代数式碰撞模型

2023.5.30

1 2d hopper

1.1 动力学方程

$$
\begin{aligned}
m\ddot{l} &= ml\dot{\theta}^2 - k(l - l_0) - mg\cos\theta + f \qquad (1a) \\
ml^2\ddot{\theta} &= mgl\sin\theta - 2ml\dot{l}\dot{\theta} + \tau \qquad (1b)
\end{aligned}
$$

如果考虑弹簧的阻尼系数，那么动力学方程如下：

$$
\begin{aligned}
m\ddot{l} &= ml\dot{\theta}^2 - k(l - l_0) - c\dot{l} - mg\cos\theta + f \\
ml^2\ddot{\theta} &= mgl\sin\theta - 2ml\dot{l}\dot{\theta} + \tau
\end{aligned}
$$

如果简化系统，忽视重力以及驱动器的作用：

$$
\begin{aligned}
m\ddot{l} &= ml\dot{\theta}^2 - k(l - l_0) \\
ml^2\ddot{\theta} &= -2ml\dot{l}\dot{\theta}
\end{aligned}
$$

$$
\begin{aligned}
m\ddot{l} - ml\dot{\theta}^2 + k(l - l_0) &= 0 \\
\frac{d}{dt}(ml^2\dot{\theta}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
m\ddot{l} + k(l - l_0) &= 0 \\
\frac{d}{dt}(ml^2\dot{\theta}) &= 0
\end{aligned}
$$

1.2 微分平坦推导

1.2.1 平坦输出

取直角坐标系中的 x 和 z 分别作为平坦输出，且有：

$$x = l\sin\theta \tag{2a}$$
$$z = l\cos\theta \tag{2b}$$

1.2.2 平坦输出 x 多阶导数

相应的存在 x 和 z 的多阶导数：

$$x = l\sin\theta \tag{3a}$$
$$\dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta \tag{3b}$$
$$\ddot{x} = \ddot{l}\sin\theta + 2\dot{l}\dot{\theta}\cos\theta + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \tag{3c}$$

注意到对动力学方程 1a两边乘上 $\sin\theta$，对 1b两边乘上 $\cos\theta$ 改写有：

$$\begin{aligned}
T1 :& m\ddot{l}\sin\theta = ml\dot{\theta}^2\sin\theta - k(l-l_0)\sin\theta - mgl\sin\theta\cos\theta + f\sin\theta \\
T2 :& ml^2\ddot{\theta}\cos\theta = mg\sin\theta\cos\theta - 2ml\dot{l}\dot{\theta}\cos\theta + \tau\cos\theta
\end{aligned}$$

继续改写，$T1$ 移项，除以 m；$T2$ 移项，除以 ml：

$$\begin{aligned}
T1 : \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta &= -\frac{k(l-l_0)\sin\theta}{m} - g\sin\theta\cos\theta + \frac{f\sin\theta}{m} \\
T2 : l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta &= g\sin\theta\cos\theta + \frac{\tau\cos\theta}{ml}
\end{aligned}$$

$T1$ 加上 $T2$ 项，得到

$$\begin{aligned}
T1 + T2 &= \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta \\
&= -\frac{k(l-l_0)\sin\theta}{m} - g\sin\theta\cos\theta + \frac{f\sin\theta}{m} + g\sin\theta\cos\theta + \frac{\tau\cos\theta}{ml} \\
&= -\frac{k(l-l_0)\sin\theta}{m} + \frac{f\sin\theta}{m} + \frac{\tau\cos\theta}{ml}
\end{aligned}$$

因此还有

$$\ddot{x} = \ddot{l}\sin\theta - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta + 2\dot{l}\dot{\theta}\cos\theta \tag{4a}$$
$$\ddot{x} = -\frac{k(l-l_0)\sin\theta}{m} + \frac{f\sin\theta}{m} + \frac{\tau\cos\theta}{ml} \tag{4b}$$

1.2.3 平坦输出 z 多阶导数

同理继续推导 z 的多阶导数：

$$z = l\cos\theta \tag{5a}$$
$$\dot{z} = \dot{l}\cos\theta - l\dot{\theta}\sin\theta \tag{5b}$$
$$\ddot{z} = \ddot{l}\cos\theta - 2\dot{l}\dot{\theta}\sin\theta - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \tag{5c}$$

