n \leq a,\ P_2 = x'_0 < \ldots < x'_m \leq b$$

such that

$$S^+(P_1) > F^+(a) - \varepsilon;\ S^+(P_2) > F^+(b) - \varepsilon,$$
$$S^-(P_1) > F^-(a) - \varepsilon;\ S^-(P_2) > F^-(b) - \varepsilon.$$

Let $P_3 := x'_0 < \ldots x'_l \leq a$, where l is the largest index i with $x'_i \leq a$, and replace both P_1 and P_3 with the refinement $P_1\#P_3$. For simplicity, call the partitions P_1 and P_2 after refinement as before. Then

$$|S^+(P_2) - S^+(P_1) - (F^+(b) - F^+(a))| < 2\varepsilon,$$
$$|S^-(P_2) - S^-(P_1) - (F^-(b) - F^-(a))| < 2\varepsilon$$

by the triangle inequality. But we can write

$$S^+(P) := S^+(P_2) - S^+(P_1),\ S^-(P) := S^-(P_2) - S^-(P_1),$$

where $P = a < x'_{l+1} < \ldots < x'_m \leq b$. That is,

$$|S^+(P) - (F^+(b) - F^+(a))| < 2\varepsilon,$$
$$|S^-(P) - (F^-(b) - F^-(a))| < 2\varepsilon.$$

By the triangle inequality again, we obtain that

$$|S^+(P) + S^-(P) - (F^+(b) - F^+(a) + F^-(b) - F^-(a))| < 4\varepsilon. \tag{3}$$

Furthermore, for any partition P of $[a, b]$, we have

$$S^+(P) \leq F^+(b) - F^+(a),\ S^-(P) \leq F^-(b) - F^-(a)$$

by construction. Thus, for partition $P' = a \leq y_0 < \ldots < y_j \leq b$ obeying

$$|S^+(P') + S^-(P') - \|F\|_{TV([a,b])}| < \varepsilon \tag{4}$$

we can again take the common refinement $P\#P'$ to conclude (3) and (4) for $P\#P'$. Combining these estimates, we obtain

$$|\|F\|_{TV([a,b])} - (F^+(b) - F^+(a) + F^-(b) - F^-(a))| < 5\varepsilon$$

by the triangle inequality. Sending $\varepsilon \to 0$ gives the second identity.

Finally, we can find $x > 0$ sufficiently large with

$$|F^+(x) - F^+(+\infty)|,\ |F^-(x) - F^-(+\infty)| < \varepsilon.$$

Taking common refinement, there exist a partition $P = x_0 < \ldots < x_n \leq x$ with

$$S^+(P) > F^+(x) - \varepsilon,\ S^-(P) > F^-(x) - \varepsilon.$$

Taking common refinement again if necessary, we may assume P to be such that

$$S^+(P) + S^-(P) > \|F\|_{TV} - \varepsilon.$$

This implies by the triangle inequality that

$$|F^+(+\infty) + F^-(+\infty) - \|F\|_{TV}| < 5\varepsilon.$$

Likewise, sending $\varepsilon \to 0$ gives the third identity. □

Exercise 75

Proof. Let $F : \mathbf{R} \to \mathbf{R}$ be locally of bounded variation. By the BV differentiation theorem, the functions $F1_{(n-1,n]}$ are differentiable a.e for every integer n. Since $F = \sum_{n \in \mathbf{Z}} F1_{(n-1,n]}$ and countable union of null sets is again a null set, F is differentiable a.e. □

Exercise 76

Proof. Suppose that $F : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous, and $[a, b]$ a compact interval. By condition, for any partition $a \leq x_0 < \ldots < x_n \leq b$, we have

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq C \sum_{i=1}^{n} |x_i - x_{i-1}| \leq C|b - a|,$$

and so $\|F\|_{TV([a,b])} \leq C|b-a| < \infty$. i.e. F is locally of bounded variation, and hence differentiable a.e. Denote by C the Lipschitz constant of F. If the derivative of F exists at x, then

$$|F'(x)| := |\lim_{y \to x} \frac{F(y) - F(x)}{y - x}| \leq \lim_{y \to x} \frac{C|y-x|}{|y-x|} \leq C.$$

The magnitude of the derivative is bounded by the Lipschitz constant. □

Exercise 78

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be a convex function, and $x \in \mathbf{R}$. Fix $y > x$, and let $z := (1-t)x + ty$. By convexity,

$$|f(z) - f(x)| \leq |(1-t)f(x) + tf(y) - f(x)| = t|f(x) - f(y)|.$$

As $t \to 0^+, z \to x^+$, hence $\lim_{z \to x^+} f(z) = f(x)$. Similarly, fix $y < x$, and let $z' := (1-t)y + tx$, then we have

$$|f(z) - f(x)| \leq |(1-t)f(y) + tf(x) - f(x)| = (1-t)|f(x) - f(y)|.$$

As $t \to 1^-$, $f(z') \to x^-$, thus $\lim_{z' \to x^-} f(z') = f(x)$. Combine the left and right limits, we conclude that f is continuous at x for every $x \in \mathbf{R}$.

