Math 245A Note 4

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1 Selected Exercises in Note 4

Exercise 2

Proof. (1). This follows from definitions of the seven modes of convergence, and the fact that if one defines $g_n(x) := f_n(x) - f(x)$, then $|f_n(x) - f(x)| = |g_n(x) - 0|$.

(2). For the first four modes of convergence, the claim follows directly from the triangle inequality $|f_n + g_n - f - g| \leq |f_n - f| + |g_n - g|$, the identity $|cf| = |c||f|$, and the fact that finite union of null sets is a null set. For almost uniform convergence, the claim follows from the uniform case and subadditivity of the measure μ. That $|f_n + g_n|$ convergences to $f + g$ in L^1 norm follows from combining the above triangle inequality with additivity of the unsigned integral, and that cf_n converges to cf in L^1 norm follows similarly from homogeneity of the unsigned integral.

Finally, if f_n and g_n converge in measure to f and g respectively, for every $\varepsilon > 0$, the measures

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon/2\}),\ \mu(\{x \in X : |g_n(x) - g(x)| \geq \varepsilon/2\})$$

converge to zero as $n \to \infty$. From the inclusion

$$\begin{aligned}
&\{x \in X : |f_n(x) + g(x) - f(x) - g(x)| \geq \varepsilon\} \subset \\
&\{x \in X : |f_n(x) - f(x)| + |g_n(x) - g(x)| \geq \varepsilon\} \subset \\
&\{x \in X : |f_n(x) - f(x)| \geq \varepsilon/2\} \cup \{x \in X : |g_n(x) - g(x)| \geq \varepsilon/2\},
\end{aligned}$$

and subadditivity of μ, we see that the measures

$$\mu(\{x \in X : |f_n(x) + g(x) - f(x) - g(x)| \geq \varepsilon\})$$

converge to zero as $n \to \infty$. That is, $f_n + g_n$ converge in measure to $f + g$. If $c = 0$, then cf_n converge trivially in measure to cf, otherwise, one has

$$\mu(\{x \in X : |cf_n(x) - cf(x)| \geq \varepsilon\}) = \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon/|c|\}),$$

which converge to zero as $n \to \infty$. Hence cf_n converge in measure to cf.

(3). For the first five modes of convergence, the claim follows directly from the squeeze test for sequence of real numbers. For convergence in L^1 norm and in measure, the claim follows from monotonicity of the unsigned integral, and of the measure μ respectively. $\square$

Exercise 3

Proof. (1). This follows directly from the definition.

(2). This follows directly from the definition.

(3). This follows directly from the definition.

(4). If f_n converges almost uniformly to f, then $\forall \varepsilon > 0$, there exists an exceptional set $E_k \in \mathcal{B}$ of measure $\mu(E) \leq \varepsilon/k$ such that f_n converges uniformly to f on the complement of E_k, and thus converges pointwise to f on the complement of E_k by part (1). Take $E := \bigcap_{k=1}^{\infty}$, then E is a null set outside of which f_n converges to f pointwise. That is, f_n converges to f pointwise almost everywhere.

(5). This follows directly from the definition.

(6). If f_n converges to f in L^1 norm, by Markov's inequality, the measures

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{\|f_n - f\|_{L^1(\mu)}}{\varepsilon}$$

converge to zero as $n \to \infty$. Thus f_n converges to f in measure.

(7). Suppose for contradiction that $f_n \nrightarrow f$ in measure. Then $\exists \delta > 0$, such that $\forall N > 0$, $\exists n \geq N$ with

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \geq \delta.$$

Since f_n converges to f almost uniformly, there exists an exceptional set $E \in \mathcal{B}$ of measure $\mu(E) \leq \delta/2$ such that f_n converges uniformly to f on the complement of E. In particular, the set $\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$ becomes empty outside of E for sufficiently large n, suggesting that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = \mu(E \cap \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \delta/2$$

for all sufficiently large n, a contradiction. $\square$

Exercise 10

Proof. (1). Suppose that f_n converges uniformly to zero, and $\varepsilon > 0$. $\exists N > 0$ such that $|A_n 1_{E_n}(x)| \leq \varepsilon$ for all $x \in X$ and $n \geq N$, which happens if and only if $|A_n| \leq \varepsilon$ for all $n \geq N$. That is, $A_n \to 0$ as $n \to \infty$. Conversely, if $A_n \to 0$ as $n \to \infty$, then for some $N > 0$ we have $|A_n| \leq \varepsilon$ for all $n \geq N$. Consequently, $|A_n 1_{E_n}(x)| \leq \varepsilon$ for all $x \in X$ and $n \geq N$. i.e. f_n converges uniformly to f.

(2). If $A_n \to 0$ as $n \to \infty$, then f_n converges to zero in L^∞ norm by (1) and Exercise 3. Conversely, if f_n converges to zero in L^∞ norm, by remark 8 we have $\|A_n 1_{E_n}\|_{L^\infty(\mu)} = A_n \to 0$ as $n \to \infty$.

(3). Suppose that f_n converges almost uniformly to zero, and A_n is bounded away from zero, where $c > 0$ is such that $A_n \geq c$ for every n. Pick any $0 < \varepsilon < c$. If we were to have $|A_n 1_{E_n}(x)| \leq \varepsilon$ for all $n \geq N$ outside of an exceptional set E, we see that E must contain E_N^*. By our assumption of almost uniform convergence, the quantity $\mu(E_N^*) \leq \mu(E)$ can be made arbitrarily small, since the tail supports E_N^* are non-increasing, this implies that $\mu(E_N^*) \to 0$ as $N \to \infty$. Consequently, we have $\min(A_n, \mu(E_n^*)) \to 0$ as $n \to \infty$.