对动力学方程 1a两边乘上 $\cos\theta$，对 1b两边乘上 $\sin\theta$ 改写有：

$$\begin{aligned}
T3 :& m\ddot{l}\cos\theta = ml\dot{\theta}^2\cos\theta - k(l-l_0)\cos\theta - mg\cos^2\theta + f\cos\theta \\
T4 :& ml^2\ddot{\theta}\sin\theta = mgl\sin^2\theta - 2ml\dot{l}\dot{\theta}\sin\theta + \tau\sin\theta
\end{aligned}$$

继续改写，$T3$ 移项，除以 m；$T4$ 移项，除以 ml：

$$\begin{aligned}
T3 : \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta &= -\frac{k(l-l_0)\cos\theta}{m} - g\cos^2\theta + \frac{f\cos\theta}{m} \\
T4 : l\ddot{\theta}\sin\theta + 2\dot{l}\dot{\theta}\sin\theta &= g\sin^2\theta + \frac{\tau\sin\theta}{ml}
\end{aligned}$$

$T3$ 减去 $T4$ 项，得到

$$\begin{aligned}
T3 - T4 &= \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta - 2\dot{l}\dot{\theta}\sin\theta \\
&= -\frac{k(l-l_0)\cos\theta}{m} - g\cos^2\theta + \frac{f\cos\theta}{m} - g\sin^2\theta - \frac{\tau\sin\theta}{ml} \\
&= -\frac{k(l-l_0)\cos\theta}{m} - g + \frac{f\cos\theta}{m} - \frac{\tau\sin\theta}{ml}
\end{aligned}$$

因此还有

$$\ddot{z} = \ddot{l}\cos\theta - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta - 2\dot{l}\dot{\theta}\sin\theta \tag{6a}$$
$$\ddot{z} = -\frac{k(l-l_0)\cos\theta}{m} - g + \frac{f\cos\theta}{m} - \frac{\tau\sin\theta}{ml} \tag{6b}$$

1.2.4 平坦输出表示状态变量

首先是对 θ 及其一阶导的表示：

$$\begin{aligned}
\tan\theta &= \frac{l\sin\theta}{l\cos\theta} = \frac{x}{z} \\
\theta &= \arctan\frac{x}{z}
\end{aligned}$$

对式 (3b) 乘以 $z = l\cos\theta$，对式 (5b) 乘 $x = l\sin\theta$：

$$\begin{aligned}
\dot{x}z &= l\dot{l}\sin\theta\cos\theta + l^2\dot{\theta}\cos^2\theta \\
\dot{z}x &= l\dot{l}\sin\theta\cos\theta - l^2\dot{\theta}\sin^2\theta
\end{aligned}$$

以上两式相减则有那么则有:

$$
\begin{aligned}
l^2\dot{\theta} &= \dot{x}z - \dot{z}x \\
\dot{\theta} &= \frac{\dot{x}z - \dot{z}x}{l^2} = \frac{\dot{x}z - \dot{z}x}{x^2 + z^2}
\end{aligned}
$$

综上有 θ 及其一阶导数关于平坦输出的表示:

$$
\begin{aligned}
\theta &= \arctan\frac{x}{z} \\
\dot{\theta} &= \frac{\dot{x}z - \dot{z}x}{x^2 + z^2}
\end{aligned}
$$

再继续求对 l 及其一阶导的表示:

$$
l = x^2 + z^2
$$

对式 (3b) 乘以 $x = l\sin\theta$,对式 (5b) 乘 $z = l\cos\theta$:

$$
\begin{aligned}
\dot{x}x &= l\dot{l}\sin^2\theta + l^2\dot{\theta}\sin\theta\cos\theta \\
\dot{z}z &= l\dot{l}\cos^2\theta - l^2\dot{\theta}\sin\theta\cos\theta
\end{aligned}
$$

以上两式相加那么则有:

$$
\begin{aligned}
l\dot{l} &= \dot{x}x + \dot{z}z \\
\dot{l} &= \frac{\dot{x}x + \dot{z}z}{l} = \frac{\dot{x}x + \dot{z}z}{\sqrt{x^2 + z^2}}
\end{aligned}
$$