$\forall a < x < b$, $x = (1-t)a + tb$ for some $0 < t < 1$. For convenience, denote by $S(L(x,y))$ the slope of the line connecting $(x, f(x))$ and $(y, f(y))$, By convexity, one can show that

$$S(L(a,x)) \leq S(L(a,b)) \leq S(L(x,b)).$$

Let $D^+(x) := \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$, $D^-(x) := \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ be the right and left derivatives at x. From the above inequality, and that bounded monotone sequence converges, we conclude that

$$D^-(a) \leq D^+(a) \leq D^-(b) \leq D^+(b)$$

for all $a < b$. Thus, if $q_x \in \mathbf{Q}$ is such that $D^-(x) < q_x < D^+(x)$, then the q_x are distinct. By the countable nature of the rationals, we see that f is differentiable almost everywhere, and that its derivative f' is equal almost everywhere to a monotone non-decreasing function. □

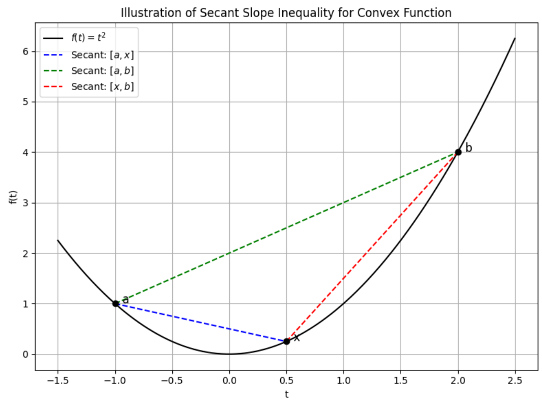

Figure 2: secant lines of a convex function

Exercise 80

Proof. Let $f : \mathbf{R} \to \mathbf{R}$ be of bounded variation. By the BV differentiation theorem, f is differentiable almost everywhere. By proposition 72, $f = g - h$, where g and h are bounded monotone non-decreasing functions, and $f' = g' - h'$ almost everywhere. By proposition 79, on any compact interval $[a, b]$,

$$\int_{[a,b]} |f'(x)|\ dx \leq \int_{[a,b]} g'(x)\ dx + \int_{[a,b]} h'(x)\ dx \leq g(b) - g(a) + h(b) - h(a)$$

Both g and h are of bounded variation, so

$$\int_{\mathbf{R}} |f'(x)|\ dx = \sum_{n \in \mathbf{Z}} \int_{[n,n+1]} |f'(x)|\ dx \leq \|g\|_{TV} + \|h\|_{TV} < \infty.$$

That is, the a.e defined f' is absolutely integrable. $\square$

Exercise 81

Proof. We extend F to all of $\mathbf{R}$ by periodic translation, then F is now a Lipschitz function on $\mathbf{R}$. As F is almost everywhere differentiable by Exercise 76, the Newton quotients

$$f_n(x) := \frac{F(x + 1/n) - F(x)}{1/n}$$

converge pointwise almost everywhere to F'. In particular, the truncations $f_n 1_{[a,b]}$ converge pointwise almost everywhere to $F' 1_{[a,b]}$. Applying dominated convergence theorem (with the dominating function $C1_{[a,b]}$, where $C > 0$ is the Lipschitz constant), we conclude that

$$\int_{[a,b]} F'(x)\ dx = \lim_{n\to\infty} \int_{[a,b]} \frac{F(x+1/n) - F(x)}{1/n}\ dx.$$

The right-hand side can be rearranged as

$$\lim_{n\to\infty} n(\int_{[a+1/n,b+1/n]} F(y)\ dy - \int_{[a,b]} F(x)\ dx)$$

which can be rearranged further as

$$\lim_{n\to\infty} n(\int_{[b,b+1/n]} F(x)\ dx - \int_{[a,a+1/n]} F(x)\ dx).$$

Since F is at most $F(b) + C/n$ on $[b, b+1/n]$ and and least $F(a) - C/n$ on $[a, a+1/n]$, the above expression is at most

$$\lim_{n\to\infty} F(b) - F(a) + \frac{2C}{n} = F(b) - F(a).$$

Similarly, as F is at least $F(b) - C/n$ on $[b, b+1/n]$ and at most $F(a) + C/n$ on $[a, a+1/n]$, the above expression is at least

$$\lim_{n\to\infty} F(b) - F(a) - \frac{2C}{n} = F(b) - F(a).$$

Therefore, the above expression is exactly $F(b) - F(a)$, and we are done. □

Exercise 82

Proof. Let $F, G : [a,b] \to \mathbf{R}$ be Lipschitz continuous functions, with Lipschitz constants $C, D > 0$ respectively. Since Lipschitz functions are continuous, and continuous functions on a compact interval are bounded, we have $|F| \leq M$ and $|G| \leq N$ for some constants M, N. Now, $\forall x, y \in [a,b]$, by the triangle inequality,

$$\begin{aligned}
|F(x)G(x) - F(y)G(y)| &\leq |F(x)G(x) - F(y)G(x)| + |F(y)G(x) - F(y)G(y)| \\
&\leq |G(x)||F(x) - F(y)| + |F(y)||G(x) - G(y)| \\
&\leq NC|x-y| + MD|x-y| \\
&= (NC + MD)|x-y|.
\end{aligned}$$

Hence the product $FG : [a,b] \to \mathbf{R}$ is Lipschitz continuous. By linearity of integration, and Exercise 81, we thus obtain

$$\begin{aligned}
\int_{[a,b]} F'(x)G(x)\ dx &= \int_{[a,b]} (FG)'(x) - G'(x)F(x)\ dx \\
&= F(b)G(b) - F(a)G(a) - \int_{[a,b]} G'(x)F(x)\ dx,
\end{aligned}$$

which is the integration by parts formula. □

Exercise 83

Proof. We argue by density, with the p.c jump functions as the dense subclass. For p.c jump functions, the claim is clear (indeed, the derivative vanishes outside of finitely many discontinuities). Now, let F be a jump function, and $a < b$. As every jump function is the uniform limit of p.c jump functions, we can find a p.c jump function F_ε such that $|F(x) - F_\varepsilon(x)| \leq \varepsilon$ for all x. Indeed, by taking F_ε to be a partial sum of the basic jump functions that make up F, we can ensure that $F - F_\varepsilon$ is also a monotone non-decreasing function. Applying proposition 79, we have

$$\int_{[a,b]} F'(x) - F'_\varepsilon(x)\ dx \leq F(b) - F_\varepsilon(b) - (F(a) + F_\varepsilon(a)),$$

which implies that

$$\int_{[a,b]} F'(x)\ dx \leq 2\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we see that $\int_{[a,b]} F'(x)\ dx = 0$. Clearly $F' \geq 0$ since F is monotone non-decreasing. Thus by the vanishing property of the unsigned integral, $F' 1_{[a,b]} = 0$ almost everywhere. As countable union of null set is again a null set, this extends to the real line, and we conclude that $F' = 0$ almost everywhere. $\square$

Exercise 84

Proof. (1).

