Provement of Differential Flatness in Stance Phase of Spring-legged Quadrotors

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1 Differential Flatness in 2D Spring-legged Quadrotors

1.1 equation of dynamics

$$m\ddot{l} = ml\dot{\theta}^2 - k(l - l_0) - mg\cos\theta + f \tag{1a}$$

$$ml^2\ddot{\theta} = mgl\sin\theta - 2ml\dot{l}\dot{\theta} + \tau \tag{1b}$$

1.2 Derivation of Differential Flatness

1.2.1 Flatness output

Choose the state variables in orthogonal coordinates $[x, z]$.

$$x = l\sin\theta \tag{2a}$$

$$z = l\cos\theta \tag{2b}$$

1.2.2 Description of higher derivation of x

$$x = l\sin\theta \tag{3a}$$

$$\dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta \tag{3b}$$

$$\ddot{x} = \ddot{l}\sin\theta + 2\dot{l}\dot{\theta}\cos\theta + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \tag{3c}$$

By multiplying both sides of dynamic equation 1a by $\sin\theta$ and both sides of 1b by $\cos\theta$, we can transform them into the following form:

$$
\begin{aligned}
T1 :& m\ddot{l}\sin\theta = ml\dot{\theta}^2\sin\theta - k(l-l_0)\sin\theta - mgl\sin\theta\cos\theta + f\sin\theta \\
T2 :& ml^2\ddot{\theta}\cos\theta = mg\sin\theta\cos\theta - 2ml\dot{l}\dot{\theta}\cos\theta + \tau\cos\theta
\end{aligned}
$$

Continuing the rearrangement, isolate $T1$ by dividing both sides by m; similarly, isolate $T2$ by dividing both sides by ml:

$$
\begin{aligned}
T1 : \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta &= -\frac{k(l-l_0)\sin\theta}{m} - g\sin\theta\cos\theta + \frac{f\sin\theta}{m} \\
T2 : l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta &= g\sin\theta\cos\theta + \frac{\tau\cos\theta}{ml}
\end{aligned}
$$

Plus $T1$ and $T2$:

$$
\begin{aligned}
T1 + T2 &= \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta \\
&= -\frac{k(l-l_0)\sin\theta}{m} - g\sin\theta\cos\theta + \frac{f\sin\theta}{m} + g\sin\theta\cos\theta + \frac{\tau\cos\theta}{ml} \\
&= -\frac{k(l-l_0)\sin\theta}{m} + \frac{f\sin\theta}{m} + \frac{\tau\cos\theta}{ml}
\end{aligned}
$$

Finally:

$$
\ddot{x} = \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta \tag{4a}
$$

$$
\ddot{x} = -\frac{k(l-l_0)\sin\theta}{m} + \frac{f\sin\theta}{m} + \frac{\tau\cos\theta}{ml} \tag{4b}
$$

1.2.3 Description of higher derivation of z

$$
z = l\cos\theta \tag{5a}
$$

$$
\dot{z} = \dot{l}\cos\theta - l\dot{\theta}\sin\theta \tag{5b}
$$

$$
\ddot{z} = \ddot{l}\cos\theta - 2\dot{l}\dot{\theta}\sin\theta - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \tag{5c}
$$

By multiplying both sides of dynamic equation 1a by $\cos\theta$ and both sides of 1b by $\sin\theta$, we can transform them into the following form:

$$
\begin{aligned}
T3 :& m\ddot{l}\cos\theta = ml\dot{\theta}^2\cos\theta - k(l-l_0)\cos\theta - mg\cos^2\theta + f\cos\theta \\
T4 :& ml^2\ddot{\theta}\sin\theta = mgl\sin^2\theta - 2ml\dot{l}\dot{\theta}\sin\theta + \tau\sin\theta
\end{aligned}
$$

Continuing the rearrangement, isolate $T3$ by dividing both sides by m; similarly, isolate $T4$ by dividing both sides by ml:

$$T3: \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta = -\frac{k(l-l_0)\cos\theta}{m} - g\cos^2\theta + \frac{f\cos\theta}{m}$$
$$T4: l\ddot{\theta}\sin\theta + 2\dot{l}\dot{\theta}\sin\theta = g\sin^2\theta + \frac{\tau\sin\theta}{ml}$$

