Discrete Math — Homework 2 Solutions

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Q1

(a) $\exists p(F(p) \wedge B(p)) \to \exists j L(j)$: If there exists a printer being out of service and busy, then then there exists a print job being lost.

(b) $\forall p B(p) \to \exists j Q(j)$: If all printers are busy, then there must be a print job being queued.

(c) $\exists j(Q(j) \wedge L(j)) \to \exists p F(p)$: If there exists a print job being lost and queued, then there exists a printer being out of service.

(d) $(\forall p B(p) \wedge \forall j Q(j)) \to \exists j L(j)$: If all printers are busy and all print jobs are queued, then there exists a print job being lost.

Q2

(a) $\neg \forall x \exists y \forall z T(x,y,z) \iff \exists x \forall y \exists z(\neg T(x,y,z))$

(b) $\neg(\forall x \forall y P(x,y) \vee \forall x \forall y Q(x,y)) \iff (\exists x \exists y \neg P(x,y)) \wedge (\exists x \exists y \neg Q(x,y))$

(c) $\neg(\forall x \exists y(P(x,y) \wedge \exists z R(x,y,z))) \iff \exists x \forall y \neg(P(x,y) \wedge \exists z R(x,y,z))$
$\iff \exists x \forall y(\neg P(x,y) \vee \forall z \neg R(x,y,z))$

Q3

$P(x,y) : 2x + y = 0$ where $x, y \in \mathbb{R}$

(a) $\forall x \exists y P(x,y)$ means for every x, there's a solution for y, which is a tautology.
$\forall y \exists x P(x,y)$ means for every y, there's a solution for x, which is a tautology as well.
Two tautologys has the same truth value all the time. Thus, they're logically equivalent.

(b) $2x + y = 0 \implies y = -2x$, let $x := 0.1$, then $y = -0.2 \notin \mathbb{Z}$, so the LHS(left hand side) is not a tautology.
$2x + y = 0 \implies x = -\frac{y}{2} \in \mathbb{R}$, so the RHS is always true.
So, the statement is not true.

(c) No. Let $P(x,y)$ be $x^2 = y$, where $x, y \in \mathbb{R}$.

Q4

Let $L(x, y)$ be "x loves y", where the domain for both x and y consists of all people in the world.

(a) Everybody loves Jerry: $\forall x L(x, \text{Jerry})$.

(b) Everybody loves somebody: $\forall x \exists y L(x, y)$.

(c) There is somebody whom everybody loves: $\exists y \forall x L(x, y)$.

(d) There is somebody whom Lydia does not love: $\exists y \neg L(\text{Lydia}, y)$.

(e) There is somebody whom no one loves: $\exists y \forall x \neg L(x, y)$.

(f) There is someone who loves no one besides himself or herself: $\exists x(L(x, x) \wedge (\forall p(x \neq p \rightarrow \neg L(x, p)))$.

Q5

(a) $(p \rightarrow r) \wedge (q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r)$

Proof:

Hypotheses:

- $p \rightarrow r \iff (\neg p \vee r)$ is true.
- $q \rightarrow r \iff (\neg q \vee r)$ is true.

Thus,

$$\begin{aligned} (p \vee q) \rightarrow r &\iff \neg(p \vee q) \vee r \iff (\neg p \wedge \neg q) \vee r \\ &\iff (\neg p \vee r) \wedge (\neg q \vee r) \end{aligned}$$

the conclusion is drawn. □

(b) $(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s) \rightarrow (p \rightarrow \neg r)$

Proof:

Hypotheses:

- $p \rightarrow q$ is true.
- $r \rightarrow s \iff \neg s \rightarrow \neg r$ is true.
- $\neg q \vee \neg s \iff q \rightarrow \neg s$ is true.

According to hypothetical syllogism, the conclusion is drawn. □

(c) $(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r) \rightarrow (q \vee s)$

Proof:

Hypotheses:

- $p \rightarrow q \iff \neg q \rightarrow \neg p$ is true.
- $r \rightarrow s$ is true.

- $p \lor r \iff \neg p \to r$ is true.

According to hypothetical syllogism, $\neg q \to s \iff q \lor s$ is true. $\square$

(d) $((w \lor r) \to v) \land (v \to (c \lor s)) \land (s \to u) \land \neg c \land \neg u \to \neg w$

Proof:

Hypotheses:

- $(w \lor r) \to v$ is true.
- $v \to (c \lor s)$ is true.
- $s \to u$ is true.
- $\neg c$ is true, which means c is false.
- $\neg u$ is true, which means u is false.

Since u is false and $s \to u$ is true, s should be false.
Then,$c \lor s$ should be false, and given $v \to (c \lor s)$ is true, we can deduce that v is false.
Since $(w \lor r) \to v$ is true, $w \lor r$ is false. Hence, w is false, which means $\neg w$ is true. $\square$

Q6

If $\forall x(P(x) \lor Q(x)), \forall x(\neg Q(x) \lor S(x)), \forall x(R(x) \to \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

Proof:

Hypotheses:

- $\forall x(P(x) \lor Q(x)) \iff \forall x(\neg P(x) \to Q(x))$ is true.
- $\forall x(\neg Q(x) \lor S(x)) \iff \forall x(Q(x) \to S(x))$ is true.
- $\forall x(R(x) \to \neg S(x)) \iff \forall x(S(x) \to \neg R(x))$ is true.
- $\exists x \neg P(x)$ is true.

Use the first 3 hypotheses and the logically syllogism, we can get:

$$\forall x(\neg P(x) \to \neg R(x))$$

Since $\exists x \neg P(x)$, use the existential instantiation rule, let x be a, then $\neg P(a)$ is true.
Then, use universal instantiation, $\neg R(a)$ is true. The conclusion was drawn. $\square$

Q7

(a) first **existential generalization**, then **universal instantiation**.

(b) first **universal instantiation**, then **modus ponens**.

Q8

Proof:
Let

- R be "It is raining."
- U be "Yvette has her umbrella."
- W be "Yvette gets wet."

Then, the **hypotheses** can be written as:

- $\neg R \vee U \iff R \to U$ is true.
- $\neg U \vee \neg W \iff U \to \neg W$ is true.
- $R \vee \neg W$ is true.

Conclusion: $\neg W$.
If R is true, then we can deduce that U is true, and then $\neg W$ is true, the conclusion was drawn.
If R is false, then since $R \vee \neg W$ is true, $\neg W$ must be true.
Summarizing, we can infer that "Yvette does not get wet.". $\square$