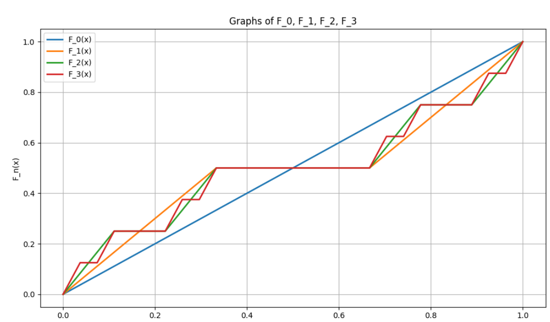

Figure 3: Cantor function construction

(2). We induct on n. By definition, the claim is true for $n = 0$. Suppose the claim holds for some $n-1,\ n > 0$. On each of $[0, 1/3], (1/3, 2/3), [2/3, 1]$, F_n either is constant or is the composition of continuous functions, thus is itself continuous. By definition, $F_n(0) := F_{n-1}(0) = \ldots = F_0(0) = 0$, and similarly $F_n(1) := \frac{1}{2} + \frac{1}{2}F_{n-1}(1) = \ldots = \frac{1}{2} + \frac{1}{2}F_0(1) = 1$. As such,

$$\lim_{x \to \frac{1}{3}^-} F_n(x) := \lim_{x \to \frac{1}{3}^-} \frac{1}{2}F_{n-1}(3x) = \frac{1}{2} = \lim_{x \to \frac{1}{3}^+} F_n(x),$$

and

$$\lim_{x \to \frac{2}{3}^-} F_n(x) := \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \lim_{x \to \frac{2}{3}^+} F_{n-1}(3x-2) = \lim_{x \to \frac{2}{3}^+} F_n(x).$$

Hence, F_n is continuous on $[0, 1]$. It remains to show that F_n is monotone non-decreasing. Let $0 \leq x < y \leq 1$. If x, y both belong to one of the three piecewise intervals, then $F_n(x) \leq F_n(y)$ by the inductive hypothesis. For $x \in [0, 1/3]$ and $y > 1/3$, by the inductive hypothesis,

$$F_n(x) \leq \frac{1}{2}F_{n-1}(3 \cdot \frac{1}{3}) = \frac{1}{2},$$

while

$$F_n(y) := \begin{cases} \frac{1}{2} & \text{if } y \in (1/3, 2/3); \\ \frac{1}{2} + \frac{1}{2}F_{n-1}(3y-2) & \text{if } y \in [2/3, 1] \end{cases}$$

In both cases, we have $F_n(x) \leq F_n(y)$. The same argument gives $F_n(x) \leq F_n(y)$ when $x \in (1/3, 2/3)$ and $y \geq 2/3$. We conclude that F_n is monotone non-decreasing.

(3). Again, we induct on n. By direct computation, $|F_1(x) - F_0(x)| \leq (1/2)^0$, which is the base case for $n = 0$. Suppose inductively that the claim holds for some $n - 1 \geq 0$. When $x \in [0, 1/3]$,

$$|F_{n+1}(x) - F_n(x)| = \frac{1}{2}|F_n(3x) - F_{n-1}(3x)| \leq \frac{1}{2} \cdot (\frac{1}{2})^{n-1} = (\frac{1}{2})^n.$$

When $x \in (1/3, 2/3)$, $|F_{n+1}(x) - F_n(x)| = 0 \leq (\frac{1}{2})^n$. Finally, when $x \in [2/3, 1]$,

$$|F_{n+1}(x) - F_n(x)| = \frac{1}{2}|F_n(3x-2) - F_{n-1}(3x-2)| \leq \frac{1}{2} \cdot (\frac{1}{2})^{n-1} = (\frac{1}{2})^n.$$

Hence we get $|F_{n+1}(x) - F_n(x)| \leq 2^{-n}$ for all $x \in [0, 1]$, establishing the claim for n. By the triangle inequality, and telescoping series, $\forall j < k$,

$$|F_j(x) - F_k(x)| \leq \sum_{i=j}^{k-1} |F_{i+1}(x) - F_i(x)| \leq (k-j)2^{-j} < \varepsilon$$

for sufficiently large j, k. Since $(\mathbf{R}, d)$ is a complete metric space, the F_n converge uniformly to a limit $F : [0, 1] \to \mathbf{R}$.

(4). As uniform limits preserve continuity, F is continuous. By properties of the limit,

$$\lim_n F_n(0) = 0, \ \lim_n F_n(1) = 1, \ \lim_n F_n(x) \leq \lim_n F_n(y)$$

whenever $0 \leq x < y \leq 1$, i.e. F is monotone non-decreasing with $F(0) = 0$ and $F(1) = 1$.

(5). With the same notations as in Exercise 10 of note 1, we show by induction that F_n is constant on $[0,1]\backslash I_n$ for $n = 1, 2, \ldots$ By computation, the claim holds for $n = 1$. Suppose it holds for some $n \geq 1$. By construction, $[0,1]\backslash I_{n+1}$ is the union of middle third intervals removed at level $n+1$. More formally,

$$[0,1]\backslash I_{n+1} = \bigcup_{a_1,\ldots,a_{n+1}\in\{0,1,2\};a_j=1 \text{ for some } j} (\sum_{i=1}^{n+1} \frac{a_i}{3^i}, \sum_{i=1}^{n+1} \frac{a_i}{3^i} + \frac{1}{3^{n+1}}).$$

If $x \in [0,1]\backslash I_{n+1}$, there are three cases. When $x \in [0, 1/3]$, then $a_1 = 0$, and

$$3x \in (\sum_{i=1}^{n} \frac{a_{i+1}}{3^i}, \sum_{i=1}^{n} \frac{a_{i+1}}{3^i} + \frac{1}{3^n}), \ a_{i+1}{=}1 \text{ for some } 1 \leq i \leq n.$$

That is, $3x \in [0,1]\backslash I_n$. By induction hypothesis, $F_{n+1}(x) = \frac{1}{2}F_n(3x)$ is thus constant. By symmetry, when $x \in [2/3, 1]$, $3x - 2 = 3(x - 2/3) \in [0,1]\backslash I_n$, and again $F_{n+1}(x) = \frac{1}{2} + \frac{1}{2}F_n(3x - 2)$ is constant. Finally, when $x \in (1/3, 2/3)$, $F_{n+1}(x) = 1/2$. Hence, F_{n+1} is constant on $[0,1]\backslash I_{n+1}$ and the claim follows.

