Discrete Math — Homework 6 Solutions

Yuquan Sun, SID 10234900421

April 8, 2025

Q1

Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.
Solution: 4,3,2,1,8,7,6,5,12,11,10,9,16,15,14,13.

Q2

What is the coefficients of x^8,x^9 and x^{10} in $(2-x)^{19}$?
Solution:

$$\begin{aligned}(2-x)^{19} = \cdots &+ \binom{19}{8} \cdot 2^{11} \cdot (-1)^8 \cdot x^8 \\ &+ \binom{19}{9} \cdot 2^{10} \cdot (-1)^9 \cdot x^9 \\ &+ \binom{19}{10} \cdot 2^{9} \cdot (-1)^{10} \cdot x^{10} \\ &+ \cdots\end{aligned}$$

Q3

Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}/2$$

Proof: From the property of combination: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, we deduce:

$$\begin{aligned}\binom{2n}{n+1} + \binom{2n}{n} &= \binom{2n+1}{n+1} \\ &= \frac{1}{2}\left[\binom{2n+1}{n} + \binom{2n+1}{n+1}\right] \\ &= \frac{1}{2}\binom{2n+2}{n+1}\end{aligned}$$

$\square$

Q4

Give a combinatorial proof that
$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

Proof: Let's count in two ways the number of ways to select a committee and to then select a leader of the committee out of n people.

First, if we consider the committee by the scale of people. For a k-people committee, then the cases are first choose k people, namely $\binom{n}{k}$;then there are k chances to pick a leader out of all of them, and thus, the answer should be $k\binom{n}{k}$. Let k iterate from 1 to n, then the total sum should be $\sum_{k=1}^{n} k\binom{n}{k}$.

From a different perspective, we first pick a leader then there are n choices in total, then for the rest $n-1$ people, they are either in the committee or not, so there are 2^{n-1} cases, put the 2 parts together, we get $n \cdot 2^{n-1}$. $\square$

Q5

In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?
Solution: Denote $F[n]$ as the answer for the $2 \times n$ case.

(I) Base case $n = 1$. $F[1] = 1$.

(II) Base case $n = 2$. $F[2] = 3$.

(III) Common case for $n \geq 3$.
If we use 1×2 blocks, then there are 2 ways. If put it **vertically**, then $F[n-2]$; if put it **horizontally**, then $F[n-1]$.
If we use 2×2 block, then $F[n-2]$.
In total, $F[n] = F[n-1] + 2F[n-2]$.

Solving it, we get $F[n] = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot (-1)^n$.

Q6

(a) Find the recurrence relation satisfied by R_n, where R_n is # regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.
Solution: It's easy to see that $R_n = R_{n-1} + n$ where $R_1 = 2$.

(b) Find R_n using iteration.
Solution: $R_n = R_{n-1} + n = R_{n-2} + (n-1) + n = R_1 + 2 + \cdots + (n-1) + n = 1 + n(n+1)/2$.

Q7

A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

(a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
Solution: Denote $F[n]$ as the ways for the n dollars case. Considering always put the newest item at the end of the sequence, then this ensures iterate all permutations but no repetition.
Then $\begin{cases} F[n] = 2F[n-1] & ,0 < n < 5 \\ F[n] = 2F[n-1] + F[n-5] & ,n \geq 5 \end{cases}$.

(b) What are the initial conditions?
Solution: $F[0] = 1$.

(c) How many ways are there to deposit \$10 for a book of stamps?
Solution: By calculation, we get $\textbf{Ans} = 1217$.

Q8

For bit strings, find a recurrence relation for the number of bit strings of length n that contain an odd number of 0s.
Solution: Denote $F[n,0]$ as the number of cases where there exists even 0s, and $F[n,1]$ for the odd case. Then we can get the recurrence relation:

$$F[n,0] = F[n-1,0] + F[n-1,1]$$

$$F[n,1] = F[n-1,0] + F[n-1,1]$$

And the initial condition, $F[1,0] = F[1,1] = 1$.

Q9

1. Given two strings A and B, we need to find the minimum number of operations which can be applied on A to convert it to B. The operations are:

   a. Edit - Change a character to another character;
   b. Delete - Delete a character;
   c. Insert - Insert a character.

   The **edit distance** of two strings is defined by the minimum # operations required to transform one string into the other. For the following two strings, their edit distance is 3.

$$GCGTATGAGGCTA-ACGC$$
$$GC-TATGCGGCTATACG$$

   Please utilize the dynamic programming to compute the edit distance between two strings A and B.

   a. Define the subproblems for DP;
   b. Find the recurrence;
   c. Implement the algorithm to return the edit distance of two strings.

Q10

In the knapsack problem we are given a set of n items, where each item i is specified by a size s_i and a value v_i. We are also given a size bound S (the size of our knapsack). The goal is to find the subset of items of maximum total value such that sum of their sizes is at most S (they all fit into the knapsack). To implement a DP algorithm to solve this problem,

(a) Define subproblems;
Denote $F[i, j]$ as the maximum value under the condition where we just use the former i items and the sum of the size doesn't exceed j.

(b) Find the recurrence relation;

(c) Solve the base cases;

(d) Implement the algorithm to return the solution of the knapsack problem.

Q11

Q12

Q13

Q14

Q15

Q16

Q17