Discrete Math — Homework 3 Solutions

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Q1

Show these statements are equivalent $\forall x \in \mathbb{Z}$

a. $3x + 2$ is even .

b. $x + 5$ is odd .

c. x^2 is even .

Proof:

Proof for a. → c. $(\forall x \in \mathbb{Z})(2 \mid (3x+2) \to 2 \mid 3x)$. Since $\gcd(2,3) = 1$, we can infer that $2 \mid x$. Thus, $2 \mid x^2 \iff x^2$ is even.

Proof for c. → b. Given that x^2 is even, x must be even. It is easy to be proved by contradiction. (Assume x is odd, then x^2 must be odd as well). Since x is even, then $x + 5$ is odd.

Proof for b. → a. $x + 5$ is odd indicates that x is even. $2 \mid x \implies 2 \mid 3x \implies 2 \mid (3x + 2)$.

According to all 3 proofs above, all statements are logically equivalent. □

Q2

Prove that at least one of the real numbers $a_1, a_2, \ldots a_n$ is greater than or equal to the average of these numbers.

Proof: For convenience, we denote $\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$.

We prove by contradiction. **Assume that the statement is false**, i.e. $a_i < \bar{a}, \forall i \in \{1, 2, \ldots n\}$. Then,

$$\bar{a} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) < \frac{1}{n}(\bar{a} + \bar{a} + \cdots + \bar{a}) = \bar{a}$$

which leads to a contradiction.

Thus, the statement is true. □

Q3

Prove that there is no positive integer n such that $n^2 + n^3 = 100$.
Proof: For all $n \in \mathbb{Z}_+$,

$$\begin{aligned} n^2 + n^3 = 100 &\iff n^2(n+1) = 100 \\ &\iff n = \frac{100}{n^2} - 1 \geq 1 \\ &\implies (1 \leq n^2 \leq 50) \wedge (n \in \mathbb{Z}_+) \\ &\implies n \in \{1, 2, 3, 4, 5, 6, 7\} \end{aligned}$$

Verify the equation for all the 7 cases, we find that there exists no solution. Thus, the statement is proved. □

Q4

Any dollar sum greater than 12 can be formed by the combination of 4 and 5 dollar coins .
Proof: Using mathematical induction.
The property we want to prove is $P(n)$: "n can be written as the combination of 4 and 5".
I'll prove by cases and for each case, using the induction method.
First prove $\forall n \in \mathbb{N}(n > 12) \wedge (n \equiv 1 \pmod 4) \to P(n)$.

Base case: $13 = 4 \times 2 + 5$

Inductive step: We assume $P(k)$ is true for $k = 4m + 1, m \in \{3, 4, \ldots\}$, and show $P(k + 4)$ is right. We first assume the Induction Hypothesis $P(k) : k = 4s + 5t,\ s, t \in \mathbb{N}$. Then, for $P(k + 4)$, it is obvious that $k + 4 = 4(s + 1) + 5t$, this implies that $P(k + 4)$ is satisfied.

From the Principle of Mathematical Induction, this implies that $\forall n \in \mathbb{N}(n > 12) \wedge (n \equiv 1 \pmod 4) \to P(n)$.
In a simillar way, a change in the base case brings out:

- $\forall n \in \mathbb{N}(n > 12) \wedge (n \equiv 2 \pmod 4) \to P(n)$.
- $\forall n \in \mathbb{N}(n > 12) \wedge (n \equiv 3 \pmod 4) \to P(n)$.
- $\forall n \in \mathbb{N}(n > 12) \wedge (n \equiv 0 \pmod 4) \to P(n)$.

Summing all the cases above, we can deduced that $\forall n \in \mathbb{N}(n > 12) \to P(n)$. □

Q5

Prove that

$$3 + 3 \times 5 + 3 \times 5^2 + \cdots + 3 \times 5^n = 3(5^{n+1} - 1)/4$$

whenever n is a nonnegative integer.

Proof: Using mathematical induction.

The property that we want to prove $P(n)$ is "$\sum_{i=0}^{n} 3 \cdot 5^i = 3(5^{n+1} - 1)/4$".

Base cases: We plug in $n = 0$ to check that $P(0)$ is true: $3 = 3$.

Inductive step: We assume that $P(k)$ is true for $k \geq 0$ and show that $P(k+1)$ is true.

We first assume the Induction Hypothesis $P(k) : \sum_{i=0}^{k} 3 \cdot 5^i = 3(5^{k+1} - 1)/4$.

Then, for $P(k+1)$, we write it as $(\sum_{i=0}^{k} 3 \cdot 5^i) + 3 \cdot 5^{k+1}$.

Using the Induction Hypothesis,

$$\begin{aligned}\frac{3 \cdot (5^{k+1} - 1)}{4} + 3 \cdot 5^{k+1} &= \frac{15 \cdot 5^{k+1} - 3}{4} \\ &= \frac{3 \cdot (5^{k+2} - 1)}{4}\end{aligned}$$

This implies $P(k+1)$ as required.

From the Principle of Mathematical Induction, this implies that $P(n)$ is true for every nonnegative integer n. $\square$

Q6

Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets s.t. $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j$$

Proof: Considering prove by contradiction.

Assume the statement is false i.e. $\exists a(a \in \bigcap_{j=1}^{n} A_j) \wedge (a \notin \bigcap_{j=1}^{n} B_j)$.

By definition, $a \in \bigcap_{j=1}^{n} A_j \implies \exists A_i \in \{A_1, A_2, \ldots, A_n\}\,(a \in A_i)$. Plus, $A_j \subseteq B_j$, we get $a \in B_i$.

However, $a \notin \bigcap_{j=1}^{n} B_j \implies \forall j \in \{1, 2, \ldots, n\}\,(a \notin B_j) \implies a \notin B_i$. and this leads to a contradiction.

Therefore, the statement is true. $\square$