Minus $T3$ by $T4$:

$$\begin{aligned}
T3 - T4 &= \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta - 2\dot{l}\dot{\theta}\sin\theta \\
&= -\frac{k(l-l_0)\cos\theta}{m} - g\cos^2\theta + \frac{f\cos\theta}{m} - g\sin^2\theta - \frac{\tau\sin\theta}{ml} \\
&= -\frac{k(l-l_0)\cos\theta}{m} - g + \frac{f\cos\theta}{m} - \frac{\tau\sin\theta}{ml}
\end{aligned}$$

Finally:

$$\ddot{z} = \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta - 2\dot{l}\dot{\theta}\sin\theta \tag{6a}$$
$$\ddot{z} = -\frac{k(l-l_0)\cos\theta}{m} - g + \frac{f\cos\theta}{m} - \frac{\tau\sin\theta}{ml} \tag{6b}$$

1.2.4 State variables description by flatness output

The polar coordinates can be formulated as follows:

$$\tan\theta = \frac{l\sin\theta}{l\cos\theta} = \frac{x}{z}$$
$$\theta = \arctan\frac{x}{z}$$

By multiplying equation (3b) by $z = l\cos\theta$, and equation (5b) by $x = l\sin\theta$, we can obtain:

$$\dot{x}z = l\dot{l}\sin\theta\cos\theta + l^2\dot{\theta}\cos^2\theta$$
$$\dot{z}x = l\dot{l}\sin\theta\cos\theta - l^2\dot{\theta}\sin^2\theta$$

By subtracting the above two equations, we obtain:

$$l^2\dot{\theta} = \dot{x}z - \dot{z}x$$
$$\dot{\theta} = \frac{\dot{x}z - \dot{z}x}{l^2} = \frac{\dot{x}z - \dot{z}x}{x^2+z^2}$$

Combining the above results, we obtain the expressions for θ and its first derivative in terms of the flat outputs:

$$\theta = \arctan\frac{x}{z}$$
$$\dot{\theta} = \frac{\dot{x}z - \dot{z}x}{x^2+z^2}$$

Further deriving the expressions for l and its first derivative, we obtain:

$$l = x^2 + z^2$$

By multiplying equation (3b) by $x = l\sin\theta$, and equation (5b) by $z = l\cos\theta$, we can obtain:

$$\begin{aligned}
\dot{x}x &= l\dot{l}\sin^2\theta + l^2\dot{\theta}\sin\theta\cos\theta \\
\dot{z}z &= l\dot{l}\cos^2\theta - l^2\dot{\theta}\sin\theta\cos\theta
\end{aligned}$$

By summing the above two equations, we obtain:

$$\begin{aligned}
l\dot{l} &= \dot{x}x + \dot{z}z \\
\dot{l} &= \frac{\dot{x}x + \dot{z}z}{l} = \frac{\dot{x}x + \dot{z}z}{\sqrt{x^2 + z^2}}
\end{aligned}$$

Finally, the expressions for $[l, \theta]$ and their first derivative in terms of the flat outputs:

$$\theta = \arctan\frac{x}{z} \tag{7a}$$

$$\dot{\theta} = \frac{z\dot{x} - x\dot{z}}{x^2 + z^2} \tag{7b}$$

$$l = x^2 + z^2 \tag{7c}$$

$$\dot{l} = \frac{x\dot{x} + z\dot{z}}{\sqrt{x^2 + z^2}} \tag{7d}$$

1.2.5 Control variables description by flatness output

By multiplying equation (4b) by $x = l\sin\theta$, and equation (6b) by $z = l\cos\theta$, we can obtain:

$$\begin{aligned}
x\ddot{x} &= -\frac{kl(l - l_0)\sin^2\theta}{m} + \frac{lf\sin^2\theta}{m} + \frac{\tau\sin\theta\cos\theta}{ml} \\
z\ddot{z} &= -\frac{kl(l - l_0)\cos^2\theta}{m} - gz + \frac{lf\cos^2\theta}{m} - \frac{\tau\sin\theta\cos\theta}{ml}
\end{aligned}$$

