Math 245A Note 3

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1 Selected Exercises in Note 3

Exercise 1

Proof. Let $E, F \in \overline{\mathcal{E}[\mathbf{R}^d]}$. By Boolean closure, $E \cup F \in \overline{\mathcal{E}[\mathbf{R}^d]}$. Also $\emptyset$, $E^c \in \overline{\mathcal{E}[\mathbf{R}^d]}$ by definition. Hence $\overline{\mathcal{E}[\mathbf{R}^d]}$ is a Boolean algebra. $\square$

Exercise 2

Proof. Since $\emptyset \cap E = \emptyset$ for any set E, we get $\emptyset \in \mathcal{B}\big|_Y$. Since $\mathcal{B}$ is a Boolean algebra, $\forall E, F \in \mathcal{B}$, we have $(E \cap Y) \cup (F \cap Y) = (E \cup F) \cap Y \in \mathcal{B}\big|_Y$. Likewise, for $E \cap Y \in \mathcal{B}\big|_Y$, one has $(E \cap Y)^c = Y \backslash (E \cap Y) = Y \cap E^c \in \mathcal{B}\big|_Y$. Hence $\mathcal{B}\big|_Y$ is a Boolean algebra on Y. If Y is $\mathcal{B}$-measurable, then $E \cap Y \in \mathcal{B}$ for any $E \in \mathcal{B}$. Hence $\mathcal{B}\big|_Y \subset \{E \subset Y : E \in \mathcal{B}\}$. Also $\forall E \subset Y, E \in \mathcal{B}$, we can write $E = E \cap Y$, so $\{E \subset Y : E \in \mathcal{B}\} \subset \mathcal{B}\big|_Y$, and we are done. $\square$

Exercise 3

Proof. Suppose that $\mathcal{A}((A_\alpha)_{\alpha \in I}) = \mathcal{A}((A'_{\alpha'})_{\alpha' \in I'})$. By condition, $\forall \alpha' \in I'$, one can write $A'_{\alpha'} = \bigcup_{\alpha \in I_{\alpha'}} A_\alpha = \bigcup_{\alpha \in I_{\alpha'}} (\bigcup_{\beta' \in I'_\alpha} A'_{\beta'})$ for some $I_{\alpha'} \subset I$ and $I'_\alpha \subset I'$. Since the atoms are disjoint, this happens if and only if $I_{\alpha'} = \{\alpha\}$ for a single $\alpha \in I$. Hence we see that $A'_{\alpha'} = A_\alpha$ for some $\alpha \in I$. Similarly, $\forall \alpha \in I'$, $A_\alpha = A'_{\alpha'}$ for some $\alpha' \in I'$. In particular, we conclude that the sets of atoms for the two algebras coincide. That is, there exists a bijection $\phi : I \to I'$ such that $A'_{\phi(\alpha)} = A_\alpha$ for all $\alpha \in I$.

Conversely, suppose that such a bijection exists. Since atomic algebras are formed by unions of the atoms, we clearly have $\mathcal{A}((A_\alpha)_{\alpha \in I}) = \mathcal{A}((A'_{\alpha'})_{\alpha' \in I'})$ by definition. $\square$

Exercise 4

Proof. Let $\mathcal{B}$ be a finite Boolean algebra, thus $\mathcal{B} = \{E_1, \ldots, E_n\}$ for some natural number n. By Boolean closure, the 2^n disjoint sets $A_1, \ldots, A_{2^n}$, each formed by intersecting n elements of the set $\{E_1, \ldots, E_n, E_1^c, \ldots, E_n^c\}$, are measurable. By construction, every measurable set E_i can be written as the union of some $A_j's$. Hence $(A_j)_{j=1}^{2^n}$ is a partition of the ambient space X, and $\mathcal{B} = \mathcal{A}((A_j)_{j=1}^{2^n})$ is a (finite) atomic algebra. In particular, if $\mathcal{B}$ is a finite Boolean algebra that is also an atomic algebra with n atoms, then $|\mathcal{B}| = 2^n$. □

Exercise 5

Proof. Suppose for contradiction that these algebras are all atomic. For the elementary algebra, since the degenerate box $\{x\} = \prod_{i=1}^{d}[x_i, x_i]$ is in $\overline{\mathcal{E}(\mathbf{R}^d)}$, the atoms must be the singleton sets $\{x\}$ for all $x \in \mathbf{R}^d$, which produces the discrete algebra instead.

Likewise, since singleton sets are both Jordan and Lebesgue measurable with measure zero, the Jordan and Lebesgue algebra must have singleton sets as their atoms, which produce the discrete algebra, a contradiction. □

Exercise 6

Proof. Let $E, F \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$. Since the $\mathcal{B}_\alpha$ are Boolean algebras, $E^c \in \mathcal{B}_\alpha$ for all $\alpha \in I$. Hence $E^c \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$. Similarly, $E \cup U,\ \emptyset \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$. By definition, $\bigwedge_{\alpha \in I} \mathcal{B}_\alpha$ is a Boolean algebra.

Let $\mathcal{A}$ be a Boolean algebra that is coarser than all of the $\mathcal{B}_\alpha$. That is, $\mathcal{A} \subset \mathcal{B}_\alpha$ for all $\alpha \in I$. By definition, we must have $\mathcal{A} \subset \bigwedge_{\alpha \in I} \mathcal{B}_\alpha := \bigcap_{\alpha \in I} \mathcal{B}_\alpha$. Therefore, $\bigwedge_{\alpha \in I} \mathcal{B}_\alpha$ is the finest such algebra. □

Exercise 7

Proof. Let $\mathcal{B}$ be the collection of boxes in $\mathbf{R}^d$. By definition, $\langle \mathcal{B} \rangle_{\text{bool}} \subset \overline{\mathcal{E}(\mathbf{R}^d)}$. If $E \in \overline{\mathcal{E}(\mathbf{R}^d)}$, then E is either elementary or co-elementary. That is, E is either a finite union of boxes or the compliment of a union of boxes. But any Boolean algebra that contains $\mathcal{B}$, must contain such unions or compliment of unions by Boolean closure. Hence $\overline{\mathcal{E}(\mathbf{R}^d)} \subset \langle \mathcal{B} \rangle_{\text{bool}}$, and we are done. □

Exercise 8

Proof. We first show that $\langle\mathcal{F}\rangle_{\text{bool}}$ is a finite Boolean algebra, and thus is atomic by Exercise 4. Let $\mathcal{F} = \{E_1, \ldots, E_n\}$, and X be the ambient space. We claim that the E_i's induce a partition of X into at most 2^n atoms. When $n = 0$, we get $2^0 = 1$ and the trivial partition using the only atom X. Suppose inductively that the claim holds for some $n > 0$, and denote the largest possible partition of X by $\{A_1, \ldots, A_{2^n}\}$. For an additional set E_{n+1}, the most atoms can be obtained if E_{n+1} has a non-empty intersection with each of the 2^n atoms induced by $\mathcal{F}$. i.e. $E_{n+1} \cap A_i \neq \emptyset$ for all $1 \leq i \leq 2^n$. This induces a partition $\{E_{n+1} \cap A_i, E_{n+1} \cap A_i^c : 1 \leq i \leq 2^n\}$ of size $2 \cdot 2^n = 2^{n+1}$, induction closed. Since $\langle\mathcal{F}\rangle_{\text{bool}}$ is the coarsest Boolean algebra that contains $\mathcal{F}$, it is finite.

By the proof of Exercise 4 and the last paragraph, $|\langle\mathcal{F}\rangle_{\text{bool}}| \leq 2^{2^n}$, and this bound is the best possible. For an example, consider a discrete ambient space such as the discrete cube $X = \{0,1\}^n$, which has cardinality 2^n, with $|2^X| = 2^{2^n}$. By the definition of generated Boolean algebras and Example 7, $|\langle\mathcal{F}\rangle_{\text{bool}}| \leq |\langle 2^X\rangle_{\text{bool}}| = |2^X| = 2^{2^n}$ for any collection $\mathcal{F}$ of sets in X. $\square$

Exercise 9

Proof. We fist show that $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ is a Boolean algebra. Let $E, F \in \bigcup_{n=0}^{\infty} \mathcal{F}_n$. Then $E \in \mathcal{F}_i$ and $F \in \mathcal{F}_j$ for some i and j. Without loss of generality, suppose that $i \leq j$. By definition, the $\mathcal{F}_n$'s are non-decreasing, so $E, F \in \mathcal{F}_j$, and $E \cup F \in \mathcal{F}_{j+1}$, giving $E \cup F \in \bigcup_{n=0}^{\infty} \mathcal{F}_n$. Similarly, $E^c \in \mathcal{F}_{i+1}$ and thus $E^c \in \bigcup_{n=0}^{\infty} \mathcal{F}_n$. Finally, the empty union $\emptyset$ is in $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ by construction. We conclude that $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ is a Boolean algebra.

Clearly, $\langle\mathcal{F}\rangle_{\text{bool}} \subset \bigcup_{n=0}^{\infty} \mathcal{F}_n$ by definition of the generated Boolean algebra. Conversely, let $E \in \bigcup_{n=0}^{\infty} \mathcal{F}_n$. Then $E \in \mathcal{F}_n$ for some natural number n. By an inductive argument, one can show that E is either the union of a finite number of sets in $\mathcal{F}_0 = \mathcal{F}$ (including the empty union $\emptyset$), or the complement of such a union. By Boolean closure $E \subset\in \mathcal{F}\rangle_{\text{bool}}$. Thus, we have $\bigcup_{n=0}^{\infty} \mathcal{F}_n \subset \langle\mathcal{F}\rangle_{\text{bool}}$, as desired. $\square$

Exercise 10

Proof. Let $\mathcal{A}((A_\alpha)_{\alpha\in I})$ be an atomic algebra, and $E_1, E_2, \ldots \in \mathcal{A}((A_\alpha)_{\alpha\in I})$. By definition, $E_n \in \bigcup_{\alpha\in J_n} A_\alpha$ for some $J_n \subset I$ and all $n \geq 1$. Hence in particular, one has

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{\alpha\in J_n} A_\alpha = \bigcup_{\alpha\in\bigcup_n J_n} A_\alpha \in \mathcal{A}((A_\alpha)_{\alpha\in I}).$$

As a result, all atomic algebras are σ-algebras. In particular, the discrete algebra and trivial algebra are σ-algebras, as are the finite algebras and the dyadic algebras on Euclidean spaces. $\square$

Exercise 11

Proof. By Lemma 10 of Note 1, $\mathcal{L}[\mathbf{R}^d]$ is a σ-algebra. By countable subadditivity, $\mathcal{N}(\mathbf{R}^d)$ is also a σ-algebra. By Exercise 1 of Note 1, the countable union of Jordan measurable sets need not be Jordan measurable, and similar for elementary sets. consequently, the elementary and Jordan algebras are not σ-algebras. $\square$

Exercise 12

Proof. By Exercise 2, $\mathcal{B}\big|_Y := \{E \cap Y : E \in \mathcal{B}\}$ is a Boolean algebra. Given that $E_1 \cap Y, E_2 \cap Y, \ldots \in \mathcal{B}\big|_Y$, we get $\bigcup_{n=1}^{\infty}(E_n \cap Y) = (\bigcup_{n=1}^{\infty} E_n) \cap Y \in \mathcal{B}\big|_Y$, since the set $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}\big|_Y$. This implies that $\mathcal{B}\big|_Y$ is a σ-algebra on the subspace Y. $\square$

Exercise 13

Proof. It is straightforward to check that $\bigwedge_{\alpha\in I} \mathcal{B}_\alpha := \bigcap_{\alpha\in I} \mathcal{B}_\alpha$ is closed under finite union, compliment, and has $\emptyset$ as an element. Hence $\bigwedge_{\alpha\in I} \mathcal{B}_\alpha$ is a Boolean algebra. Let $E_1, E_2, \ldots \in \bigwedge_{\alpha\in I} \mathcal{B}_\alpha$. Since $\mathcal{B}_\alpha$ is closed under countable union for all $\alpha \in I$, we have $\bigcup_{n=1}^{\infty} E_n \in \bigwedge_{\alpha\in I} \mathcal{B}_\alpha$. Consequently, $\bigwedge_{\alpha\in I} \mathcal{B}_\alpha$ is a σ-algebra.

