Math 245B note 0

qshuyu

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1 Selected Exercises in Note 0

Exercise 1

Proof. Let X and Y be equipped with their respective topology $\mathcal{F}_X$ and $\mathcal{F}_Y$. By continuity, $f^{-1}(U) \in \mathcal{F}_X$ whenever $U \in \mathcal{F}_Y$. If $V \subset Y$ is such that $f^{-1}(Y) \in \mathcal{F}_X$, then $Y \backslash V$ is such that $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V) \in \mathcal{F}_X$. If $V_1, V_2, \ldots \subset Y$ are such that $f^{-1}(V_n) \in \mathcal{F}_X$ for all n, then $f^{-1}(\bigcup_{n=1}^{\infty} V_n) = \bigcup_{n=1}^{\infty} f^{-1}(V_n) \in \mathcal{F}_X$. Finally, we have $f^{-1}(\emptyset_Y) = \emptyset_X \in \mathcal{F}_X$. We conclude that $f^{-1}(E) \in \mathcal{B}[\mathcal{F}_X]$ whenever $E \in \mathcal{B}[\mathcal{F}_Y]$. That is, f is Borel measurable. $\square$

Exercise 2

Proof. Let $\mathcal{B}[.]$ denote the Borel σ-algebra on the given space. By chasing the definition, one can show that $\mathcal{B}[\mathbb{R}^n] \downharpoonleft_{V_\alpha} = \mathcal{B}[V_\alpha]$ and $\mathcal{B}[M] \downharpoonleft_{U_\alpha} = \mathcal{B}[U_\alpha]$. Hence we aim to show that

$$\mathcal{B}[U_\alpha] = \pi_\alpha^{-1}(\mathcal{B}[V_\alpha]).$$

Similarly, $\pi_\alpha^{-1}(\mathcal{B}[V_\alpha])$ is a σ-algebra on U_α. Since π_α is a homeomorphism for every α, it pulls back the generators of $\mathcal{B}[V_\alpha]$ (open subsets of V_α) onto the generators of $\mathcal{B}[U_\alpha]$ (open subsets of U_α), and vice versa. It follows that $\mathcal{B}[U_\alpha] = \pi_\alpha^{-1}(\mathcal{B}[V_\alpha])$, as required.

For uniqueness, let $\mathcal{G}$ be a σ-algebra on M such that $\mathcal{G} \downharpoonleft_{U_\alpha} = \pi_\alpha^{-1}(\mathcal{B}[V_\alpha])$ for every α. Let $O \subset M$ be open. For any α, $O \cap U_\alpha \in \mathcal{B}[U_\alpha]$, so $O \cap U_\alpha = E \cap U_\alpha$ for some $E \in \mathcal{G}$. Hence $O = E \in \mathcal{G}$ and $\mathcal{B}[M] \subset \mathcal{G}$. Conversely, let $E \in \mathcal{G}$. By definition, M is second countable, i.e. $M = \bigcup_{i=1}^{\infty} O_i$ for $O_i \subset M$ open. Thus $O_i \subset U_{\alpha_i}$ for some $\alpha = \alpha_i$ and all i, thus

$$E = \bigcup_{i=1}^{\infty} E \cap O_i = \bigcup_{i=1}^{\infty} E \cap U_{\alpha_i}.$$

Since $E \cap U_{\alpha_i} \in \mathcal{B}[U_{\alpha_i}] \subset \mathcal{B}[M]$ for all i, $E \in \mathcal{B}[M]$ and thus $\mathcal{G} \subset \mathcal{B}[M]$. $\square$

Exercise 3

Proof. A σ-algebra on a finite set X is a finite, and thus an atomic algebra by Exercise 1.4.4 of "M". Combine with Exercise 1.4.3 of "M", we conclude that this algebra necessarily arises from a partition $X = \bigcup_{\alpha \in A} E_\alpha$, and is unique (up to relabeling). $\square$

Exercise 4

Proof. $\forall \alpha \in A$, let $\{U_{\alpha,i}\}_{i=1}^{\infty}$ be a countable basis of X_α. For simplicity, let $\mathcal{F}_\alpha$ be the topology on X_α for each $\alpha \in A$, $\mathcal{F}_A$ the topology on $X_A = \prod_{\alpha \in A} X_\alpha$. First we show that

$$\mathcal{B}[\mathcal{F}_A] = \langle \{\pi_\alpha^{-1}(U_{\alpha,i}) : \alpha \in A,\ i \geq 1\} \rangle.$$

By definition, $\mathcal{F}_A$ consists of sets of the form $U = \prod_{\beta \in F} U_\beta \times \prod_{\alpha \in A \setminus F} X_\alpha$, where $F \subset A$ is a finite subset and $U_\beta \subset X_\beta$ are open. For each $\beta \in F$, write $U_\beta = \bigcup_{i \in N_\beta} U_{\beta,i},\ N_\beta \subset \mathbb{N}$ as a union of base elements. Then

$$U = \bigcap_{\beta \in F} \pi_\beta^{-1}(U_\beta) = \bigcap_{\beta \in F} \bigcup_{i \in N_\beta} \pi_\beta^{-1}(U_{\beta,i}).$$

Hence LHS $\subset$ RHS. Since $\pi_\alpha^{-1}(U_{\alpha,i}) = U_{\alpha,i} \times \prod_{\beta \neq \alpha} X_\beta$, we get RHS $\subset$ LHS and the result follows.

