Discrete Math — Homework 4 Solutions

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March 22, 2025

Q1

Given three sets A, B, and C , please prove the following statements.

Let p be "$x \in A$", q be "$x \in B$", r be "$x \in C$".

(a) $(A \cap B) \subseteq A$

Proof: According to the definition, $\forall x \in A \cap B (x \in A)$, and thus we have $(A \cap B) \subseteq A$. $\square$

(b) $A \cap (B - A) = \emptyset$

Proof: For all $x \in A \cap (B - A)$, by definition, we have $(x \in A) \wedge (x \notin A)$, so there doesn't exists such x. Therefore, $A \cap (B - A) = \emptyset$. $\square$

(c) $A \cup (B - A) = A \cup B$

Proof: $A \cup (B - A) = \{x \mid p \vee (q \wedge \neg p)\} = \{x \mid (p \vee q) \wedge (p \vee \neg p)\} = \{x \mid p \vee q\} = A \cup B$. $\square$

(d) $A - B = A \cap \overline{B}$

Proof: $A - B = \{x \mid p \wedge \neg q\} = A \cap \overline{B}$. $\square$

(e) $(A \cap B) \cup (A \cap \overline{B}) = A$

Proof: $(A \cap B) \cup (A \cap \overline{B}) = \{x \mid (p \wedge q) \vee (p \wedge \neg q)\} = \{x \mid p \wedge (q \vee \neg q)\} = \{x \mid p\} = A$. $\square$

(f) $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

Proof: $\overline{A \cap B \cap C} = \{x \mid \neg(p \wedge q \wedge r)\} = \{x \mid \neg p \vee \neg q \vee \neg r\} = \overline{A} \cup \overline{B} \cup \overline{C}$. $\square$

Q2

Show the following Cartesian products are not the same.

(a) $A \times B$ and $B \times A$, unless $A = B$.

Let $A = \{1\}$, $B = \{2\}$. Then $A \times B = \{(1, 2)\} \neq \{(2, 1)\} = B \times A$.

(b) $A \times B \times C$ and $(A \times B) \times C$. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$. Then $A \times B \times C = \{(1, 2, 3)\} \neq \{((1, 2), 3)\} = (A \times B) \times C$.

Q3

Determine whether the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m,n) = m + n$ is onto.

(b) $f(m,n) = m^2 + n^2$ is **not** onto.

(c) $f(m,n) = m$ is onto.

(d) $f(m,n) = |n|$ is **not** onto.

(e) $f(m,n) = m - n$ is onto.

Q4

If f and $f \circ g$ are onto, does it follow that g is onto?
No.
Proof: Let $f\colon V \to W$.

f is onto means $\operatorname{Im} f = f(V) = R$. $f \circ g$ is onto means $\operatorname{Im} f \mid \operatorname{Im} g = R$. (the vertival bar here means restrict the domain of f within the image of g)

Once $V \subseteq \operatorname{Im} g$, then we can ensure that $f \circ g$ is onto. So to construct a counterexample, we just need to set g as a function such that $V \subseteq \operatorname{Im} g \subset$ co-domain of g. $\square$

Q5

Let S be a subset of a universal set U. The characteristic function f_S of S is the function from U to $\{0,1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be sets.

(a) $f_{A\cap B}(x) = f_A(x) \cdot f_B(x)$
Proof:

LHS, $f_{A\cap B}(x) = 1$ iff. $x \in A \cap B \iff x \in A \wedge x \in B$.

RHS, $f_A(x) \cdot f_B(x) = 1$ iff. $f_A = 1 \wedge f_B = 1 \iff x \in A \wedge x \in B$.

Thus, the 2 expressions are equivalent. $\square$

(b) $f_{A\cup B(x)} = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
Proof: Proving $f_{A\cup B(x)} = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$ is proving $f_{A\cup B(x)} + f_{A\cap B}(x) = f_A(x) + f_B(x)$. We prove by cases:

I. $x \notin A \wedge x \notin B$. LHS=0=RHS.

II. x in either set. LHS=1=RHS.

III. x in both set. LHS=2=RHS.

Thus, the equation was verified. $\square$

(c) $f_{\overline{A}}(x) = 1 - f_A(x)$
Proof: We prove by cases:

I. $x \in A$. LHS=0=RHS.

II. $x \notin A$. LHS=1=RHS.

Thus, the equation was verified. $\square$

Q6

Show that the function $f(x) = ax + b$ from $\mathbb{R}$ to $\mathbb{R}$ is invertible, where a and b are constants, with $a \neq 0$, and find the inverse of f .
Solution:

First show f is injection. $\forall m, n \in \mathbb{R}$, $f(m) = f(n) \implies am + b = an + b \implies m = n$.
Then show f is surjection. $\forall y_0 \in \mathbb{R}$, $\exists x_0 = \frac{1}{a}(y_0 - b)$ s.t. $f(x_0) = y_0$.
So, f is bijection, and thus, f is invertible. $f^{-1} = \frac{1}{a}(x - b)$.

Q7

Prove or disprove each of these statements about the floor and ceiling functions.

(a) $\forall x \in \mathbb{R}, \lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$
Proof: Write $x = n + \epsilon, n \in \mathbb{Z}, \epsilon \in [0, 1)$. $\lceil \lfloor x \rfloor \rceil = \lceil n \rceil = n = \lfloor x \rfloor$. $\square$

(b) $\forall x \in \mathbb{R}, \lfloor 2x \rfloor = 2 \lfloor x \rfloor$
False. $x = 2.5$ is an counterexample.

(c) $\forall x, y \in \mathbb{R}, \lceil xy \rceil = \lceil x \rceil \lceil y \rceil$
False. $x = y = 9.9$ is an counterexample.