If $\mathcal{B}$ is a σ-algebra that is coarser than all of the $\mathcal{B}_\alpha$, then $\mathcal{B} \subset \mathcal{B}_\alpha$ for all $\alpha \in I$. Thus $\mathcal{B} \subset \bigwedge_{\alpha\in I} \mathcal{B}_\alpha$. $\square$

Exercise 14

Proof. (1). This follows directly from the definition.

(2). By Boolean closure, any σ-algebra that contains the closed subsets of $\mathbf{R}^d$ also contains the open subsets of $\mathbf{R}^d$, and vice versa. By the definition of generated algebra, $\mathcal{B}[\mathbf{R}^d]$ is also generated by the closed sets of $\mathbf{R}^d$.

(3). Let $\mathcal{K}$ be the collection of compact subsets of $\mathbf{R}^d$ and $\mathcal{C}$ be the collection of closed subsets of $\mathbf{R}^d$. By the Heine-Borel theorem, compact sets are closed and bounded, hence by definition we have $\langle\mathcal{K}\rangle \subset \langle\mathcal{C}\rangle = \mathcal{B}[\mathbf{R}^d]$. Conversely, by expressing any closed set C as $C = \bigcup_{n=1}^{\infty} C \cap \overline{B(0,n)}$, we see by the countable union axiom of σ-algebras that any σ-algebra that contains $\mathcal{K}$ must contain $\mathcal{C}$. Hence $\langle\mathcal{C}\rangle \subset \langle\mathcal{K}\rangle$.

(4). Let $\mathcal{O}$ be the collection of open balls of $\mathbf{R}^d$. Since open balls are open sets, we have $\langle\mathcal{O}\rangle \subset \mathcal{B}[\mathbf{R}^d]$. Let $E \subset \mathbf{R}^d$ be an open subset. By definition, for every $x \in E$, there exists $r_x > 0$ such that $r_x \in \mathbf{Q}$ and $B(x, r_x) \subset E$. In particular, we can express E as a countable union of open balls as $E = \bigcup_{x \in \mathbf{Q} \cap E} B(x, r_x)$. Hence every σ-algebra that contains $\mathcal{O}$ must contain the collection of open sets, which implies that $\mathcal{B}[\mathbf{R}^d] \subset \langle\mathcal{O}\rangle$.

(5). Let $\mathcal{B}$ be the collection of boxes of $\mathbf{R}^d$. By Lemma 8 of Note 1, every open set of $\mathbf{R}^d$ can be expressed as the countable union of almost disjoint boxes. By the countable union axiom of σ-algebras, $\mathcal{B}[\mathbf{R}^d] \subset \langle\mathcal{B}\rangle$. Conversely, since every interval can be written as a countable intersection of open intervals, every box can be written as a countable intersection of open boxes. In particular, every σ-algebra that contains the open sets must contain $\mathcal{B}$. i.e. $\langle\mathcal{B}\rangle \subset \mathcal{B}[\mathbf{R}^d]$.

(6). Let $\mathcal{E}$ be the collection of elementary sets. By definition, elementary sets are finite union of boxes. By (5), this implies $\langle\mathcal{E}\rangle \subset \mathcal{B}[\mathbf{R}^d]$. Conversely, boxes are elementary sets, so every σ-algebra that contains $\mathcal{E}$ must contain the boxes, and $\mathcal{B}[\mathbf{R}^d] \subset \langle\mathcal{E}\rangle$. $\square$

Exercise 17

Proof. Assume first that $F \subset \mathbf{R}^{d_2}$ is a box. Let $P(E)$ be the property of any subset $E \subset \mathbf{R}^{d_1}$ that $E \times F \subset \mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$ is Borel measurable. For $\emptyset \subset \mathbf{R}^{d_1}$, $\emptyset \times F = \emptyset \subset \mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$ is Borel measurable. For any box $E \subset \mathbf{R}^{d_1}$, $E \times F$ is a box and thus Borel measurable by Exercise 14. If $E \times F$ is Borel measurable for some $E \subset \mathbf{R}^{d_1}$, then $(\mathbf{R}^{d_1} \backslash E) \times F = (\mathbf{R}^{d_1} \times F) \backslash (E \times F)$ is Borel measurable by Boolean closure. Finally, if $E_1, E_2, \ldots \subset \mathbf{R}^{d_1}$ are such that $E_n \times F$ are Borel measurable for all n, then $\bigcup_{n=1}^{\infty} E_n \times F$ is Borel measurable by the countable union axiom of the Borel σ-algebra. By Remark 4 and Exercise 14, $E \times F$ is Borel measurable for any Borel measurable subset $E \subset \mathbf{R}^{d_1}$.

Let $P'(F)$ be the property of any subset $F \subset \mathbf{R}^{d_2}$ that $E \times F$ is Borel measurable whenever $E \subset \mathbf{R}^{d_1}$ is Borel measurable. By the last paragraph, $P'(F)$ holds for all boxes $F \subset \mathbf{R}^{d_2}$ (including the empty box $\emptyset \subset \mathbf{R}^{d_2}$). Likewise, $P'(\mathbf{R}^{d_2} \backslash F)$ holds whenever $P'(F)$ holds for some subset $F \subset \mathbf{R}^{d_2}$. And if $F_1, F_2, \ldots \subset \mathbf{R}^{d_2}$ are such that $E \times F_n$ are Borel measurable for all n, then $\bigcup_{n=1}^{\infty} E \times F_n$ is Borel measurable by the countable union axiom of the Borel σ-algebra. Again by Remark 4 and Exercise 14, we see that $E \times F$ is Borel measurable for any Borel measurable $E \subset \mathbf{R}^{d_1}$ and $F \subset \mathbf{R}^{d_2}$. $\square$

Exercise 18

Proof. (1). By symmetry, we only need to show the claim for one slice. Fix some $x_1 \in \mathbf{R}^{d_1}$. Let $P(E)$ be the property of any subset $E \subset \mathbf{R}^{d_1+d_2}$, that the slice $\{x_2 \in \mathbf{R}^{d_2} : (x_1, x_2) \in E\}$ is a Borel measurable subset of $\mathbf{R}^{d_2}$. Clearly $P(\emptyset)$ holds trivially. If $E = \prod_{i=1}^{d_1+d_2} I_i$ is a box, then the slice is either the box $\prod_{i=d_1+1}$, then the slice is either the box $\prod_{i=d_1+1}^{d_1+d_2} I_i \subset \mathbf{R}^{d_2}$ or the empty set $\emptyset \subset \mathbf{R}^{d_2}$, which are both Borel measurable. Thus $P(E)$ hold for all boxes $E \subset \mathbf{R}^{d_1+d_2}$. Let $P(E)$ be true, then from

$$\{x_2 \in \mathbf{R}^{d_2} : (x_1, x_2) \in \mathbf{R}^{d_1+d_2} \backslash E\} = \mathbf{R}^{d_2} \backslash \{x_2 \in \mathbf{R}^{d_2} : (x_1, x_2) \in E\}$$

and Boolean closure, we see that $P(\mathbf{R}^{d_1+d_2} \backslash E)$ holds as well. Similarly, if $E_1, E_2, \ldots \subset \mathbf{R}^{d_1+d_2}$ are such that $P(E_n)$ is true for all n, then $P(\bigcup_{n=1}^{\infty} E_n)$ is true also. By Remark 4 and Exercise 14, $P(E)$ is true of any Borel measurable $E \subset \mathbf{R}^{d_1+d_2}$.

(2). Let $E \subset [0, 1]$ be the set of Proposition 12 in Note 1. By definition the Cartesian product of E with any point $x \in \mathbf{R}$ is a Lebesgue null set, but E is not Lebesgue measurable. $\square$

Exercise 19

Proof. Denote the Borel σ-algebra, the null σ-algebra and the Lebesgue σ-algebra by $\mathcal{B}[\mathbf{R}^d]$, $\mathcal{N}(\mathbf{R}^d)$ and $\mathcal{L}[\mathbf{R}^d]$ respectively. Clearly $\mathcal{B}[\mathbf{R}^d] \cup \mathcal{N}(\mathbf{R}^d) \subset \mathcal{L}[\mathbf{R}^d]$ by definition, hence $\langle \mathcal{B}[\mathbf{R}^d] \cup \mathcal{N}(\mathbf{R}^d) \rangle \subset \mathcal{L}[\mathbf{R}^d]$.

Conversely, let $E \in \mathcal{L}[\mathbf{R}^d]$. By Exercise 19 of Note 1, E is a G_δ set with a null set removed, thus $E \in \langle \mathcal{B}[\mathbf{R}^d] \cup \mathcal{N}(\mathbf{R}^d) \rangle$ by Boolean closure. Consequently, we see that $\langle \mathcal{B}[\mathbf{R}^d] \cup \mathcal{N}(\mathbf{R}^d) \rangle = \mathcal{L}[\mathbf{R}^d]$. $\square$

Exercise 20

Proof. (1). By finite additivity, $\mu(F) = \mu(E) + \mu(F \backslash E) \geq \mu(E)$.

(2). This follows from the definition of finite additive measure and induction.

(3). This follows from (2) and that some of the E_i might not be disjoint.

(4). Writing $E \cup F = (E \backslash F) \cup (F \backslash E) \cup (E \cap F)$ as a disjoint union, we get $\mu(E \cup F) = \mu(E \backslash F) + \mu(F \backslash E) + \mu(E \cap F)$ by finite additivity. Similarly, $\mu(E) + \mu(F) = \mu(E \backslash F) + \mu(F \backslash E) + 2\mu(E \cap F)$, from which the claim follows. □

Exercise 21

Proof. By Exercise 4, every finite Boolean algebra is an atomic algebra. Hence if $\mathcal{B}$ is a finite Boolean algebra, it is generated by a finite family $A_1, \ldots, A_k$ of non-empty atoms. By finite additivity, we get

$$\mu(E) = \sum_{1 \leq j \leq k : A_j \subset E} c_j$$

for any $E \in \mathcal{B}$, where $c_j = \mu(A_j)$ for all $1 \leq j \leq k$. Equivalently, if x_j is a point in A_j for each $1 \leq j \leq k$, then $\mu = \sum_{j=1}^{k} c_j \delta_{x_j}$. Furthermore, since μ is characterized by its' values on the atoms, the $c_1, \ldots, c_k$ are uniquely determined by μ. □

Exercise 22

Proof. (1). By Example 11, $c\mu$ is a finitely additive measure. If $E_1, E_2, \ldots \in \mathcal{B}$ are a countable sequence of disjoint measurable sets, since μ is a countably additive measure on $\mathcal{B}$, we get

$$c\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = c\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} c\mu(E_n),$$

so $c\mu$ is also countably additive.