Conversely, if $\min(A_n, \mu(E_n^*)) \to 0$ as $n \to \infty$, either $\mu(E_n^*) \to 0$ or $A_n \to 0$. In the first case, $\forall \varepsilon > 0$, $\exists N > 0$ such that $\mu(E_n^*) \leq \varepsilon$ for all $n \geq N$. Then $|A_n 1_{E_n}(x)| = 0$ for all $n \geq N$ outside of the exceptional set $E := E_N^*$. In the second case, we have $A_n 1_{E_n}(x) \leq A_n \to 0$ for all $x \in X$. In both cases, we have f_n converges to f almost uniformly.

(4). Suppose that f_n converges pointwise to zero and A_n is bounded away from zero. By condition, this can happen only if

$$\lim_{n\to\infty} 1_{E_n}(x) = \limsup_{n\to\infty} 1_{E_n}(x) = 0$$

for all $x \in X$. This implies that $\forall x \in X$, $\exists N > 0$ such that $x \notin E_N^*$ (so $x \notin E_n^*$ for all $n \geq N$). Therefore,

$$\bigcap_{N=1}^{\infty} E_N^* = \bigcap_{N=1}^{\infty} \bigcup_{n \geq N} E_n = \emptyset.$$

Conversely, given $A_n \to 0$ as $n \to \infty$, we get f_n converges to f pointwise by (1) and Exercise 3. Otherwise, if $\bigcap_{N=1}^{\infty} E_N^* = \emptyset$, $\forall x \in X$, $\exists N > 0$ such that $x \notin E_N^*$. Consequently, $f_n(x) = 0$ for all $n \geq N$, so f_n converges to f pointwise.

(5). Suppose that f_n converges pointwise almost everywhere to zero and A_n is bounded away from zero. As before we get

$$\lim_{n\to\infty} 1_{E_n}(x) = \limsup_{n\to\infty} 1_{E_n}(x) = 0$$

for all $x \in X\backslash E$, where E is a null set. This implies that $\forall x \in X\backslash E,\ \exists N > 0$ such that $x \notin E_N^*$, which in turn implies that $\bigcap_{N=1}^{\infty} E_N^* \subset E$, so $\bigcap_{N=1}^{\infty} E_N^*$ is a null set.

Conversely, given $A_n \to 0$ as $n \to \infty$, we get f_n converges pointwise to f by (4) and Exercise 3. Otherwise, if $E := \bigcap_{N=1}^{\infty} E_N^*$ is a null set, $\forall x \in X\backslash E,\ \exists N > 0$ such that $x \notin E_N^*$, so $1_{E_n}(x) = 0$ for all $n \geq N$. That is, f_n converges pointwise to f on $X\backslash E$.

(6). Suppose that f_n converges in measure to zero, and A_n is bounded away from zero, where $c > 0$ is such that $A_n \geq c$ for every n. Pick any $0 < \varepsilon < c$. Then we have

$$\mu(E_n) \leq \mu(\{x \in X : A_n 1_{E_n}(x) \geq \varepsilon\}) \to 0$$

as $n \to \infty$, by our construction and monotonicity. Hence $\min(A_n, \mu(E_n)) \to 0$ as $n \to \infty$.

Conversely, if $\min(A_n, \mu(E_n)) \to 0$ as $n \to \infty$, either $\mu(E_n) \to 0$ or $A_n \to 0$. In the first case, from $\{x \in X : A_n 1_{E_n}(x) \geq \varepsilon\} \subset E_n$ and monotonicity, one gets f_n converges in measure to f. In the second case, we have $A_n < \varepsilon$ for all n sufficiently large, and so $\{x \in X : A_n 1_{E_n}(x) \geq \varepsilon\} = \emptyset$ for all n sufficiently large, and f_n converges in measure to f.

(7). Suppose that f_n converges in L^1 norm to zero, then

$$\int_X |A_n 1_{E_n}(x)|\ d\mu = \int_X A_n 1_{E_n}(x)\ dx = A_n \mu(E_n) \to 0$$

as $n \to \infty$. Conversely, if $A_n \mu(E_n) \to 0$ as $n \to \infty$, the same identity above shows that f_n converges in L^1 norm to zero. □

Exercise 13

Proof. Given $0 < \mu(X) < \infty$, if f_n converges to f in L^∞ norm, $\forall \varepsilon > 0$, $\exists N > 0$, and a null set $E \in \mathcal{B}$, such that $|f_n(x) - f(x)| \leq \varepsilon/\mu(X)$ for all $x \in X\backslash E$ and $n \geq N$. By finite additivity, monotonicity, and almost everywhere equivalence of the unsigned integral, we have

$$\int_X |f_n - f|\ d\mu = \int_{X\backslash E} |f_n - f|\ d\mu + \int_E |f_n - f|\ d\mu \leq \frac{\varepsilon}{\mu(X)} \cdot \mu(X) = \varepsilon$$

for all $n \geq N$. That is, f_n converges to f in L^1 norm. □

Exercise 14

Proof. (1). In view of Exercise 3 and Egorov's theorem, we may assume that f_n converges to f in measure. Let $\lambda \in \mathbf{R}$ be a continuity point of F, hence $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $|F(\Gamma) - F(\lambda)| \le \varepsilon$ whenever $|\Gamma - \lambda| \le \delta$. Fix such a pair of (ε, δ), we aim to show that

$$F(\lambda - \delta) \le \lim_{n\to\infty} F_n(\lambda) \le F(\lambda + \delta). \tag{1}$$

