Note 0 of Math 275A Probability theory

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1 Selected Exercises in Note 0

Exercise 15

Proof. We want to show that $\mathcal{B}[\mathbf{R}^n]$, the Borel σ algebra of $\mathbf{R}^n$, is equal to $\prod_{i=1}^n \mathcal{B}[\mathbf{R}]$, the product of n copies of Borel σ-algebra of $\mathbf{R}$, which is defined by the algebra generated by the sets $(E_1 \times \mathbf{R} \times \ldots \times \mathbf{R})$, $(\mathbf{R} \times E_2 \times \ldots \times \mathbf{R})$, $\ldots$, $(\mathbf{R} \times \ldots \times E_n)$, where $E_i \in \mathcal{B}[\mathbf{R}]$ for all $1 \leq i \leq n$.

We first show that $\mathcal{B}[\mathbf{R}^n] \subset \prod_{i=1}^n \mathcal{B}[\mathbf{R}]$. By Exercise 1.4.14 of "M", $\mathcal{B}[\mathbf{R}^n]$ is generated by the boxes. Let $B = I_1 \times I_2 \times \ldots \times I_n$ be any box in $\mathbf{R}^n$. Then we see that $B = (I_1 \times \mathbf{R} \times \ldots \times \mathbf{R}) \cap (\mathbf{R} \times I_2 \times \ldots \times \mathbf{R}) \cap \ldots \cap (\mathbf{R} \times \mathbf{R} \times \ldots \times I_n) \in \prod_{i=1}^n \mathcal{B}[\mathbf{R}]$. By the definition of a generated σ-algebra, $\mathcal{B}[\mathbf{R}^n] \subset \prod_{i=1}^n \mathcal{B}[\mathbf{R}]$.

Now we show that $\prod_{i=1}^n \mathcal{B}[\mathbf{R}] \subset \mathcal{B}[\mathbf{R}^n]$. By Exercise 1.4.17 of "M", if $E_1 \subset \mathbf{R}^{d_1}$ and $E_2 \subset \mathbf{R}^{d_2}$ are Borel measurable, then $E \times F \subset \mathbf{R}^{d_1+d_2}$ is Borel measurable. Hence the generators of $\prod_{i=1}^n \mathcal{B}[\mathbf{R}]$ are all members $\mathcal{B}[\mathbf{R}^n]$. Again by the definition of a generated σ-algebra, $\prod_{i=1}^n \mathcal{B}[\mathbf{R}] \subset \mathcal{B}[\mathbf{R}^n]$. $\square$

Exercise 17

Proof. Let $f : X \to Y$ be a continuous map, and $\mathcal{B}[\mathbf{X}]$, $\mathcal{B}[\mathbf{Y}]$ be the Borel σ-algebra of X and Y respectively. Let $C := \{B \in \mathcal{B}[\mathbf{Y}] : f^{-1}(B) \in \mathcal{B}[\mathbf{Y}]\}$.

We first show that C is a σ-algebra. Clearly $\emptyset \in C$. If $E \in C$, then $Y \setminus E \in C$ since $f^{-1}(Y \setminus E) = f^{-1}(Y) \setminus f^{-1}(E) = X \setminus f^{-1}(E) \in \mathcal{B}[\mathbf{X}]$. Finally, if E_1, E_2, $\ldots \in C$, then $\bigcup_{n=1} E_n \in C$ since $f^{-1}(\bigcup_{n=1} E_n) = \bigcup_{n=1} f^{-1}(E_n) \in \mathcal{B}[\mathbf{X}]$. By definition, C is indeed a σ-algebra.

By Definition 2.5.8 of "Analysis II", the pre-image under f of any open set is still open, which implies that C contains all the generators of $\mathcal{B}[\mathbf{Y}]$. By the definition of a σ-algebra, $\mathcal{B}[\mathbf{Y}] \subset C$. By the construction of C we also have $C \subset \mathcal{B}[\mathbf{Y}]$. Hence $C = \mathcal{B}[\mathbf{Y}]$ and we're done. $\square$

Exercise 18

Proof. Let $\mathcal{B}_1$, ... ,$\mathcal{B}_n$ be the σ-algebras of R_1, ... ,R_n, respectively, and $\mathcal{F}$ be the σ-algebra of Ω. The product algebra $\prod_{i=1}^n \mathcal{B}_i$ on the Cartesian product space $\prod_{i=1}^n R_i$ is generated by sets of the form $(R_1 \times \ldots \times E_i \times \ldots \times R_n)$, where $E_i \in \mathcal{B}_i$ for all $1 \leq i \leq n$.

