Math 245B note 1

qshuyu

November 2025

1 Selected Exercises in Note 1

Exercise 1

Proof. Let $E_1, E_2 \ldots \in \mathcal{X}$ be disjoint. By the monotone convergence theorem,

$$\begin{aligned} m_f(\bigcup_{n=1}^{\infty} E_n) &:= \int_X 1_{\bigcup_{n=1}^{\infty} E_n} f \ dm = \int_X \sum_{n=1}^{\infty} 1_{E_n} f \ dm = \sum_{n=1}^{\infty} \int_X 1_{E_n} f \ dm \\ &= \sum_{n=1}^{\infty} m_f(E_n). \end{aligned}$$

Also, $m_f(\emptyset) := \int_X 1_{\emptyset} f \ dm = 0$. Hence m_f is indeed an unsigned measure.

To show that $dm_f = f dm$, if $g = 1_E$ is an indicator function, then

$$\int_X g \ dm_f = \int_X 1_E \ dm_f = m_f(E) = \int_X 1_E f \ dm = \int_X gf \ dm.$$

So the claim holds for indicator functions. If $g = \sum_{i=1}^n a_i 1_{E_i}$ is a simple function, then by linearity of the unsigned integral,

$$\int_X g \ dm_f = \sum_{i=1}^n a_i \int_X 1_{E_i} \ dm_f = \sum_{i=1}^n a_i \int_X 1_{E_i} f \ dm = \int_X gf \ dm,$$

thus the claim holds for unsigned simple g. For unsigned measurable g, let $g = \sup_n g_n$ be the sup of a sequence $g_1 \leq g_2 \leq \ldots$ of simple functions, for example by setting $g_n(x)$ to be the largest integer multiple of 2^{-n} that is less than or equal to $\min(g(x), n)$ on $g^{-1}([0, n])$, and $g_n(x) := 0$ otherwise. Then

$$\int_X g \ dm_f = \int_X \lim_n g_n \ dm_f = \lim_n \int_X g_n \ dm_f = \lim_n \int_X g_n f \ dm = \int_X gf \ dm$$

by the simple function case and the monotone convergence theorem. $\square$

Exercise 2

Proof. Suppose that $m(X) < \infty$ and $m_f = m_g$. If f and g are not equal m-almost everywhere, then either $f > g$ on a set of positive measure, or $f < g$ on a set of positive measure. Assume the former. Let $E \in \mathcal{X}$ be s.t $m(E) > 0$ with $f > g$ on E. Then

$$m_f(E) = \int_X 1_E f \; dm > \int_X 1_E g \; dm = m_g(E)$$

by monotonicity and vanishing properties of the unsigned integral, contradicting $m_f = m_g$. Conversely, we clearly have $m_f = m_g$ given $f = g$ for m-a.e x by the vanishing property of the unsigned integral.

If m is σ-finite, we partition X into countably many sets X_n of finite measure, and see that the restriction $m_f \downharpoonleft_{X_n} = m_g \downharpoonleft_{X_n}$ iff $f \downharpoonleft_{X_n} = g \downharpoonleft_{X_n}$ for m-a.e $x \in X_n$. Since countable unions of null sets are null, the claim follows. □

Exercise 3

Proof. Let $f : X \to [0, +\infty]$ be s.t $f(x) := \frac{d\mu}{dm}(x)$. If f is continuous, then $f 1_{[0,x]}$ is Riemann integrable, by compatibility of Lebesgue and Riemann integrals, and the first fundamental theorem of calculus, we see that $x \mapsto \mu([0, x])$ is differentiable, with

$$\frac{d}{dx}\mu([0,x]) = \frac{d}{dx}m_f([0,x]) = \frac{d}{dx}\int_{[0,+\infty)} 1_{[0,x]} f \; dm = \frac{d}{dx}\int_0^x f(t) \; dt = f(x)$$

for all x, as desired. □

Exercise 4

Proof. Let μ be a measure on $(X, 2^X)$, where X is at most countable. Let $f : X \to [0, +\infty]$ be s.t $f(x) := \mu(\{x\})$. By the monotone convergence theorem,

$$\int_X 1_E f \; d\# = \int_X \sum_{x \in E} 1_{\{x\}} f \; d\# = \sum_{x \in E} \int_X 1_{\{x\}} f \; d\# = \sum_{x \in E} f(x) = \mu(E)$$

for every subset $E \subset X$. Hence $\mu = \#_f$ and thus μ is differentiable w.r.t counting measure $\#$. □