By summing the above two equations:

$$\begin{aligned}
x\ddot{x} + z\ddot{z} &= -\frac{kl(l - l_0)}{m} - gz + \frac{lf}{m} \\
f &= \frac{mx\ddot{x} + mz\ddot{z} + kl(l - l_0) + mgz}{l} \\
&= k(\sqrt{x^2 + z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{l}
\end{aligned}$$

By multiplying equation (4b) by $z = l\cos\theta$, and equation (6b) by $x = l\sin\theta$, we can obtain:

$$\begin{aligned}
z\ddot{x} &= -\frac{kl(l - l_0)\sin\theta\cos\theta}{m} + \frac{lf\sin\theta\cos\theta}{m} + \frac{\tau\cos^2\theta}{m} \\
x\ddot{z} &= -\frac{kl(l - l_0)\sin\theta\cos\theta}{m} - gx + \frac{lf\sin\theta\cos\theta}{m} - \frac{\tau\sin^2\theta}{m}
\end{aligned}$$

By subtracting the above two equations:

$$z\ddot{x} - x\ddot{z} = gx + \frac{\tau}{m}$$
$$\tau = mz\ddot{x} - mx\ddot{z} - mgx$$

Finally, the expressions for control variables $[f, \tau]$ in terms of the flat outputs:

$$f = k(\sqrt{x^2+z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{\sqrt{x^2+z^2}} \tag{8a}$$
$$\tau = mz\ddot{x} - mx\ddot{z} - mgx \tag{8b}$$

2 Differential Flatness in Sagittal Spring-legged Quadrotors

The dynamic model is shown in the following equations.

$$m(\ddot{l} - l\dot{\theta}^2 - l\dot{\varphi}^2\sin^2\theta) = k(l_0 - l) + f - mg\cos\theta, \tag{9a}$$
$$m(l^2\ddot{\theta} + 2l\dot{l}\dot{\theta} - l^2\dot{\varphi}^2\sin\theta\cos\theta) = \tau_\theta + mgl\sin\theta, \tag{9b}$$
$$m(l^2\ddot{\varphi}\sin^2\theta + 2l^2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta + 2l\dot{l}\dot{\varphi}\sin^2\theta) = \tau_\varphi, \tag{9c}$$
$$M_{zz}\ddot{\psi} = \tau_\psi. \tag{9d}$$

By expressing the variables $[x, y, z]$ in the orthogonal coordinate system and their derivatives in terms of the spherical coordinates $[l, \varphi, \theta]$, we first write the expressions for x and its first/second derivatives as follows:

$$\begin{aligned}
x &= l\sin\theta\cos\varphi,\\
\dot{x} &= \dot{l}\sin\theta\cos\varphi + l\dot{\theta}\cos\theta\cos\varphi - l\sin\theta\sin\varphi\dot{\varphi},\\
\ddot{x} &= \ddot{l}\sin\theta\cos\varphi + 2\dot{l}\dot{\theta}\cos\theta\cos\varphi - 2\dot{l}\sin\theta\sin\varphi\dot{\varphi} + l\ddot{\theta}\cos\theta\cos\varphi\\
&\quad - l\dot{\theta}^2\sin\theta\cos\varphi - 2l\dot{\theta}\sin\theta\sin\varphi\dot{\varphi} - l\cos\theta\sin\varphi\ddot{\varphi} - l\sin\theta\cos\varphi\dot{\varphi}^2.
\end{aligned} \tag{10}$$

And the expressions for y and its first/second derivatives as follows:

$$\begin{aligned}
y &= l\sin\theta\sin\varphi\\
\dot{y} &= \dot{l}\sin\theta\sin\varphi + l\dot{\theta}\cos\theta\sin\varphi + l\sin\theta\cos\varphi\dot{\varphi}\\
\ddot{y} &= \ddot{l}\sin\theta\sin\varphi + 2\dot{l}\dot{\theta}\cos\theta\sin\varphi + 2\dot{l}\sin\theta\cos\varphi\dot{\varphi} + l\ddot{\theta}\cos\theta\sin\varphi\\
&\quad - l\dot{\theta}^2\sin\theta\sin\varphi + 2l\dot{\theta}\sin\theta\cos\varphi\dot{\varphi} + l\sin\theta\cos\varphi\ddot{\varphi} - l\sin\theta\sin\varphi\dot{\varphi}^2
\end{aligned} \tag{11}$$

