



華東師範大學
EAST CHINA NORMAL UNIVERSITY

线性代数

笔记

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East China Normal University

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Chapter 1

向量空间

线性代数是研究有限维向量空间上的线性映射的学问.我们最终会理解这些术语的具体含义.在本章中,我们将定义向量空间并讨论它们的基本性质.

在线性代数中,如果将复数与实数放在一起研究,就会得到更好的定理和更深刻的见解.因此,我们将从介绍复数及其基本性质开始.我们将把平面和三维空间这些例子推广到 \mathbb{R}^n 和 \mathbb{C}^n , 再进一步推广得到向量空间的概念.我们将会明白,向量空间是具有满足自然的代数性质的加法和标量乘法运算的集合.

接着,我们将讨论子空间.子空间之于向量空间,就类似于子集之于集合.最后,我们将关注子空间的和(类似于子集的并集)与子空间的直和(类似于不相交集合并集)

1.1 \mathbb{R}^n 和 \mathbb{C}^n

你应该已经熟悉了实数集合 \mathbb{R} 的基本性质.发明复数,是为了让我们可以取负数的平方根.我们的想法是,假设 -1 有平方根,将其用 i 表示,并且它遵守通常的算术规则.正式的定义如下.

Definition 1.1.1 复数(complex number) \mathbb{C} .

- 一个复数是一个有序对 (a, b) , 其中 $a, b \in \mathbb{R}$, 不过我们会把这个写成 $a + bi$
- 全体复数集合用 \mathbb{C} 表示:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- \mathbb{C} 上的加法和乘法定义为

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

其中 $a, b, c, d \in \mathbb{R}$.

Lemma 1.1.1 \mathbb{L} .

This is Lemma env.

Proposition 1.1.1 \mathbb{P} .

This is Proposition env.

Problem 1.1.1 \mathbb{PP} .

证明: $\sqrt{2} \notin \mathbb{Q}$.

Theorem 1.1.1 length of z .

Suppose $z \in \mathbb{C}$, then we have:

$$z\bar{z} = \bar{z}z = |z|^2$$

Example 1.1.1 复数的算数运算.

运用 1 中的性质, 可以算出 $(2+3i)(4+5i)$ 的值:

$$\begin{aligned}(2+3i)(4+5i) &= 2 \cdot (4+5i) + (3i) \cdot (4+5i) \\ &= -7 + 22i\end{aligned}$$

1.2 向量空间的定义

Definition 1.2.1 复数(complex number) \mathbb{C} .

- 一个复数是一个有序对 (a, b) , 其中 $a, b \in \mathbb{R}$, 不过我们会把这个写成 $a + bi$
- 全体复数集合用 \mathbb{C} 表示:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

1.3 子空间

Definition 1.3.1 子空间(subspace).

设 V 是一个向量空间. 如果 $W \subseteq V$ 并且 W 本身也是一个向量空间(使用 V 中的加法和标量乘法), 那么我们说 W 是 V 的一个子空间.

接下来的结果给出了检验向量空间的子集是否为子空间的最简单方法.

Remark 1.3.1 检验子空间的条件.

当且仅当 V 的子集 U 满足以下三个条件时, U 是 V 的子空间:

- 加法恒等元
 $0 \in U$
- 对于加法封闭
 $u, w \in U \Rightarrow u + w \in U$
- 对于标量乘法封闭
 $c \in \mathbb{F}, u \in U \Rightarrow cu \in U$

Chapter 2

Vector space with measurement

Back in middle school, we know such mapping called ‘inner product’, who receives 2 vectors and gives out one value. With inner product, we can define the length of a vector, angles of 2 vectors, and so on ... and that’s the motivation of this chapter.

2.1 Bilinear function

From the concept of inner product, we could formalize a kind of function $f : V \times V \rightarrow \mathbb{F}$, and it’s also a linear stuff.

Definition 2.1.1 Bilinear function.

For $f : V \times V \rightarrow \mathbb{F}$, if it satisfies linearity for both of the 2 variables, aka.

- $f(k\alpha_1 + \alpha_2, \beta) = kf(\alpha_1, \beta) + f(\alpha_2, \beta)$
- $f(\alpha, k\beta_1 + \beta_2) = kf(\alpha, \beta_1) + f(\alpha, \beta_2)$

then we call such f a **bilinear function** on V .

Theorem 2.1.1 Expansion of bilinear function.

Let $V = \text{span}(e_1, e_2, \dots, e_n)$, and 2 vectors in V are $\alpha = (a_1, a_2, \dots, a_n)'$, $\beta = (b_1, b_2, \dots, b_n)'$, we have

$$f(\alpha, \beta) = \sum_{i=1}^n \sum_{j=1}^n a_i b_j \cdot f(e_i, e_j)$$

Proof. Use the linearity to expand f , first α , then β . Readers can verify themselves. ■

Theorem 2.1.2 Matrix representation of bilinear function.

Let $V = \text{span}(e_1, e_2, \dots, e_n)$, $\alpha = [e_1 \ e_2 \ \dots \ e_n]x$, $\beta = [e_1 \ e_2 \ \dots \ e_n]y$, consider a matrix

$$A = \begin{pmatrix} f(e_1, e_1) & f(e_1, e_2) & \dots & f(e_1, e_n) \\ f(e_2, e_1) & f(e_2, e_2) & \dots & f(e_2, e_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(e_n, e_1) & f(e_n, e_2) & \dots & f(e_n, e_n) \end{pmatrix}$$

we conclude: $f(\alpha, \beta) = x' Ay$.

Proof. Use definition of matrix multiplication, readers can verify themselves. ■

We call matrix A a **measure matrix**. You may be confused with this naming, but later we'll explain.

However, for this matrix A we can interpret it from another perspective.

Theorem 2.1.3 Another interpretation of measure matrix.

Let $V = \text{span}\{e_1, e_2, \dots, e_n\}$, its dual space with dual basis $V' = \text{span}\{f_1, f_2, \dots, f_n\}$ and f is a bilinear function on V . Consider a mapping $\varphi : V \rightarrow V'$, where $\varphi(\beta) = f(x, \beta)$, we have

$$\varphi[e_1 \ e_2 \ \dots \ e_n] = [f_1 \ f_2 \ \dots \ f_n]A$$

which means A is the matrix of φ under a basis of V and its correspondent dual basis.

Proof. This is proof. ■

Problem 2.1.1 Basis.

List a basis for the following linear spaces with default addition and scaling operator:

- \mathbb{R}^2
- \mathbb{C}

Solution.

- $(1, 0)', (0, 1)'$
- 1

2.1.1 Here are some codes

cpp

```
1 #include <iostream>
2 using namespace std;
3
4 int main(){
5     cout<<"111"<<endl;
6     return 0;
7 }
```