Discrete Math — Homework 5 Solutions

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Q1

How many 6-element RNA sequences
Solution:
RNA sequences consists of A,U,C,G.

(a) do not contain U? **Ans** $= 3^6 = 729$.

(b) end with GU? **Ans** $= 4^4 = 256$.

(c) start with C ? **Ans** $= 4^5 = 1024$.

(d) contain only A or U? **Ans** $= 2^6 = 64$.

Q2

Find the value of each of these quantities.

(a) $P(6,4) = 6 \times 5 \times 4 \times 3 = 360$.
$P(7,5) = 7 \times 6 \times 5 \times 4 \times 3 = 2520$.

(b) $C(6,4) = \frac{P(6,4)}{4!} = 15$.
$C(7,5) = \frac{P(7,5)}{5!} = 21$.

(c) $C(6,2) = C(6,4) = 15$.
$C(7,2) = C(7,5) = 21$.

Q3

How many permutations of the letters ABCDEFG contain

(a) string BCD? **Ans** $= P(5,5) = 120$.

(b) strings BA and GF? **Ans** $= P(5,5) = 120$.

(c) strings ABC and CDE? **Ans** $= P(3,3) = 6$.

(d) strings CBA and BED? **Ans** $= 0$.

Q4

A multiple-choice test contains 10 questions. There are four possible answers for each question.

(a) In how many ways can a student answer the questions on the test if the student answers every question?
Ans $= 4^{10} = 1048576$.

(b) In how many ways can a student answer the questions on the test if the student can leave answers blank?
Ans $= 5^{10} = 9765625$.

Q5

How many positive integers less than 1000

(a) have distinct digits
Considering enumerate the digits by cases.

  I. integers within 10. In total, $\text{Ans}_{10} = 9$.
  II. integers from 10 to 99. In total, $\text{Ans}_{100} = 9 \times 9 = 81$.
  III. integers from 100 to 999. In total, $\text{Ans}_{1000} = 9 \times 9 \times 8 = 648$.

Thus, **Ans** $= \text{Ans}_{10} + \text{Ans}_{100} + \text{Ans}_{1000} = 738$.

(b) have distinct digits and are even
It's believed that the odd cases have the same capacity with the even ones. Thus, **Ans** $= 738/2 = 369$.

Q6

How many bit strings of length 10 contain

(a) exactly four 1s? **Ans** $= \frac{P(10,10)}{4! \cdot 6!} = 210$.

(b) at most four 1s? **Ans** $= \sum_{i=0}^{4} \frac{P(10,10)}{i! \cdot (10-i)!} = 386$.

(c) at least four 1s? **Ans** $= 2^{10} - 386 + 210 = 848$.

(d) an equal number of 0s and 1s? **Ans** $= \frac{P(10,10)}{5! \cdot 5!} = 252$.

Q7

Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

(a) the sets are pairwise disjoint
Pairwise disjoint means $A_i \neq A_j, \forall i \neq j \in \{1,2,3\}$. Thus, $|A_1 \cup A_2 \cup A_3| = 300$.

(b) there are 50 common elements in each pair of sets and no elements in all three sets
According to Inclusion-Exclusion Principle,
$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3| = 300 - 50 \cdot 3 + 0 = 150$

(c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
According to Inclusion-Exclusion Principle,
$|A_1 \cup A_2 \cup A_3| = \sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3| = 300 - 50 \cdot 3 + 25 = 170$

(d) the sets are equal
$|A_1 \cup A_2 \cup A_3| = |A_1| = 100$.

Q8

How many derangements are there of a set with seven elements?
Solution: Let P_i be "i is at its place", then our objective is to solve $\left|\bigcap_{i=1}^{7} \overline{P_i}\right| = \left|\overline{\bigcup_{i=1}^{7} P_i}\right|$.

$$\begin{aligned}
\left|\bigcup_{i=1}^{7} P_i\right| &= \sum |P_i| - \sum |P_i \cap P_j| + \sum |P_i \cap P_j \cap P_k| + \cdots + \sum_{|\lambda|=\#} (-1)^{1+\#} \cdot \left|\bigcap_{i\in\lambda} P_i\right| + \cdots + \left|\bigcap_{i=1}^{7} P_i\right| \\
&= \binom{7}{1} \cdot 6! - \binom{7}{2} \cdot 5! + \binom{7}{3} \cdot 4! + \cdots + \binom{7}{7} \cdot 0! \\
&= \frac{7!}{1! \cdot 6!} \cdot 6! - \frac{7!}{2! \cdot 5!} \cdot 5! + \cdots + \frac{7!}{7! \cdot 0!} \cdot 0! \\
&= 7! \cdot (\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{7!}) \\
&= 3186
\end{aligned}$$

Therefore, $\left|\overline{\bigcup_{i=1}^{7} P_i}\right| = P(7,7) - \left|\bigcup_{i=1}^{7} P_i\right| = 1854$.

