# Tutorial 4

1. Given three sets $A$, $B$, and $C$, please prove the following statements
   a. $(A \cap B) \subseteq A$;
   b. $A \cap (B - A) = \emptyset$;
   c. $A \cup (B - A) = A \cup B$;
   d. $A - B = A \cap \overline{B}$;
   e. $(A \cap B) \cup (A \cap \overline{B}) = A$;
   f. $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.
2. Show the following Cartesian products are not the same.
   a. $A \times B$ and $B \times A$, unless $A = B$;
   b. $A \times B \times C$ and $(A \times B) \times C$;

1. Determine whether the function $f : Z \times Z \to Z$ is onto if
   a. $f(m, n) = m + n$;
   b. $f(m, n) = m^2 + n^2$;
   c. $f(m, n) = m$;
   d. $f(m, n) = |n|$;
   e. $f(m, n) = m - n$.
2. If $f$ and $f \circ g$ are onto, does it follow that $g$ is onto? Justify your answer.
3. Let $S$ be a subset of a universal set $U$. The characteristic function $f_S$ of $S$ is the function from $U$ to $\{0, 1\}$ such that $f_S(x) = 1$ if $x$ belongs to $S$ and $f_S(x) = 0$ if $x$ does not belong to $S$. Let $A$ and $B$ be sets. Show that for all $x \in U$,
   a. $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$;
   b. $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$;
   c. $f_{\overline{A}}(x) = 1 - f_A(x)$.

4. Show that the function $f(x) = ax + b$ from $R$ to $R$ is invertible, where $a$ and $b$ are constants, with $a \neq 0$, and find the inverse of $f$.
5. Prove or disprove each of these statements about the floor and ceiling functions.
   a. $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real numbers $x$;
   b. $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ whenever $x \in R$;
   c. $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ for all $x, y \in R$;
   d. $\lceil \frac{x}{2} \rceil = \lfloor \frac{x+1}{2} \rfloor$ for all $x \in R$.
6. Please formulate the following problem as a formal mathematical problem via using indicator functions:

   Input:

   Universal set $U = \{u_1, u_2, \cdots, u_n\}$

   Subsets $S_1, S_2, \cdots, S_m \subseteq U$

   Goal:

   Find $k$ subsets that maximizes their total coverage, i.e., set $\bigcup_{i \in k} S_i$ contains the most elements of $U$

# Tutorial 7

1. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
   a. all bit strings not containing the bit 0;
   b. the integers that are multiples of 7;
   c. the irrational numbers between 0 and 1;
   d. the real numbers between 0 and $\frac{1}{2}$.
2. Give an example of two uncountable sets $A$ and $B$ such that $A \cap B$ is
   a. finite;
   b. countably infinite;
   c. uncountable;

3. Explain why the set $A$ is countable if and only if $|A| \leq |Z^+|$.
4. Show that if $A$ and $B$ are sets, $A$ is uncountable, and $A \subseteq B$, then $B$ is uncountable.
5. Show that if $A, B$, and $C$ are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

# Tutorial 7

1. Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$,
   a. Find $A^T$, $A^{-1}$, and $A^3$;
   b. Find $(A^{-1})^3$ and $(A^3)^{-1}$.
2. Suppose that $A \in \mathscr{R}^{n\times n}$ where $n$ is a positive integer. Show that $A + A^T$ is symmetric.
3. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Find $A \wedge B$, $A \vee B$, and $A \odot B$.
4. Let $A$ be a zero-one matrix. Show that
   a. $A \vee A = A$;
   b. $A \wedge A = A$.
5. Let $A$ be an $n \times n$ zero-one matrix. Let $I$ be the $n \times n$ identity matrix. Show that $A \odot I = I \odot A = A$.