Write the set $\{f_n(x) \le \lambda\}$ as a disjoint union

$$\{(f_n(x) \le \lambda) \wedge (|f_n(x) - f(x)| \ge \delta)\} \cup \{(f_n(x) \le \lambda) \wedge (|f_n(x) - f(x)| < \delta)\},$$

by monotonicity and our assumption of convergence in measure, we get

$$\lim_{n\to\infty} F_n(\lambda) = \lim_{n\to\infty} \mu(\{(f_n(x) \le \lambda) \wedge (|f_n(x) - f(x)| < \delta)\}).$$

Note that if $x \in X$ is such that $f(x) \le \lambda - \delta$, then $|f_n(x) - f(x)| < \delta$ implies that $f_n(x) \le \lambda$, from which one obtains that

$$F(\lambda - \delta) \le \mu(\{(f_n(x) \le \lambda) \wedge (|f_n(x) - f(x)| < \delta)\}) + \mu(\{|f_n(x) - f(x)| \ge \varepsilon\}),$$

by monotonicity and subadditivity. Taking $n \to \infty$, and using convergence in measure again, we get half of (1)

$$F(\lambda - \delta) \le \lim_{n\to\infty} F_n(\lambda).$$

On the other hand, if $x \in X$ is such that $f_n(x) \le \lambda$ and $|f_n(x) - f(x)| < \delta$, then $f(x) < f_n(x) + \delta \le \lambda + \delta$, and therefore we get the other half of (1)

$$\lim_{n\to\infty} F_n(\lambda) \le F(\lambda + \delta)$$

on taking $n \to \infty$. From (1), we conclude that

$$F(\lambda) - \varepsilon \le \lim_{n\to\infty} F_n(\lambda) \le F(\lambda) + \varepsilon,$$

and the claim follows.

(2). By Exercise 3 (and the assumption of finite measure space), it suffices to construct an example where f_n converges to f in distribution, but not in measure. Consider the measure space $([0,1], \mathcal{L}([0,1]), m)$, the unit interval with the Lebesgue measure. Define $f_n,\ f : [0,1] \to \mathbf{R}$ by setting $f_n(x) := x$ for all n, and $f(x) := 1 - f_n(x) = 1 - x$. Clearly f_n converges to f in distribution, but since $f(x) - f_n(x) = 1$ for all x and every n, it is impossible that f_n converges in measure to f.

(3). Use the construction in (2), and define $g_n,\ g : [0,1] \to \mathbf{R}$ by setting $g_n(x) := 1 - x$ for all n, and $g(x) := g_n(x) = 1 - x$. Clearly g_n converges trivially in distribution to g, but $f_n(x) + g_n(x) = 1$ does not converge in distribution to $f + g = 2 - 2x$ (say for $\lambda = 1$).

(4). Use the construction in (2), we see that $f_n(x) = x$ converges in distribution to $f(x) = 1 - x$, and also trivially in distribution to $g(x) = x$, but $f(x) = g(x)$ only at $x = 1/2$, so f and g are not equal almost everywhere. $\square$

Exercise 15

Proof. (1). First work with step functions $f_n = A_n 1_{E_n}$, and $f = 0$. For simplicity we will assume that the $A_n > 0$ are positive reals, and that the E_n have a positive measure $\mu(E_n) > 0$. We also assume that either the A_n converge to zero, or else they are bounded away from zero.

If $A_n \to 0$ as $n \to \infty$, then f_n converges to f pointwise almost everywhere by Exercise 10. Otherwise, from

$$\sum_{n=1}^{\infty} c\mu(E_n) \leq \sum_{n=1}^{\infty} A_n \mu(E_n) = \sum_{n=1}^{\infty} \|f - f_n\|_{L^1(\mu)} < \infty$$

where $c > 0$, one has $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. By the Borel-Cantelli lemma, it follows that for almost every x, $x \in E_n$ for finitely many n. Therefore $\bigcap_{N=1}^{\infty} E_N^*$ is a null set, and we have f_n converges to f pointwise almost everywhere by Exercise 10.

For general measurable functions $f_n, f : X \to \mathbf{C}$ obeying the fast L^1 convergence condition, we suppose for contradiction that $\{x \in X : f_n(x) \nrightarrow f(x)\}$ has positive measure. As countable union of null set is still a null set, this implies that $\exists k > 0$, such that the set

$$E = \bigcap_{n=1}^{\infty} E_n := \bigcap_{n=1}^{\infty} \{x \in X : |f_n(x) - f(x)| \geq 1/k\}$$

has positive measure. By construction, we see that

$$\sum_{n=1}^{\infty} \int_X \frac{1}{k} 1_{E_n}(x)\ d\mu \leq \sum_{n=1}^{\infty} \int_X |f_n - f| 1_{E_n}(x)\ d\mu < \infty.$$

By the step function case, $\bigcap_{N=1}^{\infty} E_N^*$ is a null set. As $E \subset \bigcap_{N=1}^{\infty} E_N^*$, it's a contradiction. Hence f_n converges pointwise almost everywhere to f.

Alternatively, use the Tonelli's theorem to get $\int_X \sum_{n=1}^{\infty} |f_n - f| d\mu < \infty$, which implies that the function $\sum_{n=1}^{\infty} |f_n - f|$ is finite almost everywhere and thus $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for almost every x.

(2). Fix some $\varepsilon > 0$. If $f_n = \sum_{n=1}^{\infty} A_n 1_{E_n}$ are step functions and $f = 0$, we have $\min(A_n, \mu(E_n^*)) \leq A_n \to 0$ given $A_n \to 0$ as $n \to \infty$. Otherwise A_n is bounded away from zero. $\forall \varepsilon > 0$, the fast L^1 convergence condition then shows that there exists a positive integer N_ε such that

$$\sum_{n=N_\varepsilon}^{\infty} c\mu(E_n) \leq \sum_{n=N_\varepsilon}^{\infty} A_n \mu(E_n) \leq c\varepsilon$$

where $c > 0$. By countable subadditivity, $\mu(E_{N_\varepsilon}^*) \leq \sum_{n=N_\varepsilon}^{\infty} \mu(E_n) \leq \varepsilon$, which implies that $\min(A_n, \mu(E_n^*)) \leq \mu(E_n^*) \to 0$ as $n \to \infty$. By Exercise 10, we conclude that f_n converges almost uniformly to f.