And the expressions for z and its first/second derivatives as follows:

$$
\begin{aligned}
z &= l\cos\theta \\
\dot{z} &= \dot{l}\cos\theta - l\dot{\theta}\sin\theta \\
\ddot{z} &= \ddot{l}\cos\theta - 2\dot{l}\dot{\theta}\sin\theta - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta
\end{aligned}
\tag{12}
$$

2.1 State variables description by flatness output

By employing the transformation formulas between spherical and orthogonal coordinate systems, one can readily derive the expressions for the spherical coordinate state variables in terms of the orthogonal coordinate variables.

$$l = \sqrt{x^2 + y^2 + z^2} \tag{13a}$$

$$\varphi = \arctan\frac{y}{x} \tag{13b}$$

$$\theta = \arctan\frac{\sqrt{x^2 + y^2}}{z} \tag{13c}$$

$$\tag{13d}$$

2.2 First order of state variable by flatness output

By recognizing that the three state variables in spherical coordinates are multivariate functions composed of $[x, y, z]$, and acknowledging the complexity in calculating their first-order derivatives, we propose an alternative approach to construct these derivative expressions. By constructing the following expressions based on the $[x, y, z]$ variables and their derivatives(10,11,12):

$$
\begin{aligned}
x\dot{x} &= l\dot{l}\sin^2\theta\cos^2\varphi + l^2\dot{\theta}\sin\theta\cos\theta\cos^2\varphi - l^2\dot{\varphi}\sin^2\theta\sin\varphi\cos\varphi \\
y\dot{x} &= l\dot{l}\sin^2\theta\sin\varphi\cos\varphi + l^2\dot{\theta}\sin\theta\cos\theta\sin\varphi\cos\varphi - l^2\dot{\varphi}\sin^2\theta\sin^2\varphi \\
z\dot{x} &= l\dot{l}\sin\theta\cos\theta\cos\varphi + l^2\dot{\theta}\cos^2\theta\cos\varphi - l^2\dot{\varphi}\sin\theta\cos\theta\sin\varphi \\
x\dot{y} &= l\dot{l}\sin^2\theta\sin\varphi\cos\varphi + l^2\dot{\theta}\sin\theta\cos\theta\sin\varphi\cos\varphi + l^2\dot{\varphi}\sin^2\theta\cos^2\varphi \\
y\dot{y} &= l\dot{l}\sin^2\theta\sin^2\varphi + l^2\dot{\theta}\sin\theta\cos\theta\sin^2\varphi + l^2\dot{\varphi}\sin^2\theta\sin\varphi\cos\varphi \\
z\dot{y} &= l\dot{l}\sin\theta\cos\theta\sin\varphi + l^2\dot{\theta}\cos^2\theta\sin\varphi + l^2\dot{\varphi}\sin\theta\cos\theta\cos\varphi \\
x\dot{z} &= l\dot{l}\sin\theta\cos\theta\cos\varphi + l^2\dot{\theta}\sin^2\theta\cos\varphi \\
y\dot{z} &= l\dot{l}\sin\theta\cos\theta\sin\varphi - l^2\dot{\theta}\sin^2\theta\cos\varphi \\
z\dot{z} &= l\dot{l}\cos^2\theta - l^2\dot{\theta}\sin\theta\cos\theta
\end{aligned}
$$

Through these composite forms, the first-order derivatives of the $[l, \theta, \varphi]$ variables can be further derived as follows:

$$\dot{l} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\dot{\theta} = \frac{z}{x^2 + y^2 + z^2} \cdot \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} - \dot{z} \cdot \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2}$$

$$\dot{\varphi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

2.3 Control variables description by flatness output

By multiplying equation (9a) by$\frac{\sin\theta\cos\varphi}{m}$ equation (9b) by $\frac{\cos\theta\cos\varphi}{ml}$, and equation (9c) by $\frac{\sin\varphi}{ml\sin\theta}$ can obtain:

$$\ddot{l}\sin\theta\cos\varphi - l\dot{\theta}^2\sin\theta\cos\varphi - l\dot{\varphi}^2\sin^3\theta\cos\varphi = \frac{f - k(l - l_0)}{m}\sin\theta\cos\varphi - g\sin\theta\cos\theta\cos\varphi \tag{14a}$$