最后有:

$$
\theta = \arctan\frac{x}{z} \tag{7a}
$$

$$
\dot{\theta} = \frac{z\dot{x} - x\dot{z}}{x^2 + z^2} \tag{7b}
$$

$$
l = x^2 + z^2 \tag{7c}
$$

$$
\dot{l} = \frac{x\dot{x} + z\dot{z}}{\sqrt{x^2 + z^2}} \tag{7d}
$$

1.2.5 平坦输出表示控制变量

将式 (4b) 乘以 $x = l\sin\theta$;式 (6b) 乘以 $z = l\cos\theta$

$$
\begin{aligned}
x\ddot{x} &= -\frac{kl(l - l_0)\sin^2\theta}{m} + \frac{lf\sin^2\theta}{m} + \frac{\tau\sin\theta\cos\theta}{ml} \\
z\ddot{z} &= -\frac{kl(l - l_0)\cos^2\theta}{m} - gz + \frac{lf\cos^2\theta}{m} - \frac{\tau\sin\theta\cos\theta}{ml}
\end{aligned}
$$

以上两式相加:

$$
\begin{aligned}
x\ddot{x} + z\ddot{z} &= -\frac{kl(l-l_0)}{m} - gz + \frac{lf}{m} \\
f &= \frac{mx\ddot{x} + mz\ddot{z} + kl(l-l_0) + mgz}{l} \\
&= k(\sqrt{x^2+z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{l}
\end{aligned}
$$

再将式 (4b) 乘以 $z = l\cos\theta$；式 (6b) 乘以 $x = l\sin\theta$：

$$
\begin{aligned}
z\ddot{x} &= -\frac{kl(l-l_0)\sin\theta\cos\theta}{m} + \frac{lf\sin\theta\cos\theta}{m} + \frac{\tau\cos^2\theta}{m} \\
x\ddot{z} &= -\frac{kl(l-l_0)\sin\theta\cos\theta}{m} - gx + \frac{lf\sin\theta\cos\theta}{m} - \frac{\tau\sin^2\theta}{m}
\end{aligned}
$$

以上两式相减：

$$
\begin{aligned}
z\ddot{x} - x\ddot{z} &= gx + \frac{\tau}{m} \\
\tau &= mz\ddot{x} - mx\ddot{z} - mgx
\end{aligned}
$$

最后有：

$$
f = k(\sqrt{x^2+z^2} - l_0) + \frac{mx\ddot{x} + mz\ddot{z} + mgz}{\sqrt{x^2+z^2}} \tag{8a}
$$

$$
\tau = mz\ddot{x} - mx\ddot{z} - mgx \tag{8b}
$$

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反馈线性化

$$
\begin{aligned}
f &= m(\ddot{x}_d\sin\theta + \ddot{z}_d\cos\theta + g) + k(l-l_0) - k_1(\dot{x} - \dot{x}_d) - k_2(x - x_d) \\
\tau &= ml(\ddot{x}_d\cos\theta - \ddot{z}_d\sin\theta) - k_3(\dot{z} - \dot{z}_d) - k_4(z - z_d)
\end{aligned}
$$