Remark on the Hahn decomposition theorem:

1. See also Lemma 3.2 of Folland. X_+ is totally positive, as a countable union of totally positive sets is totally positive: If $E_1, E_2, \ldots$ are totally positive sets, let $F_n := E_n \backslash \bigcup_{m=1}^{n-1} E_m$, then the F_n are disjoint and $F_n \subset E_n$ are totally positive. Hence if $A \subset X_+$, $\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap F_n) \geq 0$. From

$$\mu(X_+) = \mu(E_n) + \mu(X_+ \backslash E_n) \geq \mu(E_n)$$

for all n, we thus see that $\mu(X_+) = m_+$, which is finite by our hypothesis.

2. To see that $E = \bigcap_j E_j$ has positive measure, let $F_j := E_j \backslash E_{j+1}$, by construction we see that the F_j are disjoint with $\mu(F_j) \leq -1/n_j$ for all j. From the identity $E_1 = E \cup \bigcup_j F_j$, we get by countable additivity that

$$\mu(E_1) = \mu(E) + \sum_j \mu(F_j)$$

As $\mu(E_1)$ (hence $\mu(E)$) is finite, we obtain $\mu(E) = \mu(E_1) - \sum_j \mu(F_j) > \sum_j 1/n_j$. As the measure is signed, the tool of say downwards monotone convergence is unavailable, in particular we we no longer have $\mu(A \backslash B) = \mu(A) - \mu(B)$ whenever $B \subset A$ are measurable sets, which is used in the proof of such result in the unsigned case. Indeed, we only have one-sided finiteness in the current situation, so if $B \subset A$ is s.t $\mu(A) = \mu(B) = -\infty$, $\mu(A) - \mu(B)$ is undefined.

3. Since $\infty > \mu(E) > \sum_j 1/n_j$, we have $n_j \to \infty$. Hence if $F \subset E$ is s.t $\mu(F) \geq \mu(E) + 1/n$ for some integer n, then $\exists n_j > n$, and since $F \subset E_j$, this contradicts the construction of E_j and n_j.

4. If μ has finite positive variation, with $M := \sup\{\mu(E) : E \in \mathcal{X}\}$, then $\forall A \subset E_n \backslash E_{n-1}$:

$$\mu(A) = \mu(E_{n-1} \cup A) - \mu(E_{n-1}) \leq M - (M - 2^{-(n-1)}) = 2^{-(n-1)},$$

valid since $\mu(E_{n-1})$ is finite. i.e. any subset of $E_n \backslash E_{n-1}$ has measure $O(2^{-n})$.

$$\mu(\bigcup_{n=n_0}^{\infty} E_n) = \mu(\bigcup_{n=n_0}^{\infty} E_n \backslash \bigcup_{n'=n_0}^{n-1} E_{n'}) = \mu(E_{n_0}) + \sum_{n>n_0} \mu(E_n \backslash \bigcup_{n'=n_0}^{n-1} E_{n'})).$$

By geometric series, we thus have

$$\mu(F_{n_0}) = \mu(\bigcup_{n \geq n_0} E_n) = \mu(E_{n_0}) + O(2^{-n_0})$$

for all n_0. Now let $X_+ := \limsup_n E_n = \bigcap_{n_0=1}^{\infty} F_{n_0}$. From

$$\mu(F_{n_0}) = \mu(\bigcup_{n \geq n_0} F_n \backslash F_{n+1}) + \mu(X_+),$$

and the fact that all these quantities are finite, we get

$$\mu(X_+) = \mu(F_{n_0}) - \sum_{n \geq n_0} \mu(F_n \backslash F_{n+1}).$$

Observe that $F_n \backslash F_{n+1} = E_n \backslash \bigcup_{m \geq n+1} E_m$, and thus

$$\mu(F_n \backslash F_{n+1}) = O(2^{-n}).$$

This implies that $\mu(X_+) = \mu(E_{n_0}) + O(2^{-n_0})$ for all n_0, sending $n_0 \to \infty$ we obtain $\mu(X_+) = M$. Finally, if $A \subset X_+$, then $A \subset F_{n_0}$ for all $n_0 \geq 1$. In particular,

$$\mu(A) = \mu(F_{n_0}) - \mu(F_{n_0} \backslash A) \geq \mu(F_{n_0}) - M.$$

Sending $n_0 \to \infty$ again shows that $\mu(A) \geq 0$, as required.