By the above and Remark 1.4.15 of "M", we can show that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{B}[\mathcal{F}_A]$ for all $E_\alpha \in \mathcal{B}[\mathcal{F}_\alpha]$ and any $\alpha \in A$. Hence we get

$$\bigcup_{\alpha \in A} \pi_\alpha^*(\mathcal{B}[\mathcal{F}_\alpha]) \subset \mathcal{B}[\mathcal{F}_A],$$

where $\pi_\alpha^*(.)$ denotes the pullback σ-algebra. As $\prod_{\alpha \in A} \mathcal{B}[\mathcal{F}_\alpha]$ is the generation of this set, we get $\prod_{\alpha \in A} \mathcal{B}[\mathcal{F}_\alpha] \subset \mathcal{B}[\mathcal{F}_A]$. The other inclusion follows from the first paragraph. $\square$

Exercise 5

Proof. See (3) of Exercise 1.7.18 in "M". $\square$

Exercise 6

Proof. See Exercise 1.4.23 of "M". ☐

Exercise 7

Proof. See Exercise 1.4.26 of "M". ☐

Exercise 8

Proof. By condition, we can find a covering $(A_n)_{n=1}^{\infty}$ of E by elements in $\mathcal{A}$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(E) + \varepsilon/2.$$

As $\mu(E) < \infty$, this implies that $\sum_{n=1}^{\infty} \mu(A_n) - \mu(E) \leq \varepsilon/2$. Similarly, we can find $N \geq 1$ with

$$\sum_{n=1}^{N} \mu(A_n) \geq \sum_{n=1}^{\infty} \mu(A_n) - \varepsilon/2.$$

Let $F = \bigcup_{n=1}^{N} A_n$. By monotonicity, additivity, and countable subadditivity,

$$\mu(E \Delta F) = \mu(E \backslash F) + \mu(F \backslash E) \leq \varepsilon,$$

as desired. See also Exercise 1.4.28 of "M" (approximation by an algebra). ☐

Remark on Theorem 1:

Define the outer measure $\mu_*(E)$ of any set $E \subset X$ as the infimum of $\sum_{n=1}^{\infty} \mu(A_n)$, where $(A_n)_{n=1}^{\infty}$ ranges over all coverings of E by elements in $\mathcal{A}$. It is not hard to see that μ_* is a legit outer measure, and that μ_* agrees with μ on $\mathcal{A}$. Let $\mathcal{B}$ be the collection of all subsets of X that are Carathéodory measurable with respect to μ_*, we show that $\mathcal{B} \supset \mathcal{A}$, and thus $\mathcal{B} \supset \langle \mathcal{A} \rangle$, in this sense Theorem 1 is a special case of the full extension theorem.

For all $E \in \mathcal{A}$, we want to show that $\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \backslash E)$ for every set $A \subset X$. That LHS $\leq$ RHS follows from monotonicity of the outer measure. For any coverings $(A_n)_{n=1}^{\infty}$ of A by elements in $\mathcal{A}$, we have

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n \cap E) + \mu(A_n \backslash E) \geq \mu_*(A \cap E) + \mu_*(A \backslash E).$$

Taking infimum over all such coverings on the LHS, we get LHS $\geq$ RHS and the claim follows.

Exercise 9

Proof. Applying Proposition 1.7.11 of "M" iteratively gives the result. □

Exercise 10

Proof. See Exercise 1.2.15 and Lemma 1.2.12 of "M". □

Exercise 11

Proof. See Exercise 1.5.10 and Exercise 1.5.13 of "M". □

Remark on the Weak Carathéodory extension theorem:

1. Let $d(A, B) := \mu_*(A \Delta B)$ be the pseudometric on 2^X. The Boolean operations (union, intersection, complement, etc.) being all continuous with respect to this pseudometric means that if $d(A, A_n), d(B, B_n)$ are small, then $d(A \cup A_n, B \cup B_n), d(A \cap A_n, B \cap B_n), d(A^c, A_n^c)$ are all small, analogues to addition of vectors being continuous with respect to the Euclidean metric.

