Discrete Math — Homework 9 Solutions

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Q1

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution: Let r.v. X be the number of defectives in the purchase.

Then, we have

$$P(X=0) = C(17,2)/C(20,2) = \frac{68}{95}$$
$$P(X=1) = C(3,1)\cdot C(17,1)/C(20,2) = \frac{51}{190}$$
$$P(X=2) = C(3,2)/C(20,2) = \frac{3}{190}$$

Q2

An experiment consists of tossing 4 fair coins. Compute the probability and distribution functions for the following r.v.s.

(a) The number of heads before the first tail;
Solution: Denote the r.v. as X. Then $P(X=k) = (\frac{1}{2})^{\min\{k+1,4\}}$ where $k \in \{0,1,2,3,4\}$.

(b) The number of heads after the first tail;
Solution: Denote the r.v. as X. Then $P(X=k) = \sum_{i=1}^{4-k} C(4-i,k)\cdot(\frac{1}{2})^4$, where $k \in \{1,2,3\}$, and the core idea is to enumerate the position of the initial tail. Then for the $X=0$ case, since the first tail ensure that there must exist a tail in the consequence which cannot cover all the possible case of tossing 4 fair coins, so to make the distribution function work, we define $P(X=0) := 1 - \sum_{i=1}^{3} P(X=i)$ to ensure the property of the distribution function is well-defined.

(c) The number of heads less the number of tails.
Solution: Denote the r.v. as X. Then $P(X=-4) = P(X=4) = \frac{1}{16}$, $P(X=-2) = P(X=2) = \frac{1}{4}$, $P(X=0) = \frac{3}{8}$.

Q3

A coin is tossed three times. If X is a r.v. giving the number of head that arise, construct a table showing the probability distribution of X.
Solution: Assume the possibility of tossing a head is p, then the tail should be $1-p$. Then we have

$$P(X=k)=\binom{3}{k}\cdot p^k(1-p)^{3-k}$$

where $k\in\{0,1,2,3\}$.

Q4

An urn holds 5 white and 3 black marbles. If 2 marbles are to be drawn at random without replacement and X denotes the number of white marbles, find the probability distribution for X .
Solution: For $k\in\{0,1,2\}$,

$$P(X=k)=\binom{5}{k}\cdot\binom{3}{2-k}\cdot\binom{8}{2}^{-1}$$

Q5

Suppose that Frida selects a ball by first picking one of two boxes at random and then selecting a ball from this box at random. The first box contains two white balls and three blue balls, and the second box contains four white balls and one blue ball. What is the probability that Frida picked a ball from the first box if she has selected a blue ball?
Solution: For the first box, it has 2W3B, for the second box it has 4W1B. Denote F as pick ball from the first box, then $\overline{F}$ stands for picking from the second box. Denote B as picking a blue ball. Then we have

$$\begin{aligned}P(F|B)&=\frac{P(B|F)}{P(B|F)+P(B|\overline{F})}\\&=\frac{3/5}{3/5+1/5}=\frac{3}{4}\end{aligned}$$

Q6

The joint probability function of two r.v.s X and Y is given by $f(x,y)=c(2x+y)$, where x and y can asseme all integers such that $0\le x\le 2$, $0\le y\le 3$, and $f(x,y)=0$ otherwise.

(a) Find the value of the constant c
Solution: Use the axiom of probability, we have

$$\sum_{x=0}^{2}f_X(x)=\sum_{x=0}^{2}\sum_{y=0}^{3}f(x,y)=c\cdot\sum_{x=0}^{2}\sum_{y=0}^{3}(2x+y)=1$$

Thus, $c=\frac{1}{42}$.

(b) Find $P(X = 2, Y = 1)$
Solution: $P(X = 2, Y = 1) = f(2, 1) = \frac{5}{42}$.

(c) Find $P(X \geq 1, Y \leq 2)$
Solution:

$$P(X \geq 1, Y \leq 2) = \sum_{x=1}^{2}\sum_{y=0}^{2} f(x,y) = \sum_{x=1}^{2}\sum_{y=0}^{2} \frac{1}{42}(2x+y) = \frac{4}{7}$$

(d) Find the marginal probability functions of X and Y
Solution:

$$P(X = x) = f_X(x) = \sum_{y=0}^{3} f(x,y) = \frac{1}{42}(8x+6)$$

$$P(Y = y) = f_Y(y) = \sum_{x=0}^{2} f(x,y) = \frac{1}{42}(3y+6)$$

(e) Show that the r.v.s X and Y are dependent
Solution: Since $P(X = 0) \cdot P(Y = 0) \neq 0 = P(X = 0, Y = 0)$, X and Y are dependent.