Let $P(F)$ be the property that $F \subset R_1 \times \ldots \times R_n$ is such that $(X_1, \ldots, X_n)^{-1}(F) \in \mathcal{F}$. Note that for any i, $(X_1, \ldots, X_n)^{-1}(R_1 \times \ldots \times E_i \times \ldots \times R_n) = X_i^{-1}(E_i) \in \mathcal{F}$. So P holds on the generators of $\prod_{i=1}^n \mathcal{B}_i$. Also, if $P(F)$ is true for some $F \subset R_1 \times \ldots \times R_n$, then $P(F^C)$ is true since $(X_1, \ldots, , X_n)^{-1}(F^C)$ $= \Omega \setminus (X_1, \ldots, , X_n)^{-1}(F) \in \mathcal{F}$. And if $F_1, F_2, \ldots \subset R_1 \times \ldots \times R_n$ are such that $P(F_n)$ is true for all n, then $(X_1, \ldots, X_n)^{-1}(\bigcup_n F_n) = \bigcup_n (X_1, \ldots, X_n)^{-1}(F_n) \in \mathcal{F}$, meaning that $P(\bigcup_n F_n)$ is true. Finally, we clearly have $P(\emptyset_{R_1 \times \ldots \times R_n})$ is trivially true since the pre-image of $\emptyset_{R_1 \times \ldots \times R_n}$ is $\emptyset_\Omega$. Hence by Exercise 11, P holds on $\prod_{i=1}^n \mathcal{B}_i$, as we desired. □

Exercise 24

Proof. (1). First suppose that the sets $\{\omega \in \Omega : f(\omega) \leq t\}$ are measurable for all real t. Note that $\forall t \in \mathbf{R}$, we have $[-\infty, t) = \bigcup_{n=1}^\infty [-\infty, t - \frac{1}{n}]$, so in particular

$$f^{-1}([-\infty, t)) = f^{-1}(\textstyle\bigcup_{n=1}^\infty [-\infty, t - \frac{1}{n}]) = \bigcup_{n=1}^\infty f^{-1}([-\infty, t - \frac{1}{n}])$$

, which is measurable for any real t. Thus $\forall a < b \in \mathbf{R}$, we get

$f^{-1}((a, b)) = f^{-1}([-\infty, b) \setminus [-\infty, a]) = f^{-1}([-\infty, b)) \setminus f^{-1}([-\infty, a])$, which is measurable.

By Lemma 7.4.10 of "Analysis II", every open set can be written as an at most countable union of open boxes, and in this case, open intervals. Hence $f^{-1}(U)$ is measurable for any open $U \subset [-\infty, +\infty]$. Run the induction argument of Exercise 11, we see that f is measurable.

Conversely, suppose that f is measurable. Since $\forall t \in [-\infty, +\infty]$, the set $[-\infty, t]$ is Borel measurable in $[-\infty, +\infty]$, $f^{-1}([-\infty, t])$ is measurable in Ω. □

(2). If $\{\omega \in \Omega : f(\omega) \leq t\} = \{\omega \in \Omega : g(\omega) \leq t\}$. Assume for contradiction that $\exists \omega \in \Omega$ such that $f(\omega) \neq g(\omega)$. Without loss of generality suppose that $f(\omega) < g(\omega)$. Then we can find some $t \in \mathbf{R}$ such that $f(\omega) < t < g(\omega)$. This implies that $\omega \in \{\omega \in \Omega : f(\omega) \leq t\}$ but $\omega \notin \{\omega \in \Omega : g(\omega) \leq t\}$, a contradiction. So $f = g$. The other direction is obvious. □

(3). For any real t, we have

$\{\omega \in \Omega : \sup_n f_n(\omega) \leq t\} = \bigcap_n \{\omega \in \Omega : f_n(\omega) \leq t\} \in \mathcal{B}$,

and similarly

$\{\omega \in \Omega : \inf_n f_n(\omega) \leq t\} = \bigcup_n \{\omega \in \Omega : f_n(\omega) \leq t\} \in \mathcal{B}$.

By part (1), $\sup_n f_n$ and $\inf_n f_n$ are thus measurable. Then it follows by definition that $\limsup_{n\to\infty} f_n := \inf_{N>0} \sup_{n\geq N} f_n$ and $\liminf_{n\to\infty} f_n := \sup_{N>0} \inf_{n\ \geq N} f_n$ are both measurable. $\square$

Exercise 26

Proof. That F is non-decreasing follows from the monotonicity of the measure μ.