(2). By definition, $(\sum_{n=1}^{\infty} \mu_n)(\emptyset) = \sum_{n=1}^{\infty} \mu_n(\emptyset) = 0$. If $E_1, E_2, \ldots \in \mathcal{B}$ are a countable sequence of disjoint measurable sets, then

$$(\sum_{n=1}^{\infty} \mu_n)(\bigcup_{m=1}^{\infty} E_m) = \sum_{n=1}^{\infty} \mu_n(\bigcup_{m=1}^{\infty} E_m) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mu_n(E_m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mu_n(E_m)$$

by Tonelli's theorem for series. i.e. the sum $\sum_{n=1}^{\infty} \mu_n$ is a countably additive measure. □

Exercise 23

Proof. (1). This follows from countable additivity.

(2). Express $\bigcup_{n=1}^{\infty} E_n$ as the disjoint union $\bigcup_{n=1}^{\infty} A_n,\ A_n := E_n \backslash \bigcup_{n'=1}^{n-1} E_{n'}$, then countable additivity gives

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(E_n) - \mu(E_{n-1}) = \lim_{n\to\infty} \mu(E_n) = \sup_n \mu(E_n),$$

where we take $E_0 := \emptyset$.

(3). Suppose that $\mu(E_N) < \infty$ for some $N \geq 1$. Note that

$$E_N \backslash \bigcap_{n=N}^{\infty} E_n = (E_N \backslash E_{N+1}) \cup (E_{N+1} \backslash E_{N+2}) \cup \ldots$$

By countable additivity and the finiteness of $\mu(E_N)$, this implies

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcap_{n=N}^{\infty} E_n) = \mu(E_N) - \sum_{i=0}^{\infty} \mu(E_{N+i} \backslash E_{N+i+1})$$

The RHS then reduces to $\lim_{N\to\infty} \mu(E_N) = \inf_n \mu(E_n)$.

Consider the real line with the Lebesgue σ-algebra and the Lebesgue measure, with $E_n := (n, \infty)$ for all n, hence the downward monotone convergence claim can fail if the hypothesis that $\mu(E_n) < \infty$ for at least one n is dropped. □

Exercise 24

Proof. (1). Use the identity $1_E(x) = \liminf_{n\to\infty} 1_{E_n}(x),\ E = \bigcup_{N\geq 1}\bigcap_{n\geq N} E_n$ as shown in Exercise 13 of Note 1, E is $\mathcal{B}$-measurable by Boolean closure.

(2). By finite subadditivity and monotonicity, we have

$$\lim_{n\to\infty}\mu(E_n)-\mu(E)=\lim_{n\to\infty}\mu(E_n\Delta E)\le\lim_{N\to\infty}\mu(\bigcup_{n>N}(E_n\Delta E))$$

which by downward monotonicity becomes $\mu(\bigcap_N\bigcup_{n>N}(E_n\Delta E))$. As in (1), one can show that $E=\bigcap_N\bigcup_{n>N}E_n$. In particular, we have

$$\begin{aligned}E\backslash E&=(\bigcap_N\bigcup_{n>N}E_n)\backslash E=\bigcap_N(\bigcup_{n>N}E_n\backslash E)=\bigcap_N\bigcup_{n>N}(E_n\backslash E)\\ E\backslash E&=E\backslash(\bigcup_N\bigcap_{n>N}E_n)=\bigcap_N(E\backslash\bigcap_{n>N}E_n)=\bigcap_N\bigcup_{n>N}(E\backslash E_n).\end{aligned}$$

This implies that $\bigcap_N\bigcup_{n>N}(E_n\Delta E)=(E\backslash E)\cup(E\backslash E)=\emptyset$, which has measure zero, and we are done.

(3). We need all the E_n be contained in a set of finite measure such that the sets $A_N=\bigcup_{n>N}$ have finite measure, and the downward monotone convergence is applicable. □

Exercise 25

Proof. Given any measure μ on the measurable space $(X,2^X)$, we define $c_x:=\mu(\{x\})$ for every $x\in X$, then clearly we have $\mu=\sum_{x\in X}c_x\delta_x$ by countable or finite additivity. In particular, $\forall E\subset X$, it follows that $\mu(E)=\sum_{x\in E}c_x$. □

Exercise 26

Proof. Let $\overline{\mathcal{B}}:=\{E'\subset X:E'\triangle E\subset N$ for some $E,N\in\mathcal{B}$ with $\mu(N)=0\}$. Define a measure $\overline{\mu}$ on $\overline{\mathcal{B}}$ by $\overline{\mu}(E'):=\mu(E)$.

(**$\overline{\mathcal{B}}$ is a $\sigma-$ algebra**): Clearly $\emptyset\in\overline{\mathcal{B}}$ since $\emptyset=\emptyset\triangle\emptyset$ is a $\mathcal{B}$-subnull set. If $E'\in\overline{\mathcal{B}}$, then $(E')^c\triangle E^c=E\triangle E'$, which is a $\mathcal{B}$-subnull set by construction. i.e. $(E')^c\in\overline{\mathcal{B}}$. Let $(E'_n)_n$ be a countable sequence in $\overline{\mathcal{B}}$, with corresponding $\mathcal{B}$-measurable sequence $(E_n)_n$, and corresponding sequence of $\mathcal{B}$-null sets $(F_n)_n$. Then

$$(\bigcup_n E'_n\setminus\bigcup_m E_m)=\bigcup_n(E'_n\setminus\bigcup_m E_m)\subset\bigcup_n(E'_n\setminus E_n)\subset\bigcup_n F_n$$

Similarly we have $(\bigcup_m E_m\setminus\bigcup_n E'_n)\subset\bigcup_n F_n$. In particular, it follows that

$$(\bigcup_n E'_n)\triangle(\bigcup_m E_m)\subset\bigcup_n F_n,$$

which is a $\mathcal{B}$-null set by countable sub-additivity. Therefore, $\bigcup_n E'_n \in \overline{\mathcal{B}}$ and $\overline{\mathcal{B}}$ is a $\sigma-$ algebra.

(**$\overline{\mu}$ is well-defined**): For some $E' \subset X$, let $A, B \in \mathcal{B}$ be such that $E'\Delta A$ and $E'\Delta B$ are both $\mathcal{B}$-subnull set. Note that

$$A \backslash B = ((A \cap E') \backslash B) \cup ((A \backslash E') \backslash B) \subset (E' \backslash B) \cup (A \backslash E') \subset (E'\Delta A) \cup (E'\Delta B).$$

By symmetry, this implies that $A\Delta B$ is a $\mathcal{B}$-null set, and in particular

$$\overline{\mu}(E') = \mu(A) = \mu(B),$$

so $\overline{\mu}$ is indeed a map. By definition, $\overline{\mu}(\emptyset) := \mu(\emptyset) = 0$. Let $E'_1, E'_2, \ldots \in \overline{\mathcal{B}}$ be a disjoint sequence, with the corresponding sequence $E_1, E_2, \ldots \in \mathcal{B}$. Furthermore, we can take the $E'_i s$ to be disjoint: Consider $E_i \cap E_j$. Then either

$$E_i \cap E_j \subset E_i \backslash E'_i \text{ or } E_i \cap E_j \subset E_i \cap E'_i.$$

In the first case, $E_i \cap E_j \subset E_i \Delta E'_i$ is a null set, and hence can be removed from E_i. In the second case, it must be that $E_i \cap E_j \subset E_j \backslash E'_j$, and identical argument shows that it is a null set and thus can be removed from E_j. By the fact that $(\bigcup_n E_n)\Delta(\bigcup_n E'_n) \subset \bigcup_n (E_n \Delta E'_n)$, monotonicity, and countable subadditivity, we have

$$\overline{\mu}(\bigcup_n E'_n) = \mu(\bigcup_n E_n) = \sum_n \mu(E_n) = \sum_n \overline{\mu}(E'_n).$$

Thus we have shown that $\overline{\mu}$ is a well-defined countably additive measure.

(**$(X, \overline{\mathcal{B}}, \overline{\mu})$ is complete**): Let $N' \in \overline{\mathcal{B}}$ be such that $\overline{\mu}(N') = \mu(N) = 0$, with $N, F \in \mathcal{B}$ be such that $N' \triangle N \subset F$ and $\mu(F) = 0$. Suppose that $A \subset N'$. Then

$$A \triangle N = (A \setminus N) \cup (N \setminus A) \subset (N' \setminus N) \cup N \subset F \cup N,$$

which implies that $A \in \overline{\mathcal{B}}$. By monotonicity, $\overline{\mu}(A) \leq \overline{\mu}(N') = 0$. Consequently, the measure space $(X, \overline{\mathcal{B}}, \overline{\mu})$ is complete.

(**coarsest complete refinement**): Clearly, $\mathcal{B} \subset \overline{\mathcal{B}}$. Let $(X, \mathcal{A}, \lambda)$ be a complete refinement of $(X, \mathcal{B}, \mu)$, and $E' \in \overline{\mathcal{B}}$, with $E, N \in \mathcal{B}$ be such that $E'\Delta E \subset N$, $\mu(N) = 0$. Observe that $E' \cap E$ can be expressed as $E \backslash N$ (or $E' \backslash N$ for that matter). Since $E' \backslash E \subset N$ and $(X, \mathcal{A}, \eta)$ is complete, $\lambda(E' \backslash E) = 0$. In particular, $E' \backslash E \in \mathcal{A}$. Clearly $E \setminus N \in \mathcal{B} \subset \mathcal{A}$. Hence we get

$$E' = (E \backslash N) \cup (E' \backslash E) \in \mathcal{A}$$

and therefore $\overline{\mathcal{B}} \subset \mathcal{A}$. $\square$

Exercise 27

Proof. By part (iii) of Lemma 10 in Note 1, the Lebesgue measure space is complete. By Exercise 26, it suffices to show that

$$\mathcal{L}[\mathbf{R}^d] \subset \overline{\mathcal{B}[\mathbf{R}^d]} = \{E' \subset \mathbf{R}^d : \exists E, N \in \mathcal{B}[\mathbf{R}^d], E' \triangle E \subset N,\ m(N) = 0\}.$$

Let $A \subset \mathbf{R}^d$ be a Lebesgue measurable set. By Exercise 19 of Note 1, A is is a G_δ set with a null set removed, which implies that $A \in \overline{\mathcal{B}[\mathbf{R}^d]}$ and thus $\mathcal{L}[\mathbf{R}^d] \subset \overline{\mathcal{B}[\mathbf{R}^d]}$, as desired. □

Exercise 28

Proof. (1). Suppose that the level sets $\{x \in X : f(x) > \lambda\}$ are $\mathcal{B}$-measurable. By Lemma 8 of Note 1, it suffices that $f^{-1}(I)$ is measurable for any closed interval $I = [a, b]$. Expressing $[a, b] = \bigcap_{n=1}^{\infty}(a - \delta/n, b]$, it suffices that $f^{-1}((c, d])$ are $\mathcal{B}$-measurable for any $c \leq d$. But $(c, d] = (c, +\infty) \backslash (d, +\infty)$, from the Boolean closure the result follows. Conversely, if the function $f : X \to [0, +\infty]$ is measurable, then the level sets are clearly $\mathcal{B}$-measurable since the sets $(\lambda, +\infty]$ are open in $[0, +\infty]$.

(2). For any $\lambda \in [0, +\infty]$, either $\lambda < 0$, or $\lambda \in [0, 1]$, or $\lambda > 1$. Correspondingly, the set $1_E^{-1}((\lambda, +\infty])$ is equal to either X or E or $\emptyset$. The result then follows from (1).