(f) Compute $P(Y = 1|X = 2)$
Solution:

$$\begin{aligned} P(Y = 1|X = 2) &= \frac{P(X = 2, Y = 1)}{P(X = 2)} \\ &= \frac{f(x, 1)}{f_X(2)} \\ &= \frac{5}{22} \end{aligned}$$

Q7

Suppose that a Bayesian spam filter is trained on a set of 1000 spam messages and 400 messages that are not spam. The word “opportunity” appears in 175 spam messages and 20 messages that are not spam. Would an incoming message be rejected as spam if it contains the word “opportunity” and the threshold for rejecting a message is 0.9?
Solution: Let S be the word being in spam message and then the word and then the word not in spam is denoted as $\overline{S}$, and A be the message contains the word “opportunity”. Then the chance that the incoming message that contains “opportunity” is a spam is

$$\begin{aligned} P(S|A) &= \frac{P(A|S)}{P(A|S) + P(A|\overline{S})} \\ &= \frac{175/1000}{175/1000 + 20/400} = \frac{7}{9} \approx 0.778 < 0.9 \end{aligned}$$

Therefore, the incoming message will not be rejected.

Q8

Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that
Solution: Denote X as one in the clinic infected with HIV, then $\overline{X}$ means one in the clinic not infected with HIV. Denote Y as one test positive in a blood test, then $\overline{Y}$ means one is test negative in a blood test.

From the description of the problem, we have

- $P(X) = 8\%$
- $P(\overline{X}) = 1 - P(X) = 92\%$
- $P(Y|X) = 98\%$
- $P(Y|\overline{X}) = 3\%$

(a) a patient testing positive for HIV is infected with it?
Ans $= P(X|Y) = \frac{P(Y|X)P(X)}{P(Y|X)P(X)+P(Y|\overline{X})P(\overline{X})} = \frac{98\% \cdot 8\%}{98\% \cdot 8\% + 3\% \cdot 92\%} = \frac{196}{265}$

(b) a patient testing positive for HIV is not infected with it?
Ans $= P(\overline{X}|Y) = 1 - P(X|Y) = \frac{69}{265}$

(c) a patient testing negative for HIV is infected with it?
First, we calculate $P(Y) = P(Y|X)P(X) + P(Y|\overline{X})P(\overline{X}) = \frac{53}{500}$. Thus, $P(\overline{Y}) = 1 - P(Y) = \frac{447}{500}$. Therefore, **Ans** $= P(X|\overline{Y}) = \frac{P(\overline{Y}|X)P(X)}{P(\overline{Y})} = \frac{(1-P(Y|X))P(X)}{P(\overline{Y})} = \frac{4}{2235}$

(d) a patient testing negative for HIV is not infected with it?
Ans $= P(\overline{X}|\overline{Y}) = 1 - P(X|\overline{Y}) = \frac{2231}{2235}$

Q9

Let X be the number appearing on the first dice when two fair dice are rolled and let Y be the sum of the numbers appearing on the two dice. Show that $E(X)E(Y) \neq E(XY)$.
Solution: Since the dices are all fair, each number hold the same possibility, and thus $E(X) = \frac{1+2+3+\cdots+6}{6} = \frac{7}{2}$, $E(Y) = 2E(X) = 7$. And in a similar way, all possible outcomes after the multiplications have the same possibility $\frac{1}{36}$, so $E(XY) = \frac{\sum_{i=1}^{6}\left(i \cdot \sum_{j=1}^{6} i+j\right)}{36} = \frac{987}{36}$. Hence, $E(X)E(Y) \neq E(XY)$.

Q10

The **law of total expectation** states that if sample space Ω is the disjoint union of events $S_1, S_2, \ldots S_n$ and X is a r.v., then $E(X) = \sum_{j=1}^{n} E(X|S_j)P(S_j)$, where $E(X|S_j)$ is the **conditional expectation** of r.v. given event S_j from sample space Ω, and can be computed as $E(X|S_j) = \sum_{r \in X(\Omega)} r \cdot P(X = r|S_j)$

(a) Prove the law of total expectation

Proof:

$$\begin{aligned}
E(X) &= \sum_{r \in X(\Omega)} r \cdot P(X = r) \\
&= \sum_{r \in X(\Omega)} r \cdot \sum_{j=1}^{n} P(X = r | S_j) P(S_j) \\
&= \sum_{j=1}^{n} P(S_j) \cdot \sum_{r \in X(\Omega)} r \cdot P(X = r | S_j) \\
&= \sum_{j=1}^{n} E(X | S_j) P(S_j)
\end{aligned}$$

$\square$

(b) Use the law of total expectation to find the average weight of a breeding elephant seal, giben that 12% of the breeding elephant seals are male and the rest are female, and the expected weights of a breeding elephant is 4200 pounds for a male and 1100 pounds for a female.

Solution: **Ans** $= 12\% \cdot 4200 + 88\% \cdot 1100 = 1472$.