Provement of Differential Flatness in Stance Phase of Spring-legged Quadrotors

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1 Differential Flatness in Planar Spring-legged Quadrotors

1.1 equation of dynamics

$$m\ddot{l} = ml\dot{\theta}^2 - k(l - l_0) - mg\cos\theta + f \tag{1a}$$

$$ml^2\ddot{\theta} = mgl\sin\theta - 2ml\dot{l}\dot{\theta} + \tau \tag{1b}$$

1.2 Derivation of Differential Flatness

1.2.1 Flatness output

Choose the state variables in orthogonal coordinates $[x, z]$.

$$x = l\sin\theta \tag{2a}$$

$$z = l\cos\theta \tag{2b}$$

1.2.2 Description of higher derivation of x

$$x = l\sin\theta \tag{3a}$$

$$\dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta \tag{3b}$$

$$\ddot{x} = \ddot{l}\sin\theta + 2\dot{l}\dot{\theta}\cos\theta + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \tag{3c}$$

By multiplying both sides of dynamic equation 1a by $\sin\theta$ and both sides of 1b by $\cos\theta$, we can transform them into the following form:

$$\begin{aligned}
T1 :& m\ddot{l}\sin\theta = ml\dot{\theta}^2\sin\theta - k(l-l_0)\sin\theta - mgl\sin\theta\cos\theta + f\sin\theta\\
T2 :& ml^2\ddot{\theta}\cos\theta = mg\sin\theta\cos\theta - 2ml\dot{l}\dot{\theta}\cos\theta + \tau\cos\theta
\end{aligned}$$

Continuing the rearrangement, isolate $T1$ by dividing both sides by m; similarly, isolate $T2$ by dividing both sides by ml:

$$\begin{aligned}
T1 : \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta &= -\frac{k(l-l_0)\sin\theta}{m} - g\sin\theta\cos\theta + \frac{f\sin\theta}{m}\\
T2 : l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta &= g\sin\theta\cos\theta + \frac{\tau\cos\theta}{ml}
\end{aligned}$$

Plus $T1$ and $T2$:

$$\begin{aligned}
T1 + T2 &= \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta\\
&= -\frac{k(l-l_0)\sin\theta}{m} - g\sin\theta\cos\theta + \frac{f\sin\theta}{m} + g\sin\theta\cos\theta + \frac{\tau\cos\theta}{ml}\\
&= -\frac{k(l-l_0)\sin\theta}{m} + \frac{f\sin\theta}{m} + \frac{\tau\cos\theta}{ml}
\end{aligned}$$

Finally:

$$\ddot{x} = \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta \tag{4a}$$

$$\ddot{x} = -\frac{k(l-l_0)\sin\theta}{m} + \frac{f\sin\theta}{m} + \frac{\tau\cos\theta}{ml} \tag{4b}$$

1.2.3 Description of higher derivation of z

$$z = l\cos\theta \tag{5a}$$

$$\dot{z} = \dot{l}\cos\theta - l\dot{\theta}\sin\theta \tag{5b}$$

$$\ddot{z} = \ddot{l}\cos\theta - 2\dot{l}\dot{\theta}\sin\theta - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \tag{5c}$$

By multiplying both sides of dynamic equation 1a by $\cos\theta$ and both sides of 1b by $\sin\theta$, we can transform them into the following form:

$$\begin{aligned}
T3 :& m\ddot{l}\cos\theta = ml\dot{\theta}^2\cos\theta - k(l-l_0)\cos\theta - mg\cos^2\theta + f\cos\theta\\
T4 :& ml^2\ddot{\theta}\sin\theta = mgl\sin^2\theta - 2ml\dot{l}\dot{\theta}\sin\theta + \tau\sin\theta
\end{aligned}$$

Continuing the rearrangement, isolate $T3$ by dividing both sides by m; similarly, isolate $T4$ by dividing both sides by ml:

$$\begin{aligned}
T3: \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta &= -\frac{k(l-l_0)\cos\theta}{m} - g\cos^2\theta + \frac{f\cos\theta}{m} \\
T4: l\ddot{\theta}\sin\theta + 2\dot{l}\dot{\theta}\sin\theta &= g\sin^2\theta + \frac{\tau\sin\theta}{ml}
\end{aligned}$$

Minus $T3$ by $T4$:

$$\begin{aligned}
T3 - T4 &= \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta - 2\dot{l}\dot{\theta}\sin\theta \\
&= -\frac{k(l-l_0)\cos\theta}{m} - g\cos^2\theta + \frac{f\cos\theta}{m} - g\sin^2\theta - \frac{\tau\sin\theta}{ml} \\
&= -\frac{k(l-l_0)\cos\theta}{m} - g + \frac{f\cos\theta}{m} - \frac{\tau\sin\theta}{ml}
\end{aligned}$$