(3). Since open sets are Borel-measurable, f is measurable if $f^{-1}(E)$ is $\mathcal{B}$-measurable for every Borel-measurable subset E. Conversely, if f is measurable, let $P(E)$ be the property of sets $E \subset Y$ that $f^{-1}(E) \in \mathcal{B}$, where $Y = \mathbf{C}$ or $Y = [0, +\infty]$. By definition, $f^{-1}(U) \in \mathcal{B}$ for all open sets $U \subset Y$, and $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}$. If $f^{-1}(E) \in \mathcal{B}$, $f^{-1}(Y \backslash E) = X \backslash f^{-1}(E) \in \mathcal{B}$. Let $E_1, E_2, \ldots, \subset Y$ be such that $f^{-1}(E_n) \in \mathcal{B}$ for all n, then $f^{-1}(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} f^{-1}(E_n) \in \mathcal{B}$. By Remark 4, $f^{-1}(E)$ is $\mathcal{B}$-measurable for every Borel-measurable subset E.

(4). We identify $\mathbf{C} = \mathbf{R}^2$, and let $f : X \to \mathbf{C}$ be measurable. Then $\mathrm{Re}(f)^{-1}(U) = f^{-1}(U \times \mathbf{R})$, $\mathrm{Im}(f)^{-1}(V) = f^{-1}(\mathbf{R} \times V)$ are both $\mathcal{B}$-measurable for any open sets $U, V \subset \mathbf{R}$, and $\mathrm{Re}(f), \mathrm{Im}(f) : X \to \mathbf{R}$ are both measurable. Conversely, if $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are both measurable, let $U \subset \mathbf{R}^2$ be an open set. Let $\pi_1, \pi_2 : X \to \mathbf{R}$ be the projections onto the real and imaginary axis, respectively. Let $x \in \pi_1(U)$. That is, $\exists c = (x, y) \in U$. Since U is open, there is an open ball $B(c, r) \subset U$. In particular, $(x - r, x + r) \subset \pi_1(U)$. Hence $\pi_1(U)$ is open. Likewise, $\pi_2(V)$ is open. Therefore, $f^{-1}(U) = \mathrm{Re}(f)^{-1}(\pi_1(U)) \cap \mathrm{Im}(f)^{-1}(\pi_2(V))$ is open, and f is measurable.

(5). Let $f : X \to \mathbf{R}$ be measurable, and $\lambda \in \mathbf{R}$. For $\lambda = 0$, we have

$$f_+^{-1}((\lambda, +\infty)) = f^{-1}((0, +\infty)),\ f_-^{-1}((\lambda, +\infty)) = f^{-1}((-\infty, 0))$$

both being $\mathcal{B}$-measurable. And for $\lambda > 0$, we have

$$f_+^{-1}((\lambda, +\infty)) = f^{-1}((\lambda, +\infty)),\ f_-^{-1}((\lambda, +\infty)) = f^{-1}((-\infty, -\lambda))$$

both being $\mathcal{B}$-measurable. Finally, for $\lambda < 0$, we have

$$f_+^{-1}((\lambda, +\infty)) = f_-^{-1}((\lambda, +\infty)) = X$$

which is $\mathcal{B}$-measurable. By part (1), this implies that f_+ and f_- are measurable.

Conversely, let $f_+ := \max(f, 0),\ f_- := \max(-f, 0)$ be measurable, and $U \subset \mathbf{R}$ be open. Note that $U = (U \cap (0, +\infty]) \cup (U \cap (-\infty, 0))$. Hence it follows that $f^{-1}(U) = f_+^{-1}(U \cap [0, +\infty)) \cup f_-^{-1}(-U \cap (0, +\infty))$ is $\mathcal{B}$-measurable, and f is measurable.

(6). If f is the pointwise limit of f_n, then for every $x \in X$ one has

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \inf_{N>0} \sup_{n\geq N} f_n(x).$$

This implies that, for any λ, the set $\{x \in X : f(x) > \lambda\}$ is equal to

$$\bigcup_{M>0} \bigcap_{N>0} \{x \in X : \sup_{n\geq N} f_n(x) > \lambda + \frac{1}{M}\}$$

which in turn is equal to

$$\bigcup_{M>0} \bigcap_{N>0} \bigcup_{n\geq N} \{x \in X : f_n(x) > \lambda + \frac{1}{M}\}.$$

As each f_n is measurable, the sets $\{x \in X : f_n(x) > \lambda + \frac{1}{M}\}$ are $\mathcal{B}$-measurable by (1). Since σ-algebras are closed under countable Boolean operations, we obtain the claim. Replace $[0, +\infty]$ by $\mathbf{C}$. By (4) and (5), $\mathrm{Re}(f_n)_+,\ \mathrm{Re}(f_n)_-$ are measurable for all n, and by what we just proved, $\mathrm{Re}(f)_+,\ \mathrm{Re}(f)_-$ are thus measurable, and so $\mathrm{Re}(f)$ is measurable. Similarly, $\mathrm{Im}(f)$ is measurable. We conclude that $f : X \to \mathbf{C}$ is measurable.

(7). Let $U \subset [0, +\infty]$ be an open set. Since $\phi : [0, +\infty] \to [0, +\infty]$ is continuous, $\phi^{-1}(U) \subset [0, +\infty]$ is open. Therefore, $(\phi \circ f)^{-1}(U) = f^{-1} \circ \phi^{-1}(U)$ is measurable. That is, the function $\phi \circ f : X \to [0, +\infty]$ is measurable. Clearly, the same argument holds if $[0, +\infty]$ is replaced by $\mathbf{C}$.

(8). Let $f, g : X \to [0, +\infty]$ be measurable, and define $h : X \to \mathbf{C}$ by $h(x) := f(x) + ig(x)$, which is measurable by (4). Let $s : \mathbf{C} \to \mathbf{C}$ be the continuous map given by $s(a + bi) := a + b$, and $p : \mathbf{C} \to \mathbf{C}$ be the continuous map given by $p(a + bi) := ab$. Then by (7), $f + g = s \circ h : X \to [0, +\infty]$ and $fg = p \circ h : X \to [0, +\infty]$ are both measurable.

Using the continuous map $d : \mathbf{C} \to \mathbf{C}$ given by $d(a+bi) := a-b$, we see further that the difference of two unsigned measurable functions is measurable too. Hence for measurable functions $f, g \in \mathbf{C}$, by splitting into real and imaginary parts, and using (4) and (5), one gets that $f+g$ and fg are both measurable. □

Exercise 29

Proof. Let $f : X \to [0, +\infty]$ or $\mathbf{C}$ be a measurable function, and $A_\alpha \in \mathcal{B},\ \alpha \in I$ be an atom of X. For the sake of contradiction, assume that $\exists x,\ y \in A_\alpha$ such that $f(x) = c \neq d = f(y)$. By (3) of Exercise 28, $f^{-1}(\{c\})$ and $f^{-1}(\{d\})$ are both $\mathcal{B}$-measurable. Since X is atomic, both sets are union of some atoms. In particular, we get $A_\alpha \subset f^{-1}(\{c\}) \cap f^{-1}(\{d\})$, which implies that for all $x \in A_\alpha,\ f(x) = c$ and $f(x) = d$, a contradiction.

This implies that $f = \sum_{\alpha \in I} c_\alpha 1_{A_\alpha}$ for some constants c_α in $[0, +\infty]$ or in $\mathbf{C}$ as appropriate. And if $f \neq g$, they must differ on some atom A_α, hence the c_α are uniquely determined by f. □

Exercise 30

Proof. Fix $\varepsilon > 0$. By modifying f_n and f on a set of measure zero (that can be absorbed into E at the end of the argument) we may assume that f_n converges pointwise everywhere to f. Define

$$E_{N,m} := \bigcup_{n \geq N} \{x \in X : |f_n(x) - f(x)| > 1/m\}$$

to be the set on which $|f_n(x) - f(x)| > 1/m$ for some $n \geq N$. Therefore $\bigcap_{N>0} E_{N,m}$ is the set on which $|f_n(x) - f(x)| > 1/m$ for an arbitrarily large n. Note that these sets are all $\mathcal{B}$-measurable by (7) and (1) of Exercise 29. Since $f_n \to f$ pointwise, we have $\bigcap_{N>0} E_{N,m} = \emptyset$. Since $\mu(X) < \infty$, and the $E_{N,m}$ are decreasing in N, we can apply downward monotone convergence to conclude that

$$\lim_{N \to \infty} \mu(E_{N,m}) = \mu(\bigcap_{N>0} E_{N,m}) = 0$$

for all $m > 0$. In particular, $\exists N_m > 0$ such that $\mu(E_{N,m}) < \varepsilon/2^m$ for all $N \geq N_m$. Define $E := \bigcup_{m=1}^\infty E_{N_m,m}$. By countable subadditivity,

$$\mu(E) \leq \sum_{m=1}^\infty \varepsilon/2^m = \varepsilon.$$

By our construction of E, f_n converges uniformly to f outside of E.

From the moving bump example $f_n := 1_{[n,n+1]}$ on $\mathbf{R}$ in Remark 7 of Note 2, the claim can fail when the measure μ is not finite. $\square$

Exercise 31

Proof. By condition, the atoms $\{A_i' : 1 \leq i \leq n\}$ of $\mathcal{B}'$ partition the atoms $\{A_i : 1 \leq i \leq m\}$ of $\mathcal{B}$. That is, for each $1 \leq i \leq m$, $A_i = \bigcup_{j \in J_i; J_i \subset \{1,2,\ldots,n\}} A_j'$ for some subset J_i of $\{1, 2, \ldots, n\}$. Therefore, f can also be represented as $f = \sum_{j \in J_i : 1 \leq i \leq m} c_i A_j'$, and one obtains that

$$\mathrm{Simp} \int_X f \, d\mu := \sum_{j \in J_i : 1 \leq i \leq m} c_i \mu(A_j') = \sum_{i=1}^{m} c_i \mu(A_i)$$

from finite additivity. Hence simple integrals are unaffected by refinements. $\square$

Exercise 32

Proof. By condition, f is a measurable function that takes on finitely many values $a_1, \ldots, a_k$, and g is a measurable function that takes on finitely many values $b_1, \ldots, b_j$. Let $\mathcal{B}''$ be the finite σ-algebra generated by the preimages $A_1 := f^{-1}(\{a_1\}), \ldots, A_k := f^{-1}(\{a_k\})$, and $\mathcal{B}'''$ be the finite σ-algebra generated by the preimages $B_1 := g^{-1}(\{b_1\}), \ldots, B_j := g^{-1}(\{b_j\})$.

(1). By condition, f and g are both measurable with respect to the common refinement $\mathcal{B}'' \vee \mathcal{B}''' := \langle \mathcal{B}'' \cup \mathcal{B}''' \rangle$, which is also finite. By Exercise 4, X is partitioned into a finite number of atoms $E_1, \ldots, E_n$, then by Exercise 29, f and g each has a unique representation of the form $f = \sum_{i=1}^{n} c_i 1_{E_i}$, $g = \sum_{i=1}^{n} d_i 1_{E_i}$, where $0 \leq c_i \leq d_i$ for all i. Hence by definition and Exercise 31,

$$\mathrm{Simp} \int_X f \, d\mu = \mathrm{Simp} \int_X f \, d\mu \downharpoonleft_{\mathcal{B}'' \vee \mathcal{B}'''} := \sum_{i=1}^{n} c_i \mu(E_i),$$

which is less than or equal to

$$\mathrm{Simp} \int_X g \, d\mu = \mathrm{Simp} \int_X g \, d\mu \downharpoonleft_{\mathcal{B}'' \vee \mathcal{B}'''} := \sum_{i=1}^{n} d_i \mu(E_i).$$

(2). This follows from definition 9 and 10.

