Eigen Decomposition

Eigenvectors & Eigenvalues

- ▼ Keywords
  - ▼ Eigenvectors
  - ▼ Eigenvalues

Eigenvectors and Eigenvalues

1. Eigenvectors
   a. given a square matrix A of size (n x n)
   b. an eigenvector is a nonzero vector x of a matrix A
   c. such that
      i. $Ax = \lambda x$ for some scalar λ, which is called an eigenvalue of A
      ii. x is called an eigenvector corresponding to λ
2. Transformation perspective
   a. linear transformation $T(x) = Ax = \lambda x$
   b. computational advantage
      i. matrix multiplication needs to compute 4 times
      ii. whereas using eigenvectors only computes twice
   c. application
      i. image rotation
3. Eigenvectors and eigenvalues
   a. the equation $Ax = \lambda x$ can be rewritten as $(A - \lambda I)x = 0$
      i. which can be seen as one of the linear combination
   b. the equation has a nontrivial solution if and only if matrix $(A - \lambda I)$ is linearly dependent

Null Space & Orthogonal Complement

- ▼ Keywords
  - ▼ Null space
  - ▼ Orthogonal complement

Null Space

1. Null space
   a. usually denoted as Nul A
      i. where A is a matrix of size (m x n)
   b. the set of all solutions of a homogeneous linear system, $Ax = 0$
   c. note that
      i. orthogonal vectors are linearly independent
      ii. however, linearly independent vectors are not always orthogonal

Orthogonal Complement

1. Orthogonal complement
   a. the set of all vectors z that are orthogonal to W
2. Fundamental subspaces given by A
   a. the fundamental subspaces
      i. Nul A = orthogonal space of Row A
      ii. Nul A^T= orthogonal space of Col A
   b. rank theorem
      i. n = dim(Row A) + dim(Nul A)

Characteristic Equation

- ▼ Keywords
  - ▼ Characteristic equation

Characteristic Equation

1. Previous
   a. eigenvalues are given

b. find eigenvectors when eigenvalues are given

2. Characteristic equation
   a. linearly (in)dependent vs. (non)invertible
      i. linearly (in)dependent can be defined on both rectangular and square matrix
      ii. invertible can be defined only on the square matrix
   b. under the special condition
      i. linearly dependent is similar to non-invertible
      ii. linearly independent is similar to invertible
   c. therefore, the eigenvalue λ can be found by computing $det(A) = 0$
      i. which means non-invertible, i.e., linearly dependent
   d. $det(A) = 0$ is called characteristic equation

Eigenspace

1. Eigenspace
   a. the dimension of the eigenspace can be more than one
      i. corresponds to a particular λ
   b. any vector in the eigenspace satisfies $T(x) = Ax = \lambda x$

Diagonalization

▼ Keywords

  ▼ Diagonalizable matrix

Diagonalization

1. Diagonalization
   a. the values of diagonal entries are nonzero and the others are zero
      i. given square matrix A of size n x n
      ii. $D = V^{-1}AV$
      iii. if and only if the invertible matrix V exists

b. the matrix A is called diagonalizable matrix

c. the matrix D is called diagonal matrix

2. Finding V and D

    a. for V

        i. the vectors v of the matrix V should be eigenvectors

        ii. the matrix V should be invertible, i.e., the eigenvectors v are linearly independent

    b. for D

        i. the matrix D has eigenvalues as diagonal entries

    c. for A

        i. A should have n linearly independent eigenvectors

Eigendecomposition & Linear Transformation

▼ Keywords

  ▼ Eigendecomposition

Eigendecomposition

1. Eigendecomposition

    a. diagonalizable matrix $A = VDV^{-1}$

        i. this is called eigendecomposition of A

    b. if the matrix is diagonalizable, the matrix has eigendecomposition

        i. the two condition is equivalent

2. Linear transformation via eigendecomposition

    a. $T(x) = Ax = VDV^{-1}x = V(D(V^{-1}x))$

        i. where the linear transformation $T(x) = Ax$

        ii. the matrix A is diagonalizable, i.e., has eigendecomposition

    b. computational advantage

i. the matrix multiplication can be transformed into the summation of vectors

3. Change of basis

   a. 평행사변형 법칙을 이용하여 vector x 값을 구하는 것이 아니라 eigendecomposition 을 이용하여 구함

      i. $Vx = a$ where a vector a is a given arbitrary vector and the matrix V is a set of eigenvectors

4. Element-wise scaling

   a. $T(x) = V(D(V^{-1}x)) = V(Dy)$

   b. let $z = Dy$

      i. the vector z is simply the element-wise scaling of the vector y

      ii. b/c of the definition of the diagonal matrix

5. Dimension-wise scaling

   a. $T(x) = V(Dy) = Vz$

      i. where z is still a coordinate based on the new basis $\{v_1, v_2\}$

   b. Vz is a linear combination of v_1, v_2 using the coefficient vector z

      i. which means $Vz = v_1z_1 + v_2z_2$

6. Linear transformation via A^k

   a. consider recursive transformation A^kx

   b. $A^k = (VDV^{-1})(VDV^{-1})...(VDV^{-1}) = VD^kV^{-1}$

      i. when A is diagonalizable

Further Study

- ▼ Keywords
  - ▼ Diagonalization
  - ▼ Algebraic multiplicity and geometric multiplicity

Existence of Eigendecomposition

1. Determining whether a matrix A (n x n) is diagonalizable

  a. geometric multiplicity should be equal to algebraic multiplicity
     i. $det(A - \lambda I) = 0$
     ii. eigenvalue 값이 다르면 두 eigenspace 는 서로 linearly independent 함
  b. A has n distinct eigenvalues, A is diagonalizable
     i. e.g., $(\lambda - 1)(\lambda - 3)(\lambda - 2)(\lambda + 5)(\lambda + 2)$
2. Solve $(A - \lambda I)x = 0$ for a given eigenvalue λ