Finally:

$$\ddot{z} = \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta - 2\dot{l}\dot{\theta}\sin\theta \tag{6a}$$

$$\ddot{z} = -\frac{k(l-l_0)\cos\theta}{m} - g + \frac{f\cos\theta}{m} - \frac{\tau\sin\theta}{ml} \tag{6b}$$

1.2.4 State variables description by flatness output

The polar coordinates can be formulated as follows:

$$\begin{aligned}
\tan\theta &= \frac{l\sin\theta}{l\cos\theta} = \frac{x}{z} \\
\theta &= \arctan\frac{x}{z}
\end{aligned}$$

By multiplying equation (3b) by $z = l\cos\theta$, and equation (5b) by $x = l\sin\theta$, we can obtain:

$$\begin{aligned}
\dot{x}z &= l\dot{l}\sin\theta\cos\theta + l^2\dot{\theta}\cos^2\theta \\
\dot{z}x &= l\dot{l}\sin\theta\cos\theta - l^2\dot{\theta}\sin^2\theta
\end{aligned}$$

By subtracting the above two equations, we obtain:

$$\begin{aligned}
l^2\dot{\theta} &= \dot{x}z - \dot{z}x \\
\dot{\theta} &= \frac{\dot{x}z - \dot{z}x}{l^2} = \frac{\dot{x}z - \dot{z}x}{x^2 + z^2}
\end{aligned}$$

Combining the above results, we obtain the expressions for θ and its first derivative in terms of the flat outputs:

$$\begin{aligned}
\theta &= \arctan\frac{x}{z} \\
\dot{\theta} &= \frac{\dot{x}z - \dot{z}x}{x^2 + z^2}
\end{aligned}$$

Further deriving the expressions for l and its first derivative, we obtain:

$$l = x^2 + z^2$$

By multiplying equation (3b) by $x = l\sin\theta$, and equation (5b) by $z = l\cos\theta$, we can obtain:

$$\begin{aligned}\dot{x}x &= l\dot{l}\sin^2\theta + l^2\dot{\theta}\sin\theta\cos\theta \\ \dot{z}z &= l\dot{l}\cos^2\theta - l^2\dot{\theta}\sin\theta\cos\theta\end{aligned}$$

By summing the above two equations, we obtain:

$$\begin{aligned}l\dot{l} &= \dot{x}x + \dot{z}z \\ \dot{l} &= \frac{\dot{x}x + \dot{z}z}{l} = \frac{\dot{x}x + \dot{z}z}{\sqrt{x^2 + z^2}}\end{aligned}$$

Finally, the expressions for $[l, \theta]$ and their first derivative in terms of the flat outputs:

$$\theta = \arctan\frac{x}{z} \tag{7a}$$

$$\dot{\theta} = \frac{z\dot{x} - x\dot{z}}{x^2 + z^2} \tag{7b}$$

$$l = x^2 + z^2 \tag{7c}$$

$$\dot{l} = \frac{x\dot{x} + z\dot{z}}{\sqrt{x^2 + z^2}} \tag{7d}$$

1.2.5 Control variables description by flatness output

By multiplying equation (4b) by $x = l\sin\theta$, and equation (6b) by $z = l\cos\theta$, we can obtain:

$$\begin{aligned}x\ddot{x} &= -\frac{kl(l-l_0)\sin^2\theta}{m} + \frac{lf\sin^2\theta}{m} + \frac{\tau\sin\theta\cos\theta}{ml} \\ z\ddot{z} &= -\frac{kl(l-l_0)\cos^2\theta}{m} - gz + \frac{lf\cos^2\theta}{m} - \frac{\tau\sin\theta\cos\theta}{ml}\end{aligned}$$

By summing the above two equations:

$$\begin{aligned}x\ddot{x} + z\ddot{z} &= -\frac{kl(l-l_0)}{m} - gz + \frac{lf}{m} \\ f &= \frac{mx\ddot{x} + mz\ddot{z} + kl(l-l_0) + mgz}{l} \\ &= k(\sqrt{x^2+z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{l}\end{aligned}$$

By multiplying equation (4b) by $z = l\cos\theta$, and equation (6b) by $x = l\sin\theta$, we can obtain:

$$\begin{aligned}z\ddot{x} &= -\frac{kl(l-l_0)\sin\theta\cos\theta}{m} + \frac{lf\sin\theta\cos\theta}{m} + \frac{\tau\cos^2\theta}{m} \\ x\ddot{z} &= -\frac{kl(l-l_0)\sin\theta\cos\theta}{m} - gx + \frac{lf\sin\theta\cos\theta}{m} - \frac{\tau\sin^2\theta}{m}\end{aligned}$$

