Discrete Math — Homework 8 Solutions

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Q1

What is the **probability** of these events when we randomly select a permutation of $\{1, 2, 3, 4\}$? Let the sample space Ω be "all permutations of $\{1, 2, 3, 4\}$", then we get $|\Omega| = 4! = 24$.

(a) 1 precedes 4
Solution: Let E be the event "1 precedes 4". Then, if we switch the position of 1 and 4, then we get a case in $\overline{E}$ and thus there exists a bijection between E and $\overline{E}$, so $|E| = |\overline{E}|$. Therefore, $P(E) = \frac{1}{2}$.

(b) 4 precedes 1 and 4 precedes 2
Solution: Let E be the event "4 precedes 1 and 2". Then, $|E| = 3! + 2 = 8$. Therefore, $P(E) = \frac{|E|}{|\Omega|} = \frac{8}{24} = \frac{1}{3}$.

(c) 4 precedes 3 and 2 precedes 1
Solution: Let E be the event "4 precedes 3 and 2 precedes 1". $|E| = 2 \cdot 2 + 2 = 6$. Therefore, $P(E) = \frac{|E|}{|\Omega|} = \frac{6}{24} = \frac{1}{4}$.

Q2

What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?
Solution: Let Ω be the sample space of choosing integer within 100, then $|\Omega| = 100$. Let E_1 be event that "the number chosen is divisible by 5", and E_2 be event that "the number chosen is divisible by 7".

$$\begin{aligned}\mathbf{Ans} = P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{7} \right\rfloor - \left\lfloor \frac{100}{5 \cdot 7} \right\rfloor \\ &= 20 + 14 - 2 \\ &= 32\end{aligned}$$

Q3

For each of the following pairs of events, which are subsets of the set of all possible outcomes when a coin is tossed three times, determine whether or not they are independent.

(a) E_1: tails comes up with the coin is tossed the first time; E_2: heads comes up when the coin is tossed the second time.
Solution: Yes.

(b) E_1: the first coin comes up tails; E_2: two, and not three, heads come up in a row.
Solution: Yes.

(c) E_1: the second coin comes up tails; E_2: two, and not three, heads come up in a row.
Solution: No.

Q4

Let E_1 and E_2 be events in sample space Ω. Then we have

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof: Using Inclusion-Exclusion Principle, we get:

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

divide by sides by $|\Omega|$, we conclude that

$$P(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|\Omega|} = \frac{|E_1|}{|\Omega|} + \frac{|E_2|}{|\Omega|} - \frac{|E_1 \cap E_2|}{|\Omega|} = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$\square$

Q5

What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)
Solution: Let Ω the sample space. Let A be the event "a randomly generated bit string of length four contains at least two consecutive 0", B be " the first bit is a 1".

Then, $P(B) = \frac{|B|}{|\Omega|} = \frac{1}{2}$, $P(AB) = \frac{|A \cap B|}{|\Omega|} = \frac{3}{2^4} = \frac{3}{16}$. So, $P(A|B) = \frac{P(AB)}{P(B)} = \frac{3}{8}$.

Q6

What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?
Solution: Let A be the event that "the first flip came up heads", B be "exactly four heads appear when a fair coin is flipped five times.".

Then, $P(B) = \frac{5}{2^5}$, $P(AB) = \frac{4}{2^5}$. Therefore, $P(A|B) = \frac{P(AB)}{P(B)} = \frac{4}{5}$.

Q7

Find each of the following probabilities when a coin is flipped n times, and head appears with probability p.

(a) the probability of no failures
Solution: Let E_1 be "no failures". $P(E_1) = \binom{n}{n} \cdot p^n(1-p)^0 = p^n$.

(b) the probability of at least one failure
Solution: Let E_2 be "at least one failure". Then, $P(E_2) = 1 - P(E_1) = 1 - p^n$.

(c) the probability of at most one failure
Solution: Let E_3 be "at most one failure". Then, $P(E_3) = P(E_1) + \binom{n}{n-1} \cdot p^{n-1}(1-p) = p^n + n \cdot p^{n-1}(1-p) = np^{n-1} + (1-n)p^n$.

(d) the probability of at least two failures
Solution: Let E_4 be "at least two failures". Then, $P(E_4) = 1 - P(E_3) = 1 - np^{n-1} + (n-1)p^n$.

Q8

Suppose that E, F_1, F_2, and F_3 are events from a sample space Ω and that F_1, F_2,and F_3 are paircase disjoint and their union is Ω. Find $P(E)$ if $P(E|F_1) = 1/8$, $P(E|F_2) = 1/4$,$P(E|F_3) = 1/6$,$P(F_1) = 1/4$, $P(F_2) = 1/4$, and $P(F_3) = 1/2$.
Solution:

$$\begin{aligned}
P(E) &= P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + P(E|F_3)P(F_3) \\
&= \frac{1}{8} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{1}{2} \\
&= \frac{17}{96}
\end{aligned}$$

Q9

When a test for steroids is given to soccer players, 98% of the players taking steroids test positive and 12% of the players not taking steroids test positive. Suppose that 5% of soccer players take steroids. What is the probability that a soccer player who tests positive takes steroids?
Solution: Let A be "test positive", B be "take steroids" . Let E be "a soccer player who tests positive takes steroids".

$$\begin{aligned}
P(B|A) &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\overline{B})P(\overline{B})} \\
&= \frac{98\% \cdot 5\%}{98\% \cdot 5\% + 12\% \cdot 95\%} \\
&= \frac{49}{163} \approx 30.1\%
\end{aligned}$$