$$l\ddot{\theta}\cos\theta\cos\varphi + 2\dot{l}\dot{\theta}\cos\theta\cos\varphi - l\dot{\varphi}^2\sin\theta\cos^2\theta\cos\varphi = \frac{\tau_\theta}{ml}\cos\theta\cos\varphi + g\sin\theta\cos\theta\cos\varphi \tag{14b}$$

$$l\ddot{\varphi}\sin\theta\sin\varphi + 2l\dot{\theta}\dot{\varphi}\cos\theta\sin\varphi + 2\dot{l}\dot{\varphi}\sin\theta\sin\varphi = \frac{\tau_\varphi\sin\varphi}{ml\sin\theta} \tag{14c}$$

By adding equation (14a) to (14b) and then subtracting (14c), we obtain:

$$\ddot{x} = \frac{f - k(l - l_0)}{m}\sin\theta\cos\varphi + \frac{\tau_\theta}{ml}\cos\theta\cos\varphi - \frac{\tau_\varphi\sin\varphi}{ml\sin\theta} \tag{15}$$

By multiplying equation (9a) by$\frac{\sin\theta\sin\varphi}{m}$ equation (9b) by $\frac{\cos\theta\sin\varphi}{ml}$, and equation (9c) by $\frac{\cos\varphi}{ml\sin\theta}$ can obtain:

$$\ddot{l}\sin\theta\sin\varphi - l\dot{\theta}^2\sin\theta\sin\varphi - l\dot{\varphi}^2\sin^3\theta\sin\varphi = \frac{f - k(l - l_0)}{m}\sin\theta\sin\varphi - g\sin\theta\cos\theta\sin\varphi \tag{16a}$$

$$l\ddot{\theta}\cos\theta\sin\varphi + 2\dot{l}\dot{\theta}\cos\theta\sin\varphi - l\dot{\varphi}^2\sin\theta\cos^2\theta\sin\varphi = \frac{\tau_\theta}{ml}\cos\theta\sin\varphi + g\sin\theta\cos\theta\sin\varphi \tag{16b}$$

$$l\ddot{\varphi}\sin\theta\cos\varphi + 2l\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi + 2\dot{l}\dot{\varphi}\sin\theta\cos\varphi = \frac{\tau_\varphi\cos\varphi}{ml\sin\theta} \tag{16c}$$

By adding equation (16a), (16b) and (16c), we obtain:

$$\ddot{y} = \frac{f - k(l - l_0)}{m}\sin\theta\sin\varphi + \frac{\tau_\theta}{ml}\cos\theta\sin\varphi + \frac{\tau_\varphi\cos\varphi}{ml\sin\theta} \tag{17}$$

By multiplying equation (9a) by$\frac{\cos\theta}{m}$ equation (9b) by $\frac{\sin\theta}{ml}$ can obtain:

$$\ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\dot{\varphi}^2\sin^2\theta\cos\theta = \frac{f-k(l-l_0)}{m}\cos\theta - g\cos^2\theta \tag{18a}$$

$$l\ddot{\theta}\sin\theta + 2\dot{l}\dot{\theta}\sin\theta - l\dot{\varphi}^2\sin^2\theta\cos\theta = \frac{\tau_\theta}{ml}\sin\theta + g\sin^2\theta \tag{18b}$$

By minusing equation (18a) by (18b), we can obtain:

$$\ddot{z} = \frac{f-k(l-l_0)}{m}\cos\theta + \frac{\tau_\theta}{ml}\sin\theta - g \tag{19}$$

Through the expressions of $[x, y, z]$ and their second-order derivatives, we ultimately construct the formulation of the control variables as follows:

$$f = m\frac{x\ddot{x} + y\ddot{y} + z\ddot{z} + gz}{l} + k(\sqrt{x^2+y^2+z^2} - l_0) \tag{20a}$$

$$\tau_\theta = m\frac{x\ddot{x}z + y\ddot{y}z - (x^2+y^2)\ddot{z} - g(x^2+y^2)}{\sqrt{x^2+y^2}} \tag{20b}$$

$$\tau_\varphi = -my\ddot{x} + mx\ddot{y} \tag{20c}$$