By subtracting the above two equations:

$$
\begin{aligned}
z\ddot{x} - x\ddot{z} &= gx + \frac{\tau}{m} \\
\tau &= mz\ddot{x} - mx\ddot{z} - mgx
\end{aligned}
$$

Finally, the expressions for control variables $[f, \tau]$ in terms of the flat outputs:

$$
f = k(\sqrt{x^2 + z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{\sqrt{x^2 + z^2}} \tag{8a}
$$

$$
\tau = mz\ddot{x} - mx\ddot{z} - mgx \tag{8b}
$$

2 Differential Flatness in Sagittal Spring-legged Quadrotors

The dynamic model is shown in the following equations.

$$
m(\ddot{l} - l\dot{\theta}^2 - l\dot{\varphi}^2 \sin^2\theta) = k(l_0 - l) + f - mg\cos\theta, \tag{9a}
$$

$$
m(l^2\ddot{\theta} + 2l\dot{l}\dot{\theta} - l^2\dot{\varphi}^2 \sin\theta\cos\theta) = \tau_\theta + mgl\sin\theta, \tag{9b}
$$

$$
m(l^2\ddot{\varphi}\sin^2\theta + 2l^2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta + 2l\dot{l}\dot{\varphi}\sin^2\theta) = \tau_\varphi, \tag{9c}
$$

$$
M_{zz}\ddot{\psi} = \tau_\psi. \tag{9d}
$$

By expressing the variables $[x, y, z]$ in the orthogonal coordinate system and their derivatives in terms of the spherical coordinates $[l, \varphi, \theta]$, we first write the expressions for x and its first/second derivatives as follows:

$$
\begin{aligned}
x &= l\sin\theta\cos\varphi \\
\frac{d}{dt}(x) &= \dot{l}\sin\theta\cos\varphi + l\dot{\theta}\cos\theta\cos\varphi - l\sin\theta\sin\varphi\dot{\varphi} \\
\frac{d^2}{dt^2}(x) &= \ddot{l}\sin\theta\cos\varphi + 2\dot{l}\dot{\theta}\cos\theta\cos\varphi - 2\dot{l}\sin\theta\sin\varphi\dot{\varphi} + l\ddot{\theta}\cos\theta\cos\varphi \\
&\quad - l\dot{\theta}^2\sin\theta\cos\varphi - 2l\dot{\theta}\sin\theta\sin\varphi\dot{\varphi} \; - l\cos\theta\sin\varphi\ddot{\varphi} - l\sin\theta\cos\varphi\dot{\varphi}^2
\end{aligned}
$$

And the expressions for y and its first/second derivatives as follows:

$$\begin{aligned}
y &= l\sin\theta\sin\varphi \\
\frac{d}{dt}(y) &= \dot{l}\sin\theta\sin\varphi + l\dot{\theta}\cos\theta\sin\varphi + l\sin\theta\cos\varphi\dot{\varphi} \\
\frac{d^2}{dt^2}(y) &= \ddot{l}\sin\theta\sin\varphi + 2\dot{l}\dot{\theta}\cos\theta\sin\varphi + 2\dot{l}\sin\theta\cos\varphi\dot{\varphi} + l\ddot{\theta}\cos\theta\sin\varphi \\
&\quad - l\dot{\theta}^2\sin\theta\sin\varphi + 2l\dot{\theta}\sin\theta\cos\varphi\dot{\varphi} + l\sin\theta\cos\varphi\ddot{\varphi} - l\sin\theta\sin\varphi\dot{\varphi}^2
\end{aligned}$$

And the expressions for z and its first/second derivatives as follows:

$$\begin{aligned}
z &= l\cos\theta \\
\frac{d}{dt}(z) &= \dot{l}\cos\theta - l\dot{\theta}\sin\theta \\
\frac{d^2}{dt^2}(z) &= \ddot{l}\cos\theta - 2\dot{l}\dot{\theta}\sin\theta - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta
\end{aligned}$$

By multiplying equation (9a) by$\frac{\sin\theta\cos\varphi}{m}$ equation (9b) by $\frac{\cos\theta\cos\varphi}{ml}$, and equation (9c) by $\frac{\sin\varphi}{ml\sin\theta}$ can obtain:

$$f = k(\sqrt{x^2+z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{\sqrt{x^2+z^2}} \tag{10a}$$

$$\tau = mz\ddot{x} - mx\ddot{z} - mgx \tag{10b}$$