Discrete Math — Homework 7 Solutions

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Q1

Solve the following recurrence relations

(a) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$, and $a_2 = 88$.
Solution: The characteristic equation of $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ is $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$.

Solving it, we get: $\lambda_1 = \lambda_2 = \lambda_3 = 2$.

So the general solution of the recurrence relation should be $a_n = c_1 \cdot 2^n + c_2 \cdot n2^n + c_3 \cdot n^2 2^n$.

The initial condition means $\begin{cases} c_1 & = -5 \\ 2c_1 + 2c_2 + 2c_3 = 4 \\ 4c_1 + 8c_2 + 16c_3 = 88 \end{cases}$, then we get $\begin{cases} c_1 = -5 \\ c_2 = \dfrac{1}{2} \\ c_3 = \dfrac{13}{2} \end{cases}$.

Therefore, $a_n = \left(\frac{13}{2}n^2 + \frac{1}{2}n - 5\right) \cdot 2^n$.

(b) $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5, a_1 = -9$, and $a_2 = 15$.
Solution: The characteristic equation of $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

Solving it, we get: $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

So the general solution of the recurrence relation should be $a_n = c_1 \cdot (-1)^n + c_2 \cdot n(-1)^n + c_3 \cdot n^2(-1)^n$.

The initial condition means $\begin{cases} c_1 = 5 \\ -c_1 - c_2 - c_3 = -9 \\ c_1 + 2c_2 + 4c_3 = 15 \end{cases}$, then we get $\begin{cases} c_1 = 5 \\ c_2 = 3 \\ c_3 = 1 \end{cases}$.

Therefore, $a_n = \left(n^2 + 3n + 5\right) \cdot (-1)^n$.

(c) $a_n = 2a_{n-1} + 3 \cdot 2^n$ for $n \geq 1$ with $a_1 = 5$.
Solution: The associated LHRR is $a_n = 2a_{n-1}$. Its solution is $a_n = \alpha \cdot 2^n$, where α is a constant.

Let $p_n = \beta \cdot n2^n$ be a particular solution of the recurrence relation, where β is a constant. Then we have $\beta \cdot n2^n = 2\beta \cdot (n-1)2^{n-1} + 3 \cdot 2^n$, that is $\beta = 3$.

Consequently, $a_n = \{a_n^{(p)} + a_n^{(h)}\} = 3n \cdot 2^n + \alpha \cdot 2^n$. Using the initial conditions, it gives $\alpha = -\frac{1}{2}$. Thus, $a_n = 3n \cdot 2^n - 2^{n-1}$.

(d) $a_n = 2a_{n-1} + 2n^2$ for $n \geq 1$ with $a_1 = 2$.
Solution: The associated LHRR is $a_n = 2a_{n-1}$. Its solution is $a_n = \alpha \cdot 2^n$, where α is a constant.

Let $p_n = cn^2 + dn + e$ be a particular solution of the recurrence relation, where c, d, e are constants. Then we have $cn^2 + dn + e = 2\left[c(n-1)^2 + d(n-1) + e\right] + 2n^2$, then we have $c = -\frac{1}{2}, d = -2, e = -3$.

Consequently, $a_n = \{a_n^{(p)} + a_n^{(h)}\} = -\frac{1}{2}n^2 - 2n - 3 + \alpha \cdot 2^n$. Using the initial conditions, it gives $\alpha = \frac{15}{4}$. Thus, $a_n = -\frac{1}{2}n^2 - 2n - 3 + \frac{15}{4} \cdot 2^n$.

Q2

What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if
Solution: The associated LHRR is $a_n = 8a_{n-2} - 16a_{n-4}$. Its solution is $a_n^{(h)} = \alpha \cdot 4^n + \beta \cdot n4^n$.

(a) $F(n) = n^3$
$a_n = \{a_n^{(h)} + a_n^{(p)}\} = \alpha \cdot 4^n + \beta \cdot n4^n + c_3n^3 + c_2n^2 + c_1n + c_0$.

(b) $F(n) = (-2)^n$
$a_n = \{a_n^{(h)} + a_n^{(p)}\} = \alpha \cdot 4^n + \beta \cdot n4^n + c_0 \cdot (-2)^n$.

(c) $F(n) = n2^n$
$a_n = \{a_n^{(h)} + a_n^{(p)}\} = \alpha \cdot 4^n + \beta \cdot n4^n + (c_1n + c_0) \cdot 2^n$.

(d) $F(n) = n^2 4^n$
$a_n = \{a_n^{(h)} + a_n^{(p)}\} = \alpha \cdot 4^n + \beta \cdot n4^n + n^2(c_2n^2 + c_1n + c_0) \cdot 4^n$.

(e) $F(n) = (n^2 - 2)(-2)^n$
$a_n = \{a_n^{(h)} + a_n^{(p)}\} = \alpha \cdot 4^n + \beta \cdot n4^n + (c_2n^2 + c_1n + c_0) \cdot (-2)^n$.

(f) $F(n) = 2$
$a_n = \{a_n^{(h)} + a_n^{(p)}\} = \alpha \cdot 4^n + \beta \cdot n4^n + c_0$.

Q3

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1,-2,-2,-2, 3, 3,-4?
Solution: $a_n = (a_3n^3 + a_2n^2 + a_1n + a_0) + (b_2n^2 + b_1n + b_0) \cdot (-2)^n + (c_1n + c_0) \cdot 3^n + d_0 \cdot (-4)^n$.

Q4

Suppose that each person in a group of n people votes for exactly two people from a slate of candidates to fill two positions on a committee. The top two finishers both win positions as long as each receives more than $n/2$ votes. Devise a DC algorithm that determines whether the two candidates who received the most votes each received at least $n/2$ votes and, if so, determine who these two candidates are.
Solution:

Algorithm 1 most 2 voted candidates

```
Require: voted candidates' names v[1···n]
  procedure COUNT(voted candidates' names v[1···n], left bound l, right bound r,candidate c)
    ret ← 0
    for i ← l to r do
      if v[i][1] = c then
        ret ← ret + 1
      if v[i][2] = c then
        ret ← ret + 1
    return ret
  procedure SOLVE(voted candidates' names v[1···n], left bound l, right bound r)
    if l = r then
      return v[l]
    mid ← (l + r)/2
    can ←UNION(SOLVE(v, l, mid),SOLVE(v, mid + 1, r))
    sort can by the COUNT(v,l,r,c) foreach c in can
    return [can[1], can[2], can[3]]                    ▷ always get top 3 candidates
  res ← SOLVE(v, l, r)
  if COUNT(l, r, res[1]) > n/2 then
    output res[1]
  if COUNT(l, r, res[2]) > n/2 then
    output res[2]
```

Q5

Set up a DC recurrence relation for the number of multiplications required to compute x^n, where x is a real number and n is a positive integer;
Solution:

Algorithm 2 n power of x

```
  procedure POWER(x,n)
    if n = 0 then
      return 1
    if n = 1 then
      return x
    t ← POWER(⌊x/2⌋)
    if x mod 2 = 0 then
      return t^2
    else
      return x · t^2
```