Discrete Math — Homework 4 Solutions

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Q1

Given three sets A, B, and C , please prove the following statements.

Let p be "$x \in A$", q be "$x \in B$", r be "$x \in C$".

(a) $(A \cap B) \subseteq A$

Proof: According to the definition, $\forall x \in A \cap B(x \in A)$, and thus we have $(A \cap B) \subseteq A$. $\square$

(b) $A \cap (B - A) = \emptyset$

Proof: For all $x \in A \cap (B - A)$, by definition, we have $(x \in A) \wedge (x \notin A)$, so there doesn't exists such x. Therefore, $A \cap (B - A) = \emptyset$. $\square$

(c) $A \cup (B - A) = A \cup B$

Proof: $A \cup (B - A) = \{x \mid p \vee (q \wedge \neg p)\} = \{x \mid (p \vee q) \wedge (p \vee \neg p)\} = \{x \mid p \vee q\} = A \cup B$. $\square$

(d) $A - B = A \cap \overline{B}$

Proof: $A - B = \{x \mid p \wedge \neg q\} = A \cap \overline{B}$. $\square$

(e) $(A \cap B) \cup (A \cap \overline{B}) = A$

Proof: $(A \cap B) \cup (A \cap \overline{B}) = \{x \mid (p \wedge q) \vee (p \wedge \neg q)\} = \{x \mid p \wedge (q \vee \neg q)\} = \{x \mid p\} = A$. $\square$

(f) $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

Proof: $\overline{A \cap B \cap C} = \{x \mid \neg(p \wedge q \wedge r)\} = \{x \mid \neg p \vee \neg q \vee \neg r\} = \overline{A} \cup \overline{B} \cup \overline{C}$. $\square$

Q2

Show the following Cartesian products are not the same.

(a) $A \times B$ and $B \times A$, unless $A = B$.

Let $A = \{1\}$, $B = \{2\}$. Then $A \times B = \{(1, 2)\} \neq \{(2, 1)\} = B \times A$.

(b) $A \times B \times C$ and $(A \times B) \times C$. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$. Then $A \times B \times C = \{(1, 2, 3)\} \neq \{((1, 2), 3)\} = (A \times B) \times C$.

Q3

Determine whether the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m,n) = m + n$ is onto.

(b) $f(m,n) = m^2 + n^2$ is **not** onto.

(c) $f(m,n) = m$ is onto.

(d) $f(m,n) = |n|$ is **not** onto.

(e) $f(m,n) = m - n$ is onto.

Q4

If f and $f \circ g$ are onto, does it follow that g is onto?
No.
Proof: Let $f\colon V \to W$.

f is onto means $\operatorname{Im} f = f(V) = R$. $f \circ g$ is onto means $\operatorname{Im} f \mid \operatorname{Im} g = R$. (the vertival bar here means restrict the domain of f within the image of g)

Once $V \subseteq \operatorname{Im} g$, then we can ensure that $f \circ g$ is onto. So to construct a counterexample, we just need to set g as a function such that $V \subseteq \operatorname{Im} g \subset$ co-domain of g. $\square$

Q5

Let S be a subset of a universal set U. The characteristic function f_S of S is the function from U to $\{0,1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be sets.
Define $\beta\colon \{\text{true}, \text{false}\} \to \{0,1\}$, which maps **true** to 1 and **false** to 0. Let p be "$x \in A$", q be "$x \in B$".

(a) $f_{A\cap B}(x) = f_A(x) \cdot f_B(x)$
Proof:

LHS, $f_{A\cap B}(x) = 1$ iff. $x \in A \cap B \iff x \in A \wedge x \in B$.

RHS, $f_A(x) \cdot f_B(x) = 1$ iff. $f_A = 1 \wedge f_B = 1 \iff x \in A \wedge x \in B$.

Thus, the 2 expressions are equivalent. $\square$

(b) $f_{A\cup B(x)} = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
Proof: Proving $f_{A\cup B(x)} = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$ is proving $f_{A\cup B(x)} + f_{A\cap B}(x) = f_A(x) + f_B(x)$. We prove by cases:

I. $x \notin A \wedge x \notin B$. LHS=0=RHS.

II. x in either set. LHS=1=RHS.

III. x in both set. LHS=2=RHS.

Thus, the equation was verified. $\square$

(c) $f_{\overline{A}}(x) = 1 - f_A(x)$
Proof: We prove by cases:

I. $x \in A$. LHS=0=RHS.

II. $x \notin A$. LHS=1=RHS.

Thus, the equation was verified. $\square$

Q6

Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f .
Solution:

First show f is injection. $\forall m, n \in \mathbb{R}$, $f(m) = f(n) \implies am + b = an + b \implies m = n$.
Then show f is surjection. $\forall y_0 \in \mathbb{R}$, $\exists x_0 = \frac{1}{a}(y_0 - b)$ s.t. $f(x_0) = y_0$.
So, f is bijection, and thus, f is invertible. $f^{-1} = \frac{1}{a}(x - b)$.

Q7

Prove or disprove each of these statements about the floor and ceiling functions.

(a) $\forall x \in \mathbb{R}, \lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$
Proof: Write $x = n + \epsilon, n \in \mathbb{Z}, \epsilon \in [0, 1)$. $\lceil \lfloor x \rfloor \rceil = \lceil n \rceil = n = \lfloor x \rfloor$. $\square$

(b) $\forall x \in \mathbb{R}, \lfloor 2x \rfloor = 2 \lfloor x \rfloor$
False. $x = 2.5$ is an counterexample.

(c) $\forall x, y \in \mathbb{R}, \lceil xy \rceil = \lceil x \rceil \lceil y \rceil$
False. $x = y = 9.9$ is an counterexample.

(d) $\forall x \in \mathbb{R}, \lceil \frac{x}{2} \rceil = \lfloor \frac{x+1}{2} \rfloor$
Proof:

$\square$

Q8

Q9

Q10

Q11

Q12

Q13

Q14