Q9

How many positive integers less than 200 are

(a) either odd or the square of an integer;
Let A be the odd positive integers within 200, and B be the square numbers within 200. Then, we have $|A| = 100$, $|B| = 14$, $|A \cap B| = 7$. Therefore,

$$\textbf{Ans} = |A \cup B| = |A| + |B| - |A \cap B| = 107$$

(b) second or higher powers of integers?

Let $P_i = \{x\colon \exists t \in [2, 200) \cap \mathbb{Z} \text{ s.t. } x = t^i\}$.

Since, $\lfloor \log_2 199 \rfloor = 7$, we conclude that $\forall i \geq 8, P_i = \emptyset$. And, obviously we get

i. $|P_i| = \lfloor 199^{1/i} \rfloor - 1$.

ii. $d \mid n \implies P_n \subseteq P_d$

iii. $\gcd(i, j) = 1 \implies P_i \cap P_j = \emptyset$

By calculation, we have $(P_2, P_3, P_4, P_5, P_6, P_7) = (13, 4, 2, 1, 1, 1)$. Therefore, $\left|\bigcup_{i=2}^{7} P_i\right| = |P_2 \cup P_3 \cup P_5 \cup P_7| = 13 + 4 + 1 + 1 = 19$.

But, for the answer, we should take 1 into account. Thus, $\textbf{Ans} = 20$.

(c) either primes or second or higher powers of integers?

Let A be set of all primes within 200 and B be set of second or higher powers of integers within 200. Since, all primes cannot be a power of other integer, thus $A \cap B = \emptyset$.

Therefore, $\textbf{Ans} = |A \cup B| = |A| + |B| - |A \cap B| = 46 + 20 = 66$.

(d) not divisible by the square of an integer greater than 1?

Square of integers within 200 are listed as

$$S = \{4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196\}$$

Let $P_i = \{x\colon i \mid x, x \in [1, 199] \cap \mathbb{Z}\}$, and $S' = \{4, 9, 25, 49, 121, 169\}$. Then, we have

i. $|P_i| = \lfloor 199/i \rfloor$

ii. $d \mid n \implies P_n \subseteq P_d$

iii. $\gcd(i, j) = 1 \implies P_i \cap P_j = \emptyset$

Then,

$$\begin{aligned}
\textbf{Ans} &= 199 - \left|\bigcup_{i \in S} P_i\right| = 199 - \left|\bigcup_{i \in S'} P_i\right| \\
&= 199 - \sum_{i \in S'} |P_i| = 199 - \sum_{i \in S'} \left\lfloor \frac{199}{i} \right\rfloor \\
&= 199 - 49 - 22 - 7 - 4 - 1 - 1 \\
&= 115
\end{aligned}$$

Q10

How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

Solution: Since the pennies and nickels are identical, the answer should be 9, which correspond to pairs: $(8,0),(7,1),\ldots,(1,7),(0,8)$.

Q11

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17 \tag{1}$$

Solution: For equation

$$\sum_{i=1}^{n} x_i = m$$

where $x_i \in \mathbb{Z}^+$, the cardinality of solution set is $\binom{m-1}{n-1}$.

(a) if x_1, x_2, x_3 and x_4 are nonnegative integers?
Rewrite (1) as $(x_1+1)+(x_2+1)+(x_3+1)+(x_4+1) = 21$, then **Ans** $= \binom{20}{3} = 1140$.

(b) if x_1, x_2, x_3 and x_4 are positive integers?
It is trivial that **Ans** $= \binom{16}{3} = 560$.

(c) if $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4$ and x_4 are positive integers?
Rewrite (1) as $(x_1-1)+(x_2-2)+(x_3-3)+x_4 = 11$, then **Ans** $= \binom{10}{3} = 120$.