For general measurable functions $f_n, f : X \to \mathbf{C}$ obeying the fast L^1 convergence condition, define $E_{n,k} := \{x \in X : |f_n(x) - f(x)| > 1/k\}$. From

$$\sum_{n=1}^{\infty} \|\frac{1}{k} 1_{E_{n,k}}\|_{L^1(\mu)} \leq \sum_{n=1}^{\infty} \|f_n - f\|_{L^1(\mu)} < \infty$$

and the step function case, $\mu(E_{n,k}^*) \to 0$ as $n \to \infty$. In particular, for every positive integer k, there exists an integer $N_k > 0$ such that $\mu(E_{n,k}^*) \leq \varepsilon/2^k$ for all $n \geq N_k$. Set

$$E := \bigcup_{k=1}^{\infty} E_{N_k,k}^*,$$

where $E_{N_k,k}^*$ is the set of x such that $|f_n(x) - f(x)| > 1/k$ for some $n \geq N_k$. By our construction, $\mu(E) \leq \varepsilon$, and f_n converges uniformly to f outside of E. □

Exercise 17

Proof. Let $\varepsilon > 0$. By the assumption of convergence in measure, for each $j \geq 1$, we can choose n_j so that the set $E_j := \{x \in X : |f_{n_j}(x) - f(x)| > 1/j\}$ has measure at most $\varepsilon/2^j$. Define $E := \bigcup_{j=1}^{\infty} E_j$, then $\mu(E) \leq \varepsilon$. Moreover, for any $\varepsilon' > 0$, there exists $j \geq 1$ such that $1/j \leq \varepsilon'$, as $1/j$ is decreasing in j, we have by our construction that

$$|f_{n_i}(x) - f(x)| \leq 1/j \leq \varepsilon'$$

for all x outside of E_i whenever $i \geq j$, which of course contains all x outside of E. Hence, f_{n_j} converges almost uniformly to f. □

Exercise 18

Proof. Suppose that f_n converges in measure to f, and f_{n_j} is a subsequence of the f_n. Clearly f_{n_j} converges in measure to f as well. By Exercise 17, there exists a further subsequence $f_{n_{j_k}}$ that converges almost uniformly (and hence, pointwise almost everywhere) to f. Apply the dominated convergence theorem to $f - f_{n_{j_k}}$, we see that $f_{n_{j_k}}$ converges in L^1 norm to f. For contradiction, if f_n does not converge in L^1 norm to f, then the sequence $\|f_n - f\|_{L^1(\mu)}$ has a subsequence $\|f_{n_i} - f\|_{L^1(\mu)}$ that is bounded away from zero. In particular, any further subsequence of this subsequence will be bounded away from zero, a contradiction. Hence f_n converges in L^1 norm to f.

The converse direction is given by Exercise 3.

Alternatively, consider the sets $E_k := \{x \in X : 1/k \leq g(x) \leq k\}$. The E_k are increasing, with $\bigcup_{k=1}^{\infty} E_k = \{x \in X : 0 < g(x) < \infty\}$. Since the set on which $g = 0$ contributes nothing to the L^1 norm, and g being absolutely integrable implies that the set on which $g = \infty$ must be a null set, and thus also contribute nothing to the L^1 norm, we can locate an integer $N > 0$ by the monotone convergence theorem, such that

$$\int_X g \, d\mu - \int_X g 1_{E_N} \, d\mu = \int_{X \backslash E_N} g \, d\mu \leq \varepsilon.$$

Set $E := E_N$. By monotonicity, this implies that $\int_{X \backslash E} |f_n| \, d\mu$, $\int_{X \backslash E} |f| \, d\mu \leq \varepsilon$, and thus

$$\int_{X \backslash E} |f_n - f| \, d\mu \leq 2\varepsilon \tag{1}$$

by the triangle inequality.

On the other hand, since f_n converges to f in measure on X, it converges to f in measure on E. In particular, if we set $A_n := \{x \in E : |f_n(x) - f(x)| > \varepsilon\}$ and $B_n := E \backslash A_n$, then

$$\begin{aligned}
\int_E |f_n - f| \, d\mu &= \int_{A_n} |f_n - f| \, d\mu + \int_{B_n} |f_n - f| \, d\mu \\
&\leq \int_{A_n} 2g \, d\mu + \int_E \varepsilon \, d\mu \\
&\leq 2N\mu(A_n) + \varepsilon\mu(E).
\end{aligned}$$

Since $\mu(A_n) \to 0$ as $n \to \infty$, we have $2N\mu(A_n) \leq \varepsilon$ for sufficiently large n. It follows that

$$\int_X |f_n - f| \, d\mu \leq (1 + \mu(E))\varepsilon \tag{2}$$

for sufficiently large n. Combine the bounds (1) and (2), we see that

$$\|f_n - f\|_{L^1(\mu)} \leq C\varepsilon$$

for some absolute constant C and sufficiently large n, showing that f_n converges in L^1 norm to f. The converse direction still follows from Exercise 3. $\square$

