Tutorial 12

1. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.
2. What is the coefficients of x^8, x^9 and x^{10} in $(2-x)^{19}$?
3. Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}/2.$$

4. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$$

[Hint: Count in two ways the number of ways to select a committee and to then select a leader of the committee.]

5. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?
6. a. Find the recurrence relation satisfied by R_n, where R_n is # regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.
   b. Find R_n using iteration.
7. A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.
   a. Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
   b. What are the initial conditions?
   c. How many ways are there to deposit \$10 for a book of stamps?
8. For bit strings, find a recurrence relation for the number of bit strings of length n that contain an odd number of 0s.

1. Given two strings A and B, we need to find the minimum number of operations which can be applied on A to convert it to B. The operations are:
   a. Edit - Change a character to another character;
   b. Delete - Delete a character;
   c. Insert - Insert a character.

   The **edit distance** of two strings is defined by the minimum # operations required to transform one string into the other. For the following two strings, their edit distance is 3.

$$GC\color{red}{G}\color{black}{TATG}\color{red}{A}\color{black}{GGCTA}\color{red}{-}\color{black}{ACGC}$$

$$GC\color{red}{-}\color{black}{TATG}\color{red}{C}\color{black}{GGCTA}\color{red}{T}\color{black}{ACG}$$

   Please utilize the dynamic programming to compute the edit distance between two strings A and B.
   a. Define the subproblems for DP;
   b. Find the recurrence;
   c. Implement the algorithm to return the edit distance of two strings.

2. In the knapsack problem we are given a set of n items, where each item i is specified by a size s_i and a value v_i. We are also given a size bound S (the size of our knapsack). The goal is to find the subset of items of maximum total value such that sum of their sizes is at most S (they all fit into the knapsack).
   To implement a DP algorithm to solve this problem,
   a. Define subproblems;
   b. Find the recurrence relation;
   c. Solve the base cases;
   d. Implement the algorithm to return the solution of the knapsack problem.

Tutorial 13 Cont'd

3. Solve the following recurrence relations
   a. $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with $a_0 = 1$ and $a_1 = 0$.
   b. $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$ with $a_0 = 6$ and $a_1 = 8$.
4. The Lucas numbers satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2}$$

   and the initial conditions $L_0 = 2$ and $L_1 = 1$.
   a. Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, \cdots$, where f_n is the $n-$th Fibonacci number.
   b. Find an explicit formula for the Lucas numbers.