Exercise 5

Proof. By the Hahn decomposition theorem, there is a partition $X = X_+ \cup X_-$ such that $\mu \downharpoonleft_{X_+} \geq 0$ and $\mu \downharpoonleft_{X_-} \leq 0$. Let $\mu_+ := \mu \downharpoonleft_{X_+}$ and $\mu_- := -\mu \downharpoonleft_{X_-}$, then $\forall E \in \mathcal{X}$,

$$\mu(E) = \mu(E \cap X_+) + \mu(E \cap X_-) = \mu_+(E) - \mu_-(E).$$

Clearly μ_+ and μ_- are mutually singular, as $X_+ \cap X_- = \emptyset$.

If $\mu = \lambda_+ - \lambda_-$ and $\lambda_+ \perp \lambda_-$ are unsigned measures, then there exists partition $X = Y_+ \cup Y_-$ with $\lambda_+(Y_-) = \lambda_-(Y_+) = 0$.

$$\mu \downharpoonleft_{Y_+} (E) = \lambda_+(E \cap Y_+) \geq 0, \ \mu \downharpoonleft_{Y_-} (E) = -\lambda_-(E \cap Y_-) \leq 0.$$

By the Hahn decomposition theorem, we conclude that $X_+ = Y_+, X_- = Y_-$ modulo μ-null sets. This implies that $\mu_+(E) = \lambda_+(E)$ and $\mu_-(E) = \lambda_-(E)$, so we get uniqueness. $\square$

Note that $\mu \perp \nu$ implies that $\mu = \mu \downharpoonleft_E, \ \nu = \nu \downharpoonleft_F$ for some $E \cap F = \emptyset$. Let $A := E, \ B := X \backslash E$, then

$$\mu(B) = \mu \downharpoonleft_E (X \backslash E) = 0, \ \nu(A) = \nu \downharpoonleft_F (E) = 0.$$

Hence we get a partition $X = A \cup B$ with $\mu(B) = \nu(A) = 0$.

Exercise 6

Proof. By Jordan decomposition theorem, $-|\mu| \leq \mu \leq |\mu|$. Let ν be an unsigned measure s.t $-\nu \leq \mu \leq \nu$, and $X = X_+ \cup X_-$ be the Hahn decomposition w.r.t μ. From $\mu \leq \nu$, $\mu_+(E) = \mu(X_+ \cap E) \leq \nu(X_+ \cap E)$. From $-\nu \leq -\mu$, $-\nu(E \cap X_-) \leq \mu(X_- \cap E) = -\mu_-(E)$. Combining them gives

$$|\mu|(E) = \mu_+(E) + \mu_-(E) \leq \nu(E \cap X_+) + \nu(E \cap X_-) = \nu(E).$$

Let $(E_n)_{n=1}^{\infty}$ be a partition of E. By the triangle inequality,

$$\mu(E_n \cap X_+) - \mu(E_n \cap X_-) \geq |\mu(E_n \cap X_+) + \mu(E_n \cap X_-)|,$$

i.e. $|\mu|(E_n) \geq |\mu(E_n)|$. By countable additivity, it follows that

$$|\mu(E)| = \sum_n |\mu|(E_n) \geq \sum_n |\mu(E_n)|,$$

For $E = E_1 \cup E_2$ where $E_1 := E \cap X_+$ and $E_2 := E \cap X_-$, we have

$$|\mu|(E) := \mu_+(E) + \mu_-(E) = |\mu(E_1)| + |\mu(E_2)|,$$

so the max value over the partitions is attained by $|\mu|$. □

Exercise 7

Proof. By Jordan decomposition, $\mu = \mu_+ - \mu_-$ is finite iff the positive and negative parts μ_+ and μ_- are both finite, iff $|\mu| := \mu_+ + \mu_-$ is finite. □