By definition, $C := \bigcap_{n=1}^{\infty} I_n$, so if $x \in [0,1]\backslash C = \bigcup_{n=1}^{\infty}[0,1]\backslash I_n$, $x \in [0,1]\backslash I_n$ for some n, on which F_n is constant by what we have shown. By the nesting nature

$$[0,1]\backslash I_0 \subset [0,1]\backslash I_1 \subset \ldots,$$

and the fact that F is the uniform limit of the F_n, we see that F is constant on $[0,1]\backslash I_n$. By construction, I_n is a finite union of closed intervals, and thus $[0,1]\backslash I_n$ is open. By a.e equivalence and vanishing of the unsigned integral, we conclude that

$$\int_{[0,1]} F'(x)\, dx = \int_{[0,1]\backslash C} F'(x)\, dx = 0 \neq 1 = F(1) - F(0),$$

so that the second fundamental theorem of calculus fails for this function.

(6). We claim inductively that $F_n(\sum_{i=1}^{n} a_i 3^{-i}) = \sum_{i=1}^{n} \frac{a_i}{2} 2^{-i}$ for all $n \geq 1$ and $a_1, \ldots, a_n \in \{0, 2\}$. For $n = 1$, we see that

$$F_1(a_1 3^{-1}) = \begin{cases} 0 = \frac{a_1}{2}2^{-1} & \text{if } a_1 = 0; \\ \frac{1}{2} = \frac{a_1}{2}2^{-1} & \text{if } a_1 = 2 \end{cases}$$

Hence we get the base case $n = 1$. Suppose inductively that the claim holds for some $n \geq 1$. $\forall n \geq 1$, note that $\sum_{i=1}^{n} a_i 3^{-i} \in [0, 1/3]$ or $[2/3, 1]$ when $a_1, \ldots, a_n \in \{0, 2\}$. In the former case, $a_1 = 0$, and

$$F_{n+1}(\sum_{i=1}^{n+1} a_i 3^{-i}) = \frac{1}{2} F_n(\sum_{i=2}^{n+1} a_i 3^{-i+1}) = \frac{1}{2} \sum_{i=2}^{n+1} \frac{a_i}{2} 2^{-i+1} = \sum_{i=1}^{n+1} \frac{a_i}{2} 2^{-i}$$

by the induction hypothesis. Here we complete the pattern by adding a term $\frac{a_1}{2} 2^{-1}$ at the end. In the latter case, $a_1 = 2$, and

$$F_{n+1}(\sum_{i=1}^{n+1} a_i 3^{-i}) = \frac{1}{2} + \frac{1}{2} F_n(\sum_{i=2}^{n+1} a_i 3^{-i+1}) = \frac{1}{2} + \sum_{i=2}^{n+1} \frac{a_i}{2} 2^{-i} = \sum_{i=1}^{n+1} \frac{a_i}{2} 2^{-i}$$

by the induction hypothesis. Therefore, the claim holds for all $n \geq 1$. Since $F_n \to F$ uniformly, and the partial sum $\sum_{i=1}^{n} a_i 3^{-i} \to \sum_{i=1}^{\infty} a_i 3^{-i}$; as n tends to infinity, we obtain

$$|F_n(\sum_{i=1}^{n} a_i 3^{-i}) - F(\sum_{i=1}^{\infty} a_i 3^{-i})| = |\sum_{i=1}^{n} \frac{a_i}{2} 2^{-i} - F(\sum_{i=1}^{\infty} a_i 3^{-i})| < \varepsilon$$

for large n and arbitrary $\varepsilon > 0$. Sending $n \to \infty$ on the right hand side, we get

$$F(\sum_{n=1}^{\infty} a_n 3^{-n}) = \sum_{n=1}^{\infty} \frac{a_n}{2} 2^{-n}$$

as desired.

(7). That $|I| = 3^{-n}$ is obvious. By the monotone nature of F and the intermediate value theorem, $F(I)$ is an interval of length

$$|F(I)| = F(\sum_{i=1}^{n} a_i 3^{-i} + 3^{-n}) - F(\sum_{i=1}^{n} a_i 3^{-i}).$$

Using geometric series to write

$$\frac{1}{3^n} = \frac{1}{3^n}(\frac{2}{3} + \frac{2}{3^2} + \ldots) = \sum_{i=n+1}^{\infty} \frac{2}{3^i},$$

one obtains by (6) that

$$F(\sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}) = F(\sum_{i=1}^{\infty} \frac{a_i'}{3^i}) = \sum_{i=1}^{\infty} \frac{a_i'}{2} 2^{-i}, \; a_i' := \begin{cases} a_i & 1 \leq i \leq n; \\ 2 & i > n \end{cases}$$

In particular, we see that

$$F(\sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}) = \sum_{i=1}^{n} \frac{a_i}{2} 2^{-i} + \sum_{i=n+1}^{\infty} 2^{-i} = \sum_{i=1}^{n} \frac{a_i}{2} 2^{-i} + 2^{-n}.$$

Directly from (6), we have

$$F(\sum_{i=1}^{n}\frac{a_i}{3^i}) = \sum_{i=1}^{n}\frac{a_i}{2}2^{-i},$$

so it follows that $|F(I)| = 2^{-n}$.