Q12

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 \leq 17 \tag{2}$$

Solution: For inequation

$$\sum_{i=1}^{n} x_i \leq m$$

where $x_i \in \mathbb{Z}^+$, the cardinality of solution set is $\sum_{i=n}^{m} \binom{i-1}{n-1}$.

(a) if x_1, x_2, x_3 and x_4 are nonnegative integers?
Rewrite (2) as $(x_1+1)+(x_2+1)+(x_3+1)+(x_4+1) = 21$, then **Ans** $= \sum_{i=4}^{21} \binom{i-1}{3} = 5985$.

(b) if x_1, x_2, x_3 and x_4 are positive integers?
It is trivial that **Ans** $= \sum_{i=4}^{17} \binom{i-1}{3} = 2380$.

Q13

Find the next larger permutation in lexicographic order after each of these permutations.

(a) 1432; **Ans** = 2134.

(b) 54123; **Ans** = 54213.

(c) 12453; **Ans** = 12534.

(d) 31528764. **Ans** = 31542678.

Q14

Find the next larger 5-combination of the set $\{1, 2, 3, 4, 5, 6, 7\}$ after each of these 5-combinations

(a) $\{1, 2, 4, 5, 7\}$; **Ans** = $\{1, 2, 4, 6, 7\}$.

(b) $\{1, 4, 5, 6, 7\}$; **Ans** = $\{2, 3, 4, 5, 6\}$.

Q15

Write the pseudo-code for generating the next permutation in a reverse lexicographic order.
Solution:

Algorithm 1 generate the next permutation in a reverse lexicographic order

```
procedure NEXT PERMUTATION(a_1 a_2 ··· a_n :permutation of {1, 2, ..., n} not equal to
1 2 ··· n−1 n)
    j ← n − 1
    while a_j < a_{j+1} do                        ▷ find the max suffix that is well ordered
        j ← j − 1
    k ← n
    while a_k > a_j do                            ▷ find the first number that is smaller than a_j
        k ← k − 1
    INTERCHANGE(a_j, a_k)
    r ← n
    l ← j + 1
    while r > l do                                ▷ sort the rest part in descending order
        INTERCHANGE(a_l, a_r)
        l ← l + 1
        r ← r − 1                                 ▷ a_1 a_2 ··· a_n is now the next permutation
```

Q16

Given set $\{n, n-1, \ldots, 1\}$, write the pseudo-code for generating the next r-combination in a reverse lexicographic order.

Solution:

Algorithm 2 generate the next r-combination in a reverse lexicographic order

procedure NEXT r-COMBINATION($a_1 a_2 \cdots a_r$:proper subset of $\{1, 2, \ldots, n\}$ not equal to $\{r, r-1, \ldots 2, 1\}$ with $a_1 > a_2 > \cdots > a_r$)

  $i \leftarrow r$ ▷ *iterate from the last digit*

  while $a_i = r - i + 1$ **do** ▷ *find the first digit that can be modified*

    $i \leftarrow i - 1$

  $a_i \leftarrow a_i - 1$

  for $j \leftarrow i + 1$ **to** r **do** ▷ *follow closely to the digit*

    $a_j \leftarrow a_i - j + i$

    $j \leftarrow j + 1$ ▷ $a_1, a_2, \ldots, a_r$ *is now the next combination*

Q17

Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Solution: 5 numbers have 5 remainders, but the range of remainders is $\{0, 1, 2, 3\}$, and according to Pigeonhole Principle, there must be 2 remainders be the same.

Q18

Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Solution: It's obvious that n consecutive integers have n different remainders when divided by n, but, the remainders generated by dividing n has exactly n elements. Therefore, there exists exactly one 0 among the remainders, which means it is divisible by n.

Q19

Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of whose neighbors are boys.

Solution: Denote the 50 positions as $p_1, p_2, \ldots p_{50}$, and then divide the group into two sub-groups:$\{a_1, a_3, \ldots, a_{49}\}$ and $\{a_2, a_4, \ldots, a_{50}\}$. According to Pigeonhole Principle, either the odd or the even group will have at least $\lceil \frac{25}{2} \rceil = 13$ boys. And this ensures that at least 2 boys are adjacent to each other in this sub-group, which means they are neighbors to a person in the whole circle.