Remark on the Lebesgue–Radon-Nikodym theorem:

1. Assume that μ, $m < \infty$, and $\mu = m_f + \mu_s = m_g + \mu_r$ are two decompositions given by the LRN theorem. Subtracting (since all terms are finite) gives

$$m_f - m_g = m_{f-g} = \mu_r - \mu_s.$$

Suppose that μ_r is supported on E and μ_s is supported on F, then they are both supported on $E \cup F$. As $\mu_r, \mu_s \perp m$, this implies that

$$m_{f-g} = \mu_r - \mu_s = (\mu_r - \mu_s) \downharpoonright_{E \cup F} \perp m.$$

Since m_h cannot be mutually singular with m for any non-zero h, $f - g = 0$ almost everywhere, but then $m_f = m_g$ by Exercise 2 and we get uniqueness.

2. By the Jordan decomposition theorem, we see that

$$\mu = \mu_+ - \mu_- = (m_f - m_g) + (\mu_s - \mu_r),$$

where $f, g : X \to \mathbf{R}^+$ are measurable and $\mu_s, \mu_r \perp m$. Also, if $h := f - g$, then $h_+ = f$ and $h_- = g$ by construction. Since μ is a signed measure, it takes at most one of the values $+\infty$ or $-\infty$. If $+\infty$ is never attained, $f \in L^1(X, dm)$, if $-\infty$ is never attained, $g \in L^1(X, dm)$, so m_h is well-defined. We may thus assume that μ is unsigned.

3. Let $\lambda := \mu_s - \varepsilon m$. If $\lambda_+ \perp m$, then λ_+ is supported on an m-null set N. Thus $\forall E \in \mathcal{X} \backslash \{N\}$, Jordan decomposition gives

$$\lambda(E) = \lambda_+(E) - \lambda_-(E) = -\lambda_-(E) \leq 0.$$

That is, outside of an m-null set, $\mu_s(E) \leq \varepsilon m(E)$ for all E. Let ε be a countable sequence going to zero, we conclude that $\mu_s \leq 0$ outside of an m-null set N'. But $\mu_s \geq 0$ by construction, so we can write $\mu_s = \mu_s \downharpoonright_{N'}$. Since $m = m \downharpoonright_{X \backslash N'}$, we see that $\mu_s \perp m$.

4. By the Hahn decomposition theorem, $\lambda_+ = \lambda \downharpoonright_{X_+}$ for some set $E := X_+$. If $m(E) = 0$ then $\lambda_+ = \lambda_+ \downharpoonright_E \perp m \downharpoonright_{X \backslash E} = m$, so if λ_+ were not singular w.r.t m then $m(E) > 0$.

Exercise 8

Proof. Suppose that μ and m are σ-finite. As before, we may assume that μ is unsigned by the Jordan decomposition theorem. By condition, there exists a partition $X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n), m(X_n)$ finite for all n. Denote $\mu_n := \mu \downharpoonright_{X_n}$, $m^{(n)} := m \downharpoonright_{X_n}$. By the finite measure case, there exists a unique decomposition

$$\mu_n = m^{(n)}_{f_n} + \mu_{s,n}$$

where $f_n : X_n \to \mathbf{R}^+$ is absolutely integrable and $\mu_{s,n} \perp m^{(n)}$. It is clear that $f := \sum_n 1_{X_n} f_n : X \to \mathbf{R}$ is measurable, and if $\mu_s := \sum_n \mu_{s,n}$, then μ_s is a measure on $\mathcal{X}$ and $\mu_s \perp m$.