(3). From definition and part (1), we see that

$$\text{Simp}\int_X cf\ d\mu = \sum_{i=1}^{n} cc_i\mu(E_i) = c\sum_{i=1}^{n} c_i\mu(E_i) = c \times \text{Simp}\int_X cf\ d\mu.$$

(4). By part (8) of Exercise 28, $f+g$ is measurable, and has a unique representation of the form $f+g=\sum_{i=1}^{n}(c_i+d_i)1_{E_i}$ by part (1), thus it is a simple function by definition. It follows that

$$\text{Simp}\int_X (f+g)\ d\mu = \sum_{i=1}^{n}(c_i+d_i)\mu(E_i)$$
$$= \sum_{i=1}^{n} c_i\mu(E_i) + \sum_{i=1}^{n} d_i\mu(E_i) = \text{Simp}\int_X f\ d\mu + \text{Simp}\int_X g\ d\mu.$$

(5). By condition, $(X,\mathcal{B}',\mu')$ is a refinement of $(X,\mathcal{B}'',\mu'\downharpoonright_{\mathcal{B}''})$. Viewing $(X,\mathcal{B}'',\mu'\downharpoonright_{\mathcal{B}''})$ as an extension of $(X,\mathcal{B}'',\mu\downharpoonright_{\mathcal{B}''})$, we obtain from definition 10 and Exercise 31 that

$$\text{Simp}\int_X f\ d\mu' := \text{Simp}\int_X f\ d\mu'\downharpoonright_{\mathcal{B}''} = \text{Simp}\int_X f\ d\mu\downharpoonright_{\mathcal{B}''} = \text{Simp}\int_X f\ d\mu.$$

(6). By condition, $f \neq g$ on some sub-null set N. In part (1), we can then remove $E_i \cap N$ from E_i for all $1 \le i \le n$ since they contribute nothing to the value of the simple integral. Thus one can assume that $f = g$ on E_i for all i, or equivalently that $f = g$ for every $x \in X$, it follows directly that $\text{Simp}\int_X f\ d\mu = \text{Simp}\int_X g\ d\mu$.

(7) Suppose that $\text{Simp}\int_X f\ d\mu < \infty$. From definition 10, this implies that

$$\sum_{i=1}^{k} a_i\mu(A_i) < \infty.$$

Hence it must be that $a_i\mu(A_i) < \infty$ for all $1 \le i \le k$. In particular, for those i with $a_i \neq 0$, it must be that $\mu(A_i) < \infty$, i.e. f is supported on a set of finite measure. Furthermore, if $a_i = \infty$, then $\mu(A_i) = 0$. Since the union of null sets is again a null set, we see that it is only possible for f to be infinite on a null set, so f is finite almost everywhere.

Conversely, if $f = \sum_{i=1}^{k} a_i 1_{A_i}$ is finite almost everywhere, and is supported on a set of finite measure, then for any i, we deduce that $a_i = \infty$ implies that $\mu(A_i) = 0$, and that $\mu(A_i) = \infty$ implies that $a_i = 0$. Consequently, one has

$$\text{Simp}\int_X f\ d\mu = \sum_{i=1}^{k} a_i\mu(A_i) < \infty.$$

(8). Given that Simp $\int_X f\ d\mu = 0$, if $f > 0$ on some measurable set E with positive measure, then part (1) and (2) would have shown that Simp $\int_X f\ d\mu \geq \mu(E) > 0$, a contradiction. Conversely, if f is zero almost everywhere, then $f \leq 1_E$ for some null set E, and (1) and (2) again show that Simp $\int_X f\ d\mu = 0$. $\square$

Exercise 33

Proof. Induct on n, by finite additivity, the identity holds for $n = 1, 2$. Suppose inductively that the identity holds for some $n > 2$. Let $A := \bigcup_{i=1}^n A_i$. By our induction hypothesis,

$$\mu(\bigcup_{i=1}^{n+1} A_i) = \mu(A \cup A_{n+1}) = \mu(A) + \mu(A_{n+1}) - \mu(A \cap A_{n+1})$$

$$= \sum_{J\subset\{1,\ldots,n\}:J\neq\emptyset} (-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i) + \mu(A_{n+1}) - \mu(\bigcup_{i=1}^{n} A_i \cap A_{n+1}).$$

Replacing A_i with $A_i \cap A_{n+1}$ and using the induction hypothesis again, we get:

$$\sum_{J\subset\{1,\ldots,n\}:J\neq\emptyset} (-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i) + \mu(A_{n+1}) - \sum_{I\subset\{1,\ldots,n\}:I\neq\emptyset} (-1)^{|I|-1}\mu(\bigcap_{j\in I} A_j \cap A_{n+1}).$$

Denote the expression above by (1). Note that for any fixed $2 \leq k \leq n$, the expression $\sum_{J\subset\{1,\ldots,n+1\}:|J|=k}(-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i)$ equals

$$\sum_{J\subset\{1,\ldots,n\}:|J|=k} (-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i) - \sum_{I\subset\{1,\ldots,n\}:|I|=k-1} (-1)^{k-2}\mu(\bigcap_{j\in I} A_j \cap A_{n+1}),$$

which implies that the expression $\sum_{J\subset\{1,\ldots,n+1\}:J\neq\emptyset}(-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i)$ equals (1). Or equivalently,

$$\mu(\bigcup_{i=1}^{n+1} A_i) = \sum_{J\subset\{1,\ldots,n+1\}:J\neq\emptyset} (-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i).$$

This closes the induction and we are done.

Alternatively, note that $\mu(\bigcup_{i=1}^{n+1} A_i) = \text{Simp}\int_X 1_{\cup_{i=1}^{n+1} A_i}\ d\mu$, which equals

$$\begin{aligned}
&\text{Simp}\int_X (1-\prod_{i=1}^{n+1}(1-1_{A_i}))\ d\mu \\
&= \text{Simp}\int_X (1-\prod_{i=1}^{n}(1-1_{A_i}))\ d\mu + \text{Simp}\int_X 1_{A_{n+1}}\prod_{i=1}^{n}(1-1_{A_i})\ d\mu \\
&\overset{IH}{=} \sum_{J\subset\{1,\dots,n\}:J\neq\emptyset} (-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i) + \mu(\bigcap_{i=1}^{n} A_i^c \cap A_{n+1}) \\
&= \sum_{J\subset\{1,\dots,n\}:J\neq\emptyset} (-1)^{|J|-1}\mu(\bigcap_{i\in J} A_i) + \mu(A_{n+1}) - \mu(\bigcup_{i=1}^{n} A_i \cap A_{n+1})
\end{aligned}$$

which is exactly (1), and the rest of the argument goes the same as above. □

Exercise 34

Proof. (1). This follows from the definition and almost everywhere equivalence for the simple integrals.

(2). If $f \leq g$ μ-almost everywhere, then the set of unsigned simple functions minorizing f minorize g μ-almost everywhere, and the claim then follows from almost everywhere equivalence for the simple integrals and definition 11.

(3). The claim trivially holds for $c = 0$. Let $c > 0$. For every simple function h such that $0 \leq h \leq f$, we have from homogeneity of the simple integral that

$$c \times \text{Simp}\int_X h\ d\mu = \text{Simp}\int_X ch\ d\mu \leq \int_X cf\ d\mu.$$

Take the supremum over all such h on the LHS gives $c \times \text{Simp}\int_X f\ d\mu \leq \int_X cf\ d\mu$. Similarly, for every simple function h such that $0 \leq h \leq cf$,

$$\text{Simp}\int_X h\ d\mu = \text{Simp}\int_X c\cdot\frac{1}{c}h\ d\mu = c \times \text{Simp}\int_X \frac{h}{c}\ d\mu \leq c \times \int_X f\ d\mu.$$

Take the supremum over all such h on the LHS gives $c \times \text{Simp}\int_X f\ d\mu \geq \int_X cf\ d\mu$, as desired.

(4). For every simple function h_1 such that $0 \leq h_1 \leq f$ and every simple function h_2 such that $0 \leq h_2 \leq g$, $h_1 + h_2$ is a simple function such that $0 \leq h_1 + h_2 \leq f + g$. By finite additivity of the simple integral,

$$\text{Simp}\int_X h_1\ d\mu + \text{Simp}\int_X h_2\ d\mu = \text{Simp}\int_X h_1 + h_2\ d\mu.$$

Take the supremum first over h_1 and then over h_2 on both sides, one obtains the claim.

(5). This comes trivial with the definition.

(6). We have the trivial pointwise inequality

$$\lambda 1_{\{x\in X: f(x)\geq\lambda\}} \leq f(x).$$

From compatibility with the simple integral and monotonicity, we conclude that $\lambda\mu(\{x \in X : f(x) \geq \lambda\}) \leq \int_X f \, d\mu$, and the claim follows.

(7). For contradiction, suppose that $f = \infty$ on some set A of positive measure. From the definition and finiteness of the simple integral, this implies that $\int_X f \, d\mu = \infty$, a contradiction.

(8). The claim follows from monotonicity, the definition, and the vanishing property of the simple integral.

(9). Assume $f = \infty$ on some set E of positive measure. By monotonicity and compatibility with the simple integral,

$$\int_X f \, d\mu \geq \int_X \min(f, n) \, d\mu \geq \int_X n 1_E \, d\mu = n\mu(E).$$

Sending $n \to \infty$ gives $\lim_{n\to\infty} \int_X \min(f, n) \, d\mu = \int_X f \, d\mu = \infty$. Otherwise, f is bounded μ-almost everywhere on every set of positive measure, and specifically is bounded μ-almost everywhere. Hence $\min(f, n) = f$ μ-almost everywhere for sufficiently large n, and $\lim_{n\to\infty} \int_X \min(f, n) \, d\mu = \int_X f \, d\mu$ by almost everywhere equivalence.

(10). Let $f = \sum_{i=1}^k a_i 1_{A_i}$ be a simple function. By compatibility with the simple integral and finite additivity,

$$\int_X f 1_{E_n} \, d\mu = \mathrm{Simp} \int_X \sum_{i=1}^k a_i 1_{A_i \cap E_n} \, d\mu = \sum_{i=1}^k a_i \mu(A_i \cap E_n).$$

From condition, $A_i \cap E_1 \subset A_i \cap E_2 \subset \ldots$ for all i, so upwards monotone convergence gives

$$\lim_{n\to\infty} \int_X f 1_{E_n} \, d\mu = \sum_{i=1}^k a_i \mu(\bigcup_{n=1}^\infty E_n \cap A_i) = \int_X f 1_{\cup_{n=1}^\infty E_n} \, d\mu.$$

Thus the claim holds for simple functions.