(d) $\forall x \in \mathbb{R}, \lceil \frac{x}{2} \rceil = \lfloor \frac{x+1}{2} \rfloor$
False. $x = 6.2$ is an counterexample.

Q8

Please formulate the following problem as a formal mathematical problem via using indicator functions:

Input: Universal set $U = \{u_1, u_2, \ldots, u_n\}$. Subsets $S_1, S_2, \ldots, S_m \subseteq U$.

Goal: Find k subsets that maximizes their total coverage, i.e. , set $\bigcup_{i\in k} S_i$ contains the most elements of U.

Solution: Let $\lambda \subseteq \{1, 2, \dots, n\}$, $\delta_i = \begin{cases} 0, & i \notin \lambda \\ 1, & i \in \lambda \end{cases}$, $S = \bigcup_{i\in\lambda} S_i$, $I_S(u_i) = \begin{cases} 0, & u_i \notin S \\ 1, & u_i \in S \end{cases}$

Objective: $\max \sum_{i=1}^{n} I_S(u_i)$ s.t. $\sum_{i=1}^{n} \delta_i = 9$.

Q9

Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) all bit strings not containing the bit 0.
Countably infinite. $f\colon$ bit strings $\to \mathbb{Z}^+, str \mapsto$ length of str.

(b) the integers that are multiples of 7.
Countably infinite. Let $f\colon \mathbb{Z} \to \mathbb{Z}, x \mapsto \frac{x}{7}$, $g\colon \mathbb{Z} \to \mathbb{Z}^+, x \mapsto \begin{cases} 2x, & x \geq 0 \\ 2\,|x| - 1, & x < 0 \end{cases}$. And $g \circ f$ is what we want.

(c) the irrational numbers between 0 and 1.
Uncountable.

(d) the real numbers between 0 and $\frac{1}{2}$.
Uncountable.

Q10

Give an example of two uncountable sets A and B such that $A \cap B$ is

(a) finite. Let $A = (0, 1], B = [1, 2]$.

(b) countably infinite. Let $A = \mathbb{Z} \cup (0, 1), B = \mathbb{Z} \cup (1, 2)$.

(c) uncountable. Let $A = B = (0, 1)$.

Q11

Explain why the set A is countable if and only if $|A| \leq |\mathbb{Z}^+|$.
Solution:

If A is countable, it is trivial that there exists an injection from A to $\mathbb{Z}^+$, so $|A| \leq |\mathbb{Z}^+|$.

Then for the other case $|A| \leq |\mathbb{Z}^+|$. If $|A| < |\mathbb{Z}^+| = \aleph_0$, then A is finite, so it is countable. If $|A| = |\mathbb{Z}^+| \iff (|A| \leq |\mathbb{Z}^+|) \wedge (|A| \geq |\mathbb{Z}^+|)$, and each term ensure a injection. So according to Schröder–Bernstein theorem, there exist a bijection from A to $\mathbb{Z}^+$, and thus A is countable.

Q12

Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.
Proof:

Use contradiction. Assume B is countable, then $|B| \leq |\mathbb{Z}^+|$, there we ensure an injection f from B to $\mathbb{Z}^+$. And $A \subseteq B$, thus we ensure an injection g from A to B. Then $f \circ g$ is an injection from A to $\mathbb{Z}^+$. Threrfore, $|A| \leq |\mathbb{Z}^+|$, which means A is countable. This contradicts with "A is uncountable". So, B is uncountable. □

Q13

Show that if A, B, and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.
Proof:

Since, $|A| \leq |B|$, we ensure an injection $f\colon A \to B$. Similarly, we ensure an injection $g\colon B \to C$. Then, $g \circ f$ is an injection from A to C. Therefore, $|A| \leq |C|$. □

Q14

Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$,

(a) Find A^T, A^{-1}, and A^3.

$A^T = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$, $A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 18 \\ 9 & 37 \end{pmatrix}$.

(b) Find $(A^{-1})^3$ and $(A^3)^{-1}$.

$(A^{-1})^3 = (A^3)^{-1} = \frac{1}{125}\begin{pmatrix} -37 & 18 \\ 9 & -1 \end{pmatrix}$.

Q15

Suppose that $A \in \mathbb{R}^{n \times n}$ where n is a positive integer. Show that $A + A^T$ is symmetric.
Proof: $(A + A^T)^T = A^T + (A^T)^T = A + A^T$. □

Q16

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Find $A \wedge B$, $A \vee B$, and $A \odot B$.
Solution:

$$A \wedge B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A \vee B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, A \odot B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Q17

Let A be a zero-one matrix. Show that

(a) $A \vee A = A$

For every entry a_{ij} of matrix A, $a_{ij} \vee a_{ij} = a_{ij}$. Thus, $A \vee A = A$.

(b) $A \wedge A = A$

For every entry a_{ij} of matrix A, $a_{ij} \wedge a_{ij} = a_{ij}$. Thus, $A \wedge A = A$.

Q18

Let A be an $n \times n$ zero-one matrix. Let I be the $n \times n$ identity matrix. Show that $A \odot I = I \odot A = A$.

Proof:

Denote $A[i, j]$ as the entry of matrix A at ith row and jth column.

$(A \odot I)[i, j] = \bigvee_{k=1}^{n}(A[i, k] \wedge I[k, j]) = A[i, j]$ since $I[i, j] = 1 \iff i = j$. So, we have $A \odot I = A$.

Similarly, $(I \odot A)[i, j] = \bigvee_{k=1}^{n}(I[i, k] \wedge A[k, j]) = A[i, j]$. Thus, $I \odot A = A$.

To sum up, we have $A \odot I = I \odot A = A$. $\square$