Tutorial 20

1. What is the expected number of heads that come up when a fair coin is flipped five times?
2. What is the expected number of times a 6 appears when a fair die is rolled 10 times?
3. The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, and 25 multiple-choice questions, each worth four points. The probability that Linda answers a true/false question correctly is 0.9, and the probability that she answers a multiple-choice question correctly is 0.8. What is her expected score on the final?
4. Suppose that we roll a pair of fair dice until the sum of the numbers on the dice is seven. What is the expected number of times we roll the dice?

Tutorial 20 Cont'd

5. Let X_n be the r.v. that equals # tails minus # heads when n fair coins are flipped.
   - What is the expected value of X_n?
   - What is the variance of X_n?
6. Suppose that X_1 and X_2 are independent Bernoulli trials each with probability $1/2$, and let $X_3 = (X_1 + X_2) \bmod 2$.
   - Show that X_1, X_2, and X_3 are pairwise independent, but X_3 and $X_1 + X_2$ are not independent.
   - Show that $V(X_1 + X_2 + X_3) = V(X_1) + V(X_2) + V(X_3)$.
7. The covariance of two r.v.s X and Y on a sample space Ω, denoted by $Cov(X, Y)$, is defined to be the expected value of r.v. $(X - E(X))(Y - E(Y))$. That is, $Cov(X, Y) = E((X - E(X))(Y - E(Y)))$.
   - Show that $Cov(X, Y) = E(XY) - E(X)E(Y)$, and use this result to conclude that $Cov(X, Y) = 0$ if X and Y are independent r.v.s.
   - Show that $V(X + Y) = V(X) + V(Y) + 2Cov(X, Y)$.

Tutorial 21

1. Suppose that the number of tin cans recycled in a day at a recycling center is a random variable with an expected value of 50,000 and a variance of 10,000.
   a. Use Markov's inequality to find an upper bound on the probability that the center will recycle more than 55,000 cans on a particular day;
   b. Use Chebyshev's inequality to provide a lower bound on the probability that the center will recycle 40,000 to 60,000 cans on a certain day;
2. In n tosses of a fair coin, let X be # heads, what's the upper bound of $P(X > \frac{5n}{6})$?
   a. Given by Markov's inequality;
   b. Given by Chebyshev's inequality;

3. Let X_i be a sequence of independent and Bernoulli r.v.s with $P(X_i = 1) = p_i$. Assume that r.v. $X = \sum_{i=1}^n X_i$ and $\mu = \sum_{i=1}^n p_i$. Prove that
   a. $P(X > (1+\delta)\mu) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$;
   b. $P(X > (1+\delta)\mu) < \exp(-\mu\delta^2/3)$;
4. Let X_i be a sequence of independent identical disitribution r.v.s. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, please compute that
   a. $V(\overline{X})$;
   b. $E(X_i - \overline{X})$;
   c. $V(X_i - \overline{X})$;
   d. $E(\sum_{i=1}^n (X_i - \overline{X})^2)$;
   e. $V(\frac{X_i - E(X_i)}{V(X_i)})$.