(8). Let $x \in C = \bigcap_{n=1}^{\infty} I_n$. By definition, $\forall n \geq 1, \exists I \subset I_n$ with $x \in I$. Denoting I by $[a,b]$. From (7) and the triangle inequality, we see that

$$(\frac{3}{2})^n = |\frac{F(b)-F(a)}{b-a}| \leq |\frac{F(b)-F(x)}{b-x}| + |\frac{F(x)-F(a)}{x-a}|.$$

Sending $n \to \infty$, the one-sided derivatives at x are such that $|D^+F(x)| = +\infty$ or $|D^-F(x)| = +\infty$. Therefore, we conclude that F is not differentiable at x. $\square$

Exercise 87

Proof. (1). Let $F : \mathbf{R} \to \mathbf{R}$ be absolutely continuous, with ε and δ be as in the definition. Let $a < b$ be such that $|(a,b)| \leq \delta$. Viewed as a collection of one element, we get $|F(b) - F(a)| \leq \varepsilon$. That is, F is uniformly continuous.

(2). Let $[a,b]$ be a compact interval with $|b-a| \leq \delta$. For every increasing sequence $a \leq x_0 < \ldots < x_n \leq b$, it holds that

$$\|F\|_{TV([a,b])} := \sup_{a \leq x_0 < \ldots < x_n \leq b} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq \varepsilon < \infty$$

For a compact interval $[a,b]$ with $|b-a| > \delta$, $\exists n \in \mathbf{N}$ such that $r := |b-a|/n < \delta$. On the partition $a < a+r < \ldots < a+(n-1)r < b$, we have

$$\|F\|_{TV([a,b])} = \sum_{i=1}^{n} \|F\|_{TV([a+(i-1)r, a+ir])} \leq n\varepsilon < \infty$$

by Exercise 69. Hence, F is of bounded variation on $[a,b]$ and thus differentiable almost everywhere.

(3). Let $\varepsilon > 0$ and $F : \mathbf{R} \to \mathbf{R}$ be Lipschitz continuous, with Lipschitz constant $C > 0$. If $(a_1,b_1),\ldots,(a_n,b_n)$ is a finite collection of disjoint intervals of total length $\sum_{j=1}^{n} b_j - a_j \leq \varepsilon/C$, then

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq C\sum_{j=1}^{n} |b_j - a_j| \leq \varepsilon.$$

That is, F is absolutely continuous.

(4). $\forall C > 0,\ |\sqrt{x}| > C|x|$ whenever $0 < x < 1/C^2$, hence the function $x \mapsto \sqrt{x}$ is not Lipschitz continuous. By concavity of $\sqrt{x}$,

$$\sqrt{a+h} - \sqrt{a} \geq \sqrt{b+h} - \sqrt{b}$$

for all $a \leq b$ and $h > 0$. That is, on intervals of identical length, $\sqrt{x}$ rises more on the ones to the left. Let $\varepsilon > 0$, and $(a_j, b_j) \subset [0,1],\ 1 \leq j \leq n$ be disjoint. Suppose that the (a_j, b_j) are labeled from left to right, $a := a_1,\ b := b_n$, then:

$$\sum_{j=1}^{n} \sqrt{b_j} - \sqrt{a_j} \leq \sqrt{b} - \sqrt{a} \leq \sqrt{b-a}$$

by telescoping sum. The claim now follows by setting $\delta := \varepsilon^2$.

(5). By (4) of Exercise 84, the Cantor function F is continuous and monotone, since $[0,1]$ is compact, $F : [0,1] \to \mathbf{R}$ is also uniformly continuous. Let

$$[\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}],\ a_i \in \{0,2\}$$

be the intervals used in the n^{th} cover I_n of C, which are disjoint. By (7) of Exercise 84,

$$\sum_{a_1,\ldots,a_n \in \{0,2\}} |F(\sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}) - F(\sum_{i=1}^{n} \frac{a_i}{3^i})| = 2^n(\frac{1}{2^n}) = 1;$$

while these 2^n intervals have total length $(2/3)^n \to 0$. Hence F can not be absolutely continuous.

(6). Applying Littlewood's second principle, there is a continuous, compactly supported function $g : \mathbf{R} \to \mathbf{R}$ such that

$$\int_{\mathbf{R}} |f(x) - g(x)|\ dx \leq \frac{\varepsilon}{2}.$$

By condition, $\exists C > 0,\ |g(x)| \leq C$ for all $x \in \mathbf{R}$. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be a finite collection of disjoint intervals, then by the triangle inequality:

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq \sum_{j=1}^{n} \int_{[a_j,b_j]} |f(x) - g(x)|\ dx + \sum_{j=1}^{n} \int_{[a_j,b_j]} |g(x)|\ dx.$$

Since the (a_j, b_j) are disjoint, if $\delta \leq \frac{\varepsilon}{2C}$, we get

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

And we conclude that F is absolutely continuous. By the Lebesgue differentiation theorem, F is differentiable a.e with $F'(x) = f(x)$ for a.e x.

(7). Let $F, G : [a,b] \to \mathbf{R}$ be absolutely continuous, $(a_1, b_1), \ldots, (a_n, b_n)$ a finite collection of disjoint intervals in $[a,b]$. By condition, $\exists C > 0$ such that $|F(x)|, |G(x)| \leq C$. By the triangle inequality, we get:

$$\begin{aligned}
&\sum_{j=1}^{n} |F(b_j)G(b_j) - F(a_j)G(a_j)| \\
&\leq \sum_{j=1}^{n} |F(b_j)G(b_j) - F(a_j)G(b_j)| + |F(a_j)G(b_j) - F(a_j)G(a_j)| \\
&\leq C\sum_{j=1}^{n} |F(b_j) - F(a_j)| + C\sum_{i=1}^{n} |G(b_i) - G(a_i)|,
\end{aligned}$$

which can be made less than ε by controlling the quantity $\sum_{j=1}^{n} b_j - a_j$. i.e. the product FG is absolutely continuous. Use the triangle inequality to obtain

$$\sum_{j=1}^{n} |F(b_j) + G(b_j) - F(a_j) - G(a_j)| \leq \sum_{j=1}^{n} |F(b_j) - F(a_j)| + |G(b_j) - G(a_j)|,$$

we conclude that the sum $F + G$ is absolutely continuous. If F and G are defined on $\mathbf{R}$ instead, the sum is still absolutely continuous, but the product might not be. For a counterexample, take $F(x) = G(x) = x$, which is absolutely continuous, yet the product x^2 is not uniformly continuous, hence not absolutely continuous.