Let $(t_n)_n$ be a monotone decreasing sequence in $\mathbf{R}$ such that $t_n \to -\infty$. By the monotone nature of F, $(F(t_n))_n$ is a monotone decreasing sequence in $[0, 1]$. By bounded monotone sequences converge, $F(t_n) \to L$ for some $L \geq 0 \in \mathbf{R}$. In particular, $\forall \varepsilon > 0$, $\exists N > 0$ large enough such that $|F(t_N) - L| < \varepsilon$. Now for any sequence $(t'_n)_n$ in $\mathbf{R}$ such that $t'_n \to -\infty$ (not just monotone ones), $\exists M > 0$ large enough such that $t'n \leq t_N$ for all $n \geq M$, implying again by the monotone nature of F that $|F(t'_n) - L| \leq \varepsilon$ for all $n \geq M$. That is, $F(t'_n) \to L$. By proposition 9.3.9 of "Analysis I", $\lim_{t\to-\infty} F(t) = L$ exists.

Now we show that $L = 0$. For any monotone decreasing sequence $(t_n)_n$ converging to $-\infty$, the sets $(-\infty, t_1] \supset (-\infty, t_2] \supset \ldots$ are all measurable and $\mu((-\infty, t_1])$ finite, so by continuity from above, we have

$L = \lim_{n\to\infty} \mu((-\infty, t_n]) = \mu(\bigcap_{n=1}^{\infty}(-\infty, t_n]) = \mu(\emptyset) = 0$.

By similar argument, using continuity from below and the fact that $\mu(\mathbf{R}) = 1$, we can show that $\lim_{t\to+\infty} F(t) = 1$.

As above, it suffices to show that $\lim_{n\to\infty} F(s_n) = F(t)$, where $(s_n)_n$ is a monotone decreasing sequence converging to t from above. By continuity from above, we have

$\lim_{n\to\infty} F(s_n) = \lim_{n\to\infty} \mu((-\infty, s_n]) = \mu(\bigcap_{n=1}^{\infty}(-\infty, s_n]) = \mu((-\infty, t]) = F(t)$,

as desired. $\square$

Exercise 28

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be the probability space that models X. As before, it can be shown that $\lim_{s\to t^-} F(s) = \lim_{n\to\infty} F(s_n)$, where $(s_n)_n$ is a monotone non-decreasing sequence such that $s_n \leq t$ for all n and $\lim_{n\to\infty} s_n = t$. By continuity from below, we then have

$\lim_{n\to\infty} F(s_n) = \lim_{n\to\infty} \mu(\{\omega \in \Omega : X(\omega) \leq s_n\})$

$= \mu(\bigcup_{n=1}^{\infty}\{\omega \in \Omega : X(\omega) \leq s_n\}) = \mu(\{\omega \in \Omega : X(\omega) < t\}) := \mathbf{P}(X < t)$.

By finite additivity of μ, we get

$\mathbf{P}(X = t) := \mu(\{\omega \in \Omega : X(\omega) = t\})$

$= \mu(\{\omega \in \Omega : X(\omega) \leq t\}) - \{\omega \in \Omega : X(\omega) < t\} = F(t) - \lim_{s\to t^-} F(s)$.

Finally, by Exercise 24, F is right continuous, so $\mathbf{P}(X = t) = 0$ gives

$F(t) - \lim_{s\to t^-} F(s) = \lim_{s\to t^+} F(s) - \lim_{s\to t^-} F(s) = 0$.

That is, F is continuous. The other direction is obvious. $\square$

Exercise 29

Proof. Let X^+ and X^- be real random variables modeled by the probability space $(\Omega, \mathcal{F}, \mu)$. By Exercise 24, we need to show that the sets $\{\omega \in \Omega : X^-(\omega) \leq t\}$ and $\{\omega \in \Omega : X^+(\omega) \leq t\}$ are both measurable for any real t.

Since F is a CDF, it obeys the properties (i)-(iii) of Exercise 26. In particular, F is non-decreasing, so for any $\omega \in \Omega$,

$X^-(\omega) = \sup\{y \in \mathbf{R} : F(y) \leq U(\omega)\} = F^{-1}(U(\omega))$, which is a single element set by the definition of a CDF.

Then we see that

$\{\omega \in \Omega : X^-(\omega) \leq t\} = \{\omega \in \Omega : F^{-1}(U(\omega)) \leq t\} = \{\omega \in \Omega : U(\omega) \leq F(t)\}$, which is measurable by the fact that U is measurable.

Hence X^- is a indeed a random variable. Similarly X^+ is a random variable.