By construction, $\mu = \sum_n \mu_n = \sum_n m^{(n)}_{f_n} + \mu_s$. It remains to show that $\sum_n m^{(n)}_{f_n} = m_f$. This follows from the restriction property of the unsigned integral and the monotone convergence theorem. Note that the finiteness of either the positive or negative variation of μ constrains the corresponding variations of f. $\square$

Remark on the Radon-Nikodym theorem:

Suppose the claim holds for μ finite. Let $X = \bigcup_{n=1}^{\infty} X_n$ be a partition such that $\mu(X_n) < \infty$ for all n. Define $\mu_n := \mu \downharpoonright_{X_n}$, then $\mu_n(E) = 0$ whenever $m(E) = 0$ (so $\mu(E) = 0$ whenever $m(E) = 0$), iff $\mu_n = m_{f_n}$ for some measurable f_n for all n. Then $f := \sum_n f_n$ is measurable, and by the Jordan decomposition

$$\mu = \mu_+ - \mu_- = \sum_n (m_{f_n})_+ - \sum_n (m_{f_n})_- = \sum_n m_{(f_n)_+} - \sum_n m_{(f_n)_-},$$

where the last equality follows as the f_n are absolutely integrable. By monotone convergence,

$$\sum_n m_{(f_n)_+} = m_{f_+}, \ \sum_n m_{(f_n)_-} = m_{f_-}.$$

If the positive variation μ_+ is finite, then f_+ is absolutely integrable, similarly if the negative variation μ_- is finite. Hence m_f is well-defined, and $\mu = m_f$. It thus suffices to establish the case when μ is finite.

Exercise 9

Proof. By the Lebesgue decomposition theorem, $\mu = \mu_{ac} + \mu_s$, where $\mu_{ac} \ll m$ and $\mu_s \perp m$. It suffices to show that $\mu_s = \mu_{sc} + \mu_{pp}$. By condition, μ_s is supported on an m-null set N. Let $C := \{x \in N : \mu_s(x) \neq 0\}$, we claim that C is at most countable. Suppose otherwise. Since μ is σ-finite, so is μ_s. Let $E_n := \{x \in C : |\mu_s|(\{x\}) \geq 1/n\}$, then $C = \bigcup_{n=1}^{\infty} E_n$. From hypothesis $\exists n \geq 1$ with E_n uncountable, otherwise C will be countable. By countable additivity, this contradicts the fact that $|\mu_s|$ is σ-finite. Define

$$\mu_{sc} := \mu_s \downharpoonright_{N \setminus C}, \ \mu_{pp} := \mu_s \downharpoonright_C .$$

Both obey the given conditions, and uniqueness follows from that of the Lebesgue decomposition theorem. Finally, if μ is unsigned, then μ_{ac} and μ_s are also, which means that $\mu_{ac}, \mu_{sc}, \mu_{pp}$ are all unsigned. $\square$

Exercise 10

Proof. 1. Let $x \mapsto \mu([0, x])$ be a continuous map. Fix an x. Let ε_n be a countable sequence that goes to 0 with corresponding δ_n that goes to 0, such that $\mu([x, x + \delta_n]) \leq \varepsilon_n$ for all n. Taking limit as $n \to \infty$ shows $\mu(\{x\}) = 0$. i.e. μ is continuous. By downwards monotone convergence,

$$\mu(\{x\}) = \mu(\bigcap_n [x, x + 1/n]) = \lim_n \mu([x, x + 1/n]) = 0,$$

so $\forall \varepsilon > 0$, $\exists \delta' > 0$ such that $\mu([x, x+\delta']) \leq \varepsilon$. Similarly one can show that $\mu([x-\delta'', x]) \leq \varepsilon$ for sufficiently small δ''. Take $\delta := \min(\delta', \delta'')$, we conclude that the map $x \mapsto \mu([0,x])$ is continuous.

2. Suppose that μ is an absolutely continuous measure w.r.t m. By the Radon-Nikodym theorem and finite additivity, if $[x_1, y_1], \ldots, [x_n, y_n]$ are disjoint intervals in $[0, +\infty]$ of total length at most δ, then

$$\sum_{i=1}^{n} (\mu([0, y_i]) - \mu([0, x_i])) = \mu(\bigcup_{i=1}^{n} (x_i, y_i]) < \varepsilon.$$

That is, the function $x \mapsto \mu([0,x])$ is absolutely continuous.

