Linear System & Linear Transformation

Linear Equation

1. Linear equation is written as below:

   $a_1x_1 + a_2x_2 + ... + a_nx_n = b$

   - coefficients: $a_1, ..., a_n$
2. By using vector:

   $a^Tx = b$

Linear System

1. Linear system is a set of linear equations
   a. 주어진 여러 feature를 이용하여 최종 값 b를 잘 예측하는 것이 목표 (= regression)
   b. solution
      i. use vector or matrix
         - matrix A is a collection of the coefficients
      ii. use inverse matrix
         - the solution is uniquely obtained
      iii. use non-invertible matrix
         - the solution is infinite or not exist
2. Identity Matrix
   a. written as I_n
3. Inverse Matrix
   a. written as A^{-1}
   b. not all of the matrices have inverse matrix
      i. i.e., need a special condition to have an inverse matrix

$$AA^{-1} = A^{-1}A = I_n$$

- use determinant of A (= det A) such as $ad - bc$

ii. defined only for **square matrix**

Linear Combinations

1. Linear combination is written as below:

   $c_1v_1 + ... + c_pv_p$

   - called a linear combination of $v_1, ..., v_p$ with weights or coefficients $c_1, ..., c_p$
2. Solution of linear system
   a. by matrix

      $Ax = b$

   b. by vector

      $a_1x_1 + a_2x_2 + a_3x_3 = b$

      - called a vector equation
      - the solution exists only when b is contained in the span $\{a_1, a_2, a_3\}$
3. Span
   a. Span $\{v_1, ..., v_p\}$ is the set of all linear combinations of $v_1, ..., v_p$
   b. Span with different number of vectors
      i. 2개의 벡터를 이용해서 span 되는 3차원 공간은 전체가 될 수 있음
      ii. 3개의 벡터를 이용해서 span 되는 4차원 공간은 전체가 되기에는 살짝 무리가 있음

Linear Transformation

1. Definition
   a. domain (정의역): X
   b. co-domain (공역): Y
   c. image (상): $y = f(x)$
      i. preimage (원상): $x = f^{-1}(y)$

d. range (치역): set of all images

e. 화살표는 하나의 x에 대하여 **한번만** 가능

2. Linear transformation

a. Transformation (or mapping) T is linear if:

$$T(cu + dv) = cT(u) + dT(v)$$

- e.g., $y = 3x + 2$
- $(1, 2) \rightarrow 3 * 1 + 4 * 2 = 11 \rightarrow 3 * 11 + 2 = 35$
- $(1, 2) \rightarrow (5, 8) \rightarrow 3 * 5 + 4 * 8 = 47$
- 위 두 값이 다르기 때문에 선형변환이 아님

b. Vector of linear transformation

i. transform n-dimensional vector to m-dimensional vector

c. Matrix of linear transformation

i. written as a matrix-vector multiplication

$$T(x) = Ax, A = [T(e_1), ..., T(e_n)]$$

- matrix A는 가장 단순한 기저벡터를 넣었을 때 나온 결과물의 집합

3. Neural networks [link]

a. Affine layer (or frequent layer)

i. fc layer usually involve a bias term

ii. use linear transformation to deal with bias term

ONTO and ONE-TO-ONE

1. ONTO (전사)

a. co-domain (공역) = range (치역)

b. 어떠한 y값에 대해서도 최소한 한 개 이상의 화살표를 받아야함

c. NOT able to be ONTO

i. input dimension < output dimension

- e.g., GAN, decoder

2. ONE-TO-ONE (일대일)

a. 어떠한 y값에 대해서도 화살표를 한 개만 받아야함
b. NOT able to be ONE-TO-ONE
    i. input dimension > output dimension
c. ONE-TO-ONE = linearly independent

3. Neural networks
    a. ONE-TO-ONE
        i. unique people mapped to the same (over weighted, tall and smoking)
        ii. may have information loss
    b. ONTO
        i. (over weighted, tall and smoking) always exist