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$$
\begin{aligned}
\frac{d}{dt}(l\sin\theta\cos\phi) &= \dot{l}\sin\theta\cos\phi + l\dot{\theta}\cos\theta\cos\phi - l\sin\theta\sin\phi\dot{\phi} \\
\frac{d^2}{dt^2}(l\sin\theta\cos\phi) &= \ddot{l}\sin\theta\cos\phi + 2\dot{l}\dot{\theta}\cos\theta\cos\phi - 2\dot{l}\sin\theta\sin\phi\dot{\phi} + l\ddot{\theta}\cos\theta\cos\phi \\
&\quad - l\dot{\theta}^2\sin\theta\cos\phi - 2l\dot{\theta}\sin\theta\sin\phi\dot{\phi} \ - l\cos\theta\sin\phi\ddot{\phi} - l\sin\theta\cos\phi\dot{\phi}^2 \\
\frac{d}{dt}(l\sin\theta\sin\phi) &= \dot{l}\sin\theta\sin\phi + l\dot{\theta}\cos\theta\sin\phi + l\sin\theta\cos\phi\dot{\phi} \\
\frac{d^2}{dt^2}(l\sin\theta\sin\phi) &= \ddot{l}\sin\theta\sin\phi + 2\dot{l}\dot{\theta}\cos\theta\sin\phi + 2\dot{l}\sin\theta\cos\phi\dot{\phi} + l\ddot{\theta}\cos\theta\sin\phi \\
&\quad - l\dot{\theta}^2\sin\theta\sin\phi + 2l\dot{\theta}\sin\theta\cos\phi\dot{\phi} + l\sin\theta\cos\phi\ddot{\phi} - l\sin\theta\sin\phi\dot{\phi}^2 \\
\frac{d}{dt}(l\cos\theta) &= \dot{l}\cos\theta - l\dot{\theta}\sin\theta \\
\frac{d^2}{dt^2}(l\cos\theta) &= \ddot{l}\cos\theta - 2\dot{l}\dot{\theta}\sin\theta - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
Q = {} & 288P_{\text{seg0},4}^2 + 360P_{\text{seg0},4}P_{\text{seg0},5} + 360P_{\text{seg0},4}P_{\text{seg0},6} + 315P_{\text{seg0},4}P_{\text{seg0},7} \\
& + 360P_{\text{seg0},5}P_{\text{seg0},4} + 600P_{\text{seg0},5}^2 + 675P_{\text{seg0},5}P_{\text{seg0},6} + 630P_{\text{seg0},5}P_{\text{seg0},7} \\
& + 360P_{\text{seg0},6}P_{\text{seg0},4} + 675P_{\text{seg0},6}P_{\text{seg0},5} + 787.5P_{\text{seg0},6}^2 + 787.5P_{\text{seg0},6}P_{\text{seg0},7} \\
& + 315P_{\text{seg0},7}P_{\text{seg0},4} + 630P_{\text{seg0},7}P_{\text{seg0},5} + 787.5P_{\text{seg0},7}P_{\text{seg0},6} + 787.5P_{\text{seg0},7}^2 \\
& + 288P_{\text{seg1},4}^2 + 360P_{\text{seg1},4}P_{\text{seg1},5} + 360P_{\text{seg1},4}P_{\text{seg1},6} + 315P_{\text{seg1},4}P_{\text{seg1},7} \\
& + 360P_{\text{seg1},5}P_{\text{seg1},4} + 600P_{\text{seg1},5}^2 + 675P_{\text{seg1},5}P_{\text{seg1},6} + 630P_{\text{seg1},5}P_{\text{seg1},7} \\
& + 360P_{\text{seg1},6}P_{\text{seg1},4} + 675P_{\text{seg1},6}P_{\text{seg1},5} + 787.5P_{\text{seg1},6}^2 + 787.5P_{\text{seg1},6}P_{\text{seg1},7} \\
& + 315P_{\text{seg1},7}P_{\text{seg1},4} + 630P_{\text{seg1},7}P_{\text{seg1},5} + 787.5P_{\text{seg1},7}P_{\text{seg1},6} + 787.5P_{\text{seg1},7}^2
\end{aligned}
$$

(@1*P$_seg0_4$) + (@2 $* P_seg0_5$) + (@2 $* P_seg0_6$) + (@3 $* P_seg0_7$)

@1=576, @2=1440, @3=2880, @4=5040, @5=4800, @6=10800, @7=20160, @8=25920, @9=50400, @10=100800 @1=288, @2=360, @3=315, @4=600, @5=675, @6=630, @7=810, @8=787.5 ((@1*P$_seg0_4$) + (@2 $* P_seg0_5$) + (@2 $* P_seg0_6$) + (@3 $* P_seg0_7$) $* P_seg0_4$) + ((@2 $* P_seg0_4$) + (@4 $* P_seg0_5$) + (@5 $* P_seg0_6$) + (@6 $* P_seg0_7$) $* P_seg0_5$) + ((@2 $* P_seg0_4$) + (@5 $* P_seg0_5$) + (@7 $* P_seg0_6$) + (@8 $* P_seg0_7$) $* P_seg0_6$) + ((@3 $* P_seg0_4$) + (@6 $* P_seg0_5$) + (@8 $* P_seg0_6$) + (@8 $* P_seg0_7$) $* P_seg0_7$)+

((@1*P$_seg1_4$) + (@2 $* P_seg1_5$) + (@2 $* P_seg1_6$) + (@3 $* P_seg1_7$) $* P_seg1_4$) + ((@2 $* P_seg1_4$) + (@4 $* P_seg1_5$) + (@5 $* P_seg1_6$) + (@6 $* P_seg1_7$) $* P_seg1_5$) + ((@2 $* P_seg1_4$) + (@5 $* P_seg1_5$) + (@7 $* P_seg1_6$) + (@8 $* P_seg1_7$) $* P_seg1_6$) + ((@3 $* P_seg1_4$) + (@6 $* P_seg1_5$) + (@8 $* P_seg1_6$) + (@8 $* P_seg1_7$) $* P_seg1_7$)