For another example, take $F(x) = x,\ G(x) = \sin x$. Both F and G are Lipschitz continuous, and thus absolutely continuous. On the intervals $[a_j, b_j] := [2j\pi, 2j\pi + 1/j]$ for $j = 1, 2, \ldots$, we have

$$|F(b_j)G(b_j) - F(a_j)G(a_j)| = (2j\pi + \frac{1}{j}) \sin(\frac{1}{j}) \to 2\pi,$$

as $j \to \infty$. Thus we conclude that the product is not absolutely continuous. $\square$

Exercise 88

Proof. (1). Let $F : \mathbf{R} \to \mathbf{R}$ be absolutely continuous, and E a null set. By outer regularity, $\forall \delta > 0$, there exists an open set $U \subset \mathbf{R}$ such that

$$E \subset U,\ m(U) \leq \delta.$$

By Exercise 28, $U = \bigcup_{j=1}^{\infty} (a_j, b_j)$, where the (a_j, b_j) are disjoint non-empty intervals. By the intermediate value theorem,

$$F([a_j, b_j]) = [F(a_j'), F(b_j')],\ a_j \leq a_j' \leq b_j' \leq b_j,$$

where $F(a_j')$ and $F(b_j')$ are the min and max of F on $[a_j, b_j]$, respectively. By construction,

$$F(E) \subset \bigcup_{j=1}^{\infty} [F(a_j'), F(b_j')].$$

F is absolutely continuous, and the (a_j', b_j') are disjoint because the (a_j, b_j) are, $\forall \varepsilon > 0$, we thus have

$$\sum_{j=1}^{\infty} F(b_j') - F(a_j') = \sup_{n \geq 1} \sum_{j=1}^{n} F(b_j') - F(a_j') \leq \varepsilon,$$

provided that $\delta > 0$ is sufficiently small. Since $\varepsilon > 0$ is arbitrary, this implies that $F(E)$ is measurable with measure zero by monotonicity.

(2).$\forall x \in [0,1]$ such that $x = \sum_{j=1}^{\infty} a_j 3^{-j}$ for digits $a_1, a_2, \ldots \in \{0, 2\}$, we show that $x \in C$. By definition, intervals used in the n^{th} cover I_n of C have the form:

$$I = [\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}],\ a_1, \ldots, a_n \in \{0, 2\}.$$

In particular, we see that $x \in I_n$ for all n since

$$x \in [\sum_{j=1}^{n} \frac{a_j}{3^j}, \sum_{j=1}^{n} \frac{a_j}{3^j} + \frac{1}{3^n}] = [\sum_{j=1}^{n} \frac{a_j}{3^j}, \sum_{j=1}^{n} \frac{a_j}{3^j} + \sum_{j=n+1}^{\infty} \frac{2}{3^j}].$$

We conclude that $x \in \bigcap_{n=1}^{\infty} I_n = C$. By (6) of Exercise 84, we see that

$$F(x) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}.$$

Running through all sequences of 0's and 2's, we see that $[0,1] \subset F(C)$. Clearly $F(C) \subset [0,1]$, and thus $F(C) = [0,1]$, which is not a null set.

Alternatively, one can argue geometrically: By (7) of Exercise 84, if I and J are intervals in I_n, then $F(I)$ and $F(J)$ are intervals of length 2^{-n}. We claim that $F(I)$ and $F(J)$ are almost disjoint. As $I \cap J = \emptyset$, we can assume that I sits to the left of J. If for contradiction that $F(I) \cap F(J)$ is not a point, then

$$F(I) \cap F(J) = K,\ K \text{ an interval}.$$

In particular, $\exists a, b \in K,\ a < b$. Since $F^{-1}(b) \in I$ and $F^{-1}(a) \in J$, we have

$$F^{-1}(b) < F^{-1}(a);$$

a contradiction to the fact that F is monotone non-decreasing. Hence, by additivity we obtain:

$$m(F(I_n)) = \sum_{I \in I_n} |F(I)| = 2^n \cdot 2^{-n} = 1.$$

By construction, $I_0 \supset I_1 \supset \ldots$, so $F(I_0) \supset F(I_1) \supset \ldots$ By downward monotone convergence, we get

$$m(F(C)) = m(\bigcap_{n=1}^{\infty} F(I_n)) = 1.$$

Thus $F(C)$ can not be a null set. $\square$

Exercise 90

Proof. Suppose that $F : [a, b] \to \mathbf{R}$ is absolutely continuous. By Exercise 80, F' exists a.e and is absolutely integrable. If we set $f := F'$, and $C := F(a)$, then by theorem 89, $\forall x \in [a, b]$, we have

$$F(x) = F(x) - F(a) + F(a) = \int_{[a,x]} f(y)\ dy + C.$$

Conversely, let $F(x) = \int_{[a,x]} f(y)\ dy + C$; $f : [a, b] \to \mathbf{R}$ absolutely integrable, and C a constant. Extending f to $\mathbf{R}$ by setting $f(y) := 0, \forall y \in \mathbf{R}\backslash[a, b]$, then by Exercise 87, F is absolutely continuous. $\square$