Denote the CDF of X^- and X^+ by F_{X^-} and F_{X^+}, respectively. By what we've shown above, we get

$F_{X^-}(t) := \mathbf{P}(X^- \le t) = \mu(\{\omega \in \Omega : U(\omega) \le F(t)\}) = F_U(F(t)) = \min(\max(F(t), 0), 1) = F(t)$.

Similarly we can show that $F_{X^+}(t) = F(t)$. $\square$

Exercise 30

Proof. (1). This follows from the monotonicity of the measure μ. $\square$

(2). Let $(t_{1,k})_k = (t_{2,k})_k = \ldots = (t_{n,k})_k$ be a monotone decreasing sequence in $\mathbf{R}$ converging to $-\infty$. By the monotone nature of F, $(F(t_{1,k}, \ldots, t_{n,k}))_k$ is a monotone decreasing sequence bounded below by 0, so

$\lim_{k\to\infty} F(t_{1,k}, \ldots, t_{n,k}) = L \ge 0 \in \mathbf{R}$.

In particular, $\forall \varepsilon > 0$, $\exists K > 0$ such that $|F(t_{1,k}, \ldots, t_{n,k}) - L| \le \varepsilon$ for all $k \ge K$.

Now let $t_1, \ldots, t_n \to -\infty$. By the monotone nature of F again, $|F(t_1, \ldots, t_n) - L| \le \varepsilon$, for any $t_1, \ldots, t_n \le C$, where $C = t_{1,K} = \ldots = t_{n,K}$. Thus we get

$\lim_{t_1,\ldots,t_n\to-\infty} F(t_1, \ldots, t_n) = L$.

Finally, we show that $L = 0$. By continuity from above, we get

$L = \lim_{k\to\infty} \mu((-\infty, t_{1,k}] \times \ldots \times (-\infty, t_{n,k}]) = \mu(\bigcap_{k=1}^{\infty} (-\infty, t_{1,k}] \times \ldots \times (-\infty, t_{n,k}]) = \mu(\emptyset_{\mathbf{R}^n}) = 0$. By a similar argument, we can show that $\lim_{t_1,\ldots,t_n\to+\infty} F(t_1, \ldots, t_n) = 1$. $\square$

(3). As argued above, it suffices to evaluate the limit along the sequence $(t_1 + \frac{1}{k}, \ldots, t_n + \frac{1}{k})_{k=1}^{\infty}$. By continuity from above, we have:

$\lim_{(s_1,\ldots,s_n)\to(t_1,\ldots,t_n)^+} F(s_1, \ldots, s_n) = \lim_{k\to\infty} F(t_1 + \frac{1}{k}, \ldots, t_n + \frac{1}{k})$

$= \lim_{k\to\infty} \mu((-\infty, t_1 + \frac{1}{k}] \times \ldots \times (-\infty, t_n + \frac{1}{k}])$

$= \mu(\bigcap_{k=1}^{\infty} (-\infty, t_1 + \frac{1}{k}] \times \ldots \times (-\infty, t_n + \frac{1}{k}]) = \mu((-\infty, t_1] \times \ldots \times (-\infty, t_n]) = F(t_1, \ldots, t_n)$. i.e. F is right-continuous. $\square$

(4). For $n = 1$, the given sum equals $F(t_{1,1}) - F(t_{1,0}) = \mu((t_{1,0}, t_{1,1}]) \ge 0$. For any $n > 1$, there are 2^n sequences of 0's and 1's of length n. Divide these sequences into two groups of equal size: Those ending with 0, and those ending with 1. One can then arrange them into pairs such that each pair consists of one sequence from each group, differing only in the last digit.

Let $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ be two such sequences. Without loss of generality, suppose that α_i, $\beta_i \in \{0, 1\}$ for all $1 \leq i \leq n-1$, with $\alpha_n = 0$ and $\beta_n = 1$. By definition, $F(t_{1,\alpha_1}, \ldots, t_{n,\alpha_n}) - F(t_{1,\beta_1}, \ldots, t_{n,\beta_n})$ equals

$$\mu((-\infty, t_{1,\alpha_1}] \times \ldots \times (-\infty, t_{n,\alpha_n}]) - \mu((-\infty, t_{1,\beta_1}] \times \ldots \times (-\infty, t_{n,\beta_n}])$$

$$= \mu(\emptyset_{\mathbf{R}} \times \emptyset_{\mathbf{R}} \times \ldots \times (t_{n,\alpha_n}, t_{n,\beta_n}]) = \mu(\emptyset_{\mathbf{R}^n}) = 0.$$

Since the original sum can always be broken into the sum of such pairs, the conclusion follows for all $n \geq 1$. $\square$