For an unsigned measurable function f, we thus have

$$\begin{aligned}
\int_X f 1_{\cup_{n=1}^\infty E_n}\ d\mu &:= \sup_{0 \le h \le f 1_{\cup_{n=1}^\infty E_n}; h \text{ simple}} \text{Simp} \int_X h\ d\mu \\
&\le \sup_{0 \le h \le f; h \text{ simple}} \text{Simp} \int_X h 1_{\cup_{n=1}^\infty E_n}\ d\mu \\
&= \sup_{0 \le h \le f; h \text{ simple}} \lim_{n\to\infty} \int_X h 1_{E_n}\ d\mu \\
&= \lim_{n\to\infty} \sup_{0 \le h \le f; h \text{ simple}} \int_X h 1_{E_n}\ d\mu \\
&\le \lim_{n\to\infty} \int_X f 1_{E_n}\ d\mu.
\end{aligned}$$

On the other hand we have $\lim_{n\to\infty} \int_X f 1_{E_n}\ d\mu \le \int_X f 1_{\cup_{n=1}^\infty E_n}\ d\mu$ by monotonicity, and the claim follows.

(11). For every simple function $h : Y \to [0, +\infty]$ such that $0 \le h \le f\ \lfloor_Y$, there is an extension of h into a simple function $h' : X \to [0, +\infty]$ such that $0 \le h' \le f 1_Y$ (by setting $h' := h$ on Y and $h' := 0$ outside of Y). Conversely, for every simple function $h' : X \to [0, +\infty]$ such that $0 \le h' \le f 1_Y$, there is a restriction $h := h'\ \lfloor_Y : Y \to [0, +\infty]$ of h' to Y such that $0 \le h \le f\ \lfloor_Y$. From the definition of the simple integral, we are taking the supremum over the same set of values, and therefore $\int_X f 1_Y\ d\mu = \int_Y f\ \lfloor_Y\ d\mu\ \lfloor_Y$. □

Exercise 35

Proof. (1). Let $f = \sum_{i=1}^m c_i 1_{A_i}$ be an unsigned simple function. By definition, $\text{Simp} \int_X f\ d(c\mu) := \sum_{i=1}^m c_i c\mu(A_i) = c \times \sum_{i=1}^m c_i \mu(A_i) = c \times \text{Simp} \int_X f\ d\mu$. Hence, for an unsigned measurable f, we have

$$\begin{aligned}
\int_X f\ d(c\mu) &:= \sup_{0 \le g \le f; g \text{ simple}} \text{Simp} \int_X g\ d(c\mu) = c \times \sup_{0 \le g \le f; g \text{ simple}} \text{Simp} \int_X g\ d\mu \\
&= c \times \int_X f\ d\mu.
\end{aligned}$$

(2). Likewise, we have from Tonelli's theorem for series:

$$\text{Simp} \int_X f\ d\sum_{n=1}^\infty \mu_n := \sum_{i=1}^m \sum_{n=1}^\infty c_i \mu(A_i) = \sum_{n=1}^\infty \sum_{i=1}^m c_i \mu(A_i) = \sum_{n=1}^\infty \text{Simp} \int_X f\ d\mu_n$$

when f is simple. The claim then follows similarly from definition for unsigned measurable f. □

Exercise 36

Proof. (1). By construction, we have $\phi_*\mu(\emptyset) = \mu(\phi^{-1}(\emptyset)) = \mu(\emptyset) = 0$. Furthermore, if $E_1, E_2, \ldots \in \mathcal{C}$ are disjoint, then $\phi^{-1}(E_1), \phi^{-1}(E_2), \ldots \in \mathcal{B}$ are as well, and so

$$\begin{aligned}\phi_*\mu(\bigcup_{n=1}^{\infty} E_n) &:= \mu(\phi^{-1}(\bigcup_{n=1}^{\infty} E_n)) = \mu(\bigcup_{n=1}^{\infty} \phi^{-1}(E_n)) = \sum_{n=1}^{\infty} \mu(\phi^{-1}(E_n)) \\ &= \sum_{n=1}^{\infty} \phi_*\mu(E_n).\end{aligned}$$

Hence $\phi_*\mu$ is a measure on $\mathcal{C}$, so that$(Y, \mathcal{C}, \phi_*\mu)$ is a measure space.

(2). Let $g = \sum_{i=1}^{n} c_i 1_{A_i} : Y \to [0, +\infty]$ be a simple function such that $g \leq f$, where $A_1 \ldots, A_n$ are $\mathcal{C}$-measurable sets. Since $\phi : X \to Y$ is a measurable morphism, the sets $\phi^{-1}(A_1), \ldots, \phi^{-1}(A_n)$ are $\mathcal{B}$-measurable. Define the simple function $h : X \to [0, +\infty]$ by $h := \sum_{i=1}^{n} c_i 1_{\phi^{-1}(A_i)}$, then we see that $h \leq f \circ \phi$, and

$$\int_Y g \ d\phi_*\mu = \int_X h \ d\mu = \sum_{i=1}^{n} c_i \phi_*\mu(A_i).$$

i.e. For every unsigned simple function $g \leq f$, there is an unsigned simple function $h \leq f \circ \phi$ with the same value of integral, we get $\int_Y f \ d\phi_*\mu \leq \int_X (f \circ \phi) \ d\mu$.

On the other hand, let $h = \sum_{i=1}^{m} d_i 1_{B_i} : X \to [0, +\infty]$ be a simple function such that $h \leq f \circ \phi$, where the $B_i := h^{-1}(\{d_i\})$ are $\mathcal{B}$-measurable. Note that $B_i \subset (f \circ \phi)^{-1}([d_i, +\infty)])$ for all i, and thus we can define the simple function $g : Y \to [0, +\infty]$ by $g := \sum_{i=1}^{m} d_i 1_{E_i}$, where the $E_i := f^{-1}([d_i, +\infty])$ are $\mathcal{C}$-measurable. Then $g \leq f$, and

$$\int_X h \ d\mu = \sum_{i=1}^{m} c_i \mu(B_i) \leq \sum_{i=1}^{m} \mu(\phi^{-1}(E_i)) = \int_Y g \ d\phi_*\mu$$

from monotonicity. From this, we conclude that $\int_Y f \ d\phi_*\mu \geq \int_X (f \circ \phi) \ d\mu$ and the claim follows. $\square$

Exercise 37

Proof. By Exercise 21 of note 1 and the definition of pushforward, if $T : \mathbf{R}^d \to \mathbf{R}^d$ is invertible, then

$$T_* m(E) := m(T^{-1}(E)) = |(\det T)^{-1}| m(E) = \frac{1}{|\det T|} m(E)$$

for every Lebesgue measurable set $E \subset \mathbf{R}^d$. $\square$

Exercise 38

Proof. By Exercise 8, $f : X \to [0, +\infty]$ is trivially measurable. Assume that $f = \sum_{i=1}^n c_i 1_{A_i}$ is a simple function. If $c_i \#(A_i) = \infty$ for some i, then either $c_i > 0$ and $\#(A_i) = +\infty$, or $c_i = +\infty$ and $\#(A_i) > 0$. In both cases $\sum_{x \in X} f(x) = \sum_{i=1}^n c_i \#(A_i) = \infty$ by definition, so the claim holds for simple functions.

Let $f : X \to [0, +\infty]$ be an arbitrary unsigned function. If $\sum_{x \in X} f(x)$ is finite, then $f(x) = 0$ for all but at most countably many $x \in X$ by Exercise 3 of note 1. That is, $\sum_{x \in X} f(x) = \sum_{n=1}^{\infty} f(x_n)$. And we have

$$\int_X f \; d\# := \sup_{0 \le g \le f; g \text{ simple}} \int_X g \; d\# = \sup_{0 \le g \le f; g \text{ simple}} \sum_{x \in X} g(x) = \sum_{x \in X} f(x).$$

Otherwise, if $\sum_{x \in X} f(x) = \infty$, then $\forall M > 0$, there exists a finite set $F \subset X$ such that $\sum_{x \in F} f(x) > M$, which implies that $\sup_{0 \le g \le f; g \text{ simple}} \sum_{x \in X} g(x) > M$, and therefore $\int_X f \; d\# = \infty$. $\square$

Exercise 39

Proof. (1). The zero function is clearly absolutely integrable. Let f and g be absolutely integrable, and $c \in \mathbf{C}$. From the triangle inequality and monotonicity, one gets $\int_X |f+g| \; d\mu \le \int_X |f| \; d\mu + \int_X |g| \; d\mu < \infty$. Similarly, from homogeneity one gets $\int_X |cf| \; d\mu = \int_X |c||f| \; d\mu = |c| \int_X |f| \; d\mu < \infty$. Since both $f + g$ and cf are absolutely integrable, $L^1(X, \mathcal{B}, \mu)$ is a complex vector space.

(2). Let f and g be real-valued, absolutely integrable functions, and $c \in \mathbf{R}$. from the identity

$$f + g = (f + g)_+ - (f + g)_- = f_+ - f_- + g_+ - g_-,$$

the finite additivity and the homogeneity of the unsigned integral, one gets that $\int_X (f + g) \; d\mu = \int_X f \; d\mu + \int_X f \; d\mu + \int_X g \; d\mu$, and $\int_X cf \; d\mu = c \times \int_X f \; d\mu$ as in Exercise 19 of note 2.

If $f,\ g \in L^1(X, \mathcal{B}, \mu)$ are complex-valued, and $c \in \mathbf{C}$, then by breaking f and g into their real and imaginary components, we see that $\int_X f + g\ d\mu = \int_X f\ d\mu + \int_X g\ d\mu$. Expanding the product, and using linearity in the real-valued case, we get $\int_X cf\ d\mu = c \times \int_X f\ d\mu$. Hence the integration map $f \mapsto \int_X f\ d\mu$ is a complex-linear map from $L^1(X, \mathcal{B}, \mu)$ to $\mathbf{C}$.

(3). This follows from the proof of part (1).

(4). This follows from almost everywhere equivalence of the unsigned integral, and the fact that if $f(x) = g(x)$ for μ-almost every $x \in X$, then $\mathrm{Re}_+ f(x) = \mathrm{Re}_+ g(x)$, $\mathrm{Im}_+ f(x) = \mathrm{Im}_+ g(x)$, and $\mathrm{Re}_- f(x) = \mathrm{Re}_- g(x)$, $\mathrm{Im}_- f(x) = \mathrm{Im}_- g(x)$ for μ-almost every $x \in X$.

(5). From definition, it suffices to show that the unsigned integral is insensitive to refinement. Hence we suppose that $f \in L^1(X, \mathcal{B}, \mu)$ is unsigned. By (5) of Exercise 32, and the fact that a simple function with respect to $\mathcal{B}$ is a simple function with respect to $\mathcal{B}'$, we have $\int_X f\ d\mu \leq \int_X f\ d\mu'$.

Now we establish the other inequality. First assume that f is bounded with finite measure support (By Exercise 28, $\mathrm{supp}(f) = \{x \in X : f(x) > 0\}$ is measurable). Pick an $\epsilon > 0$, and let f_ϵ be f rounded down to the nearest integer multiple of ϵ. Since f is bounded, f_ϵ is simple (with respect to both $\mathcal{B}$ and $\mathcal{B}$'). By construction we have

$$\int_X f\ d\mu - \mathrm{Simp} \int_X f_\epsilon\ d\mu' \leq 2\epsilon\mu(\mathrm{supp}(f)).$$

Sending $\epsilon \to 0$ and using the assumption that $\mu(\mathrm{supp}(f)) < \infty$, we see that $\int_X f\ d\mu' \leq \int_X f\ d\mu$. Therefore, $\int_X f\ d\mu' = \int_X f\ d\mu$ for unsigned f when f is bounded with finite measure support. For general unsigned f, one then obtains

$$\begin{aligned} \int_X f\ d\mu' &= \lim_{n\to\infty} \lim_{k\to\infty} \int_X \min(f, n) 1_{E_k}\ d\mu' \\ &= \lim_{n\to\infty} \lim_{k\to\infty} \int_X \min(f, n) 1_{E_k}\ d\mu \\ &= \int_X f\ d\mu \end{aligned}$$

from the horizontal and vertical truncation properties, where we define $E_k := \{x \in X : f(x) > \frac{1}{k}\}$ for all natural number k. Clearly this also implies that $f \in L^1(X, \mathcal{B}', \mu')$.