Exercise 91

Proof. By theorem 89, it suffices to show that $F\phi : [a, b] \to \mathbf{R}$ is absolutely continuous. Let $\varepsilon > 0$. Let $(a_1, b_1), \ldots, (a_n, b_n) \subset [a, b]$ be a finite collection of disjoint intervals of total length at most δ, to be determined later. Using the triangle inequality

$$|F(b_j)\phi(b_j) - F(a_j)\phi(a_j)| \leq |F(b_j)||\phi(b_j) - \phi(a_j)| + |\phi(a_j)||F(b_j) - F(a_j)|,$$

the mean value theorem that $\exists x_j \in (a_j, b_j)$ with

$$\phi(b_j) - \phi(a_j) = \phi'(x_j)(b_j - a_j),$$

and continuity to bound $F,\ \phi,\ \phi'$ on $[a, b]$ by some $C > 0$, we get

$$|F(b_j)\phi(b_j) - F(a_j)\phi(a_j)| \leq C^2(b_j - a_j) + C|F(b_j) - F(a_j)|.$$

Hence, if we set δ to be such that $\delta \leq \frac{\varepsilon}{2C^2}$ and $\sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq \frac{\varepsilon}{2C}$, then:

$$\sum_{j=1}^{n} |F(b_j)\phi(b_j) - F(a_j)\phi(a_j)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $F\phi$ is absolutely continuous, as desired.

Alternatively, one could use the continuously differentiable nature of ϕ to show that it is absolutely continuous. By Exercise 87, the product of two absolutely continuous functions on an interval $[a, b]$ remains absolutely continuous. □

Exercise 94

Proof. By chain rule and product rule, F is differentiable on $[-1,1]\backslash\{0\}$. As

$$\lim_{h\to 0} \frac{F(h) - F(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h^3})}{h} = \lim_{h\to 0} h \sin(\frac{1}{h^3}) = 0,$$

$F : [-1, 1] \to \mathbf{R}$ is differentiable. Setting $F'(0) := 0$, we get

$$\int_{[-1,1]} |F'(x)|\ dx = \int_{[-1,1]} |2x \sin(\frac{1}{x^3}) - \frac{3}{x^2} \cos(\frac{1}{x^3})|\ dx.$$

Setting $2x\sin(\frac{1}{x^3})$ and $\frac{3}{x^2}\cos(\frac{1}{x^3})$ to be zero at the origin, it suffices to show

$$\int_{[0,1]} |\frac{1}{x^2} \cos(\frac{1}{x^3})|\ dx = \infty$$

since $2x\sin(\frac{1}{x^3})$ is continuous. Substituting $u = \frac{1}{x^3}$, the above integral becomes

$$\int_{[1,+\infty]} \frac{1}{3} u^{-\frac{2}{3}} |\cos u|\ du.$$

On $[2k\pi - \frac{\pi}{3}, 2k\pi + \frac{\pi}{3}],\ k \geq 1,\ |\cos u| \geq \frac{1}{2}$, and $u^{-\frac{2}{3}} \geq (2k\pi + \frac{\pi}{3})^{-\frac{2}{3}}$, hence

$$\int_{[1,+\infty]} \frac{1}{3} u^{-\frac{2}{3}} |\cos u|\ du \geq \sum_{k=1}^{N} \frac{\pi}{3} (2k\pi + \frac{\pi}{3})^{-\frac{2}{3}},\ \forall N \geq 1.$$

By the comparison test, the partial sum diverges since $\sum_{k=1}^{\infty} k^{-\frac{2}{3}}$ is a divergent p-series. We conclude that F' is not absolutely integrable. □

Exercise 95

Proof. (1). Let $f : [a, b] \to \mathbf{R}$ be Henstock-Kurzweil integrable with integrals L and L', and $\varepsilon > 0$. Then there exist gauge functions $\delta,\ \delta' : [a, b] \to (0, +\infty)$ obeying the criteria of the definition. Applying Cousin's theorem to the gauge function $\min(\delta', \delta)$, we get a partition $a = t_0 < t_1 < \ldots < t_k = b,\ k \geq 1$, and $t_j^* \in [t_{j-1}, t_j]$ for each $1 \leq j \leq k$, such that $|t_j - t_{j-1}| \leq \min(\delta', \delta)(t_j^*)$. Hence

$$|\sum_{j=1}^{k} f(t_j^*)(t_j - t_{j-1}) - L|,\ |\sum_{j=1}^{k} f(t_j^*)(t_j - t_{j-1}) - L'| \leq \varepsilon.$$

By the triangle inequality, this implies that $|L - L'| \leq 2\varepsilon$, and the claim follows.

(2). Let $f : [a, b] \to \mathbf{R}$ be Riemann integrable, and $\varepsilon > 0$. Use the notations in definition 4 of note 0, there exists $\tau > 0$ such that $|\mathcal{R}(f, \mathcal{P}) - \int_a^b f(x)\ dx| \leq \varepsilon$ for every tagged partition $\mathcal{P}$ with $\Delta(\mathcal{P}) \leq \tau$. we define $\delta : [a, b] \to (0, +\infty)$ by $\delta(x) := \tau$. Then whenever $k \geq 1$ and $a = t_0 < t_1 < \ldots < t_k = b$, and $t_1^*, \ldots, t_k^*$ are such that $t_j^* \in [t_{j-1}, t_j]$ and $|t_j - t_{j-1}| \leq \delta(t_j^*)$ for every $1 \leq j \leq k$, we get

$$|\sum_{j=1}^{k} f(t_j^*)(t_j - t_{j-1}) - \int_a^b f(x)\ dx| \leq \varepsilon$$

by definition of the Riemann sum. Hence, f is Henstock-Kurzweil integrable, with the Henstock-Kurzweil integral $\int_{[a,b]} f(x)\ dx$ equal to the Riemann integral $\int_a^b f(x)\ dx$.