Exercise 21

Proof. (1). Clearly one has $\sup_n \|f_n\|_{L^1(\mu)} = \int_X |f| \, d\mu < +\infty$, so the sequence f_n obeys a uniform bounded on the L^1 norm. Without loss of generality, we can take $M \in \mathbf{Z}^+$, and let $f_M := f 1_{|f|<M}$. For any $\varepsilon > 0$, one has

$$\mu(\{x \in X : |f_M(x) - f(x)| \geq \varepsilon\}) \leq \mu(\{x \in X : |f(x)| \geq M\}) \leq \frac{\|f\|_{L^1(\mu)}}{M}$$

by the Markov's inequality. Since $f \in L^1(X, \mathcal{B}, \mu)$, we see that f_M converges in measure to f. By Exercise 18, f_M thus converges in L^1 norm to f. i.e. $\int_X |f_M - f| \, d\mu = \int_{|f| \geq M} f \, d\mu \to 0$ as $M \to \infty$. Hence the sequence f_n obeys no escape to vertical infinity.

Finally, define $E_k := \{x \in X : |f(x)| \geq 1/k\}$, $E_1 \subset E_2 \subset \ldots$. By the monotone convergence theorem (vertical truncation), we have

$$\lim_{k \to \infty} \int_X |f| 1_{E_k} \, d\mu = \int_X |f| 1_{\bigcup_{k=1}^{\infty} E_k} \, d\mu = \int_X |f|_{|f|>0} \, d\mu = \int_X |f| \, d\mu.$$

Hence one can find an integer $N = N_\varepsilon > 0$ such that

$$\int_X |f| \, d\mu - \int_X |f| 1_{E_N} \, d\mu = \int_{X \setminus E_N} |f| \, d\mu \leq \varepsilon.$$

Again we use the fact that $\|f\|_{L^1(\mu)} < \infty$, taking $E_\varepsilon := E_N$, we see that the sequence f_n obeys no escape to width or horizontal infinity. Therefore, the constant sequence $f_n = f$ is uniformly integrable.

(2). Let f_n be a dominated sequence of measurable functions, dominated by an absolutely integrable function $g : X \to [0, +\infty]$. Then $\sup_n \|f_n\|_{L^1(\mu)} \leq \|g\|_{L^1(\mu)} < \infty$ by monotonicity. From domination, we see that

$$\{x \in X : |f_n|(x) \geq M\} \subset \{x \in X : g(x) \geq M\},$$

from which it follows by (1) that $\int_{|f_n| \geq M} |f_n| \, d\mu \leq \int_{g \geq M} g \, d\mu \to 0$ as $M \to \infty$ for all n, and thus $\sup_n \int_{|f_n| \geq M} |f_n| \, d\mu \to 0$ as $M \to +\infty$. By (1) again, for every $\varepsilon > 0$, there is a finite measure subset E_ε of X such that $\int_{X \setminus E_\varepsilon} |g| \, d\mu \leq \varepsilon$, so $\int_{X \setminus E_\varepsilon} |f_n| \, d\mu \leq \varepsilon$ for all n by monotonicity. Consequently, the dominated sequence f_n is uniformly integrable.

(3). Take the step function $f_n := n1_{[1/n^2,2/n^2]}$. For all $n \geq 1$, $n \cdot 1/n^2 = 1/n \leq 1$. As $n_j \to \infty$ along any subsequence n_j, $n_j/n_j^2 = 1/n_j^2 \to 0$. And the L^1 mass of the f_n are all trapped inside the finite measure set $[0,2]$. It follows that the sequence f_n is uniformly integrable. However, any function dominating f_n for all n needs to take at least n on $[1/n^2, 2/n^2]$ for every n. As the sum $\sum_n 1/n$ is divergent, the sequence can not be dominated by an absolutely integrable function. □

Exercise 22

Proof. Suppose the sequence f_n obeys no escape to vertical infinity, i.e.

$$\sup_n \int_{|f_n| \geq M} |f_n| \ d\mu \to 0$$

as $M \to +\infty$. Then for sufficiently large M we see that

$$\begin{aligned}\sup_n \|f_n\|_{L^1(\mu)} &= \sup_n \int_{|f_n|<M} |f_n| \ d\mu + \sup_n \int_{|f_n| \geq M} |f_n| \ d\mu \\ &\leq M\mu(X) + 1/2 < \infty.\end{aligned}$$

Hence the sequence obeys uniform bound on the L^1 norm, and is thus uniformly bounded. The converse direction is immediate. □

Exercise 23

Proof. As $\mu(X) < \infty$, it suffices to show that the L^1 masses of the f_n do not escape to vertical infinity: $\sup_n \int_{|f_n| \geq M} |f_n| \ d\mu \to 0$ as $M \to +\infty$. For $M \geq 1$, we have

$$\int_{|f_n| \geq M} |f_n|^p \ d\mu = \int_{|f_n| \geq M} |f_n|^{p-1} |f_n| \ d\mu \geq M^{p-1} \int_{|f_n| \geq M} |f_n| \ d\mu.$$

Taking the supremum over n and sending $M \to \infty$, we get the desired result. □