Conversely, assume the function $x \mapsto \mu([0,x])$ is absolutely continuous. Let $\varepsilon > 0$, with corresponding $\delta > 0$ as in the definition of absolute continuity. Let $E \subset [0, +\infty]$ be s.t $m(E) < \delta/2$. There is $U \supset E$, U open with $m(U \backslash E) < \delta/2$. Express U as a countable union of disjoint open intervals I_i, we get

$$m(U) = \sum_{i=1}^{\infty} |I_i| < \delta.$$

By assumption, this implies that $\forall N \geq 1, \sum_{i=1}^{N} \mu(I_i) = \mu(\bigcup_{i=1}^{N} I_i) < \varepsilon$. By upwards monotone convergence,

$$\mu(U) = \lim_{N \to \infty} \mu(\bigcup_{i=1}^{N} I_i) \leq \varepsilon.$$

By monotonicity, we conclude that $\mu(E) \leq \mu(U) \leq \varepsilon$, so μ is absolutely continuous w.r.t m by the Radon-Nikodym theorem. $\square$

Remark on the finitary analogue of the Lebesgue-Radon-Nikodym theorem:

1. By repeated applications of the Bolzano-Weierstrass theorem, there exist nested sequences

$$(n_j^{(k+1)})_j \subset (n_j^{(k)})_j, \ k \geq 1,$$

s.t $\lim_{j \to \infty} \mu_{n_j^{(k)}}(\{f_{n_j^{(k)}} \geq k\})$ converges for each k. Take the diagonal sequence $(n_j^{(j)})_j$, and relabel to obtain

$$\mu_j(\{f_j \geq k\}) := \mu_{n_j^{(j)}}(\{f_{n_j^{(j)}} \geq k\}).$$

Then $\mu_n(\{f_n \geq k\})$ converges for positive k to some limit c_k.

2. $\mu_{n,sc} := \mu_n - \mu_{n,ac} = \mu_n \lfloor_{\{f_n \geq k_n\}}$. By Markov's inequality,

$$\frac{|\{f_n \geq k_n\}|}{|X_n|} = m_n(\{f_n \geq k_n\}) \leq \frac{1}{k_n} \int_{X_n} f_n \; dm_n = \frac{1}{k_n}.$$

That is, $\mu_{n,sc}$ is supported on a set of size $o(|X_n|)$.

3. Let $A_{n,j} := \{x : \mu_n(\{x\}) \geq 1/j\}$, and $\varepsilon > 0$. Since $d_j \to d$, there exists $J > 0$ s.t $d_j \geq d - \varepsilon/4$ for all $j \geq J$. Since $\mu_n(A_{n,j}) \to d_j$ as $n \to \infty$, we have

$$\mu_n(A_{n,J}) \geq d - \varepsilon/2$$

for n sufficiently large. Similarly, since $\mu_n(A_{n,j_n}) \to d$ as $n \to \infty$, we have

$$\mu_n(A_{n,j_n}) \leq d + \varepsilon/2$$

for n sufficiently large. Set $E_n := A_{n,J}$, note that $|E_n| \leq J$, and we obtain

$$\mu_{n,pp}(X_n \backslash E_n) \leq \mu_n(A_{n,j_n} \backslash A_{n,J}) = \mu_n(A_{n,j_n}) - \mu_n(A_{n,J}) \leq \varepsilon$$

for n sufficiently large (in particular $j_n \geq J$, as $A_{n,j}$ is increasing in j). Now set $N := J$. (For the remaining n, one simply increases N as much as is necessary.)

Exercise 11

Proof. We claim that Theorem 3 holds in the more general setting. Let $v_n := (\mu_n)_+$ and $w_n := (\mu_n)_-$ be the positive and negative variation in the Jordan decomposition. By defining

$$\mu_{n,ac} := v_{n,ac} - w_{n,ac}, \mu_{n,sc} := v_{n,sc} - w_{n,sc}, \mu_{n,pp} := v_{n,pp} - w_{n,pp},$$

we may assume that the μ_n are unsigned. Using the Radon-Nikodym theorem, we can write $d(\mu_n)_{ac} = f_n \; dm_n$ for some unsigned $f_n \in L^1(X_n, dm_n)$.