2. Let $A, B \in \overline{\mathcal{A}}$, then $\exists A_\varepsilon, B_\varepsilon \in \mathcal{A}$ such that $\mu_*(A\Delta A_\varepsilon), \mu_*(B\Delta B_\varepsilon) < \varepsilon$. From $(A\cup B)\Delta(A_\varepsilon \cup B_\varepsilon) \subset A\Delta A_\varepsilon \cup B\Delta B_\varepsilon$, and $A^c \Delta A_\varepsilon^c = A\Delta A_\varepsilon$, we see that the closure of $\mathcal{A}$ w.r.t the pseudometric is closed under finite unions and complements. Let $(A_n)_{n=1}^\infty \in \overline{\mathcal{A}}$ be disjoint, and $A_{n,\varepsilon} \in \mathcal{A}$ be such that $\mu_*(A_n \Delta A_{n,\varepsilon}) < \varepsilon/2^{n+1}$ for all n. Since $\mu(X) < \infty$, the expression $\bigcup_{n=1}^\infty \mu(A_{n,\varepsilon})$ is absolutely summable, hence we can find large N such that $\mu_*(\bigcup_{n=1}^\infty A_{n,\varepsilon} \Delta \bigcup_{m=1}^N A_{m,\varepsilon}) = \sum_{n>N} \mu(A_{n,\varepsilon}) < \varepsilon$. It follows that

$$\mu_*(\bigcup_{n=1}^\infty A_n \Delta \bigcup_{m=1}^N A_{m,\varepsilon}) \le \mu_*(\bigcup_{n=1}^\infty A_n \Delta \bigcup_{n=1}^\infty A_{n,\varepsilon}) + \mu_*(\bigcup_{n=1}^\infty A_{n,\varepsilon} \Delta \bigcup_{m=1}^N A_{m,\varepsilon}) < \varepsilon.$$

Hence the closure of $\mathcal{A}$ is closed under countable disjoint union as well.

3. By subadditivity, $\mu_*(A) - \mu_*(B) \le \mu_*(A\Delta B)$, and similarly if we swap A and B, so $|\mu_*(A) - \mu_*(B)| \le \mu_*(A\Delta B)$ and the function $A \mapsto \mu_*(A)$ is Lipschitz continuous. Let $E, F \in \mathcal{X}$, and $E_\varepsilon, F_\varepsilon \in \mathcal{A}$ be such that

$$\mu_*(E\Delta E_\varepsilon), \mu_*(F\Delta F_\varepsilon) < \varepsilon/2.$$

Then by Lipschitz continuity of μ_*, triangle inequality, and subadditivity we get

$$\begin{aligned}
|\mu_*(E\cup F) - \mu_*(E) - \mu_*(F)| &\le |\mu_*(E\cup F) - \mu_*(E_\varepsilon \cup F_\varepsilon)| \\
&\le \mu_*(E\cup F \Delta E_\varepsilon \cup F_\varepsilon) \\
&\le \mu_*((E\Delta E_\varepsilon) \cup (F\Delta F_\varepsilon)) \\
&\le \mu_*(E\Delta E_\varepsilon) + \mu_*(F\Delta F_\varepsilon) \\
&\le \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Taking limit as $\varepsilon \to 0$ gives finite additivity of μ_* on the measurable sets.

4. We covered the finite measure case. To handle the σ-finite case, suppose that X is partitioned into $X = \bigcup_{n=1}^\infty A_n,\ A_n \in \mathcal{A},\ \mu(A_n) < \infty$. Apply the finite measure result to $(A_n, \mathcal{A}\downharpoonleft_{A_n}, \mu\downharpoonleft_{A_n})$, we can extend μ to measurable subsets of each A_n, denoted say by μ_n. $\forall E \in \langle \mathcal{A} \rangle$, we have $E \cap A_n \in \langle \mathcal{A} \rangle \downharpoonleft_{A_n} = \langle \mathcal{A}\downharpoonleft_{A_n} \rangle$, thus we define

$$\mu(E) := \sum_{n=1}^\infty \mu\downharpoonleft_{A_n} (E\cap A_n).$$

Note that if $X = \bigcup_n A_n = \bigcup_n B_n$ are two partitions, then the same construction on $X = \bigcup_{n,m} A_n \cap B_m$ and pre-countable additivity shows that the above definition is well-defined.

5. If $\mu_*(A\Delta B) < \varepsilon$ for $A, B \in \mathcal{X}$, then $\mu_1(A\Delta B) \le \mu_1(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty \mu(A_n)$ for any at most countable collections of elementary sets A_n that cover $A\Delta B$, hence

$$|\mu_1(A) - \mu_1(B)| \le \mu_1(A\Delta B) \le \mu_*(A\Delta B) < \varepsilon.$$

Similarly for μ_2. i.e. μ_1 and μ_2 are both continuous with respect to the μ_* pseudometric. Previously we have shown that the closure of the elementary sets $\overline{\mathcal{A}}$ is a σ-algebra that contains the measurable sets, so every measurable set is a limit of elementary sets in this pseudometric. That is, $\forall E \in \mathcal{X},\ \exists A_n \in \mathcal{A}$ such that

$$\mu_*(E \Delta E_n) < 1/n \to 0 \text{ as } n \to \infty.$$

By continuity of μ_1 and μ_2 w.r.t the pseudometric, this implies that

$$\begin{aligned} |\mu_1(E) - \mu_2(E)| &= |\mu_1(E) \pm \mu_1(E_n) - \mu_2(E) \pm \mu_2(E_n)| \\ &\leq |\mu_1(E) - \mu_1(E_n)| + |\mu_2(E) - \mu_2(E_n)| \end{aligned}$$

goes to 0 as $n \to \infty$. We conclude that $\mu_1 = \mu_2$ for the finite measure case.