Exercise 24

Proof. Fix some $\varepsilon > 0$, by no escape to vertical infinity, one can find an M large enough such that $\int_{|f_n|\geq M} |f_n|\ d\mu \leq \varepsilon/2$ for all n. Let $\delta > 0$ be such that $\delta \leq \varepsilon/2M$. Then, for any measurable set E with $\mu(E) \leq \delta$ and every $n \geq 1$, one has

$$\begin{aligned}\int_E |f_n|\ d\mu &= \int_{\{|f_n|\geq M\}\cap E} |f_n|\ d\mu + \int_{\{|f_n|<M\}\cap E} |f_n|\ d\mu \\ &\leq \frac{\varepsilon}{2} + M \cdot \frac{\varepsilon}{2M} \\ &= \varepsilon,\end{aligned}$$

as desired. □

Exercise 25

Proof. Since $\mu(X) = 1 < \infty$, it suffices to demonstrate no escape to vertical infinity. For $M > 0$, take the supremum over n on both sides of the Markov inequality to obtain

$$\sup_n \mu(\{x \in X : |f_n(x)| \geq M\}) \leq \frac{\sup_n \|f_n\|_{L^1}}{M},$$

which goes to zero as $M \to \infty$ since the numerator on the RHS is finite. In particular, $\forall\varepsilon > 0$, $\exists\delta > 0$ such that LHS $\leq \delta$ for sufficiently large M. By our assumption we get

$$\sup_n \int_{|f_n|\geq M} |f_n|\ d\mu \leq \varepsilon$$

for sufficiently large M, and we are done. □

Exercise 26

Proof. Let $\varepsilon > 0$. By no escape to horizontal/width infinity, there is a finite measure subset E_ε of X such that $\int_{X\setminus E_\varepsilon} |f_n|\ d\mu \leq \varepsilon/2$ for all n. It follows that

$$\begin{aligned}\int_{|f_n|\leq\delta} |f_n|\ d\mu &= \int_{\{|f_n|\leq\delta\}\cap E_\varepsilon} |f_n|\ d\mu + \int_{\{|f_n|\leq\delta\}\setminus E_\varepsilon} |f_n|\ d\mu \\ &\leq \delta\mu(E_\varepsilon) + \varepsilon/2 \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon\end{aligned}$$

for all n and a sufficiently small δ. Therefore, $\sup_n \int_{|f_n|\leq\delta} |f_n|\ d\mu \to 0$ as $\delta \to 0$, as desired. □

Exercise 27

Proof. For the "if" direction, let $\varepsilon > 0$ and $h : X \to [0, +\infty]$ be the absolutely integrable function chosen accordingly. We see that

$$\begin{aligned}\sup_n \|f_n\|_{L^1(\mu)} = \sup_n \int_X |f_n| \ d\mu = \sup_n (\int_{|f_n|>h} |f_n| \ d\mu + \int_{|f_n|\le h} |f_n| \ d\mu) \\ \le \sup_n \int_{|f_n|>h} |f_n| \ d\mu + \sup_n \int_{|f_n|\le h} h \ d\mu \\ \le \varepsilon + \|h\|_{L^1(\mu)} < \infty,\end{aligned}$$

so the sequence f_n obeys uniform bound on L^1 norm. Similarly, by decomposing the set $\{|f_n| \ge M\}$ into the disjoint union of $\{|f_n| \ge M : |f_n| > h\}$ and $\{|f_n| \ge M : |f_n| \le h\}$, we obtain

$$\begin{aligned}\sup_n \int_{|f_n|\ge M} |f_n| \ d\mu \le \sup_n \int_{\{|f_n|\ge M:|f_n|>h\}} |f_n| \ d\mu + \sup_n \int_{\{|f_n|\ge M:|f_n|\le h\}} |f_n| \ d\mu \\ \le \varepsilon + \int_{h\ge M} h \ d\mu \\ \le \varepsilon + \varepsilon = 2\varepsilon\end{aligned}$$

for M large enough, using the fact that a constant sequence of absolutely integrable functions is uniformly integrable. This shows that the sequence f_n obeys no escape to vertical infinity. Finally, let E_ε be a finite measure set such that $\int_{X\setminus E_\varepsilon} h \ d\mu \le \varepsilon$. Then we have

$$\begin{aligned}\int_{X\setminus E_\varepsilon} |f_n| \ d\mu = \int_{\{|f_n|>h\}\setminus E_\varepsilon} |f_n| \ d\mu + \int_{\{|f_n|\le h\}\setminus E_\varepsilon} |f_n| \ d\mu \\ \le \varepsilon + \int_{X\setminus E_\varepsilon} h \ d\mu \\ \le \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

and this shows that the sequence f_n obeys no escape to horizontal/width infinity. Consequently, f_n is uniformly integrable.

Conversely, if f_n is uniformly integrable, there exists an $M > 0$ such that $\sup_n \int_{|f_n|\ge M} |f_n| \ d\mu \le \varepsilon$ by no escape to vertical infinity, and there exists a finite measure set E_ε such that $\int_{X\setminus E_\varepsilon} |f_n| \ d\mu \le \varepsilon$ for all n by no escape to horizontal/width infinity. If we define the function $h : X \to [0, +\infty]$ by $h :=$

$M1_{E_\varepsilon}$, then h is absolutely integrable, and for all n,

$$\begin{aligned}\int_{|f_n|>h} |f_n|\ d\mu &= \int_{\{|f_n|>h\}\cap E_\varepsilon} |f_n|\ d\mu + \int_{\{|f_n|>h\}\setminus E_\varepsilon} |f_n|\ d\mu \\ &\le \int_{|f_n|\ge M} |f_n|\ d\mu + \int_{X\setminus E_\varepsilon} |f_n|\ d\mu \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

It follows that $\sup_n \int_{|f_n|>h} |f_n|\ d\mu \le \varepsilon$, as desired. □

Exercise 28

Proof. From Fatou's lemma and uniform integrability of the f_n, we get

$$\int_X |f|\ d\mu \le \liminf_{n\to\infty} \int_X |f_n|\ d\mu < \infty,$$

so f is absolutely integrable (and hence uniformly integrable as a sequence).

