Discrete Math — Homework 5 Solutions

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March 28, 2025

Q1

How many 6-element RNA sequences
Solution:
RNA sequences consists of A,U,C,G.

(a) do not contain U?
Ans $= 3^6 = 729$.

(b) end with GU?
Ans $= 4^4 = 256$.

(c) start with C ?
Ans $= 4^5 = 1024$.

(d) contain only A or U?
Ans $= 2^6 = 64$.

Q2

Find the value of each of these quantities.

(a) $P(6,4) = 6 \times 5 \times 4 \times 3 = 360$.
$P(7,5) = 7 \times 6 \times 5 \times 4 \times 3 = 2520$.

(b) $C(6,4) = \frac{P(6,4)}{4!} = 15$.
$C(7,5) = \frac{P(7,5)}{5!} = 21$.

(c) $C(6,2) = C(6,4) = 15$.
$C(7,2) = C(7,5) = 21$.

Q3

How many permutations of the letters ABCDEFG contain

(a) string BCD?
Ans $= P(5,5) = 120$.

(b) strings BA and GF?
Ans $= P(5,5) = 120$.

(c) strings ABC and CDE?
Ans $= P(3,3) = 6$.

(d) strings CBA and BED?
Ans $= 0$.

Q4

A multiple-choice test contains 10 questions. There are four possible answers for each question.

(a) In how many ways can a student answer the questions on the test if the student answers every question?
Ans $= 4^{10} = 1048576$.

(b) In how many ways can a student answer the questions on the test if the student can leave answers blank?
Ans $= 5^{10} = 9765625$.

Q5

How many positive integers less than 1000

(a) have distinct digits
Considering enumerate the digits by cases.

I. integers within 10. In total, $\text{Ans}_{10} = 9$.
II. integers from 10 to 99. In total, $\text{Ans}_{100} = 9 \times 9 = 81$.
III. integers from 100 to 999. In total, $\text{Ans}_{1000} = 9 \times 9 \times 8 = 648$.

Thus, Ans $= \text{Ans}_{10} + \text{Ans}_{100} + \text{Ans}_{1000} = 738$.

(b) have distinct digits and are even
It's believed that the odd cases have the same capacity with the even ones. Thus, Ans $= 738/2 = 369$.

Q6

How many bit strings of length 10 contain

(a) exactly four 1s?
Ans $= \frac{P(10,10)}{4! \cdot 6!} = 210$.

(b) at most four 1s?
Ans $= \sum_{i=0}^{4} \frac{P(10,10)}{i! \cdot (10-i)!} = 386$.

(c) at least four 1s?
Ans $= 2^{10} - 386 + 210 = 848$.

(d) an equal number of 0s and 1s?
Ans $= \frac{P(10,10)}{5!\cdot 5!} = 252$.

Q7

Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

(a) the sets are pairwise disjoint
Pairwise disjoint means $A_i \neq A_j, \forall i \neq j \in \{1,2,3\}$. Thus, $|A_1 \cup A_2 \cup A_3| = 300$.

(b) there are 50 common elements in each pair of sets and no elements in all three sets
According to Inclusion-Exclusion Principle,
$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3| = 300 - 50 \cdot 3 + 0 = 150$

(c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
According to Inclusion-Exclusion Principle,
$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3| = 300 - 50 \cdot 3 + 25 = 170$

(d) the sets are equal
$|A_1 \cup A_2 \cup A_3| = |A_1| = 100$.

Q8

How many derangements are there of a set with seven elements?
Solution: Let P_i be i is at its place, then our objective is to solve $\left|\bigcap_{i=1}^{7} \overline{P_i}\right| = \left|\overline{\bigcup_{i=1}^{7} P_i}\right|$.

$$\left|\bigcup_{i=1}^{7} P_i\right| = \sum |P_i| - \sum |P_i \cap P_j| + \sum |P_i \cap P_j \cap P_k| + \cdots + \sum_{|\lambda|=\#} (-1)^{\#} \cdot \left|\bigcap_{i\in\lambda} P_i\right| + \cdots - \left|\bigcap_{i=1}^{7} P_i\right|$$
$$=$$

Q9

How many positive integers less than 200 are

(a) either odd or the square of an integer;

(b) second or higher powers of integers?

(c) either primes or second or higher powers of integers?

(d) not divisible by the square of an integer greater than 1?

Q10

How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Q11

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$

(a) if x_1, x_2, x_3 and x_4 are nonnegative integers?

(b) if x_1, x_2, x_3 and x_4 are positive integers?

(c) if $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4$ and x_4 are positive integers?

Q12

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 \leq 17$

(a) if x_1, x_2, x_3 and x_4 are nonnegative integers?

(b) if x_1, x_2, x_3 and x_4 are positive integers?

Q13

Find the next larger permutation in lexicographic order after each of these permutations.

(a) 1432;

(b) 54123;

(c) 12453;

(d) 31528764.

Q14

Find the next larger 5-combination of the set $\{1, 2, 3, 4, 5, 6, 7\}$ after each of these 4-combinations

(a) $\{1, 2, 4, 5, 7\}$

(b) $\{1, 4, 5, 6, 7\}$

Q15

Write the pseudo-code for generating the next permutation in a reverse lexicographic order.

Q16

Given set $\{n, n-1, \ldots, 1\}$, write the pseudo-code for generating the next r-combination in a reverse lexicographic order.

Q17

Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Q18

Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Q19

Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of whose neighbors are boys.