For each positive integer k, the sequence $\mu_n(\{f_n \geq k\})$ is bounded between 0 and M, where $\mu_n(X_n) \leq M$ for all n. By the Bolzano-Weierstrass theorem, it has a convergent subsequence. Applying the usual diagonalisation argument, we may assume (after passing to a subsequence and relabeling) that $\mu_n(\{f_n \geq k\})$ converges for positive k to some limit c_k.

Clearly, the c_k are decreasing and range between 0 and M, and so converge as $k \to \infty$ to some limit $0 < c < M$. Since $\lim_{k\to\infty} \lim_{n\to\infty} \mu_n(\{f_n \geq k\}) = c$, we can find a subsequence k_n going to infinity such that $\mu_n(\{f_n \geq k_n\}) \to c$ as $n \to \infty$.

We now set $\mu_{n,ac} := (\mu_n)_{ac} \lfloor_{\{f_n<k_n\}}$, and claim the absolute continuity property 1. Indeed, for any $\varepsilon > 0$ we can find a k such that $c_k \leq c + \varepsilon/10$. For n sufficiently large, we thus have

$$\mu_n(\{f_n \geq k\}) \leq c + \varepsilon/5 \tag{1}$$

and

$$\mu_n(\{f_n \geq k_n\}) \geq c - \varepsilon/5 \tag{2}$$

and hence

$$\mu_{n,ac}(\{f_n \geq k\}) \leq 2\varepsilon/5. \tag{3}$$

If we take $\delta < \varepsilon/5k$, we thus see (for n sufficiently large) that 1. holds. (For the remaining n, one simply shrinks delta as much as is necessary.)

Write $\mu_{n,s} := \mu_n - \mu_{n,ac}$, and $A_n := \text{supp}(\mu_n)_{ac}$, then $\mu_{n,s}$ is supported on $\{f_n \geq k_n\} \cup \{f_n < k_n\}\backslash A_n$. But $\{f_n < k_n\}\backslash A_n \subset \text{supp}(\mu_n)_s$, which has zero m_n-measure. Thus $\mu_{n,s}$ is supported on a set of m_n-measure $M/k_n = o(1)$ by Markov's inequality. It remains to extract out the pure point components. This we do by a similar procedure as above. Indeed, by arguing as before we may assume (after passing to a subsequence as necessary) that the quantities $\mu_n\{x : \mu_n(\{x\}) \geq 1/j\}$ converge to a limit d_j for each positive integer j, that the d_j themselves converge to a limit d, and that there exists a sequence $j_n \to \infty$ such that $\mu_n\{x : \mu_n(\{x\}) \geq 1/j_n\}$ converges to d. If one sets $\mu_{n,sc}$ and $\mu_{n,pp}$ to be the restrictions of $\mu_{n,s}$ to the sets $\{x : \mu_n(\{x\}) < 1/j_n\}$ and $\{x : \mu_n(\{x\}) \geq 1/j_n\}$ respectively, one can verify the remaining claims by arguments similar to those already given. The only difference being that we initialize $N := J\lceil M\rceil$ and increase N accordingly. □

Notes on the finite convergence principle:

Pigeonhole principle (third version)

Proof. Applying the first version of the pigeonhole principle to the sequence $y_n := x_{2^n}$, $n = 0, 1, \ldots$, we see if $y_{M'-1}$ (for M' terms) is such that $M' \geq 1/\varepsilon + 1$, there exists $0 \leq N' < M' - 1$ such that $|y_{N'+1} - y_{N'}| \leq \varepsilon$. Converting back, let $M := 2^{M'} \geq 2^{1/\varepsilon+1}$, and $N := 2^{N'}$, we see that

$$|x_{2N} - x_N| \leq \varepsilon$$

for $1 \leq N < 2N \leq M$. Since the sequence x_n is non-decreasing, for any indices $N \leq n, m \leq 2N$,

$$|x_n - x_m| \leq |x_{2N} - x_N| \leq \varepsilon,$$

as claimed. □

Finite convergence principle

Proof. Applying the first version of the pigeonhole principle to the sequence $y_j := x_{i_j}$, where i_j is defined recursively by $i_1 := 1, i_{j+1} := i_j + F(i_j)$, we see if $M' \geq 1/\varepsilon + 1$, there exist $1 \leq N' < M'$ such that $|y_{N'+1} - y_{N'}| \leq \varepsilon$. Converting back, let $M := i_{M'} \geq i_{\lfloor 1/\varepsilon \rfloor + 1}$, and $N := i_{N'}$, we see that