(6). Let $f \in L^1(X, \mathcal{B}, \mu)$. If $\|f\|_{L^1(\mu)} := \int_X |f|\ d\mu = 0$, then $|f|$ is zero μ-almost everywhere by the vanishing property of the unsigned integral, which implies that f is zero μ-almost everywhere. Conversely, if f is zero μ-almost everywhere, then $\|f\|_{L^1(\mu)} = 0$ by almost everywhere equivalence.

(7). By the restriction property of the unsigned integral,

$$\int_Y |f\downharpoonright_Y|\ d\mu\downharpoonright_Y = \int_X |f|1_Y\ d\mu \le \int_X |f|\ d\mu < \infty,$$

so $f\downharpoonright_Y \in L^1(Y, \mathcal{B}\downharpoonright_Y, \mu\downharpoonright_Y)$. That the restriction property of the absolutely integrable integral follows from that of the real-valued case, which follows from the unsigned case, which is established in Exercise 34. □

Exercise 40

Proof. By Exercise 28, f is measurable. We want $|\int_X f_n\ d\mu - \int_X f\ d\mu| \to 0$ as $n \to \infty$. Suppose that $f_n : X \to \mathbf{C}$ are a sequence of absolutely integrable functions. From the definition, it suffices to prove the claim when the f_n are real-valued. By the triangle inequality and the fact that $\mu(X) < \infty$,

$$\int_X |f|\ d\mu \le \int_X |f - f_n|\ d\mu + \int_X |f_n|\ d\mu < \infty$$

for sufficiently large n, so $f \in L^1(X, \mathcal{B}, \mu)$. Then, by (2) of Exercise 39,

$$|\int_X f_n\ d\mu - \int_X f\ d\mu| = |\int_X f_n - f\ d\mu| \le \int_X |f_n - f|\ d\mu,$$

where the last inequality holds since for any absolutely integrable, real-valued function g,

$$\begin{aligned}
|\int_X g\ d\mu| &= |\int_X g_+\ d\mu - \int_X g_-\ d\mu| \\
&\le |\int_X g_+\ d\mu + \int_X g_-\ d\mu| \\
&= |\int_X g_+ + g_-\ d\mu| \\
&= \int_X |g|\ d\mu.
\end{aligned}$$

Let $\varepsilon > 0$. Since f_n converges to f uniformly, there exists $N > 0$ such that $|f_n(x) - f(x)| < \varepsilon$ for every $n > N$ and $x \in X$. By monotonicity, this implies that $\int_X |f_n - f|\ d\mu < \varepsilon\mu(X)$ for every $n > N$. Using the assumption that $\mu(X)$ is finite, we obtain the claim.

Now suppose that $f_n : X \to [0, +\infty]$ are a sequence of unsigned measurable functions. If $\int_X f\ d\mu < \infty$, we are back to the absolutely integrable case, so we assume that $\int_X f\ d\mu = \infty$. This implies that $\forall M > 0$, there exists a simple function h with $0 \leq h \leq f$, such that $\mathrm{Simp}\int_X h\ d\mu > M$. Pick some $\varepsilon > 0$ small enough such that $0 \leq h - \varepsilon$. By uniform convergence, we see that

$$M - \varepsilon\mu(X) < \mathrm{Simp}\int_X h - \varepsilon\ d\mu \leq \int_X f_n\ d\mu$$

for sufficiently large n. As M and ε are both arbitrary and $\mu(X)$ is finite, we get $\int_X f_n\ d\mu = \infty$, and the claim follows. $\square$

Exercise 41

Proof. Inspired by the proof of MCT, we let the partial sums escape to horizontal infinity. Let X be the real line with Lebesgue measure, define f_n recursively by setting $f_1 := 1_{[1,2]}$, $f_n := -1_{[n-1,n]} + 1_{[n,n+1]}$ for all $n > 1$. Then we see that

$$\sum_{n=1}^{\infty} f_n := \lim_{N\to\infty} \sum_{n=1}^{N} f_n = \lim_{N\to\infty} 1_{[N,N+1]} = 0,$$

so

$$\int_{\mathbf{R}} \sum_{n=1}^{\infty} f_n(x)\ dx = 0.$$

On the other hand, since $\int_{\mathbf{R}} f_n(x)\ dx = 0$ for all $n > 1$, we have

$$\sum_{n=1}^{\infty} \int_{\mathbf{R}} f_n(x)\ dx = \int_{\mathbf{R}} f_1(x)\ dx = 1.$$

Hence the corollary can fail if the f_n are assumed to be absolutely integrable rather than unsigned measurable, even if the sum $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent for each x. $\square$

Exercise 42

Proof. Apply Tonelli's theorem to the indicator functions 1_{E_n}, we obtain

$$\sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \int_X 1_{E_n}\ d\mu = \int_X \sum_{n=1}^{\infty} 1_{E_n}\ d\mu < \infty,$$

and the claim follows from the finiteness of the unsigned integral. $\square$

Exercise 43

Proof. (1). Observe that the set $N := \{x \in X : x \in E_n \text{ for infinitely many } n\}$ is equal to the set $\{x \in X : \forall N > 0,\ \exists n > N \text{ s.t } x \in E_n\}$, which is contained in the set $\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n$. Apply downwards monotone convergence to the sets $F_N := \bigcup_{n=N}^{\infty} E_n$, we see that

$$\mu(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n) = \lim_{N\to\infty} \mu(\bigcup_{n=N}^{\infty} E_n).$$

From $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, and countable subadditivity, we obtain

$$\lim_{N\to\infty} \mu(\bigcup_{n=N}^{\infty} E_n) \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) = \lim_{N\to\infty} (\sum_{n=1}^{\infty} \mu(E_n) - \sum_{n=1}^{N-1} \mu(E_n)) = 0.$$

Therefore N is a subnull set, as desired.

(2). Let X be the real line with Lebesgue measure, the idea is to partition the unit interval $[0, 1]$ into increasingly finer subintervals, so every point in $[0, 1]$ is in infinitely many such subintervals. For instance, define

$$\begin{aligned}
&E_1 := [0, \frac{1}{2}],\ E_2 := [\frac{1}{2}, 1] \\
&E_3 := [0, \frac{1}{4}], E_4 := [\frac{1}{4}, \frac{2}{4}], E_5 := [\frac{2}{4}, \frac{3}{4}], E_6 := [3/4, 1] \\
&\vdots
\end{aligned}$$

In general, one partitions $[0, 1]$ into 2^k subintervals of length $1/2^k$ at step k. Clearly, one has $\lim_{n\to\infty} m(E_n) = 0$, but $\sum_{n=1} m(E_n) = \infty$. □

Exercise 44

Proof. By modifying f_n, f on a null set, we may assume without loss of generality that the f_n converge to f pointwise everywhere rather than μ-almost everywhere, and similarly we can assume that $|f_n|$ are bounded by G pointwise everywhere rather than μ-almost everywhere.

By taking real and imaginary parts we may assume without loss of generality that f_n, f are real, thus $-G \leq f_n \leq G$ pointwise. Of course, this implies that $-G \leq f \leq G$ pointwise also. By the triangle inequality, the functions $|f_n - f|$ are bounded by $2G$. Apply dominated convergence theorem to them gives the claim. □

Exercise 45

Proof. By modifying f_n, f on a null set, we may assume without loss of generality that the f_n converge to f pointwise everywhere rather than μ-almost everywhere, and similarly we can assume that $|f_n|$ are bounded by $G + g_n$ pointwise everywhere rather than μ-almost everywhere.

By taking real and imaginary parts we may assume without loss of generality that f_n, f are real, thus $-G - g_n \leq f_n \leq G + g_n$ pointwise. Of course, this implies that $-G - \liminf_{n\to\infty} g_n \leq f \leq G + \liminf_{n\to\infty} g_n$ pointwise also. Apply Fatou's lemma to the g_n, and use the assumption $\lim_{n\to\infty} \int_X g_n \ d\mu = 0$, we see that $\int_X \liminf_{n\to\infty} g_n \ d\mu = 0$, so $\liminf_{n\to\infty} g_n = 0$ μ-almost everywhere by the vanishing property of the unsigned integral. Again, we may take $\liminf_{n\to\infty} g_n = 0$ pointwise to get $-G \leq f \leq G$ pointwise.

If we apply Fatou's lemma to the unsigned functions $f_n + G + g_n$, we see that

$$\int_X f + G \ d\mu \leq \liminf_{n\to\infty} \int_X f_n + G + g_n \ d\mu,$$

which on subtracting the finite quantity $\int_X G \ d\mu$ gives

$$\int_X f \ d\mu \leq \liminf_{n\to\infty} \int_X f_n \ d\mu.$$

Similarly, if we apply that lemma to the unsigned functions $G + g_n - f_n$, we obtain

$$‘\int_X G - f \ d\mu \leq \liminf_{n\to\infty} \int_X G + g_n - f_n \ d\mu;$$

negating this inequality and then canceling $\int_X G \ d\mu$ again we conclude that

$$\limsup_{n\to\infty} \int_X f_n \ d\mu \leq \int_X f \ d\mu.$$

The claim then follows by combining these inequalities. □

Exercise 46

Proof. By (1) of Exercise 28, the functions $\min(f_n, f)$ are measurable. Apply the dominated convergence theorem to them, we see that

$$\lim_{n\to\infty} \int_X \min(f_n, f) \ d\mu = \int_X f \ d\mu.$$

By (2) of Exercise 39 and the identity $f_n - f - |f_n - f| = 2(\min(f_n, f) - f)$, we get

$$\int_X f_n \, d\mu - \int_X f \, d\mu - \|f - f_n\|_{L^1(\mu)} = 2(\int_X \min(f_n, f) \, d\mu - \int_X f \, d\mu),$$

taking limit as $n \to 0$ and using the previous identity, we obtain the claim. $\square$

Exercise 47

Proof. By definition, $\mu_g(\emptyset) = 0$. If $E_1, E_2, \ldots \in \mathcal{B}$ are a countable sequence of disjoint measurable sets, then by Tonelli's theorem for sums and integrals, we have

$$\mu_g(\bigcup_{n=1}^{\infty} E_n) = \int_{\bigcup_{n=1}^{\infty} E_n} g \, d\mu = \int_X \sum_{n=1}^{\infty} g 1_{E_n} \, d\mu = \sum_{n=1}^{\infty} g 1_{E_n} \, d\mu = \sum_{n=1}^{\infty} \mu_g(E_n).$$

Hence $\mu_g : \mathcal{B} \to [0, +\infty]$ is a measure. $\square$

Exercise 48

Proof. Define the function $\mu : \mathcal{B} \to [0, +\infty]$ by $\mu(E) := I(1_E)$ for all $E \in \mathcal{B}$. We first verify that μ is a measure. By homogeneity, $\mu(\emptyset) := I(1_\emptyset) = I(0) = 0$. Let $E_1, E_2, \ldots$ be a sequence of disjoint $\mathcal{B}$-measurable sets, by monotone convergence and finite additivity,

$$\begin{aligned}\mu(\bigcup_{n=1}^{\infty} E_n) &:= I(1_{\bigcup_{n=1}^{\infty} E_n}) = I(\lim_{N\to\infty} 1_{\bigcup_{n=1}^{N} E_n}) = \lim_{N\to\infty} I(1_{\bigcup_{n=1}^{N} E_n}) \\ &= \lim_{N\to\infty} I(\sum_{n=1}^{N} 1_{E_n}) = \lim_{N\to\infty} \sum_{n=1}^{N} I(1_{E_n}) = \sum_{n=1}^{\infty} \mu(E_n).\end{aligned}$$