(3). Let $f : [a, b] \to \mathbf{R}$ be everywhere defined, everywhere finite, and absolutely integrable. By Exercise 21 of notes 4, f is uniformly integrable. Now let $\varepsilon > 0$. By Exercise 24 of Notes 4, we can find $\kappa > 0$ such that $\int_U |f(x)|\ dx \leq \varepsilon$ whenever $U \subset [a, b]$ is a measurable set of measure at most κ. Extend f to $\mathbf{R}$ by setting $f(x) := 0$ on $\mathbf{R} \backslash [a, b]$. By Lebesgue differentiation theorem, the function $F : [a, b] \to \mathbf{R}$ defined by $F(x) := \int_{[a,x]} f(t)\ dt$ is continuous and almost everywhere differentiable, with $F'(x) = f(x)$ for almost every $x \in [a, b]$.

Let $E \subset [a, b]$ be the set of points x where F is not differentiable, together with the endpoints a, b. Thus E is a null set. By outer regularity (or the definition of outer measure) we can find an open set U containing E of measure $m(U) < \kappa$. In particular, $\int_U |f(x)|\ dx \leq \varepsilon$.

Now define a gauge function $\delta : [a, b] \to (0, +\infty)$ as follows. If $x \in E$, we define $\delta(x) > 0$ to be small enough that the open interval $(x - \delta(x), x + \delta(x))$ lies in U. If $x \notin E$, then F is differentiable at x. We let $\delta(x) > 0$ be small enough that $|F(y) - F(x) - (y - x)F'(x)| \leq \varepsilon|y - x|$ holds whenever $|y - x| \leq \delta(x)$;

such a $\delta(x)$ exists by the definition of differentiability. We rewrite this property using big-O notation as

$$F(y) - F(x) = (y - x)f(x) + O(\varepsilon|y - x|).$$

For any partition $a = t_0 < t_1 < \ldots < t_k = b$ with $k \geq 1$, and real numbers $t_j^* \in [t_{j-1}, t_j]$ with $t_j - t_{j-1} \leq \delta(t_j^*)$ for all $1 \leq j \leq k$ (the existence of such partition is guaranteed by the Cousin's theorem), we want to estimate the size of the sum

$$\sum_{j=1}^{k} f(t_j^*)(t_j - t_{j-1}).$$

First consider those j for which $t_j^* \in E$. Then, by construction, the intervals (t_{j-1}, t_j) are disjoint in U. Since f is everywhere finite, $C := \max_{j:t_j^* \in E} f(t_j^*)$ is finite. By construction of κ, we thus have

$$\sum_{j:t_j^* \in E} f(t_j^*)(t_j - t_{j-1}) \leq C \sum_{j:t_j^* \in E} (t_j - t_{j-1})$$

and thus by finite additivity

$$\sum_{j:t_j^* \in E} f(t_j^*)(t_j - t_{j-1}) = O(\varepsilon).$$

Next, we consider those j for which $t_j^* \notin E$. By construction, for those j we have

$$F(t_j) - F(t_j^*) = (t_j - t_j^*)F'(t_j^*) + O(\varepsilon|t_j - t_j^*|)$$

and

$$F(t_j^*) - F(t_{j-1}) = (t_j^* - t_{j-1})F'(t_j^*) + O(\varepsilon|t_j^* - t_{j-1}|)$$

and thus

$$F(t_j) - F(t_{j-1}) = (t_j - t_{j-1})F'(t_j^*) + O(\varepsilon|t_j - t_{j-1}|).$$

Or equivalently, that

$$f(t_j^*)(t_j - t_{j-1}) = \int_{[t_{j-1}, t_j]} f(y)\ dy + O(\varepsilon|t_{j-1} - t_j|).$$

Summing in j, we conclude that

$$\sum_{j:t_j^* \notin E} f(t_j^*)(t_j - t_{j-1}) = \int_S f(y)\ dy + O(\varepsilon(b - a))$$

where S is the union of all the $[t_{j-1}, t_j]$ with $t_j^* \notin E$. By construction, this set is contained in $[a,b]$ and contains $[a,b]\backslash U$. Since $\int_U |f(x)|\ dx \leq \varepsilon$, we conclude that

$$\int_S f(y)\ dy = \int_{[a,b]} f(y)\ dy + O(\varepsilon).$$

Putting everything together, we conclude that

$$\sum_{j=1}^{k} f(t_j^*)(t_j - t_{j-1}) = \int_{[a,b]} f(y)\ dy + O(\varepsilon) + O(\varepsilon|b-a|).$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

(4). Let $F : [a,b] \to \mathbf{R}$ be everywhere differentiable, and $\varepsilon > 0$. Define a gauge function $\delta : [a,b] \to (0, +\infty)$ by setting $\delta(x) > 0$ small enough that $|F(y) - F(x) - (y-x)F'(x)| \leq \varepsilon|y-x|$ holds whenever $|y-x| \leq \delta(x)$.

Let $a = t_0 < t_1 < \ldots < t_k = b$ be a partition with $k \geq 1$, and real numbers $t_j^* \in [t_{j-1}, t_j]$ with $t_j - t_{j-1} \leq \delta(t_j^*)$ for all $1 \leq j \leq k$ (the existence of such partition is guaranteed by the Cousin's theorem). Express $F(b) - F(a)$ as a telescoping series

$$F(b) - F(a) = \sum_{j=1}^{k} F(t_j) - F(t_{j-1}).$$

Arguing exactly as in (3), for every $1 \leq j \leq k$, we have

$$F(t_j) - F(t_{j-1}) = (t_j - t_{j-1})F'(t_j^*) + O(\varepsilon|t_j - t_{j-1}|),$$

and thus

$$F(b) - F(a) = \sum_{j=1}^{k} F'(t_j^*)(t_j - t_{j-1}) + O(\varepsilon|b-a|).$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

(5). let $F : [a,b] \to \mathbf{R}$ and $G : [a,b] \to \mathbf{R}$ be differentiable functions. Define $H(x) := F(x) - G(x)$, then H is everywhere differentiable. By (4) and (3) (or (4) and the definition of HK integral), we have

$$H(x) = \int_{[a,x]} H'(y)\ dy + H(a) = \int_{[a,x]} 0\ dy + H(a) = H(a),$$

from which Exercise 10 follows. Proposition 92 follows directly from (3) and (4). $\square$