Linear System & Linear Transformation

Linear Equation

1. Linear equation is written as below:

   $a_1x_1 + a_2x_2 + ... + a_nx_n = b$

   - coefficients: $a_1, ..., a_n$

2. By using vector:

   $a^Tx = b$

Linear System

1. Linear system is a set of linear equations
   a. 주어진 여러 feature를 이용하여 최종 값 b를 잘 예측하는 것이 목표 (= regression)
   b. solution
      i. use vector or matrix
         - matrix A is a collection of the coefficients
      ii. use inverse matrix
         - the solution is uniquely obtained
      iii. use non-invertible matrix
         - the solution is infinite or not exist
2. Identity Matrix
   a. written as I_n
3. Inverse Matrix
   a. written as A^{-1}
   b. not all of the matrices have inverse matrix
      i. i.e., need a special condition to have an inverse matrix

$$AA^{-1} = A^{-1}A = I_n$$

- use determinant of A (= det A) such as $ad - bc$

ii. defined only for **square matrix**

Linear Combinations

1. Linear combination is written as below:

   $c_1v_1 + ... + c_pv_p$

   - called a linear combination of $v_1, ..., v_p$ with weights or coefficients $c_1, ..., c_p$

2. Solution of linear system

   a. by matrix

   $$Ax = b$$

   b. by vector

   $$a_1x_1 + a_2x_2 + a_3x_3 = b$$

   - called a vector equation
   - the solution exists only when b is contained in the span $\{a_1, a_2, a_3\}$

3. Span

   a. Span $\{v_1, ..., v_p\}$ is the set of all linear combinations of $v_1, ..., v_p$

   b. Span with different number of vectors

      i. ~~2~~ **3개의 벡터**를 이용해서 span 되는 **3차원 공간**은 전체가 될 수 있음

      ii. ~~3~~ **4개의 벡터**를 이용해서 span 되는 **4차원 공간**은 전체가 되기에는 살짝 무리가 있음

4. Column combination & Row combination

   a. Column combination (default)

      i. Coefficients are the right matrix

      ii. Left matrix is bases

   $(AX)^T$: bases A, coefficients X

   b. Row combination

      i. Coefficients are the left matrix

    ii. Right matrix is bases

$X^T A^T$: bases X^T, coefficients A^T

5. Sum of Rank-1 Outer Products
    a. Regard the matrix as the collection of vectors
    b. Perform outer product for each vectors from two different matrices
    c. Sum up all the matrices generated by outer product

Linear Independence

1. Linear independence
    a. If no v_j which is contained in the span $\{v_1, v_2, ..., v_{j-1}\}$ for $j = 1, ..., p$ is found, then $\{v_1, ..., v_p\}$ is **linearly independent**
    b. If there is only one solution for $x_1v_1 + x_2v_2 + ... + x_pv_p = 0 = X$, then $\{v_1, ..., v_p\}$ is **linearly independent**
2. Linear dependence
    a. If at least one v_j which is contained in the span $\{v_1, v_2, ..., v_{j-1}\}$ for $j = 1, ..., p$ is found, then $\{v_1, ..., v_p\}$ is **linearly dependent**
    b. If at least one x_i is nonzero, i.e., there is other nontrivial solution for the above equation, then $\{v_1, ..., v_p\}$ is **linearly dependent**

Subspace

1. Subspace H
    a. Subset closed under the linear combination
    b. Similar to span
2. Basis of a subspace (기저벡터)
    a. A set of vectors that ssatisfies both of the following:
        i. fully spans the given subspace H
        ii. linearly independent (i.e., no redundancy)
    b. Non-uniqueness

i. 지금까지는 basis가 주어지고 span을 구했지만 여기서는 span이 먼저 주어지고 basis를 찾아야함

ii. using different basis means using different coefficients

- change of basis
- eigendecomposition

c. Standard basis

$[1\ 0\ 0]^T, [0\ 1\ 0]^T, [0\ 0\ 1]^T$

3. Dimension of subspace

a. The number of basis of a subspace H is called dimension

4. Column space, Rank

a. Column space

i. column space of a matrix A is the subspace spanned by the columns of A

ii. written as Col A

b. Rank

i. rank of a matrix A is the dimension of the column space of A

ii. written as rank A

Linear Transformation

1. Definition

a. domain (정의역): X

b. co-domain (공역): Y

c. image (상): $y = f(x)$

i. preimage (원상): $x = f^{-1}(y)$

d. range (치역): set of all images

e. 화살표는 하나의 x에 대하여 **한번만** 가능

2. Linear transformation

a. Transformation (or mapping) T is linear if:

$$T(cu + dv) = cT(u) + dT(v)$$

- e.g., $y = 3x + 2$ ($c = 3, d = 4$)
- $(1,2)$→$3 * 1 + 4 * 2 = 11$→$3 * 11 + 2 = 35$
- $(1,2)$→$(5,8)$→$3 * 5 + 4 * 8 = 47$
- 위 두 값이 다르기 때문에 선형변환이 아님

b. Vector of linear transformation

i. transform n-dimensional vector to m-dimensional vector

c. Matrix of linear transformation

i. written as a matrix-vector multiplication

$$T(x) = Ax, A = [T(e_1), ..., T(e_n)]$$

- matrix A는 가장 단순한 기저벡터를 넣었을 때 나온 결과물의 집합

3. Neural networks [link]

a. Affine layer (or frequent layer)

i. fc layer usually involve a bias term

ii. use linear transformation to deal with bias term

ONTO and ONE-TO-ONE

1. ONTO (전사)

a. co-domain (공역) = range (치역)

b. 어떠한 y값에 대해서도 최소한 한 개 이상의 화살표를 받아야함

c. NOT able to be ONTO

i. input dimension < output dimension

- e.g., GAN, decoder

2. ONE-TO-ONE (일대일)

a. 어떠한 y값에 대해서도 화살표를 한 개만 받아야함

b. NOT able to be ONE-TO-ONE

i. input dimension > output dimension

c. ONE-TO-ONE = linearly independent

3. Neural networks
    a. ONE-TO-ONE
        i. unique people mapped to the same (over weighted, tall and smoking)
        ii. may have information loss
    b. ONTO
        i. (over weighted, tall and smoking) always exist

Reference

- 부분 공간 (subspace) 과 생성 (span) - [link]
- A Span is Always a Subspace - [link]
- Khan Academy - [link]