Let $f = \sum_{i=1}^{k} c_i 1_{A_i}$ be a simple function, where $c_i \in [0, +\infty]$ and the A_i are disjoint $\mathcal{B}$-measurable sets. By homogeneity and finite additivity,

$$I(f) = I(\sum_{i=1}^{k} c_i 1_{A_i}) = \sum_{i=1}^{k} c_i I(1_{E_i}) = \sum_{i=1}^{k} c_i \mu(E_i) = \int_X f \, d\mu.$$

By compatibility with measure, μ is unique. From finite additivity of I, one deduces monotonicity of I. Now, let $f \in \mathcal{U}(X, \mathcal{B})$ be an unsigned measurable function. If $\int_X f \, d\mu = \infty$, then $\forall M > 0$, there is a simple function $0 \le g \le f$ such that $\int_X g \, d\mu = I(g) > M$. By monotonicity, this implies that

$$I(f) \ge I(g) > M.$$

Since M is arbitrary, $I(f) = \infty$. Otherwise, if $\int_X f\ d\mu < \infty$, then $\forall n > 0$, there is a simple function $0 \le g_n \le f$ such that

$$\int_X f\ d\mu - 1/n < \int_X g_n\ d\mu.$$

Without loss of generality, one can take the g_n to be non-decreasing. Apply monotone convergence (both of the unsigned integral and of I), one obtains

$$\int_X f\ d\mu = \int_X \lim_{n\to\infty} g_n\ d\mu = \lim_{n\to\infty} I(g_n) = I(\lim_{n\to\infty} g_n).$$

From almost everywhere equivalence of the unsigned integral, $f = \lim_{n\to\infty} g_n$ μ-almost everywhere. By our construction and the properties of I, one deduces

$$I(f) - I(\lim_{n\to\infty} g_n) = I(f - \lim_{n\to\infty} g_n) = 0,$$

from which one gets $I(\lim_{n\to\infty} g_n) = I(f)$, and the claim follows. □

Exercise 49

Proof. As μ is complete, the upper and lower integral are well-defined for every function $f : X \to \mathbf{R}$ (otherwise they are not for $f = 1_N$ where N is a non-measurable subnull set). Pick an $\varepsilon > 0$, and denote $\underline{\int_X} f\ d\mu = \overline{\int_X} f\ d\mu := I(f)$. By condition, for every $n \in \mathbf{N}$, we can find simple functions h_n and g_n such that $h_n \le f \le g_n$ with

$$I(f) - \varepsilon/n < \int_X h_n\ d\mu \le \int_X g_n\ d\mu < I(f) + \varepsilon/n.$$

Let M be such that $|f| < M$. Without loss of generality, one can assume that h_n and g_n are bounded by M as well. By the definition of simple integral, we get

$$\int_X g_n\ d\mu - \int_X h_n\ d\mu = \int_X g_n - h_n\ d\mu < 2\varepsilon/n$$

for all n. Since $\mu(X)$ is finite, $g_n - h_n$ are bounded by the absolutely integrable function $G := 2M$ for all n. By the dominated convergence theorem, we thus have

$$\int_X \lim_{n\to\infty} (g_n - h_n)\ d\mu = 0.$$

By the vanishing property of the unsigned integral and the squeeze test, this implies that

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n = f$$

μ-almost everywhere, so f is measurable by (6) of Exercise 28 (which holds for pointwise almost everywhere convergence by the completeness assumption), as desired. □

Exercise 50

Proof. Suppose for contradiction that $(\mathbf{Z}, 2^{\mathbf{Z}}, \mathbf{P})$ is a probability space such that $\mathbf{P}$ is translation-invariant. Then $\mathbf{P}(\{n\}) = \mathbf{P}(\{0\} + n) = \mathbf{P}(\{0\}), \ \forall n \in \mathbf{Z}$. By countable additivity,

$$1 = \mathbf{P}(\mathbf{Z}) = \sum_{n \in \mathbf{Z}} \mathbf{P}(\{0\}).$$

But if $\mathbf{P}(\{0\}) = 0$, one gets $1 = 0$, and if $\mathbf{P}(\{0\}) > 0$, one gets $1 = \infty$, a contradiction. Consequently, there is no translation-invariant probability measure on the integers with the discrete σ-algebra. $\square$

Exercise 51

Proof. Suppose for contradiction that $(\mathbf{R}, \mathcal{L}[\mathbf{R}], \mathbf{P})$ is a probability space such that $\mathbf{P}$ is translation-invariant. Like before, one gets from countable additivity that

$$1 = \mathbf{P}(\mathbf{R}) = \mathbf{P}(\bigcup_{n \in \mathbf{Z}} ((0,1] + n) = \sum_{n \in \mathbf{Z}} \mathbf{P}((0,1] + n) = \mathbf{P}((0,1]).$$

A similar argument as in the previous Exercise leads to a contradiction. Consequently, there is no translation-invariant probability measure on the integers with the the Lebesgue σ-algebra. $\square$

Exercise 52

Proof. (1). For every set $E \subset X$, let $P(E)$ be the property that $\forall \epsilon > 0$, $\exists F \in \mathcal{A}$ such that $E \Delta F$ is $\langle \mathcal{A} \rangle$-measurable and $\mu(E \Delta F) < \epsilon$. Clearly $P(E)$ is true for all $E \in \mathcal{A}$ (which includes the empty set $E = \emptyset$). Since $(X \backslash E) \Delta (X \backslash F) = E \Delta F$ and the Boolean algebra is closed under complement, $P(X \backslash E)$ holds whenever $P(E)$ holds.

Finally, let $E_1, E_2, \ldots \subset X$ be such that $P(E_n)$ is true for all n. One needs to caution against taking countable union of $\mathcal{A}$-measurable sets, as $\mathcal{A}$ is only assumed to be a Boolean algebra. However, since $\mu(X) < \infty$, we can approximate such countable union by a finite union. Set $s := \mu(\bigcup_{n=1}^{\infty} E_n)$, then $0 \leq s < \infty$. By countable additivity,

$$s = \mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n \backslash \bigcup_{m<n} E_m)$$

In particular, $\forall \epsilon > 0$ there exists a natural number k such that

$$\mu(\bigcup_{n=1}^{\infty} E_n \backslash \bigcup_{n=1}^{k} E_n) = s - \sum_{n=1}^{k} \mu(E_n \backslash \bigcup_{m<n} E_m) < \epsilon.$$

By our assumption, for every $1 \leq n \leq k$, there exists $F_n \in \mathcal{A}$ such that $E_n \Delta F_n \in \langle \mathcal{A} \rangle$ and $\mu(E_n \Delta F_n) < \epsilon/k$. By the definition of symmetric difference, we see that

$$(\bigcup_{n=1}^{k} E_n)\Delta(\bigcup_{n=1}^{k} F_n) = \bigcup_{n=1}^{k}(E_n \backslash \bigcup_{m=1}^{k} F_m) \cup \bigcup_{m=1}^{k}(F_m \backslash \bigcup_{n=1}^{k} E_n) \subset \bigcup_{n=1}^{k} E_n \Delta F_n.$$

So subadditivity gives

$$\mu((\bigcup_{n=1}^{k} E_n)\Delta(\bigcup_{n=1}^{k} F_n)) < \epsilon.$$

From triangle inequality of the symmetric difference, we conclude that

$$\mu((\bigcup_{n=1}^{\infty} E_n)\Delta(\bigcup_{n=1}^{k} F_n)) < 2\epsilon,$$

and the claim follows from remark 4.

(2). Suppose that $E \in \langle \mathcal{A} \rangle$ has finite measure. As $X = \bigcup_{n=1}^{\infty} A_n$, we have

$$E = \bigcup_{n=1}^{\infty} E \cap A_n := \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty}(E_n \backslash \bigcup_{m<n} E_m) := \bigcup_{n=1}^{\infty} E'_n,$$

where we define $E_n := E \cap A_n$ and $E'_n := E_n \backslash \bigcup_{m<n} E_m$ for all n. Since the E'_n. Since the E'_n are disjoint, we get

$$s := \mu(E) = \mu(\bigcup_{n=1}^{\infty} E'_n) = \sum_{n=1}^{\infty} \mu(E'_n).$$

In particular, $\forall \epsilon > 0$ there exists a natural number k such that

$$s - \sum_{n=1}^{k} \mu(E'_n) = \mu(E \backslash E') < \epsilon,$$

where $E' := \bigcup_{n=1}^{k} E'_n$. Let $X_k := \bigcup_{n=1}^{k} A_n$. We claim that

$$\langle \mathcal{A} \downharpoonleft_{X_k} \rangle = \langle \mathcal{A} \rangle \downharpoonleft_{X_k}$$

Clearly $\langle \mathcal{A} \downharpoonleft_{X_k} \rangle \subset \langle \mathcal{A} \rangle \downharpoonleft_{X_k}$ by definition. To show the other direction, note that $\langle \mathcal{A} \rangle \downharpoonleft_{X_k} = \{E \cap X_k : E \in \langle \mathcal{A} \rangle\}$. Let $P(E)$ be the property of sets $E \subset X$

that $E \cap X_k \in \langle \mathcal{A} \downharpoonleft_{X_k} \rangle$. Then $P(\emptyset)$ is trivially true, and $P(E)$ is true for all $E \in \mathcal{A}$. If $P(E)$ is true for some $E \subset X$, namely, if $E \cap X_k \in \langle \mathcal{A} \downharpoonleft_{X_k} \rangle$, then $E^c \cap X_k = X_k \backslash E = X_k \backslash (E \cap X_k) \in \langle \mathcal{A} \downharpoonleft_{X_k} \rangle$, so $P(X \backslash E)$ is true. Finally, if $E_1, E_2, \ldots \subset X$ are such that $P(E_n)$ is true for all n, that is, if $E_n \cap X_k \in \langle \mathcal{A} \downharpoonleft_{X_k} \rangle$ for all n, then $(\bigcup_{n=1}^{\infty} E_n) \cap X_k = \bigcup_{n=1}^{\infty} (E_n \cap X_k) \in \langle \mathcal{A} \downharpoonleft_{X_k} \rangle$. By remark 4, one concludes that $P(E)$ is true for all $E \in \langle \mathcal{A} \rangle$. Equivalently, $\langle \mathcal{A} \rangle \downharpoonleft_{X_k} \subset \langle \mathcal{A} \downharpoonleft_{X_k} \rangle$.

Apply part (1) to the finite measure space $(X_k, \langle \mathcal{A} \rangle \downharpoonleft_{X_k}, \mu \downharpoonleft_{X_k})$, we see that there exists $F \in \mathcal{A} \downharpoonleft_{X_k} \subset \mathcal{A}$ such that

$$\mu(E' \Delta F) < \epsilon.$$

The claim follows then from combining these estimates for $\mu(E \backslash E')$, $\mu(E' \Delta F)$ with the triangle inequality for symmetric difference $E \Delta F \subset (E \Delta E') \cup (E' \Delta F)$. $\square$