$$|x_{N+F(N)} - x_N| \leq \varepsilon$$

for $1 \leq N < N + F(N) \leq M$. Since the sequence x_n is non-decreasing, for any indices $N \leq n, m \leq N + F(N)$,

$$|x_n - x_m| \leq |x_{N+F(N)} - x_N| \leq \varepsilon,$$

as claimed. □

The space of all bounded sequences is sequentially compact in the product topology.

Proof. Let $(x_n^{(i)})_{n=1}^\infty \in X := \prod_{n=1}^\infty [-M, M], i = 1, 2, \ldots$ be a sequence in the space X of bounded sequences. For each $n \in \mathbb{N}$, $(x_n^{(i)})_i$ is a bounded sequence of real numbers, so by the Bolzano-Weierstrass theorem, it has a convergent subsequence $(x_n^{(i_j^{(n)})})_j$ such that

$$\lim_{j \to \infty} x_n^{(i_j^{(n)})} = x_n.$$

The sequence $(x_{n+1}^{(i_j^{(n)})})_j$ is a bounded sequence, so we can apply the Bolzano-Weierstrass theorem again to obtain a convergent subsequence $(x_{n+1}^{(i_j^{(n+1)})})_j$, e.t.c. Take the diagonal sequence $(x_n^{i_j^{(j)}})_{n=1}^\infty, j = 1, 2, \ldots$ in X, observe that

$$\lim_{j \to \infty} x_n^{(i_j^{(j)})} = x_n$$

simultaneously for all n. Thus the subsequence $(x_n^{i_j^{(j)}})_{n=1}^\infty, j \geq 1$ of $(x_n^{(i)})_{n=1}^\infty, i \geq 1$ converge to the sequence $(x_n)_{n=1}^\infty$, as desired. □

Given two probability measures μ and m, by the Lebesgue-Radon-Nikodym theorem, we get $\mu = m_f + \mu_s$, where $f \in L^1(X, dm)$ is unsigned and $\mu_s \perp m$. Define $\mu_{\leq k} := m_{f_k}$, where $f_k := f \downharpoonright_{\{f \leq k\}}$, and $\mu_{>k} := \mu - \mu_{\leq k}$. Then

$$\mu_{\leq k}(E) = \int_{E \cap \{f \leq k\}} f \, dm \leq \int_E k \, dm = km(E)$$

for all measurable E. And $\mu_{>k} = (m_f - m_{f_k}) + \mu_s$ is supported on $\{f > k\}$, which has m-measure at most $1/k$ by Markov's inequality. We conclude that $\mu_{>k}$ is supported on a set of measure at most $1/k$.

Let $\phi(t) := \mu(\{f \geq t\})$, which is decreasing in t, and consider the sequence $x_n := 1 - \phi(n)$, which is increasing and bounded between 0 and 1. Applying the finite convergence principle, if M is sufficiently large depending on M and ε, then there exists $1 \leq k < k + F(k) \leq M$ such that $|x_k - x_{k+F(k)}| \leq \varepsilon$. Hence

$$|\phi(k) - \phi(k + F(k))| = \mu(\{k < f \leq k + F(k)\}) \leq \varepsilon.$$

Decompose $\mu_{>k}$ as $\mu_{>k} = \mu_{k<\cdot\leq F(k)} + \mu_{\geq F(k)}$ using the threshold parameter $F(k)$, where we define

$$\mu_{k<\cdot\leq F(k)} := m_{f1_{\{k<f\leq k+F(k)\}}},\ \mu_{\geq F(k)} := m_{f1_{\{f\geq k+F(k)\}}} + \mu_s,$$

we see that $\mu_{k<\cdot\leq F(k)}(X) \leq \mu(\{k < f \leq k + F(k)\}) \leq \varepsilon,\ k = O_{F,\varepsilon}(1)$, and $\mu_{\geq F(k)}$ is supported on a set of m-measure at most $1/F(k)$, again by Markov's inequality.