Now we show that $f_n - f$ is uniformly integrable: Clearly we have uniform bound on L^1 norm by the triangle inequality. Note the inequality

$$\sup_n \int_{|f_n-f|\ge 2M} |f_n - f|\ d\mu \le \sup_n \int_{|f_n|+|f|\ge 2M} |f_n| + |f|\ d\mu.$$

The quantity on the RHS can be bounded by

$$\sup_n \int_{|f_n|\ge M} |f_n|\ d\mu + \sup_n \int_{|f|\ge M} |f|\ d\mu + \sup_n \int_{|f_n|\ge M} |f|\ d\mu + \sup_n \int_{|f|\ge M} |f_n|\ d\mu.$$

Using the fact that f_n and f are uniformly integrable, the first two terms can be readily controlled for large M, by combining Markov's inequality and Exercise 24, the second two terms can also be controlled for large M. We conclude that

$$\sup_n \int_{|f_n-f|\ge 2M} |f_n - f|\ d\mu \to 0$$

as $M \to \infty$, showing that $f_n - f$ obeys no escape to vertical infinity. Finally, for finite measure subsets E_ε and F_ε of X such that $\int_{X\setminus E_\varepsilon} |f_n|\ d\mu \le \varepsilon$ for all n, $\int_{X\setminus F_\varepsilon} |f|\ d\mu \le \varepsilon$, the finite measure set $E_\varepsilon \cup F_\varepsilon$ is such that

$$\int_{X\setminus E_\varepsilon\cup F_\varepsilon} |f_n - f|\ d\mu \le 2\varepsilon,$$

and $|f_n - f|$ obeys no escape to horizontal/width infinity. Therefore, $|f_n - f|$ is uniformly integrable.

Having established such, for every $\varepsilon > 0$, there is a finite measure set G_ε such that $\int_{X \setminus G_\varepsilon} |f_n - f| \, d\mu \le \varepsilon/3$. One can then decompose $\|f_n - f\|_{L^1(\mu)}$ as

$$\int_X |f_n - f| \, d\mu = \int_{\{|f_n - f| \ge \delta\}} |f_n - f| \, d\mu + \int_{\{|f_n - f| < \delta\}} |f_n - f| \, d\mu,$$

with the first term on the RHS further equals to

$$\int_{\{|f_n - f| \ge \delta\} \cap G_\varepsilon} |f_n - f| \, d\mu + \int_{\{|f_n - f| \ge \delta\} \setminus G_\varepsilon} |f_n - f| \, d\mu.$$

By Exercise 26, there is a $\delta > 0$ such that $\int_{\{|f_n - f| < \delta\}} |f_n - f| \, d\mu \le \varepsilon/3$. Fix this δ. On the set G_ε, f_n converges pointwise almost everywhere to f, hence it converges almost uniformly by Egorov's theorem, and thus in measure by Exercise 3. By Exercise 24, we have $\int_{\{|f_n - f| \ge \delta\} \cap G_\varepsilon} |f_n - f| \, d\mu \le \varepsilon/3$ for sufficiently large n. Combine these inequalities, we see that

$$\|f_n - f\|_{L^1(\mu)} \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for sufficiently large n, and thus f_n converges in L^1 norm to f.

Since the f_n are absolutely integrable, we get convergence of the integral

$$\int_X f_n \, d\mu \to \int_X f \, d\mu$$

as $n \to \infty$, by the triangle inequality (or by almost dominated convergence). □

Exercise 30

Proof. Denote $f := \sup_n f_n$, by (6) of Exercise 28 in note 3, f is measurable. By Fatou's lemma, $\int_X f \, d\mu \le \sup_n \int_X f_n \, d\mu < \infty$, so f so absolutely integrable.

We claim that the sequence f_n is uniformly integrable. The condition already gives uniform bound on L^1 norm. To establish no escape to vertical infinity, note that for any M,

$$\sup_n \int_{f_n \ge M} f_n \, d\mu = \lim_{n \to \infty} \int_X f_n 1_{f_n \ge M} \, d\mu = \int_X f 1_{f \ge M} \, d\mu,$$

by the monotone convergence theorem. Sending $M \to \infty$ and observing that f is uniformly integrable as a sequence gives the result. For every $\varepsilon > 0$, there is a finite measure subset E_ε of X such that $\int_{X \setminus E_\varepsilon} f \, d\mu \le \varepsilon \le \varepsilon$, but then by monotonicity

$$\int_{X \setminus E_\varepsilon} f_n \, d\mu \le \int_{X \setminus E_\varepsilon} f \, d\mu \le \varepsilon$$

for all n. As f_n also obeys no escape to horizontal/width infinity, it is uniformly integrable.

It suffices by theorem 29 to show that f_n converges to f in measure. Let $\varepsilon > 0$ be arbitrary, and define $E_n := \{f - f_n \geq \varepsilon\}$. By the monotone nature of the sequence f_n, the E_n is non-increasing, and

$$\mu(E_n) \leq \|f - f_n\|_{L^1}/\varepsilon \leq (\|f\|_{L^1} + \|f_n\|_{L^1})/\varepsilon < \infty$$

for all n, by Markov's inequality and the triangle inequality. Hence, by downward monotone convergence of sets, we get

$$\lim_{n\to\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) = 0,$$

where we absorb the null set where $f = \infty$ into the intersection. Consequently, we see that f_n converges to f in measure. □

Exercise 31

Proof. As before, f is absolutely integrable by Fatou's lemma, and by modifying f_n, f on a null set, we may assume that f_n converges pointwise to f. The claim follows then from Exercise 46 of note 3. □

Exercise 32

Proof. (1). By Egorov's theorem, we see that $\forall n \geq 1,\ \exists m_n \geq 1$ such that

$$|f_{n,m}(x) - f_n(x)| \leq 1/n$$

for all x outside of some exceptional set E_n with $\mu(E_n) \leq 1/(n \cdot 2^{n+1})$, whenever $m \geq m_n$. By the axiom of choice, one can take the m_n to be increasing in n. For arbitrary $\varepsilon, \delta > 0$, there is an integer N_1 such that

$$1/N_1 \leq \min(\varepsilon/2, \delta/2),$$

and an integer N_2 such that

$$|f_n(x) - f(x)| \leq \delta/2$$

for all x outside of some exceptional set F with $\mu(F) \leq \varepsilon/2$, whenever $n \geq N_2$. Let $N := \max(N_1, N_2)$. By the triangle inequality, whenever $n \geq N$, we see that

$$|f_{n,m_n}(x) - f(x)| \leq |f_{n,m_n}(x) - f_n(x)| + |f_n(x) - f(x)| \leq 2 \cdot \frac{\delta}{2} = \delta,$$

for all x outside of the exceptional set $E \cup F$, where $E := \bigcup_{n \geq N} E_n$. From our construction, the size of this set is bounded by

$$\mu(E \cup F) \leq \sum_{n \geq N} \mu(E_n) + \mu(F) \leq \sum_{n \geq N} \frac{1}{(n \cdot 2^{n+1})} + \frac{\varepsilon}{2} \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

This implies that f_{n,m_n} converges uniformly to f on the complement of $E \cup F$, and thus f_{n,m_n} converges almost uniformly to f. Apply Egorov's theorem again, we conclude that $f_{m_n,n}$ converges pointwise almost everywhere to f.

(2). Let $X := \bigcup_{n=1}^{\infty} X_n$ be a countable union of finite measure sets X_n, and Y_N be such that $Y_N := \bigcup_{n=1}^{N} X_n$. From part (1), for each N there exist a sequence $m_n^{(N)}$ such that

$$f_{n,m_n^{(N)}} \longrightarrow f$$

almost uniformly as $n \to \infty$ on Y_N. In particular, for each N there is an integer M_N such that

$$|f_{n,m_n^{(N)}}(x) - f(x)| \leq 1/N$$

for all $x \in Y_N$ outside of some exceptional set E_N of measure at most $\frac{1}{N \cdot 2^N}$, whenever $n \geq M_N$. Let us define

$$m_n := m_{M_n}^{(n)};$$

taken to be increasing in n by the axiom of choice. Let $\varepsilon, \delta > 0$ be arbitrary, and N be large enough that $1/N \leq \min(\varepsilon, \delta)$. Then whenever $n \geq N$, we see that

$$|f_{n,m_n}(x) - f(x)| \leq 1/N \leq \varepsilon$$

for all $x \in \bigcup_{n \geq N} Y_n = X$, outside of an exceptional set $\bigcup_{n \geq N} E_n$ of measure at most

$$\sum_{n \geq N} \frac{1}{n \cdot 2^n} \leq \delta.$$

In other words, $f_{m_n,n}$ converges almost uniformly to f, and thus converges pointwise almost everywhere. □

Exercise 33

Proof. The "if" direction is immediate, we show the "only if" part. Let $\varepsilon > 0$ be arbitrary. By the dominated convergence theorem, f_n converges to f in L^1 norm, and thus converges in measure. This implies that $\exists N > 0$, such that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \varepsilon$$

for all $n \geq N$. Take the exceptional set E to be $\{x : |f_n(x) - f(x)| \geq \varepsilon\}$, we conclude that f_n converges almost uniformly to f. □

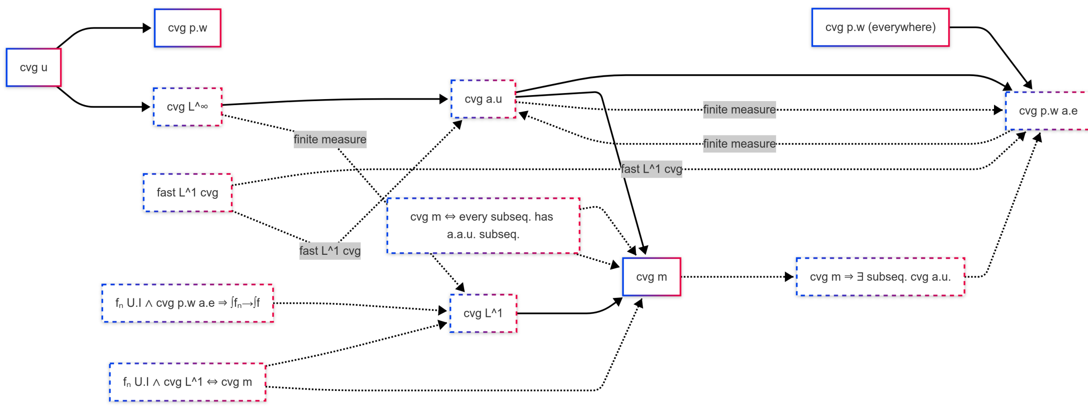

Figure 1: Modes of Convergence

Exercise 34

Proof. If f_n converges in measure f, then any subsequence f_{n_j} of the f_n also converges in measure to f. By Exercise 17, it has a further subsequence that converges almost uniformly, and hence in measure to f.

Conversely, suppose every subsequence f_{n_j} of the f_n has a further subsequence $f_{n_{j_i}}$ that converges almost uniformly to f. If for contradiction we have $f_n \not\to f$ in measure, then $\exists c,\ \varepsilon > 0$, such that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \geq c$$

for every n. Fix this ε, for every further subsequence $f_{n_{j_i}}$ of a random subsequence f_{n_j}, we see that

$$\mu(\{x \in X : |f_{n_{j_i}}(x) - f(x)| \geq \varepsilon\}) \geq c \not\to 0$$

as $i \to \infty$, a contradiction. $\square$